SPOTTED DISK AND SPHERE GRAPHS I

URSULA HAMENSTÄDT

Abstract. The disk graph of a handlebody $H$ of genus $g \geq 2$ with $m \geq 0$ marked points on the boundary is the graph whose vertices are isotopy classes of disks disjoint from the marked points and where two vertices are connected by an edge of length one if they can be realized disjointly. We show that for $m = 1$ the disk graph contains quasi-isometrically embedded copies of $\mathbb{R}^2$. The same holds true for sphere graphs of the doubled handlebody with one marked points provided that $g$ is even.

1. Introduction

The curve graph $CG$ of an oriented surface $S$ of genus $g \geq 0$ with $m \geq 0$ punctures and $3g - 3 + m \geq 2$ is the graph whose vertices are isotopy classes of essential (i.e. non-contractible and not homotopic into a puncture) simple closed curves on $S$. Two such curves are connected by an edge of length one if and only if they can be realized disjointly. The curve graph is a locally infinite hyperbolic geodesic metric space of infinite diameter [MM99].

A handlebody of genus $g \geq 1$ is a compact three-dimensional manifold $H$ which can be realized as a closed regular neighborhood in $\mathbb{R}^3$ of an embedded bouquet of $g$ circles. Its boundary $\partial H$ is an oriented surface of genus $g$. We allow that $\partial H$ is equipped with $m \geq 0$ marked points (punctures) which we call spots in the sequel. The group $\text{Map}(H)$ of all isotopy classes of orientation preserving homeomorphisms of $H$ which fix each of the spots is called the handlebody group of $H$. The restriction of an element of $\text{Map}(H)$ to the boundary $\partial H$ defines an embedding of $\text{Map}(H)$ into the mapping class group of $\partial H$, viewed as a surface with punctures [S77, W98].

An essential disk in $H$ is a properly embedded disk $(D, \partial D) \subset (H, \partial H)$ whose boundary $\partial D$ is an essential simple closed curve in $\partial H$, viewed as a surface with punctures. An isotopy of such a disk is supposed to consist of such disks.

The disk graph $DG$ of $H$ is the graph whose vertices are isotopy classes of essential disks in $H$. Two such disks are connected by an edge of length one if and only if they can be realized disjointly.

In [MS13, H19, H16] the following is shown.

Theorem 1. The disk graph of a handlebody of genus $g \geq 2$ without spots is hyperbolic.

The main goal of this work is to show that in contrast to the case of curve graphs, Theorem 1 is not true if we allow spots on the boundary.

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Theorem 2. Let \( H \) be a handlebody of genus \( g \geq 2 \) with one spot. Then the disk graph of \( H \) contains quasi-isometrically embedded copies of \( \mathbb{R}^2 \). In particular, it is not hyperbolic.

Theorem 2 implies that disk graphs cannot be used effectively to obtain a geometric understanding of the handlebody group \( \text{Map}(H) \) of a handlebody \( H \) of genus \( g \geq 3 \) paralleling the program developed by Masur and Minsky for the mapping class group [MM00]. Note that \( \text{Map}(H) \) is an exponentially distorted subgroup of the mapping class group of \( \partial H \) [HH12]. The analogue of the strategy of Masur and Minsky would consist of cutting a handlebody open along an embedded disk which yields a (perhaps disconnected) handlebody with one or two spots on the boundary and studying disk graphs in the cut open handlebody.

Theorem 2 has an analogue for geometric graphs related to the outer automorphism group \( \text{Out}(F_g) \) of the free group on \( g \) generators. Namely, doubling the handlebody \( H \) yields a connected sum \( M = \sharp_g S^2 \times S^1 \) of \( g \) copies of \( S^2 \times S^1 \) with \( m \) marked points. A doubled disk is an embedded essential sphere in \( M \), which is a sphere which is not homotopically trivial or homotopic into a marked point. The sphere graph of \( M \) is the graph whose vertices are isotopy classes of essential spheres in \( M \) and where two such spheres are connected by an edge of length one if and only if they can be realized disjointly. As before, an isotopy of spheres is required to be disjoint from the marked points. The sphere graph of a doubled handlebody without marked points is hyperbolic [HM13b].

Paralleling the result in Theorem 2 we have

Theorem 3. Let \( g \geq 2 \) and let \( M \) be a doubled handlebody of genus \( g \) with one marked point. If \( g \) is even then the sphere graph of \( M \) contains quasi-isometrically embedded copies of \( \mathbb{R}^2 \). In particular, it is not hyperbolic.

The argument in the proof of Theorem 3 uses the first part of Theorem 2 and a result in [HH15] which relates the sphere graph in a connected sum \( \sharp_g S^2 \times S^1 \) for \( g \) even to the arc graph of an oriented surface of genus \( g/2 \) with connected non-empty boundary. A corresponding result for odd \( g \) and a non-orientable surface with a single boundary component would yield the first part of Theorem 3 for odd \( g \geq 3 \), but at the moment, such a result is not available.

As in the case of disk graphs, this indicates that sphere graphs cannot be used to obtain an effective geometric understanding of \( \text{Out}(F_g) \) following the program developed in [MM00]. Theorem 3 may be related to the fact that in contrast to mapping class groups [Mo95], for \( g \geq 3 \) the Dehn functions of \( \text{Out}(F_g) \) and of the handlebody group \( \text{Map}(H) \) of a handlebody of genus \( g \) is exponential [BV12, HM13a, HH19].

2. Once spotted handlebodies

The goal of this section is to construct quasi-isometrically embedded copies of \( \mathbb{R}^2 \) in the disk graph of a handlebody with a single spot.

Thus let \( H \) be a handlebody of genus \( g \geq 2 \) with a single spot. Let \( H_0 \) be the handlebody obtained from \( H \) by removing the spot and let

\[ \Phi : H \to H_0 \]

be the spot removal map. The image under \( \Phi \) of an essential diskbounding simple closed curve in \( \partial H \) is an essential diskbounding simple closed curve in \( \partial H_0 \).
The handlebody $H_0$ without spots can be realized as an $I$-bundle over a surface $F$ with a single boundary component. If the surface $F$ is orientable, then the genus $g$ is even and the $I$-bundle is trivial. The genus of $F$ equals $g/2$, and the boundary $\partial F$ of $F$ defines an isotopy class of a separating simple closed curve $c$ on $\partial H_0$ which decomposes $\partial H_0$ into two surfaces of genus $g/2$, with a single boundary component. If the surface $F$ is non-orientable, then the $I$-bundle is non-trivial and the boundary $\partial F$ defines a non-separating simple closed curve $c$ in $\partial H_0$.

Following [H19, H16], define an $I$-bundle generator for $H_0$ to be a simple closed curve $c \subset \partial H_0$ so that $H_0$ can be realized as an $I$-bundle over a compact surface $F$ with connected boundary $\partial F$ and such that $c$ is freely homotopic to $\partial F \subset \partial H_0$. The surface $F$ is called the base of the $I$-bundle. If the $I$-bundle generator $c$ is separating, then $F$ is orientable of genus $g/2$ where $g$ is the genus of $H_0$. If $c$ is non-separating, then the surface $F$ is non-orientable, and the complement of an open annulus about $c$ in $\partial H_0$ is the orientation cover of $F$. The $I$-bundle over every essential simple embedded arc in $F$ with endpoints on $\partial F$ is an essential disk in $H_0$ which intersects $c$ in precisely two points (up to isotopy).

