GENERATING THE SPIN MAPPING CLASS GROUP BY DEHN TWISTS

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Abstract. Let φ be a $\mathbb{Z}/2\mathbb{Z}$-spin structure on a closed oriented surface Σ_g of genus $g \geq 4$. We determine a generating set of the stabilizer of φ in the mapping class group of Σ_g consisting of Dehn twists about an explicit collection of $2g+1$ curves on Σ_g. If $g = 3$ then we also determine a generating set of the stabilizer of an odd $\mathbb{Z}/4\mathbb{Z}$-spin structure consisting of Dehn twists about a collection of 6 curves.

1. Introduction

For some $r \geq 2$, a $\mathbb{Z}/r\mathbb{Z}$-spin structure on a closed surface Σ_g of genus g is a cohomology class $\varphi \in H^1(UT\Sigma_g, \mathbb{Z}/r\mathbb{Z})$ which evaluates to one on the oriented fibre of the unit tangent bundle $UT\Sigma_g \to \Sigma_g$ of Σ_g. Such a spin structure exists for all r which divide $2g-2$. If r is even, then it reduces to a $\mathbb{Z}/2\mathbb{Z}$-spin structure on Σ_g.

A $\mathbb{Z}/2\mathbb{Z}$-spin structure on Σ_g has a *parity*, either even or odd. Thus there is a notion of parity for all $\mathbb{Z}/r\mathbb{Z}$-spin structures with r even. If φ, φ' are two $\mathbb{Z}/r\mathbb{Z}$-spin structures on Σ_g so that either r is odd or r is even and the parities of φ, φ' coincide, then there exists an element of the mapping class group $\mathrm{Mod}(\Sigma_g)$ of Σ_g which maps φ to φ'. Hence the stabilizers of φ and φ' in $\mathrm{Mod}(\Sigma_g)$ are conjugate.

Spin structures naturally arise in the context of abelian differentials on Σ_g. The moduli space of such differentials decomposes into strata of differentials whose zeros are of the same order and multiplicity. Understanding the orbifold fundamental group of such strata requires some understanding of their projection to the mapping class group. If the orders of the zeros of the differentials are all multiples of the same number $r \geq 2$, then this quotient group preserves a $\mathbb{Z}/r\mathbb{Z}$-spin structure φ on Σ_g. Hence the orbifold fundamental groups of components of strata relate to stabilizers $\mathrm{Mod}(\Sigma_g)[\varphi]$ of spin structures φ on Σ_g.

To make such a relation explicit we define

Definition 1. A *curve system* on a closed surface Σ_g is a finite collection of smoothly embedded simple closed curves on Σ_g which are non-contractible and mutually not freely homotopic, and such that any two curves from this collection intersect transversely in at most one point.

Date: March 2, 2020.
AMS subject classification: 30F30, 30F60, 37B10, 37B40.

A curve system defines a *curve diagram* which is a finite graph whose vertices are the curves from the system and where two such vertices are connected by an edge if the curves intersect.

Definition 2. A curve system on Σ_g is *admissible* if it decomposes Σ_g into a collection of topological disks and if its curve diagram is a tree.

Using a construction of Thurston and Veech (see [Lei04] for a comprehensive account), admissible curve systems on Σ_g give rise to abelian differentials on Σ_g, and the component of the stratum and hence the equivalence class of a spin structure (if any) it defines can be read off explicitly from the combinatorics of the curve system. This makes it desirable to investigate the subgroup of the mapping class group generated by Dehn twists about the curves of an admissible curve system.

The main goal of this article is to present a systematic study of stabilizers of suitably chosen curves in the spin mapping class group $\mathrm{Mod}(\Sigma_g)[\varphi]$ and to use this information to build generators for this group by induction over subsurfaces. As a main application we obtain the following.

For $g \geq 3$ let $\mathcal{C}_g$ and $\mathcal{V}_g$ be the collections of $2g+1$ non-separating simple closed curves on a closed surface Σ_g of genus g shown in Figure 1. We show

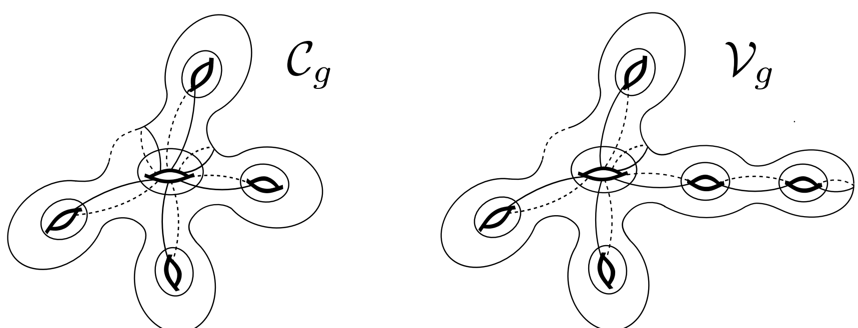

Figure 1

Theorem 3. (1) *Let φ be an odd $\mathbb{Z}/2\mathbb{Z}$-spin structure on a closed surface Σ_g of genus $g \geq 3$. Then* $\mathrm{Mod}(\Sigma_g)[\varphi]$ *is generated by the Dehn twists about the curves from the curve system $\mathcal{C}_g$.*
(2) *Let φ be an even $\mathbb{Z}/2\mathbb{Z}$-spin structure on a closed surface Σ_g of genus $g \geq 4$. Then* $\mathrm{Mod}(\Sigma_g)[\varphi]$ *is generated by the Dehn twists about the curves from the curve system $\mathcal{V}_g$.*

That the spin mapping class group can be generated by finitely many Dehn twists or and finite products of Dehn twists is due to Hirose. In [Hi02] he found for any genus $g \geq 2$ a generating set for the stabilizer of an even $\mathbb{Z}/2\mathbb{Z}$-spin structure by finitely many finite products of Dehn twists, and the stabilizer of an odd $\mathbb{Z}/2\mathbb{Z}$-spin structure is treated in [Hi05].

For surfaces of genus $g \geq 5$, Calderon [Cal19] and Calderon and Salter [CS19] identified the image of the orbifold fundamental group of most components of strata in the mapping class group by constructing a different but equally explicit generating set for the spin mapping class group. Earlier Walker [W09, W10] obtained some information on the image of the orbifold fundamental group of some strata of quadratic differentials in the mapping class group using completely different tools.

Theorem 3 does not construct generators for the stabilizer of an even $\mathbb{Z}/2\mathbb{Z}$-spin structure on a surface of genus $g = 2, 3$. Namely, in these cases there is no admissible curve system with the property that the Dehn twists about the curves from the system stabilize an even $\mathbb{Z}/2\mathbb{Z}$-spin structure and such that the Dehn twists about these curves generate a finite index subgroup of the mapping class group. This corresponds to a classification result of Kontsevich and Zorich [KZ03]: There is no component of a stratum of abelian differentials with a single zero on a surface of genus 2 and even spin structure. On a surface Σ_3 of genus 3, the component of the stratum of abelian differentials with two zeros of order two and even spin structure is hyperelliptic and hence the projection of its orbifold fundamental group to $\mathrm{Mod}(\Sigma_3)$ commutes with a hyperelliptic involution and is of infinite index.

Our results can be used to construct an explicit finite set of generators of the stabilizer of a $\mathbb{Z}/r\mathbb{Z}$-spin structure for any $r \leq 2g-2$ and any closed surface Σ_g, given by Dehn twists, positive powers of Dehn twists and products of Dehn twists about two simple closed curves forming a bounding pair. Potentially they can also be used inductively to find generators by Dehn twists about curves from an admissible curve system. We carry this program only out in a single case, which is the odd $\mathbb{Z}/4\mathbb{Z}$-spin structure on a surface of genus 3.

Consider the system $\mathcal{E}_6$ of simple closed curves on the surface Σ_3 of genus 3 shown in Figure 2 which is of particular relevance for the understanding of the stratum of abelian differentials with a single zero on Σ_3 [LM14]. We show

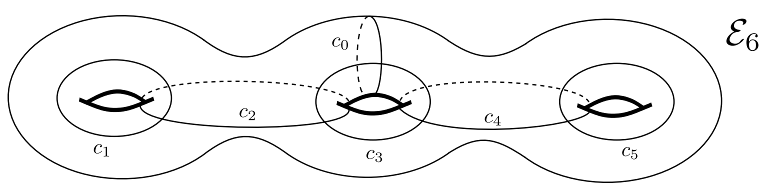

Figure 2

Theorem 4. *The subgroup of* $\mathrm{Mod}(\Sigma_3)$ *generated by the Dehn twists about the curves from the curve system* $\mathcal{E}_6$ *equals the stabilizer of an odd* $\mathbb{Z}/4\mathbb{Z}$*-spin structure on* Σ_3.

The strategy for the proofs of the main results is as follows.

For some $r \geq 2$ let us consider an arbitrary $\mathbb{Z}/r\mathbb{Z}$-spin structure φ on a compact oriented surface S of genus $g \geq 2$, perhaps with boundary. Following [HJ89] and [Sa19], the spin structure can be viewed as a $\mathbb{Z}/r\mathbb{Z}$-valued function on oriented closed curves on S which assumes the value one on the oriented boundary of an embedded disk in S. Changing the orientation of the curve changes the value of φ on the curve to its negative [HJ89, Sa19].

Define a graph $\mathcal{CG}_1^+$ as follows. Vertices are nonseparating simple closed curves c on S with $\varphi(c) = \pm 1$, and two such vertices d, e are connected by an edge if d, e can be realized disjointly and if furthermore, $S - (d \cup e)$ is connected. Thus $\mathcal{CG}_1^+$ is a subgraph of the curve graph of S. The stabilizer $\mathrm{Mod}(S)[\varphi]$ of φ in the mapping class group of S acts on $\mathcal{CG}_1^+$ as a group of simplicial automorphisms.

In Section 2 we show that for any $g \geq 3$ and $r \leq 2g - 2$ the graph $\mathcal{CG}_1^+$ is connected. We also note that for an odd $\mathbb{Z}/2\mathbb{Z}$-spin structure on a surface of genus $g = 2$, this is not true. In Section 3 we verify that the action of the group $\mathrm{Mod}(S)[\varphi]$ on the graph $\mathcal{CG}_1^+$ is transitive on vertices.

For a vertex c of $\mathcal{CG}_1^+$ we are then led to describing the intersection of $\mathrm{Mod}(S)[\varphi]$ with the stabilizer of c in $\mathrm{Mod}(S)$. Most important is the understanding of the intersection of $\mathrm{Mod}(S)[\varphi]$ with the so-called *disk pushing subgroup*, namely the kernel of the natural homomorphism of the stabilizer of c to the mapping class group of the surface obtained from $S - c$ by capping off the two distinguished boundary components of $S - c$. This is also carried out in Section 3.

In Section 4 we specialize further to a $\mathbb{Z}/2\mathbb{Z}$-spin structure φ. We find a presentation of $\mathrm{Mod}(S)[\varphi]$ as a quotient of a $\mathbb{Z}/2\mathbb{Z}$-extension of the product of two copies of the stabilizer of a vertex of $\mathcal{CG}_1^+$, amalgamated over the stabilizer of an edge of $\mathcal{CG}_1^+$. This is used to prove the first part of Theorem 3 with an argument by induction on the genus g of the closed surface Σ_g.

The proof of the second part of Theorem 3 uses similar methods and is contained in Section 5. A variation of these arguments yield the proof of Theorem 4 in Section 6.

The appendix contains a technical variation of the main result of Section 2 which is used in Section 5. Its proof follows along exactly the same line as the proof of the main result of Section 2.

This work is partially motivated by the article [Sa19] of Salter. However, aside from some simple constructions using curves, the only result from [Sa19] we use is Proposition 4.9.

Acknowledgement: I am grateful to Dawei Chen, Samuel Grushevsky, Martin Möller and Nick Salter for useful discussions. This work was completed while the author was in residence at the MSRI in Berkeley, California, in the fall semester 2019, supported by the National Science Foundation under Grant No. DMS-1440140.

2. Graphs of curves with fixed spin value

In this section we consider a compact surface S of genus $g \geq 2$, with or without boundary. For a number $r \geq 2$ we introduce $\mathbb{Z}/r\mathbb{Z}$-spin structures on S and use these structures to define various subgraphs of the curve graph of S. We then study connectedness of these graphs. Of primary interest is a graph whose vertices are nonseparating simple closed curves with spin value ± 1. We show that for all $r \leq 2g - 2$ and for all $g \geq 3$ this graph is connected. This is used in Section 3 to study the stabilizer of a spin structure in the mapping class group of S.

This section is divided into 5 subsections. We begin with summarizing some information on spin structures. Each of the remaining subsections is devoted to the investigation of a specific subgraph of the curve graph of S defined by a spin structure φ on S.

2.1. **Spin structures.** The following is taken from [HJ89], see Definition 3.1 of [Sa19]. For its formulation, denote by ι the symplectic form on $H_1(S,\mathbb{Z})$.

Definition 2.1 (Humphries-Johnson)**.** For a number $r \geq 2$, a $\mathbb{Z}/r\mathbb{Z}$-*spin structure* on S is a $\mathbb{Z}/r\mathbb{Z}$-valued function φ on isotopy classes of oriented simple closed curves on S with the following properties.

(1) (Twist linearity) Let c, d be oriented simple closed curves and let T_c be the left Dehn twist about c; then
$$\varphi(T_c(d)) = \varphi(d) + \iota(d,c)\varphi(c) \pmod r.$$
(2) (Normalization) $\varphi(\zeta) = 1$ for the oriented boundary ζ of an embedded disk $D \subset S$.

As an additional property, one obtains that whenever c^{-1} is obtained from c by reversing the orientation, then $\varphi(c^{-1}) = -\varphi(c)$ (Lemma 2.2 of [HJ89]).

Humphries and Johnson [HJ89] (see Theorem 3.5 of [Sa19]) also give an alternative description of spin structures. Namely, for some choice of a hyperbolic metric on S let UTS be the unit tangent bundle of S. It can be viewed as the quotient of the complement of the zero section in the tangent bundle of S by the multiplicative group $(0,\infty)$ and hence it does not depend on the metric.

The *Johnsson lift* of a smoothly embedded oriented simple closed curve c on S is simply the closed curve in UTS which consists of all unit tangents of c defining the given orientation. The following is Theorem 2.1 and Theorem 2.5 of [HJ89] as formulated in Theorem 3.5 of [Sa19].

Theorem 2.2 (Humphries-Johnsson)**.** *Let S be a compact surface and let ζ be the oriented fibre of the unit tangent bundle $UTS \to S$. A cohomology class $\psi \in H^1(UTS,\mathbb{Z}/r\mathbb{Z})$ with $\psi(\zeta) = 1$ determines a $\mathbb{Z}/r\mathbb{Z}$-spin structure via*
$$\alpha \to \psi(\tilde{\alpha})$$
where α is an oriented simple closed curve on S and $\tilde{\alpha}$ is its Johnson lift. This determines a 1-1 correspondence between $\mathbb{Z}/r\mathbb{Z}$-spin structures and
$$\{\psi \in H^1(UTS,\mathbb{Z}/r\mathbb{Z}) \mid \psi(\zeta) = 1\}.$$

There is another interpretation as follows; we refer to p.131 of [Hai95] for more information on this construction. Given a number $r \geq 2$ which divides $2g-2$, an application of the Gysin sequence for the Euler class of UTS yields a short exact sequence
$$0 \to \mathbb{Z}/r\mathbb{Z} \to H_1(UTS,\mathbb{Z}/r\mathbb{Z}) \to H_1(S,\mathbb{Z}/r\mathbb{Z}) \to 0. \tag{1}$$

By covering space theory, an r-th root of the tangent bundle of S, viewed as a complex line bundle for some fixed complex structure, is determined by a homomorphism $H_1(UTS, \mathbb{Z}/r\mathbb{Z}) \to \mathbb{Z}/r\mathbb{Z}$ whose composition with the inclusion $\mathbb{Z}/r\mathbb{Z} \to H_1(UTS, \mathbb{Z}/r\mathbb{Z})$ is the identity and therefore

Proposition 2.3. *There is a natural one-to-one correspondence between the r-th roots of the canonical bundle of S and splittings of the sequence (1).*

A $\mathbb{Z}/2\mathbb{Z}$-spin structure on a compact surface S of genus g with empty or connected boundary has a *parity* which is defined as follows.

A *geometric symplectic basis* for $H_1(S, \mathbb{Z})$ is a system $a_1, b_1, \ldots, a_g, b_g$ of simple closed curves on S such that a_i, b_i intersect in a single point and that $a_i \cup b_i$ is disjoint from $a_j \cup b_j$ for $i \neq j$. Then the parity of the spin structure φ equals

$$\mathrm{Arf}(\varphi) = \sum_i (\varphi(a_i) + 1)(\varphi(b_i) + 1) \in \mathbb{Z}/2\mathbb{Z}. \tag{2}$$

This does not depend on the choice of the geometric symplectic basis.

2.2. **The graph of nonseparating curves with vanishing spin value.** The *curve graph* $\mathcal{CG}$ of S is the graph whose vertices are *essential* (that is, neither nullhomotopic nor homotopic into the boundary) simple closed curves in S and where two such curves are connected by an edge if they can be realized disjointly. We can use the spin structure φ to introduce various subgraphs of $\mathcal{CG}$ and study their properties. One of the main technical ingrediences to this end is the following result of Salter (Corollary 4.3 of [Sa19]).

Lemma 2.4 (Salter)**.** *Let $\Sigma \subset S$ be an embedded one-holed torus. Then there exists a simple closed curve $c \subset \Sigma$ with $\varphi(c) = 0$.*

Denote by $\mathcal{CG}_0 \subset \mathcal{CG}$ the complete subgraph of the curve graph whose vertex set consists of nonseparating curves c with $\varphi(c) = 0$. Note that this is well defined, that is, it is independent of the choice of an orientation of c. As a fairly easy consequence of Lemma 2.4 we obtain

Lemma 2.5. *Let φ be a spin structure on a closed surface of genus $g \geq 3$. Then $\mathcal{CG}_0$ is connected.*

Proof. We use the following result of Masur-Schleimer [MS06], see Theorem 1.2 of [Put08]. Let $\mathcal{SG} \subset \mathcal{CG}$ be the complete subgraph whose vertex set consists of *separating* simple closed curves; then $\mathcal{SG}$ is connected. Note that this requires that $g \geq 3$.

Let a, b be vertices of $\mathcal{CG}_0$. Choose simple closed curves $\hat{a}, \hat{b}$ which intersect a, b in a single point; such curves exist since a, b are nonseparating. Then the boundary c, d of a tubular neighborhood of $a \cup \hat{a}$ and $b \cup \hat{b}$, respectively, is a separating simple closed curve which decomposes S into a one-holed torus containing a, b and a surface of genus $g - 1 \geq 2$ with boundary.

Connect c to d by an edge path $(c_i)_{0 \leq i \leq k} \subset \mathcal{SG}$ (here $c = c_0$ and $d = c_k$). Construct inductively an edge path $(a_i) \subset \mathcal{CG}_0$ connecting $a = a_0$ to $b = a_k$ such that for each i, a_i is disjoint from c_i, as follows. Put $a_0 = a$ and assume that we constructed already such a path for some $j < k$. Then a_j is disjoint from c_j.

If a_j also is disjoint from c_{j+1} then define $a_{j+1} = a_j$. Otherwise a_j is contained in the same component Σ of $S - c_j$ as c_{j+1}. Choose a one-holed torus $T \subset S - \Sigma$. Such a torus exists since c_j decomposes S into two surfaces of positive genus with connected boundary. By Lemma 2.4, this torus contains a nonseparating simple closed curve a_{j+1} with $\varphi(a_{j+1}) = 0$, and this curve is disjoint from both a_j and c_{j+1}. This yields the induction step. □

Remark 2.6. The proof of Lemma 2.5 extends with a bit more care to compact surfaces of genus at least 3 with connected boundary. We expect that the Lemma also holds true for $g = 2$.

2.3. **The graph of nonseparating curves with spin value ± 1 on a surface of genus 2.** Define $\mathcal{CG}_1$ to be the complete subgraph of $\mathcal{CG}$ of all nonseparating simple closed curves c on S with $\varphi(c) = \pm 1$. Note that this condition does not depend on the orientation of c and hence it is indeed a condition on the vertices of $\mathcal{CG}$. In this subsection we discuss the special case $g = 2$.

