SPOTTED DISK AND SPHERE GRAPHS I

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Abstract. The disk graph of a handlebody H of genus $g \geq 2$ with $m \geq 0$ marked points on the boundary is the graph whose vertices are isotopy classes of disks disjoint from the marked points and where two vertices are connected by an edge of length one if they can be realized disjointly. We show that for $m = 1$ the disk graph contains quasi-isometrically embedded copies of $\mathbb{R}^2$. The same holds true for sphere graphs of the doubled handlebody with one marked points provided that g is even.

1. Introduction

The *curve graph* $\mathcal{CG}$ of an oriented surface S of genus $g \geq 0$ with $m \geq 0$ punctures and $3g-3+m \geq 2$ is the graph whose vertices are isotopy classes of essential (that is, non-contractible and not homotopic into a puncture) simple closed curves on S. Two such curves are connected by an edge of length one if and only if they can be realized disjointly. The curve graph is a locally infinite hyperbolic geodesic metric space of infinite diameter [MM99].

A handlebody of genus $g \geq 1$ is a compact three-dimensional manifold H which can be realized as a closed regular neighborhood in $\mathbb{R}^3$ of an embedded bouquet of g circles. Its boundary ∂H is an oriented surface of genus g. We allow that ∂H is equipped with $m \geq 0$ marked points (punctures) which we call *spots* in the sequel. The group $\mathrm{Map}(H)$ of all isotopy classes of orientation preserving homeomorphisms of H which fix each of the spots is called the *handlebody group* of H. The restriction of an element of $\mathrm{Map}(H)$ to the boundary ∂H defines an embedding of $\mathrm{Map}(H)$ into the mapping class group of ∂H, viewed as a surface with punctures [S77, Wa98].

An *essential disk* in H is a properly embedded disk $(D, \partial D) \subset (H, \partial H)$ whose boundary ∂D is an essential simple closed curve in ∂H, viewed as a surface with punctures. An isotopy of such a disk is supposed to consist of such disks.

The *disk graph* $\mathcal{DG}$ of H is the graph whose vertices are isotopy classes of essential disks in H. Two such disks are connected by an edge of length one if and only if they can be realized disjointly.

In [MS13, H19, H16] the following is shown.

Theorem 1. *The disk graph of a handlebody of genus $g \geq 2$ without spots is hyperbolic.*

The main goal of this article is to show that in contrast to the case of curve graphs, Theorem 1 is not true if we allow spots on the boundary.

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Theorem 2. *Let H be a handlebody of genus $g \geq 2$ with one spot. Then the disk graph of H contains quasi-isometrically embedded copies of $\mathbb{R}^2$. In particular, it is not hyperbolic.*

Theorem 2 implies that disk graphs can not be used effectively to obtain a geometric understanding of the handlebody group $\mathrm{Map}(H)$ of a handlebody H of genus $g \geq 3$ paralleling the program developed by Masur and Minsky for the mapping class group [MM00]. The analogue of the strategy of Masur and Minsky would consist of cutting a handlebody open along an embedded disk which yields a (perhaps disconnected) handlebody with one or two spots on the boundary and studying disk graphs in the cut open handlebody.

A systematic study of groups to which the strategy laid out by Masur and Minsky can be applied was recently initiated by Behrstock, Hagen and Sisto [BHS17], and these groups are called *hierarchically hyperbolic*. Such groups have quadratic Dehn functions, but for $g \geq 3$ the Dehn function of $\mathrm{Map}(H)$ is exponential [HH19]. Hence $\mathrm{Map}(H)$ can not be hierarchically hyperbolic. However, the geometric mechanism behind an exponential Dehn function for $\mathrm{Map}(H)$ is not detected by the failure of being hierarchically hyperbolic in an obvious way.

Theorem 2 has an analogue for geometric graphs related to the outer automorphism group $\mathrm{Out}(F_g)$ of the free group on $g \geq 2$ generators. Namely, doubling the handlebody H yields a connected sum $M = \sharp_g S^2 \times S^1$ of g copies of $S^2 \times S^1$ with m marked points. A deep result of Laudenbach [L74] shows that $\mathrm{Out}(F_n)$ is a cofinite quotient of the groups of isotopy classes of homeomorphisms of M.

A doubled disk is an embedded essential sphere in M, which is a sphere which is not homotopically trivial or homotopic into a marked point. The *sphere graph* of M is the graph whose vertices are isotopy classes of embedded essential spheres in M and where two such spheres are connected by an edge of length one if and only if they can be realized disjointly. As before, an isotopy of spheres is required to be disjoint from the marked points. The sphere graph of a doubled handlebody without marked points is hyperbolic [HM13b].

Paralleling the result in Theorem 2 we have

Theorem 3. *Let $g \geq 2$ and let M be a doubled handlebody of genus g with one marked point. If g is even then the sphere graph of M contains quasi-isometrically embedded copies of $\mathbb{R}^2$. In particular, it is not hyperbolic.*

The argument in the proof of Theorem 3 uses Theorem 2 and a result in [HH15] which relates the sphere graph in a connected sum $\sharp_g S^2 \times S^1$ for g even to the arc graph of an oriented surface of genus $g/2$ with connected non-empty boundary. A corresponding result for odd g and a non-orientable surface with a single boundary component would yield Theorem 3 for odd $g \geq 3$, but at the moment, such a result is not available.

As in the case of disk graphs, this indicates that sphere graphs are of limited use for obtaining an effective geometric understanding of $\mathrm{Out}(F_g)$. Note that as in the case of the handlebody group, for $g \geq 3$ the Dehn function of $\mathrm{Out}(F_g)$ is exponential [BV12, HM13a].

In a sequel to this article [H12], it is shown that the disk graph of a handlebody of genus $g \geq 2$ with two spots contains quasi-isometrically embedded $\mathbb{R}^n$ for any $n \geq 1$, and the same holds true for the sphere graph of a doubled handlebody with two spots. We conjecture that the disk graph of a handlebody H with $m \geq 3$ spots is quasi-isometrically embedded into the curve graph of ∂H.

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2. Once spotted handlebodies

The goal of this section is to construct quasi-isometrically embedded copies of $\mathbb{R}^2$ in the disk graph of a handlebody with a single spot.

Thus let H be a handlebody of genus $g \geq 2$ with a single spot. Let H_0 be the handlebody obtained from H by removing the spot and let

$$\Phi : H \to H_0$$

be the spot removal map. The image under Φ of an essential (that is, not contractible or homotopic into the spot) diskbounding simple closed curve in ∂H is an essential diskbounding simple closed curve in ∂H_0.

The handlebody H_0 without spots can be realized as a fiber bundle over a surface F with non-empty connected boundary ∂F whose fiber is the closed interval $I = [0, 1]$. Such a fiber bundle is called an *I-bundle*. We summarize from Section 3 of [H16] (p.381-383) some properties of such I-bundles used in the sequel.

There are two different ways a handlebody H_0 of genus g can arise as an I-bundle over a surface F with connected boundary ∂F. In the first case, the surface F is orientable. Then the genus g of H_0 is even and the I-bundle is trivial. The genus of F equals $g/2$, and the boundary ∂F of F defines an isotopy class of a separating simple closed curve c on ∂H_0 which decomposes ∂H_0 into two surfaces of genus $g/2$, with a single boundary component.

If the surface F is non-orientable, then the I-bundle is non-trivial and the boundary ∂F defines a non-separating simple closed curve c in ∂H_0. An example is the orientable I-bundle over the connected sum of g projective planes with a disk. The complement of an open annulus about c in ∂H_0 is the orientation cover of F.

Following Definition 3.3 of [H16], define an *I-bundle generator* for H_0 to be an essential simple closed curve $c \subset \partial H_0$ so that H_0 can be realized as an I-bundle over a compact surface F with connected boundary ∂F and such that c is freely homotopic to $\partial F \subset \partial H_0$. The surface F is then called the *base* of the I-bundle.

An I-bundle generator c in ∂H_0 is *diskbusting*, which means that it has an essential intersection with every disk (see [MS13, H19]). Namely, the base F of the I-bundle is a deformation retract of H_0. Thus if γ is any essential closed curve on ∂H_0 which does not intersect c, then γ projects to an essential closed curve on F. Such a curve is not nullhomotopic in H_0 and hence it can not be diskbounding.

