Rank-one isometries of proper CAT(0)-spaces

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Abstract. Let X be a proper CAT(0)-space with visual boundary ∂X. Let G be a non-elementary group of isometries of X with limit set $\Lambda \subset \partial X$. We survey properties of the action of G on Λ under the assumption that G contains a rank-one element. Among others, we show that there is a dense orbit for the action of G on the complement of the diagonal Δ in $\Lambda \times \Lambda$ and that pairs of fixed points of rank-one elements are dense in $\Lambda \times \Lambda - \Delta$.

1. Introduction

A geodesic metric space (X, d) is called *proper* if closed balls in X of finite radius are compact. A proper CAT(0)-metric space X can be compactified by adding the *visual boundary* ∂X.

The isometry group $\mathrm{Iso}(X)$ of a proper CAT(0)-space X, equipped with the compact open topology, is a locally compact σ-compact topological group which acts as a group of homeomorphisms on ∂X. The *limit set* Λ of a subgroup G of $\mathrm{Iso}(X)$ is the set of accumulation points in ∂X of an orbit of the action of G on X. The limit set does not depend on the orbit, and it is closed and G-invariant. The group G is called *elementary* if either its limit set consists of at most two points or if G fixes a point in ∂X.

For every $g \in \mathrm{Iso}(X)$ the *displacement function* of g is the function $x \to d(x, gx)$. The isometry g is called *semisimple* if the displacement function assumes a minimum on X. If this minimum vanishes then g has a fixed point in X and is called *elliptic*, and otherwise g is called *axial*. If g is axial then the subset A of X on which the displacement function is minimal is isometric to a product space $C \times \mathbb{R}$ where C is a closed convex subset of A and where g acts on each of the geodesics $\{x\} \times \mathbb{R}$ as a translation. Such a geodesic is called an *axis* for g. We refer to the books [**1, 3, 6**] for basic properties of CAT(0)-spaces and for references.

Call an axial isometry g of X *rank-one* if there is an axis γ for g which does not bound a flat half-plane. Here by a flat half-plane we mean a totally geodesic embedded isometric copy of an euclidean half-plane in X.

Ballmann and Brin [**2**] investigated discrete groups of isometries of a proper CAT(0)-space X whose boundary ∂X contains more than two points and which act cocompactly on X. Such a group $G < \mathrm{Iso}(X)$ is necessarily non-elementary, and its limit set is the whole boundary ∂X. They showed (Theorem A of [**2**]) that

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if G contains a rank-one element then for any two non-empty open subsets U, V of ∂X there is an element $g \in G$ with $g(\partial X - U) \subset V$ and $g^{-1}(\partial X - V) \subset U$. It is possible to choose g to be rank-one. Moreover (Theorem 4.6 of [**2**]), G contains a free non-abelian subgroup.

In this note we extend this result to all non-elementary groups of isometries of a proper CAT(0)-space which contain a rank-one element. For the formulation of our result, call the action of a group G of homeomorphism of a compact space *minimal* if every orbit is dense.

Theorem 1.1. *Let X be a proper* Cat(0)*-space and let $G < \text{Iso}(X)$ be a non-elementary subgroup with limit set Λ which contains a rank-one element.*

1. *Λ is perfect, and the G-action on Λ is minimal.*
2. *Pairs of fixed points of rank-one elements are dense in the complement of the diagonal Δ of $\Lambda \times \Lambda$.*
3. *There is a dense orbit for the action of G on $\Lambda \times \Lambda - \Delta$.*
4. *G contains a free subgroup with two generators consisting of rank-one elements.*

The second part of the above theorem is contained in [**8**]. The last part of the above result is (in a slightly different context) contained in [**5**], with a different proof. The paper [**8**] also contains a proof of this last part (without the conclusion that the free subgroup consists of rank-one elements). The other parts are probably also known to the experts, however they seem to be unavailable in the literature in this form.

The organization of this paper is as follows. In Section 2 we collect some basic geometric properties of proper CAT(0)-spaces. In Section 3 we look at geodesics in proper CAT(0)-spaces. Following [**5**], we define contracting geodesics and study some of their properties. In Section 4 we investigate some geometric properties of isometries of a proper CAT(0)-space X, and in Section 5 we look at groups of isometries and prove the theorem above.

2. Basic CAT(0)-geometry

In this section we summarize some geometric properties of CAT(0)-spaces. We use the books [**1, 3, 6**] as our main references and for the discussion of a large set of examples.

