Skript zur Vorlesung Differentialgeometrie I, WS 2006/2007

Ursula Hamenstädt

1. Curves in $\mathbb{R}^n$

1.1. **Basic concepts.**

Definition 1.1.1. A *parameterized curve* in $\mathbb{R}^n$ is a continuous map $c\colon I \to \mathbb{R}^n$ where $I \subset \mathbb{R}$ is an interval.

Example.

(1) A *straight line* is a curve $c\colon \mathbb{R} \to \mathbb{R}^n$ of the form $c(t) = c_0 + tv$ for some $v \in \mathbb{R}^n - \{0\}$
(2) A *circle* about 0 of radius $r > 0$ in $\mathbb{R}^2$ is given by $c(t) = (r\cos t, r\sin t)$. Tus a circle is a *periodic* curve, i.e. we have $c(t + 2\pi) = c(t)$ for all t.
(3) A *spiral* $c(t) = (r\cos t, r\sin t, ht)$

Question: How do we understand the dependence of the curve c on its "travelling speed"?

Definition 1.1.2. Let $c\colon I \to \mathbb{R}^n$ be a parameterized curve. A parameter transformation of c is a homeomorphism $\phi\colon J \to I$; the curve $\tilde{c} = c \circ \phi\colon J \to \mathbb{R}^n$ is a *reparameterization* of c.

Note: We have $\tilde{c}(J) = c(I)$. A parameter transformation is *orientation preserving* if ϕ is increasing.

Definition 1.1.3. A *regular curve* is a curve $c\colon I \to \mathbb{R}^n$ which is continuously differentiable and such that $c'(t) \neq 0$ for all t. The vector $c'(t) \in \mathbb{R}^n$ is the *tangent* of c at t. A *reparametrization* of a regular curve $c : I \to \mathbb{R}^n$ is defined by a *diffeomorphism* $\varphi : J \to I$.

Lemma 1.1.4. If $c\colon I \to \mathbb{R}^n$ is a regular curve with $c'(t) \equiv b$ for all t then c is (part of) a straight line.

Proof. If $s_0 \in I$ is a fixed point then $c(s) = c(s_0) + \int_{s_0}^s c'(t)dt = c(s_0) + (s - s_0)b$. □

Lemma 1.1.5. Write $\mathbb{R}^2 = \{x + iy \mid x, y \in \mathbb{R}\} = \mathbb{C}$; if $c\colon I \to \mathbb{C}$ is a regular curve with $c'(t) = i\,c(t)\forall t$ then c is (part of) a circle about 0.

Proof. For $s_0 \in I$ look for the solution of the ordinary first order differential equation $d'(t) = i\, d(t)$ with initial condition $d(s_0) = c(s_0)$. The curve $d(t) = e^{i(t-s_0)}c(s_0)$ is a solution of this equation with the same initial condition and hence it coincides with c. $\square$

The *length* $\|v\|$ of a vector v $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ is defined by $\|v\| = \sqrt{\sum v_i^2}$ Recall: $\|\ \|$ is a *norm* on $\mathbb{R}^n$, i.e. a function with the following properties.

(1) $\|v\| \geq 0$, with equality iff $v = 0$.
(2) $\|av\| = |a|\|v\|$ for all $a \in \mathbb{R}$, $v \in \mathbb{R}^n$.
(3) $\|v + w\| \leq \|v\| + \|w\|$ with equality iff $w = \lambda v$ for some $\lambda \geq 0$ (triangle inequality).

Definition 1.1.6. A *polygon* in $\mathbb{R}^n$ is a curve $c\colon [a,b] \to \mathbb{R}^n$ such that there is a partition $a = t_0 < t_1 < \cdots < t_k = b$ and vectors $v_1, \dots, v_k \in \mathbb{R}^n$ such that for all $i < k$ the restriction of c to $[t_{i-1}, t_i]$ is of the form $c(t) = c(t_{i-1}) + (t - t_{i-1})v_i$. The *length* of this polygon is $\sum_{i=1}^k \|v_i\|(t_i - t_{i-1})$.

Note: For each i, the tangent $c'(t)$ of c at $t \in (t_{i-1}, t_i)$ is v_i, i. e. the length of c equals $\int_a^b \|c'(t)\|dt$. The points $c(t_i)$ are the *vertices* of the polygon.

Definition 1.1.7. The *length* of a regular curve $c\colon I \to \mathbb{R}^n$ is $\ell(c) = \int_I \|c'(t)\|dt$.

Lemma 1.1.8. The length of a regular curve does not depend on its parameterization.

Proof. Let $c\colon I \to \mathbb{R}^n$ be regular, $\phi\colon J \to I$ diffeomorphism $\Longrightarrow$

$$\int_J \|(c \circ \phi)'(t)\|dt = \int_J \|c'(\phi(t))\, \phi'(t)\|dt$$

$$= \int_J \|c'(\phi(t))\||\phi'(t)|dt = \int_I \|c'(s)\|ds \tag{1}$$

by the transformation rule for integrals. $\square$

Note: If $c\colon [a,b] \to \mathbb{R}^n$ is any curve and $a = t_0 < t_1 < \cdots < t_k = b$ is a partition of the interval $[a,b]$ then c defines a polygon $p\colon [a,b] \to \mathbb{R}^n$ with vertices $p(t_i) = c(t_i)$.

Theorem 1.1.9. Let $c\colon [0,1] \to \mathbb{R}^n$ be a regular curve. Then we have

$$\ell(c) = \inf\{\ell(d) \mid d\colon [0,1] \to \mathbb{R}^n,\ d(0) = c(0), d(1) = c(1)\}$$

if and only if c is a line segment up to parameterization.

Proof. Let $c\colon [0,1] \to \mathbb{R}^n$ be a regular curve. Then we have $c(1) - c(0) = \int_0^1 c'(s)\, ds$.

We claim that

$$\| \int_0^1 c'(s)\, ds\| \ \leq \int_0^1 \|c'(s)\| ds. \tag{2}$$

Namely, let $\epsilon > 0$. Since $c'(t)$ is continuous, there is a partition $0 = t_0 < \cdots < t_k = 1$ such that

$$\|c'(t) - c'(s)\| < \varepsilon\ \forall s, t \in [t_{i-1}, t_i] \tag{3}$$

and

$$c(t_i) = c(t_{i-1}) + (t_i - t_{i-1})c'(t_{i-1}) + o(i) \tag{4}$$

where $\|o(i)\| \leq \varepsilon |t_i - t_{i-1}|$. By the estimate (4) we have

$$\|c(1) - c(0)\| = \| \sum_{i=1}^k (c(t_i) - c(t_{i-1}))\|$$

$$\leq \sum_{i=1}^k \|c(t_i) - c(t_{i-1})\| \leq \sum_i |t_i - t_{i-1}| \|c'(t_{i-1})\| + \varepsilon.$$

On the other hand, by (2) and the triangle inequality, for $t_{i-1} \leq s \leq t_i$ we have

$$|\|c'(s)\| - \|c'(t_{i-1})\|| \leq \|c'(s) - c'(t_{i-1})\| \leq \epsilon \tag{5}$$

and consequently

$$\int_0^1 \|c'(t)\| dt \geq \sum_i \int_{t_{i-1}}^{t_i} (\|c'(ti-1)\| - \epsilon) dt = \sum_i |t_i - t_{i-1}| \|c'(t_{i-1})\| - \varepsilon. \tag{6}$$

This shows that

$$\|c(1) - c(0)\| \leq \int_0^1 \|'(t)\| dt + 2\varepsilon$$

and since $\varepsilon > 0$ was arbitrary, we conclude that

$$\|c(1) - c(0)\| \leq \int_0^1 \|c'(s)\| ds,$$

with equality for straight line segments.

Now if c is *not* a reparametrization of a straight line segment then since c is regular there is some $t_0 \in (0,1)$ such that $c(t_0)$ does not lie on the line segment connecting $c(0)$ to $c(1)$. The usual triangle inequality shows that

$$\|c(t_0) - c(0)\| + \|c(t_0) - c(1)\| > \|c(1) - c(0)\|$$

and therefore by our above consideration, we have

$$\int_0^1 c'(t) dt = \int_0^{t_0} \|c'(s)\| ds + \int_{t_0}^1 \|c'(s)\| ds > \|c(1) - c(0)\|$$

as well. □

Example.

(1) The length of the straight line segment $c\colon [0,1] \to \mathbb{R}^n, c(t) = x_0 + tv$ equals $\ell(c) = \|v\|$.
(2) The length of the circle $c(t) = (r\cos t, r\sin t), t \in [0, 2\pi]$ of radius r equals $\ell(c) = r\int_0^{2\pi} dt = 2\pi r$.

Definition 1.1.10. A regular curve c is parameterized by *arc length* if $\|c'(t)\| = 1$ for all t.

Proposition 1.1.11. *For every regular curve c there is a reparameterization by the arc length.*

Proof. Let $c\colon I \to \mathbb{R}^n$ be regular and $t_0 \in I$ Then the assignment $s \to \phi(s) = \int_{t_0}^s \|c'(t)\| dt$ is strictly increasing and continuously differentiable and hence it has a continuously differentiable inverse ψ. Now define $d(t) = c(\psi(t))$; then $\|d'(t)\| = \|c'(\psi(t))\psi'(t)\|$ and $\psi'(t) = \frac{1}{\phi'(\psi(t))} = \frac{1}{\|c'(\psi(t))\|}$ which means that we obtain parameterization as required. □

1.2. **Plane curves.**

Definition 1.2.1. A parameterized curve $c\colon I \to \mathbb{R}^2$ is called a *plane curve*. Similarly we define plane regular curves, plane curves parameterized by arc length.

