SPOTTED DISK AND SPHERE GRAPHS II

URSULA HAMENSTÄDT

Abstract. The disk graph of a handlebody H of genus $g \geq 2$ with $m \geq 0$ marked points on the boundary is the graph whose vertices are isotopy classes of disks disjoint from the marked points and where two vertices are connected by an edge of length one if they can be realized disjointly. We show that for $m = 2$ the disk graph contains quasi-isometrically embedded copies of $\mathbb{R}^2$. Furthermore, the sphere graph of the doubled handlebody of genus $g \geq 4$ with two marked points contains for every $n \geq 1$ a quasi-isometrically embedded copy of $\mathbb{R}^n$.

1. Introduction

The *curve graph* $\mathcal{CG}$ of an oriented surface S of genus $g \geq 0$ with $m \geq 0$ punctures and $3g-3+m \geq 2$ is the graph whose vertices are isotopy classes of essential (that is, non-contractible and not homotopic into a puncture) simple closed curves on S. Two such curves are connected by an edge of length one if and only if they can be realized disjointly. The curve graph is a locally infinite hyperbolic geodesic metric space of infinite diameter [MM99].

A handlebody of genus $g \geq 1$ is a compact three-dimensional manifold H which can be realized as a closed regular neighborhood in $\mathbb{R}^3$ of an embedded bouquet of g circles. Its boundary ∂H is an oriented surface of genus g. We allow that ∂H is equipped with $m \geq 0$ marked points (punctures) which we call *spots* in the sequel. The group $\mathrm{Map}(H)$ of all isotopy classes of orientation preserving homeomorphisms of H which fix each of the spots is called the *handlebody group* of H. The restriction of an element of $\mathrm{Map}(H)$ to the boundary ∂H defines an embedding of $\mathrm{Map}(H)$ into the mapping class group of ∂H, viewed as a surface with punctures [S77, Wa98].

An *essential disk* in H is a properly embedded disk $(D, \partial D) \subset (H, \partial H)$ whose boundary ∂D is an essential simple closed curve in ∂H, viewed as a surface with punctures. An isotopy of such a disk is supposed to consist of such disks.

The *disk graph* $\mathcal{DG}$ of H is the graph whose vertices are isotopy classes of essential disks in H. Two such disks are connected by an edge of length one if and only if they can be realized disjointly. Thus by identifying a disk with its boundary circle, the disk graph is a subgraph of the curve graph of ∂H which is invariant under the handlebody group $\mathrm{Map}(H)$.

If H does not have spots, then it is known that the disk graph is a quasi-convex subgraph of $\mathcal{CG}$. This means that for any two points $D, E \in \mathcal{DG}$, any geodesic in $\mathcal{CG}$ connecting ∂D to ∂E is contained in a uniformly bounded neighborhood of $\mathcal{DG}$. However, the inclusion $\mathcal{DG} \to \mathcal{CG}$ is *not* a quasi-isometric embedding [MS13]. More precisely, there are uniformly quasi-isometrically embedded subgraphs of $\mathcal{DG}$,

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so-called *holes*, whose diameter in $\mathcal{DG}$ is arbitrarily large but whose diameter in the curve graph is uniformly bounded.

A metric space X is said to have *asymptotic dimension* $\mathrm{asdim}(X) \leq n$ if for every $R > 0$ there exists a covering of X by uniformly bounded subsets of X so that any ball of radius R intersects at most $n + 1$ sets from the covering. The asymptotic dimension of a curve graph is finite [BF08] (see also [BB19] for a quantitative statement).

In [MS13, H19a, H16, H19b, H21] the following is shown.

Theorem 1. (1) *The disk graph of a handlebody of genus $g \geq 2$ without spots is hyperbolic and has finite asymptotic dimension.*
(2) *The disk graph of a handlebody of genus $g \geq 2$ with a single spot on the boundary contains quasi-isometrically embedded $\mathbb{R}^2$. In particular, it is not hyperbolic.*

The mechanism for part (2) of Theorem 1 consists in taking advantage of specific holes for the disk graph of a handlebody without spots which arise from representing the handlebody as a (possibly non-oriented) I-bundle over a compact surface F with connected boundary together with a point pushing construction about the boundary circle of F.

The first main goal of this work is to extend the second part of Theorem 1 to handlebodies with two spots.

Theorem 2. *The disk graph of a handlebody H of genus $g \geq 2$ with 2 spots on the boundary contains quasi-isometrically embedded $\mathbb{R}^2$. If g is even, then it contains quasi-isometrically embedded $\mathbb{R}^3$. In particular, it is not hyperbolic.*

The proof of Theorem 2 uses the presence of precisely two spots in an essential way, and it is not an extension of the proof of the second part of Theorem 1.

Theorem 2 shows that disk graphs can not be used effectively to obtain a geometric understanding of the handlebody group $\mathrm{Map}(H_0)$ of a handlebody H_0 of genus $g \geq 3$ with no spots paralleling the program developed by Masur and Minsky for the mapping class group [MM00].

The analogue of the strategy of Masur and Minsky would consist in taking advantage of hyperbolicity of the disk graph on which the handlebody group acts coarsely transitively. One then analyzes the point stabilizers for this action. That this is a valuable approach for the geometric study of the handlebody group follows from the fact that the stabilizer of a disk is an undistorted subgroup of the handlebody group [He21]. But the second part of Theorem 1 and Theorem 2 yield that the geometry of the stabilizer of a disk can not be studied effectively by cutting a handlebody open along an embedded disk which results in a (perhaps disconnected) handlebody with two spots on the boundary and, in an inductive procedure, using knowledge of the geometry of the disk graph of the cut open handlebody. This failure of such an inductive approach may be a witness for the fact that $\mathrm{Map}(H_0)$ is an exponentially distorted subgroup of the mapping class group of ∂H_0 [HH12], and its Dehn function is exponential [HH21].

Theorem 2 has a stronger analogue for geometric graphs related to the outer automorphism group $\mathrm{Out}(F_g)$ of the free group on g generators. Namely, doubling the handlebody H yields a connected sum $M = \sharp_g S^2 \times S^1$ of g copies of $S^2 \times S^1$ with m marked points. A doubled disk is an embedded essential sphere in M, that is, a sphere which is not homotopically trivial or homotopic into a marked point.

The *sphere graph* of M is the graph whose vertices are isotopy classes of essential spheres in M and where two such spheres are connected by an edge of length one if and only if they can be realized disjointly. As before, an isotopy of spheres is required to be disjoint from the marked points. The sphere graph of a doubled handlebody without marked points is hyperbolic [HM13b]. If g is even, then the sphere graph of a doubled handlebody with one marked point on the boundary contains quasi-isometrically embedded $\mathbb{R}^2$ [H21].

The following is the second main result of this article.

Theorem 3. *The sphere graph of a doubled handlebody of genus $g \geq 4$ with 2 marked points contains for every $n \geq 1$ a quasi-isometrically embedded copy of $\mathbb{R}^n$. In particular, it is not hyperbolic, and its asymptotic dimension is infinite.*

The sphere graph of a doubled handlebody of genus $g \geq 2$ with two marked points is isomorphic to the subgraph of the sphere graph of a doubled handlebody $\sharp_{g+1}S^1 \times S^2$ of genus $g+1$ with no marked point consisting of all spheres which are disjoint from a fixed non-separating sphere (see Section 4 for details). As in the case of disks in a handlebody and the handlebody group, for $g \geq 3$ the stabilizer of a sphere in $\sharp_g S^1 \times S^2$ is an undistorted subgroup of $\mathrm{Out}(F_g)$ [HM13a] and thus Theorem 3 may among others witness the fact that the Dehn function of $\mathrm{Out}(F_g)$ is exponential [BV12]. Note that unlike for the disk graph, it seems to be unknown whether or not for $h \geq 3$ the sphere graph of $\sharp_h S^1 \times S^2$ has finite asymptotic dimension.

The first example known to us of a geometric graph of infinite asymptotic dimension is due to Sabalka and Savchuk [SS14]. The vertices of this graph are isotopy classes of essential separating spheres in $\sharp_g S^2 \times S^1$. Two such spheres are connected by an edge of length one if and only if they can be realized disjointly. We use the main construction of [SS14] for the proof of Theorem 3.

Note that Theorem 2 and Theorem 3 do not exclude the possibility that the graph of non-separating disks or non-separating spheres in a handlebody with two spots or a doubled handlebody with two spots is hyperbolic.

The article is subdivided into four sections. In Section 2, we introduce disks in a handlebody of genus $g \geq 2$ enclosing the two spots. We show how one can pass from a disk D enclosing the two spots to another disk E enclosing the two spots in two explicit but different ways with a point pushing diffeomorphism. Namely, we can use point pushing of either of the two spots, resulting in different elements of the handlebody group. The effect on disks of such point pushing transformations can be controlled provided that they are supported in disjoint subsurfaces of the boundary of the handlebody. As a byproduct of this analysis, we obtain the following statement of independent interest.

Theorem 4. *Let Σ be a compact oriented surface with nonempty boundary $\partial\Sigma$, possibly with a finite number of interior points (punctures) removed. Let $p_1 \in \partial\Sigma$ be a fixed point and let $p_2 \in \Sigma$ be a marked interior point. Then point pushing the marked point p_2 determines a bijection between $\pi_1(\Sigma, p_2)$ and the set $\mathcal{A}(p_1, p_2)$ of isotopy classes of arcs in Σ connecting p_1 and p_2.*

The boundary of a handlebody H of even genus $2h$ contains preferred subsurfaces carrying the entire topology of the handlebody. These subsurfaces are compact surfaces F of genus h with connected boundary so that H is an orientable I-bundle over F (that is, a fiber bundle over F with fiber the interval $[0,1]$). The orientation reversing involution Ψ of H which exchanges the endpoints of the fibers maps F to a disjoint subsurface of the boundary. If we choose one of the spots p in the interior of F and assume that the second spot equals $\Psi(p)$, then point pushing of p along loops in F commutes with point pushing of $\Psi(p)$ along loops in $\Psi(F)$. This is explained in detail in Section 3. A variation of this construction extends to handlebodies of odd genus. We use this to prove Theorem 2.

In Section 4, we apply the discussion in Section 3 and the main construction of [SS14] to the double of a handlebody with two spots and establish Theorem 3.

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2. Disks enclosing the spots

The goal of this section is to introduce disks enclosing the two spots, establish some first properties of these disks and prove Theorem 4. To simplify the terminology, when we talk about disks in the sequel, we always identify disks which are isotopic in the sense explained in the introduction.

We begin with discussing briefly handlebodies of genus one. A handlebody of genus one with at most one spot on the boundary contains a single disk up to isotopy. This is used to establish

Proposition 2.1. *The disk graph of a solid torus with two spots on the boundary is a tree.*

Proof. Let H be a solid torus with two spots p_1, p_2 on the boundary. The handlebody H_1 obtained from H by removing the spot p_2 is a solid torus with one spot on the boundary. Let $\Phi_1 : \partial H \to \partial H_1$ be the natural spot removal map.

The handlebody H_1 contains a single disk D_1, and this disk is non-separating. If $D \subset H$ is any non-separating disk then $\Phi_1(\partial D) = \partial D_1$. Thus by Theorem 7.1 of [KLS09], the complete subgraph of the disk graph of H whose vertex set is the set of non-separating disks in H is a tree T. This is the Bass-Serre tree for the splitting of $\pi_1(\partial H_1, p_2) = F_2$ defined by D_1. Equivalently, it is the tree dual to the curve ∂D_1 and its images under the action of $\pi_1(\partial H_1, p_2)$.

If $D \subset H$ is a separating disk then ∂D decomposes ∂H into a disk with two spots and a torus with the interior of a closed disk removed. In particular, $\Phi_1(\partial D)$ is peripheral. There is a single disk in H which is disjoint from D, and this disk is non-separating. Thus there is a single edge in $\mathcal{DG}$ with one endpoint at D. The second endpoint is a vertex in the simplicial tree T.

As a consequence, the disk graph of H is an extension of the simplicial tree T which attaches to each vertex of T an at most countable collection of edges whose second endpoints are univalent. Thus this graph is a tree as well. The proposition follows. $\square$

Remark 2.2. The disk graph of a solid torus H with two spots on the boundary is a tree with countable valency. Namely, if D is any non-separating disk in H then cutting H open along D yields a ball with four spots on the boundary. Two of these spots are the two copies of D. Any simple closed curve which separates these two distinguished spots from the remaining two spots is the boundary of a separating disk in H disjoint from D, and any separating disk disjoint from D arises in this way. There are countably many such disks.