As established in [MS13, H16, H19], I-bundle generators play a special role for the geometry of the disk graph of H_0. Our goal is to take advantage of this fact for the understanding of the geometry of the handlebody with one spot. To this end define the *arc graph* $\mathcal{A}(X)$ of a compact surface X of genus $n \geq 1$ with connected boundary ∂X to be the graph whose vertices are isotopy classes of embedded essential arcs in X with endpoints on the boundary, and isotopies are allowed to move the endpoints of an arc along ∂X. Two such arcs are connected by an edge of length one if and only if they can be realized disjointly. The arc graph $\mathcal{A}(X)$ of X is hyperbolic [MS13].

For an I-bundle generator c in H_0 let $\mathcal{RD}(c)$ be the complete subgraph of the disk graph $\mathcal{DG}_0$ of H_0 consisting of disks which intersect c in precisely two points. The boundary of each such disk is an I-bundle over an arc in the base F of the I-bundle corresponding to c (see the discussion preceding Lemma 4.2 of [H16]). Namely, the I-bundle over an arc in F with endpoints on ∂F is an embedded disk in H_0. On the other hand, the boundary of a disk in H_0 defines the trivial element in the fundamental group of H_0. Thus if β is a diskbounding simple closed curve in ∂H_0 which intersects c in a precisely two points, then the homotopy classes relative to c of the two components of $\beta - c$ are exchanged under the orientation reversing involution of H_0 which exchanges the endpoints of a fiber in the I-bundle. As β has two essential intersections with c, this then implies that up to homotopy, the two components of $\beta - c$ trace through the two different preimages of the same points in F.

Now two disks intersecting c in precisely two points are disjoint if and only if the corresponding arcs in F are disjoint and hence we have

Lemma 2.1. *The graph $\mathcal{RD}(c)$ is isometric to the arc graph $\mathcal{A}(F)$ of F.*

The arc graph of a surface F with non-empty boundary ∂F is a complete subgraph of another geometrically defined graph, the so-called *arc and curve graph.* Its vertices are essential simple closed curves in F or arcs with endpoints on ∂F, and two such arcs or curves are connected by an edge of length one if they can be realized disjointly. The arc and curve graph contains the curve graph of F as a complete subgraph, and the inclusion of the curve graph into the arc and curve graph is known to be a quasi-isometry unless F is a sphere with at most three holes or a projective plane with at most three holes (Lemma 4.1 of [H16]). Recall that a map $\varphi : X \to Y$ be tween two metric spaces X, Y is an *L-quasi-isometric embedding* if for all $x, y \in X$ we have

$$d(x, y)/L - L \leq d(\varphi(x), \varphi(y)) \leq Ld(x, y) + L,$$

and it is called an *L-quasi-isometry* if moreover its image is L-dense, that is, for every $y \in Y$ there exists some $x \in X$ such that $d(\varphi(x), y) \leq L$.

The arc graph $\mathcal{A}(F)$ of F is 1-dense in the arc and curve graph of F, but the inclusion of $\mathcal{A}(F)$ into the arc and curve graph of F is a quasi-isometry only if the genus of X equals one [MS13] (see also [H16]).

A *coarse L-Lipschitz retraction* of a metric space (X, d) onto a subspace Y is a coarse L-Lipschitz map $\Psi : X \to Y$ (this means that $d(\Psi(x), \Psi(y)) \leq Ld(x, y) + L$ for some $L \geq 1$ and all x, y) with the additional property that there exists a number $C > 0$ with $d(\Psi(y), y) \leq C$ for all $y \in Y$. If X is a geodesic metric space then the image Y of a coarse Lipschitz retraction is a *coarsely quasi-convex* subspace of X, that is, any two points in Y can be connected by a uniform quasi-geodesic (for the metric of X) which is entirely contained in Y.

Lemma 2.2. *Let c be an I-bundle generator of the handlebody H_0. There exists a coarsely Lipschitz retraction $\Theta_0 : \mathcal{DG}_0 \to \mathcal{RD}(c)$ whose restriction to $\mathcal{RD}(c)$ is the identity.*

Proof. If c is a *separating* I-bundle generator, then the base F of the I-bundle can be identified with a component of $\partial H_0 - c$. Note that the choice of F is the choice of one of the two components of $\partial H_0 - c$, and this choice will be fixed throughout this proof.

Since the boundary ∂D of a disk D is an embedded simple closed curve in ∂H_0 and as c is diskbusting, the intersection $\partial D \cap F$ consists of a non-empty collection of pairwise disjoint simple arcs with endpoints on ∂F. The map

$$\Upsilon_0 : \mathcal{DG}_0 \to \mathcal{A}(F)$$

which associates to a disk D a component of $\partial D \cap F$ is coarsely well defined: Although it depends on choices, any other choice Υ_0' maps a disk D to an arc disjoint from $\Upsilon_0(D)$. If we denote by $Q : \mathcal{A}(F) \to \mathcal{RD}(c)$ the map which associates to an arc α in F the I-bundle over α, then the disks $Q(\Upsilon_0(D)), Q(\Upsilon_0'(D))$ are disjoint as well.

Furthermore, if D, D' are disjoint disks then the arcs $\Upsilon_0(D), \Upsilon_0(D')$ are disjoint and hence $d_{\mathcal{DG}_0}(Q\Upsilon_0(D), Q\Upsilon_0(D')) \leq 1$. This shows that $Q \circ \Upsilon_0$ is coarsely one-Lipschitz. As a disk $D \in \mathcal{RD}(c)$ intersects F in a single arc, we have $Q\Upsilon_0(D) = D$. Thus the map $Q \circ \Upsilon_0$ is indeed a one-Lipschitz retraction which completes the proof of the lemma in the case that c is separating. Note however that the relation between the two Lipschitz retractions constructed in this way from the two distinct components of $\partial H_0 - c$ is unclear.

The above argument does not extend to non-separating I-bundle generators in any straightforward way. Namely, if c is a non-separating I-bundle generator, then although up to homotopy, a disk which intersects c in precisely two points is invariant under the natural orientation reversing involution Ω of the corresponding I-bundle which exchanges the two endpoints of a fiber, the projection to F of the boundary of some other disk may have self-intersections, and hence there is no obvious projection of $\mathcal{DG}_0$ onto $\mathcal{RD}(c)$ as in the case of a separating I-bundle generator.

Our strategy is to establish instead that the inclusion $\mathcal{RD}(c) \to \mathcal{DG}_0$ is a quasi-isometric embedding. Namely, if this holds true then as $\mathcal{DG}_0$ is hyperbolic, the subspace $\mathcal{RD}(c)$ is *quasi-convex*, that is, there exists a constant $C > 0$ such that any geodesic in $\mathcal{DG}_0$ connecting two points in $\mathcal{RD}(c)$ is contained in the C-neighborhood of $\mathcal{RD}(c)$. Then a (coarsely well defined) shortest distance projection $\mathcal{DG}_0 \to \mathcal{RD}(c)$ is a coarsely Lipschitz retraction by hyperbolicity.

That the inclusion $\mathcal{RD}(c) \to \mathcal{DG}_0$ is indeed a quasi-isometric embedding follows from Theorem 10.1 of [MS13] (which can only be used indirectly as the "holes" are not precisely specified) and, more specifically, Corollary 4.6 and Corollary 4.7 of [H16]. These formulas establish that the distance in the disk graph between two disks D, E which intersect a given I-bundle generator c with base F in precisely two points equals the distance in $\mathcal{A}(F)$ between the projections of ∂D and ∂E to F up to a uniform constant not depending on c. In view of Lemma 2.1, this is what we want to show.

The details are as follows. Construct from the disk graph $\mathcal{DG}_0$ of H_0 another graph $\mathcal{EDG}_0$ with the same vertex set by adding additional edges as follows. If D, E are two disks in H_0, and if up to homotopy, D, E are disjoint from an *essential* simple closed curve in ∂H_0, that is, a simple closed curve which is not homotopic to zero, then we connect D, E by an edge in $\mathcal{EDG}_0$. This graph is called the *electrified disk graph* of H_0 [H16].

Let us denote by $\mathcal{ERD}(c)$ the subgraph of $\mathcal{EDG}_0$ whose vertex set consists of all disks which intersect the non-separating I-bundle generator c in precisely two points. Lemma 4.2 and Lemma 4.1 of [H16] show that the map which associates to an arc in the non-orientable surface F the I-bundle over F is a uniform quasi-isometry between the arc and curve graph of F and $\mathcal{ERD}(c)$. Furthermore, by Corollary 4.6 of [H16], the inclusion $\mathcal{EDR}(c) \to \mathcal{EDG}_0$ is a uniform quasi-isometric embedding. Here uniform means with constants not depending on c.

