L^p-COHOMOLOGY FOR GROUPS OF ISOMETRIES OF HADAMARD SPACES

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Abstract. We show that a discrete group Γ which admits a non-elementary isometric action on a Hadamard manifold of bounded negative curvature admits an isometric action on an L^p-space V for some $p > 1$ with $H^1(\Gamma, V) \neq 0$.

1. Introduction

A countable group Γ has property (T) if every affine isometric action of Γ on an L^2-space has a fixed point. Among the most prominent examples of such groups are lattices in higher rank simple Lie groups.

For such higher rank lattices, much more is true. Namely, any affine uniformly Lipschitz action on a Hilbert space has a fixed point [O22, dLdlS23]. Moreover, any isometric action on a uniformly convex Banach space has a fixed point [BFGM07], [dLdlS23].

On the other hand, it is known that any hyperbolic group Γ admits a proper isometric action on some L^p-space [Yu05, Ni13], in spite of the fact that many such groups have property (T). In particular, cocompact lattices in the rank one simple Lie groups $Sp(2m, 1), F_4^{-20}$ admit proper affine isometric action on an L^p-space where $p > 2$ can explicitly be estimated.

Property (T) and its strengthenings can be viewed as a vanishing result for degree one group cohomology with coefficients in a representation of Γ. The goal of this article is to point out that the construction of representations in an L^p-space with nontrivial first cohomology can be carried out for arbitrary countable groups which admit non-elementary isometric actions on Hadamard manifolds of bounded negative curvature. Here an isometric action of a group on a space Y which is hyperbolic in the sense of Gromov is *elementary* if either it has a bounded orbit or if its action on the Gromov boundary of Y has a global fixed point.

Theorem. *Let Γ be a discrete group which admits a non-elementary isometric action on a Hadamard manifold M of bounded negative curvature. Then there exists a number $p > 1$ and a representation of Γ on an L^p-space V with $H^1(\Gamma, V) \neq 0$.*

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This result is likely to be far from optimal since the requirement that Γ acts on a smooth manifold of bounded negative curvature rather than on an arbitrary Gromov hyperbolic geodesic metric space Y is very strong. However, some assumptions on Y are necessary for the statement of the theorem to hold true. Namely, there are finitely generated groups Γ which admit acylindrical and hence non-elementary actions on some hyperbolic geodesic metric space, but such that for any $p > 1$, any affine isometric action of Γ on an L^p-space has a fixed point [MO19]. We refer to the very recent article [DMcK23] for a comprehensive discussion of related results. For finitely generated groups acting properly discontinuously, a version of the Theorem is contained in [BMV05].

Corollary. *Let Γ be a countable subgroup of a simple rank one Lie group G. If Γ is not contained in a compact or parabolic subgroup of G then there exists a representation of Γ on an L^p-space V with $H^1(\Gamma, V) \neq 0$.*

If G does not have property (T), that is, if $G = SO(n, 1)$ or $G = SU(n, 1)$, then this result is well known, and we can in fact choose $p = 2$ (Theorem 2.7.2 of [BHV08]).

The proof of Theorem 1 uses an idea due to Nica [Ni13]. Namely, let M be a Hadamard manifold of bounded negative curvature, with ideal boundary ∂M. The geodesic flow Φ^t acts on the unit tangent bundle T^1M of M preserving the *Lebesgue Liouville measure* λ. This measure disintegrates to a Radon measure $\hat{\lambda}$ on the space of geodesics $\partial M \times \partial M - \Delta$ which is invariant under the action of the isometry group $\mathrm{Iso}(M)$ of M. In particular, for any $p \geq 1$, $\mathrm{Iso}(M)$ acts isometrically on $L^p(\partial M \times \partial M - \Delta, \hat{\lambda})$. For sufficiently large p we construct a cocycle for this action and show that its restriction to the subgroup Γ is unbounded provided that the action of Γ is non-elementary.

The organization of this article is as follows. In Section 2 we study actions of a group Γ on compact metric measure spaces and formulate a condition for such an action which is sufficient for the construction of a cocycle with values in an L^p-space. In Section 3 we impose some further constraints which guarantee that the cocycle yields a nontrivial cohomology class. In Section 4 we construct an Ahlfors regular distance function on the ideal boundary ∂M of a Hadamard manifold M of bounded negative curvature with the additional assumption that the covariant derivative of the curvature tensor is uniformly bounded in norm. The distance function d will in general not be a Gromov metric on ∂M, but it is contained in its coarse conformal gauge. In Section 5 we verify that the conditions formulated in Section 2 and Section 3 are fulfilled for the action of the isometry group of M on $(\partial M, d)$. This yields the proof of Theorem 1 under the additional assumption of bounded covariant derivative of the curvature tensor. Smoothing a given metric with the Ricci flow [K05] removes this additional assumption.

The appendix contains some regularity result for the shape operator of horospheres of a Hadamard manifold of bounded negative curvature and bounded covariant derivative of the curvature tensor which is used in an essential way in the construction of the Ahlfors regular metric d on ∂M and which we were unable to locate in the literature.

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2. Actions on compact metric measure spaces

In this section X denotes a compact Hausdorff space.

Definition 2.1. A *width* on X is a continuous symmetric function $\iota : X \times X \to [0,\infty)$ with $\iota(x,x) = 0$ for all $x \in X$. We call a space X equipped with a width a *width space.*

Example 2.2. If X is metrizable then a metric d on X defining the given topology is an example of a width, and the same holds true for the trivial function $X \times X \to \{0\}$. If ι is a width and if $\alpha > 0$ is arbitrary, then ι^α is a width.

A width is not required to satisfy the triangle inequality. Note however that by continuity of ι and compactness of X and hence of $X \times X$, a width is a bounded function.

Definition 2.3. Let $(X,\iota),(Y,d)$ be two width spaces. A map $F : X \to Y$ is *Lipschitz continuous* if there exists a number $L > 0$ such that $d(Fx,Fy) \leq L\iota(x,y)$ for all x,y.

If $F : X \to Y$ is Lipschitz, and if $x,y \in X$ are such that $\iota(x,y) = 0$, then $d(Fx,Fy) = 0$.

Most width spaces admit very few Lipschitz functions.

Example 2.4. Let $I \subset \mathbb{R}$ be the unit interval, equipped with the standard metric d. Then the space (I,d^2) is a width space for which any Lipschitz function $f : (I,d^2) \to \mathbb{R}$ is constant. This can be seen by noting that such a Lipschitz function is differentiable everywhere, with vanishing differential. In other words, raising a metric to a power bigger than one destroys Lipschitz functions but also the triangle inequality. Note however that for $\alpha \in (0,1)$ the snowflake (I,d^α) is a metric space.

Let Γ be a countable group which acts as a group of homeomorphisms on the width space (X,ι). We assume that the diagonal action of Γ on $X \times X$ preserves the fat diagonal $\Delta = \{(x,y) \in X \mid \iota(x,y) = 0\}$ in $X \times X$. Note that Δ is a closed subset of $X \times X$ by continuity of ι.

Assume that Γ preserves the measure class of a Borel probability measure μ on X of full support without atoms. Then for all $\varphi \in \Gamma$ and for μ-almost every $x \in X$, the Jacobian of φ is defined at x. Here by Jacobian we denote the Radon Nikodym derivative of $\varphi_*\mu$ with respect to μ. In the sequel we also speak about the Radon Nikodym derivative of φ, and we denote it by $\mathrm{RN}(\varphi)$.

Following [Ni13], for a number $Q > 0$ define

$$\nu = \iota^{-2Q}\mu \times \mu,$$

which is thought of as a measure on $X \times X - \Delta$. The following is fairly immediate from the above discussion.

Lemma 2.5. *The measure ν on $X \times X - \Delta$ is locally finite and quasi-invariant under the diagonal action of Γ. The Radon Nikodym derivative $\mathrm{RN}_\nu(\varphi)$ of φ with respect to ν equals*

$$\mathrm{RN}_\nu(\varphi)(x,y) = \iota(\varphi(x),\varphi(y))^{-2Q}\mathrm{RN}(\varphi)(x)\mathrm{RN}(\varphi)(y)\iota(x,y)^{2Q}. \tag{1}$$

Proof. Since μ is quasi-invariant under the action of Γ, and Γ preserves the fat diagonal, the same holds true for ν. Let $\varphi \in \Gamma$ and let $(x,y) \in X \times X - \Delta$. It is straightforward that the formula (1) computes the Radon Nikodym derivative $\mathrm{RN}_\nu(\varphi)(x,y)$ of φ at (x,y) with respect to the measure ν.

That the measure ν on $X \times X - \Delta$ is locally finite is immediate from continuity of the width ι and finiteness of μ. □

We now formulate a condition for a measure class preserving action of a group Γ on (X,ι,μ) which ensures that there exists a cocycle for the action with values in $L^p(\nu)$.

Condition $(*)$**:** The group Γ acts by bi-Lipschitz transformations on (X,ι), and for every $\varphi \in \Gamma$, the Radon Nikodym derivative $\mathrm{RN}(\varphi)$ of φ is a Lipschitz continuous function $(X,\iota) \to (0,\infty)$ which is bounded away from zero.

Note that under Condition $(*)$, by compactness of X and continuity, for each $\varphi \in \Gamma$ the Radon Nikodym derivative $\mathrm{RN}(\varphi)$ of $\varphi_*\mu$ with respect to μ is a bounded function on X. Moreover, since for each φ there exists a number $L > 0$ (the Lipschitz constant of φ^{-1}) such that $\iota(x,y) \leq L\iota(\varphi(x),\varphi(y))$ for all $(x,y) \in X \times X - \Delta$, Lemma 2.5 shows that the function $\mathrm{RN}_\nu(\varphi)$ is bounded as well.

As a consequence, for each $p > 1$ we obtain a representation of Γ on the space

$$L^p(\nu) = \{f : X \times X - \Delta \to \mathbb{R}, \int |f|^p d\nu < \infty\}$$

by

$$(\varphi f)(x,y) = f(\varphi^{-1}(x),\varphi^{-1}(y)). \tag{2}$$

Namely, since for each $\varphi \in \Gamma$ the function $\mathrm{RN}_\nu(\varphi)$ is pointwise uniformly bounded, via the formula (2), each $\varphi \in \Gamma$ acts as a bounded linear operator on $L^p(\nu)$ for every $p \geq 1$. The action is isometric if and only if the action of Γ on $(X \times X - \Delta, \nu)$ is measure preserving.

Following Bourdon and Pajot [BP03], for $p \geq 2Q$ define the *Besov space* $\mathfrak{B}_p(X)$ to consist of all measurable functions $f : X \to \mathbb{R}$ for which the *Besov semi-norm*

$$\|f\|_{\mathfrak{B}_p} = \big(\int\int |f(x)-f(y)|^p \iota^{-2Q}(x,y) d\mu(x) d\mu(y)\big)^{1/p}$$

is finite.