A *triangle* Δ in a geodesic metric space consists of three vertices connected by three (minimal) geodesic arcs a, b, c. A comparison triangle $\bar{\Delta}$ for Δ in the euclidean plane is a triangle in $\mathbb{R}^2$ with the same side-lengths as Δ. By the triangle inequality, such a comparison triangle exists always, and it is unique up to isometry. For a point $x \in a \subset \Delta$ the comparison point of x in the comparison triangle $\bar{\Delta}$ is the point on the side $\bar{a}$ of $\bar{\Delta}$ corresponding to a whose distance to the endpoints of $\bar{a}$ coincides with the distance of x to the corresponding endpoints of a.

Definition 2.1. A geodesic metric space (X, d) is called a CAT(0)*-space* if for every geodesic triangle Δ in X with sides a, b, c and every comparison triangle $\bar{\Delta}$ in the euclidean plane with sides $\bar{a}, \bar{b}, \bar{c}$ and for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$ we have

$$d(x, y) \leq d(\bar{x}, \bar{y}).$$

A complete CAT(0)-space is called a *Hadamard space*. In a Hadamard space X, any two points can be connected by a unique geodesic which varies continuously with the endpoints. The distance function is convex: If γ,ζ are two geodesics in X parametrized on the same interval then the function $t\to d(\gamma(t),\zeta(t))$ is convex. More generally, we call a function $f:X\to\mathbb{R}$ *convex* if for every geodesic $\gamma:J\to\mathbb{R}$ the function $t\to f(\gamma(t))$ is convex [**1**].

For a fixed point $x\in X$, the *visual boundary* ∂X of X is defined to be the space of all geodesic rays issuing from x equipped with the topology of uniform convergence on bounded sets. This definition is independent of the choice of x. We denote the point in ∂X defined by a geodesic ray $\gamma:[0,\infty)\to X$ by $\gamma(\infty)$. We also say that γ *connects* x to $\gamma(\infty)$. If X is proper then the visual boundary of X is compact [**1**]. Note that if $\gamma_1,\gamma_2:[0,\infty)\to X$ are two geodesic rays with $\gamma_1(\infty)=\gamma_2(\infty)$ then the function $t\to d(\gamma_1(t),\gamma_2(t))$ is convex and bounded (see Chapter II.8 of [**6**]) and hence it is nonincreasing.

There is another description of the visual boundary of X as follows. Let $C(X)$ be the space of all continuous functions on X endowed with the topology of uniform convergence on bounded sets. Fix a point $y\in X$ and for $x,z\in X$ define

$$b_x(y,z)=d(x,z)-d(x,y).$$

Then we have

$$b_x(y,z)=-b_x(z,y)\text{ for all }y,z\in X \tag{2.1}$$

and

$$|b_x(y,z)-b_x(y,z')|\le d(z,z')\text{ for all }z,z'\in X \tag{2.2}$$

and hence the function $b_x(y,\cdot):z\to b_x(y,z)$ is one-Lipschitz and vanishes at y. The assignment $x\to b_x(y,\cdot)$ is an embedding of X into $C(X)$. Moreover, for every $x\in X$ the function $b_x(y,\cdot)$ is convex. If $\tilde y\in X$ is another basepoint then we have

$$b_x(\tilde y,\cdot)=b_x(y,\cdot)+b_x(\tilde y,y). \tag{2.3}$$

A sequence $\{x_n\}\subset X$ *converges at infinity* if $d(x_n,x)\to\infty$ and if the functions $b_{x_n}(y,\cdot)$ converge in $C(X)$. The visual boundary ∂X of X can also be defined as the subset of $C(X)$ of all functions which are obtained as limits of functions $b_{x_n}(y,\cdot)$ for sequences $\{x_n\}\subset X$ which converge at infinity. In particular, the union $X\cup\partial X$ is naturally a closed subset of $C(X)$.

Namely, if $\gamma:[0,\infty)\to X$ is any geodesic ray then for every sequence $t_n\to\infty$ the sequence $\{\gamma(t_n)\}$ converges at infinity, and the limit function $b_{\gamma(\infty)}(y,\cdot)$ does not depend on the sequence $\{t_n\}$. The function $b_{\gamma(\infty)}(y,\cdot)$ is called a *Busemann function* at $\gamma(\infty)\in\partial X$. The Busemann function $b_{\gamma(\infty)}(\gamma(0),\cdot)$ satisfies

$$b_{\gamma(\infty)}(\gamma(0),\gamma(t))=-t\text{ for all }t\ge 0, \tag{2.4}$$

moreover it is convex.

Vice versa, if the sequence $\{x_n\}\subset X$ converges at infinity then the geodesics γ_n connecting a fixed point $x\in X$ to x_n converge locally uniformly to a geodesic ray which only depends on the limit of the functions $b_{x_n}(y,\cdot)$ (Chapter II.1 and II.2 of [**1**]).