Proposition 2.7. *Let φ be an odd $\mathbb{Z}/2\mathbb{Z}$-spin structure on a closed surface S of genus 2. Then any two simple closed nonseparating curves c, d on S with $\varphi(c) = \varphi(d) = 1$ intersect.*

Proof. Let φ be a $\mathbb{Z}/2\mathbb{Z}$-spin structure on S. Let c be a nonseparating simple closed curve on S with $\varphi(c) = 1$. Assume that there is a nonseparating simple closed curve d with $\varphi(d) = 1$ which is disjoint from c. As a surface of genus two does not admit bounding pairs, the surface $S - (c \cup d)$ is a four-holed sphere. Thus there exists a simple closed separating curve e which decomposes S into two one-holed tori T_1, T_2 such that $c \in T_1, d \in T_2$.

Denoting by ι the mod two homological intersection form on $H_1(S, \mathbb{Z}/2\mathbb{Z})$, there are two nonseparating simple closed curves $v \subset T_1, w \subset T_2$ so that

$$\iota(v, c) = 1 = \iota(w, d) \text{ and } \iota(w, c) = \iota(v, d) = 0. \tag{3}$$

The curves $a_1 = c, b_1 = w, a_2 = d, b_2 = w$ define a geometric symplectic basis for $H_1(S, \mathbb{Z})$. Since $\varphi(a_1) = \varphi(a_2) = 1$, the formula (2) for the Arf invariant shows that φ is even as claimed. □

2.4. **$\mathbb{Z}/r\mathbb{Z}$-spin structures for $r = 2, 4$ on a surface of genus $g \geq 3$.** In this subsection we study the graph $\mathcal{CG}_1$ for a $\mathbb{Z}/r\mathbb{Z}$-spin structure on a surface of genus $g \geq 3$ for $r = 2, 4$. To this end we introduce two more graphs related to simple closed curves on surfaces.

Definition 2.8. Let S be a compact surface of genus $g \geq 2$. The *graph of nonseparating pairs* $\mathcal{NS}$ is the graph whose vertices are unordered pairs of simple closed curves (c, d) on S so that $S - (c \cup d)$ is connected. Two such pairs $(c, d), (c', d')$ are connected by an edge of length one if they differ by a single component and can be realized disjointly.

For a compact surface of genus $g \geq 3$, with or without boundary, the graph $\mathcal{NS}$ of nonseparating pairs is connected (see [H14] for more details and more information on this graph).

Definition 2.9. Let S be a compact surface S of genus $g \geq 1$ with two distinguished boundary components A_1, A_2. The *nonseparating arc graph* is the graph whose vertices are isotopy classes of embedded arcs in S connecting A_1 to A_2. The endpoints of an arc may move freely along the boundary circles A_1, A_2 in such an isotopy class. Two such arcs ϵ_1, ϵ_2 are connected by an edge if ϵ_1, ϵ_2 are disjoint and $S - (\epsilon_1 \cup \epsilon_2)$ is connected.

Our next goal is to show that the nonseparating arc graph is connected. To this end we evoke an observation of Putman (Lemma 2.1 of [Put08]) which we refer to as the *Putman trick* in the sequel.

Lemma 2.10 (Putman). *Let G be a graph which admits a vertex transitive isometric action of a finitely generated group Γ and let v be a vertex of G. If for each element s of a finite generating set $\mathcal{S}$ of Γ, the vertex v can be connected to sv by an edge path in G, then G is connected.*

We apply the Putman trick to show

Lemma 2.11. *The nonseparating arc graph $\mathcal{A}(A_1, A_2)$ on a compact surface S of genus $g \geq 1$ with two distinguished boundary components A_1, A_2 is connected.*

Proof. Clearly the pure mapping class group $P\mathrm{Mod}(S)$ of S acts transitively on the vertices of $\mathcal{A}(A_1, A_2)$, so it suffices to show that there exists a generating set $\mathcal{S}$ of $P\mathrm{Mod}(S)$ and an arc $\epsilon \in \mathcal{A}(A_1, A_2)$ which can be connected to its image $\psi(\epsilon)$ by an edge path in $\mathcal{A}(A_1, A_2)$ for every element $\psi \in \mathcal{S}$.

Now $P\mathrm{Mod}(S)$ can be generated by Dehn twists T_{c_i} about the collection of simple closed curves $c_1, \dots, c_k$ shown in Figure 3 (see Section 4.4 of [FM12]).
Thus there exists two disjoint arcs ϵ_1, ϵ_2 connecting A_1 to A_2 such that $\epsilon_1 \cup \epsilon_2$

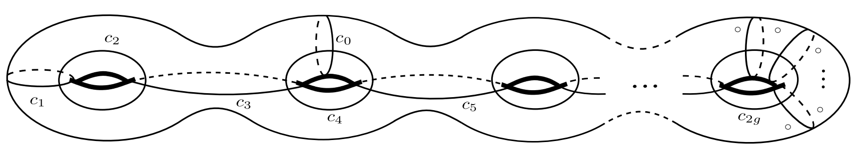

Figure 3

projects to an essential nonseparating simple closed curve in the surface obtained from S by capping off the boundary components A_1, A_2. Furthermore, ϵ_1 intersects one of the curves, say the curve c_1, in a single point and is disjoint from the remaining curves, and c_1 is disjoint from ϵ_2.

Then $T_{c_i}\epsilon_1 = \epsilon_1$ for $i \geq 2$, and ϵ_1 can be connected to $T_{c_1}(\epsilon_1)$ by the edge path $\epsilon_1, \epsilon_2, T_{c_1}\epsilon_1$. By the Putman trick this implies that $\mathcal{A}(A_1, A_2)$ is connected. ☐

Before we proceed we evoke another result of Salter [Sa19]. Namely, let c, d be disjoint simple closed curves on the compact surface S. Let ϵ be an embedded arc in S connecting c to d whose interior is disjoint from $c \cup d$. A regular neighborhood ν of $c \cup \epsilon \cup d$ is homeomorphic to a three-holed sphere. Two of the boundary components of ν are the curves c, d up to homotopy. We choose an orientation of c, d in such a way that ν lies to the left. The third boundary component $c +_\epsilon d$, oriented in such a way that ν is to its right, satisfies $[c +_\epsilon d] = [c] + [d]$ where $[c]$ denotes the homology class of the oriented curve c. The following is Lemma 3.13 of [Sa19].

Lemma 2.12 (Salter)**.** $\varphi(c +_\epsilon d) = \varphi(c) + \varphi(d) + 1$.

We use the graph of nonseparating pairs and the nonseparating arc graph as auxiliary tools to show

Proposition 2.13. *Let $r = 2, 4$ and let φ be a $\mathbb{Z}/r\mathbb{Z}$ spin structure on a compact surface S of genus $g \geq 3$, with or without boundary. Then the graph $\mathcal{CG}_1$ is connected.*

Proof. We only consider the case of a $\mathbb{Z}/4\mathbb{Z}$-spin structure, the argument for a $\mathbb{Z}/2\mathbb{Z}$-spin structure is identical.

Our strategy is to construct vertices of the graph $\mathcal{CG}_1$ from vertices of the graph $\mathcal{NS}$ of nonseparating pairs and use connectedness of $\mathcal{NS}$ to construct for any two vertices of $\mathcal{CG}_1$ a connecting edge path.

The construction of a vertex $\Lambda(c, d)$ of $\mathcal{CG}_1$ from a vertex (c, d) of $\mathcal{NS}$ is nondeterministic as follows.

If at least one of the curves c, d, say the curve c, satisfies $\varphi(c) = \pm 1$, then we choose $\Lambda(c, d) = c$. Otherwise both $\varphi(c), \varphi(d)$ are even. Connect c, d by an embedded arc ϵ in S whose interior is disjoint from $c \cup d$. Orient the curves c, d and $c +_\epsilon d$ as described in Lemma 2.12. By Lemma 2.12, we have $\varphi(c +_\epsilon d) = \varphi(c) + \varphi(d) + 1$ and hence $\varphi(c +_\epsilon d) = \pm 1$. Furthermore, as (c, d) is a nonseparating pair and $[c +_\epsilon d] = [c] + [d]$, the homology class of the oriented curve $c +_\epsilon d$ is nontrivial and therefore $c +_\epsilon d$ is nonseparating. We then can define $\Lambda(c, d) = c +_\epsilon d$. This construction uses the assumption $r = 2, 4$.

Let now c, e be two vertices of the graph $\mathcal{CG}_1$. By definition, c, e are nonseparating simple closed curves on S with $\varphi(c) = \varphi(e) = \pm 1$. Choose nonseparating simple closed curves d, f on S so that (c, d) and (e, f) are vertices in $\mathcal{NS}$. By Lemma 2.4 we may assume that $\varphi(d), \varphi(f)$ are even. This guarantees that $\Lambda(c, d) = c$ and $\Lambda(e, f) = e$. Connect (c, d) to (e, f) by an edge path $(c_i, d_i)_{0 \leq i \leq n}$ in $\mathcal{NS}$; here $(c_0, d_0) = (c, d)$ and $(c_n, d_n) = (e, f)$.

We use the edge path (c_i, d_i) in $\mathcal{NS}$ to construct inductively an edge path $(a_j)_{0 \leq j \leq m}$ in $\mathcal{CG}_1$ connecting $c = a_0$ to $e = a_m$ which passes through suitable choices of the curves $\Lambda(c_i, d_i)$. More precisely, the construction is done in such a way that there is an increasing sequence $j_0 = 0 < j_1 < \cdots < j_n = m$ such that for each $i \leq n$, the curve a_{j_i} is a possible choice for $\Lambda(c_i, d_i)$. By the choice of d, f, this path connects c to e as required. The construction is inductive, and the choices for $\Lambda(c_i, d_i)$ are determined inductively as well.

Define $a_0 = c = c_0$ and assume by induction that for some $i \geq 0$ we constructed a path $(a_s)_{s \leq j_i}$ connecting a_0 to a choice a_{j_i} for $\Lambda(c_i, d_i)$. Our goal is to construct an edge path $(a_s)_{j_i \leq s \leq j_{i+1}} \subset \mathcal{CG}_1$ for some $j_{i+1} \geq j_i + 1$ which connects a_{j_i} to some choice $a_{j_{i+1}}$ for $\Lambda(c_{i+1}, d_{i+1})$. We distinguish two cases.

Case 1. At least one of the values $\varphi(c_i)$ or $\varphi(d_i)$ equals ± 1.

By construction, up to renaming we have $a_{j_i} = c_i = \Lambda(c_i, d_i)$ in this case.

Consider the pair $(c_{i+1}, d_{i+1}) \in \mathcal{NS}$. The curves c_{i+1}, d_{i+1} are disjoint from c_i. If at least one of the values $\varphi(c_{i+1}), \varphi(d_{i+1})$ equals ± 1, say if this holds true for $\varphi(c_{i+1})$, then define $j_{i+1} = j_i + 1$ and $a_{j_{i+1}} = c_{i+1}$. Define furthermore $\Lambda(c_{i+1}, d_{i+1}) = c_{i+1}$. This is consistent with the requirements for the path (a_j). Note that we may have $a_{j_{i+1}} = a_{j_i}$.

Otherwise $\varphi(c_{i+1})$ and $\varphi(d_{i+1})$ are both even. In particular, we have $c_i \neq c_{i+1}, d_{i+1}$. Cut S open along $c_{i+1} \cup d_{i+1}$. The resulting surface is a surface T of genus $g - 2 \geq 1$ with four distinguished boundary components which glue back to c_{i+1}, d_{i+1}. It contains the curve c_i. Denote the two boundary components which project to the curve c_{i+1} by C_1, C_2, and denote the two boundary components which project to the curve d_{i+1} by D_1, D_2.

By assumption, the curve $c_i \subset S$ is nonseparating. As the curves C_1, C_2 and D_1, D_2 are identified in S, the curve c_i either is nonseparating as a curve in T, or it separates T into a surface T_1 with at least two holes and a surface T_2 with at least three holes in such a way that up to replacing C_1 by C_2, the surface T_1 contains the curve C_1 in its boundary, and T_2 contains the curves C_2 in its boundary.

As a consequence, there is an embedded arc ϵ in $T - c_i$ which connects one of the boundary components C_1, C_2 to one of the boundary components D_1, D_2. But this just means that the curve $a_{j_{i+1}} = a_{j_i+1} = c_{i+1} +_\epsilon d_{i+1}$ is disjoint from c_i, is nonseparating and satisfies $\varphi(a_{j_{i+1}}) = \pm 1$. Define $\Lambda(c_{i+1}, d_{i+1}) = a_{j_{i+1}}$. This completes the construction in Case 1.

Case 2. $\varphi(c_i)$ and $\varphi(d_i)$ are both even.

By definition of the non-deterministically chosen curve $\Lambda(c_i, d_i)$, in this case there exists an embedded arc ϵ connecting c_i to d_i such that $a_{j_i} = c_i +_\epsilon d_i$. Assume by renaming that $d_{i+1} = d_i$. The curve c_{i+1} is disjoint from c_i, d_i, but it may not be disjoint from ϵ. Furthermore, $\varphi(d_{i+1}) = \varphi(d_i)$ is even.

Cut S open along $c_i \cup d_i$. Let T be the resulting surface with four distinguished boundary components C_1, C_2 and D_1, D_2 which glue to the curves c_i, d_i. For a suitable numbering, the arc ϵ connects the boundary components C_1 and D_1 of T. We distinguish two subcases.

Subcase 2a. $\varphi(c_{i+1}) = \pm 1$.

As $\varphi(d_{i+1})$ is even we have $\Lambda(c_{i+1}, d_{i+1}) = c_{i+1}$. Thus we have to construct an edge path in $\mathcal{CG}_1$ connecting a_{j_i} to $a_{j_{i+1}} = c_{i+1}$.

We observed in Case 1 above that as c_{i+1} is nonseparating, it does not separate the pair of boundary components C_1, C_2 of T from the pair of boundary components D_1, D_2. Thus there are $p, q \in \{1, 2\}$, and there is an embedded arc η in T which is disjoint from c_{i+1} and connects C_p to D_q. If c_{i+1} does not separate the pair $\{C_1, D_2\}$ from the pair $\{C_2, D_1\}$ then we choose η in such a way that it either connects C_1 to D_1, or it connects C_2 to D_2. Choose an arc ϵ' in T which is disjoint from η and connects C_1 to D_1.

Consider the graph $\mathcal{A}(C_1, D_1)$ of nonseparating arcs in T with one endpoint on C_1 and the second endpoint on D_1. By Lemma 2.11, the graph $\mathcal{A}(C_1, D_1)$ is connected. Connect the arc ϵ to the arc ϵ' by an edge path in $\mathcal{A}(C_1, D_1)$, say the path $(\epsilon_\ell)_{0 \leq \ell \leq q}$ where $\epsilon_0 = \epsilon$ and $\epsilon_q = \epsilon'$. We construct from this system of arcs additional arcs δ_k connecting C_2 and D_2 as follows.

Let $\epsilon_\ell, \epsilon_{\ell+1}$ be two adjacent arcs in the path $(\epsilon_s) \subset \mathcal{A}(C_1, D_1)$. By definition, $T - (\epsilon_\ell \cup \epsilon_{\ell+1})$ is connected. Thus there exists an arc δ_ℓ connecting C_2 to D_2 which is disjoint from ϵ_ℓ and $\epsilon_{\ell+1}$. Replace the two arcs $\epsilon_\ell, \epsilon_{\ell+1}$ by the ordered sequence of arcs $\epsilon_\ell, \delta_\ell, \epsilon_{\ell+1}$.

Doing this construction for each ℓ yields a sequence β_u $(0 \leq u \leq 2k)$ of embedded arcs in the surface T with the following properties.

- $\beta_0 = \epsilon, \beta_{2k} = \epsilon'$.
- For each $\ell < k$ the arc $\beta_{2\ell}$ connects the boundary components C_1 and D_1, and the arc $\beta_{2\ell+1}$ connects C_2 and D_2.
- For all $u < 2k$ the arcs β_u, β_{u+1} are disjoint.

For each $u \leq 2k$ the simple closed curve $b_u = c_i +_{\beta_u} d_i$ in S is nonseparating, and as $\varphi(c_i)$ and $\varphi(d_i)$ are even we have $\varphi(b_u) = \pm 1$. Moreover, the curves b_u and b_{u+1} are disjoint. Thus $(b_i)_{0 \leq i \leq 2k}$ is a path in $\mathcal{CG}_1$ which connects $b_0 = a_{j_i}$ to $b_{2k} = c_i +_{\epsilon'} d_i$.

Recall that the arc η which is disjoint from c_{i+1} connects C_p to D_q where $p, q \in \{1, 2\}$. There are now three possibilities. In the first case, we have $p = q = 1$. Then η is a vertex in the graph $\mathcal{A}(C_1, D_1)$, and we may in fact assume that $\eta = \epsilon'$. The above construction then yields an edge path of length $2k$ in $\mathcal{CG}_1$ connecting $c +_\epsilon d$ to $c +_\eta d$. As $c +_\eta d$ is disjoint from c_{i+1}, this edge path extends to an edge path in $\mathcal{CG}_1$ of length $2k + 1$ which connects $c +_\epsilon d = a_{j_i}$ to $c_{i+1} = a_{j_i + 2k+1} = a_{j_{i+1}} = \Lambda(c_i, d_i)$ as required.

In the second case, we have $p = q = 2$. Then the curves $c_i +_{\epsilon'} d_i$ and $c_i +_\eta d_i$ are disjoint, and $c_i +_\eta d_i$ is disjoint from c_{i+1}, so we are done as before.

In the case $p = 1, q = 2$ or $p = 2, q = 1$, by assumption on η the curve c_{i+1} separates the pair $\{C_1, D_2\}$ of boundary components of T from the pair $\{C_2, D_1\}$. Then the curves $c_i +_{\epsilon'} d_i$ and $c_i +_\eta d_i$ intersect in two points, and a tubular neighborhood of $c_i +_{\epsilon'} d_i \cup c_i +_\eta d_i$ in the surface T is a four-holed sphere Y embedded in the interior of T. The surface $T - Y$ has four components, each of which contains one of the circles C_i, D_i in its boundary. As the circles C_1, C_2 and D_1, D_2 are identified in the surface S, this implies that $S - Y$ has two connected components. Since the genus of S is at least 3, one of these components, say the component Z, has genus at least one. It contains two boundary components of $S - Y$, say the circles A, B, in its boundary. The simple closed curves A, B are non-separating in S.

If for one of the two circles A, B, say for the circle A, we have $\varphi(A) = \pm 1$, then the string $c_i +_{\epsilon'} d_i, A, c_i +_\eta d_i, c_{i+1}$ defines an edge path in $\mathcal{CG}_1$ which connects $c_i +_{\epsilon'} d_i$ to c_{i+1} and we are done.

Otherwise $\varphi(A), \varphi(B)$ are both even. Since the genus g of Z is positive, using once more Lemma 2.4 we can find a non-peripheral non-separating simple closed curve $e \subset Z$ with $\varphi(e) = 0$. Connect e to the boundary circle A of Z by an arc ζ in Z and observe that $e +_\zeta A$ is disjoint from both $c_i +_{\epsilon'} d_i, c_i +_\eta d_i$ and hence can be used to construct an edge path in $\mathcal{CG}_1$ which connects $c_i +_{\epsilon'} d$ to c_{i+1} as before.

Together we constructed a path in $\mathcal{CG}_1$ which connects a_{j_i} to $c_{j+1} = a_{j_{i+1}} = \Lambda(c_{i+1}, d_{i+1})$. Observe that this construction is not possible for a surface of genus two.

Subcase 2b. $\varphi(c_{i+1}), \varphi(d_{i+1})$ are both even.

As in Subcase 2a, choose an embedded arc η in the surface $T = S - (c_i \cup d_i)$ which is disjoint from c_{i+1} and connects the boundary component C_p to the boundary component D_q for some $p, q \in \{1, 2\}$. We showed in Subcase 2a that the curve $a_{j_i} = c_i +_\epsilon d_i$ can be connected to $e = c_i +_\eta d_i$ by an edge path in $\mathcal{CG}_1$. Now $T - \eta$ is connected and contains c_{i+1} and hence there exists an embedded arc ϵ' in $T - \eta$ which connects c_{i+1} to the boundary component $D' \in \{D_1, D_2\}$ distinct from D_q. Then the curve $a_{j_{i+1}} = c_{i+1} +_{\epsilon'} d_i = \Lambda(c_{i+1}, d_{i+1})$ is disjoint from $c_i +_\eta d_i$ and hence it can be connected to a_{j_i} by an edge path passing through the curve $c_i +_\eta d_i$ (recall that $d_i = d_{i+1}$). Thus the curve $a_{j_{i+1}}$ has all the required properties to complete the induction step.