Lemma 2.6. *For each $p \geq 2Q$, the space of Lipschitz function $(X,\iota) \to \mathbb{R}$ embeds into $\mathfrak{B}_p$.*

Proof. Let $f : (X, \iota) \to \mathbb{R}$ be Lipschitz continuous. Then there exists a number $L > 0$ with $|f(x) - f(y)| \le L\iota(x, y)$ for all x, y.

By continuity of ι, for all $\epsilon > 0$ the set

$$D(\epsilon) = \{(x, y) \in X \times X \mid \iota(x, y) \ge \epsilon\}$$

is a compact subset of $X \times X - \Delta$, and $D(\epsilon) \subset D(\delta)$ for $\epsilon > \delta$, $\cup_{\epsilon>0} D(\epsilon) = X \times X - \Delta$. For all $\epsilon > 0$ we have

$$\int_{D(\epsilon)} |f(x) - f(y)|^p \iota^{-2Q}(x, y) d\mu(x) d\mu(y) \le L^p \int_{D(\epsilon)} \iota(x, y)^{p-2Q} d\mu(x) d\mu(y) \le C$$

for a universal constant $C > 0$ since $p \ge 2Q$ by assumption, since ι is a bounded function and $\mu \times \mu$ is a probability measure.

The statement now follows from Lebesgue's dominated convergence theorem, applied to the functions

$$F_\epsilon(x, y) = \begin{cases} |f(x) - f(y)|^p \text{ if } (x, y) \in D(\epsilon) \\ 0 \text{ otherwise} \end{cases}$$

and the measure ν. □

Via the map which associates to $f \in \mathfrak{B}_p$ the function $\Psi(f)(x, y) = f(x) - f(y)$, the Besov space $\mathfrak{B}_p$ embeds into $L^p(X \times X - \Delta, \nu)$.

For $\varphi \in \Gamma$ and $(x, y) \in X \times X$ define

$$c_\varphi(x, y) = \log \mathrm{RN}(\varphi)(x) - \log \mathrm{RN}(\varphi)(y). \tag{3}$$

Then c_φ is a measurable *cocycle* for the diagonal action of Γ on $X \times X$. This means that

$$c_{\varphi \circ \psi}(x, y) = c_\varphi(\psi(x, y)) + c_\psi(x, y)$$

for all $(x, y) \in X \times X$ and all $\varphi, \psi \in \Gamma$.

The following lemma explains the significance of Condition $(*)$.

Lemma 2.7. *Assume condition $(*)$. Then for $p \ge 2Q$ and each $\varphi \in \Gamma$, we have $\log \mathrm{RN}(\varphi) \in \mathfrak{B}_p$. In particular, the cocycle c_φ on $X \times X - \Delta$ consists of L^p-integrable functions with respect to ν.*

Proof. By assumption, the function $\mathrm{RN}(\varphi)$ assumes values in a compact interval $[a, b] \subset (0, \infty)$, moreover as a function $(X, \iota) \to [a, b]$, it is Lipschitz continuous.

Now the restriction of the function log to a compact interval $[a, b] \subset (0, \infty)$ is Lipschitz continuous and hence the same holds true for the composition $x \to \log(\mathrm{RN}(\varphi(x)))$ since the composition of Lipschitz functions is Lipschitz. Thus the lemma now follows from Lemma 2.6. □

Remark 2.8. By the formula (1), if Condition $(*)$ holds true then for $p \geq 2Q$ the group Γ admits an isometric action on $L^p(\nu)$ via

$$(\varphi, f)(x, y) = f(\varphi^{-1}(x), \varphi^{-1}(y))\mathrm{RN}_\nu(\varphi)^{-1/p}(x, y).$$

In other words, we obtain a representation Π of Γ into the group of linear isometries of $L^p(\nu)$.

A cocycle for this isometric action is a function $a : \Gamma \to L^p(\nu)$ such that

$$a(\varphi \circ \psi) = \Pi(\varphi)(a(\psi)) + a(\varphi).$$

Thus the cocycle c is a cocycle for this isometric representation if and only if ν is invariant under the action of Γ. A necessary condition for this to hold is that the functions $\mathrm{RN}_\nu(\varphi)$ are uniformly bounded, independent of φ. In general, it is unclear whether there exists a constant $Q > 0$ such that this boundedness condition holds true.

Example 2.9. Consider the standard projective action of the group $\Gamma = SL(n, \mathbb{Z})$ on $X = \mathbb{R}P^{n-1}$ $(n \geq 2)$. This action is by diffeomorphisms and hence if μ denotes the volume form induced by the round metric, then the Radon Nikodym derivatives of the elements of Γ are Lipschitz continuous. Thus for any $Q \geq 1$ and any $p \geq 2Q$ one obtains a cocycle c with values in the space of functions which are composed of pull-backs of functions in the Besov space $\mathfrak{B}_p$ via the first and second factor projection. For $n = 2$ and $Q = 1, p \geq 2$, this is a cocycle for an isometric action on an L^p-space which defines a nontrivial cohomology class for Γ. For $n \geq 3$ it is unclear whether there exists $Q \geq 1, p \geq 2Q$ such that this cocycle is a cocycle for an isometric action on L^p. By the main result of [BFGM07], if such numbers p, Q exist then the cocycle is a coboundary.

3. Measure preserving actions on products

The goal of this section is to find conditions which guarantee that the cocycle for the group Γ constructed in Section 2 defines a nontrivial class in the first cohomology of Γ with coefficients in the representation.

Assume for the remainder of this section that (X, d, μ) is a compact *Ahlfors regular* metric measure space of dimension $Q \geq 1$. This means that (X, d) is a metric space, and there exists a number $C > 0$ such that

$$\mu(B(\xi, r)) \in [C^{-1}r^Q, Cr^Q]$$

for all $\xi \in X$ and all $r \leq \mathrm{diam}(X)/2$, where $B(\xi, r)$ denotes the open ball of radius r about ξ. Assume furthermore that the countable group Γ acts on (X, d) as a group of bi-Lipschitz homeomorphisms. This implies that Γ preserves the measure class of μ. In particular, for every $\varphi \in \Gamma$ and μ-almost every $x \in X$ the Radon Nikodym dervative $\mathrm{RN}(\varphi)(x)$ of $\varphi_*\mu$ with respect to μ exists at x. Put

$$\langle \varphi, x \rangle = \log \mathrm{RN}(\varphi)(x);$$

then $c_\varphi(x, y) = \langle \varphi, x \rangle - \langle \varphi, y \rangle$.

A homeomorphism φ of a compact metric space (X,d) has *an attracting fixed point* x if there exists a compact neighborhood U of x such that $\varphi(U)\subset U$ and

$$\cap_{j>0}\varphi^j(U)=\{x\}.$$

Write as before $\nu=d^{-2Q}\mu\times\mu$, viewed as a measure on $X\times X-\Delta$. For a measure class preserving map φ put $\mathrm{RN}_\nu(\varphi)=d\varphi_*\nu/d\nu$.

Proposition 3.1. *Let (X,d,μ) be an Ahlfors regular metric measure space of dimension $Q\geq 1$. Assume that Γ acts on X preserving the measure class of μ. Assume moreover that there is $\varphi\in\Gamma$ and a number $C>0$ with the following properties.*

1. *φ^{-1} has an attracting fixed point.*
2. *For all $\ell\in\mathbb{Z}$ we have $\mathrm{RN}_\nu(\varphi^\ell)\in[C^{-1},C]$.*

Then for $p\geq 2Q$ and $m>0$ there is a number $k_0=k_0(\varphi,p,m)>0$ such that

$$\|c_{\varphi^k}\|^p_{L^p}\geq m$$

for all $k\geq k_0$.

Proof. Let $\varphi\in\Gamma$ and assume that φ^{-1} admits an attracting fixed point $z\in X$. Assume moreover that there exists a number $C>0$ such if we denote by $\mathrm{RN}(\varphi^\ell)$ the Radon Nikodym derivative of φ^ℓ with respect to μ then we have

$$\mathrm{RN}_\nu(\varphi^\ell)(x,y)=\mathrm{RN}(\varphi^\ell)(x)\mathrm{RN}(\varphi^\ell)(y)d(\varphi^\ell(x),\varphi^\ell(y))^{-2Q}d(x,y)^{2Q}\in[C^{-1},C]\tag{4}$$

for all $\ell\in\mathbb{Z}$ and almost all $(x,y)\in X\times X-\Delta$. By enlarging C if necessary we may moreover assume that for all $x\in X$ and all $r<\mathrm{diam}(X)$ we have

$$\mu(B(x,r))\in[C^{-1}r^Q,Cr^Q].$$

Let $b>0$ be such that the closed ball $\bar{B}(z,b)$ of radius b about z is contained in a compact neighborhood U of z as in the definition of an attracting fixed point. By assumption, there exists a number $\ell_0>0$ such that $\varphi^{-\ell_0}(U)\subset\bar{B}(z,b)$ and hence

$$\varphi^{-\ell}\bar{B}(z,b)\subset\varphi^{-\ell_0}(\varphi^{-\ell+\ell_0}U)\subset\varphi^{-\ell_0}U\subset\bar{B}(z,b)\text{ for all }\ell\geq\ell_0.$$

The same argument also shows that if $\varphi^{-j}B(z,b)\subset B(z,\delta)$ for some $\delta>0$, then $\varphi^{-k}B(z,b)\subset B(z,\delta)$ for all $k\geq\ell_0+j$.

Let $\tau=\max\{\mathrm{diam}(X)/b,(2C^2)^{1/Q}\}>2$. Then for all $r\leq b$ we have

$$\mu(B(z,r)-B(z,\tau^{-1}r))\geq C^{-1}r^Q-C\tau^{-Q}r^Q\geq C^{-1}r^Q/2.\tag{5}$$

Choose a sequence $j_m\to\infty$ such that

$$\varphi^{-j_m}B(z,b)\subset B(z,b\tau^{-8m})\tag{6}$$

for all $m\geq 1$. Note that we then have $\varphi^{-k}B(z,b)\subset B(z,b\tau^{-8m})$ for any $k\geq j_m+\ell_0$ and any m.

Let $m\geq 1$ be arbitrary. We claim that there is a universal constant $u>0$ such that for all $\ell\leq m-2$, all $x\in B(z,b\tau^{-8\ell})-B(z,b\tau^{-8\ell-2})$ and all $y\in B(z,b\tau^{-8\ell-6})-B(z,b\tau^{-8\ell-8})$ we have $\langle\varphi^{j_m},y\rangle-\langle\varphi^{j_m},x\rangle\geq u$.