An $I$-bundle generator $c$ in $\partial H_0$ is diskbusting, which means that it has an essential intersection with every disk (see [MS13, H19]). Namely, the base $F$ of the $I$-bundle is a deformation retract of $H_0$. Thus if $\gamma$ is any essential closed curve on $\partial H_0$ which does not intersect $c$ then $\gamma$ projects to an essential closed curve on $F$. Such a curve is not nullhomotopic in $H_0$ and hence it can not be diskbounding.

The arc graph $A(X)$ of a compact surface $X$ of genus $n \geq 1$ with connected boundary $\partial X$ is the graph whose vertices are isotopy classes of embedded essential arcs in $X$ with endpoints on the boundary, and isotopies are allowed to move the endpoints of an arc along $\partial X$. Two such arcs are connected by an edge of length one if and only if they can be realized disjointly. The arc graph $A(X)$ of $X$ is hyperbolic, however the inclusion of $A(X)$ into the arc and curve graph of $X$ is a quasi-isometry only if the genus of $X$ equals one [MS13] (see also [H16]).

A coarse $L$-Lipschitz retraction of a metric space $(X,d)$ onto a subspace $Y$ is a coarse $L$-Lipschitz map $\Psi : X \to Y$ (this means that $d(\Psi(x),\Psi(y)) \leq Ld(x,y) + L$ for some $L \geq 1$ and all $x,y$) with the additional property that there exists a number $C > 0$ with $d(\Psi(y),y) \leq C$ for all $y \in Y$. If $X$ is a geodesic metric space then the image $Y$ of a coarse Lipschitz retraction is a coarsely quasi-convex subspace of $X$, i.e. any two points in $Y$ can be connected by a uniform quasi-geodesic in $X$ which is entirely contained in $Y$.

For an $I$-bundle generator $c$ in $H_0$ let $\mathcal{RD}(c)$ be the complete subgraph of the disk graph $\mathcal{DG}_0$ of $H_0$ consisting of disks which intersect $c$ in precisely two points. The boundary of each such disk is an $I$-bundle over an arc in the base $F$ of the $I$-bundle corresponding to $c$. As two such disks are disjoint if and only if the corresponding arcs in $F$ are disjoint, the graph $\mathcal{RD}(c)$ is isometric to the arc graph $A(F)$ of $F$.

**Lemma 2.1.** There exists a coarsely Lipschitz retraction $\Theta_0 : \mathcal{DG}_0 \to \mathcal{RD}(c)$ whose restriction to $\mathcal{RD}(c)$ is the identity.

**Proof.** The case that $c$ is a separating $I$-bundle generator is completely elementary. Namely, in this case the base $F$ of the $I$-bundle can be identified with a component of $\partial H_0 - c$. As $c$ is diskbusting, the map

$$\Upsilon_0 : \mathcal{DG}_0 \to A(F)$$
which associates to a disk $D$ a component of $\partial D \cap F$ is coarsely well defined: Although it depends on choices, any other choice $\Upsilon'_0$ maps a disk $D$ to an arc disjoint from $\Upsilon_0(D)$. If we denote by $Q : A(F) \to \mathcal{RD}(c)$ the map which associates to an arc $\alpha$ in $F$ the $I$-bundle over $\alpha$, then the disks $Q(\Upsilon_0(D)), Q(\Upsilon'_0(D))$ are disjoint as well.

Furthermore, if $D, D'$ are disjoint disks then the arcs $\Upsilon_0(D), \Upsilon_0(D')$ are disjoint and hence $d_{\mathcal{DG}_0}(Q\Upsilon_0(D), Q\Upsilon_0(D')) \leq 1$. This shows that $Q \circ \Upsilon_0$ is coarsely one-Lipschitz. As a disk $D \in \mathcal{RD}(c)$ intersects $F$ in a single arc, we have $Q\Upsilon_0(D) = D$. This completes the proof of the lemma in the case that $c$ is separating.

The above argument does not immediately extend to non-separating $I$-bundle generators. Namely, if $c$ is a non-separating $I$-bundle generator, then the natural orientation reversing involution $\Phi$ of the corresponding $I$-bundle which exchanges the two endpoints of a fiber acts as an orientation reversing involution on the boundary $\partial H_0$ of $H_0$. This action preserves an embedded open annulus $A \subset \partial H_0$ about $c$, and the action of $\Phi$ on $\partial H_0 - A$ is free, with quotient a non-orientable surface $F$ with connected boundary $\partial F$. A disk which intersects $c$ in precisely two points then is the $I$-bundle over an embedded arc in $F$. Its boundary is a $\Phi$-invariant simple closed curve on $\partial H_0$. Thus there is no obvious projection of $\mathcal{DG}_0$ onto $\mathcal{RD}(c)$ as in the case of a separating $I$-bundle generator.

To show that the lemma holds true in this case as well, it suffices to show that the inclusion $\mathcal{RD}(c) \to \mathcal{DG}_0$ is a quasi-isometric embedding. Namely, if this holds true then as $\mathcal{DG}_0$ is hyperbolic, there exists a coarsely distance non-increasing coarsely well defined shortest distance projection $\mathcal{DG}_0 \to \mathcal{RD}(c)$, and such a projection is a coarsely Lipschitz retraction.

That the inclusion $\mathcal{RD}(c) \to \mathcal{DG}_0$ is indeed a quasi-isometric embedding follows from Theorem 10.1 of [MS13] (which can only be used indirectly as the “holes” are not precisely specified) and, more specifically, Corollary 4.6 and Corollary 4.7 of [H16].

To be more precise, in [H16] we constructed from the disk graph $\mathcal{DG}_0$ of $H_0$ another graph $\mathcal{EDG}_0$ with the same vertex set by adding additional edges as follows. If $D, E$ are two disks in $H_0$, and if up to homotopy, $D, E$ are disjoint from an essential simple closed curve in $\partial H_0$, then we connect $D, E$ by an edge in $\mathcal{EDG}_0$. The graph is called the electrified disk graph of $H_0$.

Let us denote by $\mathcal{ERD}(c)$ the subgraph of $\mathcal{EDG}_0$ whose vertex set consists of all disks which intersect the non-separating $I$-bundle generator $c$ in precisely two points. Lemma 4.2 of [H16] shows that the map which associates to an arc in the non-orientable surface $F$ the $I$-bundle over $F$ is a two-quasi-isometry between the arc and curve graph of $F$ and $\mathcal{ERD}(c)$. Furthermore, by Corollary 4.6 of [H16], the inclusion $\mathcal{ERD}(c) \to \mathcal{EDG}_0$ is a uniform quasi-isometric embedding (here uniform means with constants not depending on $c$).

Let $\zeta : [0, m] \to \mathcal{ERD}(c)$ be a geodesic. Then $\zeta$ is a uniform quasi-geodesic in $\mathcal{EDG}_0$. Define the enlargement $\zeta_2$ of $\zeta$ to be the edge path in $\mathcal{ERD}(c)$ obtained from $\zeta$ by replacing each edge $\zeta[k, k+1]$ by an edge path $\zeta_2[i_k, i_{k+1}]$ with the same endpoints as follows.