From now on let X be a *proper* (i.e. complete and locally compact) CAT(0)-space. Then $X\cup\partial X$ is compact. A subset $C\subset X$ is *convex* if for $x,y\in C$ the geodesic connecting x to y is contained in C as well. For every closed convex set $C\subset X$ and every $x\in X$ there is a unique point $\pi_C(x)\in C$ of smallest distance to x (Proposition II.2.4 of [**6**]). Now let $J \subset \mathbb{R}$ be a closed connected set and let $\gamma : J \to X$ be a geodesic arc. Then $\gamma(J) \subset X$ is closed and convex and hence there is a shortest distance projection $\pi_{\gamma(J)} : X \to \gamma(J)$. Then $\pi_{\gamma(J)}(x)$ is the unique minimum for the restriction of the function $b_x(y, \cdot)$ to $\gamma(J)$. By equality (2.3), this does not depend on the choice of the basepoint $y \in X$. The projection $\pi_{\gamma(J)} : X \to \gamma(J)$ is distance non-increasing.

For $\xi \in \partial X$ the function $t \to b_\xi(y, \gamma(t))$ is convex. Let $\overline{\gamma(J)}$ be the closure of $\gamma(J)$ in $X \cup \partial X$. If $b_\xi(y, \cdot)|\gamma(J)$ assumes a minimum then we can define $\pi_{\gamma(J)}(\xi) \subset \overline{\gamma(J)}$ to be the closure in $\overline{\gamma(J)}$ of the connected subset of $\gamma(J)$ consisting of all such minima. If $b_\xi(y, \cdot)|\gamma(J)$ does not assume a minimum then by continuity the set J is unbounded and by convexity either $\lim_{t\to\infty} b_\xi(y, \gamma(t)) = \inf\{b_\xi(y, \gamma(s)) \mid s \in J\}$ or $\lim_{t\to-\infty} b_\xi(y, \gamma(t)) = \inf\{b_\xi(y, \gamma(s)) \mid s \in J\}$. In the first case we define $\pi_{\gamma(J)}(\xi) = \gamma(\infty) \in \partial X$, and in the second case we define $\pi_{\gamma(J)}(\xi) = \gamma(-\infty)$. Then for every $\xi \in \partial X$ the set $\pi_{\gamma(J)}(\xi)$ is a closed connected subset of $\overline{\gamma(J)}$ (which may contain points in both X and ∂X).

The following simple observation will be useful several times in the sequel.

LEMMA 2.2. *Let $\gamma : J \to X$ be a geodesic, let $\xi \in \partial X$ and assume that $\pi_{\gamma(J)}(\xi) \cap X \neq \emptyset$. If $c : [0, \infty) \to X$ is a geodesic ray connecting a point $c(0) \in \pi_{\gamma(J)}(\xi)$ to $c(\infty) = \xi$ then $\pi_{\gamma(J)}(c(t)) = c(0)$ for all $t \geq 0$.*

PROOF. If $c : [0, \infty) \to X$ connects $c(0) \in \pi_{\gamma(J)}(\xi)$ to ξ then property (2.4) of Busemann functions implies that $b_\xi(c(0), c(R)) = -b_\xi(c(R), c(0)) = -R$ for all $R > 0$. Moreover, $b_\xi(c(R), \cdot)$ is one-Lipschitz and hence if $z = \pi_{\gamma(J)}(c(R)) \neq c(0)$ then $d(z, c(R)) < R$ and consequently $b_\xi(c(R), z) < R$. However, this implies that

$$b_\xi(c(0), z) = b_\xi(c(0), c(R)) + b_\xi(c(R), z) < 0$$

which violates the assumption that $c(0) \in \pi_{\gamma(J)}(\xi)$. The lemma is proven. $\square$

We also note the following easy fact.

LEMMA 2.3. *Let $\gamma : J \to X$ be a geodesic and let $(x_i) \subset X$ be a sequence converging to some $\xi \in \partial X$. Then up to passing to a subsequence, the sequence $\pi_{\gamma(J)}(x_i)$ converges to a point in $\pi_{\gamma(J)}(\xi)$.*

PROOF. Since the closure $\overline{\gamma(J)}$ of $\gamma(J)$ in $X \cup \partial X$ is compact, up to passing to a subsequence the sequence $\pi_{\gamma(J)}(x_i)$ converges to a point $z \in \overline{\gamma(J)}$. On the other hand, the functions $b_{x_i}(x, \cdot)$ converge as $i \to \infty$ locally uniformly to the Busemann function b_ξ. Now if $z \in \gamma(J)$ then this implies that z is a minimum for $b_\xi(x, \cdot)$ and hence $z \in \pi_{\gamma(J)}(\xi)$ by definition.