To see that this is the case, note that since $\ell \leq m-2$, we have

$$\varphi^{j_m}x, \varphi^{j_m}y \in X - B(z,b)$$

and hence $d(z,\varphi^{j_m}x) \in [b,\tau b], d(z,\varphi^{j_m}y) \in [b,\tau b]$ (recall that $\mathrm{diam}(X) \leq \tau b$). Now let us assume for the moment that the Radon Nikodym derivative of φ^{j_m} exists a z. Then together with the assumption (4) and the fact that $\varphi^{j_m}z = z$ for all m, we conclude that

$$\begin{aligned}(7)\qquad &\frac{\mathrm{RN}(\varphi^{j_m})(z)\mathrm{RN}(\varphi^{j_m})(y)d(\varphi^{j_m}(z),\varphi^{j_m}(y))^{-2Q}d(z,y)^{2Q}}{\mathrm{RN}(\varphi^{j_m})(z)\mathrm{RN}(\varphi^{j_m})(x)d(\varphi^{j_m}(z),\varphi^{j_m}(x))^{-2Q}d(z,x)^{2Q}}\\ \sim&\frac{\mathrm{RN}(\varphi^{j_m})(y)d(z,y)^{2Q}}{\mathrm{RN}(\varphi^{j_m})(x)d(z,x)^{2Q}} \sim \text{const.}\end{aligned}$$

Here the first $\sim$ in the estimate means equality up to a factor contained in the interval $[\tau^{-4Q},\tau^{4Q}]$ (by erasing two factors in the numerator and denominator using the above estimate), and the second $\sim$ means that value of the ratio is contained in the interval $[C^{-2}\tau^{-4Q},C^2\tau^{4Q}]$ (which follows from the fact that by (4), the ratio of numerator and denominator in the first term of the expression (7) is at most C^2). Now $\frac{d(z,y)^{2Q}}{d(z,x)^{2Q}} \leq \tau^{-8Q}$, and as $\tau^{-8Q}C^2\tau^{4Q} \leq \tau^{-3Q}$, taking the logarithm yields the claim.

The Radon Nikodym derivative of φ^{j_m} with respect to μ may not exist at z. However, it exists almost everywhere and hence by Ahlfors regularity of μ, we can find a point $\hat{z}$ arbitrarily close to z which is mapped by φ^{j_m} into an arbitrarily small neighborhood of z and such that the Radon Nikodym derivative of φ^{j_m} exists at $\hat{z}$. Replacing z by $\hat{z}$ in formula (7) then yields the statement we were looking for.

On the other hand, by (5), and Ahlfors regularity of μ, there exists a universal constant $\kappa > 0$ such that

$$\nu((B(\xi,b\tau^{-8\ell}) - B(\xi,b\tau^{-8\ell-2})) \times (B(\xi,b\tau^{8\ell-6}) - B(\xi,b\tau^{8\ell-8}))) \geq \kappa$$

for all $\ell \in [2,m]$. Since the value of $c_{\varphi^{j_m}}$ on these sets is bounded from below by $u > 0$, we have

$$\int |c_{\varphi^{j_m}}|^p d\nu \geq (m-2)u^p\kappa.$$

As the right hand side of this inequality tends to ∞ as $m \to \infty$, the proposition follows. $\square$

4. An Ahlfors regular metric on the boundary of a Hadamard manifold

In this section we consider an n-dimensional simply connected complete Riemannian manifold ($n \geq 2$) of sectional curvature contained in the interval $[-b^2,-a^2]$ for numbers $0 < a < b < \infty$. We also require that there is a universal upper bound for the norm of the covariant derivative ∇R of the curvature tensor R of M. Our goal is to construct a metric d on the *ideal boundary* ∂M of M with the following properties.

(1) d is Ahlfors regular.

(2) The Ahlfors regular measure μ defined by d is contained in the Lebesgue measure class.
(3) Isometries of M act by bi-Lipschitz transformations on $(\partial M, d)$.

Note that for general Hadamard manifolds of bounded negative curvature, it is not clear whether (2) makes sense, and it is to this end that we shall use the assumption that $|\nabla R|$ is bounded.

A point $\xi \in \partial M$ determines a *Busemann function* b_ξ at ξ. Its level sets are the *horospheres* at ξ. By the assumption on M, these Busemann functions are of class C^3, and their gradients $\operatorname{grad} b_\xi$ are C^2-vector fields on M [Shc83].

Let T^1M be the unit tangent bundle of M. The metric on M induces a natural metric on T^1M, the *Sasaki metric*. This metric defines a distance function and hence a Hölder structure for functions on T^1M. The canonical projection

$$\Pi : T^1M \to M$$

is a Riemannian submersion.

For a point $x \in M$ and a unit tangent vector $v \in T^1_x M$ let $m(v) > 0$ be the mean curvature at x of the horosphere in M whose outer normal field passes through v. That is, the horosphere is a level set of the Busemann function defined by the ideal boundary point $\gamma_v(-\infty) \in \partial M$, where γ_v denotes the geodesic with initial velocity $\gamma_v'(0) = v$.

The following result is perhaps well known. We provide a proof in the appendix (Corollary A.2).

Proposition 4.1. *The function $m : v \to m(v)$ on T^1M is Hölder continuous.*

We next observe

Lemma 4.2. *There exists a number $C_0 > 0$ with the following property. Let $\gamma \subset M$ be any geodesic and let $t > 0$; then*

$$|\int_0^t m(\gamma'(s))ds - \int_0^t m(-\gamma'(s))ds| \leq C_0.$$

Proof. The Lebesgue Liouville measure λ on T^1M is the measure defined by the volume form of the Sasaki metric. This volume form, again denoted by λ, is invariant under the geodesic flow $\Phi^t : v \to \gamma_v'(t)$. It can be described as follows.

The tangent bundle TT^1M of T^1M has an orthogonal decomposition as $TT^1M = \mathcal{H} \oplus \mathcal{V}$ where $\mathcal{V}$ is the *vertical tangent bundle*, that is, the tangent bundle of the fibers of the fibration $T^1M \to M$, and where $\mathcal{H}$ is the *horizontal bundle* defined by the Levi Civita connection. Then

$$\lambda = \omega_{\mathcal{H}} \wedge \omega_{\mathcal{V}}$$

where $\omega_{\mathcal{H}}$ is an n-form which annihilates $\mathcal{V}$, $\omega_{\mathcal{V}}$ is an $(n-1)$-form which annihilates $\mathcal{H}$ and such that $\omega_{\mathcal{V}}$, $\omega_{\mathcal{H}}$ are defined by a choice of an orientation and the Riemannian metric on $\mathcal{H}, \mathcal{V}$.

The contraction $\iota_X \lambda$ of λ with the generator X of the geodesic flow Φ^t is a smooth $(2n-2)$-form ω which is invariant under Φ^t and which equals $\iota_X \omega_{\mathcal{H}} \wedge \omega_{\mathcal{V}}$. This $(2n-2)$-form then defines a Radon measure on the space of geodesics $\partial M \times \partial M - \Delta$ which is invariant under the action of $\mathrm{Iso}(M)$.

There exists another natural $2n-2$-form on T^1M which is defined as follows. For a given vector $v \in T^1M$ we can consider the submanifolds $W^{ss}(v), W^{su}(v)$ of T^1M defined by the inner and outer normal field of the horosphere through $\Pi(v)$ at $\gamma_v(\infty)$ and $\gamma_v(-\infty)$, respectively. By the assumption on M, this construction defines two continuous foliations W^{ss}, W^{su} of T^1M, with leaves of class C^2. Thus the tangent bundles TW^{ss}, TW^{su} of these foliations are defined, and by Proposition A.1, they are Hölder continuous subbundles of TT^1M of dimension $n-1$ which do not intersect.

Namely, for $v \in T^1M$ the orthogonal complement $v^\perp$ of v in $T_{\Pi(v)}M$ has a natural isometric identification with both the orthogonal complement of X in $\mathcal{H}_v$ and the fiber $\mathcal{V}_v$. A tangent vector Y of TW^{su} at $v \in T^1M$ decomposes into

$$Y = Y^h + Y^v \text{ where } Y^h \in X^\perp \subset \mathcal{H}_v, Y^v \in \mathcal{V}_v.$$

With respect to the natural isometric identification of $X^\perp \subset \mathcal{H}_v$ and $\mathcal{V}_v$, the linear map which sends Y^h to Y^v is given by the shape operator for the outer normal field of the horosphere defined by $\gamma_v(-\infty)$, and this shape operator is a symmetric linear operator whose eigenvalues are bounded from above and below by universal positive constants.

Similarly, a tangent vector Z of TW^{ss} at v decomposes as

$$Z = Z^h + Z^v \text{ where } Z^h \in \mathcal{H}_v, Z^v \in \mathcal{V}_v$$

and the linear map $X^\perp \subset \mathcal{H}_v \to \mathcal{V}_v$ which sends Z^h to Z^v is the shape operator for the inner normal field of the horosphere defined by $\gamma_v(\infty)$, and this shape operator is a symmetric linear operator whose eigenvalues are bounded from above and below by universal negative constants. Using Proposition A.1, this shows that the bundles TW^{ss}, TW^{su} are Hölder continuous and furthermore, the angle with respect to the Sasaki metric between a nonzero vector of TW^{ss} and a nonzero vector of TW^{su} over a point $v \in T^1M$ is bounded from below by a universal positive constant not depending on v.

Thus we obtain another continuous $(2n-2)$-form $\tilde{\omega}$ on T^1M by defining

$$\tilde{\omega} = \omega^{su} \wedge \omega^{ss}$$

where the $n-1$-form ω^{ss} annihilates $TW^{su} \oplus \mathbb{R}X$ and restricts to the volume form on the leaves of the foliation W^{ss} which is induced from the pull-back of the metric on horospheres in M, and similarly for ω^{su}. The $(2n-2)$-form $\tilde{\omega}$ annihilates the generator of the geodesic flow and hence it can be represented as $\iota_X \tilde{\lambda}$ for a $2n-1$-form $\tilde{\lambda}$. Then $\tilde{\lambda} = \kappa\lambda$ for a continuous function $\kappa : T^1M \to \mathbb{R}$. By the above discussion, there exists a constant $C > 0$ such that $\kappa(T^1M) \subset [C^{-1}, C]$.