Let $\zeta : [0, m] \to \mathcal{ERD}(c)$ be a geodesic. Then ζ is a uniform quasi-geodesic in $\mathcal{EDG}_0$. Define the *enlargement* ζ_2 of ζ to be the edge path in $\mathcal{ERD}(c)$ obtained from ζ by replacing each edge $\zeta[k, k+1]$ by an edge path $\zeta_2[i_k, i_{k+1}]$ with the same endpoints as follows.

If the disks $\zeta(k), \zeta(k+1)$ are disjoint, then the edge path $\zeta_2[i_k, i_{k+1}]$ just consists of the edge connecting these two points. Otherwise $\zeta(k), \zeta(k+1)$ intersect, but they are disjoint from an essential simple closed curve in ∂H_0. As each disk $\zeta(j)$ is an I-bundles over an arc $\alpha(j)$ in the surface F, this means that there is an essential simple closed curve $\beta \subset F$ disjoint from both $\zeta(k), \zeta(k+1)$. We refer to Lemma 4.2 of [H16] for a detailed explanation.

An essential subsurface of F containing ∂F is a component of $F - \xi$ where ξ is a collection of pairwise disjoint mutually not freely homotopic essential non-boundary parallel simple closed curves in F. If $\zeta(k), \zeta(k+1)$ are disjoint from an essential simple closed curve in F, then the subsurface $\hat{X}$ of F *filled* by $\zeta(k), \zeta(k+1)$, defined to be the intersection of all essential subsurfaces of F which contain $\zeta(k)$, $\zeta(k+1), \partial F$, is not all of F.

Let $X \subset \partial H_0$ be the preimage of $\hat{X}$ in ∂H_0. Then X is an essential subsurface of ∂H_0 which contains the boundaries of the disks $\zeta(k), \zeta(k+1)$ and is invariant under the orientation reversing involution Ω. No component of its boundary is diskbounding, and it contains c as an I-bundle generator. Furthermore, no essential simple closed curve in X (here essential means non-peripheral) is disjoint from all disks with boundary in X. This follows from the fact that no essential simple closed curve in X is disjoint from both $\zeta(k)$ and $\zeta(k+1)$ as $\zeta(k), \zeta(k+1)$ are invariant under Ω and their projection to F fill the projection $\hat{X}$ of X. A subsurface X of ∂H_0 with these properties is called *thick* in [H16].

The complete subgraph $\mathcal{EDG}(X)$ of $\mathcal{EDG}_0$ whose vertex set is the set of all disks with boundary in X is an electrified disk graph for X. By Corollary 4.6 of [H16], its subgraph $\mathcal{ERD}(c, X)$ of all disks which intersect c in precisely two points is uniformly quasi-isometrically embedded in the electrified disk graph of X. Note that Corollary 4.6 of [H16] only states that this graph is uniformly quasi-convex, however Corollary 2.8 of [H16] shows that indeed, the inclusion of each of these graphs into the electrified disk graph of X is a uniform quasi-isometric embedding. Furthermore, by Lemma 4.2 of [H16], the graph $\mathcal{ERD}(c, X)$ is 4-quasi-isometric to the arc and curve graph of $\hat{X}$ where we require arcs to have endpoints on the distinguished boundary component c of $\hat{X}$.

Now if $\hat{X}$ is the complement of an orientation reversing simple closed curve disjoint from c, then X is the complement in ∂H_0 of an essential simple closed curve. In this case we define $\zeta_2[i_k, i_{k+1}]$ to be the path in $\mathcal{ERD}(c, X)$ connecting $\zeta(k)$ to $\zeta(k+1)$ which consists of I-bundles over arcs in $\hat{X}$ defined by a geodesic in the arc and curve graph of $\hat{X}$. That is, from a geodesic in the arc and curve graph of $\hat{X}$ we construct first an edge path of at most twice the length with the property that among two consecutive vertices, at least one is an arc, and then we view this edge path as an edge path in the graph $\mathcal{ERD}(c, X)$. Thus by Corollary 4.6 of [H16], $\zeta_2[i_k, i_{k+1}]$ is a uniform quasi-geodesic in $\mathcal{EDG}(X)$. If the complement of $\hat{X}$ contains an orientation preserving simple closed curve which does not bound a Möbius band, then the complement of X in ∂H_0 contains at least two disjoint simple closed curves and we define $\zeta_2[i_k, i_{k+1}]$ to be the edge between $\zeta(k)$ and $\zeta(k+1)$.

The resulting edge path ζ_2 in $\mathcal{ERD}(c)$ has the property that two consecutive vertices, which are disks D, E intersecting c in two points, are either disjoint, or their boundaries lie in the same proper thick Ω-invariant subsurface X of ∂H_0 containing c as an I-bundle generator. Moreover, D, E are connected by an edge in the graph $\mathcal{ERD}(c, X)$. In particular, the complement of the subsurface of ∂H_0 filled by D, E contains at least two disjoint essential simple closed curves.

Let $\mathcal{EDG}(2, \partial H_0)$ be the graph whose vertex set is the set of disks and where two disks are connected by an edge if either they are disjoint, or if they are disjoint from a multicurve consisting of at least two non-homotopic components. By Theorem 5.5 of [H16], the graph $\mathcal{EDG}(2, \partial H_0)$ is hyperbolic, and it is an electrification of the disk graph of H_0. This means that it has the same vertex set as the disk graph of H_0, and it is obtained from this disk graph by adding edges.

Theorem 5.5 of [H16] also shows that the path ζ_2 is a uniform quasi-geodesic in $\mathcal{EDG}(2, \partial H_0)$. Namely, following Section 5 of [H16], define a simple closed curve $\gamma \subset \partial H_0$ to be *admissible* if γ is neither diskbounding nor diskbusting. Each such curve defines a thick subsurface of ∂H_0. Write $\mathcal{EDG}(\partial H_0 - \gamma)$ to denote the electrified disk graph of $\partial H_0 - \gamma$ and let $\mathcal{F}(\gamma)$ to be the complete subgraph of $\mathcal{EDG}(2, \partial H_0)$ whose vertex set consists of all disks which are disjoint from γ. A disk $D \subset \mathcal{F}(\gamma)$ defines a vertex in $\mathcal{EDG}(\partial H_0 - \gamma)$.

Following Section 2 of [H16], define the *enlargement* of a uniform quasi-geodesic $\eta : [0, n] \to \mathcal{EDG}_0$ with no backtracking as follows. Assume that $\eta(j), \eta(j+1) \in \mathcal{EDG}(\partial H_0 - \gamma)$ for some admissible simple closed curve γ and some $j < n$; then replace the edge $\eta[j, j+1]$ by a geodesic (or uniform quasi-geodesic) in $\mathcal{EDG}(\partial H_0 - \gamma)$. Note that if $\eta(j), \eta(j+1)$ are disjoint from an essential simple closed curve in $\partial H_0 - \gamma$, then there is an edge between $\eta(j), \eta(j+1)$ in $\mathcal{EDG}(\partial H_0 - \gamma)$. Theorem 5.5 of [H16] states that enlargements of uniform quasi-geodesics in $\mathcal{EDG}_0$ are uniform quasi-geodesics in $\mathcal{EDG}(2, \partial H_0)$.

Now the above construction takes as input a geodesic in $\mathcal{RD}(c)$ and associates to it an enlargement, chosen in such a way that this enlargement consists of disks whose boundaries intersect c in precisely two points. Using once more Theorem 5.5 of [H16], this shows that inclusion defines a quasi-isometric embedding of the complete subgraph of $\mathcal{EDG}(2, \partial H_0)$ of disks which intersect c in precisely two points into the graph $\mathcal{EDG}(2, \partial H_0)$.

This construction can be iterated. In the next step, we modify the path ζ_2 to a path ζ_3 by replacing suitable edges by edge paths as follows. Consider two consecutive vertices $\zeta_2(k), \zeta_2(k+1)$ of ζ_2. These are disks which intersect c in precisely two points. If they are not disjoint, then there exists an essential simple closed curve $\gamma \subset F$ which is disjoint from both $\zeta_2(k), \zeta_2(k+1)$. If γ is orientation preserving and does not bound a Möbious band, then γ has two disjoint preimages γ_1, γ_2 in ∂H_0, and the complement of these preimages is an Ω-invariant thick subsurface X of ∂H_0 containing c as an I-bundle generator. Replace $\zeta_2[k, k+1]$ by a geodesic in $\mathcal{EDG}(X)$ with the same endpoints. This geodesic can be chosen to be the preimage of a geodesic in the arc and curve graph of $F - \gamma$. Proceed in the same way if the complement of the subsurface of F filled by $\zeta(k) \cap F, \zeta(k+1) \cap F$ only contains orientation reversing primitive simple closed curves.