Together this shows the proposition. $\square$

2.5. **$\mathbb{Z}/r\mathbb{Z}$-spin structures on a surface of genus $g \geq 4$.** In this subsection we investigate the graph $\mathcal{CG}_1$ on a surface of genus $g \geq 4$ for an arbitrary $r \geq 2$. To show connectedness we use the following auxiliary graph $\mathcal{PS}$. The vertices of $\mathcal{PS}$ are pairs of disjoint separating curves (c, d) which each decompose S into a surface of genus $g - 1$ and a one-holed torus. Thus $S - (c \cup d)$ is the disjoint union of two one-holed tori and a surface of genus $g - 2$. Two such pairs (c_1, d_1) and (c_2, d_2) are connected by an edge if up to renaming, $c_1 = c_2$ and d_2 is disjoint from c_1, d_1. Then $S - (c_1 \cup d_1 \cup d_2)$ is the disjoint union of a surface of genus $g - 3$ with at least three holes and three one-holed tori. In particular, the graph $\mathcal{PS}$ is only defined if the genus of S is at least three.

We use the Putman trick to show

Lemma 2.14. *For a compact surface S of genus $g \geq 4$, perhaps with boundary, the graph $\mathcal{PS}$ is a connected* $\mathrm{Mod}(S)$*-graph.*

Proof. The mapping class group $\mathrm{Mod}(S)$ of the surface S clearly acts on $\mathcal{PS}$, furthermore this action is vertex transitive. Namely, for any two vertices (a_1, b_1) and (a_2, b_2) of $\mathcal{PS}$, the complement $S - (a_i \cup b_i)$ is the union of two one-holed tori and a surface of genus $g - 2$ with $k + 2$ boundary components where $k \geq 0$ is the number of boundary components of S. Hence there exists $\varphi \in \mathrm{Mod}(S)$ with $\varphi(a_1, b_1) = (a_2, b_2)$.

Consider again the curve system $\mathcal{H}$ shown in Figure 3 with the property that the Dehn twists about these curves generate the mapping class group. Choose a pair of disjoint separating simple closed curves (a, b) which decompose S into a surface of genus $g - 1$ and a one-holed torus $X(a), X(b)$ and such that a curve $c \in \mathcal{H}$ intersects at most one of the curves a, b. If it intersects one of the curves a, b, then this intersection consists of precisely two points. For example, we can choose a to be the boundary of a small neighborhood of $c_1 \cup c_2$, and b to be the boundary of a small neighborhood of $c_5 \cup c_6$.

Now let $c \in \mathcal{H}$ and let T_c be the left Dehn twist about c. If c is disjoint from $a \cup b$, then $T_c(a, b) = (a, b)$ and there is nothing to show. Thus assume that c intersects a.

The image $T_c(a)$ of a is a separating simple closed curve contained in a small neighborhood Y of $X(a) \cup c$. By assumption on c, this surface is a one-holed torus disjoint from b. As $g \geq 4$, the genus of $S - (Y \cup X(b))$ is at least one and hence there is a separating curve $e \subset S - (Y \cup X(b))$ which decomposes $S - (Y \cup X(b))$ into a one-holed torus and a surface S'. But this means that (a, b) can be connected to $T_c(a, b) = (T_c a, b)$ by the edge path $(a, b) \to (e, b) \to (T_c a, b)$. As the roles of a and b can be exchanged, the lemma now follows from the Putman trick. $\square$

We are now ready to show

Proposition 2.15. *Let φ be an r-spin structure ($r \geq 2$) on a compact surface S of genus $g \geq 4$. Then the graph $\mathcal{CG}_1$ is connected.*

Proof. Let S be a compact surface of genus $g \geq 2$ and consider the graph $\mathcal{PS}$. To each of its vertices, viewed as a disjoint pair (c, d) of separating simple closed curves, we associate in a non-deterministic way a vertex $\Lambda(c, d)$ of $\mathcal{CG}_1$ as follows.

Denote by Σ_c, Σ_d the one-holed torus bounded by c, d. If one of the tori Σ_c, Σ_d contains a simple closed curve a with $\varphi(a) = \pm 1$ then define $\Lambda(c, d) = a$.

Now assume that none of the tori Σ_c, Σ_d contains a simple closed curve a with $\varphi(a) = \pm 1$. By Lemma 2.4, there are simple closed non-separating curves $a \subset \Sigma_c, b \subset \Sigma_d$ so that $\varphi(a) = 0 = \varphi(b)$. Since the tori Σ_c, Σ_d are disjoint, the pair (a, b) is non-separating, that is, $S - (a \cup b)$ is connected. Choose an embedded arc ϵ in S connecting a to b. By Lemma 2.12, the curve $\Lambda(c, d) = a +_\epsilon b$ satisfies $\varphi(a +_\epsilon b) = \pm 1$, furthermore it is nonseparating.

Let a be any vertex of $\mathcal{CG}_1$ and let b be any simple closed curve which intersects a in a single point. Such a curve exists since a is nonseparating. Then a tubular neighborhood of $a\cup b$ is a torus containing a. Let c be the boundary curve of this torus and choose a second separating simple closed curve d so that $(c,d)\in\mathcal{PS}$.

Let $e\in\mathcal{CG}_1$ be another vertex. Construct as above a vertex $(p,q)\in\mathcal{PS}$ so that e is contained in the one-holed torus cut out by p. Connect (c,d) to (p,q) by an edge path $(c_i,d_i)_{0\leq i\leq k}$ in $\mathcal{PS}$. We use this edge path to construct an edge path $(a_j)\subset\mathcal{CG}_1$ connecting a to e which passes through suitable choices a_{j_i} $(i\leq k)$ of the curves $\Lambda(c_i,d_i)$.

Define $a_0=a$ and by induction, let us assume that we constructed already the path $(a_j)_{0\leq j\leq j_i}$ for some $i\geq 0$. We distinguish two cases.

Case 1: One of the tori $\Sigma_{c_i},\Sigma_{d_i}$ contains a curve f with $\varphi(f)=\pm 1$.

By construction, in this case we may assume by renaming that $f=a_{j_i}\subset\Sigma_{c_i}$.

If $c_i\in\{c_{i+1},d_{i+1}\}$ then define $a_{j_i+1}=a_{j_{i+1}}=a_{j_i}=\Lambda(c_{i+1},d_{i+1})$ and note that this is consistent with the requirements for the induction step.

Thus we may assume now that $c_i\not\in\{c_{i+1},d_{i+1}\}$. If one of the tori $\Sigma_{c_{i+1}},\Sigma_{d_{i+1}}$, say the torus $\Sigma_{c_{i+1}}$, contains a curve h with $\varphi(h)=\pm 1$, then as Σ_{c_i} is disjoint from $\Sigma_{c_{i+1}}$, the curve h is disjoint from a_{j_i} and we can define $a_{j_i+1}=h=a_{j_{i+1}}=\Lambda(c_{i+1},d_{i+1})$.

Thus assume that neither $\Sigma_{c_{i+1}}$ nor $\Sigma_{d_{i+1}}$ contains such a curve. Since Σ_{c_i} and $\Sigma_{c_{i+1}},\Sigma_{d_{i+1}}$ are pairwise disjoint, we can find an embedded arc ϵ in $S-\Sigma_{c_i}$ connecting a simple closed curve $u\subset\Sigma_{c_{i+1}}$ with $\varphi(u)=0$ to a curve $h\subset\Sigma_{d_{i+1}}$ with $\varphi(h)=0$. We then can define $a_{j_i+1}=u+_\epsilon h=\Lambda(c_{i+1},d_{i+1})=a_{j_{i+1}}$.

Case 2: None of the tori $\Sigma_{c_i},\Sigma_{d_i}$ contains a curve f with $\varphi(f)=\pm 1$.

In this case there are simple closed curves $f\subset\Sigma_{c_i},h\subset\Sigma_{d_i}$ with $\varphi(f)=\varphi(h)=0$, and there is an embedded arc ϵ connecting f to h so that

$$a_{j_i}=\Lambda(c_{i+1},d_{i+1})=f+_\epsilon h.$$

Assume without loss of generality that $d_i=d_{i+1}$.

Let us in addition assume for the moment that the arc ϵ is disjoint from c_{i+1}. If furthermore there exists a simple closed curve $u\subset\Sigma_{c_{i+1}}$ with $\varphi(u)=\pm 1$, then this curve is a choice for $\Lambda(c_{i+1},d_{i+1})$ which is disjoint from a_{j_i} and we are done.

Otherwise cut S open along the simple closed curve $h\subset\Sigma_{d_i}=\Sigma_{d_{i+1}}$ and let H_1,H_2 be the two boundary components of $S-h$. By renaming, assume without loss of generality that ϵ connects the boundary component H_1 to the curve f, i.e. it leaves the curve h from the side corresponding to H_1. Now note that $M=S-h-\epsilon-\Sigma_{c_i}$ is a connected surface of genus $g-2\geq 2$ with two distinguished boundary circles, one of which is the curve H_2, and $M\supset\Sigma_{c_{i+1}}$. Therefore there exists an embedded arc $\epsilon'\subset M$ connecting H_2 to a simple closed curve $u\subset\Sigma_{c_{i+1}}$ with $\varphi(u)=0$. Define $a_{j_i+1}=h+_{\epsilon'}u$ and note that this definition is consistent with all requirements. This construction completes the induction step under the additional assumption that arc ϵ is disjoint from c_{i+1}.

We are left with the case that ϵ is *not* disjoint from $\Sigma_{c_{i+1}}$. Cut S open along $f \cup h$ and note that the resulting surface Z has genus $g-2 \geq 2$ and four distinguished boundary components, say the components F_1, F_2, H_1, H_2. Assume that ϵ connects F_1 to H_1.

Consider the nonseparating arc graph $\mathcal{A}(F_1, H_1)$ in Z of arcs connecting F_1 to H_1. By Lemma 2.11, this graph is connected. Let ϵ_i be a path in $\mathcal{A}(F_1, H_1)$ which connects ϵ to an arc ϵ' disjoint from $\Sigma_{c_{i+1}}$. For any two consecutive of such arcs, say the arcs $\epsilon_j, \epsilon_{j+1}$, the surface $Z - (\epsilon_1 \cup \epsilon_2)$ is connected and hence we can find a disjoint arc δ_j connecting F_2 to H_2. The curves $f +_{\epsilon_j} h, f +_{\delta_j} h, f +_{\epsilon_{j+1}} h$ are disjoint and yield a path connecting $f +_\epsilon h$ to a curve $f +_{\epsilon'} h$ which is disjoint from $\Sigma_{c_{i+1}}$. We then can apply the construction for the case that the arc connecting f to h is disjoint from $\Sigma_{c_{i+1}}$. This completes the proof of the proposition. $\square$

For technical reasons we need a stronger version of Proposition 2.13 and Proposition 2.15. Consider a $\mathbb{Z}/r\mathbb{Z}$-spin structure φ on a compact surface S of genus g (with or without boundary) for an arbitrary number $r \geq 2$. We introduce another graph $\mathcal{CG}_1^+$ as follows. The vertices of $\mathcal{CG}_1^+$ coincide with the vertices of $\mathcal{CG}_1$. Any two such vertices c, d are connected by an edge if c, d are disjoint and if furthermore $S - (c \cup d)$ is connected. Thus $\mathcal{CG}_1^+$ is obtained from $\mathcal{CG}_1$ by removing some of the edges. In particular, if $\mathcal{CG}_1^+$ is connected then then same holds true for $\mathcal{CG}_1$. We use connectedness of $\mathcal{CG}_1$ to establish connectedness of $\mathcal{CG}_1^+$.

Lemma 2.16. *If the genus g of S is at least 3 then the graph $\mathcal{CG}_1^+$ is connected provided that $\mathcal{CG}_1$ is connected.*

Proof. Let $c, d \in \mathcal{CG}_1$ be two vertices which are connected by an edge in $\mathcal{CG}_1$ and which are not connected by an edge in $\mathcal{CG}_1^+$. This means that c, d are disjoint, and $S - (c \cup d)$ is disconnected. We have to show that c, d can be connected in $\mathcal{CG}_1^+$ by an edge path.

To this end recall that c, d are nonseparating and therefore the disconnected surface $S - (c \cup d)$ has two connected components S_1, S_2. The surface S_1 has genus $g_1 \geq 1$ and at least two boundary components, and the surface S_2 has genus $g_2 = g - g_1 - 1 \geq 0$ and at least two boundary components.

Choose a simple closed curve $d_i \subset S_i$ $(i = 1, 2)$ which bounds with $c \cup d$ a pair of pants P_i. Write $\Sigma_i = S_i - P_i$; the genus of Σ_i equals g_i. Glue P_1 to P_2 along $c \cup d$ so that the resulting surface Σ_0 is a two-holed torus containing $c \cup d$ in its interior. Choose a nonseparating simple closed curve $e \subset \Sigma_0$ which intersects both c, d in a single point. Since $\varphi(c) = \pm 1$ we have $\varphi(T_c e) = \varphi(e) \pm 1$ where T_c is the left Dehn twist about c. Thus via replacing e by $T_c^k e$ for a suitable choice of $k \in \mathbb{Z}$ we may assume that $\varphi(e) = 1$. In other words, we may assume that e is a vertex of $\mathcal{CG}_1$.

Assume for the moment that $g_2 \geq 1$. By Lemma 2.4, there exist simple closed curves $a \subset \Sigma_1, b \subset \Sigma_2$ with $\varphi(a) = \varphi(b) = 0$. Connect a to b by an embedded arc ϵ which is disjoint from $c \cup e$ (and crosses through the curve d). The curve $a +_\epsilon b$ satisfies $\varphi(a +_\epsilon b) = 1$, and it is disjoint from both c and e. Moreover, the surfaces $S - (c \cup a +_\epsilon b)$ and $S - (e \cup a +_\epsilon b)$ are connected. As a consequence, c can be connected to e by an edge path in $\mathcal{CG}_1^+$ of length two which passes through $a +_\epsilon b$.

By symmetry of this construction, e can also be connected to d by an edge path in $\mathcal{CG}_1^+$ and hence c can be connected to d by such a path. This completes the proof in the case that the genus g_2 of S_2 is positive.

If the genus of S_2 vanishes then the genus of S_1 equals $g_1 = g-1 \geq 2$. Any nonseparating curve in S_1 forms with both c, d a nonseparating pair. To find such a curve e with $\varphi(e) = 1$, note that S_1 contains two disjoint one-holed tori T_1, T_2, and by Lemma 2.4, there are embedded simple closed curves $a_i \in T_i$ which satisfy $\varphi(a_i) = 0$. Then for any arc ϵ in S_1 connecting a_1 to a_2, the curve $e = a_1 +_\epsilon a_2$ is nonseparating, and it is connected with both c, d by an edge in $\mathcal{CG}_1^+$. This is what we wanted to show. $\square$

Corollary 2.17. *Let φ be a $\mathbb{Z}/r\mathbb{Z}$-spin structure on a closed surface Σ of genus $g \geq 3$. Then the graph $\mathcal{CG}_1^+$ is connected.*

Remark 2.18. The proof of Corollary 2.17 is fairly involved. The main difficulty is the case $g = 3$ where we did not find an easier argument.

3. The action of $\mathrm{Mod}(S)[\varphi]$ on geometrically defined graphs

In this section we consider an arbitrary $\mathbb{Z}/r\mathbb{Z}$-spin structure φ on a compact surface S of genus $g \geq 3$, possibly with boundary, for some number $r \geq 2$. Our goal is to gain some information on the stabilizer $\mathrm{Mod}(S)[\varphi]$ of φ through its action on the graph $\mathcal{CG}_1^+$ introduced in Section 2.

We begin with some information on the stabilizer of a spin structure φ on a compact surface S with boundary. Fix a boundary component C of S. Denote by $P\mathrm{Mod}(S)$ the subgroup of the mapping class group $\mathrm{Mod}(S)$ of S which fixes the boundary component C. Thus we have $P\mathrm{Mod}(S) = \mathrm{Mod}(S)$ if and only if the boundary of S consists of one or two components. Write $P\mathrm{Mod}(S)[\varphi]$ to denote the stabilizer of φ in $P\mathrm{Mod}(S)$. This is a subgroup of $P\mathrm{Mod}(S)$ of finite index. Let Σ be the surface obtained from S by attaching a disk to C. There is an embedding $S \to \Sigma$ which induces a surjective homomorphism

$$\Pi : P\mathrm{Mod}(S) \to \mathrm{Mod}(\Sigma).$$

By a result of Johnson, extending earlier work of Birman (see Section 4.2.5 of [FM12]), there is an exact sequence

$$1 \to \mathbb{Z} \to \ker(\Pi) \xrightarrow{\Upsilon} \pi_1(\Sigma) \to 1 \tag{4}$$

where $\mathbb{Z}$ is the infinite cyclic central subgroup of $P\mathrm{Mod}(S)$ generated by the Dehn twists about C and where $\pi_1(\Sigma)$ is a so-called point pushing group.

For the formulation of the following lemma, recall that the integral homology $H_1(\Sigma, \mathbb{Z})$ of a compact surface Σ of genus $g \geq 2$, possibly with boundary, is a free abelian group $\mathbb{Z}^h$ for some $h \geq 4$. In fact, $h = 2g$ if the boundary of Σ is empty or connected, and in this case this group is generated by the homology classes of non-separating simple closed curves on Σ. If the boundary of Σ is disconnected, then it is still true that $H_1(\Sigma, \mathbb{Z})$ is generated by simple closed possibly peripheral curves.

Let $\zeta : \pi_1(\Sigma) \to H_1(\Sigma, \mathbb{Z})$ be the natural projection. Then for some $m \geq 1$ the preimage under the homomorphism ζ of the lattice in $H_1(\Sigma, \mathbb{Z})$ which is generated by m times the simple loop generators of $H_1(\Sigma, \mathbb{Z})$ is a subgroup Λ_m of $\pi_1(\Sigma)$ of finite index. Using the notations from the previous paragraph we have

Lemma 3.1. *Assume that the boundary circle C is equipped with the orientation induced from the orientation of S.*

(1) *If $\varphi(C) = -1$ then $\Upsilon(\ker \Pi \cap P\mathrm{Mod}(S)[\varphi]) = \pi_1(\Sigma)$.*
(2) *If $\varphi(C) = 1$, then $\Upsilon(\ker \Pi \cap P\mathrm{Mod}(S)[\varphi]) = \Lambda_m$ where $m = r/2$ if r is even, and $m = r$ otherwise.*

Proof. Choose a basepoint p for $\pi_1(\Sigma)$ in the interior of the attached disk. Let $\alpha \subset \Sigma$ be a simple non-separating loop through the basepoint p. Up to homotopy, the oriented boundary of a tubular neighborhood of α consists of two simple closed curves c_1, c_2 which enclose the circle C. In other words, together with C the curves c_1, c_2 bound a pair of pants P in S. We equip the curves c_i with the orientation as boundary curves of P.

By Proposition 3.8 of [Sa19], we have

$$\varphi(C) + \varphi(c_1) + \varphi(c_2) = -1 \tag{5}$$

and hence if $\varphi(C) = -1$ then $\varphi(c_1) + \varphi(c_2) = 0$.

Let as before T_d be the left Dehn twist about a simple closed curve d. Let $\beta \subset S$ be an oriented simple closed curve which crosses through the pair of pants P. As c_1, c_2 are disjoint, we have $\iota(T_{c_2}^{-1}(\beta), c_1) = \iota(\beta, c_1)$ and therefore Definition 2.1 shows that

$$\begin{aligned} \varphi(T_{c_1}T_{c_2}^{-1}(\beta)) &= \varphi(T_{c_2}^{-1}(\beta)) + \iota(\beta, c_1)\varphi(c_1) \\ &= \varphi(\beta) + \iota(\beta, c_1)\varphi(c_1) - \iota(\beta, c_2)\varphi(c_2). \end{aligned} \tag{6}$$

On the other hand, as $c_1 + c_2$ is homologous to the boundary curve C, the homological intersection number fulfills $\iota(\beta, c_1 + c_2) = 0$. Hence from (5) we conclude that if $\varphi(C) = -1$ then $\varphi(T_{c_1}T_{c_2}^{-1}(\beta)) = \varphi(\beta)$. Since β was an arbitrary simple closed curve, this shows that $T_{c_1}T_{c_2}^{-1} \in P\mathrm{Mod}(S)[\varphi]$. But $T_{c_1}T_{c_2}^{-1} \in P\mathrm{Mod}(S)$ is just the point-pushing map about α and therefore α is contained in $\Upsilon(P\mathrm{Mod}(S)[\varphi])$. We refer to [FM12] for a comprehensive discussion of the various versions of the Birman exact sequence.