The volume form $\tilde{\lambda}$ is in general not invariant under the geodesic flow Φ^t, but it is quasi-invariant, and its Lie derivative $\mathcal{L}_X \tilde{\lambda}$ in direction of the generator X of Φ^t equals

$$\mathcal{L}_X \tilde{\lambda}(v) = (m(v) - m(-v))\tilde{\lambda}$$

since by construction and the properties of the mean curvature of horospheres we have

$$(\frac{d}{dt}\omega^{su} \circ \Phi^t|_{t=0})(v) = m(v)\omega^{su}(v) \text{ and } (\frac{d}{dt}\omega^{ss} \circ \Phi^t|_{t=0})(v) = -m(-v)\omega^{ss}(v).$$

As a consequence, for $v \in T^1M$ and $t > 0$ the logarithm at v of the Jacobian of Φ^t with respect to the measure $\tilde{\lambda}$ can be represented as

$$\int_0^t m(\Phi^s v)ds - \int_0^t m(-\Phi^s v)ds. \tag{8}$$

On the other hand, the measure λ is invariant under Φ^t, and it equals a uniformly bounded multiple of the measure $\tilde{\lambda}$. Thus the Jacobian of Φ^t with respect to $\tilde{\lambda}$ is uniformly bounded, independent of the basepoint and t. Then the same holds true for the integral (8) which shows the lemma. □

For $v \in T^1M$ define

$$f(v) = \frac{1}{2}(m(v) + m(-v)).$$

Since the function m is Hölder continuous, the same holds true for the function f. Furthermore, f takes values in a fixed interval $[c, d]$ for $0 < c < d$ since the mean curvature of horospheres is bounded from above and below by a positive constant only depending on the curvature bounds.

We now use a construction which is well known in the case that the manifold M is the universal covering of a closed manifold, see for example the articles [L95] and [H97].

For $x \in M$ we shall define a Gromov type product $(|)_x$ on ∂M based at x. To this end consider first a point $\xi \in \partial M$ and two points $x, y \in M$. There are unique geodesic rays $\gamma, \eta : [0, \infty) \to M$ connecting x, y to ξ, that is, such that $\gamma(0) = x, \gamma(\infty) = \xi$ and $\eta(0) = y, \eta(\infty) = \xi$. Given the parameterization for γ, there exists a unique preferred parameterization $\hat{\eta}$ of η (extended to a geodesic line) with $\hat{\eta}(u) = \eta(0)$ for some $u \in \mathbb{R}$ and so that $\gamma, \hat{\eta}$ are *strongly asymptotic*, that is,

$$\lim_{t\to\infty} d(\gamma(t), \hat{\eta}(t)) = 0.$$

We have

Lemma 4.3. *The limit*

$$q_\xi(x, y) = \lim_{t\to\infty} \big(\int_0^t f(\gamma'(s))ds - \int_u^t f(\hat{\eta}'(s))ds\big)$$

exists.

Proof. Let d be the Sasaki metric on T^1M. If $-a^2 < 0$ is an upper curvature bound for M then we have

$$d(\gamma'(s), \hat{\eta}'(s)) \leq C_0 e^{-as}$$

for some $C_0 > 0$ depending on γ, η.

Since by Proposition 4.1 the function f is Hölder continuous, there exist numbers $C_1 > 0, \alpha > 0$ so that

$$|f(\gamma'(s)) - f(\hat\eta'(s))| \leq C_1 d(\gamma'(s), \hat\eta'(s))^\alpha \leq C_1 C_0^\alpha e^{-\alpha a s} \text{ for all } s.$$

As the function $s \to e^{-\alpha a s}$ is integrable on $[0,\infty)$, this yields the existence of the limit

$$\lim_{t\to\infty}\Big(\int_0^t f(\gamma'(s))ds - \int_u^t f(\hat\eta'(s))ds\Big).$$

□

Note that $q_\xi(x,y) = -q_\xi(y,x)$.

For $x \in M$ and two points $\xi \neq \eta \in \partial M$ let ρ be the geodesic connecting ξ to η. Choose a point $y \in \rho$ and define

$$(\xi \mid \eta)_x = \frac{1}{2}(q_\xi(x,y) + q_\eta(x,y)).$$

Thus $(\xi \mid \eta)_x = (\eta \mid \xi)_x$ for all ξ, η. We have

Lemma 4.4. $(\xi \mid \eta)_x$ *does not depend on the choice of* $y \in \rho$.

Proof. Parameterize ρ in such a way that $\rho(0) = y$ and $\rho(\infty) = \xi$. Replacing y by $\rho(t)$ for some $t > 0$ adds the integral $\int_0^t f(\rho'(s))ds$ to $q_\xi(x,y)$ and adds the integral $-\int_0^t f(-\rho'(s))ds$ to $q_\eta(x,y)$. Since f is invariant under the flip $v \to -v$, the lemma follows. □

As a consequence of Lemma 4.4, $(\xi \mid \eta)_x$ only depends on ξ, η, x. We use these Gromov type products to define a *cross ratio* $[,,,]_x$ on ∂M by

$$[\xi_1, \xi_2, \xi_3, \xi_4]_x = (\xi_1 \mid \xi_3)_x + (\xi_2 \mid \xi_4)_x - (\xi_1 \mid \xi_4)_x - (\xi_2 \mid \xi_3)_x.$$

We have

Lemma 4.5. $[,,,]_x$ *does not depend on* x.

Proof. Let $y \in M$ be another point and let $\xi, \eta \in \partial M$. Then we have

$$(\xi \mid \eta)_y = (\xi \mid \eta)_x + q_\xi(y,x) + q_\eta(y,x).$$

This formula yields an expression for $[\xi_1, \xi_2, \xi_3, \xi_4]_y - [\xi_1, \xi_2, \xi_3, \xi_4]_x$ as a sum of terms $q_{\xi_i}(y,x)$ $(i = 1, \ldots, 4)$. Each of these terms appears twice in this expression, with opposite signs, and hence the terms cancel. □

By construction, for any isometry φ of M, for any quadruple $(\xi_1, \xi_2, \xi_3, \xi_4)$ of distinct points in ∂M and any $x \in M$ we have

$$[\varphi(\xi_1), \varphi(\xi_2), \varphi(\xi_3), \varphi(\xi_4)]_{\varphi(x)} = [\xi_1, \xi_2, \xi_3, \xi_4]_x$$

and therefore we obtain

Corollary 4.6. *The isometry group of* M *preserves* $[,,,]_x$.

Fix again a point $x \in M$. We have

Lemma 4.7. *There exists a number $\kappa > 0$ such that the product $(|)_x$ satisfies the κ-ultrametric inequality*

$$(\xi \mid \eta)_x \geq \min\{(\xi \mid \zeta)_x, (\zeta \mid \eta)_x\} - \kappa.$$

Proof. The proof is standard and we only give a sketch. Namely, consider pairwise distinct points $\xi, \zeta, \eta \in \partial M$. These points define a geodesic triangle which is δ_0-thin for some $\delta_0 > 0$ (a side is contained in the δ_0-neighborhood of the union of the other sides) since M is hyperbolic in the sense of Gromov.

As the function f is uniformly Hölder continuous and bounded from above and below by uniform positive constants, we know that there exists a number $m > 0$ with the following property. Let y be the shortest distance projection of x into the geodesic θ connecting ξ to η and let $\rho : [0, T] \to M$ be the geodesic connecting x to y; then

$$|(\xi \mid \eta)_x - \int_0^T f(\rho'(s))ds| \leq m.$$

Namely, for this choice of a point $y \in \theta$, we have $|q_\xi(x, y) - \int_0^T f(\rho'(s))ds| \leq m$ for a universal constant $m > 0$, and the same estimate also holds true for $q_\eta(x, y)$.

On the other hand, we also know that y is contained in the δ_0-neighborhood of one of the geodesics connecting ξ to ζ or connecting η to ζ, say the geodesic γ connecting ξ to ζ. Let $y' \in \gamma$ be such that $d(y, y') \leq \delta_0$. Let z be the shortest distance projection of x into γ. Assume without loss of generality that y' is contained in the subray of γ connecting z to ξ.

Consider the geodesic triangle T with vertices x, z, y'. It has a right angle at z. Let $\rho_1 : [0, T_1] \to M, \rho_2 : [0, T_2] \to M$ be the sides of T with vertex x and second vertex z, y', respectively, Using once more Hölder continuity of f and the fact that f is bounded from above and below by a universal positive constant, we conclude that

$$\int_0^{T_1} f(\rho_1'(s))ds \leq \int_0^{T_2} f(\rho_2'(s))ds + \ell$$

for a universal constant $\ell > 0$. But this means

$$(\xi \mid \eta)_x - (\xi \mid \zeta)_x \geq -2m - \ell$$

which is what we wanted to show. □

A *quasimetric* on a space X is a symmetric function $q : X \times X \to [0, \infty)$ which vanishes only for $x = y$ and satisfies for some $K > 0$

$$q(x, y) \leq K(q(x, z) + q(z, y)) \text{ for all } x, y, z.$$

As a consequence of Lemma 4.7, if for $\epsilon > 0$, we define

$$\delta_{x,\epsilon}(\xi, \eta) = e^{-\epsilon(\xi|\eta)_x}$$

then we have

Lemma 4.8. *$\delta_{x,\epsilon}$ is a quasimetric on ∂M.*

Quasimetrics with multiplicative constant sufficiently close to 1 are known to be bi-Lipschitz equivalent to metrics and hence we have

Lemma 4.9. *For sufficiently small $\epsilon > 0$ the function $\delta_{x,\epsilon}$ is bi-Lipschitz equivalent to a metric d_x.*

Let now λ_x be the measure on ∂M which is the image of the Lebesgue measure on the round sphere $T_x M$ under the natural homeomorphism $T_x^1 M \to \partial M$. The following is the main result of this section.

Proposition 4.10. *The metric measure space $(\partial M, d_x, \lambda_x)$ is Ahlfors regular of dimension $1/\epsilon$.*

Proof. Since the diameter of d_x is finite, the total mass of the measure λ_x is finite, and furthermore d_x is bi-Lipschitz equivalent to $\delta_x = \delta_{x,\epsilon}$, it suffices to find numbers $C > 0, r_0 > 0$ such that

$$\lambda_x(B(\xi, r)) \in [Cr, C^{-1}r] \tag{9}$$

for all $\xi \in \partial M$ and all $r \leq r_0$, where $B(\xi, r) = \{\eta \mid e^{-(\xi|\eta)_x} < r\}$.

Let exp be the exponential map of the Riemannian manifold M. Using the uniform curvature bound, the map $v \to \exp(10v)$ maps the unit sphere $T_x^1 M$ onto the distance sphere $S(x, 10)$ of radius 10 about x, and its Jacobian for the standard metric on $T_x^1 M$ and the metric on $S(x, 10)$ induced from the metric on M is a smooth function h with values in $[\kappa_1, \kappa_2]$ for constants $0 < \kappa_1 < \kappa_2$ not depending on x. Thus it suffices to show the estimate (9) for the measure $h\lambda_x$.

Let $r_0 > 0$ be sufficiently small that $\int_0^{10} m(\gamma'(t))dt \leq -\log r_0$ for every geodesic γ in M. We next claim that for $r \leq r_0$ and $\xi \in \partial M$ the ball $B(\xi, r)$ can be understood as follows.