Otherwise assume that $J \supset [a, \infty)$ for some $a \in \mathbb{R}$ and that $\pi_{\gamma(J)}(x_i) = \gamma(t_i)$ where $t_i \to \infty$. Then by convexity, for every $s \in (a, \infty)$, every $t > s$ and every i which is sufficiently large that $t_i > t$ we have $b_{x_i}(x, \gamma(t)) \leq b_{x_i}(x, \gamma(s))$. Since $b_{x_i}(x, \cdot) \to b_\xi(x, \cdot)$ locally uniformly, we also have $b_\xi(x, \gamma(t)) \leq b_\xi(x, \gamma(s))$ and hence indeed $\lim_{t\to\infty} b_\xi(x, \gamma(t)) = \inf\{b_\xi(x, \gamma(s)) \mid s \in J\}$. This shows the lemma. $\square$

3. Contracting geodesics

In this section we discuss some geometric properties of geodesics in a proper CAT(0)-space X. As a convention, geodesics are always defined on closed connected subsets of $\mathbb{R}$. We begin with the following definition which is due to Bestvina and Fujiwara (Definition 3.1 of [**5**]).

Definition 3.1. A geodesic arc $\gamma : J \to X$ is *B-contracting* for some $B > 0$ if for every closed metric ball K in X which is disjoint from $\gamma(J)$ the diameter of the projection $\pi_{\gamma(J)}(K)$ does not exceed B.

We call a geodesic *contracting* if it is B-contracting for some $B > 0$. As an example, every geodesic in a CAT(κ)-space for some $\kappa < 0$ is B-contracting for a number $B = B(\kappa) > 0$ only depending on κ.

The next lemma (Lemma 3.2 and 3.5 of [**5**]) shows that a connected subarc of a contracting geodesic is contracting and that a triangle containing a B-contracting geodesic as one of its sides is uniformly thin.

Lemma 3.2. (1) *Let $\gamma : J \to X$ be a B-contracting geodesic. Then for every closed connected subset $I \subset J$, the subarc $\gamma(I)$ of γ is B+3-contracting.*
(2) *Let $\gamma : [a, b] \to X$ be a B-contracting geodesic. If $x \in X$ is such that $\pi_{\gamma[a,b]}(x) = a$ then for every $t \in [a, b]$ the geodesic connecting x to $\gamma(t)$ passes through the $3B + 1$-neighborhood of $\gamma(a)$.*

Lemma 3.2 implies that for a B-contracting biinfinite geodesic $\gamma : \mathbb{R} \to X$ and for $\xi \in \partial X - \{\gamma(-\infty), \gamma(\infty)\}$ the projection $\pi_{\gamma(\mathbb{R})}(\xi)$ is a compact subset of $\gamma(\mathbb{R})$ of diameter at most $6B + 4$.

Lemma 3.3. *For some $B > 0$ let $\gamma : \mathbb{R} \to X$ be a biinfinite B-contracting geodesic. Then for $\xi \in \partial X - \{\gamma(-\infty), \gamma(\infty)\}$ the restriction to $\gamma(\mathbb{R})$ of a Busemann function b_ξ at ξ is bounded from below and assumes a minimum. If $T \in \mathbb{R}$ is such that $\gamma(T) \in \pi_{\gamma(\mathbb{R})}(\xi)$ then*

$$|T - t| - 3B - 2 \le b_\xi(\gamma(T), \gamma(t)) \le |T - t| \text{ for all } t \in \mathbb{R}.$$

Proof. Let $\gamma : \mathbb{R} \to X$ be a biinfinite B-contracting geodesic and let $\xi \in \partial X - \{\gamma(\infty), \gamma(-\infty)\}$. Let $c : [0, \infty) \to X$ be the geodesic ray which connects $\gamma(0)$ to ξ. For $R > 0$ let $t_R \in \mathbb{R}$ be such that $\pi_{\gamma(\mathbb{R})}(c(R)) = \gamma(t_R)$. By Lemma 3.2, the geodesic c passes through the $3B + 1$-neighborhood of $\gamma(t_R)$. By triangle comparison, the geodesic segment $\gamma[0, t_R]$ is contained in the $3B + 1$-neighborhood of $c[0, \infty)$. Thus if there is a sequence $R_i \to \infty$ such that $t_{R_i} \to \infty$ (or $t_{R_i} \to -\infty$) then the geodesic ray $\gamma[0, \infty)$ (or $\gamma(-\infty, 0]$) is contained in the $3B+1$-neighborhood of $c[0, \infty)$ and hence $c = \gamma[0, \infty)$ (or $c = \gamma(-\infty, 0]$) which is impossible.