As the point pushing group $\pi_1(\Sigma)$ is generated by point pushing maps along simple loops, this shows the first part of the lemma.

To show the second part of the lemma, assume now that $\varphi(C) = 1$. Equation (5) shows that $\varphi(c_1) + \varphi(c_2) = -2$ and hence by Formula (6) we have

$$\varphi(T_{c_1}T_{c_2}^{-1}(\beta)) = \varphi(\beta) + \iota(\beta, c_1)\varphi(c_1) + \iota(\beta, c_2)(\varphi(c_1) + 2).$$

Now let us assume that the oriented simple closed curve β crosses a single time through c_1, say when it enters P. Then $\iota(\beta, c_1) = -1, \iota(\beta, c_2) = 1$ and hence

$$\varphi(T_{c_1}T_{c_2}^{-1}(\beta)) = \varphi(\beta) - \varphi(c_1) + \varphi(c_1) + 2 = \varphi(\beta) + 2. \tag{7}$$

Using this formula $r/2$ times if r is even, and r times if r is odd, we conclude that the point pushing map about α is not contained in $\mathrm{Mod}(S)[\varphi]$, but it is the case for its $r/2$-th power or r-th power, respectively. Namely, putting $m = r/2$ if r is even and $m = r$ otherwise, it follows from the above discussion that we have $\varphi((T_{c_1}T_{c_2}^{-1})^m(\beta)) = \varphi(\beta)$ for every simple closed curve β which either is disjoint from P or which crosses through P precisely once. As such curves span the first homology of Σ, we conclude that the pull-back of φ under $(T_{c_1}T_{c_2}^{-1})^m$ coincides with φ on a collection of simple closed curves which span $H_1(S,\mathbb{Z})$. Corollary 2.6 of [HJ89] then shows that indeed, $(T_{c_1}T_{c_2}^{-1})^m \in P\mathrm{Mod}(S)[\varphi]$. Moreover, by equation (7), we know that $(T_{c_1}T_{c_2}^{-1})^k \notin P\mathrm{Mod}(S)[\varphi]$ if k is not a multiple of m.

On the other hand, by Lemma 3.15 of [Sa19], Dehn twists about separating simple closed curves in S are contained in $\mathrm{Mod}(S)[\varphi]$. As the commutator subgroup of $\pi_1(\Sigma)$ is generated by simple closed separating curves, and for each such curve α both Dehn twists T_{c_1}, T_{c_2} about the boundary curves of a tubular neighborhood of α as above are contained in $P\mathrm{Mod}(S)[\varphi]$, this yields the second part of the lemma. □

Consider again an arbitrary compact surface S of genus $g \geq 2$, equipped with a $\mathbb{Z}/r\mathbb{Z}$-spin structure φ for some $r \geq 2$. We use Lemma 3.1 to analyze the action of $\mathrm{Mod}(S)[\varphi]$ on the graph $\mathcal{CG}_1^+$. We begin with the investigation of the stabilizer of a vertex c of $\mathcal{CG}_1^+$ in $\mathrm{Mod}(S)[\varphi]$. As $\mathrm{Mod}(S)[\varphi]$ is a subgroup of $\mathrm{Mod}(S)$ of finite index, the stabilizer $\mathrm{Stab}(c)[\varphi]$ of c in $\mathrm{Mod}(S)[\varphi]$ is a subgroup of finite index of the stabilizer $\mathrm{Stab}(c)$ of c in $\mathrm{Mod}(S)$.

The group $\mathrm{Stab}(c)$ can be described as follows. Cut S open along c. The result is a surface Σ^2 of genus $g-1$ with two distinguished boundary components C_1, C_2. These components are equipped with an orientation as subsets of the oriented boundary of Σ^2. To simplify notations, let $\mathrm{Mod}(\Sigma^2)$ be the subgroup of the mapping class group of Σ^2 which preserves the subset $C_1 \cup C_2$ of the boundary. We allow that an element of $\mathrm{Mod}(\Sigma^2)$ exchanges C_1 and C_2. The stabilizer $\mathrm{Stab}(c)$ of c in the mapping class group $\mathrm{Mod}(S)$ of S can be identified with the quotient of the group $\mathrm{Mod}(\Sigma^2)$ by the relation $T_{C_1}T_{C_2}^{-1} = 1$ where T_{C_i} denotes the left Dehn twist about the boundary circle C_i (Theorem 3.18 of [FM12]). In short, we have

$$\mathrm{Stab}(c) = \mathrm{Mod}(\Sigma^2)/\mathbb{Z}.$$

The infinite cyclic subgroup of $\mathrm{Stab}(c)$ generated by the Dehn twist about c is central. The quotient group $\mathrm{Stab}(c)/\mathbb{Z}$ can naturally be identified with the mapping class group $\mathrm{Mod}(\Sigma_2)$ of a surface of genus $g-1$ with two punctures and perhaps with boundary if the boundary of S is non-trivial. We refer to [FM12] for a comprehensive discussion of these facts.

Let Σ be the surface obtained from Σ_2 by forgetting the punctures. Alternatively, Σ is obtained from Σ^2 by attaching a disk to each boundary component. The group $\mathrm{Mod}(\Sigma_2) = \mathrm{Stab}(c)/\mathbb{Z}$ fits into the *Birman exact sequence*

$$1 \to \pi_1(C(\Sigma,2)) \xrightarrow{\rho} \mathrm{Stab}(c)/\mathbb{Z} \to \mathrm{Mod}(\Sigma) \to 1 \tag{8}$$

where $\pi_1(C(\Sigma,2))$ is the *surface braid group*, that is, the fundamental group of the configuration space of two unordered distinct points in Σ. In particular, $\pi_1(C(\Sigma,2))$ is a normal subgroup of $\mathrm{Stab}(c)/\mathbb{Z} = \mathrm{Mod}(\Sigma_2)$.

The surjective homomorphism

$$\theta : \mathrm{Stab}(c) \to \mathrm{Stab}(c)/\mathbb{Z} = \mathrm{Mod}(\Sigma_2)$$

restricts to a homomorphism $\mathrm{Stab}(c)[\varphi] \to \mathrm{Mod}(\Sigma_2)$. The next proposition gives some first information on its image under the assumption that φ is a $\mathbb{Z}/2\mathbb{Z}$-spin structure and $\varphi(c) = 1$.

Proposition 3.2. *Let φ be a $\mathbb{Z}/2\mathbb{Z}$-spin structure on S and let c be a simple closed curve with $\varphi(c) = 1$. Then $\rho(\pi_1(C(\Sigma,2))) \subset \theta(\mathrm{Stab}(c)[\varphi])$.*

Proof. Let $\pi_1(PC(\Sigma,2))$ be the intersection of the fibre of the Birman exact sequence (8) with the subgroup of $\mathrm{Mod}(\Sigma_2)$ which fixes each of the two distinguished punctures. Following Section 4.2.5 of [FM12], the group $\pi_1(PC(\Sigma,2))$ can be described as follows.

Let C_1, C_2 be the distinguished boundary components of the surface $\Sigma^2 = S - c$. Let Σ^1 be the surface obtained from Σ^2 by attaching a disk to the boundary circle C_1. Let $P\mathrm{Stab}(c)$ and $P\mathrm{Mod}(\Sigma^2)$ be the index two subgroup of $\mathrm{Stab}(c)$ and $\mathrm{Mod}(\Sigma^2)$ which preserves each of the two boundary components C_1, C_2 of $S - c$. The inclusion $\Sigma^2 \to \Sigma^1$ induces a surjective homomorphism

$$\Xi : P\mathrm{Stab}(c)/\mathbb{Z} \to \mathrm{Mod}(\Sigma^1)/\mathbb{Z}$$

where as before $\mathrm{Mod}(\Sigma^1)$ is required to fix the boundary component C_2 of Σ^1 and where the group $\mathbb{Z}$ acts as the group of Dehn twists about c and about C_2. The kernel $\ker(\Xi)$ of this homomorphism is isomorphic to $\pi_1(\Sigma^1)$ (see [FM12] for more information on this version of the Birman exact sequence).

The spin structure φ pulls back to a spin structure $\hat{\varphi}$ on Σ^2. Since φ is a $\mathbb{Z}/2\mathbb{Z}$-spin structure on S and $\varphi(c) = 1$, the value of $\hat{\varphi}$ on each of the two boundary circles C_1, C_2 coincides with the value of a spin structure on the boundary of an embedded disk. This implies that $\hat{\varphi}$ induces a spin structure φ' on Σ^1. Or, equivalently, $\hat{\varphi}$ is the pull-back of a spin structure φ' on Σ^1 via the inclusion $\Sigma^2 \to \Sigma^1$. By Lemma 3.1, the group $\ker(\Xi) = \pi_1(\Sigma^1)$ stabilizes $\hat{\varphi}$, that is, we have $\ker(\Xi) \subset \mathrm{Mod}(\Sigma^2)[\hat{\varphi}]$.

Apply Lemma 3.1 a second time to the homomorphism $\mathrm{Mod}(\Sigma^1)/\mathbb{Z} \to \mathrm{Mod}(\Sigma)$ where Σ is obtained from Σ^1 by attaching a disk to C_2. As the group $\pi_1(PC(\Sigma,2))$ can be described as the quotient by its center $\mathbb{Z}^2$ of the kernel of the homomorphism $P\mathrm{Mod}(\Sigma^2) \to \mathrm{Mod}(\Sigma)$ which is obtained by applying the Birman exact sequence twice, first to a map which caps off the boundary component C_1, followed by the map which caps off C_2, this shows that $\pi_1(PC(\Sigma,2)) \subset \theta(\mathrm{Stab}(c)[\varphi]$. As exchanging C_1 and C_2 also preserves $\hat{\varphi}$ the proposition follows. ☐

We are now ready to give a complete description of the stabilizer in $\mathrm{Mod}(S)[\varphi]$ of a nonseparating simple closed curve c on S with $\varphi(c) = 1$ where as before, φ is a $\mathbb{Z}/2\mathbb{Z}$-spin structures on a compact surface S of genus $g \geq 3$.

Cut S open along c and write $\Sigma^2 = S - c$. The spin structure φ of S pulls back to a $\mathbb{Z}/2\mathbb{Z}$-spin structure $\hat{\varphi}$ on Σ^2. Denote as before by Σ the surface of genus $g-1$ with empty or connected boundary obtained from Σ^2 by capping off the two distinguished boundary components. We have

Proposition 3.3. *The $\mathbb{Z}/2\mathbb{Z}$-spin structure φ on S induces a $\mathbb{Z}/2\mathbb{Z}$-spin structure φ_c on Σ whose parity coincides with the parity of φ. If $\Pi : \mathrm{Stab}(c)/\mathbb{Z} \to \mathrm{Mod}(\Sigma)$ denotes the surjective homomorphism induced by the inclusion $S - c \to \Sigma$ then*

$$\Pi^{-1}\mathrm{Mod}(\Sigma)[\varphi_c] = \mathrm{Stab}(c)[\varphi]/\mathbb{Z}.$$

Proof. As φ is a $\mathbb{Z}/2\mathbb{Z}$-spin structure, the value of φ on a boundary circle of $S-c$ corresponding to a component of c coincides with the value of a $\mathbb{Z}/2\mathbb{Z}$-spin structure on the boundary of a disk. Thus φ induces a spin structure φ_c on Σ.

To compare the parities of the spin structures φ and φ_c, assume that Σ is obtained from $S - c$ by attaching disks D_1, D_2 to the two boundary components of S which correspond to the two copies of c. Choose a geometric symplectic basis $a_1, b_1, \dots, a_{g-1}, b_{g-1}$ for Σ, consisting of simple closed oriented curves which do not intersect the disks D_1, D_2. Then $a_1, b_1, \dots, a_{g-1}, b_{g-1}$ can be viewed as a system of curves in $\Sigma^2 = \Sigma - (D_1 \cup D_2)$ which maps to a curve system with the same properties in S by the map $\Sigma^2 \to S$. This curve system can be extended to a geometric symplectic basis for S containing the curve c, equipped with any orientation. As $\varphi(c) = 1$ we have $\varphi(c) + 1 = 0$. The claim now follows from the fact that $\varphi_c(u) = \varphi(\hat{u})$ for $u \in \{a_1, b_1, \dots, a_{g-1}, b_{g-1}\}$ where $\hat{u}$ is the image of u under the inclusion $\Sigma^2 \to S$, together with the formula (2) for the Arf invariant.

We are left with showing that $\mathrm{Stab}(c)[\varphi]/\mathbb{Z} = \Pi^{-1}\mathrm{Mod}(\Sigma)[\varphi_c]$. Observe first that as φ_c is induced from φ, we have $\Pi\mathrm{Stab}(c)[\varphi]/\mathbb{Z} \subset \mathrm{Mod}(\Sigma)[\varphi_c]$.

To show that in fact equality holds let Σ_2 be the surface obtained from $S - c$ by replacing the boundary components by punctures. The group $\mathrm{Stab}(c)[\varphi]/\mathbb{Z}$ can be identified with a subgroup Γ_c of $\mathrm{Mod}(\Sigma_2)$. We view the punctures of Σ_2 as marked points p_1, p_2 in Σ.

Let θ be any diffeomorphism of Σ which preserves φ_c. Then θ is isotopic to a diffeomorphism of Σ which equals the identity on a disk $D \subset \Sigma$ containing both points p_1, p_2. Thus θ lifts to a diffeomorphism θ' of Σ_2 which preserves the pull-back of φ_c to a spin structure on Σ_2.

The boundary circle ∂D of D can be viewed as a simple closed curve in $S - c$. Via the projection $S - c \to S$ which identifies the two distinguished boundary components of $S - c$, the curve ∂D projects to a separating simple closed curve in S which decomposes S into a one-holed torus T containing c and a surface of genus $g - 1$ with connected boundary. The diffeomorphism θ' lifts to a diffeomorphism Θ of S which is the identity on T.

Then $\Theta^*\varphi$ is a spin structure on S which defines the same function on $H_1(S, \mathbb{Z})$ as φ. Using once more the result of Humphries and Johnson [HJ89] (see Theorem 3.9 of [Sa19]), this implies that Θ stabilizes φ. As Θ projects to the mapping class of Σ defined by the diffeomorphism θ, this shows surjetivity of the homomorphism $\Pi : \mathrm{Stab}(c)[\varphi]/\mathbb{Z} \to \mathrm{Mod}(\Sigma)[\varphi_c]$.

On the other hand, by Proposition 3.2 the kernel of the homomorphism Π also is contained in $\mathrm{Stab}(c)[\varphi]/\mathbb{Z}$. Together this completes the proof of the proposition. □

The next observation uses Proposition 4.9 of [Sa19]. For its formulation, recall from Section 2 the definition of the graph $\mathcal{CG}_1^+$. Its vertices are nonseparating simple closed curves with prescribed value ± 1 of the spin structure. The graph $\mathcal{CG}_1{}^+$ is well defined if the genus g of S is at least two although it may not have edges.

Proposition 3.4. *Let φ be a $\mathbb{Z}/r\mathbb{Z}$-spin structure on a compact surface S of genus $g \geq 2$ with empty or connected boundary and where $r \leq 2g$. Then for any two directed edges e_1, e_2 of the graph $\mathcal{CG}_1^+$ there exists a mapping class $\zeta \in \mathrm{Mod}(S)[\varphi]$ with $\zeta(e_1) = e_2$. In particular, the action of* $\mathrm{Mod}(S)[\varphi]$ *on $\mathcal{CG}_1^+$ is vertex transitive.*

Proof. The proof consists of an adjustment of the argument in the proof of Proposition 4.9 of [Sa19].

Recall that a geometric symplectic basis for S is a set $\{a_1, b_1, \ldots, a_{2g}, b_{2g}\}$ of simple closed curves on S such that a_i, b_i intersect in a single point, and $a_i \cup b_i$ is disjoint from $a_j \cup b_j$ for $j \neq i$.

A vertex of $\mathcal{CG}_1^+$ is a simple closed curve c on S with $\varphi(c) = \pm 1$. In the sequel we always orient such a vertex c in such a way that $\varphi(c) = 1$. For a given directed edge e of $\mathcal{CG}_1^+$ with ordered endpoints c, d, we aim at constructing a geometric symplectic basis $\mathcal{B}(e)$ such that $a_1 = c, a_2 = d, \varphi(a_i) = 0$ for $i \geq 3$, $\varphi(b_i) = 0$ for $i \leq g - 1$ and $\varphi(b_g) = 0$ or 1 as predicted by the parity of φ. If such a basis $\mathcal{B}(e_1), \mathcal{B}(e_2)$ can be found for any two directed edges e_1, e_2 of $\mathcal{CG}_1^+$ with ordered endpoints c_1, d_1 and c_2, d_2 then there exists a diffeomorphism ζ of S which maps $\mathcal{B}(e_1)$ to $\mathcal{B}(e_2)$ and maps c_1, d_1 to c_2, d_2. The pullback $\zeta^*\varphi$ of φ is a spin structure on S whose values on $\mathcal{B}(e_1)$ coincide with the values of φ. By a result of Humphries and Johnson [HJ89], see Theorem 3.9 of [Sa19], this implies that $\zeta^*\varphi = \varphi$ and hence the isotopy class of ζ is contained in $\mathrm{Mod}(S)[\varphi]$ and maps the directed edge e_1 to the directed edge e_2.

To simplify further, choose any geometric symplectic basis

$$\mathcal{B} = \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$$

for S with $\alpha_1 = c$, $\alpha_2 = d$. A small tubular neighborhood of $\alpha_i \cup \beta_i$ is a one-holed torus T_i embedded in S. By Lemma 2.4, for all $i \geq 3$ we may replace α_i by an oriented simple closed curve in T_i, again denoted by α_i, which satisfies $\varphi(\alpha_i) = 0$.

Assume that β_i $(i = 1, 2)$ is oriented in such a way that $\iota(\beta_i, \alpha_i) = 1$ where ι is the symplectic form. As $\varphi(T_{\alpha_i}(\beta_i)) = \varphi(\beta_i) + 1$, via perhaps replacing β_i by its image under a suitably chosen power of a Dehn twist about α_i we may assume that $\varphi(\beta_i) = 0$. Therefore for the construction of a geometric symplectic basis $\mathcal{B}(e)$ with the required properties, it suffices to modify successively the curves β_i $(i \geq 3)$

while keeping α_j $(j \geq 1)$ and β_k for $k < i$ fixed such that φ assumes the prescribed values on the modified curves.

We follow the proof of Proposition 4.9 of [Sa19]. For $1 \leq i \leq g$ let δ_i be the boundary curve of the torus T_i which is a small tubular neighborhood of $\alpha_i \cup \beta_i$, equipped with the orientation as an oriented boundary circle of $S - T_i$ $(i \geq 1)$. By homological coherence (Proposition 3.8 of [Sa19]), we have $\varphi(\delta_i) = 1$ for all i.

Thus if ϵ is an embedded arc in S connecting β_3 to δ_4 whose interior is disjoint from α_3 and all δ_j for $j \neq 3$, then $\varphi(\beta_3 +_\epsilon \delta_4) = \varphi(\beta_3) + 2$. Moreover, $\beta_3 +_\epsilon \delta_4$ is disjoint from δ_j for all $j \neq 3$.

Repeat this construction with an arc connecting $\beta_3 +_\epsilon \delta_4$ to δ_5 whose interior is disjoint from all δ_j for $j \neq 3$. As there are $g - 1$ of the curves δ_j $(j \neq 3)$ and as $r \leq 2g$, in this way we can find a simple closed curve β_3' intersecting α_3 in a single point and disjoint from the curves δ_j for $j \neq 3$ so that $\varphi(\beta_3') \in \{0, 1\}$.

Let δ_3' be the boundary of a tubular neighborhood of $\alpha_3 \cup \beta_3'$. Then δ_3' is disjoint from all the curves δ_j for $j \neq 3$. As in the proof of Proposition 4.9 of [Sa19], repeat this procedure with the curve β_4 and the curves $\delta_1, \delta_2, \delta_3', \dots, \delta_g$. In finitely many steps we can change the geometric symplectic basis $\mathcal{B}$ to a geometric symplectic basis $\mathcal{B}' = \{\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3', \dots, \alpha_g, \beta_g'\}$ which fulfills $\varphi(\beta_j') = 0$ or 1 for all $3 \leq j \leq g$.