Example.

(1) A *straight line* $c(t) = x_0 + tv$ is the unique solution of the differential equation $c'' = 0$, $c(0) = x_0$, $c'(0) = v$.
(2) A circle $c(t) = x_0 + (r\cos t,\, r\sin t)$ is the solution of differential equation $c''(t) = -c(t) + x_0$, $c(0) = x_0 + (r, 0)$, $c'(0) = (0, r)$

Recall: For every unit vector $x = (x_1, x_2) \in \mathbb{R}^2 = \mathbb{C}$, $x_1^2 + x_2^2 = 1$, there is a unique unit vector $y = (y_1,\, y_2) \in \mathbb{R}^2 = \mathbb{C}$ such that

$$< x, y >= x_1 y_1 + x_2, y_2 = 0, \qquad \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = 1$$

and this vector is the *oriented normal vector* $y = (-x_2, x_1) = ix$ (where $\mathbb{R}^2 = \mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$). In particular, if $c : I \to \mathbb{R}^2$ is a regular curve parameterized by arc length and of class C^2 then the *normal field* $n(t) = i\,c'(t)$ is defined. For each t, the pair $(c'(t), n(t))$ is a *positive* orthonormal basis of $\mathbb{R}^2$.

Lemma 1.2.2. Let $c\colon I \to \mathbb{R}^2$ be parameterized by arc length and of class C^2. Then we have $c''(t) = \kappa(t) \cdot n(t)$ for some continuous function $\kappa\colon I \to \mathbb{R}$, the *curvature* of c. Moreover, the following holds.

(1) $\kappa \equiv 0$ if and only if c is a straight line segment.
(2) $\kappa \equiv r$ if and only if c is a segment of a circle of radius $\frac{1}{|r|}$.

Proof. Write $< c'(t), c'(t) >\equiv 1 = \|c'(t)\|^2$. Then

$$\frac{d}{dt} < c'(t), c'(t) >= \frac{d}{dt}(c_1'(t)^2 + c_2'(t)^2) = 2 < c'(t), c''(t) >\equiv 0$$

and therefore

$$c''(t) = \kappa(t)\, n\,(t)$$

for some $\kappa(t) \in \mathbb{R}$. The function $t \to \kappa(t)$ is continuous.

Now assume that $\kappa \equiv 0$. Then we have $c'' \equiv 0$ and hence $c'(t)$ does not depend on t. Thus $c(t) = x_0 + tv$ for x_0, $v \in \mathbb{R}^2$ where $\|v\| = 1$ since c is parameterized by arc length by assumption.

If $\kappa \equiv r > 0$ then we use the *complex* notation and derive that $c''(t) = ric'(t)$. Now the curve $c(t) = x_0 + e^{irt}v/r$ for $v \in \mathbb{R}^2$, $\|v\| = 1$ satisfy $c'(t) = ie^{irt}v$, $c''(t) = -re^{irt}v$ and hence they satisfy our differential equation with an arbitrary initial condition $c(0) = x_0 + v/r, c'(0) = iv$. The lemma now follows as before from uniqueness of the solution of differential equations of second order with given initial conditions of first order. $\square$

If $c\colon I \to \mathbb{R}^2$ is parameterized by arc length then the tangent curve $t \to c'(t)$ has its image on the unit circle $S^1 = \{| x| \|x\| = 1\}$.

Lemma 1.2.3. Let $c\colon I \to \mathbb{R}^2$ be parameterized by the arc length. Then there exists a continuous curve $\theta\colon I \to \mathbb{R}$, $c'(t) = (\cos\theta(t), \sin\theta(t))$. If $\tilde{\theta}$ is another such curve then $\tilde{\theta} = \theta + 2\pi k$, for some $k \in \mathbb{Z}$.

Proof. The function $f\colon t \to e^{it}\colon \mathbb{R} \to S^1$ is surjective, i. e. for every $x_0 \in S^1$ there exists $t_0 \in \mathbb{R}$, $e^{it_0} = x_0$. If $t_1 in \mathbb{R}$ is another such point *then* $t_1 = t_0 + 2\pi k$ for some $k \in \mathbb{Z}$. Now choose t_0 such that $e^{it_0} = \acute{c}(\alpha)$, $\alpha \in I$. If $U = \{y \in S^1 | \measuredangle(y, \acute{c}(\alpha)) < \pi/2\}$ then $f^{-1}U = \coprod_{i\in\mathbb{Z}} V_i$ (disjoint union) where each V_i is an open interval of length π which is mapped by f diffeomorphically onto U. But by continuity, there is some $\varepsilon > 0$ such that $\acute{c}(\alpha - \varepsilon, \alpha + \varepsilon) \subset U$ and consequently $\acute{c} \mid (\alpha - \varepsilon, \alpha + \varepsilon)$ admits a *unique* lift into some V_i which is continuous. This shows the lemma. $\square$

Definition 1.2.4. If $c\colon \mathbb{R} \to \mathbb{R}^2$ is parameterized by arc length and if $c(t + \rho) = c(t)$ for all t and some fixed $\rho > 0$ then we call c *periodic* of period ρ. Choose $\theta\colon \mathbb{R} \to \mathbb{R}$ such that $c'(t) = e^{i\theta(t)}$; then $\frac{1}{2\pi}(\theta(t + \rho) - \theta(t))$ does not depend on t and is called the *rotation number* of c. We denote this rotation number by n_c

As a justification for this definition, if $\tilde{\theta}$ is another choice of a function such that $c'(t) = e^{i\tilde{\theta}(t)}$ then we have $\tilde{\theta}(t) = \theta(t) + 2\pi k$ for all t. As a consequence, we have $\tilde{\theta}(t + \rho) - \tilde{\theta}(t) = \theta(t + \rho) - \theta(t)$ for all t.

On the other hand, $c'(t + \rho) = c'(t)$ for all t and hence the function $t \to \theta(t + \rho) - \theta(t)$ assumes values in $2\pi\mathbb{Z}$ for all t. Consequently this function is constant by continuity.

Example.

(1) $c(t) = e^{it}$ is periodic of period 2π, with rotation number 1
(2) The rotation number of a periodic curve c is the rotation of c' about the origin.

Lemma 1.2.5. Let $c\colon \mathbb{R} \to \mathbb{R}^2$ be *smooth* (of class C^2) and periodic of period ρ, parameterized by the arc length; then $n_c = \frac{1}{2\pi}\int_0^\rho \kappa(t)dt$ where $\kappa(t)$ is the curvature.

Proof. Write $c'(t) = e^{i\theta(t)}$; then $c''(t) = \kappa(t)ie^{i\theta(t)} = \theta'(t)ie^{i\theta(t)} \implies \theta'(t) = \kappa(t)$ and $n_c = \frac{1}{2\pi}\int_0^\rho \theta'(t)\,dt = \frac{1}{2\pi}\int_0^\rho \kappa(t)dt$. $\square$

The last result in this section relates the curvature of a plane curve to some of its topological properties.

Theorem 1.2.6. *For every periodic curve which is parameterized by arc length without self intersection* $c : I \longrightarrow \mathbb{R}^2$*, we have* $|n_c| = 1$.

To prove this theorem, we need a lemma closely related to Lemma 1.2.3.

Definition 1.2.7. Let $X \subset \mathbb{R}^n$ and $x_0 \in X$. We call X star-shaped with respect to x_0, if for all $x \in X$ and all $t \in [0, 1]$ we have $tx + (1 - t)x_0 \in X$.

Lemma 1.2.8. *Let* $X \subset \mathbb{R}^n$ *be star-shaped with respect to some* $x_0 \in X$ *and let* $e : X \longrightarrow S^1$ *be continuous. Then there exists a continuous curve* $\theta : X \longrightarrow \mathbb{R}$ *with* $e(x) = (\cos\theta(x), \sin\theta(x))$*, uniquely determined by* $\theta(x_0) = \theta_0$.

Proof. Let $x \in X$. As $e_x(t) = e(tx+(1-t)x_0)$ defines a continuous curve $e_x : [0, 1] \longrightarrow S^1$, we get a unique continuous function $\theta_x : [0, 1] \longrightarrow \mathbb{R}$ with $\theta_x(0) = \theta_0$ and $e_x(t) = (\cos\theta_x(t), \sin\theta_x(t))$ by Lemma 1.2.3 (use e_x instead of c'). Define $\theta(x) = \theta_x(1)$, uniquely determined by $\theta(x_0) = \theta_0$ because of Lemma 1.2.3.
We have $e(x) = e_x(1) = (\cos\theta_x(1), \sin\theta_x(1)) = (\cos\theta(x), \sin\theta(x))$.
It remains to show that the function θ is continuous in every $x \in X$. Let $c : [0, 1] \longrightarrow X$ be any curve parameterized by arc length, with $c(0) = x$. The curve $e \circ c : [0, 1] \longrightarrow S^1$ allows a unique continuous lift θ^c with $e \circ c(t) = (\cos\theta^c(t), \sin\theta^c(t))$ and $\theta^c(0) = \theta(x)$. By uniqueness we have $\theta^c(t) = \theta(c(t))$. If y_n is any sequence along $c([0, 1])$, converging to x, there is a sequence $t_n \to 0$ such that $y_n = c(t_n)$. Hence $\theta(y_n) = \theta^c(t_n) \to x$ and θ is continuous. □

Now we can prove Theorem 1.2.6.

