

# CANNON-THURSTON MAPS AND MEASURES

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ABSTRACT. Consider a closed hyperbolic 3-manifold  $M$  which fibers over the circle, with fiber a closed surface  $S$ . We show that a measure class on the ideal boundary  $S^2 = \partial\mathbb{H}^3$  of hyperbolic 3-space which is invariant and totally ergodic under the action of the fundamental group of  $M$  is singular with respect to the push-forward by a Cannon-Thurston map of a stationary measure on  $S^1$  of a random walk with finite first moment on the fundamental group of  $S$ .

## 1. INTRODUCTION

The fundamental group  $\Gamma$  of a closed hyperbolic 3-manifold which fibers over the circle contains a normal subgroup  $\Gamma_0$  which is isomorphic to a surface group. The subgroup  $\Gamma_0$  is finitely generated but exponentially distorted in  $\Gamma$ . In other words, the relation between the geometry of  $\Gamma_0$  and of the ambient group  $\Gamma$  is very complicated.

However, the groups  $\Gamma_0$  and  $\Gamma$  are dynamically related in the following way. The group  $\Gamma_0$  admits an embedding as a lattice into  $\mathrm{PSL}(2, \mathbb{R})$ , and via such an embedding, it acts on the boundary  $S^1 = \partial\mathbb{H}^2$  of the hyperbolic plane  $\mathbb{H}^2$ . Similarly, the group  $\Gamma$  comes with a conjugacy class of an embedding into  $\mathrm{PSL}(2, \mathbb{C})$  and hence it acts on the boundary  $S^2$  of hyperbolic 3-space  $\mathbb{H}^3$  as a group of Möbius transformations. By a result of Cannon and Thurston [CT07], these two actions are related: there exists a continuous, surjective, finite-to-one  $\Gamma_0$ -equivariant map  $F : S^1 \rightarrow S^2$  (see also [CL26] for more details on this construction).

The map  $F$  allows to compare the dynamics of the action of  $\Gamma_0$  on  $S^1$  and the dynamics of the action of  $\Gamma$  on  $S^2$ . An instance of such a comparison was recently established in [GMPU26] (for measures with exponential moment) and shortly later extended in [KZ25] to the following statement. In its formulation, we call a probability measure on a discrete group *admissible* if its support generates the group.

**Theorem 1** (Corollary 1.6 of [KZ25]). *Suppose that  $\Gamma$  is the fundamental group of a closed hyperbolic 3-manifold  $M$  that fibers over the circle, and  $\Gamma_0$  its fiber group. Then the push-forward to  $S^2$  by a Cannon-Thurston map of the stationary measure on  $S^1$  of a random walk on  $\Gamma_0$  defined by an admissible probability measure of finite first moment is singular to the stationary measure of a random walk on  $\Gamma$  defined by an admissible probability measure of finite first moment.*

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Here the moment condition is for the action of  $\Gamma_0$  and  $\Gamma$  on  $\mathbb{H}^3$ , and the result in fact applies to arbitrary infinite index subgroups of groups with sufficient hyperbolicity. The subtlety of moment conditions is illustrated by Proposition 6.1 of [KZ25] which shows that the measure class of a stationary measure on  $S^2$  for a symmetric random walk on  $\Gamma$  of finite first moment can in fact be realized by a stationary measure for a symmetric random walk on  $\Gamma_0$ , however the moment condition seems to be at most logarithmic.

Our goal is to give a short proof of the following purely dynamical version of Theorem 1. For its formulation, we consider a  $\Gamma_0$ -invariant ergodic measure class  $[\mu]$  on  $S^1$ , that is, a measure class for which the action of  $\Gamma_0$  is nonsingular. We say that  $[\mu]$  is *induced by a current* if there exists a *geodesic current*, that is, a  $\Gamma_0$ -invariant Radon measure  $\nu$  on  $(S^1 \times S^1) \setminus \Delta$ , which defines the measure class of the product  $[\mu] \times [\mu]$ .