Remark 2.3. It also follows from similar considerations that the disk graph of a handlebody H with two spots is connected (for which the usual surgery argument [H19a] is problematic as surgery may lead to peripheral disks). Namely, denote by H_1 the handlebody obtained from H by removing the spot p_2, and let $\Phi_1 : H \to H_1$ be the spot forgetful map. By Theorem 7.1 of [KLS09], for any non-separating disk $D \subset H$, the preimage of $\Phi_1(D) \subset H_1$ under the map Φ_1 is the Bass Serre tree for the graph of groups decomposition of $\pi_1(\partial H_1, p_2)$ defined by D and the point $p_2 \in \partial H_1 - \Phi_1(D)$. In particular, this preimage is connected. On the other hand, any disk in H is disjoint from a disk which projects onto a non-separating disk in H_1. As the disk graph $\mathcal{DG}_1$ of H_1 is easily seen to be connected and furthermore there is a simplicial embedding $\Lambda : \mathcal{DG}_1 \to \mathcal{DG}$ so that $\Phi_1 \circ \Lambda = \mathrm{Id}$ (see Section 7 of [KLS09]), this yields that $\mathcal{DG}$ is indeed connected.

In the sequel we always denote by H a handlebody of genus $g \geq 2$ with two spots p_1, p_2 on the boundary, and we denote by H_0 the handlebody obtained from H by removing the spots. There is a natural spot removal map

$$\Phi : H \to H_0.$$

We say that a disk D *encloses the spots* if $\Phi(D) \subset H_0$ is contractible. Equivalently, the boundary of D is a simple closed curve in ∂H which bounds a twice punctured disk $\tilde{D} \subset \partial H$, with punctures at p_1, p_2.

Let $E \subset H$ be another disk which encloses the two spots p_1, p_2. Assume that E is in minimal position with respect to D. This means in particular that the boundaries $\partial D, \partial E$ intersect in the minimal number of points among all representatives in their isotopy classes.

The simple closed curves $\partial D, \partial E$ are the boundaries of unique disks $\tilde{D}, \tilde{E} \subset \partial H$ containing the two spots (thus if we think of the spots as missing points, then $\tilde{D}, \tilde{E}$ should be viewed as twice punctured disks). If D is not isotopic to E then the intersection $\tilde{D} \cap \tilde{E}$ consists of two disjoint disks A_1, A_2 with one spot at p_1, p_2, respectively, and a disjoint union of rectangles. In particular, $\partial D \cap \partial E$ consists of at least four points.

Denote as before by H_1 the handlebody obtained from H by removing the spot p_2 and let $\Phi_1 : H \to H_1$ be the spot forgetful map. Use the ordered pair of disks $(\tilde{D}, \tilde{E})$ to construct a loop $\gamma \subset (\partial H \cup \{p_2\}, p_2)$ based at p_2 as follows. First, connect the point p_1 to the point p_2 by an oriented arc α, that is, the image of the closed interval $[0, 1]$ under a topological embedding, whose interior is embedded in $\tilde{D}$. The endpoints of α are the two spots of H, and they are precisely the spots of the disk $\tilde{E}$. Thus there is an arc β in $\tilde{E}$ connecting p_1 to p_2. The loop γ is homotopic to the concatenation of α^{-1} with β (which we move off the spot p_1 with a small deformation). Note that the inverse of the loop γ is constructed with exactly the same procedure, but with the roles of the disks D, E exchanged. Furthermore, the homotopy class of γ as a loop in ∂H_1 based at p_2 is uniquely determined by the ordered pair (D, E) up to the precomposition with the homotopy class of a loop in $\tilde{D}$ based at p_2 which surrounds the marked point p_1.

The fundamental group $\pi_1(\partial H_1, p_2)$ of ∂H_1 is the free group in $2g$ generators. Let $c \in \pi_1(\partial H_1, p_2)$ be the element which can be represented by a loop in $\tilde{D}$ surrounding the marked point p_1. If we write composition from left to right, then the above discussion shows that each ordered pair (D, E) of disks in H enclosing the two spots determines uniquely the right coset of a homotopy class in $\pi_1(\partial H_1, p_2)$ by the infinite cyclic group generated by c.

To avoid working with cosets, we now replace the spot p_1 in ∂H_1 by a boundary component. Let Σ be the resulting bordered surface. Attaching to the boundary $\partial\Sigma$ of Σ a disk yields the boundary of the handlebody H_0 without spot. Fix a point p_1 in $\partial\Sigma$ and denote by $\mathcal{A}(p_1, p_2)$ the set of isotopy classes of arcs in Σ with fixed endpoints p_1, p_2. Such an arc can be viewed as an arc in ∂H with endpoints at the spots, and a thickening of such an arc defines a disk $\tilde{E} \subset \partial H$ enclosing the two spots. Vice versa, any disk in H enclosing the two spots determines an arc in $\mathcal{A}(p_1, p_2)$, unique up to the ambiguity of Dehn twisting the spot p_1 about the boundary of Σ. Thus understanding isotopy classes of disks enclosing the two spots amounts to understanding $\mathcal{A}(p_1, p_2)$.

More generally, let for the moment Σ be any compact surface with non-empty boundary $\partial\Sigma$ and perhaps a finite number of points removed. Choose a point p_2 in the interior of Σ. This choice determines a subgroup of the mapping class group $\mathrm{Mod}(\Sigma - \{p_2\})$ of $\Sigma - \{p_2\}$ which is isomorphic to the fundamental group $\pi_1(\Sigma, p_2)$ of the surface Σ. This group is the fiber group of the *Birman exact sequence*

$$0 \to \pi_1(\Sigma, p_2) \to \mathrm{Mod}(\Sigma - \{p_2\}) \to \mathrm{Mod}(\Sigma) \to 0$$

obtained from the map $\Sigma - \{p_2\} \to \Sigma$ which forgets the spot p_2. Its elements are called *point pushing maps.* If $\gamma \subset \Sigma$ is a based loop at p_2, then the point pushing map along γ can be represented by a diffeomorphism supported in an arbitrarily small neighborhood of γ.

Let as before $\mathcal{A}(p_1, p_2)$ be the set of isotopy classes of arcs with endpoints p_1, p_2. For ease of exposition, we view such arcs α, β as arcs connecting p_1 to p_2. Then the composition $\alpha^{-1} \circ \beta$ (read from left to right) is a based loop at p_2 whose homotopy class we denote by $q(\alpha, \beta)$. Note that $q(\alpha, \beta) \in \pi_1(\Sigma, p_2)$ only depends on the isotopy class of α, β. For easier exposition, in the sequel we always represent isotopy classes of arcs by actual arcs and note that the statements we make do not depend on the choice of such representatives.

Lemma 2.4. *Let $\alpha, \beta \in \mathcal{A}(p_1, p_2)$. Then β is the image of α under the element of the mapping class group of $\Sigma - \{p_2\}$ obtained by pushing the point p_2 along a loop in the homotopy class $q(\alpha, \beta)$.*

Proof. Let $\gamma \subset \Sigma \cup \{p_2\}$ be a loop based at p_2 constructed as above from the ordered pair (α, β) of arcs, and moved off $\Sigma \ni p_1$.

Assume first that the loop γ is simple, that is, it does not have self-intersections. Thus γ is embedded in Σ and hence there is an embedded annulus $A \subset \Sigma$ with core curve γ, disjoint from p_1. Up to isotopy, the arc α intersects A in a single embedded segment α_0 with one endpoint p_2. Furthermore, we may assume that α_0 intersects the core curve γ of A in the unique point p_2.

The point pushing homeomorphism of p_2 along γ equals the identity on $(\Sigma - A) \cup \gamma$. In each of the two components of $A - \gamma$, it is a Dehn twist about the core curve (which is freely homotopic to γ), in one component positive, in the second negative. As α_0 and hence α meets only one component of $A - \gamma$, this homeomorphism transforms the homotopy class of α with fixed endpoint by concatenation with γ. As a consequence, the image of the arc α by the point pushing map along γ is homotopic to the concatenation of α with γ (read from left to right). This arc is homotopic with fixed endpoints to β, and in fact isotopic to β.

This construction extends to the case that the curve γ has self-intersections. Namely, parameterize the arc β on the interval $[0,1]$. Assuming that β is in minimal position with respect to α, let $0 = t_0 < t_1 < \cdots < t_k = 1$ be such that $\beta(t_i)$ are the intersection points of β with α, with the endpoints included. Let γ_1 be the concatenation of α^{-1} with an arc ζ_1 connecting p_1 to p_2 which is composed of $\beta[0,t_1]$ and the subarc α_1 of α connecting $\beta(t_1)$ back to p_2. With a small homotopy with fixed endpoints, the arc ζ_1 can be pushed off the interior of α. The resulting loop based at p_2 is simple and can be pushed off p_1. Denote this loop again by γ_1.

In a second step, define a based loop γ_2 at p_2 as follows. Let $2 \leq j \leq k$ be such that $\beta(t_j)$ is the first intersection point of $\beta(t_1,1]$ with the subarc α_1 of α (perhaps this is the endpoint p_2 of β). Let γ_2 be the concatentation of ζ_1^{-1}, the subarc $\beta[0,t_j]$ of β and the subarc α_2 of α connecting $\beta(t_j)$ back to p_2. This loop is homotopic with fixed endpoints to the concatenation of α_1^{-1}, the arc $\beta[t_1,t_j]$ and the arc α_2 and hence it is simple. Furthermore, it can also be described as the concatentation of ζ_1^{-1} and an arc ζ_2 connecting p_1 to p_2.

Proceeding inductively, define for $\ell \leq m$ (where $m \leq k$ is a number computed from the order in which the points of $\alpha \cap \beta$ are passed through by β) a based loop γ_ℓ at p_2 composed of the arc $\zeta_{\ell-1}^{-1}$ and an arc ζ_ℓ connecting p_1 to p_2 which is a concatenation of a subarc $\beta[0,t_{j(\ell)}]$ of β and the subarc of α connecting $\beta(t_{j(\ell)})$ back to p_2. Up to homotopy, the loops γ_ℓ are simple, and we have $\gamma = \gamma_1 \circ \cdots \circ \gamma_m$ (read from left to right).

By the first part of this proof, the image of α under point pushing along γ_1 is the arc ζ_1. By the above construction, the arc ζ_1 intersects γ_2 only at the point p_2. Thus using again the first part of this proof, point pushing of ζ_1 along the loop γ_2 yields the arc ζ_2. As the point pushing group is a group, we also know that ζ_2 is the image of α under point pushing along $\gamma_1 \circ \gamma_2$ (read from left to right). In $m \leq k$ such steps we deduce that indeed, the arc β is isotopic to the image of α under pointpushing of p_2 along γ. $\square$

As a corollary, we obtain Theorem 4 from the introduction.

Corollary 2.5. *Let $\alpha \in \mathcal{A}(p_1,p_2)$ be a fixed basepoint. Then the map which associates to an element $\gamma \in \pi_1(\Sigma,p_2)$ the image of α under point pushing along γ defines a bijection $\pi_1(\Sigma,p_2) \to \mathcal{A}(p_1,p_2)$.*

Proof. Lemma 2.4 shows that up to homotopy, any arc $\beta \in \mathcal{A}(p_1,p_2)$ can be obtained from the fixed arc α by point pushing along a loop γ. Since two point pushing homeomorphisms along homotopic loops are isotopic, the isotopy class of the resulting arc only depends on the homotopy class of the loop γ. Thus the map Λ which associates to a homotopy class $[\gamma] \in \pi_1(\Sigma,p_2)$ the isotopy class of the image of α by point pushing along a based loop in the class $[\gamma]$ is surjective.

On the other hand, using again Lemma 2.4 and its proof, if β is obtained from α by point pushing along a based loop γ, then the homotopy class of β with fixed endpoints equals the homotopy class of the concatenation $\alpha \circ \gamma$. This means that point pushing along non-homotopic loops gives rise to arcs in different homotopy classes with fixed endpoints and hence to arcs which are not isotopic. This shows that the map Λ is injective as well. $\square$

Corollary 2.5 needs to be slightly modified to obtain a statement about disks in the handlebody H enclosing the two spots, or, equivalently, arcs with fixed endpoints on a compact oriented surface, where the endpoints are interior points of the surface. Namely, looking again at two disks $\tilde{D}, \tilde{E}$ enclosing the two spots p_1, p_2 and arcs $\alpha \subset \tilde{D}, \beta \subset \tilde{E}$ connecting the two spots, point pushing $\tilde{D}$ along $\gamma = \alpha^{-1} \circ \beta$ requires moving γ off p_1 which depends on a choice. Two choices differ by the homotopy class of a based loop at p_2 which is entirely contained in $\tilde{D}$ and encloses the spot p_1. The next lemma shows that the disk E is independent of the choice made.

Lemma 2.6. *The isotopy class of the disk $\tilde{D}$ is fixed by point pushing p_2 along a loop ζ based at p_2 which is entirely contained in the disk $\tilde{D}$ and encircles p_1.*

Proof. Up to homotopy with fixed basepoint, the loop ζ is embedded in a disk $\hat{D} \subset \tilde{D}$ which is isotopic to $\tilde{D}$, in particular, it contains the two marked points p_1, p_2 in its interior, and whose closure in ∂H is contained in the interior of $\tilde{D}$. Point pushing of p_2 along ζ can be represented by a diffeomorphism which fixes $\partial H - \hat{D}$ and hence $\partial \tilde{D}$ pointwise. As a consequence, point pushing of p_2 along ζ preserves the disk $\hat{D}$. ☐

Remark 2.7. Replacing one of the spots by a boundary component and marking a point on this component removes the ambiguity in the point pushing construction.