Proof. Let $c = (c_1, c_2) : I \longrightarrow \mathbb{R}^2$ be periodic with period ρ and let $x_0 = \max\{c_1(t) \mid t \in \mathbb{R}\}$. The maximum is attained since $c([0, \rho])$ is compact. Let $p = (p_1, p_2) \in c([0, \rho])$ with $p_1 = x_0$. Let $G = \{(x, p_2) \mid x \in \mathbb{R}\}$. There are no points (x, y) in $G \cap c([0, 1])$ with $x > x_0$. After some parametertransformation $t \mapsto \pm(t + t_0)$ we may assume $c(0) = p$ and $c'(0) = (0, 1)$. This transformation does not change $|n_c|$.

Let $X = \{(s,t) \in \mathbb{R}^2 \,|\, 0 \leq s \leq t \leq \rho\}$, this set is star-shaped with respect to $(0,0)$. Let

$$e(s,t) = \begin{cases} \frac{c(t)-c(s)}{||c(t)-c(s)||} & \text{for } s < t,\ (s,t) \neq (0,\rho) \\ c'(s) & \text{for } s = t \\ -c'(0) & \text{for } (s,t) = (0,\rho) \end{cases}.$$

Due to the assumptions on c, this function is well definid and continuous. The previous lemma gives a continuous $\theta : X \longrightarrow \mathbb{R}$ with $e(s,t) = (\cos\theta(s,t), \sin\theta(s,t))$. We have $e(t,t) = c'(t)$, therefore

$$2\pi n_c = \theta(\rho,\rho) - \theta(0,0) = \underbrace{\theta(\rho,\rho) - \theta(0,\rho)}_{=B} + \underbrace{\theta(0,\rho) - \theta(0,0)}_{=A}.$$

As there are no points of $c([0,1])$ in G lying to the right of p and as $(0,1) \perp c'(0)$, we have $(\cos(2\pi k), \sin(2\pi k)) = (1,0) \notin e(0,[0,\rho])$, hence $\theta(0,[0,\rho]) \subset (2\pi k, 2\pi(k+1))$ for some integer k. We compute $A = \theta(0,\rho) - \theta(0,0) = (\frac{3}{2}\pi + 2\pi k) - (\frac{1}{2}\pi + 2\pi k) = \pi$. An analog computation shows $B = \pi$, too.
Together, we have $n_c = 1$. □

2. Surfaces in $\mathbb{R}^3$

2.1. **Basic concepts.**

Definition 2.1.1. A subset $S \subset \mathbb{R}^3$ is a *surface* if for every point $p \in S$ there is an open set $U \subset \mathbb{R}^2$ and an open set $W \subset \mathbb{R}^3$ so that there is a homeomorphism $\varphi : U \longrightarrow S \cap W$. The map φ is called a chart. The collection of all those charts $\varphi : U \longrightarrow W \cap S$ (where $W \subset \mathbb{R}^3$ and $U \subset \mathbb{R}^2$ are open) is an *atlas* for S.

Example.

(1) A *plane*

$$E = \{x_0 + tv_1 + sv_2 \,|\, s,t \in \mathbb{R}, v_1,\ v_2 \in \mathbb{R}^3 \text{ linear independent }\}$$

is a surface with $\varphi(s,t) = x_0 + tv_1 + sv_2$.
A different representation of the plane is given as follows. If $E_0 = \text{span}\{v_1,\ v_2\} \subset \mathbb{R}^3$, then $E = x_0 + E_0 = \{x_0 + v \,|\, v \in E_0\}$ and if $n \in E_0^\perp = \{z \,|\, < z,\, x >= 0 \text{ for all } x \in E_0\}$ is a unit vector, then $E = f^{-1}(0)$, where $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is defined by $f(x_0 + tn + v) := t$ for $v \in E_0$.

(2) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be continuous, then the graph of f,

$$S = \{(x, f(x)) \mid x \in \mathbb{R}^2\} \subset \mathbb{R}^3,$$

is a surface with $\varphi(x) = (x, f(x))$.

(3) The 2-dimensional unit sphere

$$S^2 := \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$$

can be represented as follows. Draw a circle in the (x, y)-plane, parameterized by $\theta \to (\cos\theta, \sin\theta, 0)$ and rotate a point on the circle in orthogonal direction. This can be realized by the map $\varphi : [0, 2\pi] \times [-\pi/2, \pi/2] \longrightarrow S$, defined by

$$\varphi(\theta, \alpha) = (\cos\theta, \cos\alpha, \sin\theta\cos\alpha, \sin\alpha).$$

This map is onto, but it is not injective and its domain of definition is not open. Therefore we restrict φ to the set

$$U := \left\{(\theta, \alpha) \,\middle|\, 0 < \theta < 2\pi,\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right\} \subset \mathbb{R}^2.$$

Indeed, U is open and $\varphi|_U$ is injective, but it does not cover S^2. In fact, its image is S^2 with a half great circle deleted.

Let now

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(3).$$

Then A is a linear isomorphism of $\mathbb{R}^3$ which preserves S^2, the map $\psi := A \circ \varphi : U \longrightarrow \psi(U) \subset S^2$ is a chart and $\varphi(U) \cup \psi(U)$ is S^2 with the northpole and the southpole deleted.

Finally we consider

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \in SO(3).$$

and obtain a chart $\rho = B \circ \varphi : U \longrightarrow \rho(U) \subset S^2$.

The three charts φ, ψ and ρ cover S^2, i.e. $\varphi(U) \cup \psi(U) \cup \rho(U) = S^2$. Hence, S^2 is a surface.

(4) The double cone

$$S := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\} \subset \mathbb{R}^3$$

is not a surface. Namely otherwise there is a chart $\varphi : U \longrightarrow \varphi(U)$ with $0 \in U$, $\varphi(0) = 0$ and $\varphi(U) = S \cap W$, where W is some open ball about 0 in $\mathbb{R}^3$. Then there is a disc $D \subset U$ about 0 in U and points $x, y \in D$ with $\varphi(x) = (x_1, y_1, z_1)$ where $z_1 > 0$, and $\varphi(y) = (x_2, y_2, z_2)$, where $z_2 < 0$. Now choose a curve c in D connecting x to y which does not pass through 0. Then $\varphi \circ c$ is a curve in $W \cup S$ connecting $\varphi(x)$ to $\varphi(y)$, which does not pass through $0 \in \mathbb{R}^3$. This, however, is not possible, since the third coordinate function z is continuous along $\varphi \circ c$, i.e., $\varphi \circ c$ necessarily passes through a point with $z = 0$.

If $\varphi : U \longrightarrow S$ and $\psi : V \longrightarrow S$ are charts and if $\varphi(U) \cap \psi(V) \neq \emptyset$, then

$$\varphi^{-1} \circ \psi\Big|_{\psi^{-1}(\varphi(U)\cap\psi(V))} : \psi^{-1}(\varphi(U) \cap \psi(V)) \longrightarrow U \cap \varphi^{-1}(\psi(U))$$

is a homeomorphism which we refer to as a 'change of charts' or a 'coordinate transformation'.

Definition 2.1.2. A chart $\varphi : U \longrightarrow \varphi(U) \subset S$ is *regular* if the map φ is smooth with differential $d\varphi$ of maximal rank. Here $d\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is a linear map and if $d\varphi$ is of maximal rank, then the rank of $d\varphi$ is 2 everywhere.
A *regular surface* is a surface with an atlas of regular coordinate charts.

Often we only write S for a regular surface, but sometimes we will, more precisely, write the regular surface as the pair $(S, \mathcal{A})$, where $S \subset \mathbb{R}^3$ and $\mathcal{A}$ denotes the regular atlas in consideration.

Example. A plane is smooth, the sphere S^2 is smooth, and a graph of $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is smooth if and only if f is smooth.

As we will see later on, one is often interested in surfaces, whose coordinate transformations are all smooth. The next proposition provides examples of such surfaces.

Proposition 2.1.3. *Let $(S, \mathcal{A})$ be a regular surface with (smooth) coordinate charts $\varphi_i : U_i \longrightarrow S$. Then the transition maps*

$$\varphi_i^{-1} \circ \varphi_j : \varphi_j^{-1}(\varphi_j(U_j) \cap \varphi_i(U_i)) \longrightarrow \varphi_i^{-1}(\varphi_j(U_j) \cap \varphi_i(U_i))$$

are smooth.

Proof. Let $(U_i, \varphi_i), (U_j, \varphi_j) \in \mathcal{A}$ and $p \in \varphi_i(U_i) \cap \varphi_j(U_j) \subset S$ be arbitrary. All we have to prove is that

$$\varphi_i^{-1} \circ \varphi_j : \varphi_j^{-1}(\varphi_j(U_j) \cap \varphi_i(U_i)) \longrightarrow \varphi_i^{-1}(\varphi_j(U_j) \cap \varphi_i(U_i))$$

is smooth.
Let therefore $x_0 := \varphi_i^{-1}(p) \in U_i$. Since $(S, \mathcal{A})$ is regular, the set $T_pS = d\varphi_i(\mathbb{R}^2) \subset \mathbb{R}^3$ is a 2-dimensional subspace of $\mathbb{R}^3$.