Call a nonsingular ergodic action of a group  $G$  on a measure space *totally ergodic* if the action of every finite index subgroup of  $G$  is still ergodic. This is equivalent to stating that the action does not admit any finite factors. The following is our main result.

**Theorem 2.** *Under the assumption of Theorem 1, the push-forward by a Cannon-Thurston map of a  $\Gamma_0$ -invariant measure class on  $S^1$  induced by a current is singular to any  $\Gamma$ -invariant measure class on  $S^2$  for which the  $\Gamma$ -action is totally ergodic.*

Since a stationary measure on  $S^1$  and  $S^2$  of a symmetric random walk on  $\Gamma_0$  and  $\Gamma$  generated by a probability measure of finite first moment whose support generates the group is totally ergodic and induced by a current [K94], we obtain as an immediate corollary the statement of Theorem 1 with a slightly stronger moment condition for the measure on  $\Gamma_0$ . We do not use random walks in our approach, and Theorem 2 can not be derived from the results in [KZ25].

We also obtain the following application.

**Corollary 3.** *A  $\Gamma_0$ -invariant measure class  $[\mu]$  on  $S^1$  constructed from the Lebesgue measure on  $S^2$  and a Cannon-Thurston map has the following properties.*

- (1) *The measure class  $[\mu] \times [\mu]$  is ergodic under the diagonal action of  $\Gamma_0$ .*
- (2) *There is no geodesic current in the measure class of  $[\mu] \times [\mu]$ .*

In view of the results in Section 5 of [FMP26] and the results in [KZ25], we believe that Corollary 3 holds true for all stationary measures of random walks on  $\Gamma$  with finite first moment, but we do not carry out such an extension. The proof of Theorem 2 only uses dynamical tools, and it is completely different from the approach in [GMPU26] as well as from the tools used in [KZ25].

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## 2. THE CIRCLE ACTION OF $\text{Mod}(S_{g,1})$

Denote by  $S_{g,1}$  a surface of genus  $g \geq 2$  with one marked point, and by  $S_g$  the closed surface of genus  $g$  with no marked point. Let  $\text{Mod}(S_{g,1})$  and  $\text{Mod}(S_g)$  be the *mapping class group* of  $S_{g,1}$  and  $S_g$ , respectively, which can be identified with the group of automorphisms and outer automorphisms of  $\pi_1(S)$ . These groups fit into the *Birman exact sequence*

$$(1) \quad 0 \rightarrow \pi_1(S_g) \rightarrow \text{Mod}(S_{g,1}) \xrightarrow{\Pi} \text{Mod}(S_g) \rightarrow 0.$$

This sequence identifies  $\text{Mod}(S_{g,1})$  with the automorphism group of  $\pi_1(S_g)$ .

The *Teichmüller space*  $\mathcal{T}(S_{g,1})$  of  $S_{g,1}$  is the bundle over the Teichmüller space  $\mathcal{T}(S_g)$  of marked hyperbolic structures on  $S_g$  whose fiber over a marked hyperbolic surface  $X \in \mathcal{T}(S_g)$  is the universal covering of  $X$ , which is naturally identified with the hyperbolic plane  $\mathbb{H}^2$ . The mapping class group  $\text{Mod}(S_{g,1})$  acts properly discontinuously from the left on  $\mathcal{T}(S_{g,1})$  as a group of bundle diffeomorphisms. The subgroup  $\pi_1(S_g)$  preserves each fiber and acts cocompactly on it. Thus  $\pi_1(S_g) \backslash \mathcal{T}(S_{g,1})$  is a fiber bundle over  $\mathcal{T}(S_g)$  whose fiber over a point  $X$  is just the hyperbolic surface  $X$ . In other words, this quotient is the *universal curve* over  $\mathcal{T}(S_g)$ .