It remains to further alter β_j' for $3 \leq j \leq g - 1$ to a nonseparating simple closed curve β_j'' with $\varphi(\beta_j'') = 0$, and to alter β_g' to a simple closed curve β_g'' with $\varphi(\beta_g'') = 0$ or 1 depending on the parity of the $\mathbb{Z}/r\mathbb{Z}$-spin structure φ. This construction is carried out in detail in the proof of Proposition 4.9 of [Sa19] and will not be presented here as it would require the introduction of a significant amount of new notation. It takes place in a subsurface of S of genus $g-2$ which is disjoint from $\alpha_1, \beta_1, \alpha_2, \beta_2$ and contains α_i, β_i for $3 \leq i \leq g$. The resulting geometric symplectic basis has the properties we are looking for. □

Remark 3.5. The proof of Proposition 3.4 can also be used to show the following. Under the assumption of the proposition, let $c, d \subset S$ be two non-separating simple closed curves with $\varphi(c) = \varphi(d) = 0$; then there exists some $\zeta \in \mathrm{Mod}(S)[\varphi]$ with $\zeta(c) = d$. In fact, this case is more explicitly covered by Proposition 4.2 and Proposition 4.9 of [Sa19].

The next statement is an extension of Proposition 3.4 to surfaces with more than one boundary component under some restrictions on the spin structure.

Corollary 3.6. *For $r \leq 2g$ let φ be a $\mathbb{Z}/r\mathbb{Z}$-spin structure on a compact surface S of genus $g \geq 2$ with non-empty boundary which is induced from a spin structure φ' on a compact surface Σ of genus g with empty or connected boundary by an inclusion $S \to \Sigma$ which maps each boundary component of S to the boundary of an embedded disk in Σ. Then for any two vertices c, d of $\mathcal{CG}_1^+$ there exists a mapping class $\zeta \in \mathrm{Mod}(S)[\varphi]$ with $\zeta(c) = d$. In particular, the action of $\mathrm{Mod}(S)[\varphi]$ is transitive on the vertices of $\mathcal{CG}_1^+$.*

Proof. Let $\Psi : S \to \Sigma$ be the natural embedding. Let c, d be vertices of the graph $\mathcal{CG}_1^+$ for the spin structure φ on S. Then c, d are nonseparating simple closed curves and hence their images $\Psi(c), \Psi(d)$ are nonseparating simple closed curves on Σ. Furthermore, as φ is the pull-back of a spin structure φ' on Σ, we have $\varphi'(\Psi(c)) = \varphi'(\Psi(d)) = 1$.

By Proposition 3.4, there exists a mapping class $\theta \in \mathrm{Mod}(\Sigma)(\varphi')$ which maps $\Psi(c)$ to $\Psi(d)$. We can choose a diffeomorphism of Σ representing θ which equals the identity on each component of $\Sigma - S$. Thus there exists a lift Θ of θ to a mapping class of S. This mapping class is contained in $\mathrm{Mod}(S)[\varphi]$, and it maps the simple closed curve c to a simple closed curve d' whose image under Ψ is isotopic to $\Psi(d)$.

Using once more the Birman exact sequence, this implies that there exists a mapping class β in the kernel of the homomorphism $\mathrm{Mod}(S) \to \mathrm{Mod}(\Sigma)$ which maps d' to d. But by an iterated application of Lemma 3.1, this kernel is contained in $\mathrm{Mod}(S)[\varphi]$ and hence c can be mapped to d by an element of $\mathrm{Mod}(S)[\varphi]$. □

The *augmented Teichmüller space* $\overline{\mathcal{T}}(S)$ of the compact surface S is the union of the Teichmüller space with so-called *boundary strata*. Each of these boundary strata is defined by a non-empty system $\mathcal{C}$ of pairwise disjoint essential simple closed curves. The stratum defined by such a curve system can be thought of as the Teichmüller space of the surface obtained from S by shrinking each component of $\mathcal{C}$ to a node. In other words, such a stratum is a complex manifold which is naturally biholomorphic to the Teichmüller space of the surface obtained by cutting S open along the components of $\mathcal{C}$ and replacing each boundary component of the resulting bordered surface by a puncture.

Using Fenchel Nielsen coordinates, the augmented Teichmüller space can be equipped with a natural topology. For this topology, the usual Teichmüller space embeds into $\overline{\mathcal{T}}(S)$ as an open dense subset. Furthermore, the inclusion of the Teichmüller space of a punctured surface defined by the curve system $\mathcal{C}$ onto a boundary stratum of $\overline{\mathcal{T}}(S)$ also is an embedding. We refer to [Wol10] for an detailed description and for a discussion of the following

Theorem 3.7. *The augmented Teichmüller space $\overline{\mathcal{T}}(S)$ is a non locally compact stratified space. The mapping class group* $\mathrm{Mod}(S)$ *of S acts on $\overline{\mathcal{T}}(S)$, with quotient the Deligne Mumford compactification of the moduli space of curves of genus g.*

Fix again a $\mathbb{Z}/2\mathbb{Z}$-spin structure φ on a surface S of genus $g \geq 2$. Define the *spin Teichmüller space* $\mathcal{T}_{\mathrm{spin}}(S)$ to be the Teichmüller space of S together with this spin structure. The group $\mathrm{Mod}(S)[\varphi]$ acts on $\mathcal{T}_{\mathrm{spin}}(S)$ as a group of biholomorphic transformations, with quotient the *spin moduli space* $\mathcal{M}_\varphi = \mathcal{T}(S)/\mathrm{Mod}(S)[\varphi]$.

We can define an augmented spin Teichmüller space $\overline{\mathcal{T}}_{\mathrm{spin}}(S)$ as the union of spin Teichmüller space with all strata of augmented Teichmüller space which are defined by systems of nonseparating simple closed curves c on S with $\varphi(c) = 1$. Equipped with the subspace topology, this is a subspace of $\overline{\mathcal{T}}(S)$ which is invariant under the action of the spin mapping class group. As a corollary of the discussion in this section, we have

Corollary 3.8. *The quotient $\overline{\mathcal{T}}_{\rm spin}(S)/{\rm Mod}(S)[\varphi]$ is a partial bordification of the spin moduli space $\mathcal{T}_{\rm spin}(S)/{\rm Mod}(S)[\varphi]$. Its boundary contains the spin moduli space of the same parity on a surface of genus $g-1$ with two marked points (punctures) as an open dense subset.*

Remark 3.9. Corollary 3.8 can be thought of as describing a specific subset of a Deligne Mumford compactification of the moduli space of curves with a fixed spin structure. Such a Deligne Mumford compactification was constructed by Cornalba [Co89].

4. Structure of the spin mapping class group of odd parity

The goal of this section is to prove Theorem 3.

We begin with some additional information on the spin mapping class group. Fix a $\mathbb{Z}/r\mathbb{Z}$-spin structure φ on a closed surface Σ_g of genus g for some $r \geq 2$. For a simple closed curve c on Σ_g with $\varphi(c) = \pm 1$, this spin structure restricts to a spin structure on the surface Σ_{g-1}^2 of genus $g-1$ with two boundary circles c_1, c_2 obtained by cutting Σ_g open along c. We denote this spin structure again by φ. Define the group Γ_{g-1}^2 to be the following quotient of the spin mapping class group ${\rm Mod}(\Sigma_{g-1}^2)[\varphi]$.

The group ${\rm Mod}(\Sigma_{g-1}^2)[\varphi]$ contains a rank two free abelian central subgroup generated by the r-th powers of the left Dehn twists T_{c_1}, T_{c_2} about the boundary circles c_1, c_2 of Σ_{g-1}^2. Define $\Gamma_{g-1}^2 = {\rm Mod}(\Sigma_{g-1}^2)[\varphi]/\mathbb{Z}$ where the infinite cyclic subgroup $\mathbb{Z}$ is generated by $T_{c_1}^r T_{c_2}^{-r}$. Then Γ_{g-1}^2 is isomorphic to the stabilizer in ${\rm Mod}(\Sigma_g)[\varphi]$ of the curve c. Note that up to isomorphism, the group Γ_{g-1}^2 does not depend on the vertex $c \in \mathcal{CG}_1$. Namely, by Proposition 3.4, the stabilizers in ${\rm Mod}(\Sigma_g)[\varphi]$ of nonseparating simple closed curves c with $\varphi(c) = \pm 1$ are all conjugate and hence isomorphic.

Observe that the group Γ_{g-1}^2 is an infinite cyclic central extension of a finite index subgroup of the mapping class group of a surface $\Sigma_{g-1,2}$ of genus $g-1$ with two punctures. Thus it makes sense to talk about its action on isotopy classes of essential curves on the surfaces $\Sigma_{g-1,2}$ and Σ_{g-1}^2. The map $\Sigma_{g-1}^2 \to \Sigma_{g-1,2}$ which contracts each boundary component to a puncture defines a bijection on such isotopy classes.

We have

Proposition 4.1. *Let φ be a $\mathbb{Z}/r\mathbb{Z}$-spin structure on a closed surface Σ_g of genus $g \geq 3$. There is a commutative diagram*

$$\begin{tikzcd} \Gamma_{g-1}^2 \arrow[r, "\iota_1"] \arrow[dr, "\iota_2"'] & \Gamma_{g-1}^2 *_A \Gamma_{g-1}^2 \rtimes \mathbb{Z}/2\mathbb{Z} \arrow[d, "\rho"] \\ & {\rm Mod}(\Sigma_g)[\varphi] \end{tikzcd} \tag{9}$$

where the homomorphisms ι_1, ι_2 are inclusions, and the homomorphism ρ is surjective. The subgroup A of Γ_{g-1}^2 is the stabilizer in Γ_{g-1}^2 of a nonseparating simple *closed curve d on Σ^2_{g-1} with $\varphi(d)=\pm 1$. The group $\mathbb{Z}/2\mathbb{Z}$ acts on $\Gamma^2_{g-1} *_A \Gamma^2_{g-1}$ by exchanging the two factors, and it acts as an automorphism on A.*

Proof. Fix a pair of nonseparating simple closed disjoint curves c,d on Σ_g with $\varphi(c)=\varphi(d)=\pm 1$ which are connected by an edge in the graph $\mathcal{CG}_1^+$, that is, so that $\Sigma_g-(c\cup d)$ is connected. Let $\Gamma_c,\Gamma_d\subset \mathrm{Mod}(\Sigma_g)[\varphi]$ be the stabilizers of c,d in the spin mapping class group of Σ_g. By Corollary 3.6, these groups are naturally isomorphic to the group Γ^2_{g-1}, and they intersect in the index two subgroup $A=\Gamma_c\cap\Gamma_d$ of the stabilizer of $c\cup d$ in $\mathrm{Mod}(\Sigma_g)[\varphi]$ consisting of all elements which preserve both c,d individually. The full stabilizer of $c\cup d$ in $\mathrm{Mod}(\Sigma_g)[\varphi]$ is a $\mathbb{Z}/2\mathbb{Z}$ extension of $\Gamma_c\cap\Gamma_d$, where the generator Φ of $\mathbb{Z}/2\mathbb{Z}$ acts as involution on $A=\Gamma_c\cap\Gamma_d$ exchanging c and d. This involution extends to an involution of $\Gamma_c *_A\Gamma_d$ exchanging the two subgroups Γ_c,Γ_d.

By the universal property of free amalgamated products, there is a homomorphism

$$\rho:\Gamma=\Gamma_c *_A \Gamma_d \rtimes \mathbb{Z}/2\mathbb{Z}\to \mathrm{Mod}(\Sigma_g)[\varphi].$$

All we need to show is that ρ is surjective, that is, that $\rho(\Gamma)=\mathrm{Mod}(\Sigma_g)[\varphi]$.

As $\mathrm{Mod}(\Sigma_g)[\varphi]$ acts transitively on the vertices of the graph $\mathcal{CG}_1^+$, for this it suffices to show that its subgroup $\rho(\Gamma)$ acts transitively on the vertices of $\mathcal{CG}_1^+$ as well. Namely, by construction, the stabilizer of the vertex c of $\mathcal{CG}_1^+$ in $\rho(\Gamma)$ coincides with its stabilizer in $\mathrm{Mod}(\Sigma_g)[\varphi]$. As $\rho(\Gamma)$ is a subgroup of $\mathrm{Mod}(\Sigma_g)[\varphi]$, this then implies equality.

To show transitivity of the action of $\rho(\Gamma)$ on the vertices of $\mathcal{CG}_1^+$ let $v\in\mathcal{CG}_1^+$ be any vertex. By Proposition 2.13, and Corollary 2.17, the graph $\mathcal{CG}_1^+$ is connected and hence we can find an edge path $(c_i)\subset\mathcal{CG}_1^+$ connecting $c_0=c$ to $c_k=v$. We also may assume that $c_1=d$.

By the assumption $\varphi(d)=\pm 1$, for one of the two boundary components d_1,d_2 of Σ_g-d, equipped with the orientation as a boundary component of Σ_g-d, say the component d_1, we have $\varphi(d_1)=-1$. Thus we can attach a disk D to c_1 and obtain a surface Σ' with spin structure φ' which induces the spin structure φ on Σ_g-d. As a consequence, the restriction of φ to Σ_g-d fulfills the hypothesis in Corollary 3.6. As $c=c_0$ and c_2 are nonseparating simple closed curves in Σ_g-d with $\varphi(c)=\varphi(c_2)=\pm 1$, Corollary 3.6 shows that there exists an element $\Psi_1\in\Gamma_d\subset\rho(\Gamma)$ such that $\Psi_1(c)=c_2$. Then the stabilizer of c_2 in $\mathrm{Mod}(\Sigma_g)[\varphi]$ equals $\Psi_1\Gamma_c\Psi_1^{-1}$ and hence it is contained in $\rho(\Gamma)$. Thus we can apply Corollary 3.6 to $\Psi_1\Gamma_c\Psi_1^{-1}$ and find an element $\Psi_2\in\rho(\Gamma)$ which maps c_1 to c_3. Proceeding inductively and using the fact that Γ_c is conjugate to Γ_d in $\rho(\Gamma)$ by the generator of the subgroup $\mathbb{Z}/2\mathbb{Z}$, this completes the proof of the proposition. $\square$

Recall from the introduction the definition of an admissible curve system on a closed surface Σ_g of genus $g\geq 2$. The mapping class group of Σ_g naturally acts on the family of all admissible curve systems on Σ_g. Recall also that the curve diagram of an admissible curve system is a finite tree.

Since the curve diagram of an admissible curve system $\mathcal{C}$ is connected, each curve $c \in \mathcal{C}$ intersects at least one other simple closed curve on Σ_g transversely in a single point and hence it is non-separating.

We need some technical information on admissible curve systems. To this end let $\mathcal{C}$ be any admissible curve system on an oriented surface S. We require that the boundary of S is empty, but we allow for the moment that S has punctures. For admissibility, we require that all complementary components of $\mathcal{C}$ are either topological disks or once punctured topological disks.

The union $\cup\{c \mid c \in \mathcal{C}\}$ is an embedded graph G in S whose vertices are the intersection points between the curves from $\mathcal{C}$. Choose a basepoint $x \in G$ which is contained in the interior of an edge of G. This edge is contained in a simple closed curve $c_0 \in \mathcal{C}$ which defines a distinguished vertex v_0 in the curve diagram of $\mathcal{C}$.

Construct inductively a family L of homotopy classes of loops in G based at x as follows. Let L_0 be the family consisting of the two based loop which go once around the simple closed curve $c_0 \in \mathcal{C}$ containing x in either direction. Assume by induction that for some $k \geq 1$ we defined a system of based loops L_{k-1}. Let $\{c_{k_1}, \dots, c_{k_s}\} \subset \mathcal{C}$ be the curves in $\mathcal{C}$ whose distance in the curve diagram to the distinguished vertex v_0 equals k. Define

$$L_k = \{T_{c_{k_u}}^{\pm 1} d \mid u \leq s, d \in L_{k-1}\}$$

and let $L = L_b$ where $b \geq 1$ is the maximal distance of a vertex in the curve diagram of $\mathcal{C}$ to the distinguished vertex v_0. We have

Lemma 4.2. *The loops from the system L generate the fundamental group $\pi_1(S, x)$ of S.*

Proof. Let T be the curve diagram of $\mathcal{C}$ and let $\zeta : [0, p] \to T$ be a path without backtracking in T which connects the base vertex v_0 to a vertex v. Then $\cup_j \zeta(j)$ is an embedded chain in S, that is, a string of simple closed curves whose curve diagram is a line segment. The basepoint x is contained in the curve $\zeta(0)$.

We show by induction on $\ell \geq 1$ that the curve system L_ℓ contains a system of based loops supported on the subchain $\cup_{j \leq \ell} \zeta(j)$ which generate the fundamental group of $\cup_{j \leq \ell} \zeta(j)$, viewed as an embedded graph in S. Note that this fundamental group is just the free group in ℓ generators. The case $\ell = 0$ is clear since in this case the chain consists of a single simple closed curve, so assume that the claim holds true for some $\ell - 1 \geq 0$.

For $j \leq p - 1$ let $y_j = \zeta(j) \cap \zeta(j+1)$. By construction, the loop system $L_{\ell-1}$ contains a loop α supported in $\cup_{j \leq \ell-1} \zeta(j)$ which passes precisely once through $y_{\ell-1}$. Then α is a concatenation of two paths. The first path α^1 connects x to $y_{\ell-1}$, and the second path α^2 connects $y_{\ell-1}$ back to x. The based loop which is the concatenation of α^1, the loop $\zeta(\ell)$, based at $y_{\ell-1}$, and the arc α^2 is the image of α under the Dehn twist about $\zeta(\ell)$ and hence it is contained in the loop system L_ℓ. By induction assumption, the loops from $L_{\ell-1}$ which are supported in the subgraph $\cup_{j \leq \ell-1} \zeta(j)$ generate the fundamental group of $\cup_{j \leq \ell-1} \zeta(j)$. Since the graph $\cup_{j \leq \ell} \zeta(j)$ is obtained from $\cup_{j \leq \ell-1} \zeta(j)$ by attaching the loop $\zeta(\ell)$, we conclude that the fundamental group of $\cup_{j\leq\ell}\zeta(j)$ is generated by those loops from the system L_ℓ which are supported in $\cup_{j\leq\ell}\zeta(j)$. This completes the induction step.

As a consequence, the loops from the loop system L generate the fundamental group of the graph G. Thus they also generate the fundamental group of the subsurface of S filled by G which is just a thickening of G. But by definition of an admissible system, the inclusion $G \to S$ induces a surjection on fundamental groups. The lemma follows. $\square$

As a consequence we obtain

Lemma 4.3. *Let $\mathcal{C}$ be an admissible curve system on a surface S, possibly with punctures. Let p be a puncture of S and assume that there are two curves $c_1, c_2 \in \mathcal{C}$ which bound a once punctured annulus, with p as puncture. Then the subgroup Γ of $\mathrm{Mod}(S)$ generated by the Dehn twists about the curves from the curve system $\mathcal{C}$ contains the kernel of the homomorphism $\mathrm{Mod}(S) \to \mathrm{Mod}(\Sigma)$ where Σ is obtained from S by forgetting p.*

Proof. Let c be the common projection of the curves c_1, c_2 to Σ. We assume that c passes through p. The point pushing map about the curve c is just the concatentation $T_{c_1}T_{c_2}^{-1}$, and this element is contained in Γ by assumption.

On the other hand, if u is the image of c under a Dehn twist about any of the curves $d \in \mathcal{C} - \{c_1, c_2\}$, then the point pushing map about u is just the concatentation $T_{T_d c_1}T_{T_d c_2}^{-1} = T_d T_{c_1} T_{c_2}^{-1} T_d^{-1}$ and hence this element also is contained in Γ. Thus the lemma follows from Lemma 4.2. $\square$

For a closed surface Σ_g of genus $g \geq 2$ consider the system $\mathcal{S}_g$ of $3g-2$ simple closed curve on Σ_g shown in Figure 4. Note that for $g = 2$, the system $\mathcal{S}_g$ is just a

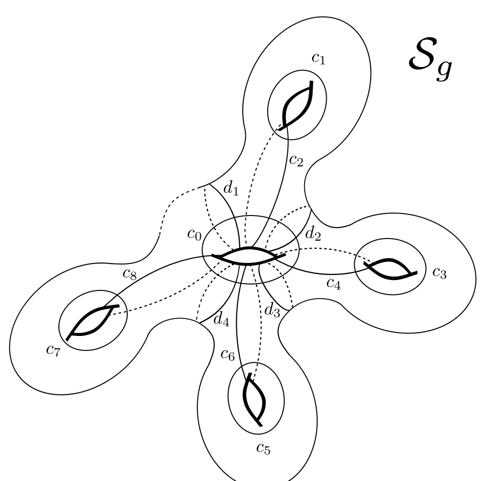

Figure 4

chain of 4 curves which are invariant under the hyperelliptic involution.