Let us summarize what we obtained so far. Let $D \subset H$ be a disk enclosing the two spots p_1, p_2. Its boundary ∂D is a simple closed curve in ∂H. It determines the homotopy class of an arc in ∂H_0 with endpoints p_1, p_2. Let us choose such an arc $\alpha \subset \partial H_0$, oriented in such a way that it connects p_1 to p_2, and let a be its homotopy class with fixed endpoints as an arc in ∂H_0.

Let $E \subset H$ be another such disk which determines the homotopy class b of an oriented arc $\beta \subset \partial H_0$ connecting p_1, p_2. The concatenation $\alpha^{-1} \circ \beta$ (read from left to right) is a loop based at p_2. It defines a right coset $a^{-1} \cdot b \in \langle c \rangle \backslash \pi_1(\partial H_1, p_2)$ where c is a loop encircling the spot p_1 which is entirely contained in the twice punctured disk $\tilde{D} \subset \partial H$ whose boundary equals the boundary of D.

Move as before $\alpha^{-1} \circ \beta$ off p_1. Point pushing p_2 along the this defined loop $\alpha^{-1} \circ \beta$ in ∂H_1 defines the isotopy class of a homeomorphism of ∂H. It can be represented by a homeomorphism which equals the identity on the complement of a small neighborhood of $\alpha^{-1} \circ \beta$, and we may assume that the point p_1 is not contained in this neighborhood. The image of the disk D under this homeomorphism is the disk E. Thus Lemma 2.4 describes an algorithm which begins with a pair of disks (D, E) enclosing the two spots and determines from this pair up to an ambiguity arising from choosing how to move off the spot p_1 a point pushing homeomorphism of ∂H which transforms D into E. As this ambiguity corresponds to the choice of a representative of the corresponding coset $\langle c \rangle \backslash \pi_1(\partial H \cup \{p_2\}, p_2)$, we conclude the following

Corollary 2.8. *The point pushing map induces a bijection from the set of cosets $\langle c \rangle \backslash \pi_1(\partial H \cup \{p_2\}, p_2)$ onto the set of isotopy classes of disks in H enclosing the two spots.*

The same construction is also valid for point pushing the point p_1. In this case, the point pushing map constructed in the above fashion which transforms D into E is point pushing of p_1 along the based loop $\alpha \circ \beta^{-1}$, moved off p_2.

3. I-bundles and disk graphs

While Lemma 2.4 and Corollary 2.5 do not use specific properties of disks and can be viewed as statements about essential simple closed curves on ∂H which become homotopically trivial after closing the spots, we now turn to the study of disks enclosing the two spots as vertices of the disk graph of H. Note that the description of a disk E enclosing the two spots as the image of a disk D by point pushing along an element in $\pi_1(\partial H \cup \{p_2\}, p_2)$ established in Lemma 2.4 coarsely determines a path in the curve graph of ∂H connecting ∂D to ∂E (that is, consecutive vertices intersect in uniformly few points and hence are uniformly close in the curve graph), but such a path may be highly inefficient. In fact, as the curve graph of ∂H is hyperbolic and such paths are also paths in the disk graph of H, it can be deduced from the proof of Theorem 2 below that this is indeed the case.

Following [H19a, H16], define an *I-bundle generator* for H_0 to be a simple closed curve $c \subset \partial H_0$ so that H_0 can be realized as an I-bundle over a compact surface F with connected boundary ∂F and such that c is the core curve of the vertical boundary of the I-bundle.[1] This vertical boundary is an annulus bounded by the two preimages of ∂F. The surface F is called the *base* of the I-bundle. If the I-bundle generator c is separating, then F is orientable of genus $g/2$ where g is the genus of H_0. If c is non-separating, then the surface F is non-orientable, and the complement of an open annulus about c in ∂H_0 is the orientation cover of F. The I-bundle over every essential arc in F with endpoints on ∂F is an essential disk in H_0 which intersects c in precisely two points (up to isotopy).

An I-bundle generator c in ∂H_0 is *diskbusting*, which means that it has an essential intersection with every disk (see [MS13, H19a]). Namely, the base F of the I-bundle is a deformation retract of H_0. Thus if γ is any essential closed curve on ∂H_0 which does not intersect c then γ projects to an essential closed curve on F. Such a curve is not nullhomotopic in H_0 and hence it can not be diskbounding.

The *arc graph* $\mathcal{A}(X)$ of a compact surface X of genus $n \geq 1$ with connected boundary ∂X and possibly marked points (punctures) in the interior of X is the graph whose vertices are isotopy classes of essential arcs in X with endpoints on the boundary, and isotopies are allowed to move the endpoints of an arc along ∂X. Two such arcs are connected by an edge of length one if and only if they can be realized disjointly. The arc graph $\mathcal{A}(X)$ of X is hyperbolic, however the inclusion of $\mathcal{A}(X)$ into the arc and curve graph of X is a quasi-isometry only if X is of genus one, with at most one marked point [MS13] (see also [H16]).

For an I-bundle generator c in ∂H_0 let $\mathcal{RD}(c)$ be the complete subgraph of the disk graph $\mathcal{DG}_0$ of H_0 consisting of disks which intersect c in precisely two points. Each such disk is an I-bundle over an arc in the base F of the I-bundle corresponding to c. We refer to the discussion preceding Lemma 2.1 of [H21] for details. As two such disks are disjoint if and only if the corresponding arcs in F are disjoint, the graph $\mathcal{RD}(c)$ is isometric to the arc graph $\mathcal{A}(F)$ of F. The following is a consequence of [MS13, H16, H19a]. We refer to Lemma 2.2 of [H21] for a detailed proof.

[1] The assumption of connectedness of the boundary of the base of the I-bundle is not used in [H19a, H16] but it is needed here to ensure that the I-bundle generator is diskbusting.

Lemma 3.1. *For each I-bundle generator c of ∂H_0, the inclusion $\mathcal{RD}(c) \to \mathcal{DG}_0$ is a quasi-isometric embedding.*

As the I-bundle over an arc with endpoints on ∂F intersects the curve c in precisely two points, such I-bundles over arcs with large distance in the arc graph of F give rise to disks with large distance in the disk graph of H_0. However, the distance in the curve graph of their boundary circles is at most 4. In [MS13], such a subspace of the disk graph is called a hole.

Let as before F be the base of the I-bundle determined by the I-bundle generator c. Let Ψ be the involution of H_0 which exchanges the two endpoints of the intervals in the interval bundle. Its fixed point set intersects ∂H_0 in the I-bundle generator c. Let $\tilde{F} \subset \partial H_0$ be the preimage of F; this is the complement of an annulus in ∂H_0 with core curve c. Choose a point $p_1 \in \partial \tilde{F}$, let $p_2 = \Psi(p_1)$ and define $H = H_0 - \{p_1, p_2\}$. The boundary of the thickening of the interval with endpoints p_1, p_2 is the boundary of a disk D enclosing the two spots. A disk in $\mathcal{RD}(c)$ which is disjoint from p_1, p_2 defines a disk in H which is invariant under Ψ up to isotopy and is disjoint from D.

Push the points $p_1, p_2 = \Psi(p_1)$ slightly into the interior of $\tilde{F}$ so that $\tilde{F}$ can be thought of as a two-sheeted cover of a surface F^+ with connected boundary and one marked point (spot) in its interior. Let $\mathcal{A}(F^+)$ be the arc graph of the surface F^+, and let $\mathcal{RD}^+(c)$ be the subgraph of the disk graph $\mathcal{DG}$ of H whose vertices are Ψ-invariant disks, with boundaries intersecting the fixed point set of Ψ in precisely two points.

For some $\ell \geq 1$, a *coarse ℓ-Lipschitz retraction* of a geodesic metric space X onto a subspace $Y \subset X$ is a coarse ℓ-Lipschitz map $E : X \to Y$ (that is, a map which is ℓ-Lipschitz up to a uniform additive constant) such that there exists a number $k > 0$ with $d(Ey, y) \leq k$ for all $y \in Y$. The subspace $Y \subset X$ is quasi-isometrically embedded in X. Since X is a length space and quasi-geodesics do not have to be continuous, this is equivalent to stating that any two points in Y can be connected by a uniform quasi-geodesic in X which is entirely contained in Y. In analogy to Lemma 3.1, we have

Lemma 3.2. *There exists a coarse two-Lipschitz retraction $\Omega : \mathcal{DG} \to \mathcal{RD}^+(c)$. Furthermore, $\mathcal{RD}^+(c)$ is isometric to the arc graph $\mathcal{A}(F^+)$.*

Proof. A disk which is Ψ-invariant and whose boundary intersects the fixed point set of Ψ in precisely two points is the I-bundle over an arc in F^+, and two such disks are disjoint if and only if their defining arcs in F^+ are disjoint. Furthermore, the I-bundle over any nontrivial arc α in F^+ is a disk in H. Thus the map which associates to an arc $\alpha \in \mathcal{A}(F^+)$ the I-bundle over α is an isometry of $\mathcal{A}(F^+)$ onto $\mathcal{RD}^+(c)$.

Let $d_{\mathcal{DG}}$ be the distance in the disk graph $\mathcal{DG}$ of H. To construct a coarse Lipschitz retraction $\Omega : \mathcal{DG} \to \mathcal{RD}^+(c)$ we proceed along the lines of the proof of Lemma 2.2 of [H21]. Assume first that the I-bundle generator c is separating. Then F^+ can be identified with a subsurface of ∂H. There are two different choices for such an identification, and we pick one of them.

Let $D \subset H$ be any disk. Since the two marked points on ∂H are contained in different components of $\partial H - c$ and the image of c under the spot forgetful map $\Phi : H \to H_0$ is diskbusting, the boundary of D intersects c and hence F^+. Take any component of $F^+ \cap \partial D$ and associate to D the Ψ-invariant disk $\Omega(D)$ which is the I-bundle over this intersection component. The disk $\Omega(D)$ coarsely does not depend on choices: Another choice of intersection arc gives rise to a disjoint disk. Moreover, the images under Ω of disjoint disks are disjoint and hence this construction defines a coarse one-Lipschitz map $\Omega : \mathcal{DG} \to \mathcal{RD}^+(c)$ which satisfies $\Omega(D) = D$ for every $D \in \mathcal{RD}^+(c)$. Thus Ω is a coarse one-Lipschitz retraction, and the inclusion $\mathcal{RD}^+(c) \to \mathcal{DG}$ is indeed a quasi-isometric embedding. This completes the proof for separating I-bundle generators.

This construction can be modified to cover the case of a non-separating I-bundle generator c as well, that is, when F^+ is a non-orientable surface with one marked point. Namely, a non-orientable surface F^+ of Euler characteristic $-2h$ with connected boundary ∂F and one marked point can be represented as the connected sum of an orientable surface of genus h with connected boundary and one marked point in the interior and a projective plane. Equivalently, F^+ contains an orientable subsurface $F_0^+ \subset F^+$ with two boundary components $\partial F^+, e$, and F^+ is obtained from F_0^+ by gluing a Möbius band to the boundary component e. The fundamental group of F_0^+ is an index two subgroup of the fundamental group of F^+. As the surface F_0^+ is oriented, its preimage in the orientation cover $\tilde{F}^+$ of F^+ consists of two disjoint copies of F_0^+, and $\tilde{F}^+$ is obtained from these two copies of F_0^+ by connecting the two components e_1, e_2 of the preimage of e with an annulus (which is the orientation cover of the Möbius band). The oriented I-bundle over F^+ contains the trivial I-bundle over the bordered subsurface F_0^+ as a submanifold.

Identify F_0^+ with a subsurface of ∂H containing the marked point p_1. The boundary of F_0^+ consists of a simple closed curve c_1 isotopic to c and a component e_1 of the preimage of the boundary of the Möbius band. Denote as before by Ψ the orientation reversing involution of H defined by the I-bundle. Let $D \in \mathcal{DG}$ be any disk. Since an I-bundle generator in H_0 is diskbusting and the surface F_0^+ contains one but not both of the marked points, the boundary curve ∂D of D intersects the surface F_0^+ nontrivially in a collection of pairwise disjoint arcs. If there is such an arc α with both endpoints on c_1 then the I-bundle over α is a disk. Define $\Omega(D)$ to be this disk. Its intersection with F_0^+ is disjoint from ∂D. The disk $\Omega(D)$ is coarsely well defined, that is, choosing another component of $\partial D \cap F_0^+$ with both endpoints on c_1 gives rise to a disjoint disk.

Assume next that ∂D does not contain an arc with both endpoints on c_1 but that there is such an intersection arc α with one endpoint on c_1 and the second endpoint y on e_1. Let $\hat{\alpha}$ be the projection of α to F^+. Connect the projection of $y \in e_1$ to the core curve of the Möbius band, attach an arc making one full turn around the core curve of the Möbius band (in either direction), connect back to the projection of y and backtrack to ∂F^+ with the inverse of $\hat{\alpha}$. Up to homotopy, this construction defines an embedded arc in F^+ with both endpoints on ∂F^+. Define $\Omega(D) \in \mathcal{RD}(c)^+$ to be the I-bundle over this arc. The disk $\Omega(D)$ depends on the choice of the component α of $\partial D \cap F_0^+$ and on the choice of the direction of the loop around the core curve of the Möbius band. However, any two distinct choices give rise to disks whose boundaries intersect in at most two points contained in the preimage of the Möbius band. Such disks are disjoint from the I-bundle over some arc in F_0^+ with both endpoints on c_1 and hence their distance in $\mathcal{DG}$ is at most two.