The unit tangent bundle  $T^1\mathbb{H}^2$  of  $\mathbb{H}^2$  is a  $\pi_1(S_g)$ -invariant circle bundle on the fiber of  $\mathcal{T}(S_{g,1})$  over  $X$ . The union of these circle bundles defines a smooth circle bundle over  $\mathcal{T}(S_{g,1})$  which is invariant under the action of  $\pi_1(S_g)$  and descends to the *vertical unit tangent bundle* of the fibers of the universal curve. By naturality of this construction, equivalently by the Birman exact sequence, the circle bundle over  $\mathcal{T}(S_{g,1})$  is in fact invariant under the whole mapping class group  $\text{Mod}(S_{g,1})$ , and it descends to the vertical tangent bundle of the universal curve over the moduli space of pointed hyperbolic surfaces of genus  $g$ .

It was discovered by Morita [Mo88] that this circle bundle is *flat*, that is, there exists a homomorphism  $\rho : \text{Mod}(S_{g,1}) \rightarrow \text{Homeo}(S^1)$  such that the circle bundle can be represented as  $\text{Mod}(S_{g,1}) \backslash \mathcal{T}(S_{g,1}) \times S^1$  where the action of  $\text{Mod}(S_{g,1})$  on  $S^1$  is via the homomorphism  $\rho$ . We next review the construction of the homomorphism  $\rho$  following Section 3 of [H21] which takes a more geometric viewpoint than Morita's article.

Let  $D \subset \mathbb{C}$  be the unit disk, viewed as the Poincaré model of the hyperbolic plane  $\mathbb{H}^2$ . A marked hyperbolic surface  $X \in \mathcal{T}(S_g)$  is defined by a discrete faithful homomorphism  $\psi_X : \pi_1(S_g) \rightarrow \text{PSL}(2, \mathbb{R})$ . Thus  $X$  is the quotient of  $D$  by the action of the image group  $\psi_X(\pi_1(S_g))$ .

Let  $x \in X$  be the projection of  $0 \in D$ . The group  $\text{Mod}(S_{g,1})$  can also be identified with the group of isotopy classes of diffeomorphisms of  $S_g$  fixing the marked point  $x \in X = S_g$ , where the isotopy is required to fix  $x$  as well. In this way, the fiber

subgroup  $\pi_1(S_g)$  of  $\text{Mod}(S_{g,1})$  of the Birman exact sequence (1), consisting of inner automorphisms of  $\pi_1(S_g)$ , is identified with the *point pushing group*.

Any diffeomorphism of  $S_g$  fixing  $x$  is a bi-Lipschitz map for the hyperbolic structure  $X$  on  $S_g$  and hence it lifts to a  $\pi_1(S_g, x)$ -equivariant bi-Lipschitz map  $\tilde{f} : \mathbb{H}^2 = D \rightarrow \mathbb{H}^2 = D$ . Since  $f$  fixes  $x$ , this lift can be chosen to fix 0. Such a bi-Lipschitz map maps geodesic rays starting at 0 to uniform quasi-geodesic rays, and any such quasi-geodesic ray is at uniformly bounded distance from a geodesic ray. As the circle  $S^1 = \partial D$  can naturally be identified with the space of geodesic rays starting at 0, this construction then associates to a bi-Lipschitz homeomorphism  $f$  of  $X$  fixing  $x$  a homeomorphism  $\rho(f)$  of  $S^1$ , and this homeomorphism only depends on the isotopy class of  $f$  fixing  $x$ . Moreover, it is Hölder continuous. This construction then defines the homomorphism  $\rho : \text{Mod}(S_{g,1}) \rightarrow \text{Homeo}(S^1)$ . Its restriction to  $\pi_1(S_g) = \pi_1(X)$  is just the standard action of  $\pi_1(X)$  on the ideal boundary of the universal covering  $\mathbb{H}^2$  of  $X$  (see Section 3 of [H21]). If  $\psi \in \text{Mod}(S_{g,1}) = \text{Aut}(\pi_1(S_g))$  is arbitrary, then  $\psi$ , viewed as an isotopy class of a diffeomorphism of  $X$ , conjugates the action of  $\pi_1(X, x)$  on  $S^1$  by the Hölder continuous homeomorphism  $\rho(\psi)$ .