Lemma 4.4. *The Dehn twists about the curves from the system $\mathcal{S}_g$ preserve an odd $\mathbb{Z}/2\mathbb{Z}$-spin structure on Σ_g.*

Proof. There exists a cyclic subgroup G of the diffeomorphism group of Σ_g of order $g-1$ which preserves $\mathcal{S}_g$ and acts freely on Σ_g as a group of rotations about the center curve c_0. The group G cyclically permutes the complementary components of $\mathcal{S}_g$.

As a consequence, the curve system $\mathcal{S}_g$ descends to a curve system on a closed surface Σ_2 of genus 2. The curve diagram of this system is just a line segment of length 4 and hence the Dehn twists about these curves preserve an odd spin structure on Σ_2. This spin structure lifts to a spin structure on Σ_g which is invariant under the Dehn twist about the curves from $\mathcal{S}_g$. The parity of this spin structure is odd, as can also easily be checked explicitly using the formula (2). This is what we wanted to show. □

We use Lemma 4.3 and Proposition 4.1 to show

Proposition 4.5. *Let φ be an odd $\mathbb{Z}/2\mathbb{Z}$-spin structure on a surface Σ_g of genus $g \geq 2$. Then the group $\mathrm{Mod}(\Sigma_g)[\varphi]$ is generated by the Dehn twists about the curves from the curve system $\mathcal{S}_g$.*

Proof. Lemma 4.4 shows that the subgroup Γ of $\mathrm{Mod}(\Sigma_g)$ generated by the Dehn twists about the curves from the curve system $\mathcal{S}_g$ is a subgroup of $\mathrm{Mod}(\Sigma_g)[\varphi]$. We have to show that it coincides with $\mathrm{Mod}(\Sigma_g)[\varphi]$.

We proceed by induction on the genus, beginning with genus 2. We observed above that in this case, the system $\mathcal{S}_g$ is just a chain of 4 simple closed curves invariant under the hyperelliptic involution. The Dehn twists about these curves are well known to generate the stabilizer $\mathrm{Mod}(\Sigma_2)[\varphi]$ of an odd spin structure φ on Σ_2 (see [FM12]).

Thus let us assume that the proposition is known for some $g-1 \geq 2$. Consider the curve system $\mathcal{S}_g$ on a surface of genus g. Using the labeling from Figure 4, let a_1 be the simple closed curve on Σ_g which intersects the curve c_1 in a single point and is disjoint from any other curve from $\mathcal{S}_g$. We know that $\varphi(a_1) = 1$. We aim at showing that $\Gamma \cap \mathrm{Stab}(a_1) = \mathrm{Mod}(\Sigma_g)[\varphi] \cap \mathrm{Stab}(a_1)$.

To this end cut Σ_g open along a_1. The resulting surface is a surface Σ_{g-1}^2 of genus $g-1$ with two boundary components. Replace these two boundary components by punctures and let $\Sigma_{g-1,2}$ be the resulting twice punctured surface. As before, the spin structure φ decends to a spin structure, again denoted by φ, on the surface Σ_{g-1} obtained by closing the punctures, and to a spin structure on $\Sigma_{g-1,2}$. The curve system $\mathcal{S}_g$ descends to the curve system $\mathcal{S}_{g-1}$ on Σ_{g-1}.

By induction hypothesis, the Dehn twists about the curves from the curve system $\mathcal{S}_{g-1}$ generate the spin mapping class group $\mathrm{Mod}(\Sigma_{g-1})[\varphi]$. On the other hand, we can apply Lemma 4.3 to each of the two punctures of $\Sigma_{g-1,2}$ as each of these two punctures is contained in a once punctured annulus bounded by two curves from the restriction of $\mathcal{S}_g$ to $\Sigma_{g-1,2}$. We conclude that the point pushing maps about these punctures are contained in the group $\Gamma \cap \mathrm{Stab}(a_1)$. As a consequence, the group $\Gamma \cap \mathrm{Stab}(a_1)$ surjects onto the index two subgroup of $\mathrm{Mod}(\Sigma_{g-1,2})$ which fixes each of the two punctures.

We have to show that there also is an element of $\Gamma \cap \mathrm{Stab}(a_1)$ which exchanges the two boundary components of $\Sigma_g - a_1$. For this it suffices to find an element of Γ which fixes the curves c_1, c_2 and exchanges d_1, d_2.

If $g = 3$ then consider the hyperelliptic involution of the surface Σ_2 obtained by cutting Σ_3 open along the simple closed curve a_1 and removing the punctures. This element can be represented as an explicit word in the Dehn twists about the curve c_2, c_0, c_4, c_3 (or, rather, their projection to Σ_2). The mapping class ψ, viewed as an element of the mapping class group of Σ_3, preserves the curves c_i and exchanges d_1 and d_2.

For $g \geq 4$ the same argument can be used. Namely, the element ψ still acts as an involution on Σ_g which preserves the curves c_1, c_2 and exchanges d_1 and d_2. However this involution does not preserve the curve system $\mathcal{S}_g$.

To summarize, we showed so far that Γ surjects onto $\mathrm{Stab}(a_1)[\varphi]/\mathbb{Z}$. Thus to show that $\Gamma \cap \mathrm{Stab}(a_1) = \mathrm{Mod}(\Sigma_g)[\varphi] \cap \mathrm{Stab}(a_1)$ it suffices to show that Γ contains the square $T_{a_1}^2$ of the Dehn twist about a_1. For an application of Proposition 4.1, we have to show furthermore that Γ contains an involution Ψ which exchanges the curve a_1 with a curve disjoint from a_1. We show first that Γ contains an involution which maps a_1 to a_2.

To this end consider again first the case $g = 3$. The curve system $\mathcal{S}_3$ contains a curves system $\mathcal{E}_6 \subset \mathcal{S}_3$ obtained from $\mathcal{S}_3$ by deleting the curve d_2. This is the curve system shown in Figure 2 in the introduction. By Theorem 1.4 of [Ma00], there exists an explicit word $c(E_6)$ in the Dehn twists about the curves from the system $\mathcal{E}_6$, the image of the so-called *Garside element* of the Artin group of type E_6, which acts as a reflection on the curve diagram of $\mathcal{E}_6$ exchanging the curves c_1 and c_3. Then this reflection exchanges a_1 and a_2 and hence it has the desired properties.

As before, this reasoning extends to any $g \geq 4$. Namely, the element $c(E_6)$, viewed as an element of the mapping class group of Σ_g, still acts as an involution on Σ_g which exchanges a_1 and a_2 and preserves the subsurface of Σ_g filled by the curves $c_1, c_2, c_0, c_4, c_3, d_2$.

For an application of Proposition 4.1, we are left with showing that the square of the Dehn twist about a_1 is contained in Γ. By the above discussion, we know that $\Gamma \cap \mathrm{Stab}(a_1)$ surjects onto $\mathrm{Mod}(\Sigma_{g-1,2})[\varphi]$. In particular, Γ contains $T_{a_2}^2$, viewed as an element of $\mathrm{Stab}(a_1) \subset \mathrm{Mod}(\Sigma_g)$. Since a_1 is the image of a_2 under an involution contained in Γ, it follows that $T_{a_1}^2 \in \Gamma$.

To summarize, we showed that $\Gamma \cap \mathrm{Stab}(a_1) = \mathrm{Mod}(\Sigma_g)[\varphi] \cap \mathrm{Stab}(a_1)$, furthermore Γ contains an involution Ψ which exchanges a_1 and a_2. Proposition 4.1 now shows that $\Gamma = \mathrm{Mod}(\Sigma_g)[\varphi]$. This completes the proof of the Proposition. □

We use Proposition 4.5 as the base case for the proof of Theorem 3 from the introduction. The curve system $\mathcal{C}_g$ is shown in Figure 1 in the introduction. Note that we have $\mathcal{C}_3 = \mathcal{S}_3$.

Theorem 4.6. *Let φ be an odd $\mathbb{Z}/2\mathbb{Z}$-spin structure on a surface Σ_g of genus $g \geq 3$. Then the group $\mathrm{Mod}(\Sigma_g)[\varphi]$ is generated by the Dehn twists about the curves from the curve system $\mathcal{C}_g$.*

Proof. The curve system $\mathcal{C}_g$ is obtained from the curve system $\mathcal{S}_g$ by deleting the curves $d_3, \dots, d_{g-1}$. Let Γ be the subgroup of $\mathrm{Mod}(\Sigma_g)[\varphi]$ generated by the Dehn twists about the curves from the curve system $\mathcal{C}_g$. By Proposition 4.5, it suffices to show that the Dehn twists T_{d_i} for $i = 3, \dots, g-1$ are contained in Γ. Moreover, as $\mathcal{D}_3 = \mathcal{C}_3$, we may assume that $g \geq 4$.

Let a_i be the simple closed curve which intersects c_{2i-1} in a single point and does not intersect any other curve from $\mathcal{S}_g$. We claim that $T_{a_1}^2 \in \Gamma$.

To show the claim consider the subsurface Σ_2^1 of Σ_g which is filled by the curves $a_1, c_0, c_1, c_2, d_1, d_2$. This is a surface of genus 2 with connected boundary. The curves d_1, d_2 bound a one-holed annulus containing the boundary circle C of Σ_2^1.

By homological coherence (Proposition 3.8 of [Sa19]), we have $\varphi(C) = 1$. Thus the spin structure φ descends to a spin structure on Σ_2^1, on the surface $\Sigma_{2,1}$ obtained from Σ_2^1 by replacing the boundary component by a puncture and on the surface Σ_2 obtained from $\Sigma_{2,1}$ by forgetting the puncture, again denoted by φ. The curves of the curve system $\mathcal{C}_g$ which are contained in Σ_2^1 define a curve system $\mathcal{F}$ on Σ_2^1 which descends to a curve system on Σ_2. The curve diagram of this system is just a line segment of length 4. As a consequence, the Dehn twists about the curves from $\mathcal{F}$ project onto $\mathrm{Mod}(\Sigma_2)[\varphi]$.

On the other hand, $\mathcal{F}$ also contains two simple closed curves which enclose the boundary component of $\Sigma_{2,1}$. It now follows from Lemma 4.3 that the subgroup of $\mathrm{Mod}(\Sigma_{2,1})$ generated by the Dehn twists about the curves from $\mathcal{F}$ equals $\mathrm{Mod}(\Sigma_{2,1})$. In particular, this group contains $T_{a_1}^2$ and therefore $T_{a_1}^2 \in \Gamma$.

We claim next that $T_{a_2}^2 \in \Gamma$. To this end consider the subsurface Σ_3^1 of Σ_g which is filled by the system of curves $\mathcal{G} = \{c_1, c_2, c_0, c_4, c_3, d_1, d_2, d_3\}$. This is a surface of genus 3 with connected boundary. The curves d_1, d_3 bound a one-holed annulus containing the boundary circle A of Σ_3^1.

The subsurface Σ_3^1 of Σ_g contains the curves $c_1, c_2, c_0, c_4, c_3, d_2$ whose curve diagram is the Dynkin diagram of type E_6 (see Figure 2 in the introduction). There is an involution of Σ_3^1 which fixes the curves c_0, d_2 and exchanges c_2, c_4 and a_1, a_2. By Theorem 1.4 of [Ma00], this involution is contained in the subgroup of the mapping class group of Σ_3^1 which is generated by the Dehn twists about the curves $c_1, c_2, c_0, c_4, c_5, d_2$. As a consequence, there is an element of Γ which exchanges a_1 and a_2. This implies that $T_{a_2}^2 \in \Gamma$.

By the chain relation for Dehn twists of surfaces (see p.108 of [FM12]), we have $(T_{a_2}^2 T_{c_3} T_{c_4})^3 = T_{d_2} T_{d_3}$. Since $T_{d_2} \in \Gamma$, we conclude that $T_{d_3} \in \Gamma$.

Now repeat this argument, replacing the curves c_j by c_{j+2} and the curve a_i by a_{i+1} where the first step discussed above is the case $i = 1$. In finitely many such steps we find that indeed $T_{d_i} \in \Gamma$ for all i. This is what we wanted to show. $\square$

5. Structure of the spin mapping class group of even parity

The goal of this section is to prove the second part of Theorem 3. Our strategy is to reduce this result to the first part of Theorem 3 by a change of parity construction.

Consider for the moment an arbitrary $\mathbb{Z}/r\mathbb{Z}$-spin structures φ on a compact surface S of genus $g \geq 4$. In the appendix we introduce a graph $\mathcal{CG}_2^+$ whose vertices are ordered pairs (a, b) of nonseparating simple closed curves which intersect in a single point and hence they fill a one-holed torus $T(a, b)$. Furthermore, it is required that $\varphi(a) = 2$ and $\varphi(b) = 0$. The spin structure on S restricts to a spin structure $\hat{\varphi}$ on $\Sigma(a, b) = S - T(a, b)$.

By homological coherence (Proposition 3.5 of [Sa19]), if we orient the boundary circle c of $\Sigma(a, b)$ as the oriented boundary of $\Sigma(a, b)$ then we have $\varphi(c) = 1$. Thus if $r = 2$ then φ descends to a spin structure $\hat{\varphi}$ on the surface Σ obtained from $\Sigma(a, b)$ by capping off the boundary. This spin structure $\hat{\varphi}$ has a parity, either even or odd.

Lemma 5.1. *A $\mathbb{Z}/2\mathbb{Z}$-spin structure φ on S induces a $\mathbb{Z}/2\mathbb{Z}$-spin structure $\hat{\varphi}$ on the surface Σ whose parity is opposite to the parity of φ.*

Proof. Choose a geometric symplectic basis $a_1, b_1, \ldots, a_{g-1}, b_{g-1}$ for Σ. This basis then lifts to a curve system on the surface $\Sigma(a, b) = S - T(a, b)$. Using the inclusion $\Sigma(a, b) \to S$, this basis can be extended to a geometric symplectic basis of S by adding a, b. As $\varphi(a) = \varphi(b) = 0$, the parity of φ is opposite to the parity of $\hat{\varphi}$. $\square$

The next observation is an analog of Proposition 3.4. Note that we only require $g \geq 3$ here.

Proposition 5.2. *Let φ be a $\mathbb{Z}/r\mathbb{Z}$-spin structure on a compact surface S of genus $g \geq 3$ with empty or connected boundary and where $r \leq 2g$. Then for any two vertices c, d of the graph $\mathcal{CG}_2^+$ there exists a mapping class $\zeta \in$ Mod$(S)[\varphi]$ with $\zeta(c) = d$. In particular, the action of* Mod$(S)[\varphi]$ *is transitive on the vertices of the graph $\mathcal{CG}_2^+$.*

Proof. The proof is very similar to the proof of Proposition 3.4.

Recall that a geometric symplectic basis for S is a set $\{a_1, b_1, \ldots, a_{2g}, b_{2g}\}$ of simple closed curves on S such that a_i, b_i intersect in a single point, and $a_i \cup b_i$ is disjoint from $a_j \cup b_j$ for $j \neq i$.

Let us consider a vertex (a, b) of $\mathcal{CG}_2^+$. It consists of a pair of simple closed curves which intersect in a single point and such that $\varphi(a) = 2$ and $\varphi(b) = 0$. Our goal is to construct for any such a pair a geometric symplectic basis $\mathcal{B}(a, b) =$

$\{a, b, \alpha_2, \beta_2, \dots, \alpha_g, \beta_g\}$ such that $\varphi(\alpha_i) = 0$ for all $i \geq 2$, $\varphi(\beta_j) = 0$ for $2 \leq j \leq g-1$ and $\varphi(\beta_g) = 0$ or 1 as predicted by the parity of the spin structure φ. By the discussion in the proof of Proposition 3.4, this suffices for the proof of the proposition.

For the construction of a geometric symplectic basis $\mathcal{B}(a,b)$ with the requested property we proceed as in the proof of Proposition 3.4. Namely, extend a, b in an arbitrary way to a geometric symplectic basis $\{a, b, \alpha_2, \beta_2, \dots, \alpha_g, \beta_g\}$ and modify this basis in such a way that $\varphi(\alpha_i) = 0$ for all $i \geq 2$. Let $\mathcal{B}$ be the resulting geometric symplectic basis. As in the proof of Proposistion 3.4 our task is now to modify the curves β_i $(i \geq 2)$ while keeping a, b, α_i fixed in such a way that φ assume the prescribed values on the modified curves.

This is done exactly as in the proof of Proposition 3.4, following the argument of Salter [Sa19]. For $1 \leq i \leq g$ let δ_i be the boundary curve of the torus T_i which is a small tubular neighborhood of $\alpha_i \cup \beta_i$, equipped with the orientation as an oriented boundary circle of $S - T_i$ $(i \geq 1)$. By homological coherence (Proposition 3.8 of [Sa19]), we have $\varphi(\delta_i) = 1$ for all i.

Thus if ϵ is an embedded arc in S connecting β_2 to δ_3 and disjoint from α_2 and all δ_j for $j \neq 2$, then $\varphi(\beta_2 +_\epsilon \delta_3) = \varphi(\beta_2) + 2$. Moreover, $\beta_2 +_\epsilon \delta_3$ is disjoint from δ_j for all $j \neq 2$.

Repeat this construction with an arc connecting $\beta_2 +_\epsilon \delta_3$ to δ_4 disjoint from all δ_j for $j \neq 2, 4$. As there are $g-1$ of the curves δ_j $(j \neq 2)$ and as $r \leq 2g$, with this construction we can find a simple closed curve β_2' intersecting α_2 in a single point and disjoint from the curves δ_j for $j \neq 2$ so that $\varphi(\beta_1') \in \{0, 1\}$.

Let δ_2' be the boundary of a tubular neighborhood of $\alpha_2 \cup \beta_2'$. Then δ_2' is disjoint from all the curves δ_j for $j \neq 2$. As in the proof of Proposition 3.4, repeat this procedure with the curve β_3 and the curves $\delta_1, \delta_2', \dots, \delta_g$. In finitely many steps we can change the geometric symplectic basis $\mathcal{B}$ to a geometric symplectic basis $\mathcal{B}' = \{\alpha_1, \beta_1, \alpha_2, \beta_2', \dots, \alpha_g, \beta_g'\}$ which fulfills $\varphi(\beta_j') = 0$ or 1 for all $2 \leq j \leq g$. The remaining step is identical to the argument in the proof of Proposition 4.9 of [Sa19] and will be omitted. □

Consider again a $\mathbb{Z}/r\mathbb{Z}$-spin structure φ on a closed surface Σ_g of genus $g \geq 3$. Let c be a separating simple closed curve on Σ_g which is the boundary of a small neighborhood of a vertex $(a, b) \in \mathcal{CG}_2^+$. Then c decomposes Σ_g into a one holed torus Σ_1^1 and a surface Σ_{g-1}^1 of genus $g-1$ with connected boundary. The spin structure restricts to a spin structure on Σ_1^1. If r is even then this spin structure has a parity, and this parity is odd.

Since c is separating, the group $\mathrm{Mod}(\Sigma_{g-1}^1)[\varphi] \times \mathrm{Mod}(\Sigma_1^1)[\varphi]$ contains a rank two free abelian central subgroup generated by the left Dehn twists T_{c_1}, T_{c_2} about the boundary circles c_1, c_2 of $\Sigma_{g-1}^1, \Sigma_1^1$. Define

$$\Gamma_{g-1,2}^2 = \mathrm{Mod}(\Sigma_{g-1}^1)[\varphi] \times \mathrm{Mod}(\Sigma_1^1)[\varphi]/\mathbb{Z}$$

where the infinite cyclic subgroup $\mathbb{Z}$ is generated by $T_{c_1} T_{c_2}^{-1}$. Then $\Gamma_{g-1,2}^2$ is isomorphic to the stabilizer in $\mathrm{Mod}(\Sigma_g)[\varphi]$ of the curve c. Note that up to isomorphism, the group $\Gamma^2_{g-1,2}$ does not depend on c since by Proposition 5.2, the stabilizers in $\mathrm{Mod}(\Sigma_g)[\varphi]$ of vertices of $\mathcal{CG}_2^+$ are all conjugate and hence isomorphic.

Observe that the group $\Gamma^2_{g-1,2}$ is an infinite cyclic central extension of the product of a finite index subgroup of the mapping class group of a surface $\Sigma_{g-1,1}$ of genus $g-1$ with one punctures and a once punctured torus $\Sigma_{1,1}$. Thus it makes sense to talk about its action on isotopy classes of essential curves on the surfaces $\Sigma_{g-1,1}$ and $\Sigma_{1,1}$. The map $\Sigma^1_{g-1}\times\Sigma^1_1\to\Sigma_{g-1,1}\times\Sigma_{1,1}$ which contracts each boundary component to a puncture defines a bijection on such isotopy classes.