Let $\gamma_\xi : [0, \infty) \to M$ be the geodesic ray connecting $x = \gamma_\xi(0)$ to ξ. Let $R > 10$ be such that

$$\int_0^R m(\gamma'(t))dt = -\log r.$$

As the function m is positive and bounded from below by a positive number, such a number $R > 10$ exists and is unique.

For $q > 0$ consider the ball

$$B^S(\gamma_\xi(R), q) \subset S(x, R)$$

of radius q about $\gamma_\xi(R)$ in the distance sphere $S(x, R)$, equipped with the intrinsic path metric. Let moreover $A(\xi, R, q) \subset \partial M$ be the set of all endpoints of geodesic rays starting at x which cross through $B^S(\gamma_\xi(R), q)$. We claim that there exist numbers $0 < q < u$ not depending on ξ and r such that $A(\xi, R, q) \subset B(\xi, r) \subset A(\xi, R, u)$.

Before we prove the claim we show that it implies the proposition. Namely, using the assumption that $R \geq 10$, comparison shows that the volumes of the balls $B^S(\gamma_\xi(R), q), B^S(\gamma_\xi(R), u)$ for the induced metric on $S(x, R)$ are contained in the interval $[\chi, \chi^{-1}]$ for a universal constant $\chi > 0$. Thus to establish the desired lower and upper bound for the $h\lambda_x$-volume of $B(\xi, r)$, it suffices to show that there exists a number $C > 0$ such that for each $y \in B^S(\gamma_\xi(R), u)$ the Jacobian of the radial projection $\pi_{R,10} : S(x, R) \to S(x, 10)$ at y is contained in $[C^{-1}r, Cr]$.

The negative of the logarithm of the Jacobian of the map $\pi_{R,10}$ at $y = \gamma_\eta(R)$ for some $\eta \in \partial M$ can be computed as an integral

$$\int_{10}^{R} m_t^S(\gamma_\eta'(t))dt$$

where $m_t^S(\gamma_\eta'(t))$ is the mean curvature of the distance sphere $S(x, t)$ at the point $\gamma_\eta(t)$.

By Lemma A.3, we have

$$|m_t^S(\gamma_\eta'(t)) - m(\gamma_\eta'(t))| \le e^{-\alpha t}$$

for a universal constant $\alpha > 0$ and therefore

$$|\int_{10}^{R} m_t^S(\gamma_\eta'(t))dt - \int_{10}^{R} m(\gamma_\eta'(t))dt| \le C_0$$

where $C_0 > 0$ is a universal constant. Thus we are left with showing that for each $\eta \in \partial M$ with $\gamma_\eta(R) \in B^S(\gamma_\xi(R), u)$ we have

$$|\int_{10}^{R} m(\gamma_\eta'(t))dt - \int_{10}^{R} m(\gamma_\xi'(t))dt| \le C_1 \tag{10}$$

for a universal constant $C_1 > 0$.

However, comparison shows that for such a point η we have

$$d(\gamma_\eta'(t), \gamma_\xi'(t)) \le C_2 e^{a(t-R)} \text{ for all } 10 \le t \le R$$

for a universal constant $C_2 > 0$ and consequently the estimate (10) follows once more from Hölder continuity of m.

It remains to show the inclusion $A(\xi, R, q) \subset B(\xi, r) \subset A(\xi, R, u)$ for universal constants $0 < q < u$. To this end recall from the proof of Lemma 4.7 that there exists a number $z > 0$ with the following property. Let $\eta \in \partial M$ and let γ_η be the geodesic ray connecting x to η. Let ρ be the geodesic connecting ξ to η and let y be the shortest distance projection of x into ρ. Let $\zeta : [0, T] \to M$ be the geodesic connecting x to y and let $\ell = \int_0^T f(\zeta'(t))dt$. If $\ell > -\log R + z$ then $\eta \in B(\xi, r)$, and if $\ell < -\log R - z$ then $\eta \notin B(\xi, r)$.

The containments then follow from Hölder continuity of the function f and Lemma 4.2. □

Corollary 4.11. *For $x, y \in M$, the measures λ_x, λ_y on ∂M are absolutely continuous, with uniformly bounded Radon Nikodym derivative.*

Proof. By construction, for $x \neq y$ the metrics d_x, d_y on ∂M are bi-Lipschitz equivalent. Since by Proposition 4.10 the metric measure spaces $(\partial M, d_x, \lambda_x)$ and $(\partial M, d_y, \lambda_y)$ are Ahlfors regular, of dimension $1/\epsilon$, absolute continuity of the measures λ_x, λ_y is immediate. □

5. Groups of isometries of Hadamard manifolds

The goal of this section is the proof of the theorem from the introduction. We begin with a group Γ which admits a non-elementary isometric action on a Hadamard manifold M of bounded negative curvature and such that the covariant derivative of the curvature tensor of M is uniformly bounded. We then say that the curvature tensor of M is bounded of first order.

Let $\epsilon > 0$ be sufficiently small that the quasimetric $\delta = \delta_{x,\epsilon}$ is bi-Lipschitz equivalent to a metric d. Such a constant exists by Lemma 4.9. For a fixed point $x \in M$ we then obtain a multiplicative cross ratio on ∂M by defining

$$\mathrm{Cr}(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{\delta(\xi_1, \xi_3)\delta(\xi_2, \xi_4)}{\delta(\xi_1, \xi_4)\delta(\xi_2, \xi_3)}.$$

By Lemma 4.5, the cross ratio Cr does not depend on x and is invariant under the action of the isometry group $\mathrm{Iso}(M)$ of M. The following is an analog of Lemma 6 of [Ni13].

Lemma 5.1. *For every $\varphi \in \mathrm{Iso}(M)$ there exists a positive continuous function $|\varphi'|$ on ∂M with the property that for all $\xi, \eta \in \partial M$, we have*

$$\delta^2(\varphi\xi, \varphi\eta) = |\varphi'|(\xi)|\varphi'|(\eta)\delta^2(\xi, \eta).$$

Proof. We copy the short proof from Lemma 6 of [Ni13] for completeness. Let x, u, v be a triple of distinct points in ∂M. Since φ preserves Cr, for any fourth distinct point y we have

$$\frac{\delta(\varphi x, \varphi y)}{\delta(x, y)} \frac{\delta(\varphi u, \varphi v)}{\delta(u, v)} = \frac{\delta(\varphi x, \varphi v)}{\delta(x, v)} \frac{\delta(\varphi y, \varphi u)}{\delta(y, u)}.$$

When $y \to x$ one obtains

$$\lim_{y\to x} \frac{\delta(\varphi x, \varphi y)}{\delta(x, y)} = \frac{\delta(\varphi x, \varphi v)}{\delta(x, v)} \frac{\delta(\varphi x, \varphi u)}{\delta(x, u)} \frac{\delta(u, v)}{\delta(\varphi u, \varphi v)}. \tag{11}$$

Let $|\varphi'|_{u,v}$ denote the right-hand side of equation (11), viewed as a function of x. Then $|\varphi'|_{u,v}$ is a positive continuous function on $X - \{u, v\}$. Since the left-hand side of equation (11) does not depend on u, v, by replacing u, v by a different pair of points we can extend $|\varphi'|_{u,v}$ to a positive function $|\varphi'|$ on all of X. The formula in the lemma is now immediate from the definition and invariance of Cr (see the proof of Lemma 6 of [Ni13]). □

Recall that the notion of a Lipschitz map makes sense for the quasimetric δ. Since there exists a constant $c > 0$ such that $c\delta \le d \le \delta$, Lipschitz maps for d are precisely the Lipschitz maps for δ.

The following corollary can readily be checked directly but is an immediate consequence of Lemma 5.1.

Corollary 5.2. $\mathrm{Iso}(M)$ *acts on $(\partial M, d)$ as a group of bi-Lipschitz transformations.*

Lemma 5.3. *For each $\varphi \in \mathrm{Iso}(M)$ the function $|\varphi'|$ is Lipschitz continuous for d.*

Proof. We follow p.779 of [Ni13]. Namely, let $\xi,\eta\in\partial M$ and choose a point ζ with $d(\xi,\zeta)\geq \mathrm{diam}(\partial M)/2$. Since d satisfies the triangle inequality, we have

$$\begin{aligned}\frac{d(\varphi\xi,\varphi\zeta)}{d(\xi,\zeta)}-\frac{d(\varphi\eta,\varphi\zeta)}{d(\eta,\zeta)} &\leq \frac{d(\varphi\xi,\varphi\eta)}{d(\xi,\zeta)}+\frac{d(\varphi\eta,\varphi\zeta)}{d(\xi,\zeta)}-\frac{d(\varphi\eta,\varphi\zeta)}{d(\eta,\zeta)}\\ &\leq \frac{d(\varphi\xi,\varphi\eta)}{d(\xi,\zeta)}+\frac{d(\varphi\eta,\varphi\zeta)}{d(\eta,\zeta)}\frac{d(\xi,\eta)}{d(\xi,\zeta)} \qquad (12)\end{aligned}$$

Now if $L>1$ is the Lipschitz constant for φ then $d(\varphi\xi,\varphi\eta)\leq Ld(\xi,\eta)$ and additionally $\frac{d(\varphi\eta,\varphi\zeta)}{d(\eta,\zeta)}\leq L$. Hence the term (12) in the above expression is bounded from above by $LCd(\xi,\eta)$ for a universal constant $C>0$.

On the other hand, the formula for $|\varphi'|$ in Lemma 5.1 shows that

$$\sqrt{|\varphi'|}(\xi)-\sqrt{|\varphi'|}(\eta)=\frac{1}{\sqrt{|\varphi'|}(\zeta)}\Big(\frac{d(\varphi\xi,\varphi\zeta)}{d(\xi,\zeta)}-\frac{d(\varphi\eta,\varphi\zeta)}{d(\eta,\zeta)}\Big).$$

Together this yields Lipschitz continuity for $\sqrt{|\varphi'|}$ and hence for $|\varphi'|$. □

By construction, we also have

Lemma 5.4. *For $\varphi,\psi\in\Gamma$ and all $\xi\in\partial M$ we have*

$$|(\varphi\circ\psi)'|(\xi)=|\varphi'|(\psi(\xi))|\psi'|(\xi).$$

Proof. The lemma is immediate from Lipschitz continuity of $|\varphi'|,|\psi'|$ and Lemma 5.1. □

Let as before λ be the Lebesgue Liouville measure on the unit tangent bundle T^1M of M. It disintegrates to an $\mathrm{Iso}(M)$-invariant Radon measure $\hat{\lambda}$ on the space of geodesics $\partial M\times\partial M-\Delta$.