Finally if every component of $\partial D \cap F_0^+$ is an arc with both endpoints on e_1 then up to homotopy, ∂D is disjoint from the simple closed curve c in ∂H whose projection to ∂H_0 is diskbusting. As the projection of ∂D to ∂H_0 is diskbounding, this implies that the projection of ∂D to H_0 is contractible, in other words, D encloses the two spots. In this case let α be an arc in F_0^+ disjoint from ∂D which connects the spot $p_1 \in F_0^+$ to a point $y \in e_1$. Let $\hat{\alpha}$ be the projection of α into F^+. It connects the spot in F^+ to the boundary e of the Möbius band. As in the previous paragraph, connect the projection of $y \in e_1$ to the core curve of the Möbius band, attach an arc making one full turn around the core curve of the Möbius band (in either direction) and connect back to the spot with the inverse of $\hat{\alpha}$. Up to homotopy, this defines an embedded essential loop in F^+ based at the spot which lifts to a Ψ-invariant arc in $\tilde{F}^+$ connecting the two spots. Define $\Omega(D)$ to be the disk enclosing the two spots whose boundary is the boundary of the thickening of this arc. Note that for a fixed choice of $\hat{\alpha}$, this construction only depends on the choice of the direction of the arc going around the Möbius band, that is, any two distinct choices intersect in precisely four points near the spots and are uniformly close in the disk graph.

Let $\Omega : \mathcal{DG} \to \mathcal{RD}^+(c)$ be the coarsely defined map constructed above. We claim that Ω is coarsely two-Lipschitz.

To show that this is the case it suffices to show that if D, D' are disjoint, then their images intersect in at most two points. That this holds indeed true for disks whose intersections with F_0^+ contain an arc with at least one endpoint on c_1 is immediate from the above discussion. If this is not the case for say the disk D, then D encloses the two spots. Then D' does not enclose the two spots as two disks enclosing the two spots are not disjoint. But then for suitable choices made in the above construction, the boundaries of the disks $\Omega(D), \Omega(D')$ intersect in at most four points, and these points are contained in the preimage of the Möbius band. Once again, the distance between $\Omega(D), \Omega(D')$ is at most two.

To complete the proof of the lemma it now suffices to show that if $D \in \mathcal{RD}^+(c)$ then the distance between D and $\Omega(D)$ is uniformly bounded. To this end note that if $\partial D \cap F_0^+$ contains an arc with both endpoints on c_1 then $\Omega(D) = D$.

If $\partial D \cap F_0^+$ contains an arc with one endpoint on c_1 and the second endpoint on e_1, then the two arcs $\alpha, \beta \subset F^+$ which define $\partial D, \partial\Omega(D)$ may not be disjoint. However, by construction, up to homotopy these arcs only intersect in the interior of the Möbius band. As we can find an arc in the surface F^+ which is disjoint from both α, β as well as the Möbius band, we conclude that the distance in $\mathcal{A}(F^+)$ between α and β is at most two. As a consequence, the distance between D and $\Omega(D)$ in $\mathcal{RD}^+(c)$ is at most two.

We are left with looking at disks in $\mathcal{RD}^+(c)$ enclosing the two spots. Now Ω is a coarsely two-Lipschitz map, and every disk in $\mathcal{RD}^+(c)$ which encloses the two spots is disjoint from a disk which is the I-bundle over an arc in the surface F_0^+, and such a disk is mapped to itself by Ω. Thus by the triangle inequality, we obtain that indeed $d_{\mathcal{DG}}(D, \Omega(D)) \leq 3$ for all $D \in \mathcal{RD}^+(c)$. The lemma follows. □

Remark 3.3. The construction in the proof of Lemma 3.2 yields in fact two distinct coarse Lipschitz retractions. If the I-bundle generator c is separating, then we obtain one such retraction for each of the two components of $\partial H - c$. If the I-bundle generator is non-separating, then there is one retraction for each choice of a component of the preimage of the boundary of an embedded Möbius band in the base of the I-bundle.

Let as before $c \subset \partial H$ be an I-bundle generator, with base F^+ and involution Ψ, and let $F_0^+ \subset \partial H$ be a once punctured subsurface whose boundary either is isotopic to c if c is separating, or consists of two connected components, say c_1, e_1, where c_1 is isotopic to c and e_1 is a preimage of the boundary of a Möbius band in the base of the I-bundle otherwise.

Denote by $\mathcal{A}(F_0^+)$ the subgraph of the arc graph of F_0^+ consisting of arcs with both endpoints on the boundary component c_1 of ∂F_0^+. If c is separating then $\mathcal{A}(F_0^+)$ equals the arc graph of F_0^+, and it is isometric to the arc graph of the surface F^+. In the case that c is non-separating, then as the projection of F_0^+ into F^+ is an embedding onto a subsurface of F^+ containing the boundary, this subsurface is a hole for the arc graph $\mathcal{A}(F^+)$ of F^+ in the sense of [MS13]. As a consequence, the arc graph $\mathcal{A}(F_0^+)$ quasi-isometrically embeds into the arc graph $\mathcal{A}(F^+)$.

The following statement is in some sense an inverse of Lemma 3.2. In its formulation, we write $\approx$ to denote an equality up to a universal multiplicative constant. Furthermore, write $d_{\mathcal{DG}}$ to denote the distance in the disk graph, and let $d_{\mathcal{A}(F_0^+)}$ be the distance in the arc graph of F_0^+. Let p_1 be the spot contained in F_0^+. For two disks D, E whose intersections with F_0^+ contain at least one arc with both endpoints on c_1 we write $d_{\mathcal{A}(F_0^+)}(\partial D \cap F_0^+, \partial E \cap F_0^+)$ to denote the distance in $\mathcal{A}(F_0^+)$ between any two such arcs. This is coarsely well defined.

Lemma 3.4. *Let D be a disk with the property that $\partial D \cap F_0^+$ contains an arc with both endpoints on c_1 and let E be a disk which is obtained from the disk D by point pushing p_1 along a loop in $F_0 = F_0^+ \cup \{p_1\}$ based at p_1. Then*

$$d_{\mathcal{DG}}(D, E) \approx d_{\mathcal{A}(F_0^+)}(\partial D \cap F_0^+, \partial E \cap F_0^+).$$

Proof. Let $\Omega_1 : \mathcal{DG} \to \mathcal{RD}^+(c)$ be the coarse two-Lipschitz retraction constructed in Lemma 3.2 from the surface F_0^+ (see Remark 3.3). By assumption, the intersection with F_0^+ of ∂D contains an arc α_0 with both endpoints on c_1, and the intersection of F_0^+ with ∂E contains the image ζ_0 of α_0 under point pushing along p_1. Thus we may assume that the images of D, E under the map Ω_1 are Ψ-invariant disks $\hat{D}, \hat{E}$ which intersect F_0^+ in the arcs α_0, ζ_0. By Lemma 3.2, we have $d_{\mathcal{CG}}(D, E) \geq \frac{1}{2} d_{\mathcal{DG}}(\hat{D}, \hat{E}) - m$ where $m > 0$ is a universal constant. Since the isometry $\mathcal{RD}^+(c) \to \mathcal{A}(F^+)$ associates to a disk in $\mathcal{RD}^+(c)$ the projection of its boundary into F^+, this implies that $d_{\mathcal{DG}}(D, E)$ is not smaller than a fixed positive multiple of $d_{\mathcal{A}(F_0^+)}(\partial D \cap F_0^+, \partial E \cap F_0^+)$. Thus it remains to show that $d_{\mathcal{DG}}(D, E)$ is bounded from above by a fixed positive multiple of $d_{\mathcal{A}(F_0^+)}(\alpha_0, \zeta_0)$.

We say that an arc in F_0^+ *encloses the spot* p_1 if it is the boundary of a thickening of an arc connecting the preferred boundary component c_1 of F_0^+ to the spot p_1. Any two such arcs which are not isotopic intersect. Let $\alpha_i \subset \mathcal{A}(F_0^+)$ be a geodesic connecting $\alpha_0 \subset \partial D \cap F_0^+$ to $\alpha_m = \zeta_0 \subset \partial E \cap F_0^+$. We begin with replacing this path by a path β_j of length at most $2m$ such that for all j, the arc β_{2j+1} encloses the spot p_1. To this end it suffices to proceed as follows. If $i < m$ is such that both α_i, α_{i+1} do not enclose the spot, then as α_i, α_{i+1} are disjoint, there exists an arc $\hat{\beta}$ enclosing the spot p_1 which is disjoint from both α_i, α_{i+1}. Replace the edge in $\mathcal{A}(F_0^+)$ connecting α_i to α_{i+1} by the path of length two with the same endpoints which passes through $\hat{\beta}$. Since no two adjacent vertices in the path α_i can enclose the spot p_1, this yields a path β_j as required.

By the analogue of Corollary 2.5, the arc β_{2j+1} is obtained from β_{2j-1} by point pushing p_1 along a loop $\gamma_j \subset F_0^+$ based at p_1. Since both β_{2j-1} and β_{2j+1} are disjoint from β_{2j}, the same holds true for γ_j. By concatenation, for each j, the arc β_{2j-1} is obtained from $\beta_0 = \alpha_0$ by point pushing with the loop $\hat{\gamma}_j = \gamma_1 \cdot \cdots \cdot \gamma_{j-1}$ (read from left to right). As the arc β_{2j} is disjoint from the image of α_0 by point pushing along $\hat{\gamma}_{j-1}$, it is the image under point-pushing along $\hat{\gamma}_{j-1}$ of an arc $\hat{\beta}_{2j}$ disjoint from α_0. As such an arc $\hat{\beta}_{2j}$ is contained in the boundary of a disk in $\mathcal{RD}^+(c)$ which is disjoint from α_0, the arc β_{2j} is contained in the boundary of a disk as well, and this disk is disjoint from the disk obtained from D by point pushing along $\hat{\gamma}_{j-1}$.

As a consequence, the path β_j is a projection to F_0^+ of a path in $\mathcal{DG}$ connecting D to E. This shows that indeed, $d_{\mathcal{DG}}(D,E)$ does not exceed $2d_{\mathcal{A}(F_0^+)}(\alpha_0,\zeta_0) = 2d_{\mathcal{A}(F_0^+)}(\partial D\cap F_0^+, \partial E\cap F_0^+)$. This completes the proof of the lemma. $\square$

For the formulation of the following corollary, note that the interval in the vertical boundary of an I-bundle with generator c whose endpoints are the two spots defines a basepoint for the arc graph of the annulus A with core curve c, and it defines a disk D enclosing the two spots. We call a simple closed curve $a\subset \partial H$ *untwisted* if the subsurface projection of a into this annulus has distance at most two to this basepoint. Note that this subsurface projection measures the amount of twisting relative to a base arc of an arc in the annulus A connecting two points in the distinct boundary components of A.

If c is a separating I-bundle generator then define $\mathcal{DG}(c)\subset\mathcal{DG}$ to be the set of all disks which intersect c in precisely two points. If c is a non-separating I-bundle generator then let $\mathcal{DG}(c)\subset\mathcal{DG}$ be the set of all disks which intersect c in precisely two points and are disjoint from the preimage of the boundary of a Möbius band in the base of the I-bundle.

Corollary 3.5. *For any I-bundle generator $c\subset\partial H$, the inclusion $\mathcal{DG}(c)\to\mathcal{DG}$ is a quasi-isometric embeddeding. Furthermore, if D,D' are two such disks whose intersections with the annulus A with core curve c are untwisted, then*

$$d_{\mathcal{DG}}(D,D') \approx \max\{d_{\mathcal{A}(F^+)}(\Omega_1(D),\Omega_1(D')), d_{\mathcal{A}(\Psi(F^+))}(\Omega_2(D),\Omega_2(D'))\}$$

where Ω_1,Ω_2 are the two distinct coarse Lipschitz retractions $\mathcal{DG}\to\mathcal{RD}^+(c)$.

Proof. Let $\Omega_1,\Omega_2:\mathcal{DG}\to\mathcal{RD}^+(c)$ be the two coarse Lipschitz retractions defined by the choices of the surfaces $F_0^+,\Psi(F_0^+)\subset\partial H$ as before. Let $D\in\mathcal{DG}(c)$ be untwisted. We observed in Lemma 3.4 and its proof that any disk $D_1\in\mathcal{RD}^+(c)$ whose intersection with the annulus A is untwisted (or, more generally, untwisted with respect to D) and which encloses the two spots can be obtained from D by point pushing the point p_1 along a loop α in F_0^+ and point pushing $p_2=\Psi(p_1)$ along a loop β in $\Psi(F_0^+)$. Since these point pushing operations clearly commute, the disk D_1 does not depend on the order in which these point pushing transformations are carried out.