The following observation is the analog of Proposition 4.1.

Proposition 5.3. *Let φ be a $\mathbb{Z}/r\mathbb{Z}$-spin structure on a closed surface Σ_g of genus $g\geq 4$. There is a commutative diagram*

$$\begin{tikzcd} \Gamma^2_{g-1,2} \arrow[r, "\iota_1"] \arrow[dr, "\iota_2"'] & \Gamma^2_{g-1,2} *_A \Gamma^2_{g-1,2} \rtimes \mathbb{Z}/2\mathbb{Z} \arrow[d, "\rho"] \\ & \mathrm{Mod}(\Sigma_g)[\varphi] \end{tikzcd} \tag{10}$$

where the homomorphisms ι_1,ι_2 are inclusions, and the homomorphism ρ is surjective. The subgroup A of $\Gamma^2_{g-1,2}$ is the stabilizer in $\Gamma^2_{g-1,2}$ of a separating simple closed curve d on Σ^2_{g-1} which is defined by a vertex of the graph $\mathcal{CG}_2^+$. The curve d decomposes Σ^1_{g-1} into a one-holed torus and a surface of genus $g-2$ with two boundary components. The group $\mathbb{Z}/2\mathbb{Z}$ acts on $\Gamma^2_{g-1,2} *_A \Gamma^2_{g-1,2}$ by exchanging the two factors, and it acts as an automorphism on A.

Proof. Fix a pair of vertices of the graph $\mathcal{CG}_2^+$ which are connected by an edge. These two vertices then determine a pair of disjoint separating simple closed curves c,d on Σ_g which cut from Σ_g a one-holed torus each. These tori are disjoint. Let $\Gamma_c,\Gamma_d\subset\mathrm{Mod}(\Sigma_g)[\varphi]$ be the stabilizers of c,d in the spin mapping class group of Σ_g. By Corollary 3.6, these groups are naturally isomorphic to the group $\Gamma^2_{g-1,2}$, and they intersect in the index two subgroup $A=\Gamma_c\cap\Gamma_d$ of the stabilizer of $c\cup d$ in $\mathrm{Mod}(\Sigma_g)[\varphi]$ consisting of all elements which preserve both c,d individually. The full stabilizer of $c\cup d$ in $\mathrm{Mod}(\Sigma_g)[\varphi]$ is a $\mathbb{Z}/2\mathbb{Z}$ extension of $\Gamma_c\cap\Gamma_d$, where the generator Φ of $\mathbb{Z}/2\mathbb{Z}$ acts as involution on $A=\Gamma_c\cap\Gamma_d$ exchanging c and d. This involution extends to an involution of $\Gamma_c *_A \Gamma_d$ exchanging the two subgroups Γ_c,Γ_d.

By the universal property of free amalgamated products, there is a homomorphism

$$\rho:\Gamma=\Gamma_c *_A \Gamma_d\rtimes\mathbb{Z}/2\mathbb{Z}\to\mathrm{Mod}(\Sigma_g)[\varphi].$$

All we need to show is that ρ is surjective, that is, that $\rho(\Gamma)=\mathrm{Mod}(\Sigma_g)[\varphi]$.

As $\mathrm{Mod}(\Sigma_g)[\varphi]$ acts transitively on the vertices of the graph $\mathcal{CG}_2^+$, for this it suffices to show that its subgroup $\rho(\Gamma)$ acts transitively on the vertices of $\mathcal{CG}_2^+$ as well. Namely, by construction, the stabilizer of the vertex c of $\mathcal{CG}_1^+$ in $\rho(\Gamma)$ coincides with its stabilizer in $\mathrm{Mod}(\Sigma_g)[\varphi]$. As $\rho(\Gamma)$ is a subgroup of $\mathrm{Mod}(\Sigma_g)[\varphi]$, this then implies equality.

To show transitivity of the action of $\rho(\Gamma)$ on the vertices of $\mathcal{CG}_2^+$ let $v \in \mathcal{CG}_2^+$ be any vertex. By Proposition 2.13, and Corollary 2.17, the graph $\mathcal{CG}_2^+$ is connected and hence we can find an edge path $(c_i) \subset \mathcal{CG}_2^+$ connecting $c_0 = c$ to $c_k = v$. We also may assume that $c_1 = d$.

By Proposition 5.2, there exists an element $\Psi_1 \in \Gamma_d \subset \rho(\Gamma)$ such that $\Psi_1(c_0) = c_2$. Then the stabilizer of c_2 in $\mathrm{Mod}(\Sigma_g)[\varphi]$ equals $\Psi_1\Gamma_c\Psi_1^{-1}$ and hence it is contained in $\rho(\Gamma)$. Thus we can apply Corollary 3.6 to $\Psi_1\Gamma_c\Psi_1^{-1}$ and find an element $\Psi_2 \in \rho(\Gamma)$ which maps c_1 to c_3. Proceeding inductively and using the fact that Γ_c is conjugate to Γ_d in $\rho(\Gamma)$ by the generator of the subgroup $\mathbb{Z}/2\mathbb{Z}$, this completes the proof of the proposition. $\square$

For a surface S of genus $g \geq 3$ consider the following system $\mathcal{U}_g$ of $3g-2$ simple closed curve on S. Note that for $g = 3$, the system $\mathcal{S}_g$ is just a chain of 7 curves

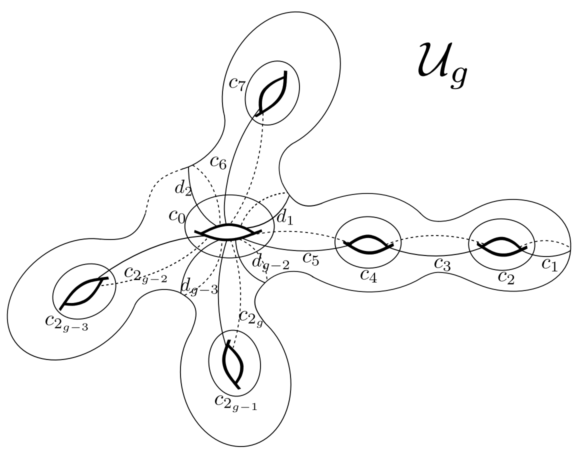

Figure 5

which are invariant under a hyperelliptic involution. It follows from the discussion in Section 4 that the Dehn twists about these curves preserve an even $\mathbb{Z}/2\mathbb{Z}$-spin structure on Σ_g.

We use Lemma 4.3 and Proposition 4.1 to show

Proposition 5.4. *Let φ be an even $\mathbb{Z}/2\mathbb{Z}$-spin structure on a surface Σ_g of genus $g \geq 4$. Then the group $\mathrm{Mod}(\Sigma_g)[\varphi]$ is generated by the Dehn twists about the curves from the curve system $\mathcal{U}_g$.*

Proof. We observed above that the subgroup Γ of $\mathrm{Mod}(\Sigma_g)$ generated by the Dehn twist about the curves from the curve system $\mathcal{U}_g$ is a subgroup of $\mathrm{Mod}(\Sigma_g)[\varphi]$. We have to show that it coincides with $\mathrm{Mod}(\Sigma_g)[\varphi]$.

To this end we proceed by induction on the genus, beginning with genus 4. Let a be the separating simple closed curve which intersects c_3 in two points and is disjoint from the remaining curve from the system $\mathcal{U}_4$. It decomposes $\mathcal{U}_4$ into a one holed torus Σ_1^1 containing the curves c_1, c_2, and a surface Σ_3^1 of genus 3 with connected boundary which contains the curve system $\mathcal{S}_3$. As we are looking at a $\mathbb{Z}/2\mathbb{Z}$-spin structure we know that the pair (c_1, c_2) and hence the curve a defines a vertex in $\mathcal{CG}_2^+$. The spin structure φ induces a spin structure on Σ_3^1 and Σ_1^1, again denoted by φ. It also induces a spin structure on the closed surface Σ_3 of genus 3 obtained from Σ_3^1 by capping off the boundary, again denoted by φ.

It is well known that the mapping class group of one holed tori is generated by a pair of Dehn twists about simple closed curves which intersect in a single point. Thus we have $\mathrm{Mod}(\Sigma_1^1) \subset \Gamma \cap \mathrm{Stab}(a)$.

On the other hand, by Proposition 4.5, the Dehn twists about the curves from the system $\mathcal{S}_3$ generate the spin mapping class group $\mathrm{Mod}(\Sigma_3)[\varphi]$ of Σ_3. Thus the projection of Γ to $\mathrm{Mod}(\Sigma_3)[\varphi]$ is surjective.

To apply Proposition 5.3 we have to show that the point pushing group of $\mathrm{Mod}(\Sigma_{3,1})[\varphi]$ is contained in the projection of $\Gamma \cap \mathrm{Stab}(a)$. Following Lemma 4.3, to this end it suffices to show that the point pushing map along a single nonseparating simple closed curve is contained in Γ.

Consider the curves c_0, c_7, c_6, d_1, c_5 which defines a curve system of type D_5 on the surface $\Sigma_{3,1}$ whose curve diagram is the Dynkin diagram D_5. By Theorem 1.5 of [Ma00], there exists an explicit word in the Dehn twists about these curves which defines the product $T_{a_4}^3 T_{a_4'}$ where a_4 is simple closed curve in Σ_3^1 which intersects c_4 in a single point and is disjoint from all other curves and where a_4' is the simple closed curve which bounds together with a_4 a once punctured annulus in Σ_3^1.

On the other hand, we know by the chain relation [FM12] that $T'_{a_4} T_{a_4} = (T_{c_1} T_{c_2} T_{c_3})^4$. Since $T_{a_4}, T_{a_4'}$ commute we deduce that $(T'_{a_4})^{-2} T_{a_4}^{-2} T_{a_4}^3 T_{a_4'} = T_{a_4} T_{a_4'}^{-1} \in \Gamma$.

If α is an embedded simple closed curve containing the marked point, then the point pushing map along α is just the product $T_{a_1}^{-1} T_{a_2}$ where a_1, a_2 are the boundary circles of a once punctured annulus containing α.

By Proposition 5.3 we are left with finding an element $\Psi \in \Gamma$ which maps a to a curve disjoint from a. However, the curve system $\mathcal{U}_4$ contains a subsystem consisting of the curves c_i $(i = 0, \dots, 7)$. The Dehn twists about these curves are well known to generate the *hyperelliptic mapping class group*, that is, the subgroup of the mapping class group which commutes with a hyperelliptic involution. The hyperelliptic mapping class group contains an element ψ which maps a to a disjoint curve, e.g. the boundary of a small neighborhood of $c_0 \cup c_5$. The proposition for $g = 4$ now follows from Proposition 5.3.

By induction, let us now assume by that the proposition is known for some $g - 1 \geq 4$. Consider the curve system $\mathcal{U}_g$ on a surface of genus g. Using the labeling from Figure 5, let a_7 be the simple closed curve on Σ_g which intersects the curve c_7 in a single point and is disjoint from any other curve from $\mathcal{U}_g$. We know that $\varphi(a_7) = 1$. We aim at showing that $\Gamma \cap \mathrm{Stab}(a_1) = \mathrm{Mod}(\Sigma_g)[\varphi] \cap \mathrm{Stab}(a_7)$.

To this end cut Σ_g open along a_1. The resulting surface is a surface Σ_{g-1}^2 of genus $g - 1$ with two boundary components. Replace these two boundary components by punctures and let $\Sigma_{g-1,2}$ be the resulting twice punctured surface. As before, the spin structure φ descends to a spin structure, again denoted by φ, on the surface Σ_{g-1} obtained by closing the punctures, and to a spin structure on $\Sigma_{g-1,2}$. The curve system $\mathcal{U}_g$ descends to the curve system $\mathcal{U}_{g-1}$ on Σ_{g-1}.

By induction hypothesis, the Dehn twists about the curves from the curve system $\mathcal{U}_{g-1}$ generate the spin mapping class group $\mathrm{Mod}(\Sigma_{g-1})[\varphi]$. On the other hand, we can apply Lemma 4.3 to each of the two punctures of $\Sigma_{g-1,2}$ as each of these two punctures is contained in a once punctured annulus bounded by two curves from the restriction of $\mathcal{U}_g$ to $\Sigma_{g-1,2}$. We conclude that the point pushing maps about these punctures are contained in the group $\Gamma \cap \mathrm{Stab}(a_1)$. As a consequence, the group $\Gamma \cap \mathrm{Stab}(a_1)$ surjects onto $\mathrm{Mod}(\Sigma_{g-1,1})$.

To summarize, we showed so far that Γ surjects onto $\mathrm{Stab}(a_1)[\varphi]/\mathbb{Z}$ where $\mathbb{Z}$ is the intersection of $\mathrm{Mod}(\Sigma_g)[\varphi|$ with the infinite cyclic group of Dehn twists about a_1. Thus to show that $\Gamma \cap \mathrm{Stab}(a) = \mathrm{Mod}(\Sigma_g)[\varphi] \cap \mathrm{Stab}(a)$ it suffices to show that Γ contains the square $T_{a_1}^2$ of the Dehn twist about a_1 as well as an involution Ψ which exchanges a_1 with a simple closed curve disjoint from a_1.

Consider again first the case $g = 4$. The curve system $\mathcal{U}_3$ contains a curve system $\mathcal{E}_6 \subset \mathcal{U}_3$ obtained from $\mathcal{U}_4$ by deleting the curves d_2, c_1, c_2. By Theorem 1.4 of [Ma00], there exists an explicit word $c(E_6)$ in the Dehn twists about the curves from the system $\mathcal{E}_6$, the image of the so-called *Garside element* of the Artin group of type E_6, which acts as a reflection on the curve diagram of $\mathcal{E}_6$ exchanging the curves c_1 and c_3. Then this reflection exchanges a_1 and a_2 and hence it has the desired properties.

As before, this reasoning extends to any $g \geq 5$. Namely, the element $c(E_6)$, viewed as an element of the mapping class group of Σ_g, still acts as an involution on Σ_g which exchanges a_1 and a_2 and preserves the subsurface of Σ_g filled by the curves $c_1, c_2, c_0, c_4, c_3, d_2$.

For an application of Proposition 4.1, we are left with showing that the square of the Dehn twist about a_1 is contained in Γ. By the above discussion, we know that $\Gamma \cap \mathrm{Stab}(a_1)$ surjects onto $\mathrm{Mod}(\Sigma_{g-1,2})[\varphi]$. In particular, Γ contains $T_{a_2}^2$, viewed as an element of $\mathrm{Stab}(a_1) \subset \mathrm{Mod}(\Sigma_g)$. Since a_1 is the image of a_2 under an involution contained in Γ, it follows that $T_{a_1}^2 \in \Gamma$.

To summarize, we showed that $\Gamma \cap \mathrm{Stab}(a_1) = \mathrm{Mod}(\Sigma_g)[\varphi] \cap \mathrm{Stab}(a_1)$, furthermore Γ contains an involution Ψ which exchanges a_1 and a_2. Proposition 4.1 now shows that $\Gamma = \mathrm{Mod}(\Sigma_g)[\varphi]$. This completes the proof of the Proposition. $\square$

We use Proposition 4.5 as the base case for the proof of Theorem 3 from the introduction. The curve system $\mathcal{V}_g$ is defined as in the Theorem 3. Note that we have $\mathcal{V}_3 = \mathcal{U}_3$.

Theorem 5.5. *Let φ be an even $\mathbb{Z}/2\mathbb{Z}$-spin structure on a surface Σ_g of genus $g \geq 4$. Then the group $\mathrm{Mod}(\Sigma_g)[\varphi]$ is generated by the Dehn twists about the curves from the curve system $\mathcal{V}_g$.*

Proof. The curve system $\mathcal{V}_g$ is obtained from the curve system $\mathcal{U}_g$ by deleting the curves $d_2, \dots, d_{g-2}$.

Let Γ be the subgroup of $\mathrm{Mod}(\Sigma_g)[\varphi]$ generated by the Dehn twists about the curves from the curve system $\mathcal{V}_g$. By Proposition 4.5, it suffices to show that the Dehn twists T_{d_i} for $i = 2, \dots, g-2$ are contained in Γ.

To see that $T_{d_2} \in \Gamma$ note that d_2 is the image of d_1 under the hyperelliptic involution of the surface of genus 3 with connected boundary filled by the curves $d_1, d_2, c_0, c_1, c_2, c_3, c_4$.

Consider the surface S filled by $c_1, \dots, c_6, d_1, d_2, d_3$. This is a surface of genus 4 with connected boundary. The system $\mathcal{V}_g$ intersects S in a curve system of type $\mathcal{U}_4$. Thus by what we proved so far, the group of Dehn twists generated by this system surjects onto the spin mapping class group of the surface obtained by capping off the boundary. In particular, if we denote by e_1, e_3 the nonseparating simple closed curves which intersect c_4 in a single point, do not intersect any other curve and form a bounding pair, then $T_{e_1} T_{e_2}^{-1} \in \Gamma$.

Now by Theorem 1.4 of [Ma00], the stabilizer in Γ of the surface of genus 3 with two boundary components obtained by removing the one-holed torus filled by c_1, c_2 contains a half-twist which exchanges the two boundary components of the surface. As the full point pushing group about one of the components is contained in Γ by the above and Lemma 4.3, the same holds true for the full pointpushing group about the other.

In other words, we have $T_{d_2} T_{d_3}^{-1} \in \Gamma$. As $T_{d_2} \in \Gamma$, we conclude that the same holds true for T_{d_3}. To generate the remaining twists about the curves d_i we argue as in the proof of Theorem 5.5. $\square$

6. Generating the $\mathbb{Z}/4\mathbb{Z}$-spin mapping class group in genus 3

The goal of is to prove Theorem 4 from the introduction. Our strategy is similar to the strategy used in Section 4. We first introduce one more graph of curves which will be useful to this end.

Consider an odd $\mathbb{Z}/2\mathbb{Z}$-spin structure φ on a surface Σ_3 of genus 3. A separating simple closed curve a on Σ_3 decomposes Σ_3 into a one-holed torus T and a surface Σ_2^1 of genus 2 with connected boundary. By homological coherence (Proposition 3.15 of [Sa19]), we have $\varphi(c) = 1$. In particular, φ induces a spin structure on the surface Σ_2^1 which has a parity. Define a to be *odd* if this parity is odd. Note that a vertex of the graph $\mathcal{CG}_2^+$ defines a separating simple closed curve which is *even*, that is, it is not odd.

Let $\mathcal{S}$ be the graph whose vertices are odd separating simple closed curves on (Σ_3, φ) and where two such curves are connected by an edge if they are disjoint. Let Φ be a $\mathbb{Z}/4\mathbb{Z}$-spin structure on Σ_3 whose $\mathbb{Z}/2\mathbb{Z}$-reduction equals φ. The stabilizer $\mathrm{Mod}(\Sigma_3)[\varphi]$ and its subgroup $\mathrm{Mod}(\Sigma_3)[\Phi]$ act on $\mathcal{S}$ as a group of simplicial automorphisms. The following observation is similar to Proposition 3.4. It uses some special properties of $\mathbb{Z}/4\mathbb{Z}$-spin structures.

Lemma 6.1. (1) *The group* $\mathrm{Mod}(\Sigma_3)[\Phi]$ *acts transitively on the vertices of* $\mathcal{S}$.
(2) *Let* $a \in \mathcal{S}$ *be any vertex. Then the stabilizer of* a *in* $\mathrm{Mod}(\Sigma_3)[\Phi]$ *acts transitively on the edges of* $\mathcal{S}$ *issuing from* a.

Proof. A vertex a of $\mathcal{S}$ decomposes Σ_3 into a one-holed torus T and a surface $\Sigma_3 - T$ of genus 2 with connected boundary and odd spin structure. Since the parity of the spin structure of φ on Σ_3 is odd, the torus T contains a simple closed curve c with $\varphi(c) = 1$ and hence $\Phi(c) = \pm 1$. Via perhaps changing the orientation for c we may assume that $\Phi(c) = 1$, furthermore there is a simple closed curve d in T which intersects c in a single point and satisfies $\Phi(d) = 0$.

By homological coherence (Proposition 3.15 of [Sa19]), if we orient a as the oriented boundary of the surface $V = \Sigma_3 - T$ then we have $\Phi(a) = 1$. Since the spin structure induced on V is odd, a geometric symplectic basis for V consists of simple closed curves a_1, b_1, a_2, b_2 with $\varphi(a_1) = 1$ and hence $\Phi(a_1) = \pm 1$ (up to ordering). A tubular neighborhood T' of $a_1 \cup b_1$ is an embedded bordered torus in V. Choose an orientation for a_1 so that $\Phi(a_1) = 1$. After perhaps replacing b_1 by its image under a multiple of a Dehn twist about a_1 we may assume that $\Phi(b_1) = 0$.