Now we fix an $n \in \mathbb{R}^3$ which is orthogonal to $d\varphi_i(\mathbb{R}^2)$ ($n \in T_pS^\perp$), and define the map $\psi_i : U_i \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\psi_i(x,t) \ := \ \varphi_i(x) \ + \ t \cdot n \qquad \text{for all } x \in U_i,\ t \in \mathbb{R}.$$

This map is smooth, and its differential in $(x_0, 0)$ is of maximal rank. The inverse function theorem thus yields that the restriction of ψ_i to some (sufficiently small) neighborhood $V \subset U_i \times \mathbb{R}$ of $(x_0, 0)$ is invertible with smooth inverse $\psi_i^{-1} : \psi_i(V) \longrightarrow V$. Moreover we find $\psi_i^{-1}(\psi_i(V) \cap S) \subset U_i \times \{0\}$. Thus $\psi_i^{-1} \circ \varphi_j|_{\varphi_j^{-1}(\psi_i(V) \cap \varphi_j(U_j))}$ is smooth and our claim follows, since $\psi_i^{-1} \circ \varphi_j = \varphi_i^{-1} \circ \varphi_j$. □

The following proposition provides a method to obtain regular surfaces.

Proposition 2.1.4. *Let $S \subset \mathbb{R}^3$ be such that for each $p \in S$ there exists an open neighborhood W of p in $\mathbb{R}^3$ and a smooth function $f : W \longrightarrow \mathbb{R}$ such that*

$$\begin{aligned} (i) &\quad S \cap W \ = \ \{(x,y,z) \,|\, f(x,y,z) \ = \ 0\} \quad \textit{and} \\ (ii) &\quad d_p f \ \neq \ 0. \end{aligned}$$

Then there exists a regular atlas $\mathcal{A}$ for S, such that $(S, \mathcal{A})$ is a regular surface.

Proof. From the implicit function theorem we deduce that for $p \in S$ and a smooth function $f : W \longrightarrow \mathbb{R}$ with $p \in W$ and $d_pf \neq 0$ there exist an open neighborhood $V \subset W$ of p, an open neighborhood U of 0 in $\mathbb{R}^2$ and a smooth function $g : U \longrightarrow \mathbb{R}$, such that

$$f^{-1}(0) \cap V \ = \ \{(x, z, g(x,y)) \,|\, (x,y) \in U\}.$$

We thus obtain a regular chart $\varphi : U \longrightarrow V$ around p via

$$\varphi(x,y) \ := \ (x, y, g(x,y)) \qquad \forall x, y \in U.$$

□

As we have seen in Proposition 2.1.3, regular surfaces have the property that their coordinate transformations are smooth. As we will explore next, this property is essential in order to define the notion of smooth functions between surfaces.

Definition 2.1.5. Let $(S,\mathcal{A})$ and $(S',\mathcal{A}')$ be regular surfaces. A map $f : S \longrightarrow S'$ is said to be smooth in $p \in S$, if there exist charts $(U,\varphi) \in \mathcal{A}$, $p \in \varphi(U)$ and $(V,\psi) \in \mathcal{A}'$, $f(p) \in \psi(V)$, such that the map

$$\psi^{-1} \circ f \circ \varphi : \ \varphi^{-1}\Big(f^{-1}(\psi(V)) \cap \varphi(U)\Big) \ \longrightarrow \ V$$

is smooth in $\varphi^{-1}(p)$.
The map f is said to be smooth, if f is smooth in all $p \in S$.

This definition seems a priori to depend on the existence of a special chart. The next proposition tells us, that, in fact, any chart is as good as any other in order to decide whether a map between regular surfaces is smooth.

Proposition 2.1.6. *Let $(S,\mathcal{A})$ and $(S',\mathcal{A}')$ be regular surfaces and $f : S \longrightarrow S'$ be a map. Then f is smooth (in $p \in S$) if and only if for all charts $(U,\varphi) \in \mathcal{A}$ with $p \in \varphi(U)$ and $(V,\psi) \in \mathcal{A}'$ with $f(p) \in \psi(S)$ the map*

$$\psi^{-1} \circ f \circ \varphi : \ \varphi^{-1}\Big(f^{-1}(\psi(V)) \cap \varphi(U)\Big) \ \longrightarrow \ V$$

is smooth (in $\varphi^{-1}(p)$).

Proof. We have to prove that for each two pairs of charts $(U_1,\varphi_1),(U_2,\varphi_2) \in \mathcal{A}$ around p and $(V_1,\psi_1),(V_2,\psi_2) \in \mathcal{A}'$ around p' the map $\psi^{-1} \circ f \circ \varphi_1$ is smooth in $\varphi_1^{-1}(p)$ if and only if the map $\psi_2^{-1} \circ f \circ \varphi_2$ is smooth in $\varphi^{-1}(p)$. This follows from the identity

$$\psi_2^{-1} \circ f \circ \varphi_2 \ = \ \psi_2^{-1} \circ \psi_1 \circ \psi_1^{-1} \circ f \circ \varphi_1 \circ \varphi_1^{-1} \circ \varphi_2$$

with Proposition 2.1.3, since the coordinate transformation $\psi_2^{-1} \circ \psi_1$ and $\varphi_1^{-1} \circ \varphi_2$ are smooth. $\square$

Thus, in general the statement about a map $f : S \longrightarrow S'$ between surfaces S and S' to be of class C^k makes sense for surfaces whose coordinate transformations are of class C^k.

2.2. **The Tangent Space, Normals and Orientability.**

Definition 2.2.1. Let $S \subset \mathbb{R}^3$ be a regular surface and let $p \in S$. The tangent space T_pS of S in p is the set of all tangent vectors of smooth curves on S through p, i.e.,

$$T_pS \ := \ \left\{ v \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} \text{there exist } \epsilon > 0 \text{ and a smooth, parameterized curve} \\ c : (-\epsilon,\epsilon) \longrightarrow S \text{ with } c(0) = p \text{ and } \dot{c}(0) = v \end{array} \right\}.$$

From this definition it is á priori not clear that this set is indeed a subspace of $\mathbb{R}^3$. This, however, follows from the following identity.

Proposition 2.2.2. *Let $S \subset \mathbb{R}^3$ be a regular surface, $p \in S$ and $\varphi : U \longrightarrow \varphi(U)$ be a regular chart with $p \in \varphi(U)$. Then*

$$T_pS \;=\; d_o\varphi(\mathbb{R}^2), \qquad \textit{where } o := \varphi^{-1}(p).$$

Proof. We prove the two inclusions seperately.
(1) $d_o\varphi(\mathbb{R}^2) \subset T_pS$.
Let $v \in d_o\varphi(\mathbb{R}^2)$. Then there exists a $w \in \mathbb{R}^2$ with $v = d_o\varphi(w)$. Now define the curve $c(t) := \varphi(o+tw)$. Note that for $\epsilon > 0$ sufficiently small, we have $o + tw \in U$ whenever $|t| < \epsilon$. Moreover, we find

$$\begin{aligned} c(0) \;&=\; \varphi(o) \;=\; p \qquad \text{as well as} \\ \dot{c}(0) \;&=\; \tfrac{d}{dt}\varphi(o+tw)\Big|_{t=0} \;=\; d_o\varphi(w) \;=\; v. \end{aligned}$$

Thus, we deduce $v \in T_pS$.
(2) $T_pS \subset d_o\varphi(\mathbb{R}^2)$.
Let first $\gamma : (-\epsilon, \epsilon) \longrightarrow U$ be smooth with $\gamma(0) = o$. Then the curve $c := \varphi \circ \gamma : (-\epsilon, \epsilon) \longrightarrow \varphi(U) \subset S \subset \mathbb{R}^3$ is smooth and it holds

$$\dot{c}(0) \;=\; (\varphi \circ \gamma)^{\cdot}(0) \;=\; d_o\varphi(\dot{\gamma}(0)) \;\in\; d_o\varphi(\mathbb{R}^2).$$

In the proof of Proposition 2.1.3 we have seen that on a sufficiently small neighborhood W of p in $\mathbb{R}^3$ there is a diffeomorphism $\psi : W \longrightarrow \psi(W)$ with $\psi(W \cap S) = U \times \{0\}$, where $\psi(W)$ is a neighborhood of $(\varphi^{-1}(p), 0)$ in $U \times \mathbb{R}$. Thus, each smooth curve c in S is locally of the form $c = \varphi \circ \gamma$, which completes the proof. $\square$

Now we know how to associate to each point p on a regular surface S a linear space, namely the tangent space T_pS. Thus, as we are used to from real analysis, we can try to associate to each differentiable map $f : S \longrightarrow S'$ between regular surfaces, a linear map in each point $p \in S$; namely the differential $d_pf : T_pS \longrightarrow T_{f(p)}S'$ of f in p.