If the disks $\zeta(k), \zeta(k+1)$ are disjoint, then the edge path $\zeta_2[i_k, i_{k+1}]$ just consists of the edge connecting these two points. Otherwise $\zeta(k), \zeta(k+1)$ are disjoint from an essential simple closed curve in $\partial H_0$. As each disk $\zeta(j)$ is an $I$-bundles over an arc $\alpha(j)$ in the surface $F$, this means that there is an essential simple closed curve $\beta \subset F$
disjoint from both \(\alpha(k), \alpha(k + 1)\). We refer to Lemma 4.2 of [H16] for a detailed explanation. Let \(X \subset \partial H_0\) be the component of the complement of the preimage of \(\beta\) in \(\partial H_0\) which contains \(c\). Then \(X\) is an essential subsurface in \(\partial H_0\) which contains the boundaries of the disks \(\zeta(k), \zeta(k + 1)\). No component of its boundary is diskbounding, and it contains \(c\) as an \(I\)-bundle generator. Furthermore, no essential simple closed curve in \(X\) (here essential means non-peripheral) is disjoint from all disks with boundary in \(X\). A subsurface \(X\) of \(\partial H_0\) with these properties is called thick in [H16].

By Corollary 4.6 of [H16], the set of all disks with boundary in \(X\) defines an electrified disk graph for \(X\). Its subgraph of all disks which intersect \(c\) in precisely two points is uniformly quasi-isometrically embedded in the electrified disk graph of \(X\). Furthermore, it is \(2\)-quasi-isometric to the arc and curve graph of \(F - \beta\). Define \(\zeta_2[i_k, i_{k + 1}]\) to be the path in \(\mathcal{ERD}(c)\) connecting \(\zeta(k)\) to \(\zeta(k + 1)\) which consists of \(I\)-bundles over arcs in \(F - \beta\) defined by a geodesic in the arc and curve graph of \(F - \beta\). That is, from a geodesic in the arc and curve graph of \(F - \beta\) we construct first an edge path of at most twice the length with the property that among two consecutive vertices, at least one is an arc, and then we view this edge path as an edge path in the subgraph \(\mathcal{ERD}(c, X)\) of the electrified disk graph of \(X\) consisting of disks which intersect \(c\) in precisely two points.

The resulting edge path \(\zeta_2\) in \(\mathcal{ERD}(c)\) has the property that two consecutive edges are either disjoint, or their boundaries lie in the same proper thick subsurface \(X\) of \(\partial H_0\) containing \(c\) as an \(I\)-bundle generator, and they are connected by an edge in the graph \(\mathcal{ERD}(c, X)\).

By the main result of [H16], the path \(\zeta_2\) is a uniform quasi-geodesic in the graph \(\mathcal{EDG}(2, H_0)\) whose vertex set is the set of disks and where two disks are connected by an edge if either they are disjoint, or if they are disjoint from a multicurve consisting of at least two components. Furthermore, the graph \(\mathcal{EDG}(2, H_0)\) is hyperbolic, and it is an electrification of the disk graph of \(H_0\).

This construction can be iterated. In the next step, we inspect two consecutive vertices \(\zeta_2(k), \zeta_2(k + 1)\) of \(\zeta_2\). These are disks which intersect \(c\) in precisely two points. If they are not disjoint, then they are consecutive vertices of one of the edge paths inserted into \(\zeta\) to construct \(\zeta_2\). That is, their boundaries are contained in the same thick subsurface \(X\) of \(\partial H_0\) containing \(c\) as an \(I\)-bundle generator, and they are disjoint from the preimage in \(\partial H_0\) of a simple closed curve \(\beta\) in the subsurface \(F_0\) of \(F\) which defines \(X\).

The curve \(\beta\) determines a new thick subsurface \(\hat{X} \subset \partial H_0\) containing \(c\) as an \(I\)-bundle generator, and this subsurface can be used to connect \(\zeta_2(k)\) to \(\zeta_2(k + 1)\) by an edge path. In finitely many steps we construct in this way a path in the graph \(\mathcal{RD}(c)\) connecting the endpoints of \(\zeta\). Its length roughly equals the sum of the subsurface projections of its endpoints into all thick subsurfaces of \(\partial H_0\) containing \(c\) as an \(I\)-bundle generator. In particular, by the distance formula in Corollary 4.7 of [H16], its length is uniformly equivalent to the distance in \(\mathcal{DG}_0\) between its endpoints. This also follows as by the main result of [H16], the so-called hierarchy paths, constructed from a geodesic in \(\mathcal{EDG}_0\) in the above fashion, are uniform quasi-geodesics in the disk graph.

As a consequence, taking the \(I\)-bundle over an arc in \(F\) defines an isometry between the arc graph of \(F\) and the graph \(\mathcal{RD}(c)\), and this graph is quasi-isometrically embedded in \(\mathcal{DG}_0\). This is what we wanted to show. \(\square\)
Our goal is to use \( I \)-bundle generators in \( \partial H_0 \) to construct quasi-isometrically embedded euclidean planes in the disk graph of \( H \). In analogy to [H19], we define an \( I \)-bundle generator for the spotted handlebody \( H \) to be a simple closed curve in \( \partial H \) whose image under the map \( \Phi \) is an \( I \)-bundle generator in \( \partial H_0 \).

Let \((c_1, c_2) \subset \partial H\) be a pair of non-isotopic disjoint \( I \)-bundle generators so that \( \partial H - \{c_1 \cup c_2\} \) has a connected component which is an annulus containing the spot in its interior. Then up to isotopy, \( \Phi(c_1) = \Phi(c_2) = c \) for an \( I \)-bundle generator \( c \) in \( H_0 \).

The following construction is due to Kra; we refer to [KLS09] for details and for some applications. For its formulation, for a pair \((c_1, c_2)\) of disjoint \( I \)-bundle generators on \( \partial H \) as in the previous paragraph let \( \mathcal{RD}(c_1, c_2) \) be the complete subgraph of the disk graph \( \mathcal{DG} \) of \( H \) whose vertex set consists of all disks which intersect each of the curves \( c_1, c_2 \) in precisely two points. Note that if \( D \in \mathcal{RD}(c_1, c_2) \) then the image of \( D \) under the spot removing map \( \Phi \) is contained in \( \mathcal{RD}(c) \) where \( c = \Phi(c_i) \).

In the next lemma we denote by abuse of notation the map \( \mathcal{DG} \rightarrow \mathcal{DG}_0 \) induced by the spot forgetful map \( \Phi \) again by \( \Phi \). Furthermore, for the remainder of this section we represent a disk by its boundary, i.e. we view the disk graph as the complete subgraph of the curve graph of \( \partial H \) whose vertex set is the set of diskbounding curves.

**Lemma 2.2.** Let \((c_1, c_2)\) be a pair of \( I \)-bundle generators bounding a punctured annulus and let \( c = \Phi(c_1) = \Phi(c_2) \). There exists a simplicial embedding \( \iota : \mathcal{DG}_0 \rightarrow \mathcal{DG} \) with the following properties.

1. \( \Phi \circ \iota \) is the identity.
2. \( \iota \) maps \( \mathcal{RD}(c) \) into \( \mathcal{RD}(c_1, c_2) \).

**Proof.** Note first that there is a natural orientation reversing involution \( \rho_0 \) of \( \partial H_0 \) which exchanges the endpoints of the fibres of the interval bundle over the base \( F \). This involution fixes \( c \) and preserves up to isotopy each diskbounding simple closed curve which intersects \( c \) in precisely two points.

