**Skript zur Vorlesung Differentialgeometrie, WS 04/05**

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# 1. Längenräume

1.1. **1.3, 10.11.04** .

**Def:** A *topological group* is a group $G$ equipped with a topology such that the following holds.

(1) The group composition $G \times G \to G, (g, h) \to gh$ is continuous.
(2) The map $G \to G$ which maps $g$ to its inverse $g^{-1}$ is continuous.

**Example:** 1) Every group $G$ with the discrete topology is a topological group.
2) The group $SL(2, \mathbb{R})$ can be viewed as a subset of $\mathbb{R}^4$. This defines a topology on $SL(2, \mathbb{R})$ which provides $SL(2, \mathbb{R})$ with the structure of a topological group.

**Def:** Let $G$ be a topological group and let $X$ be a topological space. The group $G$ *acts on $X$ as a continuous transformation group* (or *as a continuous group of isometries*) if $G$ acts on $X$ as a group of homeomorphisms (or isometries) via a *continuous* map $\Phi : G \times X \to X$.

**Example:** 1) If $G$ is a group equipped with the discrete topology and if $G$ acts on $X$ as a group of homeomorphisms then $G$ acts on $X$ as a continuous transformation group.

2) The isometry group $O(n) \times \mathbb{R}^n$ of $\mathbb{R}^n$ acts on $\mathbb{R}^n$ as a continuous transformation group.

3) The group $PSL(2, \mathbb{R})$ acts on the upper half plane $H$ as a continuous transformation group via $\big((a, b, c, d), z\big) \to \frac{az+b}{cz+d}$.

Our above definition requires that a group $G$ which acts on a space $X$ is already equipped with a topology. In the case that $X$ is a *proper* metric space and $G$ is a group of homeomorphisms of $X$ there is a natural structure of a topological group on $G$ such that $G$ acts on $X$ as a continuous transformation group. This structure is given as follows.

**Def:** Let $X$ be a proper metric space. The *compact-open* topology for a group $G$ of homeomorphisms of $X$ is defined as follows. A neighborhood basis for the topology consists of sets of the form $U(K, \epsilon, g)$ where $g \in G, K \subset X$ is a non-empty compact set, $\epsilon > 0$ and $U(K, \epsilon, g) = \{h \in G \mid d(gx, hx) < \epsilon \text{ for all } x \in K\}$.

In other words, a set $U \subset G$ is defined to be open if for every $g \in U$ there is some compact set $K \subset X$ and some $\epsilon > 0$ such that $U(g, K, \epsilon) \subset U$. It is easy to see that this defines indeed a topology on a group $G$ of homeomorphisms of $X$. With respect to this topology, a sequence $g_i \subset G$ *converges* to $g$ if and only if for every compact subset $K$ of $X$ the restriction $g_i|K$ converges *uniformly* to $g|K$ (this is equivalent to saying that for every $\epsilon > 0$ there is some $i_0 > 0$ such that $g_i \in U(K, \epsilon, g)$ for all $i \geq i_0$.

**Lemma 1.1.** *Let $X$ be a proper metric space and let $G$ be a group of homeomorphisms of $X$. Then The compact open topology define the structure of a topological group for $G$.*

*Proof:* To show that the composition map $G \times G \to G$ is continuous, let $U \subset G$ be an open set and let $g, h \in G$ be such that $gh \in U$. Since $U$ is open there is a compact set $K$ and a number $\epsilon > 0$ such that $U(K, u, \epsilon) \subset U$. Since $h$ is continuous, the set $hK$ is compact and the same is true for the closed $\epsilon/2$-neighborhood $C$ of $hK$. Let $\delta > 0$ be sufficiently small that $d(gz, gy) < \epsilon/2$ whenever $z, y \in C$ and $d(z, y) < \delta$; such a number exists since $g$ is uniformly continuous on the compact set $C$. Let $h' \in U(K, h, \delta)$, $g' \in U(C, g, \epsilon/2)$ and let $y \in K$. Then $hz \in C$ and $d(g'h'z, ghz) \leq d(g'(h'z), g(h'z)) + d(g(h'z), ghz) \leq \epsilon$ since $d(h'z, hz) < \delta$. This shows continuuity of the composition map.

Continuity of the map which assigns to a homeomorphis $g$ its inverse $g^{-1}$ can be seen in the same way. ☐

The following lemma is immediate.