Definition 2.2.3. Let $S, S' \subset \mathbb{R}^3$ be regular surfaces, let $f : S \longrightarrow S'$ be a smooth map and let $p \in S$. The differential of f in $p \in S$ is the map

$$d_pf : T_pS \longrightarrow T_{f(p)}S'$$

which is defined as follows. To each $v \in T_pS$ choose some smooth, parameterized curve $c : (-\epsilon, \epsilon) \longrightarrow S$ with $c(0) = p$ and $\dot{c}(0) = v$. Then define

$$d_pf(v) \;:=\; \frac{d}{dt}(f \circ c)|_{t=0} \in T_{f(p)}S'.$$

Proposition 2.2.4. *The differential in Definition 2.2.3 is well defined, i.e., $d_pf(v)$ only depends on v and not on the particular choice of smooth curve c. Moreover, d_pf is linear.*

Proof. Let $\varphi : U \longrightarrow \varphi(U)$ be a chart around p and $\psi : V \longrightarrow \psi(V)$ be a chart around $f(p)$. W.l.o.g. we assume that $f(S \cap \varphi(U)) \subset \psi(V)$. Then we define

$$\tilde{f} \;:=\; \psi^{-1} \circ f \circ \varphi : U \longrightarrow V$$

and set $o := \varphi^{-1}(p) \in U$.
Let now $c : (-\epsilon, \epsilon) \longrightarrow S$ with $c(0) = p$ and $\dot{c}'(0) = v$ w.l.o.g. be given such that $c((-\epsilon, \epsilon)) \subset \varphi(U)$ and $\gamma := \varphi^{-1} \circ c : (-\epsilon, \epsilon) \longrightarrow U$ defines a smooth, parameterized curve. Then

$$d_o\varphi(\acute{\gamma}(0)) \;=\; \frac{d}{dt}(\varphi \circ \gamma)\Big|_{t=0} \;=\; \acute{c}(0) \;=\; v,$$

and we obtain

$$\begin{aligned} d_pf(v) \;&=\; \frac{d}{dt}(f \circ c)\Big|_{t=0} \\ &=\; \frac{d}{dt}(f \circ \varphi \circ \gamma)\Big|_{t=0} \\ &=\; \frac{d}{dt}(\psi \circ \tilde{f} \circ \gamma)\Big|_{t=0} \\ &=\; d_{\gamma(o)}(\psi \circ \tilde{f})(\dot{\gamma}(0)) \;=\; d_o(\psi \circ \tilde{f}) \circ [d_o\varphi]^{-1}(v). \end{aligned}$$

Hence, $d_pf(v)$ indeed only depends on v. Moreover, being a composition of linear maps, d_pf is linear itself. □

Definition 2.2.5. Let $S \subset \mathbb{R}^3$ be a regular surface. A unit normal n of S in $p \in S$ is a unit vector $n \in \mathbb{R}^3$, which is orthogonal to T_pS.

Facts:

- In each point $p \in S$ there are exactly two such unit normals.
- Each chart $\varphi : U \longrightarrow S$ determines such a normal via

$$n_p \;:=\; \frac{d_o\varphi(e_1) \times d_o\varphi(e_2)}{||d_o\varphi(e_1) \times d_o\varphi(e_2)||}, \qquad o := \varphi^{-1}(p).$$

- Let $\tilde{\varphi} : \tilde{U} \longrightarrow S$ be another chart around p and let $\Phi := \varphi^{-1} \circ \tilde{\varphi}$ denote the change of charts and $\tilde{o} := \tilde{\varphi}^{-1}(p)$, then

$$d_{\tilde{o}}\tilde{\varphi}(\tilde{e}_1) \times d_{\tilde{o}}\tilde{\varphi}(\tilde{e}_2) \;=\; \det \begin{pmatrix} \frac{\partial \Phi^1}{\partial \tilde{u}}\Big|_{\tilde{o}} & \frac{\partial \Phi^1}{\partial \tilde{v}}\Big|_{\tilde{o}} \\ \frac{\partial \Phi^2}{\partial \tilde{u}}\Big|_{\tilde{o}} & \frac{\partial \Phi^2}{\partial \tilde{v}}\Big|_{\tilde{o}} \end{pmatrix} \Big[d_o\varphi(e_1) \times d_o\varphi(e_2)\Big].$$

Thus, φ and $\tilde{\varphi}$ define the same unit normals, if and only if the Jacobian of Φ is positive.

Definition 2.2.6. An orientable surface $S \subset \mathbb{R}^3$ is a regular surface endowed with an atlas all coordinate transformations of which have positive Jacobian.

The following Lemma is a simple consequence of our considerations above.

Lemma 2.2.7. *An orientable surface has a unit normal field which is canonically determined by its charts.*

Fact: A regular surface S is orientable if and only if S admits a smooth (continuous) unit normal field.

Example. (1) Affine planes are orientable. (2) $S^2 = \{x \in \mathbb{R}^3 \,|\, ||x|| = 1\}$ is orientable with unit normal field $n(x) = x$. (3) The torus of rotation, given by

$$\begin{aligned} T^2 \;=\; \Big\{ \;\; & \Big((1 + r\cos\theta)\cos\alpha, (1 + r\cos\theta)\sin\alpha, r\sin\theta\Big) \;\Big| \\ & r < 1\,\text{fixed},\; \alpha, \theta \in [0.2\pi]\Big\} \end{aligned}$$

is orientable. (4) The Möbiusstrip is not orientable.

2.3. **The First Fundamental Form, Length and Area.** Let $S \subset \mathbb{R}^3$ be a regular surface. Then for all $p \in S$ the tangent space T_pS of S in p is a two-dimensional subspace of $\mathbb{R}^3$. Thus, in each point $p \in S$, we can restrict the scalar product $< \cdot, \cdot >$ on $\mathbb{R}^3$ to T_pS.

Definition 2.3.1. The map, which associates to each $p \in S$, the restriction

$$I_p \;:=\; g_p \;:= < \cdot, \cdot > |_{T_pS \times T_pS}$$

is called the first fundamental form of S.

Let (U, φ) be a chart around $p \in \varphi(U)$. Then we can express g_p in T_pS w.r.t. the basis $\{d_o\varphi(e_1), d_o\varphi(e_2)\}$ of T_pS, where $o := \varphi^{-1}(p)$. The corresponding martix coefficients are given by

$$g_{ij}(o) \;:=\; g_p(d_o\varphi(e_i), d_o\varphi(e_j)) \;= < \frac{\partial \varphi}{\partial u^i}\Big|_o, \frac{\partial \varphi}{\partial u^j}\Big|_o > .$$

This matrix is symmetric and positive definite. Moreover, we see that the coefficients g_{ij} vary smoothly in U. The following notation is often used in the literature:

$$E := g_{11}, \qquad F := g_{12} = g_{21}, \qquad G := g_{22}.$$

Definition 2.3.2. Let $S \subset \mathbb{R}^3$ be a regular surface and let $c : I \longrightarrow S$ be a smooth curve. Then the length $l(c)$ of c is given by

$$l(c) \ := \ \int\limits_I \sqrt{g_{c(t)}(\dot{c}(t), \dot{c}(t))}dt \ = \ \int\limits_I ||\dot{c}(t)||dt.$$

Fact: (compare your homework)
Let S be a connected, regular surface. then the function $d_S : S \times S \longrightarrow \mathbb{R}_0^+$, given through

$$d_S(p,q) := \inf \left\{ l(c) \;\middle|\; \begin{array}{l} c : [a,b] \longrightarrow S \text{ continuous und piecewise} \\ \text{differentiable with } c(a) = p, \text{ und } c(b) = q \end{array} \right\},$$

defines a metric on S. Here $l(c)$ is just the sum of the length of the smooth pieces of c.

Definition 2.3.3. A smooth curve $c : I \longrightarrow S$ on a regular surface S is said to be a *minimal geodesic* on S, if for all $s, t \in I$, $s < t$,

$$l(c|_{[s,t]}) \ = \ t - s \ = \ d_s(c(s), c(t)).$$

Lemma 2.3.4. *Let $S \subset \mathbb{R}^3$ be a regular surface such that there exist $\epsilon > 0$, $x_0 \in S$ and $v \in \mathbb{R}^3$, $||v|| = 1$ with $c(t) := x_0 + tv \in S$ for $t \in (-\epsilon, \epsilon)$. Then c is a minimal geodesic on S.*

Proof. This directly follows from $d_S(x,y) \geq d(x,y) := ||x - y||$ for all $x, y \in S$. □

Example. (1) The minimal geodesics on the affine plane

$$E \ = \ \{x_0 + tv_1 + sv_2 \,|\, s, t \in \mathbb{R}, v_1, \ v_2 \in \mathbb{R}^3 \text{ linear independent } \}$$

are precisely the arc length parameterized pieces of straight lines.
(2) The minimal geodesics on

$$S^2 \ := \ \{x \in \mathbb{R}^3 \,|\, ||x|| = 1\}$$

are precisely the arc length parameterized pieces of the great halfcircles on S^2.
In order to verify this, we are going to argue in terms of metric geometry only. First of all note that due to the compactness of S^2 the existence of geodesics between any two points follows by the very definition of d_{S^2}.