Consider the pair of curves a_2, b_2. Since the spin structure on V is odd, we have $\varphi(a_2) = \varphi(b_2) = 0$ and hence $\Phi(a_2), \Phi(b_2) \in \{0, 2\}$. Our goal is to modify a_2, b_2 so that Φ vanishes on the modified curves. Thus assume without loss of generality that $\Phi(a_2) = 2$. Connect a_2 to the boundary curve a of V by an embedded arc ϵ which is disjoint from T' and b_2, and connect b_2 to the boundary δ of T' by an embedded arc η which is disjoint from ϵ and a_2. Since $\Phi(a) = 1$ for the orientation as a boundary curve of V, we obtain that $\Phi(a_2 +_\epsilon a) = 0$, furthermore this curve is disjoint from T' and intersects b_2 in a single point. Replace a_2 by $a_2 +_\epsilon a$. Similarly, if $\Phi(b_2) = 2$ then we replace b_2 by $b_2 +_\eta \delta$. This process yields a geometric symplectic basis for Σ_3 consisting of simple closed curves disjoint from a.

Given any other odd separating curve a' on Σ_3 we can find in the same way a geometric symplectic basis for Σ_3 consisting of curves disjoint from a'. Then there is a mapping class which maps a to a' and identifies the geometric symplectic bases in such a way that the values of Φ on these curves match up. By the result of Humphries and Johnson [HJ89], this implies that this mapping class is contained in $\mathrm{Mod}(\Sigma_3)[\Phi]$. In other words, there is an element of $\mathrm{Mod}(\Sigma_3)[\Phi]$ which maps a to a'. This shows the first part of the lemma.

The proof of the second part of the lemma is completely analogous but easier and will be omitted. □

Lemma 6.2. *The graph* $\mathcal{S}$ *is connected.*

Proof. Consider the curve system $\mathcal{C}_3$ on the surface Σ_3. There is an odd separating simple closed curve a which intersects the curve c_2 in two points and is disjoint from the remaining curves from the system $\mathcal{C}_3$. Using the Putman trick, Theorem 5.5 and the first part of Lemma 6.1, all we need to show is that the curve a can be connected to $T_{c_2}(a)$ by an edge path in $\mathcal{S}$.

However, the curve a' which intersects the curve c_4 in two points and is disjoint from the remaining curves from the system $\mathcal{C}_3$ is separating and odd, and it is disjoint from both a and $T_{c_2}(a)$. Thus $a, a', T_{c_2}(a)$ is an edge path in $\mathcal{S}$ which connects a to $T_{c_2}(a)$. $\square$

Using the notations from Figure 2 from the introduction, let d be the separating simple closed curve on Σ_3 which intersects the curve c_2 in two points and is disjoint from the remaining curves from the system $\mathcal{E}_6$. We show

Lemma 6.3. *The subgroup Γ of $\mathrm{Mod}(\Sigma_3)$ which is generated by the Dehn twists about the curves from the curve system $\mathcal{E}_6$ equals the stabilizer $\mathrm{Mod}(\Sigma_3)[\Phi]$ of an odd $\mathbb{Z}/4\mathbb{Z}$-spin structure Φ on Σ_3 if and only if its intersection with $\mathrm{Stab}(d)$ coincides with $\mathrm{Stab}(d) \cap \mathrm{Mod}(\Sigma_3)[\Phi]$.*

Proof. Since Γ is a subgroup of $\mathrm{Mod}(\Sigma_3)[\Phi]$, the condition is clearly necessary, so we have to show sufficiency. Thus assume that $\Gamma \cap \mathrm{Stab}(d) = \mathrm{Mod}(\Sigma_3)[\Phi] \cap \mathrm{Stab}(d)$.

By Lemma 6.2, the graph $\mathcal{S}$ whose vertices are the odd separating curves and where two such curves are connected by an edge if they are disjoint is connected. Moreover, by Lemma 6.1, the group $\mathrm{Mod}(\Sigma_3)[\Phi]$ acts transitively on the directed edges of $\mathcal{S}$ as a group of simplicial automorphisms. The curve d is odd and hence a vertex of $\mathcal{S}$.

By Theorem 1.4 of [Ma00], the group Γ contains an involution which induces a reflection in the curve diagram of the curve system $\mathcal{E}_6$ at the edge connecting the vertices c_0 and c_3. It maps the simple closed curve d to the separating simple closed curve d' which intersects c_4 in two points and is disjoint from all other curves from the system. Since d is odd, the same is true for d'.

We use this as follows. Let e be any vertex of $\mathcal{S}$ and let $d = d_0, d_1, d_2, \dots, d_m = e$ be an edge path in $\mathcal{S}$ which connects d to e. We may assume that $d_1 = d'$. Since there exists an element of Γ which maps d to d', the stabilizer of d' in Γ is conjugate to the stabilizer of d and hence by our assumption, it coincides with the stabilizer of d' in $\mathrm{Mod}(\Sigma_3)[\Phi]$. In particular, by the second part of Lemma 6.1, there exists an element of Γ which fixes d' and maps d_0 to d_2. Arguing inductively as in the proof of Proposition 4.1, we conclude that Γ acts transitively on the odd separating curves in Σ_3. As Γ is a subgroup of $\mathrm{Mod}(\Sigma_3)[\Phi]$ and furthermore the stabilizer of a vertex in Γ coincides with its stabilizer in $\mathrm{Mod}(\Sigma_3)[\Phi]$, it has to coincide with $\mathrm{Mod}(\Sigma_3)[\Phi]$. The lemma follows. $\square$

Our next goal is to show that the group Γ fulfills the assumption in Lemma 6.3. To this end let a_1, a_5 be the non-separating simple closed curves on Σ_3 which intersect c_1, c_5 in a single point and are disjoint from the remaining curves from the system $\mathcal{E}_6$. We have $\Phi(a_j) = \pm 1$, in particular, by Lemma 3.13 of [Sa19], the intersection of $\mathrm{Mod}(\Sigma_3)[\Phi]$ with the infinite cyclic group of Dehn twists about the curve a_j is generated by $T_{a_j}^4$.

Lemma 6.4. *For $j = 1, 5$, the group Γ contains $T_{a_j}^4$.*

Proof. Consider the subsystem $\mathcal{D}_5^j$ $(j=1,5)$ obtained from the curve system $\mathcal{E}_6$ by removing the curve c_j. By Theorem 1.3 (d) of [Ma00], the mapping class $T_{a_j}^4$ can be represented as an explicit word in the Dehn twists about the curves from this curve system. Thus we have $T_{a_j}^4\in\Gamma$. □

Lemma 6.5. *The stabilizer in* Γ *of the curve* d *coincides with the stabilizer of* d *in* $\mathrm{Mod}(\Sigma_3)[\Phi]$.

Proof. Let T be the one-holed torus component of Σ_3-d. The stabilizer $\mathrm{Stab}(d)[\Phi]$ of d in $\mathrm{Mod}(\Sigma_3)[\Phi]$ is the quotient of the product of two subgroups G_1,G_2 by an infinite cyclic central subgroup. The group G_1 is the group of all isotopy classes of diffeomorphisms of Σ_3 which fix the bordered surface $S=\Sigma_3-T$ pointwise and preserve the spin structure Φ. It is isomorphic to the subgroup of the mapping class group of a one-holed torus which preserves the spin structure Φ. The group G_2 is the group of all isotopy classes of diffeomorphisms of Σ_3 which fix T pointwise and preserve the spin structure Φ. The center of $\mathrm{Stab}(d)[\Phi]$ is generated by a Dehn twist T_d about d.

Consider the curve system $\mathcal{A}_4\subset\mathcal{E}_6$ which consists of the curves c_0,c_3,c_4,c_5. It is contained in the subsurface $\Sigma_2^1=\Sigma_3-T$ of Σ_3 of genus 2 which is bounded by d. The Dehn twists about these curves generate a subgroup $\mathcal{A}(\mathcal{A}_4)$ of $\Gamma\cap G_2$ which is isomorphic to the braid group in five strands (see [FM12] or [Ma00] for the last statement). By Theorem 1.4 of [Ma00], the Dehn twist T_d can be represented as an explicit word in the Dehn twists about the curves from the curve system $\mathcal{A}(A_4)$. In particular, we have $T_d\in\Gamma$.

Let as before a_1 be the simple closed curve which intersects c_1 in a single point and is disjoint from the remaining curves from the system $\mathcal{E}_6$. We observed before that $T_{a_1}^\ell\in\mathrm{Mod}(\Sigma_3)[\Phi]$ if and only if ℓ is a multiple of 4. Using the fact that the mapping class group of a bordered torus is the group $SL(2,\mathbb{Z})$, it follows that the group G_1 is generated by the elements $T_{a_1}^4,T_{c_1},T_d$. By Lemma 6.4 and the above discussion, these elements are contained in Γ and therefore $G_1\subset\Gamma$,

Let $\Sigma_{2,1}$ be the surface obtained from $\Sigma_2^1=\Sigma_3-T$ by replacing the boundary component by a puncture, and let Σ_2 be obtained from $\Sigma_{2,1}$ by forgetting the puncture. Let φ be the $\mathbb{Z}/2\mathbb{Z}$-reduction of the spin structure Φ. The spin structure φ induces an odd spin structure on $\Sigma_{2,1}$ and Σ_2, again denoted by φ. The subgroup $\mathcal{A}(\mathcal{A}_4)$ of $\Gamma\cap G_1$ surjects onto the spin mapping class group $\mathrm{Mod}(\Sigma_2)$ [FM12]. Consequently the restriction of the puncture forgetful homomorphism $G_2\to\mathrm{Mod}(\Sigma_2)[\varphi]$ to $\Gamma\cap G_2$ is surjective.

By homological coherence, if we orient d as the oriented boundary of the surface Σ_3-T, then we have $\Phi(d)=1$. Thus by Lemma 3.1, the intersection of the pointpushing group $\pi_1(\Sigma_2)$ with the stabilizer of Φ in $\mathrm{Mod}(\Sigma_{2,1})$ is the preimage of the sublattice Λ of $H_1(\Sigma_2,\mathbb{Z})$ generated by squares of primitive homology classes of oriented simple closed curves under the natural homomorphism $\pi_1(\Sigma_2)\to H_1(\Sigma_2,\mathbb{Z})$. Or, equivalently, it equals the kernel of the surjective homomorphism $\pi_1(\Sigma_2)\to H_1(\Sigma_2,\mathbb{Z}/2\mathbb{Z})$. In particular, $\mathrm{Mod}(\Sigma_{2,1}[\Phi]\cap\pi_1(\Sigma_2)$ contains the commutator subgroup of $\pi_1(\Sigma_2)$.

We claim first that the square of the point pushing map along a simple closed curve α with $\Phi(\alpha) = \pm 1$ is contained in Γ. To this end note that as $\Phi(\alpha) = \pm 1$ if and only if we have $\varphi(\alpha) = 1$ where φ is the $\mathbb{Z}/2\mathbb{Z}$-reduction of Φ, the group $\mathrm{Mod}(\Sigma_2)[\varphi]$ and hence Γ acts transitively on these curves. Thus by equivariance, it suffices to verify this claim for a single such curve.

Consider again the simple closed curve $a_5 \subset \Sigma_{2,1}$ with $\Phi(a_5) = \pm 1$ which intersects c_5 in a single point and is disjoint from all other curves from the curve system $\mathcal{E}_6$. Let a' be the simple closed curve which bounds with a_5 and the boundary circle C of $\Sigma_{2,1}$ a pair of pants, that is, a_5 and a' bound a holed annulus in Σ_2^1. By the chain relation in the mapping class group (see [FM12]), we have

$$(T_{c_0} T_{c_3} T_{c_4})^6 = T_{a_5} T_{a'} = \zeta \in \Gamma.$$

On the other hand, Lemma 6.4 shows that $T_{a_5}^4 \in \Gamma$. As T_{a_5} and $T_{a'}$ commute, we have $T_{a_5}^{-4}\zeta^2 = T_{a_5}^{-2} T_{a'}^2 \in \Gamma$, and this is just the square of the point pushing transformation (via replacing the boundary circle C by a puncture) along a_5. Thus the square of the point pushing transformation about a_5 is contained in Γ, which is what we wanted to show.

Now the sublattice $\Lambda \subset H_1(\Sigma_2, \mathbb{Z})$ is additively generated by elements of the form $2b$ where b is an oriented simple closed curve with $\varphi(b) = 1$ and hence we conclude that $\Gamma \cap \pi_1(\Sigma_2)$ surjects onto Λ.

We are left with showing that the point pushing map along any element in the commutator subgroup of $\pi_1(\Sigma_2)$ is contained in Γ. As the commutator subgroup of $\pi_1(\Sigma_2)$ is generated by separating simple closed curves, and as $\mathrm{Mod}(\Sigma_2)[\varphi]$ acts transitively on the separating simple closed curves, it suffices to show the following. There exists a separating simple closed curve e in Σ_2 such that the point pushing map along e in Σ_2 is contained in Γ.

Now by Theorem 1.4 of [Ma00], the Dehn twist about the separating simple closed curve d' which intersects c_4 in two points and is disjoint from the remaining curves from $\mathcal{E}_6$ is contained in Γ. This separating curve is odd in the sense described above. The second separating curve which bounds together with the boundary circle C and d' a pair of pants is the boundary of a tubular neighborhood of $c_0 \cup c_1$. As the Dehn twists about c_0, c_1 are contained in Γ, the same holds true for the Dehn twist about that curve. We conclude that the point pushing maps about separating simple closed curves is contained in Γ.

To summarize, the quotient of $\Gamma \cap G_2$ by the infinite cyclic group of Dehn twists about the boundary curve d contains a generating set for the point pushing subgroup of $G_2/\mathbb{Z}$ and hence it contains this point pushing subgroup. As $\Gamma \cap G_2$ surjects onto the quotient $G_2/\mathbb{Z}$ by the point pushing subgroup, we conclude that Γ surjects onto $G_2/\mathbb{Z}$. But Γ contains the infinite cyclic center of G_2 and hence $\Gamma \cap G_2 = G_2$. Together with the beginning of this proof, we conclude that indeed, $\Gamma \cap \mathrm{Stab}(d) = \mathrm{Mod}(\Sigma_3)[\Phi] \cap \mathrm{Stab}(d)$. $\square$

Remark 6.6. Theorem 4 classifies connected components of the preimage in the Teichmüller space of abelian differentials of the odd component of the stratum of abelian differentials on a surface Σ_3 of genus 3 with a single zero. Those components correspond precisely to odd $\mathbb{Z}/4\mathbb{Z}$-spin structures on Σ_3.

Remark 6.7. The results in this article give a general recipe for finding generators of spin mapping class groups. This recipe is motivated by the recent work on compactifications of strata of abelian differentials in [BCGGM18] and the goal to obtain a topological interpretation of this compactification.

Appendix A. Additional graphs of nonseparating curves with fixed spin value

In this appendix we complement the main result in Section 2 by studying connectedness of some additional geometrically defined graphs related to spin structures. The proofs do not use new ideas. We use the assumptions and notations from Section 2.

We begin with adding more constraints to the graph $\mathcal{CG}_1^+$. Define a graph $\mathcal{CG}_1^{++}$ as follows. Vertices of $\mathcal{CG}_1^{++}$ are ordered pairs (c,d) of nonseparating simple closed curves c,d such that $\varphi(c)=\pm 1, \varphi(d)=0$ and that c,d intersect in a single point. Then $c\cup d$ fills a one-holed torus $T(c,d)\subset S$. Two such pairs $(c,d),(c',d')$ are connected by an edge if and only if the tori $T(c,d)$ and $T(c',d')$ are disjoint. We use Lemma 2.16 to show

Lemma A.1. *For $g\geq 4$ the graph $\mathcal{CG}_1^{++}$ is connected.*

Proof. Let $(a,b),(c,d)$ be two vertices in the graph $\mathcal{CG}_1^{++}$. Then a,c are vertices in the graph $\mathcal{CG}_1^+$. Connect a to c by an edge path (a_i) in $\mathcal{CG}_1^+$; this is possible by Lemma 2.16. Our goal is to construct inductively a path $(c_j,d_j)\subset \mathcal{CG}_1^{++}$ connecting (a,b) to (c,d) which passes through vertices (c_{j_i},d_{j_i}) with $c_{j_i}=a_i$.

To this end observe that if the curve b is disjoint from c_1, then we can find a curve $\hat{d}_1$ which intersects c_1 in a single point and is disjoint from (a,b). In particular, $a\cup b$ is disjoint from $(c_1,\hat{d}_1)$.

We can not expect in general that $\varphi(d_1)=0$. However, as before, there exists some $k\in\mathbb{Z}$ such that $\varphi(T_{c_1}^k(\hat{d}_1))=0$. Define $d_1=T_{c_1}^k(\hat{d}_1)$ and note that d_1 is disjoint from $a\cup b$ and intersects c_1 in a single point. Thus the pair (c_1,d_1) is a vertex in $\mathcal{CG}_1^{++}$ which is connected to (a,b) by an edge.

Let us now assume that b is not disjoint from c_1. Since b intersects a in a single point, it determines a vertex in the nonseparating arc graph $\mathcal{A}(A_1,A_2)$ of $S-a$; here A_1,A_2 are the two boundary components of $S-a$ which glue back to a. Denote this arc by b_0.

Connect b_0 to an arc b' disjoint from c_1 by an edge path (b_i) in $\mathcal{A}(A_1,A_2)$. Cut S open along $a\cup b$. The result is a surface of genus $g-1\geq 3$ with connected boundary, and $S-(b\cup b_1)$ is a surface of genus $g-2\geq 2$ with two boundary components. Recall to this end that by definition of $\mathcal{A}(A_1,A_2)$, this surface is connected.

A surface of genus at least 2 contains a non-separating curve u with $\varphi(u)=1$, and in fact it contains a pair $(u,v)\in\mathcal{CG}_1^{++}$. In other words, there exists a vertex of $\mathcal{CG}_1^{++}$ which is disjoint from a,b,b_1. Connect (a,b) to (a,b_1) by the edge path $(a,b)\to(u,v)\to(a,b_1)$ and proceed by induction. $\square$

Define a graph $\mathcal{D}$ as follows. Vertices are ordered pairs (x, y) where x is a vertex in $\mathcal{CG}_1^{++}$ and where y is a disjoint simple closed non-separating cuve with $\varphi(y) = 0$. Two such pairs are connected by an edge if they can be realized disjointly. The following observation is a straighforward application of Lemma A.1 and the tools used so far. Its proof will be omitted.

Lemma A.2. *For $g \geq 4$ the graph $\mathcal{D}$ is connected.*

Define now a graph $\mathcal{CG}_2^+$ as follows. Vertices are pairs (x, y) where x is a non-separating simple closed curve on S with $\varphi(x) = 2$ and where y is a simple closed curve with $\varphi(y) = 0$ intersecting x in a single point. Two such vertices are connected by an edge of length one if and only if they can be realized disjointly.

We use this the above constructions to show

Proposition A.3. *For $g \geq 4$ the graph $\mathcal{CG}_2^+$ is connected.*

Proof. Given a pair of disjoint simple closed curves (c, d) with $\varphi(c) = \pm 1$ and $\varphi(d) = 0$, cut S open along c, d and denote the boundary components of the resulting surface by C_1, C_2, D_1, D_2. For one of the two choices of C_1, C_2, say for C_1, the curve $c +_\epsilon d$ defined by any embedded arc ϵ connecting C_1 to either of D_1, D_2 satisfies $\varphi(c +_\epsilon d) = \pm 2$.

As a consequence, to any vertex $(c, d) \in \mathcal{D}$ we can associate in a non-deterministic way a vertex in $\mathcal{CG}_2^+$ by replacing the simple closed curve a with $\varphi(a) = \pm 1$ in the pair which defines a vertex of $\mathcal{CG}_1^{++}$ to the simple closed curve component of the pair which defines a vertex in $\mathcal{D}$.

Adjacent vertices may not give rise to disjoint curves, but this issue can be resolved using a path in the nonseparating arc graph. Using the fact that the surface obtained by removing from S a torus and cutting the resulting surface open along a nonseparating simple closed curve has genus at least 2, we find for any two such arcs a disjoint curve e with $\varphi(e) = \pm 1$. Connect b to this curve with a disjoint arc. □

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