In finitely many steps we construct in this way a path in the graph $\mathcal{RD}(c)$ connecting the endpoints of ζ. Its length roughly equals the sum of the subsurface projections of the projection of its endpoints to F, where the sum is over all essential subsurfaces of F containing the boundary ∂F. In particular, by the distance formula in Corollary 6.3 of [H16], its length does not exceed a uniform multiple of the distance in $\mathcal{DG}_0$ between its endpoints. The statement also follows as by the main result of [H16], the so-called hierarchy paths, constructed from a geodesic in $\mathcal{EDG}_0$ in the above inductive fashion, are uniform quasi-geodesics in the disk graph.

As a consequence, taking the I-bundle over an arc in F defines an isometry between the arc graph of F and the graph $\mathcal{RD}(c)$, and this graph is quasi-isometrically embedded in $\mathcal{DG}_0$. This is what we wanted to show. □

Our goal is to use I-bundle generators in ∂H_0 to construct quasi-isometrically embedded euclidean planes in the disk graph of H. In analogy to [H19], we define an *I-bundle generator* for the spotted handlebody H to be a simple closed curve in ∂H whose image under the spot forgetful map Φ is an I-bundle generator in ∂H_0.

Let $(c_1, c_2) \subset \partial H$ be a pair of non-isotopic disjoint I-bundle generators so that $\partial H - \{c_1 \cup c_2\}$ has a connected component which is an annulus containing the spot in its interior. Then up to isotopy, $\Phi(c_1) = \Phi(c_2) = c$ for an I-bundle generator c in H_0.

The following construction is due to Kra; we refer to [KLS09] for details and for some applications. For its formulation, for a pair (c_1, c_2) of disjoint I-bundle generators on ∂H as in the previous paragraph let $\mathcal{RD}(c_1, c_2)$ be the complete subgraph of the disk graph $\mathcal{DG}$ of H whose vertex set consists of all disks which intersect each of the curves c_1, c_2 in precisely two points. Note that if $D \in \mathcal{RD}(c_1, c_2)$ then the image of D under the spot removing map Φ is contained in $\mathcal{RD}(c)$ where $c = \Phi(c_i)$.

In the next lemma we denote by abuse of notation the map $\mathcal{DG} \to \mathcal{DG}_0$ induced by the spot forgetful map Φ again by Φ. Furthermore, for the remainder of this section we represent a disk by its boundary, that is, we view the disk graph as the complete subgraph of the curve graph of ∂H whose vertex set is the set of diskbounding curves.

Lemma 2.3. *Let (c_1, c_2) be a pair of I-bundle generators bounding a punctured annulus and let $c = \Phi(c_1) = \Phi(c_2)$. There exists a simplicial embedding $\iota : \mathcal{DG}_0 \to \mathcal{DG}$ with the following properties.*

(1) *$\Phi \circ \iota$ is the identity.*
(2) *ι maps $\mathcal{RD}(c)$ into $\mathcal{RD}(c_1, c_2)$.*

Proof. Note first that there is a natural orientation reversing involution ρ_0 of ∂H_0 which exchanges the endpoints of the fibres of the interval bundle over the base F. This involution fixes c and preserves up to isotopy each diskbounding simple closed curve which intersects c in precisely two points. We refer to the discussion before Lemma 2.1 for this fact.

Choose a hyperbolic metric g_0 on ∂H_0 which is invariant under ρ_0 and let $\hat{c}$ be the geodesic representative of c. This makes sense since the geodesic representative of a simple closed curve is simple. Choose a point $p \in \hat{c}$ not contained in any diskbounding simple closed geodesic; this is possible since each diskbounding simple closed geodesic intersects $\hat{c}$ transversely in finitely many points and hence the set of all points of $\hat{c}$ contained in a diskbounding closed geodesic is countable. View p as a marked point on ∂H_0; then the geodesic representative of a diskbounding curve α in ∂H_0 is a diskbounding curve $\iota(\alpha)$ in $\partial H_0 - \{p\}$. Via identification of a disk with its boundary, this construction defines a simplicial embedding

$$\iota : \mathcal{DG}_0 \to \mathcal{DG}$$

with the property that $\Phi \circ \iota$ equals the identity. Note that ι is simplicial and hence one-Lipschitz because the geodesic representatives of two disjoint simple closed curves are disjoint. Furthermore, we clearly have $\iota(\mathcal{RD}(c)) \subset \mathcal{RD}(c_1, c_2)$. □

The situation in the following discussion is illustrated in Figure A. Let B be the connected component of $\partial H - \{c_1, c_2\}$ containing the spot (this is a once spotted annulus). Let Λ be a diffeomorphism of ∂H which preserves the complement of B (and hence the boundary of B) pointwise and which pushes the spot in B one full turn around a central loop in B. The isotopy class of Λ is contained in the kernel of the homomorphism $\mathrm{Mod}(\partial H) \to \mathrm{Mod}(\partial H_0)$ induced by the spot removal map Φ. The map Λ extends to a diffeomorphism of the handlebody H. This can be seen as in the case of point-pushing in a surface: Identify the image of B under the spot removal map Φ with a closed annulus A. Choose a neighborhood N of the punctured annulus B in H which is homeomorphic to $A \times [0, 1]$, with one interior point removed from $A \times \{0\}$. Gradually undo the rotation of the marked point as one moves towards $A \times \{1\} \cup \partial A \times [0, 1]$. Therefore the diffeomorphism Λ generates an infinite cyclic group of simplicial isometries of $\mathcal{RD}(c_1, c_2)$ which we denote again by Λ. With this notation, $\Phi \circ \Lambda = \Phi$.

Let $\Theta_0 : \mathcal{DG}_0 \to \mathcal{RD}(c)$ be as in Lemma 2.2. Define

$$\Theta = \Theta_0 \circ \Phi : \mathcal{DG} \to \mathcal{RD}(c). \tag{1}$$

Observe that $\Theta(\iota(D)) = \Theta_0(D)$ for all disks $D \in \mathcal{DG}_0$. This then implies that $\Theta(\iota(D)) = D$ for all $D \in \mathcal{RD}(c)$. Furthermore, Θ is coarsely Lipschitz. Namely, the puncture forgetful map Φ is simplicial and hence one-Lipschitz, and the map Θ_0 is a coarse Lipschitz retraction by Lemma 2.2. Moreover, we have

$$\Theta(\Lambda(D)) = \Theta(D)$$

for all disks D.

Recall from Lemma 2.1 that $\mathcal{RD}(c)$ is isometric to the arc graph $\mathcal{A}(F)$ of F. Define a distance d_0 on $\mathcal{RD}(c) \times \mathbb{Z}$ by

$$d_0((\alpha, a), (\beta, b)) = d_{\mathcal{RD}(c)}(\alpha, \beta) + |a - b|$$

where $d_{\mathcal{RD}(c)}$ denotes the distance in $\mathcal{RD}(c)$. Let moreover

$$\Omega = \cup_k \Lambda^k \iota(\mathcal{RD}(c)),$$

equipped with the restriction of the distance function of $\mathcal{DG}$.

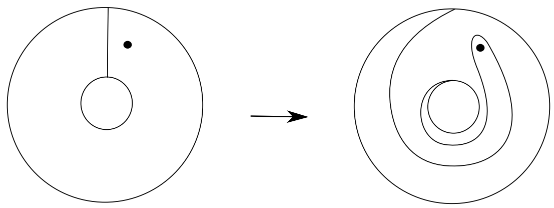
In the following lemma, the fact that the map Ψ is well defined is part of the claim which is established in its proof.

Lemma 2.4. *The map $\Psi : \Omega \to \mathcal{RD}(c) \times \mathbb{Z}$ which maps $D \in \Lambda^k \iota(\mathcal{RD}(c))$ to $\Psi(D) = (\Theta(D), k)$ is a bijective quasi-isometry.*

Proof. Recall that $\Theta(D) = \Theta(\Lambda^k(D))$ for all disks D and all k and that furthermore the restriction of Θ to $\iota(\mathcal{RD}(c))$ is an isometry. In particular, if D_0, E_0 are distinct disks in $\mathcal{RD}(c)$ then $\Theta(\iota(D_0)) \neq \Theta(\iota(E_0))$ and hence $\iota(D_0) \neq \Lambda^k(\iota(E_0))$ for all k.

We claim that for every disk $D \in \Omega$ the following hold true.