By Lemma 3.2, the distance between D_1 and D is proportional to the maximum of the distance in the arc graph $\mathcal{A}(F^+)$ of the projections of the intersections $\partial D\cap F_0^+,\partial D_1\cap F_0^+\in\mathcal{A}(F_0^+)$ and $\partial D\cap\Psi(F_0^+),\partial D_1\cap\Psi(F_0^+)\in\mathcal{A}(\Psi(F_0^+))$. As the set of disks enclosing the two spots is one-dense in $\mathcal{DG}(c)$, this yields the statement of the corollary. $\square$

Corollary 3.6. (1) *If $g = 2h$ is even then $\mathcal{DG}$ contains quasi-isometrically embedded $\mathbb{Z}^3$.*
(2) *If g is odd then $\mathcal{DG}$ contains quasi-isometrically embedded $\mathbb{Z}^2$.*

Proof. Let $g = 2h$ be even and let $c \subset \partial H_0$ be a separating I-bundle generator. Using the above notations, let α be a bi-infinite quasi-geodesic in $\mathcal{A}(F_0^+)$ starting at $\alpha(0)$ where $\alpha(0)$ is the intersection of F_0^+ with the boundary of the disk $D \in \mathcal{RD}^+(c)$. Such quasi-geodesics exists since the arc graph $\mathcal{A}(F_0^+)$ is of infinite diameter. By what we showed so far, for all s, t there exists a disk $\zeta(s, t)$ whose projection to F_0^+ equals $\alpha(s)$ and whose projection to $\Psi(F_0^+)$ is of distance at most two to $\Psi(\alpha(t))$. Corollary 3.5 then shows that

$$d_{\mathcal{DG}}(\zeta(s,t), \zeta(s',t')) \approx \max\{s - s', t - t'\}.$$

Thus the image of $\zeta : \mathbb{Z}^2 \to \mathcal{DG}$ is uniformly quasi-isometric to $\mathbb{Z}^2$ equipped with the maximum norm. As this norm is quasi-isometric to the standard Euclidean norm, we conclude that $\mathcal{DG}$ contains quasi-isometrically embedded $\mathbb{Z}^2$.

Let $A \subset \partial H_0$ be an annulus neighborhood about c. For a fixed pair of points on the boundary of A (say the marked points p_1, p_2) the curve graph $\mathcal{CG}(A)$ of A can be identified with the set of homotopy classes of arcs connecting the two boundary components of A where a homotopy is not allowed to cross through the points p_1, p_2. There exists a coarsely well defined *subsurface projection* $\Pi : \mathcal{CG}(\partial H) \to \mathcal{CG}(A)$ which maps a simple closed curve crossing through A to a component of its intersection with A. Here as before, $\mathcal{CG}(X)$ is the curve graph of the surface X.

Since c is separating by assumption, any diskbounding simple closed curve has an essential intersection with A. Hence the restriction to $\mathcal{DG}$ of the subsurface projection $\mathcal{CG}(H) \to \mathcal{CG}(A) = \mathbb{Z}$ is a Lipschitz retraction into $\mathbb{Z}$ which commutes with the two Lipschitz retractions defined by the surfaces F_0^+ and $\Psi(F_0^+)$. On the other hand, iterated point pushing one of the points p_i about the core curve of the annulus keeping the second point fixed shows that the image of this projection is all of $\mathbb{Z}$. Together this shows that there are embedded $\mathbb{Z}^3$ in $\mathcal{DG}$.

Now let g be odd. By Corollary 3.5 and the discussion in the beginning of this proof, we only have to observe once more that the diameter of $\mathcal{A}(F_0^+) \subset \mathcal{A}(F^+)$ is infinite, which is well known (see [MS13]). □

4. Free splittings and sphere graphs

This section is devoted to the proof of Theorem 3. We begin with looking again at a handlebody H of genus $g \geq 2$ with two spots p_1, p_2 on the boundary. Let H_0 be the handlebody of genus g without spots and let $\Phi : H \to H_0$ be the spot removing map. Recall that a disk D in H encloses the two spots p_1, p_2 if $\Phi(D) \subset H_0$ is homotopic to a point.

We next use the two spots to add a handle to H. The resulting manifold is a handlebody H' of genus $g + 1$ with one spot. To this end slightly enlarge the two spots p_1, p_2 to two small compact disjoint disks B_1, B_2 in ∂H with $p_i \in \partial B_i$. Identifying these two disks with an orientation reversing diffeomorphism $B_1 \to B_2$ which maps p_1 to p_2 yields a handlebody H' of genus $g + 1$. We may view the common image of the points p_1, p_2 as a spot $p \in \partial H'$. The fundamental group of H' is the free group $\mathfrak{F}_{g+1}$ with $g + 1$ generators. We choose the spot p of H' as the basepoint for the fundamental group of H'.

The following simple observation will be used several times later on.

Lemma 4.1. *A disk D in H which encloses the two spots p_1, p_2 and the choice of one of the spots p_i determines a free splitting $\pi_1(H', p) = \mathfrak{F}_{g+1} = \mathfrak{F}_g * \mathbb{Z}$. Changing the spot changes the splitting by conjugation with a generator of the $\mathbb{Z}$-factor.*

Proof. Up to isotopy, we may assume that the disk D is disjoint from the two closed disks B_1 and B_2 used in the construction of H'. Thus D determines a separating disk D' in H' which only depends on D. This disk cuts H' into a handlebody of genus g with fundamental group $\mathfrak{F}_g$ and a solid torus T with fundamental group $\mathbb{Z}$ which contains the basepoint p.

Van Kampen's theorem now shows that D' defines a free splitting

$$\pi_1(H', p) = \mathfrak{F}_{g+1} = \mathfrak{F}_g * \mathbb{Z},$$

unique up to conjugation with an element of the free factor $\mathbb{Z}$. Namely, the basepoint p is contained in the solid torus T. Thus the splitting of $\pi_1(H', p)$ obtained by van Kampen's theorem is determined by D' up to conjugation with an element of $\pi_1(T)$.

To see that if we fix one of the spots p_i then we obtain in fact a uniquely determined splitting, it suffices to observe that the solid torus T is obtained by identifying two disks in the boundary of a ball. This ball is fixed, but the disks are allowed to move within a fixed subdisk D of this boundary. As a disk is contractible, moving the two disks B_1, B_2 freely in D gives rise to the same splitting and hence there is no ambiguity in the construction (in other words, the fundamental group of the solid torus T appears only after the gluing). □

The construction in Lemma 4.1 can be reversed. Namely, observe that the handlebody H' contains a distinguished non-separating disk V which is the image of the two disks in ∂H used in the construction. The spot of H' is contained in the boundary of V. If $E \subset H'$ is any separating disk disjoint from V which decomposes H' into a solid torus $T \supset V$ and a handlebody of genus g, then E is the image of a disk in H enclosing the two spots under the gluing construction.

Remark 4.2. By Lemma 4.1, each disk D in H enclosing the two spots defines a free splitting $\mathfrak{F}_{g+1} = \mathfrak{F}_g * \mathbb{Z}$. Here the $\mathfrak{F}_g$-factor in the free product is identified with the fundamental group $\pi_1(H \cup \{p_2\}, p_2)$. Lemma 2.4 immediately implies the following. Let $a \in \mathfrak{F}_{g+1}$ be the generator of the $\mathbb{Z}$-factor in the free splitting of $\mathfrak{F}_{g+1}$ defined by the disk D and the choice of the basepoint p_1, where a is viewed as a homotopy class with fixed endpoints of an arc connecting p_1 to p_2. Let E be the image of D by point pushing p_2 along a loop in the homotopy class $q(D, E) \in \pi_1(\partial H_0, p_2)$ defined as the class of the concatenation of an arc α connecting p_2 to p_1 which is disjoint from D with an arc β connecting p_2 to p_1 which is disjoint from E (compare Lemma 2.4). Via the inclusion $\partial H_0 \to H_0$, this homotopy class defines a homotopy class $\iota_* q(D, E) \in \pi_1(H \cup \{p_2\}, p_2) = \pi_1(H_0, p_2)$ (here the last equation is the identification of fundamental groups under the spot closing map). Then the $\mathbb{Z}$-factor defined by p_1 and the disk E is generated by $a \cdot \iota_* q(D, E)$ (read from left to right).

From now on we fix a disk D enclosing the two spots in H which is a thickening of an interval in an I-bundle over a compact surface F with connected boundary. If the genus of H is even then we assume that F is orientable. This disk defines a free splitting $\mathfrak{F}_{g+1} = \mathfrak{F}_g * \mathbb{Z}$ where the free factor $\mathbb{Z}$ is generated by an element a

obtained from an embedded oriented arc in the twice spotted disk $\tilde{D}$ in ∂H with the same boundary as D which connects p_1 to p_2.

Double the handlebody H to a connected sum M of g copies of $S^1 \times S^2$ with two spots. This defines an embedding $\mathfrak{J} : H \to M$. Any disk E in H doubles to an essential sphere $\Pi(E) \subset M$, and E encloses the two spots if and only if this holds true for $\Pi(E)$. Furthermore, disjoint disks give rise to disjoint spheres. Hence the doubling map $\Pi : \mathcal{DG} \to \mathcal{SG}$ is simplicial, where $\mathcal{SG}$ is the sphere graph of M. The following observation shows that the map Π is not bilipschitz onto its image.

Lemma 4.3. *For each I-bundle generator $c \subset \partial H$, the image of the subgraph $\mathcal{RD}^+(c)$ under the map Π has diameter at most two in the sphere graph $\mathcal{SG}$.*

Proof. As any disk in $\mathcal{RD}^+(c)$ is at distance one from a disk $E \in \mathcal{RD}^+(c)$ enclosing the two spots, it suffices to show that two disks $D, E \in \mathcal{RD}^+(c)$ enclosing the two spots are mapped by Π to the same sphere enclosing the two spots.

A disk $E \in \mathcal{RD}^+(c)$ enclosing the two spots is the image of the base disk D by point pushing the point p_2 along a based loop $\zeta \in \pi_1(F_0, p_2)$, followed by point pushing p_1 along the based loop $\Psi(\zeta) \in \pi_1(\Psi(F_0), p_1)$.

Let α be an arc connecting p_1 to p_2 which is disjoint from D. Lemma 2.4, applied to both the point pushing of p_1 and of p_2, shows that up to an ambiguity arising from clearing intersections with the spots, an arc connecting p_1 to p_2 which is disjoint from E is homotopic with fixed endpoints to $\Psi(\zeta)^{-1} \circ \alpha \circ \zeta$ (read from left to right).

Note that the reflection Ψ acts as the identity on the fundamental group of H, taken at a fixed point for Ψ. Thus the loop obtained by connecting the fixed point q for Ψ in α to p_2, concatenating with ζ and going back to q along α is homotopic to its image under Ψ. Now a sphere in the doubled handlebody M enclosing the two spots is the boundary of the thickening of an arc connecting the two spots. By the above discussion, the arc defining the sphere $\Pi(E)$ is homotopic to the arc which defines the sphere $\Pi(D)$. Informally, a sphere S enclosing the two spots p_1, p_2 determines uniquely an isomorphism of the fundamental group of M based at p_1 with the fundamental group of M based at p_2 by connecting p_1 to p_2 with an arc not crossing through S. With this identification, point pushing S along loops at p_1, p_2 defining the same element in the fundamental group of M gives rise to the same sphere. Note that the loops ζ and $\alpha^{-1} \circ \psi(\zeta) \circ \alpha$ are contained in the boundary ∂H of the handlebody, and they are not homotopic as elements of the fundamental group of ∂H. This shows the lemma. $\square$

We shall construct spheres as doubles of disks in the handlebody H and keep track of distances using Lemma 4.3. Let again D be a disk which is the thickening of an interval in the I-bundle defined by the I-bundle generator c. It defines the generator a of the $\mathbb{Z}$-factor of the splitting $\mathfrak{F}_{g+1} = \mathfrak{F}_g * \mathbb{Z}$ defined by D.

Let $d_{\mathcal{SG}}$ be the distance in the sphere graph. For ease of bookkeeping, define a function $d_{\widehat{\mathcal{DG}}}$ on pairs of disks in H by

$$d_{\widehat{\mathcal{DG}}}(E, F) = d_{\mathcal{SG}}(\Pi(E), \Pi(F)).$$

Note that $d_{\widehat{\mathcal{DG}}}$ is symmetric and fulfills the triangle inequality, but by Lemma 4.3 and by Lemma 3.2, it is *not* comparable to the distance on $\mathcal{DG}$. Furthermore, we have $d_{\widehat{\mathcal{DG}}}(E, F) \leq d_{\mathcal{DG}}(E, F)$ for all disks E, F since the map Π is one-Lipschitz.

Recall that point pushing one of the two spots in H is a diffeomorphism of H which extends to a diffeomorphism of M and hence induces an isometry on the sphere graph $\mathcal{SG}$. In other words, such a map preserves the function $d_{\widehat{\mathcal{DG}}}$. Informally, we say that such a map is an isometry for $d_{\widehat{\mathcal{DG}}}$.