From now on we denote by  $\Gamma_0$  the fundamental group of the hyperbolic surface  $X$ , with universal covering  $\mathbb{H}^2$ . Let  $\psi \in \text{Mod}(S_{g,1})$  be an element so that  $\Pi(\psi)$  is a pseudo-Anosov mapping class. Then there is a pair  $\xi_+, \xi_-$  of *measured geodesic laminations* and a number  $a > 1$  so that  $\psi\xi_+ = a\xi_+$  and  $\psi\xi_- = a^{-1}\xi_-$ . This means the following. The measured lamination  $\xi_+$  is a *geodesic current*, that is, a  $\Gamma_0$ -invariant Radon measure on the space  $S^1 \times S^1 \setminus \Delta$  of geodesics in  $\mathbb{H}^2$  whose support consists of *simple* geodesics, equipped with a *transverse measure*. The mapping class  $\psi$  preserves this support and multiplies the transverse measure by  $a$ .

Note that  $\text{Mod}(S_g)$  acts on the space  $\text{Curr}$  of geodesic currents, equipped with the weak\* topology, as a group of transformations. There exists a continuous, symmetric,  $\text{Mod}(S_g)$ -invariant *intersection form*  $\iota : \text{Curr} \times \text{Curr} \rightarrow [0, \infty)$ , see [Bo88].

**Lemma 2.1.** *If  $\nu$  is a current defining the measure class of a product measure, then  $\iota(\nu, \nu) > 0$ .*

*Proof.* A geodesic current  $\xi$  with  $\iota(\xi, \xi) = 0$  is well known to be a measured lamination. But a measured lamination does not define the measure class of a product [Bo88].  $\square$

Given a hyperbolic metric  $X$  on  $S_g$ , we can consider the geodesic flow  $\Phi^t$  on the unit tangent bundle  $T^1X$  of  $X$ . A  $\Phi^t$ -invariant finite Borel measure on  $T^1X$  locally is a product of the Lebesgue measure on the orbits and a measure on a transversal, and this construction defines a homeomorphism between the space of  $\Phi^t$ -invariant finite Borel measures on  $T^1X$ , equipped with the weak\*-topology, and the space of geodesic currents. It maps ergodic invariant measures to ergodic currents. The classical Hopf argument shows (see also Theorem 1.3 and Theorem 1.4 of [K94] which can be used as a translation of a more general statement).

**Lemma 2.2.** *A current  $\nu$  defining the measure class of the product  $[\mu] \times [\mu]$  of a  $\Gamma_0$ -invariant ergodic measure class  $[\mu]$  on  $S^1$  is ergodic under the diagonal action of  $\Gamma_0$ .*

*Proof.* Let  $\hat{\nu}$  be a  $\Phi^t$ -invariant probability measure on  $T^1X$  whose corresponding current defines the measure class of  $[\mu] \times [\mu]$ . Then  $\hat{\nu}$  has a local product structure with respect to the strong stable and strong unstable foliation. The classical Hopf argument therefore implies that every  $\Phi^t$ -invariant  $\hat{\nu}$ -integrable function is constant almost everywhere. Hence  $\hat{\nu}$  is ergodic, equivalently the current defined by  $\hat{\nu}$  is ergodic under the diagonal action of  $\Gamma_0$ .  $\square$

If  $\nu$  is a current defining the measure class  $[\mu] \times [\mu]$ , then  $\psi_*\nu$  is a current defining the measure class  $[\psi_*\mu] \times [\psi_*\mu]$ . By invariance, we have  $\iota(\psi_*\nu, \psi_*\nu) = \iota(\nu, \nu)$ .