Figure A



(1) $D \neq \Lambda^k(D)$ for all $k \neq 0$.
(2) If $D \in \iota(\mathcal{RD}(c))$ then $\Lambda^k D \notin \iota(\mathcal{RD}(c))$ for all $k \neq 0$.
(3) The disks D and $\Lambda(D)$ can be realized disjointly.
(4) Two disks $D \in \Lambda^k \iota(\mathcal{RD}(c)), E \in \Lambda^\ell \iota(\mathcal{RD}(c))$ are disjoint only if $|k-\ell| \leq 1$.

To show the claim let $D \in \Omega$ and for $k \in \mathbb{Z}$ let $D_k = \Lambda^k(D)$. Figure A shows that for $\ell \geq 1$, the disk $D_{k+\ell}$ has precisely $2\ell-2$ essential intersections with D_k, and these intersection points are up to isotopy contained in the annulus B. This yields part (3) of the above claim, and part (4) follows from the same argument. Furthermore, the twist parameter k can be recovered from the geometric intersection numbers between $\Lambda^k(D)$ and $\Lambda^{-1}(D), D, \Lambda(D)$. For example, if $k \geq 2$ then these intersection numbers equal $2k, 2k-2, 2k-4$, respectively, and if $k \leq -2$ then these intersection numbers are $-2k-4, -2k-2, -2k$. This establishes part (1) of the above claim, and part (2) follows from part (1) and the fact that the map ι is an embedding. In particular, $\Omega = \sqcup_k \Lambda^k \iota(\mathcal{RD}(c))$ (disjoint union).

As a consequence, there exists a map Ψ as claimed in the statement of the lemma, and this map is a bijection. Now $\Omega \subset \mathcal{RD}(c_1, c_2)$ and the restriction of the map

Θ to $\mathcal{RD}(c_1, c_2)$ is just the map induced by the spot forgetful map and hence it is one-Lipschitz. Part (4) of the above claim implies that the map Ψ is two-Lipschitz.

As $\Lambda^k \iota(\mathcal{RD}(c))$ is isometric to $\mathcal{A}(F)$ for all k, the inverse of Ψ which associates to a pair $(D, k) \in \mathcal{RD}(c) \times \mathbb{Z}$ the disk $\Lambda^k(\iota(D))$ is coarsely one-Lipschitz. This shows that indeed, the map Ψ is a quasi-isometry. $\square$

The following proposition is the main remaining step towards a proof of Theorem 2.

Proposition 2.5. *There is a coarse Lipschitz retraction $\mathcal{DG} \to \cup_k \Lambda^k \iota(\mathcal{RD}(c)) = \Omega$. Moreover, Ω is a coarsely quasi-convex subset of $\mathcal{DG}$.*

Proof. For the construction of the Lipschitz retraction, we take advantage of the fact that any free homotopy class on a complete hyperbolic surface of finite area can be represented by a unique closed geodesic.

As in the proof of Lemma 2.3, let ρ_0 be an orientation reversing involution of ∂H_0 which fixes the I-bundle generator c pointwise. This involution determines an involution ρ of the complement in ∂H of the interior $\mathrm{int}(B)$ of the annulus B which exchanges the curves c_1 and c_2. Write as before $\Omega = \cup_k \Lambda^k \iota(\mathcal{RD}(c))$.

Choose a complete finite area hyperbolic metric on ∂H (so that the marked point becomes a puncture) with the property that the involution ρ of $\partial H - \mathrm{int}(B)$ is an isometry for this metric which maps the geodesic representative $\hat{c}_1$ of c_1 to the geodesic representative $\hat{c}_2$ of c_2. This metric restricts to a hyperbolic metric on the once punctured annulus B with geodesic boundary. We use this hyperbolic metric to determine for each pair of points $x_i \in \hat{c}_i$ $(i = 1, 2)$ a sample arc in B connecting these two points as follows.

Choose a shortest geodesic arc α connecting the two boundary components of B. By perhaps pulling back the hyperbolic metric with a diffeomorphism of B which preserves the boundary of B pointwise, we may assume that α is contained in the geodesic representative of one of the curves from $\iota(\mathcal{RD}(c))$. Cutting B open along α yields a once punctured right angled rectangle R with geodesic sides, where two distinguished sides come from the arc α. For any pair of points x_1, x_2 on the remaining two sides, choose a shortest geodesic arc $\alpha(x_1, x_2)$ in R connecting these two points. Such an arc is simple, but it may not be unique. By convexity, $\alpha(x_1, x_2)$ is disjoint from α if its endpoints are disjoint from the endpoints of α. Note that as the spot of ∂H is a puncture for the hyperbolic metric, the geodesic arcs $\alpha(x_1, x_2)$ are disjoint from the spot, and $\alpha(x_1, x_2)$ is not necessarily a shortest arc in B with fixed endpoints.

This construction yields for any pair of points $x_1 \in \hat{c}_1, x_2 \in \hat{c}_2$ an oriented geodesic arc $\alpha(x_1, x_2) \subset B$ with endpoints x_1, x_2 such that any two of these arcs connecting distinct pairs of points on $\hat{c}_1, \hat{c}_2$ intersect in at most two points. Furthermore, each of these arcs intersects a geodesic representative of a curve in $\iota(\mathcal{RD}(c))$ in at most two points.

The geodesic arcs $\alpha(x_1, x_2)$ serve as a base marking to measure the twisting of a diskbounding simple closed curve relative to a simple closed curve in the set $\iota(\mathcal{RD}(c)) \subset \mathcal{DG}$. This is reminiscent to the definition of a twist parameter for a simple closed curve crossing through c relative to a fixed marking of the surface ∂H_0. As we have to measure twisting about the puncture, we have to take care of a pair of twist parameters about the simple closed curves c_1, c_2. Our strategy to this end is to put the intersection of a simple closed diskbounding curve β with

$\partial H - B$ into a normal form and use this normal form and the a priori chosen arcs $\alpha(x_1, x_2)$ to determine a twisting datum for β. We next construct such a normal form for the intersection of β with $\partial H - B$ using hyperbolic geometry.

Thus let β be a diskbounding simple closed curve on ∂H. The intersection of β with $\partial H - \mathrm{int}(B)$ consists of a non-empty collection ζ of finitely many pairwise disjoint simple arcs with endpoints on $\hat{c}_1, \hat{c}_2$. Each such arc is freely homotopic relative to $\hat{c}_1, \hat{c}_2$ to a unique geodesic arc which meets $\hat{c}_1, \hat{c}_2$ orthogonally at its endpoints.

We claim that the components of the thus defined collection $\hat{\zeta}$ of geodesic arcs are pairwise disjoint. However, some of these arcs may have nontrivial multiplicities as $\beta \cap (\partial H - \mathrm{int}(B))$ may contain several components which are homotopic relative to the boundary. To verify the claim, double each component X of the hyperbolic surface $\partial H - \mathrm{int}(B)$ along its boundary. The resulting, possibly disconnected, closed hyperbolic surface S admits an isometric involution σ preserving the components of S whose fixed point set is precisely the image C of the boundary of $\partial H - \mathrm{int}(B)$ in the doubled manifold. The double of the above collection ζ of arcs is a collection of simple closed curves on S which are invariant under σ.

The free homotopy classes of these closed curves are σ-invariant and hence the same holds true for their geodesic representatives: Namely, if γ is the geodesic representative of such a free homotopy class, then γ intersects the geodesic multicurve C in precisely two points. Let γ_1 be the component of $\gamma - C$ of smaller length. Then $\gamma_1 \cup \sigma(\gamma_1)$ is a simple closed curve freely homotopic to γ, and its length is at most the length of γ. But γ is the unique simple closed curve of minimal length in its free homotopy class and hence $\gamma = \gamma_1 \cup \sigma(\gamma_1)$. Thus γ intersects C orthogonally, and $\gamma \cap X$ is a component of the arc system $\hat{\zeta}$. The claim now follows from the well known fact that the geodesic representative of a simple closed multicurve on a hyperbolic surface is a simple closed multicurve.

As a consequence of the above discussion, the order of the endpoints of the components of $\beta - \mathrm{int}(B)$ on $\hat{c}_1 \cup \hat{c}_2$ coincides with the order of the endpoints of the collection of geodesic arcs $\hat{\zeta}$ which meet $\hat{c}_1 \cup \hat{c}_2$ orthogonally at their endpoints and are freely homotopic to the components of $\beta - \mathrm{int}(B)$. This implies that a diskbounding simple closed curve β on ∂H can be homotoped to a curve $\hat{\beta}$ of the following form.