**Lemma 1.22:** *Let $G$ be a group of homeomorphisms on a proper metric space $X$. Then $G$ equipped with the compact open topology acts as a continuous transformation group on $X$.*

*Proof:* Let $\Phi : G \times X \to X$ be the map which describes the action of the group $G$. Let $U \subset X$ be an open set; we have to show that $\Phi^{-1}(U) \subset G \times X$ is open. To see this let $(g, z) \in \Phi^{-1}(x)$; then $\Phi(g, z) = x$. Since $U$ is open there is a number $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. Choose an open neighborhood $V$ of $z$ in $X$ such that $gV \subset B(x, \epsilon/2)$. Then $\Phi(U(\overline{V}, \epsilon/2, g), V) \subset U$ and hence $\Phi$ is continuous. ☐

We specialize now to groups of isometries of a proper metric space $X$. We have.

**Proposition 1.23:** *Let $X$ be a proper metric space $X$.*

1. $\mathrm{Iso}(X)$ *equipped with the compact open topology is locally compact and metrizable.*
2. *A closed subset $B$ of* $\mathrm{Iso}(X)$ *is compact if and only if there is a compact subset $K$ of $X$ such that $gK \cap K \neq \emptyset$ for all $g \in B$.*

*Proof:* We show first that $\mathrm{Iso}(X)$ is metrizable. For this choose a point $x_0 \in X$ and let $g, h \in \mathrm{Iso}(X)$. Since $g, h$ are isometries, for $z \in B(x_0, r)$ we have $d(gz, hz) \leq d(gz, gx_0) + d(gx_0, hx_0) + d(hx_0, hz) \leq 2r + d(gx_0, hx_0)$ and therefore the series $\sum_{i=1}^{\infty} 2^{-i} \sup\{d(gz, hz) \mid z \in B(x_0, i)\}$ converges. As a consequence, we can define a distance $D$ on $\mathrm{Iso}(X)$ by

$$D(g, h) = \sum_{i=1}^{\infty} 2^{-i} \sup\{d(gz, hz) \mid z \in B(x_0, i)\}.$$

We claim that the identity map as a map from the metric space $(\mathrm{Iso}(X), D)$ to $\mathrm{Iso}(X)$ equipped with the compact open topology is continuous. Namely, let $U \subset \mathrm{Iso}(X)$ be open for the compact open topology and let $g \in U$. Then there is some compact set $K$ and some $\epsilon > 0$ such that $U(K, g, \epsilon) \subset U$. Choose a number $m > 0$ such that $K \subset \overline{B(x_0, m)}$ and let $V$ be the $\epsilon/2^m$-ball about $g$ in $\mathrm{Iso}(X), D)$. If $h \in V$ then $d(gy, hy) \leq \epsilon$ for every $y \in \overline{B(x_0, m)}$ and therefore $h \in U(K, g, \epsilon)$.

Similarly it can be seen that the identity is also continuous as a map from $\mathrm{Iso}(X)$ equipped with the compact oepn topology into $(\mathrm{Iso}(X), D)$.

Let $B \subset \mathrm{Iso}(X)$ be a closed subset and let $K \subset X$ be compact. We show that $A = \{g \in B \mid gK \cap K \neq \emptyset\}$ is compact. For this it is enough to show that every sequence $g_i \subset A$ has a subsequence which converges uniformly on compact sets to an isometry $g$.

For this observe that if $K$ is a compact subset of a metric space then $K$ has a *countable dense subset.* This means that there is a countable subset $Q$ of $K$ such that $\overline{Q} = K$. To construct such a set, observe that by compactness, for every $j \geq 1$ there is a *finite* set $Q_j = \{y_1^j, \dots, y_{k_j}^j\}$ of $Y$ such that $K \subset \cup_{i=1}^{k_j} B(y_i^j, 1/2)$. Then $\cup_j Q_j$ is a countable dense subset of $K$.