Choose a hyperbolic metric \( g_0 \) on \( \partial H_0 \) which is invariant under \( \rho_0 \) and let \( \hat{c} \) be the geodesic representative of \( c \). Choose a point \( p \in \hat{c} \) not contained in any diskbounding simple closed geodesic; this is possible since each diskbounding simple closed geodesic intersects \( \hat{c} \) transversely in finitely many points and hence the set of all points of \( \hat{c} \) contained in a diskbounding closed geodesic is countable. View \( p \) as a marked point on \( \partial H_0 \); then the geodesic representative of a diskbounding curve \( \alpha \) in \( \partial H_0 \) is a diskbounding curve \( \iota(\alpha) \) in \( \partial H_0 - \{p\} \). Via identification of a disk with its boundary, this construction defines a simplicial embedding

\[
\iota : \mathcal{DG}_0 \rightarrow \mathcal{DG}
\]

with the property that \( \Phi \circ \iota \) equals the identity. Furthermore, we clearly have \( \iota(\mathcal{RD}(c)) \subset \mathcal{RD}(c_1, c_2) \).

The situation in the following discussion is illustrated in Figure A. Let \( B \) be the connected component of \( \partial H - \{c_1, c_2\} \) containing the spot (this is a once spotted annulus). Let \( \Lambda \) be a diffeomorphism of \( \partial H \) which preserves the complement of \( B \) (and hence the boundary of \( B \)) pointwise and which pushes the spot in \( B \) one full turn around a central loop in \( B \). The isotopy class of \( \Lambda \) is contained in the kernel of the homomorphism \( \text{Mod}(\partial H) \rightarrow \text{Mod}(\partial H_0) \) induced by the spot removal map \( \Phi \). The map \( \Lambda \) extends to a diffeomorphism of the handlebody \( H \). This can be
seen as in the case of point-pushing in a surface: Identify the image of \( B \) under the spot removal map \( \Phi \) with a closed annulus \( A \). Choose a neighborhood \( N \) of the punctured annulus \( B \) in \( H \) which is homeomorphic to \( A \times [0, 1] \), with one interior point removed from \( A \times \{0\} \). Gradually undo the rotation of the marked point as one moves towards \( A \times \{1\} \cup \partial A \times [0, 1] \). Therefore the diffeomorphism \( \Lambda \) generates an infinite cyclic group of simplicial isometries of \( \mathcal{RD}(c_1, c_2) \) which we denote again by \( \Lambda \). With this notation, \( \Phi \circ \Lambda = \Phi \).

![Figure A](image)

Let \( \Theta_0 : \mathcal{DG}_0 \to \mathcal{RD}(c) \) be as in Lemma 2.1. Define
\[
\Theta = \Theta_0 \circ \Phi : \mathcal{DG} \to \mathcal{RD}(c).
\]
Observe that \( \Theta(\iota(D)) = \Theta_0(D) \) for all disks \( D \in \mathcal{DG}_0 \). This then implies that \( \Theta(\iota(D)) = D \) for all \( D \in \mathcal{RD}(c) \). Furthermore, \( \Theta \) is coarsely Lipschitz (compare the proof of Lemma 2.1 for a detailed explanation), and we have
\[
\Theta(\Lambda(D)) = \Theta(D)
\]
for all disks \( D \).
Recall that \( \mathcal{RD}(c) \) is isometric to the arc graph \( \mathcal{A}(F) \) of \( F \). Define a distance \( d_0 \) on \( \mathcal{RD}(c) \times \mathbb{Z} \) by
\[
d_0((\alpha, a), (\beta, b)) = d_{\mathcal{RD}(c)}(\alpha, \beta) + |a - b|
\]
where \( d_{\mathcal{RD}(c)} \) denotes the distance in \( \mathcal{RD}(c) \). Let moreover
\[
\Omega = \cup_k \Lambda^k(\mathcal{RD}(c)).
\]

**Lemma 2.3.** The map \( \Psi : \Omega \to \mathcal{RD}(c) \times \mathbb{Z} \) which maps \( D \in \Lambda^k(\mathcal{RD}(c)) \) to \( \Psi(D) = (\Theta(D), k) \) is a bijective quasi-isometry.

**Proof.** Recall that \( \Theta(D) = \Theta(\Lambda^k(D)) \) for all disks \( D \) and all \( k \) and that furthermore the restriction of \( \Theta \) to \( \iota(\mathcal{RD}(c)) \) is an isometry. In particular, if \( D_0, E_0 \) are distinct disks in \( \mathcal{RD}(c) \) then \( \Theta(\iota(D_0)) \neq \Theta(\iota(E_0)) \) and hence \( \Psi(\iota(D_0)) \neq \Psi(\Lambda^k(\iota(E_0))) \) for all \( k \).

We claim that for every disk \( D \in \Omega \) the following hold true.

1. \( D \neq \Lambda^k(D) \) for all \( k \neq 0 \).
2. The disks \( D \) and \( \Lambda(D) \) can be realized disjointly.
3. Two disks \( D \in \Lambda^k(\mathcal{RD}(c)), E \in \Lambda^\ell(\mathcal{RD}(c)) \) are disjoint only if \( |k - \ell| \leq 1 \).

To end let \( D \in \Omega \) and for \( k \in \mathbb{Z} \) let \( D_k = \Lambda^k(D) \). Figure A shows that for \( \ell \geq 1 \), the disk \( D_{k+\ell} \) has precisely \( 2\ell - 2 \) essential intersections with \( D_k \), and these intersection points are up to isotopy contained in the annulus \( B \). This yields part (2) of the above claim, and part (3) follows from the same argument. Furthermore, the twist parameter \( k \) can be recovered from the geometric intersection numbers between \( \Lambda^k(D) \) and \( \Lambda^{-1}(D), D, \Lambda(D) \). For example, if \( k \geq 2 \) then these intersection numbers equal \( 2k, 2k - 2, 2k - 4 \), respectively, and if \( k \leq -2 \) then these intersection numbers are \( -2k - 4, -2k - 2, -2k \). This establishes part (1) of the above claim.

Part (1) of the above claim together with the beginning of this proof yields that the map \( \Psi \) is well defined and a bijection. Now \( \Omega \subset \mathcal{RD}(c_1, c_2) \) and the restriction of the map \( \Theta \) to \( \mathcal{RD}(c_1, c_2) \) is just the map induced by the spot forgetful map and hence it is one-Lipschitz. Part (3) of the above claim implies that the map \( \Psi \) is two-Lipschitz.

As \( \Lambda^k(\mathcal{RD}(c)) \) is isometric to \( \mathcal{A}(F) \) for all \( k \), the inverse of \( \Psi \) which associates to a pair \( (D, k) \in \mathcal{RD}(c) \times \mathbb{Z} \) the disk \( \Lambda^k(\iota(D)) \) is coarsely one-Lipschitz. This shows that indeed, the map \( \Psi \) is a quasi-isometry.

The following proposition is the main remaining step towards a proof of Theorem 2.

**Proposition 2.4.** There is a coarse Lipschitz retraction \( \mathcal{DG} \to \cup_k \Lambda^k(\mathcal{RD}(c)) = \Omega \). Moreover, \( \Omega \) is a coarsely quasi-convex subset of \( \mathcal{DG} \).

**Proof.** As in the proof of Lemma 2.2, let \( \rho_0 \) be an orientation reversing involution of \( \partial H_0 \) which fixes the \( I \)-bundle generator \( c \) pointwise. This involution determines an involution \( \rho \) of the complement in \( \partial H \) of the interior \( \text{int}(B) \) of the annulus \( B \) which exchanges the curves \( c_1 \) and \( c_2 \). Write as before \( \Omega = \cup_k \Lambda^k(\mathcal{RD}(c)) \).