**Lemma 2.3.** *If the measure classes  $[\mu]$  and  $[\psi_*\mu]$  are not singular, then  $\psi_*\nu = \nu$ .*

*Proof.* By ergodicity and  $\Gamma_0$ -invariance of the measure class  $[\mu]$ , either  $[\psi_*\mu] = [\mu]$ , or  $[\mu]$  and  $[\psi_*\mu]$  are singular. Thus assume that  $[\mu] = [\psi_*\mu]$ . Then the currents  $\nu$  and  $\psi_*\nu$  are absolutely continuous and hence they define finite  $\Phi^t$ -invariant measures on  $T^1X$  in the same measure class.

By Lemma 2.2, these measures are ergodic under the geodesic flow and hence they coincide up to a multiplicative constant, implying that  $\psi_*\nu = b\nu$  for a constant  $b > 0$ . But  $b = 1$  as  $\iota(b\nu, b\nu) = b^2\iota(\nu, \nu)$  and  $\iota(\psi_*\nu, \psi_*\nu) = \iota(\nu, \nu) > 0$ .  $\square$

We use Lemma 2.3 to show.

**Proposition 2.4.** *Let  $[\mu]$  be a measure class on  $S^1$  which is invariant and ergodic under  $\Gamma_0$  and induced by a current. Then for any pseudo-Anosov mapping class  $\psi$ , the measure classes  $[\mu]$  and  $[\psi_*\mu]$  are singular.*

*Proof.* We know that  $[\psi_*\mu]$  is invariant and ergodic under the action of  $\Gamma_0$ . Thus by Lemma 2.3, if  $[\mu]$  and  $[\psi_*\mu]$  are not singular, then a current  $\nu$  defining the measure class of  $[\mu] \times [\mu]$  coincides with  $\psi_*\nu$ .

Let  $\xi$  be a measured lamination so that  $\psi_*\xi = a\xi$  for some  $a > 1$ . Then we have

$$\iota(\xi, \nu) = \iota(\psi_*\xi, \psi_*\nu) = \iota(a\xi, \nu) = a\iota(\xi, \nu)$$

and hence  $\iota(\xi, \nu) = 0$ . Since  $\psi$  is a pseudo-Anosov mapping class, the support of the measured lamination  $\xi$  fills the surface  $S$ , and hence  $\iota(\xi, \nu) = 0$  implies that  $\nu$  is supported in  $\xi$ . But then  $\iota(\nu, \nu) = 0$ , which contradicts Lemma 2.1.  $\square$

## 3. CANNON-THURSTON MAPS

Consider now a closed hyperbolic 3-manifold  $M$  which fibers over  $S^1$ , with fiber a closed surface  $S_g$  of genus  $g$ . The fundamental group  $\Gamma$  of  $M$  fits into an exact sequence

$$(2) \quad 0 \rightarrow \pi_1(S_g) \rightarrow \Gamma \rightarrow \mathbb{Z} \rightarrow 0.$$

The monodromy of the fibration, which is the outer automorphism of  $\pi_1(S_g)$  determined by a generator of the quotient group  $\mathbb{Z}$ , is a pseudo-Anosov mapping class  $\psi$ . Thus Proposition 2.4 shows.

**Corollary 3.1.** *If the measure class  $[\mu]$  on  $S^1$  is induced by a current then the measure classes  $[\mu]$  and  $[\psi_*\mu]$  are singular.*

The group  $\Gamma$  is hyperbolic and quasi-isometric to hyperbolic 3-space and hence its Gromov boundary equals the two-sphere  $S^2$ . In spite of the fact that the subgroup  $\Gamma_0 = \pi_1(S_g)$  of  $\Gamma$  is exponentially distorted, Cannon and Thurston showed that there is a continuous,  $\Gamma_0$ -equivariant, finite-to-one map  $F : S^1 \rightarrow S^2$  [CT07]. See also [CL26] for a comprehensive account.