(i) The restriction of $\hat{\beta}$ to $\partial H - \mathrm{int}(B)$ consists of a finite collection of pairwise disjoint geodesic arcs which meet $\hat{c}_i$ orthogonally at their endpoints. Some of these arcs may occur more than once.
(ii) The restriction of $\hat{\beta}$ to the once punctured annulus B consists of a finite non-empty collection of arcs connecting $\hat{c}_1$ to $\hat{c}_2$ and perhaps a finite number of arcs which go around the puncture and return to the same boundary component of B. Distinct such arcs have disjoint interiors.

The curve $\hat{\beta}$ is uniquely determined by β and the choice of the hyperbolic metric on ∂H up to a homotopy of the components of $\hat{\beta} \cap B$ with fixed endpoints (note that the above construction does not determine uniquely the intersection of $\hat{\beta}$ with B). This completes the construction of a normal form for a diskbounding simple closed curve β on ∂H.

The goal is to use this normal form to construct a Lipschitz retraction of $\mathcal{DG}$ as stated in the proposition by associating to a diskbounding simple closed curve

β in $\mathcal{DG}$ a pair $\Psi^{-1}(\Theta(\beta),k)$ where Ψ is as in Lemma 2.4, where Θ is as in (1) and where k is a twist parameter, read off from the intersection of the normal form with the once punctured annulus B. We first check compatibility of this twist parameter construction with the twist parameter stemming from the decomposition $\Omega=\cup_k\Lambda^k\iota(\mathcal{RD}(c))$.

By construction of the map ι, if $\beta=\iota(\beta')\in\iota\mathcal{RD}(c)$ then $\hat\beta\cap\partial H-\mathrm{int}(B)$ is just the lift of the geodesic representative of β' to $\partial H-\mathrm{int}(B)$ for the following hyperbolic metric on ∂H_0-c. Recall that the metric on ∂H was chosen in such a way that there exists an orientation reversing involution ρ which maps $\hat c_1$ to $\hat c_2$. Cutting $\mathrm{int}(B)$ off ∂H and gluing c_1 to c_2 with the isometric involution ρ constructs from $\partial H-\mathrm{int}(B)$ a hyperbolic surface which can be viewed as a hyperbolic metric on ∂H_0. Using this metric for the construction of the embedding $\iota:\mathcal{RD}(c)\to\mathcal{RD}(c_1,c_2)$, we conclude that the intersections with B of the representatives $\hat\beta$ of the elements $\beta\in\iota\mathcal{RD}(c)$ are pairwise disjoint.

Define a map

$$\Xi:\mathcal{DG}\to\mathbb{Z}$$

as follows. Let $\hat\beta$ be a closed piecewise geodesic curve with properties (i),(ii) above which is constructed from the simple closed diskbounding curve β. Let b be one of the components of $\hat\beta\cap B$ with endpoints on $\hat c_1$ and $\hat c_2$, oriented in such a way that it connects $\hat c_1$ to $\hat c_2$. Such a component exists since otherwise the image of β under the spot removal map is homotopic to a curve disjoint from the diskbusting curve c on ∂H_0. Let x_1,x_2 be the endpoints of b on $\hat c_1,\hat c_2$.

Let $a=\alpha(x_1,x_2)$; then b,a are simple arcs in B with the same endpoints which intersect some core curve of the annulus B in precisely one point. Assume that $\hat c_1,\hat c_2$ are oriented and define the boundary orientation of B. Then b is homotopic with fixed endpoints to the arc $\hat c_1^k\cdot a\cdot\hat c_2^\ell$ for unique $k,\ell\in\mathbb{Z}$ (read from left to right). In other words, if we denote by τ_i the positive Dehn twist about $\hat c_i$, viewed as a diffeomorphism of the punctured disk B with fixed boundary, then b is homotopic with fixed endpoints to the arc $\tau_1^k\tau_2^{-\ell}a$. Define $\Xi(\beta)=k$.

Observe that although this definition depends on the choice of the arcs $\alpha(x_1,x_2)$ and on the choice of the component b of $B\cap\hat\beta$, the map Ξ is coarsely well defined. Namely, let b' be a second component of $\hat\beta\cap B$, with endpoints x_1',x_2' on $\hat c_1,\hat c_2$ and distinct from b. Then the interior of b' is disjoint from the interior of b. In particular, if a' is an arc in B with the same endpoints as b' whose interior is disjoint from a, then b' is homotopic with fixed endpoints to $\tau_1^q\tau_2^{-r}a'$ for $|q-k|\leq 1,|r-\ell|\leq 1$. On the other hand, the arcs $a=\alpha(x_1,x_2),\alpha(x_1',x_2')$ do not have an essential intersection with a fixed arc connecting $\hat c_1$ to $\hat c_2$ and hence $a'=\tau_1^s\tau_2^{-u}\alpha(x_1',x_2')$ for some $|s|\leq 1,|u|\leq 1$. This shows that the multiplicity k' of the curve $\hat c_1$ in the description of b' relative to $\alpha(x_1',x_2')$ satisfies $|k-k'|\leq 2$. The same reasoning yields that the map Ξ is coarsely two-Lipschitz. Furthermore, we have $\Xi(\iota(\mathcal{RD}(c)))\subset[-2,2]$. Namely, recall that we chose the geodesic arc α in the beginning of this proof to be contained in one of the curves $\iota(\mathcal{RD}(c))$ (which is nothing else but a normalization assumption).

To summarize, the map

$$(\Theta,\Xi):\mathcal{DG}\to\mathcal{RD}(c)\times\mathbb{Z}$$

is coarsely Lipschitz, and its composition with the inverse of the map Ψ from Lemma 2.4 is a coarse Lipschitz retraction of $\mathcal{DG}$ onto Ω provided that the map Ξ maps a point in $\Lambda^k\iota(\mathcal{RD}(c))$ into a uniformly bounded neighborhood of k.

However, if $\beta_0 \in \iota(\mathcal{RD}(c))$ and if $\beta = \Lambda^k(\beta_0) \in \Lambda^k\iota(\mathcal{RD}(c))$, then the intersections with $H-\mathrm{int}(B)$ of the representatives $\hat\beta, \hat\beta_0$ of β, β_0 constructed above coincide. This implies that up to homotopy with fixed endpoints, $\hat\beta \cap B = \Lambda^k(\hat\beta_0 \cap B)$.

On the other hand, point-pushing along a simple closed curve γ based at p descends to conjugation by γ in $\pi_1(\partial H_0, p)$. Therefore the image under the map Λ of a simple arc b in B with endpoints on the two distinct components of ∂B is homotopic with fixed endpoints to c_1bc_2 (recall that we oriented c_1, c_2 so that they define the boundary orientation of B). As $\Xi(\iota(\mathcal{RD}(c))) \subset [-2, 2]$, it follows that $|\Xi(\beta) - k| \leq 2$. This shows the proposition. ☐

To summarize, we obtain

Corollary 2.6. *The disk graph of a handlebody H of genus $g \geq 2$ with one spot contains quasi-isometrically embedded copies of* $\mathbb{R}^2$.

Proof. A subgraph Γ of a metric graph G is uniformly quasi-isometrically embedded if there exists a coarsely Lipschitz retraction $G \to \Gamma$. Proposition 2.5 shows that for any I-bundle generator c in ∂H_0, there is a coarsely Lipschitz retraction of $\mathcal{DG}$ onto its subgraph $\Omega = \cup_k \Lambda^k\iota(\mathcal{RD}(c))$, and by Lemma 2.4, Ω is quasi-isometric to the direct product $\mathcal{RD}(c) \times \mathbb{Z}$. Thus as by Lemma 2.1, $\mathcal{RD}(c)$ is quasi-isometric to the arc graph of the base F of the I-bundle determined by c and hence has infinite diameter, the product of any biinfinite geodesic in $\mathcal{RD}(c)$ and $\mathbb{Z}$ defines a quasi-isometrically embedded $\mathbb{Z}^2$ in $\mathcal{DG}$. ☐

Remark 2.7. In [H19] we showed that in contrast to handlebodies without spots, the disk graph of a handlebody H with a single spot on the boundary is *not* a quasi-convex subgraph of the curve graph of ∂H. We do not know whether $\mathcal{DG}$ contains quasi-isometrically embedded euclidean spaces of dimension bigger than two.

3. Once spotted doubled handlebodies

In this section we consider the connected sum $M = \sharp_g S^2 \times S^1$ of an even number $g = 2n \geq 2$ of copies of $S^2 \times S^1$ with one spot (marked point). We explain how the construction that led to the proof of Theorem 2 can be used to show Theorem 3: The sphere graph of M contains quasi-isometrically embedded copies of $\mathbb{R}^2$.