Choose a complete finite area hyperbolic metric on \( \partial H \) (so that the marked point becomes a puncture) with the property that the involution \( \rho \) of \( \partial H - \text{int}(B) \) is an isometry for this metric which maps the geodesic representative \( \tilde{c}_1 \) of \( c_1 \) to the geodesic representative \( \tilde{c}_2 \) of \( c_2 \). This metric restricts to a hyperbolic metric on the once punctured annulus \( B \) with geodesic boundary.
Choose a geodesic arc $\alpha$ connecting the two boundary components of $B$ which is contained in the geodesic representative of one of the curves from $\iota(RD(c))$. Cutting $B$ open along $\alpha$ yields a once punctured rectangle with geodesic sides, where two distinguished sides come from the arc $\alpha$. For any pair of points $x_1, x_2$ on the remaining two sides, choose a simple arc in $B$ connecting these two points which does not cross through $\alpha$ and let $\alpha(x_1, x_2) \subset B$ be the geodesic representative of this arc. By convexity, $\alpha(x_1, x_2)$ is disjoint from $\alpha$ if its endpoints are disjoint from the endpoints of $\alpha$.

This construction yields for any pair of points $x_1 \in \hat{c}_1, x_2 \in \hat{c}_2$ an oriented geodesic arc $\alpha(x_1, x_2) \subset B$ with endpoints $x_1, x_2$ such that any two of these arcs connecting distinct pairs of points on $\hat{c}_1, \hat{c}_2$ intersect in at most two points. Furthermore, each of these arcs intersects a geodesic representative of a curve in $\iota(RD(c))$ in at most two points.

We use these oriented arcs as follows. Let $\beta$ be a diskbounding simple closed curve on $\partial H$. The intersection of $\beta$ with $\partial H - \text{int}(B)$ consists of a non-empty collection $\zeta$ of finitely many pairwise disjoint simple arcs with endpoints on $\hat{c}_1, \hat{c}_2$. Each such arc is freely homotopic relative to $\hat{c}_1, \hat{c}_2$ to a unique geodesic arc which meets $\hat{c}_1, \hat{c}_2$ orthogonally at its endpoints.

We claim that the components of the thus defined collection $\hat{\zeta}$ of geodesic arcs are pairwise disjoint. However, some of these arcs may have nontrivial multiplicities as $\beta \cap (\partial H - \text{int}(B))$ may contain several components which are homotopic relative to the boundary. To verify the claim, double each component $X$ of the hyperbolic surface $\partial H - \text{int}(B)$ along its boundary. The possibly disconnected resulting closed hyperbolic surface $S$ admits an isometric involution $\sigma$ preserving the components of $S$ whose fixed point set is precisely the image $C$ of the boundary of $\partial H - \text{int}(B)$ in the doubled manifold. The double of the above collection $\zeta$ of arcs is a collection of simple closed curves on $S$ which are invariant under $\sigma$.

The free homotopy classes of these closed curves are $\sigma$-invariant and hence the same holds true for their geodesic representatives: Namely, if $\gamma$ is the geodesic representative of such a free homotopy class, then $\gamma$ intersects the geodesic multicurve $C$ in precisely two points. Let $\gamma_1$ be the component of $\gamma - C$ of smaller length. Then $\gamma_1 \cup \sigma(\gamma_1)$ is a simple closed curve freely homotopic to $\gamma$, and its length is at most the length of $\gamma$. But $\gamma$ is the unique simple closed curve of minimal length in its free homotopy class and hence $\gamma = \gamma_1 \cup \sigma(\gamma_1)$. Thus $\gamma$ intersects $C$ orthogonally, and $\gamma \cap X$ is a component of the arc system $\hat{\zeta}$. The claim now follows from the well known fact that the geodesic representative of a simple closed multicurve on a hyperbolic surface is a simple closed multicurve.

As a consequence of the above discussion, the order of the endpoints of the components of $\beta - \text{int}(B)$ on $\hat{c}_1 \cup \hat{c}_2$ coincides with the order of the endpoints of the collection of geodesic arcs $\hat{\zeta}$ which meet $\hat{c}_1 \cup \hat{c}_2$ orthogonally at their endpoints and are freely homotopic to the components of $\beta - \text{int}(B)$. This implies that a diskbounding simple closed curve $\beta$ on $\partial H$ can be homotoped to a curve $\hat{\beta}$ of the following form. The restriction of $\hat{\beta}$ to $\partial H - \text{int}(B)$ consists of a finite collection of pairwise disjoint geodesic arcs which meet $\hat{c}_1$ orthogonally at their endpoints. Some of these arcs may occur more than once. The restriction of $\hat{\beta}$ to the once punctured annulus $B$ consists of a finite non-empty collection of arcs connecting $\hat{c}_1$ to $\hat{c}_2$ and perhaps a finite number of arcs which go around the puncture and return to the same boundary component of $B$. Distinct such arcs have disjoint interiors. The
curve \( \hat{\beta} \) is uniquely determined by \( \beta \) up to a homotopy with fixed endpoints of the components of \( \hat{\beta} \cap B \). By construction of the map \( \iota \), if \( \beta = \iota(\beta') \in \mathcal{RD}(c) \) then \( \hat{\beta} \cap \partial H - \text{int}(B) \) is just the lift of the geodesic representative of \( \beta' \) to \( \partial H - \text{int}(B) \) for the following hyperbolic metric on \( \partial H_0 - c \). Recall that the metric on \( \partial H \) was chosen in such a way that the geodesics \( \hat{c}_1, \hat{c}_2 \) have the same length. Then \( \partial H - B \) can be glued along the two boundary components to a hyperbolic surface which can be viewed as a hyperbolic metric on \( \partial H_0 \). This metric depends on the choice of a twist parameter, but its restriction to the complement of the geodesic representative of the curve \( c \) does not. In particular, the intersections with \( B \) of the representatives \( \hat{\beta} \) of the elements \( \beta \in \mathcal{RD}(c) \) are pairwise disjoint.

We use this normal form for diskbounding simple closed curves to define a map

\[
\Xi : \mathcal{DG} \to \mathbb{Z}
\]

as follows. Let \( \hat{\beta} \) be a closed curve constructed from the simple closed diskbounding curve \( \beta \) as in the previous paragraph. Let \( b \) be one of the components of \( \hat{\beta} \cap B \) with endpoints on \( \hat{c}_1 \) and \( \hat{c}_2 \), oriented in such a way that it connects \( \hat{c}_1 \) to \( \hat{c}_2 \). Such a component exists since otherwise the image of \( \beta \) under the spot removal map is homotopic to a curve disjoint from the diskbounding curve \( c \) on \( \partial H_0 \). Let \( x_1, x_2 \) be the endpoints of \( b \) on \( \hat{c}_1, \hat{c}_2 \).

Let \( a = \alpha(x_1, x_2) \); then \( b, a \) are simple arcs in \( B \) with the same endpoints which intersect some core curve of the annulus \( B \) in precisely one point. Assume that \( \hat{c}_1, \hat{c}_2 \) are oriented and define the boundary orientation of \( B \). Then \( b \) is homotopic with fixed endpoints to the arc \( \hat{c}_1^k \cdot a \cdot \hat{c}_2^\ell \) for unique \( k, \ell \in \mathbb{Z} \) (read from left to right).