Consider as before an embedded subsurface F_0 of ∂H_0 determined by an I-bundle generator c. If g is even then we assume that c is separating and F_0 is isotopic to a component of $\partial H_0 - c$. If g is odd then F_0 is a component of the complement of the preimage of the core curve of a Möbius band in the orientation cover of the non-orientable base of a non-separating I-bundle generator $c \subset \partial H_0$. Let Ψ be the orientation reversing involution of H_0 determined by the I-bundle generator c. Let $p_2 \in F_0$ be a point in the interior of F_0 and let $p_1 = \Psi(p_2)$. Let $D \in \mathcal{RD}^+(c)$ be a disk isotopic to the thickening of a fiber arc of the I-bundle with endpoints p_1, p_2, thought of as being constructed by slightly pushing a point on ∂F_0 inside F_0.

Let us denote by $[a, u]_2$ the disk obtained from D by point pushing p_2 along a loop γ based at p_2 in the surface F_0, in the homotopy class $u \in \pi_1(F_0, p_2)$, and by $[a^{-1}, u]_1$ the disk obtained from D by point pushing p_1 along the based loop $\Psi(\gamma) \subset \Psi(F)$ in the homotopy class $\Psi(u)$. With this notation, for all $u \in \pi_1(F_0, p_2)$ we have

$$d_{\widehat{\mathcal{DG}}}([a, u]_2, [a^{-1}, u^{-1}]_1) \leq 2. \tag{1}$$

Namely, point pushing p_1 along the loop $\Psi(\gamma)$ is an isometry for $d_{\widehat{\mathcal{DG}}}$, and the image of the disk $[a, u]_2$ under this point pushing map is a disk which is invariant under Ψ and hence contained in the subspace $\mathcal{RD}^+(c)$. The image of this space under the projection map Π has diameter two by Lemma 4.3.

Now if we write composition from left to right, then as point pushing along based loops at p_1 preserves $d_{\widehat{\mathcal{DG}}}$, for homotopy classes $b_1, b_2, c \in \pi_1(F_0, p_2)$ we obtain from (1) that

$$\begin{aligned} d_{\widehat{\mathcal{DG}}}([a, c\cdot b_1]_2, [a, c\cdot b_2]_2) &\leq d_{\widehat{\mathcal{DG}}}([a^{-1}, b_1^{-1}\cdot c^{-1}]_1, [a^{-1}, b_2^{-1}\cdot c^{-1}]_1) + 4 \\ &= d_{\widehat{\mathcal{DG}}}([a^{-1}, b_1^{-1}]_1, [a^{-1}, b_2^{-1}]_1) + 4 \leq d_{\widehat{\mathcal{DG}}}([a, b_1]_2, [a, b_2]_2) + 8. \end{aligned} \tag{2}$$

The free factor $\mathfrak{F}_g$ in the free splitting $\mathfrak{F}_{g+1} = \mathfrak{F}_g * \mathbb{Z}$ defined by a disk enclosing the two spots is naturally isomorphic to $\pi_1(H \cup p_2, p_2)$. Thus a free basis $\mathcal{A} = \{a_1, \dots, a_g\}$ of $\mathfrak{F}_g = \pi_1(H \cup p_2, p_2)$ extends to a free basis $\hat{\mathcal{A}} = \{a_1, \dots, a_g, a\}$ of $\mathfrak{F}_{g+1}$.

We now use a device from [SS14]. Define the *Whitehead graph* $\Gamma_{\mathcal{A}}(x)$ of a word $x \in \mathfrak{F}_g$ in a free basis $\mathcal{A} \cup \mathcal{A}^{-1}$ of $\mathfrak{F}_g$ as follows. The set of vertices of $\Gamma_{\mathcal{A}}(x)$ is identified with the set $\mathcal{A} \cup \mathcal{A}^{-1}$. Each pair of consecutive letters $a_i a_j$ in the word x contributes one edge from the vertex a_i to the vertex a_j^{-1}. Thus if the length of x equals n then $\Gamma_{\mathcal{A}}(x)$ has $n-1$ edges, and $\Gamma_{\mathcal{A}}(x)$ has a cut vertex if $x \in \mathcal{A}$. Furthermore, if $\Gamma_{\mathcal{A}}(x)$ has a cut vertex, then the same holds true for the unique reduced word which defines the same element of $\mathfrak{F}_g$ as x.

Following [SS14], define the *simple* $g+1$*-length*

$$|w|_{g+1}^{simple}$$

of any reduced word w in the free basis $\mathcal{A} = \{a_1, \dots, a_g\}$ of $\mathfrak{F}_g$ to be the greatest number t such that w is of the form $w_1 w_2 \cdots w_t$ where the Whitehead graph of w_j with respect to the basis $\mathcal{A}$ has no cut vertex for each $j = 1, \dots, t$. If the Whitehead graph of w has a cut vertex then the simple $g+1$-length of w is defined to be zero. We have that $|w|^{simple}_{g+1}$ is bounded from above by the word length of the reduced word w with respect to the basis $\mathcal{A}$. Furthermore, $|w^{-1}|^{simple}_{g+1} = |w|^{simple}_{g+1}$. The terminology here is taken from [SS14] although it is not well adapted to the situation at hand.

The following statement combines Lemma 4.6 and Lemma 4.7 of [SS14],

Lemma 4.4. (1) $|u|^{simple}_{g+1} \geq |v|^{simple}_{g+1}$ *whenever v is a subword of u.*

(2)
$$|w|^{simple}_{g+1} \leq |u|^{simple}_{g+1} + |v|^{simple}_{g+1} + 1$$
if u, v are freely reduced words in the letters $\mathcal{A} \cup \mathcal{A}^{-1}$ and $w = uv$.

Proof. The statement of Lemma 4.7 of [SS14] shows the second part of the lemma only in the case that $w = uv$ is freely reduced. To show that it is true as stated, assume that $|v|^{simple}_{g+1} = 0$ and that w is the reduced word representing uv. Then w is obtained from uv by erasing some letters at the end of u and the beginning of v. In particular, by the first part of the lemma, the Whitehead graph of the subword of v which is contained in w has a cut vertex.

As a consequence, if $w = w_1 \cdots w_t$ where the Whitehead graph of w_i does not have a cut vertex, then as u is reduced, $w_1 \cdots w_{t-1}$ is a subword of u. Then $t - 1 \leq |u|^{simple}_{g+1}$ by the first part of the lemma and hence $|w|^{simple}_{g+1} \leq |u|^{simple}_{g+1} + 1$ as claimed.

The general case follows from a rather straightforward modification of this argument and will be omitted. Only the case that $|v|^{simple}_{g+1} = 0$ is used in the sequel. □

The next lemma relates simple $g + 1$-length to the sphere graph $\mathcal{SG}$ of M. To simplify the notation, in the sequel we call a sequence (S_i) of spheres in M a *path* in $\mathcal{SG}$ if for all i the sphere S_i is disjoint from S_{i+1}. Thus such a sequence is the set of integral points on a simplicial path in $\mathcal{SG}$ connecting its endpoints. Recall from Lemma 2.4 and its analogue for spheres that an ordered pair (S, U) of spheres enclosing the two spots p_1, p_2 determines uniquely an element $b(S, U) \in \pi_1(M, p_2)$, which is represented by the concatenation of an arc connecting p_2 to p_1 not crossing through S with an arc connecting p_1 to p_2 not crossing through U.

Lemma 4.5. *Let $(S_i)_{0 \leq i \leq n}$ be a path in $\mathcal{SG}$ which begins and ends with a sphere enclosing the two spots p_1, p_2. Let $w = b(S_0, S_n) \in \pi_1(M, p_2)$; then*
$$|w|^{simple}_{g+1} \leq 2n.$$

Proof. Assume without loss of generality that the path (S_i) connecting S_0 to S_n is of minimal length in $\mathcal{SG}$. First we modify inductively the sequence (S_i) without increasing its length in such a way that each of the spheres S_i $(1 \leq i \leq n-1)$ either is non-separating or encloses the spots p_1, p_2.

The construction proceeds in two steps. In a first step, we replace each separating sphere S_{2i-1} with odd index by a sphere which either is non-separating or encloses the two spots. We do not change the spheres S_{2i} with even index. In a second step, we then modify the spheres with even index and preserve those with odd index.

To carry out the first step, let $\ell \leq n/2$ and assume that the sphere $S_{2\ell-1}$ is separating and does not enclose the spots; otherwise there is nothing to do. If $S_{2\ell-2}, S_{2\ell}$ are contained in distinct components of $M - S_{2\ell-1}$ then they are disjoint. In this case we can remove $S_{2\ell-1}$ from the path (S_i) and obtain a shorter path with the same endpoints. Since the path (S_i) has minimal length this is impossible.

Thus $S_{2\ell-2}, S_{2\ell}$ are contained in the same component W of $M - S_{2\ell-1}$. Since $S_{2\ell-1}$ does not enclose the spots, neither of the two components of $M - S_{2\ell-1}$ is a ball with two balls (or points) removed from the interior. Since M has precisely two spots, this implies that the image of the fundamental group of each of the two components of $M - S_{2\ell-1}$ in the fundamental group of M is non-trivial. Now each component of $M - S_{2\ell-1}$ is a connected sum of $S^1 \times S^2$ with some balls removed and therefore the component $M - W$ of $M - S_{2\ell-1}$ contains a non-separating sphere $\tilde{S}_{2\ell-1}$. Replace $S_{2\ell-1}$ by $\tilde{S}_{2\ell-1}$.

Replace in this way any sphere $S_{2\ell-1}$ with an odd index which is separating but does not enclose p_1, p_2 by a non-separating sphere without modifying the spheres S_{2i} with even index. This implements the first step of the construction. The second step is exactly identical after exchanging the roles of even and odd index. To summarize, we may assume from now on that every separating sphere in the path (S_i) encloses the two spots.

From the path (S_i) we next construct a path $(U_k)_{0 \leq k \leq 2u}$ of spheres connecting S_0 to S_n whose length $2u$ is at most four times the length n of the path (S_i) and such that for each $j \leq u$, the sphere U_{2j} encloses the spots p_1, p_2 and the sphere U_{2j-1} is non-separating.

To this end recall that any two distinct spheres which enclose the spots p_1, p_2 intersect. This means that if the sphere S_i from the above sequence encloses the spots then the spheres S_{i-1}, S_{i+1} are non-separating. Thus for the construction of the path (U_k) it now suffices to replace any consecutive pair S_i, S_{i+1} of disjoint non-separating spheres by a path of length at most four with the same endpoints whose vertices alternate between non-separating spheres and spheres enclosing the spots.

Let $i < n-1$ be such that the spheres S_i, S_{i+1} are both non-separating. If $M - (S_i \cup S_{i+1})$ is connected then there is a sphere B which encloses the spots p_1, p_2 and which is disjoint from $S_i \cup S_{i+1}$. Such a sphere can be obtained by thickening an arc in $M - (S_i \cup S_{i+1})$ which connects p_1 to p_2. Replace the consecutive pair S_i, S_{i+1} by the path S_i, B, S_{i+1} of length two.

If $M - (S_i \cup S_{i+1})$ is disconnected and if the spots p_1, p_2 are both contained in the same component of $M - (S_i \cup S_{i+1})$, then we can proceed as in the previous paragraph. Otherwise there is a component of $M - (S_i \cup S_{i+1})$ which is a connected sum of $h \geq 1$ copies of $S^1 \times S^2$ with three points or open balls removed. One of the holes is a spot, the other two holes are bounded by spheres which glue to the spheres S_i, S_{i+1}. Hence there is a non-separating sphere B which is disjoint from $S_i \cup S_{i+1}$ and such that $M - (S_i \cup B)$ and $M - (S_{i+1} \cup B)$ are both connected. Replace the consecutive pair S_i, S_{i+1} by a path of length 4 of the form $S_i, A_1, B, A_2, S_{i+1}$ where the spheres A_1, A_2 both enclose the spots p_1, p_2. This completes the construction of the sequence (U_k).

For each j the sphere U_{2j} defines a free splitting of $\mathfrak{F}_{g+1}$ of the form $\mathfrak{F}_g * \mathbb{Z}$. If a is the generator of the free factor $\mathbb{Z}$ for the splitting defined by S_0 (in the sense as before, namely we think of a as a homotopy class of an arc in M which connects p_1 to p_2 and does not intersect the sphere S_0, and this homotopy class determines the free factor $\mathbb{Z}$ in the free splitting defined by S_0), then for each j the free factor $\mathbb{Z}$ for the free splitting defined by U_{2j} is generated by $a \cdot b(S_0, U_{2j})$ where $b(S_0, U_{2j}) \in \mathfrak{F}_g$.

Let $w_j = b(S_0, U_{2j})^{-1} b(S_0, U_{2j+2}) \in \mathfrak{F}_g$. By construction, the sphere U_{2j+1} in H is non-separating and disjoint from U_{2j}, U_{2j+2}. The set of all loops in $M \cup \{p_2\}$ with basepoint p_2 which do not intersect U_{2j+1} defines a free factor Q of $\mathfrak{F}_g$ of corank one. Since U_{2j+1} is disjoint from U_{2j} and U_{2j+2}, the element w_j is contained in the free factor Q.