Consider the double $M_0 = \sharp_g S^2 \times S^1$ of a handlebody H_0 of genus $g \geq 2$ without spots. Let M be the manifold M_0 equipped with a marked point p. As before, we call p a spot in M. There is a natural spot removing map $\Phi : M \to M_0$.

The vertices of the *sphere graph* $\mathcal{SG}$ of M are isotopy classes of embedded spheres in M which are disjoint from the spot and not isotopic into the spot. Isotopies are required to be disjoint from the spot as well. Two such spheres are connected by an edge of length one if they can be realized disjointly. Similarly, let $\mathcal{SG}_0$ be the sphere graph of M_0.

Choose an embedded oriented surface $F_0 \subset M_0$ of genus n with connected boundary such that the inclusion $F_0 \to M_0$ induces an isomorphism $\pi_1(F_0) \to \pi_1(M_0)$. We may assume that the oriented I-bundle H_0 over F_0 is an embedded handlebody $H_0 \subset M_0$ whose double equals M_0. Thus every embedded essential arc α in F_0 with boundary in ∂F_0 determines a sphere $\Upsilon_0(\alpha)$ in M_0 as follows. The interval bundle over α is an embedded essential disk in H_0, with boundary in ∂H_0, and we let $\Upsilon_0(\alpha)$ be the double of this disk. By construction, the sphere $\Upsilon_0(\alpha)$ intersects the surface F_0 precisely in the arc α. By Lemma 4.17 of [HH15], distinct arcs give rise to non-isotopic spheres, furthermore the map Υ_0 preserves disjointness and hence Υ_0 is a simplicial embedding of the arc graph $\mathcal{A}(F_0)$ of F_0 into the sphere graph $\mathcal{SG}_0$ of M_0.

Now mark a point p on the boundary ∂F_0 of F_0 and view the resulting spotted surface F as a surface in the spotted manifold M. The *arc graph* $\mathcal{A}(F)$ of F is the graph whose vertices are isotopy classes of essential simple arcs in F with endpoints on the complement of p in the boundary of F. Here we exclude arcs which are homotopic with fixed endpoints to a subarc of ∂F containing the base point p, and we require that an isotopy preserves the marked point p and hence endpoints of arcs can only slide along $\partial F - \{p\}$. Two such arcs are connected by an edge if they can be realized disjointly. Associate to an arc α in F the double $\Upsilon(\alpha)$ of the I-bundle over α.

The spot removal map $\Phi : M \to M_0$ induces a simplicial surjection $\mathcal{SG} \to \mathcal{SG}_0$, again denoted by Φ for simplicity. Similarly, if we let $\varphi : F \to F_0$ be the map which forgets the marked point $p \in \partial F$, then φ induces a simplicial surjection $\mathcal{A}(F) \to \mathcal{A}(F_0)$, denoted as well by φ. We then obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{A}(F) & \xrightarrow{\varphi} & \mathcal{A}(F_0) \\ \downarrow{\scriptstyle \Upsilon} & & \downarrow{\scriptstyle \Upsilon_0} \\ \mathcal{SG} & \xrightarrow{\Phi} & \mathcal{SG}_0 \end{array} \tag{2}$$

Similar to the case of the handlebody M_0 without spots and the map Υ_0, we obtain

Lemma 3.1. *The map Υ is a simplicial embedding of the arc graph $\mathcal{A}(F)$ into the sphere graph.*

Proof. We have to show that the map Υ is injective. As Υ_0 is injective and as the diagram (2) commutes, it suffices to show the following. Let $\alpha \neq \beta \in \mathcal{A}(F)$ be such that $\varphi(\alpha) = \varphi(\beta)$; then $\Upsilon(\alpha) \neq \Upsilon(\beta)$.

Now $\varphi(\alpha) = \varphi(\beta)$ means that up to exchanging α and β, there exists a number $k > 0$ such that β can be obtained from α by k half Dehn twists about the boundary ∂F of F. Here the half Dehn twist $T(\alpha)$ of α is defined as follows.

The orientation of F induces a boundary orientation for ∂F which in turn induces an orientation on $\partial F - \{p\}$. With respect to the order defined by this orientation, let x be the bigger of the two endpoints x, y of α. Slide x across p to obtain a new arc $T(\alpha)$, with endpoints x', y. This arc is not homotopic to α. To see this it suffices to show that the double $DT(\alpha)$ of $T(\alpha)$ in the double DF of F (which is a surface with one puncture) is not freely homotopic to the double $D(\alpha)$ of α. This follows since $D(\alpha)$ and $DT(\alpha)$ can be homotoped in such a way that they bound a once punctured annulus in DF.

The same reasoning also shows that the sphere $\Upsilon(T(\alpha))$ is not homotopic to the sphere $\Upsilon(\alpha)$. Namely, let $\chi \subset \partial F \cup \{p\}$ be the oriented embedded arc connecting the intersection point x of α with ∂F to the point x'. This arc contains p in its interior. Then the sphere $\Upsilon(T(\alpha))$ is a connected sum of the sphere $\Upsilon(\alpha)$ with the boundary of a punctured ball which is a thickening of χ. Thus $\Upsilon(\alpha)$ and $\Upsilon(T(\alpha))$

can be isotoped in such a way that they bound a subset of M homeomorphic to the complement of an interior point of $S^2 \times [0,1]$.

The above construction, applied to the sphere $\Upsilon(T(\alpha))$ instead of the sphere $\Upsilon(\alpha)$ and where the point y takes on the role of the point x in the above discussion, shows that $\Upsilon(T^2(\alpha))$ is obtained from $\Upsilon(\alpha)$ by point-pushing along the oriented loop ∂F with basepoint p. This is a diffeomorphism of M which leaves the complement of a small tubular neighborhood of ∂F pointwise fixed and pushes the basepoint p along ∂F. As in the proof of Lemma 2.4, this argument can be iterated. It shows that the sphere $\Upsilon(T^k(\alpha))$ intersects the sphere $\Upsilon(\alpha)$ in $k-1$ intersection circles. These circles are essential since they cut both $\Upsilon(T^k(\alpha))$ and $\Upsilon(\alpha)$ into two disks and $k-2$ annuli, where a disk component of $T^k(\alpha) - T(\alpha)$ bounds together with a disk component of $T(\alpha) - T^k(\alpha)$ an embedded sphere enclosing the spot. Invoking the proof of Lemma 2.4, we conclude that indeed, for $k \neq \ell$, $\Upsilon(T^k(\alpha))$ is not homotopic to $\Upsilon(T^\ell(\alpha))$.

We showed so far that the map Υ is injective. To complete the proof of the lemma, it suffices to observe that disjoint arcs are mapped to disjoint spheres. But this is immediate from the construction. $\square$

Proposition 4.18 of [HH15] shows that there is a one-Lipschitz retraction

$$\Psi_0 : \mathcal{SG}_0 \to \Upsilon_0(\mathcal{A}(F_0))$$

which is of the form $\Psi_0 = \Upsilon_0 \circ \Theta_0$ (read from right to left) where $\Theta_0 : \mathcal{SG}_0 \to \mathcal{A}(F_0)$ is a one-Lipschitz map. In particular, $\Upsilon_0(\mathcal{A}(F_0))$ is a quasi-isometrically embedded subgraph of $\mathcal{SG}_0$ which is quasi-isometric to $\mathcal{A}(F_0)$. Our goal is to show that there also is a coarse Lipschitz retraction of $\mathcal{SG}$ onto $\Upsilon(\mathcal{A}(F))$ of the form $\Psi = \Theta \circ \Upsilon$ where $\Theta : \mathcal{SG} \to \mathcal{A}(F)$ is a coarse Lipschitz map. This then yields Theorem 3 from the introduction.

To construct the map Θ we use the method from [HH15]. We next explain how this method can be adapted to our needs.

Let as before $F \subset M$ be an embedded oriented surface with connected boundary ∂F so that M is the double of the trivial I-bundle over F. We assume that the marked point p is contained in the boundary ∂F of F. Furthermore, we assume that the boundary ∂F of F is a smoothly embedded circle in $M \cup \{p\}$ (i.e. an embedded compact one-dimensional submanifold). We use the marked point p as the basepoint for the fundamental group of M. Then ∂F equipped with its boundary orientation defines a homotopy class $\beta \in \pi_1(M,p) = \pi_1(F,p) = \mathcal{F}_{2g}$ (the free group in $2g$ generators). Since β is the oriented boundary curve of F, it is an iterated commutator in a standard set of generators of $\mathcal{F}_{2g}$ and hence β is not contained in any free factor (Whitehead graphs are a convenient tool to verify this fact). Thus ∂F intersects every sphere in M. Namely, for any given sphere S in M, the subgroup of $\pi_1(M,p)$ of all homotopy classes of loops which do not intersect S is a proper free factor of $\pi_1(M,p)$.