In other words, if we denote by \( \tau_i \) the positive Dehn twist about \( \hat{c}_i \), viewed as a diffeomorphism of the punctured disk \( B \) with fixed boundary, then \( b \) is homotopic with fixed endpoints to the arc \( \tau_1^k \tau_2^\ell a \). Define \( \Xi(\beta) = k \).

Observe that although this definition depends on the choice of the arcs \( \alpha(x_1, x_2) \) and on the choice of the component \( b \) of \( B \cap \hat{\beta} \), the map \( \Xi \) is coarsely well defined. Namely, let \( b' \) be a second component of \( \hat{\beta} \cap B \), with endpoints \( x'_1, x'_2 \) on \( \hat{c}_1, \hat{c}_2 \) and distinct from \( b \). Then the interior of \( b' \) is disjoint from the interior of \( b \). In particular, if \( a' \) is an arc in \( B \) with the same endpoints as \( b' \) whose interior is disjoint from \( a \), then \( b' \) is homotopic with fixed endpoints to \( \tau_1^q \tau_2^r a' \) for \( |q - k| \leq 1, |r - \ell| \leq 1 \).

On the other hand, both arcs \( a, \alpha(x'_1, x'_2) \) do not intersect a fixed arc connecting \( \hat{c}_1 \) and hence \( a' = \tau_1^s \tau_2^u a(x'_1, x'_2) \) for some \( |s| \leq 1, |u| \leq 1 \). This shows that the multiplicity \( k' \) of the curve \( \hat{c}_1 \) in the description of \( b' \) relative to \( \alpha(x'_1, x'_2) \) satisfies \( |k - k'| \leq 2 \). The same reasoning yields that the map \( \Xi \) is coarsely two-Lipschitz.

Furthermore, we have \( \Xi(\iota(\mathcal{RD}(c))) \subset [-2, 2] \).

To summarize, the map

\[
(\Theta, \Xi) : \mathcal{DG} \to \mathcal{RD}(c) \times \mathbb{Z}
\]

is coarsely Lipschitz, and its composition with the inverse of the map \( \Psi \) from Lemma 2.3 is a coarse Lipschitz retraction of \( \mathcal{DG} \) onto \( \Omega \) provided that the map \( \Xi \) maps a point in \( \Lambda^k(\mathcal{RD}(c)) \) into a uniformly bounded neighborhood of \( k \).

However, if \( \beta_0 \in \mathcal{RD}(c) \) and if \( \beta = \Lambda^k(\beta_0) \in \Lambda^k(\mathcal{RD}(c)) \), then the intersections with \( H - \text{int}(B) \) of the representatives \( \hat{\beta}, \hat{\beta}_0 \) of \( \beta, \beta_0 \) constructed above coincide. This implies that up to homotopy with fixed endpoints, \( \hat{\beta} \cap B = \Lambda^k(\beta_0 \cap B) \).

On the other hand, point-pushing along a simple closed curve \( \gamma \) based at \( p \) descends to conjugation by \( \gamma \) in \( \pi_1(\partial H_0, p) \). Therefore the image under the map
A of a simple arc $b$ in $B$ with endpoints on the two distinct components of $\partial B$ is homotopic with fixed endpoints to $c_1bc_2$ (recall that we oriented $c_1, c_2$ so that they define the boundary orientation of $B$). As $\Xi(\iota(RD(e))) \subset [-2,2]$, it follows that $|\Xi(\beta) - k| \leq 2$. This shows the proposition. 

To summarize, we obtain

**Corollary 2.5.** The disk graph of a handlebody $H$ of genus $g \geq 2$ with one spot contains quasi-isometrically embedded copies of $\mathbb{R}^2$.

**Remark 2.6.** In [H19] we showed that in contrast to handlebodies without spots, the disk graph of a handlebody $H$ with a single spot on the boundary is *not* a quasi-convex subgraph of the curve graph of $\partial H$.

### 3. Once spotted doubled handlebodies

In this section we explain how the construction that led to the proof of Theorem 2 can be used to show the first part of Theorem 3.

Namely, consider the double $M_0 = \sharp_n S^2 \times S^1$ of a handlebody $H_0$ of genus $g \geq 2$ without spots. Let $M$ be the manifold $M_0$ equipped with a marked point $p$. As before, we call $p$ a spot in $M$. There is a natural spot removing map $\Phi : M \to M_0$.

Let $SG$ be the sphere graph of $M$ whose vertices are isotopy classes of embedded spheres in $M$ which are disjoint from the spot and not isotopic into the spot. Isotopies are required to be disjoint from the spot as well. Two such spheres are connected by an edge of length one if they can be realized disjointly. Similarly, let $SG_0$ be the sphere graph of $M_0$.

Assume from now on that $g = 2n$ for some $n \geq 1$. Choose an embedded oriented surface $F_0 \subset M_0$ of genus $n$ with connected boundary such that the inclusion $F_0 \to M_0$ induces an isomorphism $\pi_1(F_0) \to \pi_1(M_0)$. We may assume that the oriented $I$-bundle $H_0$ over $F_0$ is an embedded handlebody $H_0 \subset M_0$ whose double equals $M_0$. Thus every embedded essential arc $\alpha$ in $F_0$ with boundary in $\partial F_0$ determines a sphere $\Upsilon_0(\alpha)$ in $M_0$ as follows. The interval bundle over $\alpha$ is an embedded essential disk in $H_0$, with boundary in $\partial H_0$, and we let $\Upsilon_0(\alpha)$ be the double of this disk. By construction, the sphere $\Upsilon_0(\alpha)$ intersects the surface $F_0$ precisely in the arc $\alpha$. By Lemma 4.17 of [HH15], distinct arcs give rise to non-isotopic spheres, furthermore the map $\Upsilon_0$ preserves disjointness and hence $\Upsilon_0$ is a simplicial embedding of the arc graph $A(F_0)$ of $F_0$ into the sphere graph $SG_0$ of $M_0$.

Now mark a point $p$ on the boundary $\partial F_0$ of $F_0$ and view the resulting spotted surface $F$ as a surface in the spotted manifold $M$. The arc graph $A(F)$ of $F$ is the graph whose vertices are isotopy classes of essential arcs in $F$ with endpoints on the complement of $p$ in the boundary of $F$. Here we exclude arcs which are homotopic with fixed endpoints to a subarc of $\partial F$ containing the base point $p$, and we require that an isotopy preserves the marked point $p$ and hence endpoints of arcs can only slide along $\partial F - \{p\}$. Two such arcs are connected by an edge if they can be realized disjointly. Associate to an arc $\alpha$ in $F$ the double $T(\alpha)$ of the $I$-bundle over $\alpha$.

The spot removal map $\Phi : M \to M_0$ induces a simplicial surjection $SG \to SG_0$, again denoted by $\Phi$ for simplicity. Similarly, if we let $\varphi : F \to F_0$ be the map which forgets the marked point $p \in \partial F$, then $\varphi$ induces a simplicial surjection.
\[ \mathcal{A}(F) \to \mathcal{A}(F_0) \], denoted as well by \( \varphi \). We then obtain a commutative diagram

\[
\begin{array}{ccc}
\mathcal{A}(F) & \xrightarrow{\varphi} & \mathcal{A}(F_0) \\
\downarrow \Psi & & \downarrow \Psi_0 \\
\mathcal{SG} & \xrightarrow{\Phi} & \mathcal{SG}_0
\end{array}
\]

(1)

Similar to the case of the handlebody \( M_0 \) without spots and the map \( \Psi_0 \), we obtain

**Lemma 3.1.** The map \( \Psi \) is a simplicial embedding of the arc graph \( \mathcal{A}(F) \) into the sphere graph.

**Proof.** We have to show that the map \( \Psi \) is injective. As \( \Psi_0 \) is injective and as the diagram (1) commutes, it suffices to show the following. Let \( \alpha \neq \beta \in \mathcal{A}(F) \) be such that \( \varphi(\alpha) = \varphi(\beta) \); then \( \Psi(\alpha) \neq \Psi(\beta) \).