For $x\in M$ consider the Lebesgue measure λ_x on ∂M defined as the push forward of the Lebesgue measure on T^1_xM by the natural homeomorphism $T^1_xM\to\partial M$. Let $\epsilon>0$ be the constant which enters the definition of the metric d. We have

Proposition 5.5. *There exists a uniformly bounded function β such that $\hat{\lambda}=\beta\delta^{-2/\epsilon}\lambda_x\times\lambda_x$.*

Proof. By Corollary 4.11, the measure class of the measure λ_x on ∂M does not depend on x.

Let $p(x,y,\xi)$ be the Radon Nikodym derivative of λ_y with respect to λ_x at ξ (whenever this exists). It follows from Proposition 4.10 and its proof, using Lemma A.3, Corollary A.2 and the definition of the function δ, that there exists a universal constant $C_0>0$ such that the following holds true. Let $\xi\neq\eta\in\partial M$ and let y be the shortest distance projection of x into the geodesic connecting ξ to η; then

$$\delta^{2/\epsilon}(\xi,\eta)p(x,y,\xi)p(x,y,\eta)\in[C_0^{-1},C_0].$$

Thus we are left with showing the following. The measure $\hat{\lambda}$ is absolutely continuous with respect to $\lambda_x \times \lambda_x$, and there exists a number $C_1 > 0$ such that for any geodesic $\gamma \subset M$ and any $x \in \gamma$, the Radon Nikodym derivative at $(\gamma(-\infty), \gamma(\infty))$ of $\hat{\lambda}$ with respect to $\lambda_x \times \lambda_x$, if it is exists, is contained in the interval $[C_1^{-1}, C_1]$.

To this end recall from the proof of Lemma 4.2 the definition of the $(2n-2)$-form $\tilde{\omega}$ on T^1M. This form restricts to a volume form on a local smooth transversal N for the geodesic flow, defining a measure $\tilde{\lambda}$ on N of the form $\kappa\hat{\lambda}$ where κ is a continuous function with values in a compact subinterval of $(0, \infty)$ not depending on N.

Let $x \in N$; then by perhaps decreasing the size of N, we may assume that $\lambda_x \times \lambda_x$ descends to a measure on N by viewing N as an (open) subset of the space of geodesics. It now suffices to show that $\lambda_x \times \lambda_x$ is contained in the measure class of $\tilde{\lambda}$, with density near x bounded from above and below by a universal positive constant. However this follows from Proposition 4.10 and its proof. As this is a standard argument in smooth dynamics (which does not rely on an underlying dynamical system), carefully laid out in Chapter III of [Mn87], we omit a more detailed discussion.

Together this completes the proof of the proposition. $\square$

Let us summarize now what we achieved so far. For $x \in M$ consider the measure λ_x on ∂M. Its measure class is invariant under the action of $\mathrm{Iso}(M)$. By Lemma 5.1 and Lemma 5.3, for each $\varphi \in \Gamma$ there exists a positive Lipschitz continuous function $|\varphi'|$ on ∂M which coincides with $\mathrm{RN}(\varphi)^{1/\epsilon}$ up to a universal constant, where $\mathrm{RN}(\varphi)$ is taken with respect to the measure λ_x.

More precisely, the function $|\varphi'|(x)$ measures the infinitesimal dilatation of φ with respect to the quasimetric $\delta = \delta_x$. Lemma 5.4 shows that its logarithm defines a cocycle c_φ for Γ by

$$c_\varphi(x,y) = \log|\varphi'(x)| - \log|\varphi'(y)|.$$

As the measure λ_x is Ahlfors regular for d_x, via comparing the cocycle defined by the logarithm of Radon Nikodym derivatives to the cycycle c_φ, condition $(*)$ from Section 2 is fulfilled. In particular, for sufficiently large p, this cocycle has values in $L^p(\partial M \times \partial M - \Delta, \hat{\lambda})$.

Furthermore, there is a bounded function β such that $\beta\delta_x^{-1/\epsilon}\lambda_x \times \lambda_x$ is invariant under the action of $\mathrm{Iso}(X)$. Now if $\varphi \in \mathrm{Iso}(M)$ is a loxodromic element then φ acts on ∂M with north-south dynamics and hence the assumptions stated in Proposition 3.1 are fulfilled for φ. As a consequence, the cocycle is unbounded in $L^p(\partial M \times \partial M - \Delta, \hat{\lambda})$ for all sufficiently large p and hence it can not be a coboundary. Together we have shown

Theorem 5.6. *Let Γ be a discrete group which admits a non-elementary isometric action on a Hadamard manifold M of bounded negative curvature, with first order bounded curvature tensor. Then there exists a number $p > 1$ and an isometric action of Γ on an L^p-space V such that $H^1(\Gamma, V) \neq 0$.*

For the proof of the Theorem from the introduction, we are left with removing the assumption on the first order bound of the covariant derivative of M.

Proof of the main theorem. Let $\Gamma \subset \mathrm{Iso}(M)$ acting on a Hadamard manifold (M, g) of bounded negative curvature. By the main proposition of [K05], there exists a positive number $\epsilon > 0$ with the following properties.

(1) The Ricci flow g_t started at $g_0 = g$ exists on the time interval $[0, 2\epsilon]$.
(2) The curvature of g_ϵ is contained in $[-b^2, -a^2]$ for some numbers $0 < a < b$ depending on the curvature of M, the dimension and ϵ.
(3) $\|\nabla R_\epsilon\| \leq C$ where $C > 0$ is a fixed constant and R_ϵ is the curvature tensor of g_ϵ.

Since the Ricci flow commutes with isometries, the group Γ admits a non-elementary isometric action on (M, g_ϵ). Thus we can apply Theorem 5.6 to the action of Γ on (M, g_ϵ) and complete the proof. □

Since via a factor projection, a lattice in a semi-simple Lie group G of non-compact type acts on each rank one factor of G in a non-elementary fashion we conclude

Corollary 5.7. *Let Γ be a lattice in a semisimple Lie group G of noncompact type containing at least one factor which is of rank one. Then Γ admits an isometric action on an L^p-space V with $H^1(\Gamma, V) \neq 0$.*

Remark 5.8. The lower bound on p we obtain from the proof of Theorem 5.6 is not sharp. For example, if M is hyperbolic space of dimension $n \geq 2$, then the bound we find equals $p = 2n - 2$ while it is known that $p = 2$ is possible.

Appendix A. Mean curvature of horospheres

This appendix is independent of the rest of this article, and we keep the exposition self-contained.

Throughout, we consider a smooth Hadamard manifold M of bounded negative curvature, that is, the curvature is contained in an interval $[-b^2, -a^2]$ for numbers $0 < a \leq b$. Denote by ∂M the ideal boundary of M.

Let $\Pi : T^1M \to M$ be the unit tangent bundle of M. The Levi Civita connection on TM determines a splitting

$$TT^1M = \mathcal{H} \oplus \mathcal{V}$$

where the *vertical bundle* $\mathcal{V}$ is the tangent bundle of the fibers, and such that the restriction of $d\Pi$ to the *horizontal bundle* $\mathcal{H}$ is a fiberwise isomorphism. The decomposition is orthogonal with respect to the Sasaki metric on T^1M. For this metric, the projection Π is a Riemannian submersion.

For $x \in M$ and a unit vector $v \in T^1M$ denote by γ_v the geodesic with initial velocity $\gamma_v'(0) = v$. The *geodesic spray* X is the generator of the geodesic flow $\Phi^t : T^1M \to T^1M, v \to \Phi^t v = \gamma_v'(t)$. It is a section of the bundle $\mathcal{H}$. Thus $\mathcal{H}$

decomposes as an orthogonal direct sum $\mathcal{H} = E \oplus \mathbb{R}X$ where the fiber of E at v is mapped by $d\Pi$ onto $v^\perp$.

Let $A(v)$ be the shape operator at x of the *horosphere* through x defined by the boundary point $\gamma_v(-\infty) \in \partial M$. Since horospheres are of class C^2, this is a well defined symmetric linear endomorphism of $v^\perp$. Using the identification of the fiber E_v of E at v with $v^\perp$, we view $A(v)$ as a symmetric linear endomorphism of E_v. Thus $v \to A(v)$ is a symmetric section of the bundle $E^* \otimes E$. The bundle $E^* \otimes E$ in turn is naturally equipped with a smooth Riemannian metric constructed from the Levi Civita connection of the Sasaki metric.

The following is the main result of this appendix. It was established in [Ho40] for surfaces, that is, in the case that the dimension n of M equals 2. It is also well known under the assumption that M admits a cocompact isometry group (where boundedness of ∇R is automatic), using tools from smooth dynamics. We refer to [Mn87] for details.

Proposition A.1. *If the covariant derivative ∇R of the curvature tensor is uniformly bounded in norm then the section $v \to A(v)$ of $E^* \otimes E$ is Hölder continuous.*

For $v \in T^1M$, the mean curvature $m(v)$ of the horosphere defined by v equals the trace of the shape operator $A(v)$. Thus as an immediate corollary of Proposition A.1 we obtain

Corollary A.2. *If the covariant derivative ∇R of the curvature tensor is uniformly bounded in norm then the function $m : v \to m(v)$ on T^1M is Hölder continuous.*

The strategy of proof consists in comparing the shape operator of horospheres with the shape operator of hypersurfaces depending smoothly on the defining data. More precisely, for $R > 0$ let $A_R(v)$ be the shape operator at $x = \gamma_v(0)$ of the hypersurface

$$N_R(v) = \{y \mid d(y, \exp(\gamma_v'(-R)^\perp)) = R\}$$

where exp denotes the exponential map of M, and define similarly $A_R^S(v)$ to be the shape operator at $x = \gamma_v(0)$ of the distance sphere of radius R about $\gamma_v(-R)$.

In the statement of the following lemma, norms are taken with respect to the natural Riemannian metric on $E^* \otimes E$. Only bounded negative curvature on M is necessary for Lemma A.3 and Lemma A.4 to hold true, that is, no assumption on the covariant derivative of the curvature tensor is required.

Lemma A.3. *There exist numbers $C_0 > 0, \alpha > 0$ only depending on the curvature bounds such that*

$$|A(v) - A_R(v)| \le C_0 e^{-\alpha R} \text{ and } |A(v) - A_R^S(v)| \le C_0 e^{-\alpha R}$$

for all $v \in T^1M$ and $R \ge 10$.

Proof. Let $x \in M, v \in T_x^1M$ and let H be the horosphere passing though x which is determined by the ideal boundary point $\gamma_v(-\infty) \in \partial M$. The shape operator of H at x can be computed as follows. Let $X \in T_xH = v^\perp$ be a tangent vector of H at x. Then X determines uniquely a Jacobi field J_X along γ_v with $J_X(0) =$

X and $\lim_{t\to-\infty}\|J_X(t)\| = 0$. The shape operator $A(v)$ then equals the linear endomorphism $X \to \nabla J_X(0)$ of the euclidean vector space $T_xH = v^\perp$. Here ∇J_X denotes the covariant derivative of J_X along the geodesic γ_v.