Let $(g_i) \subset B$ be any sequence and let $m > 0$ be an arbitrary positive integer. Then there is a number $r > 0$ such that $K \subset B(x_0, m)$. Since each of the maps $g_i$ is an isometry and $g_i K \cap K \neq \emptyset$ we have $g_i B(x_0, m) \subset \overline{B(x_0.r + 2m)}$ for all $i$. Let $Q = \{q_1, \dots,\}$ be a countable dense subset of $\overline{B(x_0, m)}$. Then for each $q_j \in Q$, the sequence $(g_i q_j)_i$ is contained in a *compact* subset of $X$ and therefore it admits a convergent subsequence. Using a diagonal procedure we may assume after passing to a subsequence that $(g_i q_j)$ converges for each $j$ to some $gq_j$ and $d(gq_i, gq_j) = d(q_i, q_j)$ for all $i, j$. Now if $x \in \overline{B(x_0, m)}$ is arbitrary then there is a sequence $(q_j)_j \subset Q$ converging to $x$. Since the maps $g_i$ are isometries, the sequence $(gq_j)$ is a Cauchy sequence and hence convergent. As a consequence, we may assume that that $g_i \to g \in \mathrm{Iso}(X)$. Now $m > 0$ was arbitrary and using a second diagonal sequence we conclude that $(g_i) \subset B$ has a subsequence which converges in $\mathrm{Iso}(X)$ to an isometry $g$.

On the other hand, if $B \subset \mathrm{Iso}(X)$ is compact then for any sequence $(g_i) \subset B$ there is a convergent subsequence and hence for every $x \in X$ the sequence $(g_i x) \subset X$ is bounded. In particular, there is some compact set $K$ such that $gK \cap K \neq \emptyset$ for all $g \in B$. This completes the proof of the proposition. □

**Corollary 1.24:** *Let $X$ be a proper metric space. Then a subgroup $G \subset \mathrm{Iso}(X)$ operates properly discontinuously on $X$ if and only if $G$ is* discrete, *i.e. its induced topology is discrete.*

*Proof:* Let $G \subset \mathrm{Iso}(X)$ be any discrete subgroup and let $K \subset X$ be compact. Then $\{g \in \mathrm{Iso}(X) \mid gK \cap K \neq \emptyset\}$ is a compact subset of $\mathrm{Iso}(X)$ and hence its intersection with $G$ is finite. Thus $G$ operates properly discontinuously on $X$.

Now let $G \subset \mathrm{Iso}(X)$ be a group which acts properly discontinuously on $X$. Let $B \subset \mathrm{Iso}(X)$ be a compact subset. Then there is a compact subset $K$ of $X$ such that $gK \cap K \neq \emptyset$ for all $g \in B$, Thus $G \cap B$ is finite and $G \subset \mathrm{Iso}(X)$ is discrete. □

**Example:** The group $PSL(2, \mathbb{Z})$ is a discrete subgroup of $PSL(2, \mathbb{R})$ and hence it acts properly discontinuously on the upper half-plane $H$.

**Example: The hyperbolic plane** Let $H = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$ be the upper half-plane. Define $\mathcal{W}$ to be the collection of all reparametrizations of *piecewise smooth* paths $c\colon [a, b] \to H$. Clearly $\mathcal{W}$ is an admissible path family. For $\gamma \in W$ define $\ell(\gamma) = \int_a^b (\|\gamma'(t)\|/\mathrm{Im}(\gamma(t)))dt$.

*Claim:* The resulting length structure on $H$ defines a metric on $H$.

By Lemma 1.19 we only have to show positivity. For this let $z_0 \in H$ be arbitrary; then there are $\varepsilon > 0$, $r > 0$ such that $B(z_0, \varepsilon) = \{z \in \mathbb{C} \mid |z - z_0| < \epsilon\} \subset \{r < \mathrm{Im}(z) < 1/r\} \subset H$.

Let $y \neq z_0 \in H$ and let $\gamma\colon [a, b] \to H$ be any piecewise smooth path connecting $z_0$ to $y$. Assume first that $\gamma[a, b] \subset B(z_0, \epsilon)$; then $\mathrm{Im}(\gamma(t)) \leq 1/r$ for every $t \in [a, b]$ and hence $\ell(\gamma)$ is not smaller than $r$ times the euclidean length of $\gamma$, i.e. $\ell(\gamma) \geq r|z_0 - y|$.

On the other hand, if there is some $t \in (a, b]$ such that $|\gamma(t) - z_0| = \epsilon$ then the same argument implies that $\ell(\gamma) \geq r\epsilon$. As a consequence, we have $\ell(\gamma) \geq \min\{r|y - z_0|, r\epsilon\}$ which shows positivity.

Notice that the same argument also shows that the topology defined by the length structure $\ell$ is just the euclidean topology.

*Claim: $Iso(H)$* contains the semi-direct product $PSL(2, \mathbb{R}) \ltimes \mathbb{Z}_2$, where $PSL(2, \mathbb{C})$ acts on $H$ as the group of biholomorphic automorphisms via $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = \frac{az+b}{cz+d}$ and the generator $\iota$ of $\mathbb{Z}_2$ acts by $\iota(z) = -\overline{z}$.