Now \( \varphi(\alpha) = \varphi(\beta) \) means that up to exchanging \( \alpha \) and \( \beta \), there exists a number \( k > 0 \) such that \( \beta \) can be obtained from \( \alpha \) by \( k \) half Dehn twists about the boundary \( \partial F \) of \( F \). Here the half Dehn twist \( T(\alpha) \) of \( \alpha \) is defined as follows.

The orientation of \( F \) induces a boundary orientation for \( \partial F \) which in turn induces an orientation on \( \partial F - \{p\} \). With respect to the order defined by this orientation, let \( x \) be the bigger of the two endpoints \( x, y \) of \( \alpha \). Slide \( x \) across \( p \) to obtain a new arc \( T(\alpha) \), with endpoints \( x', y \). This arc is not homotopic to \( \alpha \). To see this it suffices to show that the double \( DT(\alpha) \) of \( T(\alpha) \) in the double \( DF \) of \( F \) (which is a surface with one puncture) is not freely homotopic to the double \( D(\alpha) \) of \( \alpha \). This follows since \( D(\alpha) \) and \( DT(\alpha) \) can be homotoped in such a way that they bound a subset of \( M \) homeomorphic to the complement of an interior point of \( S^2 \times [0, 1] \).

The above construction, applied to the sphere \( \Psi(T(\alpha)) \) instead of the sphere \( \Psi(\alpha) \) and where the point \( y \) takes on the role of the point \( x \) in the above discussion, shows that \( \Psi(T^2(\alpha)) \) is obtained from \( \Psi(\alpha) \) by point-pushing along the oriented loop \( \partial F \) with basepoint \( p \). This is a diffeomorphism of \( M \) which leaves the complement of a small tubular neighborhood of \( \partial F \) pointwise fixed and pushes the basepoint \( p \) along \( \partial F \). As in the proof of Lemma 2.3, this argument can be iterated. It shows that the sphere \( \Psi(T^k(\alpha)) \) intersects the sphere \( \Psi(\alpha) \) in \( k - 1 \) intersection circles. These circles are essential since they cut both \( \Psi(T^k(\alpha)) \) and \( \Psi(\alpha) \) into two disks and \( k - 2 \) annuli, where a disk component of \( T^k(\alpha) - T(\alpha) \) bounds together with a disk component of \( T(\alpha) - T^k(\alpha) \) an embedded sphere enclosing the spot. Invoking the proof of Lemma 2.3, we conclude that indeed, for \( k \neq \ell \), \( \Psi(T^k(\alpha)) \) is not homotopic to \( \Psi(T^\ell(\alpha)) \).

We showed so far that the map \( \Psi \) is injective. To complete the proof of the lemma, it suffices to observe that disjoint arcs are mapped to disjoint spheres. But this is immediate from the construction. \( \square \)
Proposition 4.18 of [HH15] shows that there is a one-Lipschitz retraction
\[ \Psi_0 : \mathcal{S}G_0 \to \mathcal{Y}_0(\mathcal{A}(F_0)) \]
which is of the form \( \Psi_0 = \mathcal{Y}_0 \circ \Theta_0 \) (read from right to left) where \( \Theta_0 : \mathcal{S}G_0 \to \mathcal{A}(F_0) \) is a one-Lipschitz map. In particular, \( \mathcal{Y}_0(\mathcal{A}(F_0)) \) is a quasi-isometrically embedded subgraph of \( \mathcal{S}G_0 \) which is quasi-isometric to \( \mathcal{A}(F_0) \). Our goal is to show that there also is a coarse Lipschitz retraction of \( \mathcal{S}G \) onto \( \mathcal{Y}(\mathcal{A}(F)) \) of the form \( \Psi = \Theta \circ \mathcal{Y} \) where \( \Theta : \mathcal{S}G \to \mathcal{A}(F) \) is a coarse Lipschitz map. This then yields the first part of Theorem 3 from the introduction.

To construct the map \( \Theta \) we use the method from [HH15]. We next explain how this method can be adapted to our needs.

Let as before \( F \subset M \) be an embedded oriented surface with connected boundary \( \partial F \) so that \( M \) is the double of the trivial \( I \)-bundle over \( F \). We assume that the marked point \( p \) is contained in the boundary \( \partial F \) of \( F \). Furthermore, we assume that the boundary \( \partial F \) of \( F \) is a smoothly embedded circle in \( M \cup \{p\} \) (i.e. an embedded compact one-dimensional submanifold). As before, we use the marked point \( p \) as the basepoint for the fundamental group of \( M \). Then \( \partial F \) equipped with its boundary orientation defines a homotopy class \( \beta \in \pi_1(M,p) = \mathbb{F}_{2g} \). As \( \beta \) is not contained in any free factor, \( \partial F \) intersects every sphere in \( M \). Namely, for any given sphere \( S \) in \( M \), the subgroup of \( \pi_1(M,p) \) of all homotopy classes of loops which do not intersect \( S \) is a proper free factor of \( \pi_1(M,p) \).

As in [HH15] and similar to the construction in Lemma 2.1, the strategy is to associate to a sphere \( S \) in \( M \) a component of the intersection \( F \cap S \). However, unlike in the case of curves on surfaces, there is no suitable normal form for intersections of spheres with the surface \( F \), and the main work in [HH15] consists of overcoming this difficulty by introducing a relative normal form which allows one to associate to a sphere in \( M_0 \) an intersection arc with \( F_0 \) so that the resulting map \( \mathcal{S}G_0 \to \mathcal{A}(F_0) \) is one-Lipschitz.

For the remainder of this section we outline the main steps in this construction, adapted to the sphere graph \( \mathcal{S}G \) of \( M \) and the arc graph \( \mathcal{A}(F) \) of \( F \). This requires modifying spheres with isotopies not crossing through \( p \), and modifying the surface \( F \) with homotopies leaving the boundary \( \partial F \) pointwise fixed.

For convenience, we record some definitions from [HH15] (the following combines Definition 4.7 and Definition 4.9 of [HH15]).

**Definition 3.2.** Let \( \Sigma \) be a sphere or a sphere system.

1. \( \partial F \) intersects \( \Sigma \) minimally if \( \partial F \) intersects \( \Sigma \) transversely and if no component of \( \partial F - \Sigma \) not containing the basepoint \( p \) is homotopic with fixed endpoints into \( \Sigma \).
2. \( F \) is in minimal position with respect to \( \Sigma \) if \( \partial F \) intersects \( \Sigma \) minimally and if moreover each component of \( \Sigma \cap F \) is a properly embedded arc which either is essential or homotopic with fixed endpoints to a subarc of \( \partial F \) containing the marked point.