Similarly, for $R \geq 0$ the shape operator $A_R(v)$ is computed as the linear map $X \to \nabla J_X^R(0)$ where J_X^R is the Jacobi field along γ_v with $J_X^R(0) = X \in v^\perp$ and $\nabla J_X^R(-R) = 0$. It now suffices to show that

$$\|\nabla J_X(0) - \nabla J_X^R(0)\| \leq Ce^{-\alpha R}\|X\| \tag{13}$$

for constants $\alpha > 0, C > 0$ only depending on the curvature bounds.

For each X consider the Jacobi field $\hat{J}_X = J_X - J_X^R$ along γ_v. It vanishes at $t = 0$. The Rauch comparison theorem shows that

$$\|\hat{J}_X(-R)\| \geq \sinh aR\,\|\nabla\hat{J}_X(0)\| \tag{14}$$

(here as before, $-a^2$ is an upper curvature bound for M). Furthermore, there exists a constant $C_0 > 0$ only depending on the curvature bounds such that

$$\|\nabla\hat{J}_X(-R)\| \geq C_0\|\hat{J}_X(-R)\| \text{ for all } R. \tag{15}$$

On the other hand, we have $\nabla\hat{J}_X(-R) = \nabla J_X(-R)$. Hence using once more comparison, we obtain

$$\|\nabla\hat{J}_X(-R)\| \leq C_1e^{-aR}\|X\| \tag{16}$$

for a universal constant $C_1 > 0$. For $R > 10$ the estimates (14), (15) and (16) together yield that indeed,

$$\|\nabla J_X(0) - \nabla J_X^R(0)\| \leq C_2e^{-aR}\|\hat{J}_X(-R)\| \leq C_2C_0^{-1}C_1e^{-2aR}\|X\|$$

for a universal constant $C_2 > 0$.

This shows the first estimate stated in the lemma. The second estimate follows from exactly the same argument, replacing the condition $\nabla\hat{J}_X(-R) = \nabla J_X(-R)$ by the condition $\hat{J}_X(-R) = J_X(-R)$. The lemma follows. $\square$

The principal bundle $\mathcal{P} \to M$ of orthonormal frames in TM is equipped with the Levi Civita connection which defines a decomposition $T\mathcal{P} = \mathcal{H} \oplus \mathcal{V}$ where $\mathcal{V}$ is the tangent bundle of the fibers (note that this splitting is related to the splitting of TT^1M, but the fiber spaces are different. We nevertheless use the same notation here to keep the notations simple). This splitting determines a smooth Riemannian metric on $\mathcal{P}$ with the following properties. The fibers are isometric to the orthogonal group with the bi-invariant metric defined by the Killing form, the decomposition $T\mathcal{P} = \mathcal{H} \oplus \mathcal{V}$ is orthogonal, and $\mathcal{P} \to M$ is a Riemannian submersion. We denote by $d_{\mathcal{P}}$ the distance for this metric.

Let $x \in M$ and let $v \neq w \in T_x^1M$ be two unit tangent vectors based at x. Denote by $\angle(v, w)$ the euclidean angle between v, w and assume that $\angle(v, w) < \pi/4$. Choose an orthonormal basis $P(0, 0) = (e_1, \ldots, e_{n-1}, v)$ of TM at x with the property that e_{n-1} is contained in the plane spanned by v, w. Let $s \to \chi(s) \in T_x^1M$

($s \in [0, \angle(v,w)]$) be the shortest geodesic connecting v to w (which is contained in the plane spanned by v, w) and define

$$\zeta(s,t) = \gamma_{\chi(s)}(t).$$

Thus ζ is a variation of geodesics through the point x.

Let $Q(s,t)$ be the frame over $\zeta(s,t)$ obtained from $P(0,0)$ by parallel transport along $\zeta_s(t) = \zeta(s,t)$. The following lemma is probably well known and included here for completeness. In its formulation, norms of tangent vectors are taken with respect to the Riemannian metrics on $\mathcal{P}$ and M.

Lemma A.4. *There exists a number $C_1 > 0$ such that $\|\frac{\partial}{\partial s} Q(s,t)\| \leq C_1 \|\frac{\partial}{\partial s}\zeta(s,t)\|$ for all s, t.*

Proof. The map $(s,t) \to Q(s,t)$ is a variation of horizontal geodesics in $\mathcal{P}$ with the same starting point $Q(0,0)$. For each t, the path $Q_t : s \to Q(s,t)$ is a lift to $\mathcal{P}$ of the path $\zeta_t : s \to \zeta(s,t)$ in M. It is smooth but may not be horizontal. Its tangent can be decomposed as

$$Q_t'(s) = Q'_{t,\mathcal{H}}(s) + Q'_{t,\mathcal{V}}(s)$$

into a horizontal and vertical component. Since $\|Q'_{t,\mathcal{H}}(s)\| = \|\zeta_t'(s)\|$ we have to show the existence of a number $C > 0$ such that

$$\|Q'_{t,\mathcal{V}}(s)\| \leq C\|\zeta_t'(s)\| \tag{17}$$

for all s, t. That this holds indeed true can be seen as follows.

Let ω be the connection 1-form on $\mathcal{P}$. This is a one-form on $\mathcal{P}$ with values in the Lie algebra $\mathfrak{so}(n)$ of the structure group $O(n)$ of $\mathcal{P}$ which vanishes on the horizontal bundle $\mathcal{H}$. The $\mathfrak{so}(n)$-valued curvature form $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$ is horizontal, that is, it is annihilated by $\mathcal{V}$. If $W \to M$ denotes the vector bundle of antisymmetric linear endomorphisms of TM then Ω descends to the W-valued 2-form on M defined by the Riemannian curvature tensor.

Let $s \geq 0$ be fixed. For small $h > 0$ consider the piecewise smooth loop $\zeta_{s,h,t}$ in M based at $\zeta(s,t)$ which is the concatentation of the subsegment $u \to \zeta(s+u,t)$ of ζ_t connecting $\zeta_t(s)$ to $\zeta_t(s+h)$, the geodesic arc $u \to \zeta(s+h, t-u+h)$ and the geodesic arc $u \to \zeta(s, u-t-h)$. Let $Q_{s,h,t}$ be the lift of $\zeta_{s,h,t}$ to the (bordered) surface

$$Q = \{Q(s,t) \mid 0 \leq s \leq \angle(v,w), 0 \leq t\}.$$

This is a piecewise smooth closed curve in $\mathcal{P}$ which bounds a subsurface of Q.

Since the connection form ω vanishes on $\mathcal{H}$ and the arcs $t \to Q(s,t)$ are horizontal geodesics, the integral of ω over this piecewise smooth loop equals the element $a(s,h) = \int_s^{s+h} \omega(Q'_{t,\mathcal{V}}(u))du \in \mathfrak{so}(n)$. Since Q is a smooth map, for sufficiently small h we have

$$h\|Q'_{t,\mathcal{V}}(s)\| \leq 2\|a(s,h)\| \tag{18}$$

(here the norm is taken in the Lie algebra $\mathfrak{so}(n)$).

By comparison, for small enough h the area of the sector $(u,z) \to \zeta(u,z)$ ($s \leq u \leq s+h, 0 \leq z \leq t$) with respect to the pull-back of the metric on M is bounded from above by $C_0 h\|\zeta_t'(s)\|$ where $C_0 > 0$ is a constant only depending on the curvature bounds. Since the curvature tensor, viewed as a symmetric bundle map $W \to W$, is pointwise uniformly bounded in norm, the integral of the norm of the curvature tensor over this surface is bounded from above by $C_1 h\|\zeta_t'(s)\|$ for a universal constant $C_1 > 0$.

On the other hand, ω vanishes on $\mathcal{H}$ and the curves $t \to Q(s,t) \in \mathcal{P}$ are horizontal geodesics. Thus we have $\Omega|Q = d\omega|Q$. Using Stokes's theorem, this implies that $\|a(s,h)\| \leq C_2 h\|\zeta_t'(s)\|$ where $C_2 > 0$ is a universal constant. Using the inequality (18) and taking the limit as $h \searrow 0$ yields the estimate (17). From this the lemma follows. $\square$

Let $P(s,0) = (e_1, \dots, e_{n-2}, e_{n-1}(s), \chi(s))$ be the frame obtained by rotating the plane spanned by v, w keeping the orthogonal complement pointwise fixed. The frame $P(s,0)$ extends by parallel transport along the geodesics $\zeta_s : t \to \zeta(s,t)$ to a section $(s,t) \to P(s,t)$ of $\mathcal{P}$ over ζ. The map $(s,t) \to P(s,t) \in \mathcal{P}$ is a smooth embedding. Each of the curves $t \to P(s,t)$ is a horizontal geodesic. Furthermore, by the definition of the Sasaki metric, we have

$$d_{\mathcal{P}}(Q(s,t), P(s,t)) = s \text{ for all } s,t$$

and hence Lemma A.4 shows that

$$\|\frac{\partial}{\partial s} P(s,t)\| \leq 1 + C_1 \|\frac{\partial}{\partial s}\zeta(s,t)\|. \tag{19}$$

We are now ready to complete the proof of Proposition A.1. We now need to assume that the curvature of M is bounded between two negative constants and that the covariant derivative ∇R of the curvature tensor is uniformly bounded in norm.

Proof of Proposition A.1. By the assumption on M, for each $\xi \in \partial M$ the gradient field $\operatorname{grad} b_\xi$ of a Busemann function b_ξ at ξ is a section of T^1M of class C^2, with uniformly bounded first and second covariant derivatives [Shc83]. For $x \in M$ and $v = \operatorname{grad} b_\xi(x)$, the shape operator $A(v)$ at x of the horosphere $b_\xi^{-1}(b_\xi(x))$ equals the linear map

$$X \in v^\perp \to \nabla_X \operatorname{grad} b_\xi$$

where in all computations, we normalize shape operators to be positive semi-definite. Thus the restriction of the section A of $E^* \otimes E$ to the image of the section $\operatorname{grad} b_\xi$ is of class C^1, with pointwise uniformly bounded differential with respect to the Sasaki metric, and hence it is Hölder continuous.