As in [HH15] and similar to the construction in Lemma 2.2, the strategy is to associate to a sphere S in M a component of the intersection $F \cap S$. However, unlike in the case of curves on surfaces, there is no suitable normal form for intersections of spheres with the surface F, and the main work in [HH15] consists of overcoming this difficulty by introducing a relative normal form which allows to associate to a sphere in M_0 an intersection arc with F_0 so that the resulting map $\mathcal{SG}_0 \to \mathcal{A}(F_0)$ is one-Lipschitz.

For the remainder of this section we outline the main steps in this construction, adapted to the sphere graph $\mathcal{SG}$ of M and the arc graph $\mathcal{A}(F)$ of F. This requires modifying spheres with isotopies not crossing through p, and modifying the surface F with homotopies leaving the boundary ∂F pointwise fixed.

For convenience, we record some definitions from [HH15] (the following combines Definition 4.7 and Definition 4.9 of [HH15]).

Definition 3.2. Let Σ be a sphere or a sphere system.

(1) *∂F intersects Σ minimally* if ∂F intersects Σ transversely and if no component of $\partial F - \Sigma$ not containing the basepoint p is homotopic with fixed endpoints into Σ.
(2) *F is in minimal position with respect to* Σ if ∂F intersects Σ minimally and if moreover each component of $\Sigma \cap F$ is a properly embedded arc which either is essential or homotopic with fixed endpoints to a subarc of ∂F containing the marked point.

A version of the easy Lemma 4.6 of [HH15] states that any closed curve containing the basepoint can be put into minimal position relative to a sphere system Σ as defined in the first part of Definition 3.2. The following is a version of Lemma 4.12 of [HH15]. For its formulation, call a sphere system Σ *simple* if it decomposes M into a simply connected components.

Lemma 3.3. *Let Σ be a simple sphere system in M. Suppose that F is in minimal position with respect to Σ. Let σ' be an embedded sphere disjoint from Σ and let Σ' be a simple sphere system obtained from Σ by either adding σ', or removing one sphere $\sigma \in \Sigma$. Then F can be homotoped leaving p fixed to a surface F' which is in minimal position with respect to Σ'.*

Proof. As in the proof of Lemma 4.12 of [HH15], removing a sphere preserves minimal position, so only the case of adding a sphere has to be considered.

Thus let Σ be a simple sphere system and let σ' be a sphere disjoint from Σ. Assume that F is in minimal position with respect to Σ. Let W_Σ be the complement of Σ in M, that is, W_Σ is a compact (possibly disconnected) manifold whose boundary consists of $2k$ boundary spheres $\sigma_1^+, \sigma_1^-, \cdots, \sigma_k^+, \sigma_k^-$. The boundary spheres σ_i^+ and σ_i^- correspond to the two sides of a sphere $\sigma_i \in \Sigma$. The surface F intersects W_Σ in a collection of embedded surfaces with boundaries. Each such surface is a polygonal disk P_i $(i = 1, \ldots, m)$. The sides of each such polygon alternate between subarcs of ∂F and arcs contained in Σ. There is at most one bigon, that is, a polygon with two sides, and this polygon then contains the point p in one of its sides. Each rectangle, if any, is homotopic into ∂F.

The proof of Lemma 4.12 of [HH15] now proceeds by studying the intersection of each polygonal component of $F - \Sigma$ with the sphere σ'. This is done by contracting each such polygonal component P to a ribbon tree $T(P)$ in such a way that the boundary components in Σ are contracted to single points in $T(P)$. If P is not a rectangle or bigon, then $T(P)$ has a single vertex which is not univalent. As such ribbon trees are one-dimensional objects, they can be homotoped with fixed endpoints on ∂W_Σ to trees which are in minimal position with respect to σ'. This construction applies without change to rectangles and perhaps the bigon which can be represented by an interval with one endpoint at p and the second endpoint on a component of Σ. We refer to the proof of Lemma 4.12 of [HH15] for details. No adjustment of the argument is necessary. $\square$

The above construction is only valid for simple sphere systems Σ and not for individual spheres. Furthermore, it is known that the arc system on $F\cap\Sigma$ obtained by putting F into minimal position with respect to Σ is not uniquely determined by Σ. To overcome this difficulty, the work of [HH15] uses as an auxiliary datum a maximal system A_0 of pairwise disjoint essential arcs on the surface F_0. Here maximal means that any arc which is disjoint from A_0 is contained in A_0. The system A_0 then *binds* F_0, that is, $F - A_0$ is a union of topological disks. Furthermore, ∂F_0 and each arc $\alpha \in A_0$ is equipped with an orientation.

Choose an arc system A for F which binds F. If $F \subset M$ is in minimal position with respect to Σ, then a homotopy assures that no arc from the arc system A intersects a component of $F - \Sigma$ which is a rectangle or a bigon. Then Lemma 4.12 of [HH15] and its proof applies without modification and shows that with a homotopy, F can be put into normal form with respect to the arc system A, called *A-tight minimal position* with respect to Σ. This then yields the statement of Lemma 4.16 of [HH15]: if F is in A-tight minimal position with respect to the simple sphere system Σ, then the binding arc system $\Sigma \cap F$ is determined by Σ. In particular, two distinct spheres from Σ intersect F in disjoint essential arcs. There may in addition be inessential arcs, i.e. arcs which are homotopic with fixed endpoints to a subsegment of ∂F containing the basepoint p, but these will be unimportant for our purpose.

Now let σ be an essential sphere in M. Let Σ be a simple sphere system in M containing σ as a component. We put F into A-tight minimal position with respect to Σ. Then $\sigma \cap F$ consists of a non-empty collection of essential arcs and perhaps some additional non-essential arcs. Choose one of the essential intersection arcs α and define $\Theta(\sigma) = \alpha$. As in [HH15] and Proposition 2.5 we now obtain

Proposition 3.4. *The map Θ is a coarsely Lipschitz map. For each arc $\alpha \in \mathcal{A}(F)$, we have $\Theta(\Upsilon(\alpha)) = \alpha$. As a consequence, if $g = 2n$ is even then the sphere graph $\mathcal{SG}$ of M contains quasi-isometrically embedded copies of $\mathbb{R}^2$.*

Proof. Given the above discussion, the proof that Θ is a coarsely Lipschitz map is identical to the proof that the map Θ_0 is a coarsely Lipschitz map in Proposition 4.18 of [HH15] and will be omitted. Moreover, as for $\alpha \in \mathcal{A}(F)$, the sphere $\Upsilon(\alpha)$ intersects F in the unique arc α, we have $\Theta(\Upsilon(\alpha)) = \alpha$.

As a consequence, $\Theta|\Upsilon(\mathcal{A}(F))$ is a Lipschitz bijection, with inverse Υ. Then the subgraph $\Upsilon(\mathcal{A}(F))$ of $\mathcal{SG}$ is bilipschitz equivalent to $\mathcal{A}(F)$. Furthermore, the map $\Upsilon \circ \Theta$ is a Lipschitz retraction of $\mathcal{SG}$ onto $\Upsilon(\mathcal{A}(F))$. Then $\Upsilon(\mathcal{A}(F))$ is a quasi-isometrically embedded subgraph of $\mathcal{SG}$ which is moreover quasi-isometric to $\mathcal{A}(F)$.

Let as before F_0 be the surface obtained from F by removing the spot. We are left with showing that $\mathcal{A}(F)$ is quasi-isometric to $\mathcal{A}(F_0) \times \mathbb{Z}$. However, this was shown in Lemma 2.4. Namely, in the terminology used before, the boundary ∂F is an I-bundle generator in the trivial interval bundle H over F, and associating to an arc α the I-bundle over α defines an isomorphism of $\mathcal{A}(F)$ with the subgraph Ω of the disk graph of H used in Lemma 2.4. The statement now follows from Lemma 2.4. □

Remark 3.5. Most likely Proposition 3.4 holds true as well in the case that $g = 2n + 1$ is odd, and furthermore this can be deduced with the above argument using non-orientable surfaces. However, the analogue of Proposition 4.18 of [HH15] for non-orientable surfaces is not available, and we leave the verification of these claims to other authors.

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