Since $w_j \in Q$, by Theorem 2.4 of [S00] the Whitehead graph of w_j has a cut vertex. But this just means that the simple $g+1$-length of w_j vanishes. An inductive application of the second part of Lemma 4.4 now shows that the simple $g+1$-length of the word $b(S_0, S_n) \in \mathfrak{F}_g$ is at most $u \leq 2n$. This is what we wanted to show. □

Now we are ready to show Theorem 2 from the introduction.

Theorem 4.6. *For every $g \geq 4$ and every $n \geq 1$, the sphere graph of a connected sum $\sharp_g S^1 \times S^2$ with two spots contains quasi-isometrically embedded copies of $\mathbb{R}^n$.*

Before we provide the proof, we outline the strategy for the argument which is adapted from [SS14]. View as before the manifold M as the double of a handlebody H, and represent the handlebody H as an I-bundle over a compact surface F with boundary. A sphere E in M enclosing the two spots is obtained from a base sphere S enclosing the two spots by point pushing the spot p_2 along a loop in F, based at p_2.

Point pushing induces an isometry of the sphere graph, which yields that for any homotopy classes $u, b_1, b_2 \in \pi_1(F, p_2)$ we have

$$d_{\mathcal{SG}}([a, b_1]_2, [a, b_2]_2) = d_{\mathcal{SG}}([a, b_1 \cdot u]_2, [a, b_2 \cdot u]_2)$$

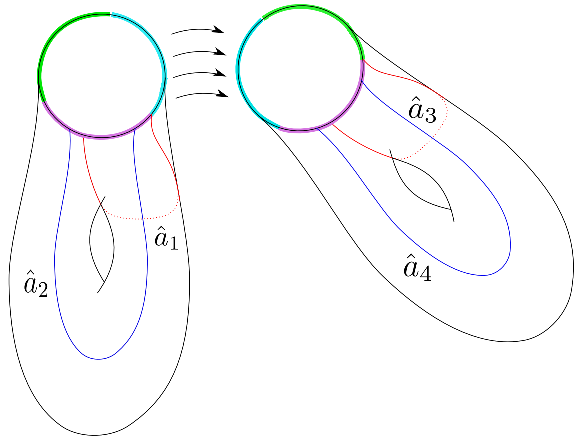
(recall that we write concatenation from the left to the right). On the other hand, the estimate (2) shows that a similar relation holds for *precomposition* with a fixed homotopy class of a based loop in (F, p_2), which distinguishes the sphere graph of M from the disk graph of the handlebody H. This allows to estimate from above the distance in $\mathcal{SG}$ of two spheres enclosing the two spots which are obtained by point pushing the base sphere enclosing the two spots along suitably chosen loops. A lower bound for such distances was established in Lemma 4.5. An explicit construction of point pushing loops, taken from [SS14], whose effect on the base sphere can be controlled using this mechanism leads to the proof of the theorem.

Proof of Theorem 4.6. As before, we view M as the double of the handlebody H. Assume first that the genus $g = 2h$ of H is even. We will explain at the end of this proof how to adjust the argument to the case that g is odd.

Let $F \subset \partial H_0$ be an embedded oriented surface with connected boundary ∂F such that H_0 equals the I-bundle over F. Let Ψ be the orientation reversing involution of H_0 which exchanges the endpoints of the intervals which make up the interval bundle. Fix a point $p_2 \in \partial F$ and let $p_1 = \Psi(p_2)$.

Arrange the $h = g/2 \geq 2$ handles of the surface F cyclically around ∂F. Choose for each handle of F two oriented disjoint non-homotopic essential arcs in the handle with endpoints on ∂F. We may assume that ∂F is partitioned into h segments $I_1, \ldots, I_h$ with disjoint interior, ordered cyclically along ∂F (that is, if $h \geq 3$ then $I_j \cap I_{j+1}$ consists of a single point for all j) so that each of these segments I_j is contained in the boundary of one of the handles and contains all four endpoints of the arcs $\hat{a}_{2j-1}, \hat{a}_{2j}$ which are embedded in that handle. The figure shows how this can be done.

A small neighborhood of the union of these $2h$ arcs and the boundary of F is a ribbon graph, that is, a planar surface $F_0 \subset F$. We require that the inclusion $F_0 \to F$ induces a surjection on fundamental groups. This is equivalent to stating



that F can be obtained from F_0 by attaching a disk to each component of the boundary of F_0 distinct from ∂F.

If $h \geq 3$ then let p_2 be the intersection $I_h \cap I_1$ and let $x = I_{h-1} \cap I_h$. If $h = 2$ then we require that $\{p_2, x\} = I_1 \cap I_2$. Slide the endpoints of the arcs $\hat{a}_i$ which define the ribbon graph F_0 along ∂F to p_2 in such a way that this sliding operation does not cross through x. The image of each of the arcs $\hat{a}_i$ under this homotopy is a based oriented loop a_i at p_2. The union of these loops is an embedded rose R with vertex p_2 (the rose R does not contain the boundary circle of F). As H_0 is an I-bundle over F, the inclusion $R \to H \cup \{p_2\}$ induces an isomorphism of $Q = \pi_1(R, p_2)$ onto the group $\pi_1(H \cup \{p_2\}, p_2)$ which is isomorphic to the fundamental group of H. Thus if we write $H_2 = H \cup \{p_2\}$ then we have $\pi_1(H_2, p_2) = Q$. In the sequel we think of the based loops a_i $(i = 1, \dots, 2h)$ as generators of the fundamental group Q of R.

As on p.592 in Subsection 5.2 of [SS14], we consider for $t \geq 1$ the element

$$b_t = a_1^{t+1} a_2^{t+1} \cdots a_g^{t+1} a_1^{t+1} a_2^{t+1} a_1^{t+1} \in Q.$$

Let D be the disk in H enclosing the two spots which is a thickening of the fiber of the interval bundle with endpoints p_1, p_2. We claim that for every $t \geq 1$ the image of D under the point pushing map of p_2 along b_t has distance at most 6 to D in the disk graph $\mathcal{DG}$ of H.

We show the claim first in the case that the genus $g = 2h$ of H is at least 6 and hence the genus of F is at least three. Then $b_t = uv$ where $u = a_1^{t+1} \cdots a_{g-2}^{t+1}$ and $v = a_{g-1}^{t+1} a_g^{t+1} a_1^{t+1} a_2^{t+1} a_1^{t+1}$. The word u does not contain the letters $a_{g-1}, a_{g-1}^{-1}, a_g, a_g^{-1}$, and the word v does not contain the letters $a_{g-3}, a_{g-3}^{-1}, a_{g-2}, a_{g-2}^{-1}$ since $g - 3 \geq 3$.

As a consequence, the word u is represented by a loop in the rose R whose image in the ribbon graph F_0 is disjoint from the arcs with endpoints in I_h. Hence up to homotopy, this loop is disjoint from the I-bundle over each of these two arcs. Then the same holds true for the image $\psi_u(D)$ of the disk D under the point pushing map ψ_u along u. In particular, the distance between D and $\psi_u(D)$ in the disk graph $\mathcal{DG}$ is at most two (see Lemma 3.4). Similarly, the image $\psi_v(D)$ of D under the point pushing map ψ_v along v is disjoint from an I-bundle over an arc with endpoints in the interval I_{h-1} and hence $d_{\mathcal{DG}}(D, \psi_v(D)) \leq 2$. But the point pushing map ψ_v acts on the disk graph as a simplicial isometry and consequently $d_{\mathcal{DG}}(\psi_v(D), \psi_v(\psi_u(D))) \leq 2$. As $b_t = uv$, together with the triangle inequality this yields

$$d_{\mathcal{DG}}(D, \psi_{b_t}(D)) \leq 4$$

(here words are read from left to right).

If $g = 4$ then write $b_t = uvw$ where $u = a_1^{t+1}a_2^{t+1}$, where $v = a_3^{t+1}a_4^{t+1}$ and $w = a_1^{t+1}a_2^{t+1}a_1^{t+1}$. Then there is a loop in R representing u, v, w which is disjoint from an arc with endpoints in I_2, I_1, I_2. As in the previous paragraph, we conclude that $d_{\mathcal{DG}}(D, \psi_s(D)) \leq 2$ for $s = u, v, w$. Thus by the triangle inequality, we have $d_{\mathcal{CG}}(D, \psi_{b_t}(D)) \leq 6$.

This argument can be used inductively and shows the following. For all $t \geq 1$ and each $k \geq 1$, we have

$$d_{\mathcal{DG}}(D, \psi_{b_t^k}(D)) \leq 6k. \tag{3}$$

Recall that the disk D defines a free splitting $\pi_1(H', p) = \mathfrak{F}_g * \mathbb{Z}$ where as before, $p \in \partial H'$ is the point obtained by identification of p_1 and p_2. Let a be the generator of the infinite cyclic group $\mathbb{Z}$, defined by the homotopy class of the arc α in ∂H connecting p_2 to p_1 which is disjoint from the boundary of D. As explained in the discussion preceding Remark 4.2, if $u \in Q$ is arbitrary, then the image of D under the point-pushing map ψ_u is a disk $\psi_u(D)$ enclosing the two spots which defines the free splitting of $\mathfrak{F}_{g+1}$ where the infinite cyclic free factor in the splitting is generated by $a \cdot \iota_* q(D, \psi_u(D))$. By the definition of the point pushing map, if we identify Q with $\pi_1(H, p_2) = \mathfrak{F}_g < \pi_1(H', p)$ as described in the beginning of this proof, the generator of this infinite cyclic free factor is just the element au. We refer to the discussion before Lemma 4.1 for more details.

Using the above notations, we follow Section 5.2 of [SS14]. For an arbitrary integer $n \geq 1$, define a map $\Lambda : \mathbb{Z}^n \to \mathcal{DG}$ which associates to $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ the image of the disk D under point-pushing of p_2 along the loop $b_1^{k_1}b_2^{k_2}\cdots b_n^{k_n} \in Q$ based at p_2. We claim that

$$d_{\widehat{\mathcal{DG}}}(\Lambda(k_1, \ldots, k_n), \Lambda(\ell_1, \ldots, \ell_n)) \leq 6\sum_{i=1}^{n}(|k_i - \ell_i| + 8). \tag{4}$$

To see this we adapt an argument from p.594 of [SS14]. Our goal is to transform the disk $\Lambda(k_1, \ldots, k_n) = \psi_{b_1^{k_1}\cdots b_n^{k_n}}(D)$ to the disk $\Lambda(\ell_1, \ldots, \ell_n) = \psi_{b_1^{\ell_1}\cdots b_n^{\ell_n}}(D)$ in a controlled way. These disks are determined by the homotopy classes $b_1^{k_1}b_2^{k_2}\cdots b_n^{k_n} \in Q$ and $b_1^{\ell_1}b_2^{\ell_2}\cdots b_n^{\ell_n} \in Q$, respectively, provided that the base disk D is fixed. To take full advantage of this fact we will now consider pairs of disks (E, V) where we view V as a basepoint, and E as a modification of the basepoint. With this viewpoint, our goal will be to transform the pair $(\psi_{b_1^{k_1}\cdots b_n^{k_n}}(D), D)$ to the pair $(\psi_{b_1^{\ell_1}\cdots b_n^{\ell_n}}(D), D)$ in a way which enables us to estimate the function $d_{\widehat{\mathcal{DG}}}$.

To simplify the discussion, let us resume the following notation. For an element $u \in Q$, represented up to homotopy by a unique reduced edge path in the rose R, let us denote by $[a \cdot u]_2$ the disk $\psi_u(D)$ obtained from D by point pushing p_2 along u, and denote by $[a^{-1} \cdot u]_1$ the disk obtained from D by point pushing $p_1 = \Psi(p_2)$ along $\Psi(u)$. Inequality (1) shows that

$$d_{\widehat{\mathcal{DG}}}([a, u]_2, [a^{-1}, u^{-1}]_1) \leq 2. \tag{5}$$

We first claim that

$$d_{\widehat{\mathcal{DG}}}(\psi_{b_1^{k_1}\cdots b_{n-1}^{k_{n-1}}b_n^{k_n}}(D),\psi_{b_1^{k_1}\cdots b_{n-1}^{k_{n-1}}b_n^{\ell_n}}(D))\leq 6|\ell_n-k_n|+8. \tag{6}$$

Namely, the estimate (3) and the fact that point-pushing of p_2 induces an isometry for $d_{\widehat{\mathcal{DG}}}$ on $\mathcal{DG}$ imply that

$$d_{\widehat{\mathcal{DG}}}(\psi_{b_n^{k_n}}(D),\psi_{b_n^{\ell_n}}(D))\leq 6|\ell_n-k_n|.$$

We now use the inequality (2). As $\psi_{b_n^u}(D)=[a\cdot b_n^u]_2$ for all u, the estimate (5) shows that

$$d_{\widehat{\mathcal{DG}}}([a^{-1}\cdot b_n^{-k_n}]_1,[a^{-1}\cdot b_n^{-\ell_n}]_1)\leq 6|k_n-\ell_n|+4.$$

Apply to both disks $[a^{-1}\cdot b_n^{-k_n}]_1,[a^{-1}\cdot b_n^{-\ell_n}]_1$ point-pushing of the point p_1 along a loop based at p_1 representing the homotopy class $\Psi(b_{n-1}^{-k_{n-1}}\cdots b_1^{-k_1})$. As point-pushing induces an isometry on the disk graph (and composition is read from left to right), we obtain

$$d_{\widehat{\mathcal{DG}}}([a^{-1}\cdot b_n^{-k_n}b_{n-1}^{-k_{n-1}}\cdots b_1^{-k_1}]_1,[a^{-1}\cdot b_n^{-\ell_n}b_{n-1}^{-k_{n-1}}\cdots b_1^{-k_1}]_1)\leq 6|k_n-\ell_n|+4.$$

Using again the estimate (5), this yields the estimate (6) we wanted to show.