To show our claim, let $z \in H$ be arbitrary. Since the action of $A \in PSL(2, \mathbb{R})$ is holomorphic, is differential at $z$ is the multiplication with the complex number

$$dA_z = \frac{a(cz + d) - (az + b)c}{(cz + d)^2} = \frac{1}{(cz + d)^2},$$

moreover we have

$$\mathrm{Im}\, z = \frac{z - \overline{z}}{2i}, \quad \mathrm{Im}(Az) = \frac{Az - \overline{Az}}{2i} = \frac{z - \overline{z}}{2i|cz + d|^2} = \frac{\mathrm{Im}\, z}{|cz + d|^2} \quad .$$

As a consequence, for every tangent vector $X \in T_zH$ be a tangent vector at $z$ we have

$$\frac{\|dA_zX\|}{\operatorname{Im} Az} = \frac{\|X\|}{|cz+d|^2} \cdot \frac{|cz+d|^2}{\operatorname{Im} z} = \frac{\|X\|}{\operatorname{Im} z}.$$

As an immediate consequence, for every piecewise smooth curve $\gamma$ in $H$ and every $A \in PSL(2,\mathbb{R})$ we have $\ell(\gamma) = \ell(A\gamma)$. In particular, the group $PSL(2,\mathbb{R}) \ltimes \mathbb{Z}_2$ acts on $H$ as a group of isometries.

*Claim:* For $0 < s < t$, the distance in $H$ between the horizontal lines $\{z \mid \operatorname{Im}(z) = s\}$ and $\{z \mid \operatorname{Im}(z) = t\}$ equals $|\log t - \log s|$.

Namely, if $\gamma\colon [a,b] \to H$ is any piecewise smooth curve connecting a point $\gamma(a) \in \{z \mid \operatorname{Im}(z) = s\}$ to a point $\gamma(b) \in \{z \mid \operatorname{Im}(z) = t\}$ then $l(\gamma) = \int_a^b \frac{\|\gamma'(t)\|}{\operatorname{Im}\gamma(t)} dt \geq |\int_a^b \frac{(\operatorname{Im}\gamma)'(t)}{Im\,\gamma(t)} dt| = |\log t - \log s|$. The discussion of equality in this calculation also implies that for every $a \in \mathbb{R}$ the curve $\tau \to \tau i + a$ ($\tau \in [s,t]$) is (up to reparameterization) the unique geodesic connecting $si + a$ to $ti + a$.

As a consequence, we conclude that $H$ is complete and hence proper. Namely, for $z_0 \in H$ and $r > 0$ the closed ball $\overline{B(z_0,r)}$ of radius $r$ about $z_0$ is contained in the strip $\{z \in H \mid \operatorname{Im}(z_0)e^{-r} \leq \operatorname{Im}(z) \leq \operatorname{Im}(z_0)e^r\})$ and the same is true for every curve of almost minimal length connecting $z_0$ to a point $y \in B(z_0,r)$. However, for such a curve $\gamma$ we have $\ell(\gamma) \geq (e^r \operatorname{Im}(z_0))^{-1}|\gamma(a) - \gamma(b)|$ and hence the ball $\overline{B(z_0,r)}$ is contained in a compact subset of $H$ and is compact.

Since $H$ is proper, $H$ is geodesic and therefore any two points in $H$ can be connected by a geodesic. To compute these geodesics we show first that $PSL(2,\mathbb{R})$ operates *transitively* on $H$. This means that for every $z \in H$ there is some $A \in PSL(2,\mathbb{R})$ with $Ai = z$. Namely, the subgroup $G = \{\left(\begin{smallmatrix} e^t & \\ & e^{-t}\end{smallmatrix}\right) | t \in \mathbb{R}\}$ of $PSL(2,\mathbb{R})$ maps $z \in H$ to $\left(\begin{smallmatrix} e^t & \\ & e^{-t}\end{smallmatrix}\right) z = e^{2t}z$ and hence the subset $\{z \mid \operatorname{Im}(z) = 1\} \subset H$ of $H$ intersects each orbit of this group. Moreover, the subgroup $H = \{\left(\begin{smallmatrix} 1 & t \\ & 1\end{smallmatrix}\right) | t \in \mathbb{R}\}$ acts on $H$ as the group of translations $z \to z + t\,(t \in \mathbb{R})$ and hence every point $z \in H$ with $\operatorname{Im}(z) = 1$ is mapped by such a translation to $i$.