- The orthogonal group $O(3)$ acts as a group of isometries (distance preserving bijections) on $(\mathbb{R}^3, ||\cdot||)$ and preserves S^2. It follows that $O(3)$ also acts as group of isometries on (S^2, d_{S^2}).
- For $x \in S^2$ let $O_x := \{A \in O(3) \,|\, Ax = x\}$. Then O_x preserves the sections of S^2 with affine planes orthogonal to x.
- These sections cut S^2 in a family of circles K and the degenerate circles $\{x\}$ and $\{-x\}$.
- The group O_x acts transitively on each such circle K, i.e., for all $y, z \in K$ there exists $A \in O_x$ with $Ay = z$.
- Since O_x acts as a group of isometries on (S^2, d_{S^2}), all points on K have the same distance to x.
- For each such K there exists a unique $c = c(K) \in (-1, 1)$ with $< x, y >= c$ for all $y \in K$, where $< \cdot, \cdot >$ denotes the scalar product on $\mathbb{R}^3$. In particular, the distance $d_{S^2}(y, z)$ of any two point $y, z \in S^2$ only depends on $< y, z >$.
- Let $x \in S^2$ and $y \in S^2 \setminus \{x, -x\}$. Then there is a unique two-dimensional subspace P of $\mathbb{R}^3$ with $x, y \in P$. Let $z \in P \cap S^2$ be such that $< x, z >=< y, z >> 0$, i.e., z is exactly the midpoint of x and y in $P \cap S^2$.
- Let H_x and H_y be the affine planes orthogonal to x and y which contain z. The circles $K_x := H_x \cap S^2$ and $K_y := H_y \cap S^2$ are 'distance spheres' of x and y in (S^2, d_{S^2}). Moreover, they precisely intersect in $\{z\}$.
- It follows that a minimal geodesic connecting x and y can intersect both, K_x and K_y, only in $\{z\}$. Since the restriction of a minimal geodesic is a minimal geodesic, the same arguments as above yield that the image of the minimal geodesic between x and y has to remain in P. The claim follows.

Let S be a regular surface in $\mathbb{R}^3$ and $\phi\colon U \subset \mathbb{R}^2 \to \phi(U) \subset S$ a regular chart. Then for every $p \in U$, the image $d\phi_p(\mathbb{R}^2) = T_{\phi_{(p)}}S$ of $\mathbb{R}^2$ under the differential $d\phi_p$ of ϕ at p is a 2-dimensional subspace of $\mathbb{R}^3$. The euclidean inner product $\langle , \rangle$ defines the *first fundamental form* of S.

Choose an orthonormal basis e_1, e_2 of $\mathbb{R}^2$ and X_1, X_2 of $T_{\phi(p)}S$. With respect to this basis, the linear map $d\phi_p$ can be represented by a $(2, 2)$ matrix $A(p)$ whose determinant $\det A(p)$ is defined. If Y_1, Y_2 is another ON-basis of $T_{\phi(p)}S$ then the matrix $B(p)$ which corresponds to $d\phi_p$ with respect to this new basis is of the form $B(p) = O \circ A(p)$ where $O \in S0(2)$. In particular, we have $\det B(p) = \det A(p)$.

Definition 2.3.5. For a continuous function $f\colon S \to \mathbb{R}$ define

$$\int_{\phi(U)} f d\mathrm{vol} = \int_U f \circ \phi(p) |\mathrm{det} A(p)| dp.$$

The *area* of U equals $\mathrm{vol}(U) = \int_{\phi(U)} d\mathrm{vol} > 0$

Lemma 2.3.6. $\int_{\phi(U)} f \, d\mathrm{vol}$ is independent of the choice of ϕ.

Proof. Let $\psi\colon V \to \psi(V) = \phi(U)$ be another regular chart. Then we can write $\psi = \phi \circ (\phi^{-1} \circ \psi)$ where $\phi^{-1} \circ \psi\colon V \to U$ is a diffeomorphism. Then the chain rule shows

$$d\psi = d\phi \circ d(\phi^{-1} \circ \psi)$$

and for any orthonormal basis e_1, e_2 of $\mathbb{R}^2$, X_1, X_2 of $T_{\psi(U)}S$ the matrix $B(u)$ of the linear map $d\psi_u$ can be written in the form $B(u) = A(\phi^{-1}\psi(u)) \circ C(u)$ where $C(u)$ is the matrix of $d(\phi^{-1}\psi)_u$ with respect to the standard basis of $\mathbb{R}^2$. The tansformation rule for integrals then shows that

$$\begin{aligned} &\int_v f \circ \psi(u) |\det B(u)| du \\ = &\int_V f \circ \phi(\phi^{-1}\psi(u)) \, |\mathrm{det} A(\phi^{-1}\Psi(u))| |\mathrm{det} C(u)| du \\ = &\int_U f \circ \phi(p) |\mathrm{det} A(p)| dp. \end{aligned}$$

This shows the lemma. ☐

Definition 2.3.7. Let $S \subset \mathbb{R}^3$ be a regular surface. The *area* of S is defined as follows:
Cover S with charts $\phi_i\colon U_i \to S$. Define inductively $\mathrm{area}(S) = \sum_{i=1}^{\infty} \int_{\phi_i(U_i) - \bigcup_{j=1}^{i-1} \phi_j(U_j)} d\,vol$.
By the above, this is independent of the charts.

Example. 1) The area of $\mathbb{R}^2 \subset \mathbb{R}^3$ is infinite.

2) Let $U = (0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}$ and define $F : U \to \mathbb{R}^3$,

$$F(\phi, \theta) = (\cos\phi \cos\theta, \sin\phi \cos\theta, \sin\theta).$$

The map F is a smooth chart whose image is the complement in S^2 of two half-circles. Therefore for the purpose of calculation of the area, the complement of the image can be neglected. We have

$$\begin{aligned} dF(\phi,\theta)(e_1) &= (-\sin\phi \cos\theta, \cos\phi \cos\theta, 0) \\ dF(\phi,\theta)(e_2) &= (-\cos\phi \sin\theta, -\sin\phi \sin\theta, \cos\theta) \end{aligned}$$

and hence

$$|dF_{(\phi,\theta)}(e_1)| = \cos\theta, < dF_{(\phi,\theta)}(e_1),\, dF_{(\phi,\theta)}(e_2) >= 0, |dF_{(\phi,\theta)}(e_2)| = 1.$$

Thus the area of the 2-sphere equals

$$\mathrm{vol}(S^2) = \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \, d\theta \, d\phi = 4\pi.$$

Another interpretation: Let

$$g_{11} =< dF(e_1), dF(e_1) >, g_{12} =< dF(e_1), dF(e_2) >= g_{21},$$
$$g_{22} =< dF(e_2), dF(e_2) >;$$

then

$$\det(dF^t \circ dF) = det \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{pmatrix} = det\,(g_{ij})$$

and for every function f on $F(U)$ we have

$$\int_{F(U)} f \, d\,vol = \int_U (f \circ F)\,(u) \sqrt{det\, g_{ij}}\; du.$$

3. Geometry of space curves

3.1. **Total curvature and convex sets.**

Definition 3.1.1. A parameterized curve $c\colon I \to \mathbb{R}^3$ is a *space curve.* The curve is *regular* if c is smooth and $c'(t) \neq 0$ for all t, and *parameterized by the arc length* if $\|c'(t)\| \equiv 1$ for all t. Recall: If $c\colon I \to \mathbb{R}^2$ is a regular plane curve parameterized by arc length then $c''(t) = \kappa(t)\, n(t)$ where $n(t) \perp c'(t)$ and $det\,(c'(t), n(t)) = 1$ and where $\kappa(t)$ is the *curvature* of c in t.

Now: It is not possible to define the curvature for space curves in this way since there is no well defined normal to the tangent of c. But we have $c''(t) \perp c'(t)$ so that $c''(t) = \kappa(t)\, n(t)$ where $n(t) = \frac{c''(t)}{\|c''(t)\|}$ if $c''(t) \neq 0$ and $\kappa(t) \geq 0$ is the *curvature* of $c \in t$, i. e. $\kappa(t) = \|c''(t)\|$.

Example.

i) $\kappa(t) \equiv 0$ if and only if c is a straight line segment (since then $c''(t) = 0$ for all t).
ii) $c(t) = (\sin t, \cos t, 0)$ has curvature 1 everywhere, since $c'(t) = (\cos t, -\sin t, 0)$ and $c''(t) = (-\sin t, -\cos t, 0)$.

iii) For the spiral $c(t) = (\cos(t/\sqrt{2}), \sin(t/\sqrt{2}),\, t/\sqrt{2})$ we have $c'(t) = (-\frac{1}{\sqrt{2}}\ \sin(t/\sqrt{2}),\ \frac{1}{\sqrt{2}}\cos(t/\sqrt{2}), \frac{1}{\sqrt{2}})$ as well as $c''(t) = (-\frac{1}{2}\cos(t/\sqrt{2}),\, \frac{1}{2}\sin(t/\sqrt{2}),\, 0)$. Hence, $\kappa(t) \equiv \frac{1}{2}$ is constant.

Definition 3.1.2. Let $c\colon [0,\ell] \to \mathbb{R}^3$ be a closed regular space-curve; then $\int_0^\ell \kappa(t)dt = \kappa(c)$ is the *total curvature* of c.