Point-pushing of the point p_2 along the loop $b_n^{-\ell_n}$ transforms the *pair* of disks $(\psi_{b_1^{k_1}\cdots b_{n-1}^{k_{n-1}}b_n^{\ell_n}}(D),D)$ to the pair $(\psi_{b_1^{k_1}\cdots b_{n-1}^{k_{n-1}}}D,\psi_{b_n^{-\ell_n}}(D))$. As point-pushing acts as an isometry for $d_{\widehat{\mathcal{DG}}}$, we view this operation as a change of basepoints which does not change distances.

In a second step, we use the reasoning which led to the estimate (6) to deduce that

$$d_{\widehat{\mathcal{DG}}}(\psi_{b_1^{k_1}\cdots b_{n-2}^{k_{n-2}}b_{n-1}^{k_{n-1}}}(D),\psi_{b_1^{k_1}\cdots b_{n-2}^{k_{n-2}}b_{n-1}^{\ell_{n-1}}}(D))\leq 6|\ell_{n-1}-k_{n-1}|+8.$$

As a next step, we change the basepoint again. Using point-pushing of the point p_2 along the loop $b_2^{-\ell_{n-1}}$, the pair $(\psi_{b_1^{k_1}\cdots b_{n-2}^{k_{n-2}}b_{n-1}^{\ell_{n-1}}}(D),\psi_{b_n^{-\ell_n}}D)$ transforms to the pair

$$(\psi_{b_1^{k_1}\cdots b_{n-2}^{k_{n-2}}}(D),\psi_{b_n^{-\ell_n}b_{n-1}^{-\ell_{n-1}}}D).$$

Proceeding inductively, in n steps we transform the pair $(\psi_{b_1^{k_1}b_2^{k_2}\cdots b_n^{k_n}}(D),D)$ to the pair $(D,\psi_{b_n^{-\ell_n}\cdots b_1^{-\ell_1}}(D))$, changing the value of the function $d_{\widehat{\mathcal{DG}}}$ by at most $\sum_i(6|\ell_i-k_i|+8)$.

Now apply one last time point-pushing of the point p_2 along the loop $b_1^{\ell_1}\cdots b_n^{\ell_n}$ to the pair

$$(D,\psi_{b_n^{-\ell_n}\cdots b_1^{-\ell_1}}(D))$$

and obtain the pair $(\psi_{b_1^{\ell_1}\cdots b_n^{\ell_n}}(D),D)$. Using again that point pushing preserves $d_{\widehat{\mathcal{DG}}}$, we conclude that

$$d_{\mathcal{SG}}(\Pi\Lambda(k_1,\ldots,k_n),\Pi\Lambda(\ell_1,\ldots,\ell_n))\leq\sum_i(6|\ell_i-k_i|+8)$$

as claimed.

Now Lemma 4.15 of [SS14] and the discussion on the bottom of p.592 and on the top of p.594 in [SS14] shows that there is a number $c>0$ such that

$$\sum_{i=1}^n|k_i-\ell_i|\leq c|b_n^{-k_n}b_{n-1}^{-k_{n-1}}\cdots b_1^{-k_1}b_1^{\ell_1}b_2^{\ell_2}\cdots b_n^{\ell_n}|_{g+1}^{simple}. \tag{7}$$

We give a short summary of the proof of this fact as found in [SS14]. Namely, following Definition 4.9 of [SS14], we say that a word w in the letters $\mathcal{A}\cup\mathcal{A}^{-1}$ has *conjugate reduced length at most* k if there exist freely reduced words $v_1,\dots,v_\ell,u_1,\dots,u_\ell$ such that.

(a) $w = v_1^{u_1}v_2^{u_2}\cdots v_\ell^{u_\ell}$, where $v_j^{u_j} = u_j^{-1}v_ju_j$, and
(b) $k = (\ell-1) + |v_1|_{g+1}^{simple} + \cdots + |v_\ell|_{g+1}^{simple}$.

The number k is called the *conjugate reduced* $g+1$*-length associated to the decomposition.* The minimal number k for which such a decomposition exists is called the *conjugate reduced length of* w, and it is denoted by $|w|^{cr}$.

The easy Lemma 4.15 of [SS14] states that $|w|_{g+1}^{simple} \geq |w|^{cr}$, so it suffices to estimate $|w|^{cr}$ from below for $w = b_n^{-k_n}b_{n-1}^{-k_{n-1}}\cdots b_1^{-k_1}b_1^{\ell_1}b_2^{\ell_2}\cdots b_n^{\ell_n}$.

Definition 4.10 of [SS14] is geared to this end. A *cancelling pair* in the reduced word w is a pair of subwords of the form u,u^{-1}. A *nested family* $\mathcal{F}$ *of cancelling pairs* is a finite collection of disjoint cancelling pairs so that if $v,v^{-1}\in\mathcal{F}$ and $u,u^{-1}\in\mathcal{F}$ then v occurs between u,u^{-1} if and only if this is true for v^{-1}. For such a family $\mathcal{F}$ of cancelling pairs let $w-\mathcal{F}$ be the finite collection of subwords of w obtained by erasing the words from $\mathcal{F}$. Define

$$|w-\mathcal{F}|_{g+1}^{simple} = |\mathcal{F}| + \sum_{w'\in w-\mathcal{F}} |w'|_{g+1}^{simple}.$$

The required estimate follows from Lemma 4.11 of [SS14] which states that

$$|w|^{cr} \geq \min_{\mathcal{F}}(\max\{\frac{|\mathcal{F}|}{2}-1, \frac{1}{5}|w-\mathcal{F}|_{g+1}^{simple}-3\}). \tag{8}$$

To apply this estimate to the above word w, let $\mathcal{F}$ be a nested family of cancelling pairs for w which minimizes the expression on the right hand side of equation (8) and write $d=\sum|k_i-\ell_i|$. If $|\mathcal{F}|\geq d/10$ then we immediately obtain the required estimate. Otherwise note that by removing a cancelling pair we can at most delete a subword of a string of the form $b_i^{\min\{k_i,\ell_i\}}$. Furthermore it is easy to see that $|b_t^s|_{g+1}^{simple} \geq |s|$ for all s. Thus if $|\mathcal{F}|\leq d/10$ then a rough counting of the simple norm of the subsegments of $w-\mathcal{F}$ as carried out in detail on p.593 of [SS14] yields again the required estimate.

On the other hand, by Lemma 4.5, we have

$$d_{\mathcal{SG}}(\Pi\Lambda(k_1,\dots,k_n),\Pi\Lambda(\ell_1,\dots,\ell_n)) \geq \frac{1}{2}|b_n^{-k_n}b_{n-1}^{-k_{n-1}}\cdots b_1^{-k_1}b_1^{\ell_1}b_2^{\ell_2}\cdots b_n^{\ell_n}|_{g+1}^{simple}. \tag{9}$$

The estimates (4), (7) and (9) together show that the distance in $\mathcal{SG}$ of the images of $\Pi(D)$ under the point pushing of p_2 along $b_1^{k_1}b_2^{k_2}\cdots b_n^{k_n}$ and by $b_1^{\ell_1}b_2^{\ell_2}\cdots b_n^{\ell_n}$ is bounded from above and below by a fixed positive multiple of $\sum_{i=1}^n|k_i-\ell_i|$. Thus the map $\Lambda:\mathbb{Z}^n\to\mathcal{SG}$ is a quasi-isometric embedding. The theorem in the case that g is even follows,

The argument can be adjusted to the case that g is odd as follows. Let H_0 be a handlebody of odd genus $g\geq 5$. Choose a non-separating I-bundle generator c. Then H_0 is the oriented I-bundle over a non-orientable surface F with connected boundary $\partial F = c$. The surface F can be obtained from an orientable surface F_0 of genus $(g-1)/2\geq 2$ whose boundary consists of 2 connected components c_0,c_1 by attaching a Möbius band to c_1. The orientation cover of F equals the complement in ∂H_0 of an open annulus with core curve c, and the preimage of F_0 consists of two copies of F_0 which are glued along an annulus. The fundamental group of F_0 is a free group in g generators, and the inclusion of a component of its preimage in ∂H_0 into H_0 defines an isomorphism on fundamental groups.

The argument in the beginning of this proof now applies verbatim using the surface F_0 instead of F and noting that we may choose disjoint generating arcs for the fundamental group of $F_0 \subset F$ with endpoints on the boundary of F with the property that there is a partition of ∂F into $(g-1)/2+1 \geq 2$ disjoint intervals, each containing the endpoints of one or two arcs. This suffices to control the distance in the disk graph of a disk obtained from the base disk D by point pushing p_2 along a loop defined by the word b_t in the corresponding generators. The rest of the argument is identical to the argument for connected sums of an even number $g \geq 4$ of copies of $S^1 \times S^2$ with two spots. □

References

[BF08] G. Bell, K. Fujiwara, *The asymptotic dimension of the curve graph is finite*, J. Lond. Math. Soc. 77 (2008), 33–50.

[BB19] M. Bestvina, K. Bromberg, *On the asymptotic dimension of the curve complex*, Geom. Topol. 23 (2019), 2227–2276.

[BV12] M. Bridson, K. Vogtmann, *The Dehn function of* $\mathrm{Out}(F_n)$ *and* $\mathrm{Aut}(F_n)$, Ann. Inst. Fourier 62 (2012), 1811-1817.

[H16] U. Hamenstädt, *Hyperbolic relatively hyperbolic graphs and disk graphs*, Groups, Geom. Dyn. 10 (2016), 1–41.

[H19a] U. Hamenstädt, *Asymptotic dimension and the disk graph I*, J. Topol. 12 (2019), 658–673.

[H19b] U. Hamenstädt, *Asymptotic dimension and the disk graph II*, J. Topol. 12 (2019), 674–683.

[H21] U. Hamenstädt, *Spotted disk and sphere graphs I*, New York J. Math. 27 (2021), 881–902.

[HH12] U. Hamenstädt, S. Hensel, *The geometry of the handlebody groups I: Distortion*, J. Top. & Anal. 4 (2012), 71–97.

[HH15] U. Hamenstädt, S. Hensel, *Spheres and projections for* $\mathrm{Out}(F_n)$, J. Topol. 8 (2015), 65–92.

[HH21] U. Hamenstädt, S. Hensel, *Geometry of the handlebody group II: Dehn functions*, Michigan Math. J. 70 (2021), 23–53.

[HM13a] M. Handel, L. Mosher, *Lipschitz retraction and distortion for* $\mathrm{Out}(F_n)$, Geom. Top. 17 (2013), 1535–1580.

[HM13b] M. Handel, L. Mosher, *The free splitting complex of a free group I: Hyperbolicity*, Geom. Top. 17 (2013), 1581–1670.

[He21] S. Hensel, *(Un)distorted stabilisers in the handlebody group*, J. Topol. 14 (2021), 460–487.

[KLS09] R. Kent, C. Leininger, S. Schleimer, *Trees and mapping class groups*, J. reine angew. Math. 637 (2009), 1–21.

[MM99] H. Masur, Y. Minsky, *Geometry of the complex of curves I: Hyperbolicity*, Invent. Math. 138 (1999), 103–149.

[MM00] H. Masur, Y. Minsky, *Geometry of the complex of curves II: Hierachical structure*, Geom. Funct. Anal. 10 (2000), 902–974.

[MS13] H. Masur, S. Schleimer, *The geometry of the disk complex*, J. Amer. Math. Soc. 26 (2013), 1–62.

[SS14] L. Sabalka, D. Savchuk, *On the geometry of the edge splitting complex*, Groups, Geom. Dyn. 8 (2014), 565–598.

[S77] S. Suzuki, *On homeomorphisms of a 3-dimensional handlebody*, Canad. J. Math. 29 (1977), 111-124.

[S00] J. Stallings. *Whitehead graphs on handlebodies*, Geometric group theory down under (Canberra 1996), de Gruyter, Berlin, 1999, 317–333.

[Wa98] B. Wajnryb, *Mapping class group of a handlebody*, Fund. Math. 158 (1998), 195–228.

MATH. INSTITUT DER UNIVERSITÄT BONN, ENDENICHER ALLEE 60, 53115 BONN, GERMANY
e-mail: ursula@math.uni-bonn.de