The fibers of  $F$  can be understood explicitly. Namely, consider the equivalence relation  $\sim$  on  $S^1$  defined by  $x \sim y$  if and only if  $F(x) = F(y)$ . Following [CL26], this relation is the union of two disjoint (no perfect fits) *laminar* relations in the sense of Definition 1.16 of [CL26]: Its equivalence classes are closed subsets of  $S^1$ , the relation is closed in the space of unordered distinct pairs of points, and distinct equivalence classes are *unlinked*. As the relation is invariant under the action of  $\Gamma_0$ , this can be thought of as follows. If  $F(x) = F(y)$  with  $x \neq y \in S^1$ , then the geodesic joining  $x$  to  $y$  projects to a simple geodesic in  $S_g = \Gamma_0 \backslash \mathbb{H}^2$  (for any discrete embedding of  $\Gamma_0$  into  $\mathrm{PSL}(2, \mathbb{R})$ ). Hence  $F$  is injective on the Borel set  $\Omega \subset S^1$  of all points which are *not* endpoints of lifts of *simple* geodesics in  $S_g$ . We need the following proposition.

**Proposition 3.2.** *Let  $[\mu]$  be a  $\Gamma_0$ -invariant ergodic measure class on  $S^1$  induced by a current. Then  $[\mu]$  gives measure zero to the set of endpoints of lifts of simple geodesics in  $S_g$ .*

*Proof.* By Lemma 2.2, a current  $\nu$  inducing  $[\mu]$  is ergodic under the action of  $\Gamma_0$ , equivalently, the invariant measure it defines on  $T^1S$  is ergodic under the action of the geodesic flow  $\Phi^t$ . Then  $[\mu]$ -almost every  $z \in S^1$  is the endpoint of a geodesic whose tangent line projects to a Birkhoff regular orbit for the geodesic flow in  $T^1S_g$ , equipped with the finite invariant probability measure  $\hat{\nu}$  corresponding to (a multiple of)  $\nu$ . We have to show that such a point cannot be the endpoint of a lift of a simple geodesic on  $S_g$ .

To see this assume otherwise. Then there are geodesics  $\gamma_1, \gamma_2 \subset \mathbb{H}^2$  with the same endpoint  $z \in S^1$  so that the tangent line  $\gamma'_1$  of  $\gamma_1$  is Birkhoff regular for the measure  $\hat{\nu}$ , and the projection of the geodesic  $\gamma_2$  to  $S_g$  is simple. Thus, for suitable parameterizations of  $\gamma_1, \gamma_2$  by arc length and the Sasaki metric  $d_S$  on  $T^1\mathbb{H}^2$ , it holds

$$d_S(\gamma'_1(t), \gamma'_2(t)) \leq Ce^{-t}$$

for a constant  $C > 0$ , and the same holds true for the projections of  $\gamma'_1, \gamma'_2$  to  $T^1S_g$ .

Let  $P : \mathbb{H}^2 \rightarrow X$  be the canonical projection. Since  $P\gamma_2$  is simple, its closure  $\mathcal{L}$  in  $S_g$  is a geodesic lamination. The subset  $T^1\mathcal{L}$  of  $T^1S_g$  of all unit tangent vectors to geodesics in  $\mathcal{L}$  is a closed subset of  $T^1S_g$ , intersecting the fiber of  $T^1S_g$  over a point in  $\mathcal{L}$  in precisely two antipodal points, and it is the closure of the unit tangents (for both orientations) of the geodesic  $P\gamma_2$ . In particular, it is compact. As a consequence of the previous paragraph, for any  $\alpha > 0$  there then exists some  $t(\alpha) > 0$  such that whenever  $t > t(\alpha)$ , the Sasaki distance between  $P\gamma'_1(t)$  and the compact set  $T^1\mathcal{L}$  is at most  $\alpha$ .