Next we compute the *isotropy subgroup* of the point $i$ for the group $PSL(2,\mathbb{R})$, i.e. the subgroup of all $A \in PSL(2,\mathbb{R})$ with $Ai = i$. For this notice that

$$Ai = i \Leftrightarrow \frac{ai+b}{ci+d} = i \Leftrightarrow ai + b = -c + di,\ ad - bc = 1 \tag{1}$$

$$\Leftrightarrow b = -c, a = d, ad - bc = 1 \Rightarrow A \in SO(2) \tag{2}$$

To calculate the geodesics in $H$, we use the following.

**Lemma 1.2.** A *circle* in $\hat{\mathbb{C}}$ is either a circle of the form $\{|z - a| = r\} \subset \mathbb{C}$ *or* a line in $\mathbb{C}$. The group $PSL(2,\mathbb{C})$ maps circles in $\hat{\mathbb{C}}$ to circles and its subgroup $PSL(2,\mathbb{R})$ maps circles which intersect the real line $\mathbb{R} = \{z \mid \operatorname{Im}(z) = 0\}$ orthogonally to circles with the same property.

*Proof:* The claim is obvious for translations of the form $z \to z + a$ ($a \in \mathbb{C}$) and for homotheties of the form $z \to az$ ($a \in \mathbb{C}^*$). The *inversion* is the map $z \to \frac{1}{z}$. If $K = \{|z+c|^2 = r\}$ is a circle then $z\overline{z}+z\overline{c}+\overline{z}c+\overline{c}\overline{c} = r \Leftrightarrow z\overline{z}+z\overline{c}+\overline{z}c+\overline{z}\overline{c}+(c\overline{c}-r) = 0$ for $(c\overline{c} - r) \in \mathbb{R}$ and the family of equations for circles can be multiplied with the real number $\alpha \neq 0$ to get $\alpha z\overline{z} + \overline{\alpha c}z + \overline{z}\alpha c + d = 0$ for $\alpha, d \in \mathbb{R}$, $c\overline{c} > \alpha d$ (includes all lines).

But: From $\alpha z\overline{z} + cz + \overline{cz} + d = 0$ and $w = \frac{1}{z}$ we have $w\overline{w}(\alpha z\overline{z} + cz + \overline{cz} + d) = \alpha + c\overline{w} + \overline{c}w + dw\overline{w} = 0$ which is an equation of the same type. Thus the inversion maps circles to circles.

On the other hand, *every* element $A \in PSL(2, \mathbb{C})$ can be written as a composition of maps of the above form. Namely, let $Az = \frac{az+b}{cz+d}$; in the case $c = 0$ we have $Az = \frac{a}{d}z + \frac{b}{d}$ and hence $A$ is of the required form. So assume that $c \neq 0$. Then

$$z \to z + \frac{d}{c} \to (z + \frac{d}{c})^{-1} = \frac{c}{cz+d} \to \frac{bc-ad}{c^2}(\frac{c}{cz+d}) = \frac{bc-ad}{c(cz+d)}$$

$$\to \frac{bc-ad}{c(cz+d)} + \frac{a}{c} = \frac{bc-ad+a(cz+d)}{c(cz+d)} = \frac{bc-ad+acz+ad}{c(cz+d)} = \frac{az+b}{cz+d}.$$

and this determines a decomposition as required. □

**Corollary 1.3.** A geodesic in $H$ is up to parameterization a circle which intersects $\{z \mid \operatorname{Im} z = 0\}$ orthogonally.

*Proof:* Notice first that the group $PSL(2, \mathbb{R})$ maps circles or lines which intersect the line $\operatorname{Im} = 0$ orhogonally to circles and lines with the same property. This simply follows from the fact that every element of $PSL(2, \mathbb{R})$ is holomorphic and hence its differential preserves the euclidean angles, moreover the group preserves the line $\mathbb{R} \cup \infty$.