Example. The curve $c(t) = (\sin t, \cos t, 0)$ has total curvature $\kappa(c) = 2\pi$. A *simply closed plain* curve c has total curvature $\kappa(c) \geq 2\pi$ with equality if and only if the curvature of c vanishes *nowhere*, i. e. if c is *convex*. *Here:* A regular simple closed curve c is *convex* if the following holds:

Let $t \in I$ and L be the line trough $c(t)$ spanned by $c'(t)$. Performing a linear tranformation and a translation, if necessary, we may assume that $c(t) = 0,\ L = \{(s,0) \mid s \in \mathbb{R}\}$, so we can write $c(t) = (c_1(t), c_2(t))$ with $c'(t) = (1,0)$ such that $c''(t) = (0,\alpha)$ for some $\alpha > 0$ since c is *convex* and therefore near t, c lies in the closed half-space $\{(*,y) \mid y \geq 0\}$. Now if the function $t \to d(c(t),\, L) = c_2(t)$ has a critical point at t_0 then $c_2'(t) = 0$ and the same argument as above shows that this point is a strict local maximum or a strict local minimum. It follows that the function has *precisely* two critical points, one is the maximum and the other is the minimum.

Definition 3.1.3. A subset A of $\mathbb{R}^n$ is *convex* if for all $x, y \in A$ the line segment $\{sx + (1-s)\,y \mid s \in [0,1]\}$ is contained in A.

Example.

1) For every linear functional $\alpha\colon \mathbb{R}^n \to \mathbb{R}$ the half-space $\{\alpha \geq 0\} \subset \mathbb{R}^n$ is convex.
2) For a family $A_i \subset \mathbb{R}^n$ of convex sets the intersection $\bigcap_i A_i$ is convex.
3) A *ball* $\{x \in \mathbb{R}^n \mid \|x - b\| \leq r\}$, $b \in \mathbb{R}^n$, is convex.

Proposition 3.1.4. *Let $c\colon [0,T] \to \mathbb{R}^2$ be a simple closed smooth curve with curvature $\kappa > 0$, then c bounds a convex disc in $\mathbb{R}^2$*

Proof. Let $c\colon [0,T] \to \mathbb{R}^2$ be a simple closed curve, parameterized by arc length with $\kappa > 0$. For $t \in [0,T]$ let $H(t) \subset \mathbb{R}^2$ be the closed half-plane in $\mathbb{R}^2$ bounded by the tangent line $\{c(t) + sc'(t) \mid s \in \mathbb{R}\} = L(t)$ which contains $c(t-\varepsilon,\, t+\varepsilon)$ for some $\varepsilon > 0$. $\square$

Claim: $c[0,T] \subset H(t)$ for all t. Namely, consider the function $\varphi(s) = \mathrm{dist}(c(s), L(t))^2$, then, if $n(t)$ is the oriented normal for $c'(t)$, we have

$$\varphi(s) = (\langle c(s), n(t)\rangle - \langle c(t), n(t)\rangle)^2$$

and φ is differentiable. If $s_0 \in [0,T]$ is a *critical point* for φ then

$$\varphi'(s_0) = 0 \Leftrightarrow 2\langle c'(s_0), n(t)\rangle (\langle c(s_0), n(t)\rangle - \langle c(t), n(t)\rangle) = 0$$

and therefore *either* $\langle c'(s_0), n(t)\rangle = 0$, i. e. $c'(s_0) = \pm c'(t)$, or $c'(s_0) \in L(t)$. But there is *precisely* one point $s_0 \neq t$ such that $c'(s_0) = \pm c'(t)$ and consequently φ has a *unique* maximum and $c[0,T] \subset H(t)$. Then $c[0,T] \subset \bigcap_t H(t) = D$, and the boundary of $H(t)$ equals $c[0,T]$ since $c(t) \in \partial H(t)$.

Lemma 3.1.5. A smooth closed curve $c\colon S^1 \to \mathbb{R}^2$ bounds a convex disc if and only if for each $b \in S^1$ the function $t \to \langle c(t), b\rangle$ assumes a local maximum in a unique connected subset of S^1. In particular, if a maximum is *non-degenerate* (i. e. if the second derivative does not vanish) then it is unique.

3.2. **The Fary Milnor Theorem.** *Recall:* If $c\colon [0,T] \to \mathbb{R}^3$ is smooth and parameterized by arc length then $\int_0^T \kappa(t)\, dt$ is the *length* of $c'\colon t \to S^2$.

Definition 3.2.1. Let $c\colon [0,T] \to \mathbb{R}^n$ be any smooth curve of length $\ell(c) = \int_0^T \|c'(t)\| dt$. A C^0-approximation of c by curves γ_k is given by a family $\{\gamma_k\colon [0,T] \to \mathbb{R}^n\}$ of curves such that $\sup ||\gamma_k(t) - c(t)|| \xrightarrow{k\to\infty}$ for all $t \in [0,T]$.

Definition 3.2.2. If $\kappa(t) \neq 0$ then $n(t) = c''/\|c''(t)\|$ is a *normal* vector.

Proposition 3.2.3. *Let $c\colon [0,T] \to \mathbb{R}^n$ be continuously differentiable and let $\{\gamma_k\}$ be a C^0-approximation of c. Then* $\lim_{k\to\infty} \inf l(\gamma_k) = l(c)$.

Proof. The curve c' is continuous. Therefore, for $\varepsilon > 0$ there exists $\rho > 0$ such that

$$|s-t| < \rho \qquad \text{implies} \qquad \|c'(s) - c'(t)\| < \varepsilon$$

as well as

$$\|c(t) - c(s)\| \geq |t-s|\|c'(s)\| - \varepsilon\,|t-s|.$$

Choose a partition $0 = t_0 < \cdots < t_k = T$ such that $|t_i - t_{i-1}| < \rho$ for all $i = 1, ..., k$. Let $p\colon [0,T] \to \mathbb{R}^n$ be the polygon with vertices $p(t_i) = c(t_i)$; then $l(p) \leq l(c)$ and

$$\begin{aligned} l(p) &= \sum_{i=1}^{k} \|c(t_i) - c(t_{i-1})\| \geq \sum_{i=1}^{k} (t_i - t_{i-1})\|c'(t_{i-1})\| - \varepsilon T \\ &\geq \int_0^T \|c'(s)\| ds - 2\varepsilon T. \end{aligned}$$

On the other hand, if $\{\gamma_k\}$ is a C^0- approximation of c then, for l sufficiently large, we have $\|\gamma_k(t_i) - c(t_i)\| < \varepsilon T/2k$. It follows that

$$l(\gamma_l) \geq \sum_{i=1}^{k} \|\gamma_l(t_i) - \gamma_l(t_{i-1})\| \geq \sum_{i=1}^{k} \|c(t_i) - c(t_{i-1})\| - \varepsilon T$$

and hence

$$\lim_{l\to\infty} \inf\, l(\gamma_l) \geq l(c) - 3\varepsilon T.$$

Since $\varepsilon > 0$ was arbitrary, the lemma follows. $\square$

Now let $c\colon [0,T] \to \mathbb{R}^3$ be a smooth curve parameterized by arc length. Then c' is a smooth curve on S^2 with length $\int_0^T \kappa(t)\,dt$. For a partition $\mathcal{P} = 0 = t_0 < \cdots < t_k = T$ let $p = p(\mathcal{P})$ be the polygon with the vertices $p(t_i) = c(t_i)$. Then for each i, the left derivative $\lim_{t \searrow t_i^-} p'(t) = \alpha(t_i)$ is defined. We use the vertices $\alpha(t_i)/\|\alpha(t_i)\| \in S^2$ to define a curve $c(\mathcal{P})\colon [0,T] \to S^2$ as follows. The restriction of $c(\mathcal{P})$ to $[t_{i-1}, t_i]$ is the great circle arc connecting $\alpha(t_{i-1})/\,\|\alpha(t_{i-1})\|$ to $\alpha(t_i)/\,\|\alpha(t_i)\|$.

The *width* of a partition is the maximum of the distances $|t_i - t_{i-1}|$. We have

Lemma 3.2.4. Let $c\colon [0,T] \to \mathbb{R}^n$ be a smooth curve parameterized by arc length and let $\mathcal{P} = \{\mathcal{P}_k\}$ be a family of partitions of $[0,T]$ whose width converges to 0 as $k \to 0$. Then $c(\mathcal{P}_i)$ is a C^0-approximation of c'.

Proof. Since c' is continuous, for each $\varepsilon > 0$ there is some $\delta > 0$ such that $\|c'(s) - c'(t)\| < \varepsilon$ whenever $|s - t| < \delta$ and that, moreover, $\|c(t) - c(s) - (t-s)\, c'(s)\| < \varepsilon|t-s|$. Now, if $\mathcal{P}$ is a partition of $[0,T]$ and if $|t_i - t_{i-1}| < \delta$, then the tangent $\alpha(t_{i-1})$ of the arc $p[t_{i-1},\, t_i]$ connecting $c(t_{i-1})$ to $c(t_i)$ equals $\frac{c(t_i) - c(t_{i-1})}{t_i - t_{i-1}}$ and hence $\|\alpha(t_{i-1}) - c'(t_{i-1})\| < \varepsilon$. Thus

$$\begin{aligned} \left\|\frac{\alpha(t_{i-1})}{\|\alpha(t_{i-1})\|} - c'(t_{i-1})\right\| &\le \left\|\frac{\alpha(t_{i-1})}{\|\alpha(t_{i-1})\|} - \alpha(t_{i-1})\right\| + \varepsilon \\ &= \Big|\|\alpha(t_{i-1})\| - 1\Big| + \varepsilon \;<\; 2\varepsilon. \end{aligned}$$

Now we also have $\|c'(t) - c'(t_{i-1})\| < \varepsilon$ for every $t \in [t_{i-1}, t_i]$. In particular we have

$$\|c'(t_i) - c'(t_{i-1})\| < \varepsilon$$

and hence the distance between $\alpha(t_{i-1})/\|\alpha(t_{i-1})\|$ and $\alpha(t_i)/\|\alpha(t_i)\|$ does not exceed 5ε. Then the distance between $p(t)$ and $\alpha(t_{i-1})/\|\alpha(t_{i-1})\|$ for all $t \in [t_{i-1}, t_i]$ is bounded from above by a multiple of ε. Since $\varepsilon > 0$ was arbitrary, the lemma follows. $\square$

The same procedure: Approximate c' by spherical polygons p_k whose vertices are points on c', then obtain that $l(p_k) \to l(c') = \int \kappa$.