A version of the easy Lemma 4.6 of [HH15] states that any closed curve containing the basepoint can be put into minimal position relative to a sphere system \( \Sigma \) as defined in the first part of Definition 3.2. The following is a version of Lemma 4.12 of [HH15]. For its formulation, call a sphere system \( \Sigma \) simple if it decomposes \( M \) into a simply connected components.
Lemma 3.3. Let $\Sigma$ be a simple sphere system in $M$. Suppose that $F$ is in minimal position with respect to $\Sigma$. Let $\sigma'$ be an embedded sphere disjoint from $\Sigma$ and let $\Sigma'$ be a simple sphere system obtained from $\Sigma$ by either adding $\sigma'$, or removing one sphere $\sigma \in \Sigma$. Then $F$ can be homotoped leaving $p$ fixed to a surface $F'$ which is in minimal position with respect to $\Sigma'$.

Proof. As in the proof of Lemma 4.12 of [HH15], removing a sphere preserves minimal position, so only the case of adding a sphere has to be considered.

Thus let $\Sigma$ be a simple sphere system and let $\sigma'$ be a sphere disjoint from $\Sigma$. Assume that $F$ is in minimal position with respect to $\Sigma$. Let $W_\Sigma$ be the complement of $\Sigma$ in $M$, that is, $W_\Sigma$ is a compact (possibly disconnected) manifold whose boundary consists of $2k$ boundary spheres $\sigma_1^+, \sigma_1^-, \ldots, \sigma_k^+, \sigma_k^-$. The boundary spheres $\sigma_i^+$ and $\sigma_i^-$ correspond to the two sides of a sphere $\sigma_i \in \Sigma$. The surface $F$ intersects $W_\Sigma$ in a collection of embedded surfaces with boundaries. Each such surface is a polygonal disk $P_i$ ($i = 1, \ldots, m$). The sides of each such polygon alternate between subarcs of $\partial F$ and arcs contained in $\Sigma$. There is at most one bigon, that is, a polygon with two sides, and this polygon then contains the point $p$ in one of its sides. Each rectangle, if any, is homotopic into $\partial F$.

The proof of Lemma 4.12 of [HH15] now proceeds by studying the intersection of each polygonal component of $F - \Sigma$ with the sphere $\sigma'$. This is done by contracting each such polygonal component $P$ to a ribbon tree $T(P)$ in such a way that the boundary components in $\Sigma$ are contracted to single points in $T(P)$. If $P$ is not a rectangle or bigon, then $T(P)$ has a single vertex which is not univalent. As such ribbon trees are one-dimensional objects, they can be homotoped with fixed endpoints on $\partial W_\Sigma$ to trees which are in minimal position with respect to $\sigma'$. This construction applies without change to rectangles and perhaps the bigon which can be represented by an interval with one endpoint at $p$ and the second endpoint on a component of $\Sigma$. We refer to the proof of Lemma 4.12 of [HH15] for details. No adjustment of the argument is necessary. \qed

The above construction is only valid for simple sphere systems $\Sigma$ and not for individual spheres. Furthermore, it is known that the arc system on $F \cap \Sigma$ obtained by putting $F$ into minimal position with respect to $\Sigma$ is not uniquely determined by $\Sigma$. To overcome this difficulty, the work of [HH15] uses as an auxiliary datum a maximal system $A_0$ of pairwise disjoint essential arcs on the surface $F_0$. Here maximal means that any arc which is disjoint from $A_0$ is contained in $A_0$. The system $A_0$ then binds $F_0$, that is, $F - A_0$ is a union of topological disks. Furthermore, $\partial F_0$ and each arc $\alpha \in A_0$ is equipped with an orientation.

Choose an arc system $A$ for $F$ which binds $F$. If $F \subset M$ is in minimal position with respect to $\Sigma$, then a homotopy assures that no arc from the arc system $A$ intersects a component of $F - \Sigma$ which is a rectangle or a bigon. Then Lemma 4.12 of [HH15] and its proof applies without modification and shows that with a homotopy, $F$ can be put into normal form with respect to the arc system $A$, called $A$-tight minimal position with respect to $\Sigma$. This then yields the statement of Lemma 4.16 of [HH15]: if $F$ is in $A$-tight minimal position with respect to the simple sphere system $\Sigma$, then the binding arc system $\Sigma \cap F$ is determined by $\Sigma$. In particular, two distinct spheres from $\Sigma$ intersect $F$ in disjoint essential arcs. There may in addition be inessential arcs, i.e. arcs which are homotopic with fixed
endpoints to a subsegment of $\partial F$ containing the basepoint $p$, but these will be unimportant for our purpose.

Now let $\sigma$ be an essential sphere in $M$. Let $\Sigma$ be a simple sphere system in $M$ containing $\sigma$ as a component. We put $F$ into $A$-tight minimal position with respect to $\Sigma$. Then $\sigma \cap F$ consists of a non-empty collection of essential arcs and perhaps some additional non-essential arcs. Choose one of the essential intersection arcs $\alpha$ and define $\Theta(\sigma) = \alpha$. As in [HH15] and Proposition 2.4 we now obtain

**Proposition 3.4.** The map $\Theta$ is a coarsely Lipschitz map. For each arc $\alpha \in \mathcal{A}(F)$, we have $\Theta(\Upsilon(\alpha)) = \alpha$. As a consequence, if $g = 2n$ is even then the sphere graph $SG$ of $M$ contains quasi-isometrically embedded copies of $\mathbb{R}^2$.

**Proof.** Given the above discussion, the proof that $\Theta$ is a coarsely Lipschitz map is identical to the proof that the map $\Theta_0$ is a coarsely Lipschitz map in Proposition 4.18 of [HH15] and will be omitted. Moreover, as for $\alpha \in \mathcal{A}(F)$, the sphere $\Upsilon(\alpha)$ intersects $F$ in the unique arc $\alpha$, we have $\Theta(\Upsilon(\alpha)) = \alpha$.

As a consequence, $\Theta \Upsilon(\mathcal{A}(F))$ is a Lipschitz bijection, with inverse $\Upsilon$. Then the subgraph $\Upsilon(\mathcal{A}(F))$ of $SG$ is bilipschitz equivalent to $\mathcal{A}(F)$. Furthermore, the map $\Upsilon \circ \Theta$ is a Lipschitz retraction of $SG$ onto $\Upsilon(\mathcal{A}(F))$. Then $\Upsilon(\mathcal{A}(F))$ is a quasi-isometrically embedded subgraph of $SG$ which is moreover quasi-isometric to $\mathcal{A}(F)$.

Let as before $F_0$ be the surface obtained from $F$ by removing the spot. We are left with showing that $\mathcal{A}(F)$ is quasi-isometric to $\mathcal{A}(F_0) \times \mathbb{Z}$. However, this was shown in Lemma 2.3. Namely, in the terminology used before, the boundary $\partial F$ is an $I$-bundle generator in the trivial interval bundle $H$ over $F$, and associating to an arc $\alpha$ the $I$-bundle over $\alpha$ defines an isomorphism of $\mathcal{A}(F)$ with the subgraph $\Omega$ of the disk graph of $H$ used in Lemma 2.3. The statement now follows from Lemma 2.3. $\square$

**Remark 3.5.** Most likely Proposition 3.4 holds true as well in the case that $g = 2n + 1$ is odd, and furthermore this can be deduced with the above argument using non-orientable surfaces. However, the analogue of Proposition 4.18 of [HH15] for non-orientable surfaces is not available, and we leave the verification of these claims to other authors.

**References**


MATH. INSTITUT DER UNIVERSITÄT BONN, ENDENICHER ALLEE 60, 53115 BONN, GERMANY

e-mail: ursula@math.uni-bonn.de