Choose a small compact arc  $I \subset \gamma_2$  and consider the set  $A \subset T^1S_g$  of all unit tangent vectors with footpoint in  $I$  which make an angle of at least  $\pi/4$  to the tangent of  $\gamma_2$ . This set is a compact transversal for the geodesic flow which is disjoint from  $T^1\mathcal{L}$ . As a consequence, there is a number  $\epsilon > 0$  so that the compact set  $B = \cup_{-\epsilon \leq t \leq \epsilon} \Phi^t A$  is disjoint from the compact set  $T^1\mathcal{L}$  as well and hence has positive Sasaki distance  $\alpha > 0$  from  $T^1\mathcal{L}$ .

Now note that the invariant probability measure  $\hat{\nu}$  corresponding to the current  $\nu$  which defines the measure class of  $\mu \times \mu$  gives positive measure to every open subset of  $T^1S_g$ . As  $B$  contains an open subset of  $T^1S_g$  and  $\gamma'_1(0)$  is a Birkhoff regular point for the geodesic flow  $\Phi^t$  on  $T^1S_g$  with respect to  $\hat{\nu}$ , the forward  $\Phi^t$ -orbit of  $(P\gamma_1)'(t(\alpha))$ , which is just the tangent line  $(P\gamma_1)'[t(\alpha), \infty)$ , is dense in  $T^1S_g$  and hence it intersects  $B$  for arbitrarily large time. This is a contradiction which complete the proof.  $\square$

Together this yields Theorem 2 and Corollary 3.

**Proposition 3.3.** *Let  $[\mu]$  be a  $\Gamma_0$ -invariant ergodic measure class on  $S^1$  induced by a current. Then the push-forward  $F_*[\mu]$  of  $[\mu]$  under a Cannon-Thurston map  $F$  is singular with respect to any  $\Gamma$ -invariant totally ergodic measure class on  $S^2$ .*

*Proof of Theorem 2.* As a consequence of Proposition 3.2, the measure class  $[\mu]$  gives full mass to the Borel set of points on  $S^1$  on which the map  $F$  is injective, and the same holds true for  $[\psi_*\mu]$ . By Corollary 3.1, the measure classes  $F_*\mu$  and  $F_*\psi_*\mu$  are singular. Thus, if  $\eta$  is a  $\Gamma$ -invariant ergodic measure class on  $S^2$  which is not singular with respect to  $F_*[\mu]$ , then the finite index subgroup of  $\Gamma$  generated by  $\Gamma_0$  and  $\psi^2$  preserves a nontrivial measurable set and hence indeed, such a measure is not totally ergodic, that is, it admits a finite factor.  $\square$

**Corollary 3.4.** *The Lebesgue measure on  $S^2$  defines a  $\Gamma_0$ -invariant measure class  $[\mu]$  on  $S^1$  so that  $[\mu] \times [\mu]$  is ergodic under the diagonal action of  $\Gamma_0$ , but there is no geodesic current defining  $[\mu] \times [\mu]$ .*

*Proof.* The Lebesgue measure class  $[\lambda]$  on  $S^2$  is well known to be totally ergodic under the action of  $\Gamma$ , moreover the measure class  $[\lambda] \times [\lambda]$  is ergodic under the diagonal action of  $\Gamma_0$  [C01, H02].

By Proposition 5.5 of [FMP26],  $[\lambda]$  gives full measure to the set of points in  $S^2$  with only one preimage under  $F$ . This allows to pull the measure back to a measure

on  $S^1$  whose measure class  $[F^*\lambda]$  is invariant under the action of  $\Gamma_0$ . We refer to Section 5 of [FMP26] for details. By ergodicity of  $[\lambda] \times [\lambda]$  under the diagonal action of  $\Gamma_0$ , the same holds true for the measure class  $[F^*\lambda] \times [F^*\lambda]$ . However, by Proposition 3.3,  $[\lambda]$  can not be induced by a current.  $\square$

**Remark 3.5.** We believe that Corollary 3.4 extends to measure classes on  $S^2$  which are defined by Gibbs equilibrium states for the geodesic flow on the unit tangent bundle of  $M$ , with the same proof. However, we used Proposition 5.5 of [FMP26] whose proof extends, but which is not available in this generality in the literature (see however [KZ25]).

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