University Bonn — WS 2015/16	Ursula Hamenstädt
Globale Analysis I	Sebastian Hensel
Exercise Sheet 2	27. Oktober 2015

2.1. (Not Biholomorphic)

i) Show that the unit disk $\Delta = \{z \in \mathbb{C}, |z| < 1\}$ is not biholomorphic to $\mathbb{C}$.

ii) Show that the punctured unit disk

$$\Delta^* = \{z \in \mathbb{C}, 0 < |z| < 1\}$$

is not biholomorphic to $\mathbb{C} \setminus \{0\}$.

2.2. (Antiholomorphic Involutions)

i) Let $U \subset \mathbb{C}$ be a domain. A function $f : U \to \mathbb{C}$ is called *antiholomorphic* if it satisfies the equations

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

where $f(x + iy) = u(x, y) + iv(x, y)$. Show that the composition of two antiholomorphic functions is holomorphic; show that the composition of an antiholomorphic function with a holomorphic function is antiholomorphic.

Convince yourself that the notion of antiholomorphic function makes sense on a Riemann surface.

ii) Show there there is a function $\iota : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with the properties:

i. ι is antiholomorphic.

ii. ι is an involution, i.e. $\iota(\iota(z)) = z$ for all $z \in \hat{\mathbb{C}}$.

iii. ι is the identity on the unit circle: $\iota(z) = z$ for all $z \in \mathbb{C}$ with $|z| = 1$.

(*Hint: start with* $z \mapsto 1/z$)

iii) Show the following version of the *Schwarz reflection principle*: suppose that $f : \{z \in \mathbb{C}, |z| \leq 1\} \to \mathbb{C}$ is a function which is continuous and holomorphic on the unit disk $\{z \in \mathbb{C}, |z| < 1\}$. Then $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined by

$$F(z) = \begin{cases} f(z) & \text{for} |z| \leq 1 \\ \iota(f(\iota(z))) & \text{otherwise} \end{cases}$$

is a holomorphic extension of f to the Riemann sphere.

iv) Show that the *pillowcase surface* X (as defined in class) is biholomorphic to the Riemann sphere $\hat{\mathbb{C}}$. To get started, show that X admits an antiholomorphic involution exchanging the "front" and "back" side. Then, argue that there is a biholomorphic map which sends the "front" rectangle to the unit cicle, and use this to construct the desired map.

2.3. (Complex Curves Are Not Compact) Show that any complex algebraic curve

$$\{(z,w) \in \mathbb{C}^2 \mid P(z,w) = 0\}$$

is not compact.

2.4. (Algebraic Curve Unions) Show that a finite union of complex algebraic curves is a complex algebraic curve.