Now for $v \in S^2$ define $f_v(t) = \langle c(t),\, v\rangle$ and

$$m(c,\, v) := \sharp\Big\{\text{local maxima in } [0,T] \text{ of } t \to \langle c(t),\, v\rangle = f_v(t)\Big\} \in \mathbb{N}\cup\{\infty\}.$$

Let $\kappa(c)$ be the total curvature of c, where, if c is a polygon, this is defined as the sum of the angles at the breakpoints between the tangents of the line segments. We have

Proposition 3.2.5. *Let $c\colon [0,T] \to \mathbb{R}^3$ be a closed polygon; then*

$$\frac{1}{area\,(S^2)} \int_{S^2} m(c,\, v)\, d\,vol(v) \;=\; \frac{\kappa(c)}{2\pi}.$$

Proof. Let $c\colon [0,T] \to \mathbb{R}^3$ be a closed polygon parameterized by arc length with vertices $c(t_i)$, $0 = t_0 < \cdots < t_k \,=\, T$. Let $v \in S^2$, $S_v = S^2 \cap v^\perp \,=\, \{x \in S^2 | \langle x, v\rangle \,=\, 0\}$. For each $v \in S^2$ which is *not* orthogonal to one of the $c'(t_i)^+$, a maximum or minimum for $\langle c(t),\, v\rangle$ is a vertex $c(t_i)$; namely for $s \in [t_{i-1}, t_i]$ we have

$$c(s) \;=\; c(t_{i-1}) \;+ s\, c'(t_{i-1})^+$$

and hence

$$\frac{d}{ds}\langle c(s),\, v\rangle|_{s=s_0} \;=\; \langle c'(t_{i-1})^+,\, v\rangle.$$

Moreover, $c(t_i)$ is a maximum or minimum if and only if the sign of $\langle v,\, c'(t_{i-1})^+\rangle$ and $\langle v,\, c'(t_i)^+\rangle$ are distinct. Therefore the spherical great circle arc α_i connecting $c'(t_{i-1})^+$ to $c'(t_i)^+$ crosses S_v.

Now let $A_i = \{v \in S^2 \alpha_i \text{ crosses } S_v\}$. Then $A_i = -A_i$ and the points of A_i which are contained in the hemisphere determined by the mid-point of α_i are contained in a spherical shell of width α_i so that the total volume $= \frac{2\alpha_i}{2\pi}$ $(= \frac{\alpha_i}{\pi})$-th-times $\operatorname{vol} S^2$.

$$\operatorname{vol}\Big\{v \mid t \to \langle c(t), v\rangle \text{ has a local extremum at } c(t_i)\Big\} \;=\; \frac{\alpha_i}{\pi}\operatorname{vol}(S^2),$$

where α_i is the angle between $c'(t_{i-1})^+$ and $c'(t_i)^+$.

Therefore,

$$\int_{S^2} (m(c,v) \;+\; m(c,-v))\, d\,vol = \frac{vol\, S^2}{\pi} \sum_j \alpha_j \,=\, vol\,(S^2)\,\kappa(c)/\pi$$

and hence

$$\int_{S^2} m(c,v)\, d\,vol \;=\; \frac{vol\, S^2}{2\pi}\,\kappa(c)$$

$\square$

Remember: A piecewise smooth curve is a continuous mapping $c\colon [0,T] \to \mathbb{R}^3$ with subdivision $0 = t_0 < \cdots < t_k = T$ such that $c|_{(t_{i-1},t_i)}$ is smooth. If c is parameterized by arc length then $\kappa(c) = \int_0^T \|c''(t)\|\,dt + \sum_{i=1}^k \alpha_i$, where α_i denotes the angle between the left and the right tangent at t_i.

Lemma 3.2.6. Let $c\colon [0,T] \to \mathbb{R}^3$ be a piecewise smooth closed curve parameterized by arc length and $p\colon [0,T] \to \mathbb{R}^3$ be a closed polygon with vertices $p(t_i) = c(t_i)$; then $\kappa(p) \le \kappa(c)$.

Proof. The claim is clear if p is the constant polygon, so assume that the claim holds for all polygons inscribed in all piecewise smooth curves with at most k vertices. Let p be any closed polygon with vertices $p(t_i)$, $0 = t_0 < \cdots < t_k = T$ and let $\tilde{p}$ be the polygon which is obtained from p by adding an arbitrary vertex $\tilde{p}\,(\tau)$, $\tau \in (t_1, t_2)$. We claim that $\kappa(\tilde{p}) \ge \kappa(p)$ with equality if and only if either

$\alpha)$ $p(t_1)$, $\tilde{p}(\tau)$, $p(t_2)$ are contained in a line or
$\beta)$ $p(t_0)$, $p(t_1)$, $\tilde{p}(\tau), p(t_2), p(t_3)$ are contained in a plane.

Namely, let β_1 and β_2 denote the angles of $\tilde{p}(\tau) - p(t_1)$ with $p(t_1) - p(t_0)$ and $p(t_2) - p(t_1)$, respectively. In the same way let β_3 and β_4 denote the angles of $p(t_2) - \tilde{p}(\tau)$ with $p(t_2) - p(t_1)$ and $p(t_3) - p(t_2)$, respectively. If $\alpha)$ is satisfied, then equality holds. Otherwise the points $p(t_1)$, $\tilde{p}(\tau)$, $p(t_2)$ span a plane. The angles satisfy $\alpha_1 \le \beta_1 + \beta_2$, with equality only if $p(t_0)$, $p(t_1)$, $\tilde{p}(\tau)$ and $p(t_2)$ lie in a plane. Similarly $\alpha_2 \le \beta_3 + \beta_4$ with equality only if $p(t_1)$, $\tilde{p}(\tau)$, $p(t_2)$ and $p(t_3)$ lie in a plane. With γ being the angle of $\tilde{p}(\tau) - p(t_1)$ with $p(t_2) - \tilde{p}(\tau)$, we find $\pi - \gamma = \pi - \beta_2 - \beta_3$ so that $\gamma = \beta_2 + \beta_3$. This shows the claim and the proof of the lemma follows by induction. □

Korollar 3.2.7. Let $c\colon [0,T] \to \mathbb{R}^3$ be a simple closed space-curve. Then $\kappa(c) \ge 2\pi$ with equality if and only if c is a convex plane curve.

Proof. The above argument shows that the total curvature is at least 2π. The equality is only possible if c is planar. □

Korollar 3.2.8. Define the *crookedness* $m(c)$ of a closed smooth curve c as $m(c) := \min\{m\,(c,v) | v \in S^1\}$; then $m(c) = 1$ if $\kappa(c) < 4\pi$.

Definition 3.2.9. An *isotopy* of $\mathbb{R}^3$ is a continuous map $\Phi\colon [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$ such that for all $t \in [0,1], \Phi(t,\,\cdot)\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism. Two simple closed embedded curves $c_0,\, c_1\colon S^1 \to \mathbb{R}^3$ are *isotopic* if there exists an isotopy $\Phi\colon [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$ with $\Phi(1,\, c_0(S^1)) = c_1(S^1)$.

Lemma 3.2.10. Isotopy is an equivalence relation for simple closed curves $c\colon S^1 \to \mathbb{R}^3$

Proof. Clear! □

Definition 3.2.11. A curve $c_0\colon S^1 \to \mathbb{R}^3$ is *unknotted* if c_0 is isotopic to the plane curve $c_1(t) = (\cos 2\pi\, t,\, \sin 2\pi\, t,\, 1)$

Theorem 3.2.12. *Let $c\colon [0,T] \to \mathbb{R}^3$ be a simple closed curve with $\kappa(c) < 4\pi$; then c is unknotted.*

Proof. Let $c\colon [0,1] \to \mathbb{R}^3$ be regular and $\kappa(c) < 4\pi$.

(1) Approximate c by inscribed polygons p such that the resulting curve is simple, closed and isotopic to c; $\kappa(p) < 4\pi$ *Know:* There exists $v \in S^2$ such that $f_v\, t \to \langle p(t),\, v\rangle$ has precisely one maximum and one minimum. Thus, for each $s \in (\tau_{min}, \tau_{max})$, have *precisely* two insection points with $E_s = \{\langle v, \cdot\rangle = s\}$.

□