On the other hand, for every point $z \in H$ and every tangent vector $0 \neq X \in T_zH$ there is a unique circle intersecting $\mathbb{R} \cup \infty$ orthogonally which passes through $z$ and is tangent to $X$. As a consequence, the action of $PSL(2, \mathbb{R})$ on those circles is transitive. Namely, if $\gamma$ is any such circle then there is some $A \in PSL(2, \mathbb{R})$ such that $A\gamma(0) = i$. The differential at $i$ of an element of the isotropy group $SO(2)$ at $i$ acts on the tangent space at $i$ by multiplication with a complex number of the form $\frac{1}{(-\sin\theta i + \cos\theta)^2} = \frac{1}{(\cos\theta)^2 - (\sin\theta)^2 - 2i\cos\theta\sin\theta} = \cos 2\theta + i \sin 2\theta$ and hence this differential acts by a rotation with the angle $2\theta$. In particular, the differentials of this group act *transitively* on the space of unit vectors of length 1 and hence this group acts transitively on the set of circles passing through $i$ which intersect $\mathbb{R} \cup \infty$ orthogonally. This means that the action of $PSL(2, \mathbb{R})$ on the collection $\mathcal{C}$ of circles which intersect $\mathbb{R}$ orthogonally is transitive.

Now let $\gamma : [a, b] \to H$ be any non-degenerate geodesic. Then there is a unique circle $cin\mathcal{C}$ passing through $\gamma(a), \gamma(b)$. With an element $A \in PSL(2, \mathbb{R})$ we can map this circle $c$ to the line $\operatorname{Re} = 0$. However, the subsegment of $Ac$ connecting $A\gamma(a)$ to $A\gamma(b)$ is the *unique* geodesic connecting these two points and therefore $\gamma \subset c$. This shows our corollary. □

Next we look at the *modular group* $PSL(2,\mathbb{Z}) \subset PSL(2,\mathbb{R})$. We have.

**Lemma 1.4.** *The modular group acts properly discontinuously on $H$.*

*Proof:* We only have to show that the topology on $PSL(2,\mathbb{R}$ induced by the eulidean metric on $\mathbb{R}^4$ coincides with the compact open topology for $PSL(2,\mathbb{R}$ viewed as a subgroup of the isometry group of $H$. For this notice that the subgroup of $PSL(2,\mathbb{R})$ which is generated by the group $G = \{\begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} \mid t \in \mathbb{R}\}$ and the group of translations $z \to z+t$ $(t \in \mathbb{R})$ acts *simply transitive* on $H$, i.e. that for every $z \in H$ there is a *unique element* $g$ in this group with $gi = z$. However, this was shown above. The lemma now follows. $\square$

Notice that the modular group $PSL(2,\mathbb{Z}) \subset PSL(2,\mathbb{R})$ contains the *translation* $T : Z \to z+1$ which correspond to $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ and the inversion $S(z) = -\frac{1}{z}$ which is defined by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Notice that $S$ is the composition of two reflections, one reflection $a+ib \to -a+ib$ along $\{z \mid \mathrm{Re}(z) = 0$ and the second reflection $\beta(z) = 1/\overline{z}$ at the circle $|z| = 1$ (for $|z| = 1$ we have $z\overline{z} = 1 \Rightarrow z = 1/\overline{z}$ and hence for $re^{i\theta}$ we have $\beta(re^{i\theta}) = e^{i\theta}/r$.

**Lemma 1.5.** *Let $B = \{z \in H| \, |\mathrm{Rez}| < \frac{1}{2}, \, |\mathrm{z}| > 1\}$; if $A \in PSL(2,\mathbb{Z}) - \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ then $A(B) \cap B = \emptyset$.*

**Proof:** Write $Az = \frac{az+b}{cz+d}$ for $a,b,c,d \in \mathbb{Z}, ad-bc = 1 \Rightarrow |cz+d|^2 = (cz+d)(c\overline{z}+d) = c^2|z|^2 + 2Re(z)cd + d^2 > c^2 + d^2 - |cd| = (|c|-|d|)^2 + |cd|$ if $c \neq 0$ since $|z| > 1, |Re\, z| < \frac{1}{2}$. But $c = 0$ is only possible for $A = \mathrm{Id}$ since $cd - bc = 1$ adn $a,b,c,d$ are integers. Hence the right hand side of the inequality is strictly positive and it is not smaller than 1 since $c,d$ are intergers $\Rightarrow |cz+d|^2 > 1$ and $\mathrm{Im}\, Az = \frac{\mathrm{Im}\, z}{|cz+d|^2} < \mathrm{Im}\, z$ for <u>all</u> $z \in B$. The same holds for $A^{-1}$ at point the $Az$ if $Az \in B$. As a consequence, we have $A(B) \cup B = \emptyset$ if $A \neq \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$. $\square$