For $v \in \operatorname{grad} b_\xi$, the tangent space of T^1M at v is a direct sum of the vertical tangent space $\mathcal{V}_v$, that is, the tangent space of the fibers of the fibration $T^1M \to M$, and the tangent space $T_v \operatorname{grad} b_\xi$ of the C^2-submanifold $\operatorname{grad} b_\xi$. Due to the fact that the eigenvalues of the shape operators of horospheres are bounded from above and below by universal positive constants, this decomposition is well adapted to the Sasaki metric. By this we mean that the angle between a vector of $\mathcal{V}_v$ and a vector of $T_v \operatorname{grad} b_\xi$ is bounded from below by a universal positive constant. Moreover, any two points $v, w \in T^1M$ can be connected by a piecewise smooth path which consists of finitely many segments alternating between segments in submanifolds $\operatorname{grad} b_\xi$ for some $\xi \in \partial M$ and segments in fibers of the fibration $T^1M \to M$ and whose length is bounded from above by a universal constant times the distance in T^1M between v,w. It therefore suffices to show the existence of numbers $C>0,\kappa\in(0,1)$ with the following property. Let $x\in M$ and let $v,w\in T^1_xM$; then

$$|A(v)-A(w)|\leq C\angle(v,w)^\kappa$$

where as before, $\angle(v,w)$ is the Euclidean angle between the unit vectors v,w and the norm is taken as the norm of a symmetric linear endomorphism of the Euclidean vector space T_xM. For ease of notations, we shall show $|A(-v)-A(-w)|\leq C\angle(v,w)^\kappa$.

To see that this indeed holds true let $v\neq w\in T^1_xM$ and assume without loss of generality that $\angle(v,w)<1/2$ and hence $\log\angle(v,w)<0$. We use now the constructions and notations from the beginning of this appendix and consider the minimal geodesic $\chi:[0,\angle(v,w)]\to T^1_xM$ which connects v to w and the corresponding variation of geodesics $\zeta(s,t)=\gamma_{\chi(s)}(t)$ with variation Jacobi fields J_s, determined by the initial condition $J_s(0)=0$ and $\nabla J_s(0)=\frac{d}{ds}\chi(s)$. Since $\|\nabla J_s(0)\|=1$ for all s, standard Jacobi field estimates show that

$$\|J_s(-R)\|\in[\sinh aR,\sinh bR]. \tag{20}$$

Let $R>0$ be such that $\int_0^{\angle(v,w)}\|\zeta_R'(s)\|ds=1$. By the estimate (20) we have

$$\angle(v,w)\sinh aR\leq 1\leq\angle(v,w)\sinh bR$$

and consequently

$$R\in[-\log\angle(v,w)/b,-\log\angle(v,w)/a+C_2]. \tag{21}$$

for a universal constant $C_2>0$.

Let $r=aR/10b$. Using the estimate (20), we have

$$d(\gamma_v(r),\gamma_w(r))\leq e^{br}\angle(v,w).$$

But $e^{br}\leq e^{aR/10}\leq e^{C_2/10}\angle(v,w)^{-1/10}$ by the estimate (21) and therefore

$$e^{br}\angle(v,w)\leq e^{C_2/10}\angle(v,w)^{1-1/10}. \tag{22}$$

By hyperbolicity and the estimate (19), we also have

$$d_{\mathcal{P}}(P(0,t),P(\angle(v,w),t))\leq C_3\angle(v,w)^{9/10} \tag{23}$$

for all $t\in[0,r]$ and a universal constant $C_3>0$, where $P(s,t)\in\mathcal{P}$ is as in the construction preceding this proof.

Using the trivialization of $TM|\zeta(s,t)$ defined by the frames $P(s,t)$, the Jacobi equation translates into the Riccati equation

$$A_s'(t)+A_s^2(t)+R_s(t)=0 \tag{24}$$

for the shape operators $A_s(t)$ of the hypersurfaces of distance t to $\exp(\zeta_s'(r)^\perp)$. Here $R_s(t)Y=R(Y,\zeta_s'(r-t))\zeta_s'(r-t)$ and the equation is thought of as an ODE for symmetric $(n-1,n-1)$-matrices written with respect to the parallel orthonormal trivialization $t\to P(s,t)$ of $TM|\zeta_s$. The solution we are looking for is determined by the initial condition $A_s(0)=0$. Note that for these expressions, we invert the time of the geodesics $t\to\zeta(s,t)$ and use a time shift so that $t=0$ corresponds to $\zeta(s,r)$ for all s.

The symmetric linear operators $R_s(t)$ of the euclidean vector spaces $\zeta_s'(r-t)^\perp$ are uniformly bounded and uniformly negative definite. Since the covariant derivative ∇R is pointwise uniformly bounded, it follows from the estimate (23) that there exists a number $C_4 > 0$ not depending on (v, w) such that

$$R_0(t)(1 + C_4\angle(v, w)^{9/10}) \leq R_{\angle(v,w)}(t)$$

for all $0 \leq t \leq r$. This means that $R_{\angle(v,w)}(t) - R_0(t)(1 - C_4\angle(v, w)^{9/10})$ is non-negative definite.

Namely, by the inequality (23), in the frame bundle $\mathcal{P}$, the length of the lift to $\mathcal{P}$ of the path $s \to \zeta_s(t)$ is bounded from above by $C_3\angle(v, w)^{9/10}$. The curvature form on $\mathcal{P}$ is a smooth 2-form on $\mathcal{P}$ with values in the Lie algebra $\mathfrak{so}(n)$ of the fiber group $SO(n)$. The natural Riemannian metric on $\mathcal{P}$ induces a metric on the bundle of $\mathfrak{so}(n)$-valued 2-forms on $\mathcal{P}$. Since the covariant derivative of the curvature tensor of M is pointwise uniformly bounded, the same holds true for the covariant derivative of this $\mathfrak{so}(n)$-valued 2-form. As a consequence, given the value of the form at the frame $P(0, t)$, the value at the frame $P(s, t)$ differs in norm from the value at the frame obtained by parallel transport along the lift of the path $s \to \zeta(s, t)$ by a uniform multiple of the length of the path, that is, by at most $C\angle(v, w)^{9/10}$ for a fixed number $C > 0$. Viewing the curvature as a symmetric endomorphism of the bundle of two-forms on M, if the restriction of such an endomorphism Λ to a codimension one subspace V is negative definite, with largest eigenvalue bounded from above by a fixed constant $-a < 0$, then for any symmetric endomorphism T such that the operator norm of $\Lambda - T$ is sufficiently small, the restriction of $\Lambda(1+C\|\Lambda-T\|)-T$ to V is nonpositive definite provided that $C > 0$ is sufficiently large.

As a consequence, the symmetric matrix

$$(1 + C_4\angle(v, w)^{9/10})R_0(t) - R_{\angle(v,w)}(t)$$

is nonpositive definite for all $0 \leq t \leq r$, where we use the frames $P(s, t)$ to identify the vector spaces $\zeta_s'(t)^\perp$ with a single euclidean vector space of dimension $n - 1$.

Denote by $\mathcal{J}(t)$ the matrix in the parallel frame $P(0, t)$ defining the Jacobi fields J_X along the geodesic $t \to \gamma_v(r - t)$ with initial condition $J_X(0) = X, \nabla J_X(0) = 0$ where $X \in \gamma_v'(r)^\perp$. The matrix valued curve $t \to \mathcal{J}(t)$ consists of invertible matrices starting at the identity and hence it can be written as $t \to \exp(V(t))$ where exp is the exponential map of the Lie group $GL(n-1, \mathbb{R})$ (for right invariant vector fields) and where $V(t) \in \mathfrak{gl}(n - 1, \mathbb{R})$.

Put $q = 1+C_4\angle(v, w)^{9/10}$ and $Q(t) = \exp(qV(t))$. Then $Q'(t) = qV'(t)Q(t)$ and consequently

$$B(t) = Q'(t)Q(t)^{-1} = q\mathcal{J}'(t)\mathcal{J}(t)^{-1}. \tag{25}$$

Since $\mathcal{J}'(t)\mathcal{J}(t)^{-1} = A_0(t)$, we have

$$B'(t) = qA_0'(t) = Q''(t)Q^{-1}(t) - (Q'(t)Q(t)^{-1})^2 = Q''(t)Q^{-1}(t) - q^2A_0(t)^2$$

and hence

$$B'(t) + B(t)^2 = qA_0'(t) + q^2A_0^2(t) \geq q(A_0'(t) + A_0^2(t)) = -qR_0(t). \tag{26}$$

Here the inequality in the expression (26) stems from the fact that the matrix $A_0(t)$ is positive semi-definite for all t.

Since $qR_0(t) \leq R_{\angle(v,w)}(t)$ for all t, we conclude from comparison of solutions of the Riccati equation (the main theorem of [EH90]) with the same initial condition that

$$B(t) \geq A_{\angle(v,w)}(t) \text{ for } 0 \leq t \leq r.$$

The equation (25) shows that $B(t) = qA_0(t)$ for all t and hence

$$A_{\angle(v,w)}(r) \leq qA_0(r).$$

As $A_{\angle(v,w)}(r) = A_r(w)$ and $A_0(r) = A_r(v)$ (via the change of coordinates defined by the frames $P(0,0)$ and $P(\angle(v,w),0)$) and as for $r > 10$ the eigenvalues of the shape operators $A_r(u)$ are bounded from above and below by a universal positive constant, exchanging the roles of v, w yields that

$$|A_r(v) - A_r(w)| \leq C_5 \angle(v,w)^{9/10}$$

for a universal constant $C_5 > 0$.

On the other hand, Lemma A.3 shows that $|A(v) - A_r(v)| \leq e^{-\alpha r}$ provided that $r > 10$. Since $r \geq -a \log \angle(v,w)/10b^2$, together we obtain

$$\begin{aligned} |A(v) - A(w)| &\leq |A(v) - A_r(v)| + |A_r(v) - A_r(w)| + |A_r(w) - A(w)| \\ &\leq C_6 \angle(v,w)^\chi \end{aligned}$$

for universal constants $C_6 > 0, \chi > 0$ which is what we wanted to show. □

Remark A.5. By Proposition A.1, for any $x \in M$ the shape operator of the horosphere defined by $v \in T^1_x M$, viewed as symmetric linear operator on $v^\perp$, depends in a Hölder continuous fashion on v. This is equivalent to stating that the subbundle TW^{su} of TT^1M whose fiber at $v \in T^1M$ equals the tangent space of the submanifold $\operatorname{grad} b_\xi$ ($\xi = \gamma_v(-\infty)$) is Hölder continuous. We refer to the explicit description of TW^{su} in the proof of Lemma 4.2 which immediately yields this equivalence. In the case that M is a surface, this is the result established in [Ho40].

For Hölder continuity of TW^{su} to hold true, the assumption that ∇R is uniformly bounded can not be omitted. Namely, it was shown in [BBB87] that for every $\alpha > 0$ and every $\epsilon > 0$ there exists a surface S_0 with a smooth metric of finite volume and curvature in $[-1-\epsilon, -1+\epsilon]$ and with the property that the subbundle TW^{su} of TT^1S_0 is not Hölder continuous with exponent α [BBB87].

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