As a consequence, the set $\{t_R \mid R \ge 0\} \subset \mathbb{R}$ is bounded and therefore there is a number $T \in \mathbb{R}$ and a sequence $R_i \to \infty$ such that $t_{R_i} \to T$ $(i \to \infty)$. By Lemma 3.2, for sufficiently large i and all $t \in \mathbb{R}$ the geodesic connecting $\gamma(t)$ to $c(R_i)$ passes through the $3B + 2$-neighborhood of $\gamma(T)$. On the other hand, as $i \to \infty$ these geodesics converge locally uniformly to the geodesic connecting $\gamma(t)$ to ξ. Together with (2.1), (2.2) and (2.4) above, this implies that

$$|T - t| - 3B - 2 \le b_\xi(\gamma(T), \gamma(t)) \le |T - t|$$

as claimed in the lemma. In particular, the restriction of the function $b_\xi(\gamma(T), \cdot)$ to $\gamma(\mathbb{R})$ is bounded from below by $-3B-2$, and if $|T-t| > 3B+2$ then $b_\xi(\gamma(T), \gamma(t)) > 0$ and hence $\gamma(t) \notin \pi_{\gamma(\mathbb{R})}(\xi)$. □

Remark: Lemma 3.2 and Lemma 3.3 and their proofs are valid without the assumption that the space X is proper.

Two points $\xi \ne \eta \in \partial X$ are connected in X by a geodesic if there is a geodesic $\gamma : \mathbb{R} \to X$ with $\gamma(\infty) = \xi, \gamma(-\infty) = \eta$. Unlike in a proper hyperbolic geodesic metric space, such a geodesic need not exist. Therefore we define.

Definition 3.4. A point $\xi \in \partial X$ is called a *visibility point* if for every $\zeta \neq \xi \in \partial X$ there is a geodesic connecting ξ to ζ.

Lemma 3.2 and Lemma 3.3 are used to show (compare also Lemma 23 of [**15**]).

Lemma 3.5. *Let $\gamma : [0,\infty) \to X$ be a contracting geodesic ray. Then $\gamma(\infty) \in \partial X$ is a visibility point.*

Proof. Let $\gamma : [0,\infty) \to X$ be a B-contracting geodesic ray for some $B > 0$ and let $\xi \in \partial X - \gamma(\infty)$. By Lemma 3.3 (or, rather, its obvious modification for geodesic rays) the projection $\pi_{\gamma[0,\infty)}(\xi)$ is a compact subset of $\gamma[0,\infty)$ of diameter at most $6B + 4$.

Let $r \geq 0$ be such that $\gamma(r) \in \pi_{\gamma[0,\infty)}(\xi)$. Let $c : [0,\infty) \to X$ be the geodesic ray connecting $c(0) = \gamma(r)$ to ξ. By Lemma 2.2, for every $t > 0$ we have $\pi_{\gamma[0,\infty)}(c(t)) = \gamma(r)$. Together with Lemma 3.2, this shows that for every $t > 0$ the geodesic ζ_t connecting $\gamma(t)$ to $c(t)$ passes through the $3B + 1$-neighborhood of $\gamma(r)$. Since X is proper, up to reparametrization and up to passing to a subsequence we may assume that the geodesics ζ_t converge uniformly on compact sets as $t \to \infty$ to a geodesic ζ. By construction and by convexity, ζ connects $\gamma(\infty)$ to $c(\infty) = \xi$. Since $\xi \in \partial X - \gamma(\infty)$ was arbitrary, this shows the lemma. □

For fixed $B > 0$, B-contracting geodesics are stable under limits.

Lemma 3.6. *Let $B > 0$ and let $\gamma_i : J_i \to X$ be a sequence of B-contracting geodesics converging locally uniformly to a geodesic $\gamma : J \to X$. Then γ is B-contracting.*

Proof. Assume to the contrary that there is a sequence $(\gamma_i : J_i \to X)$ of B-contracting geodesics in X converging locally uniformly to a geodesic $\gamma : J \to X$ which is not B-contracting. Then there is a compact metric ball K which is disjoint from $\gamma(J)$ and such that the diameter of $\pi_{\gamma(J)}(K)$ is bigger than B. In other words, there are two points $x, y \in K$ with $d(\pi_{\gamma(J)}(x), \pi_{\gamma(J)}(y)) > B$.

Since $\gamma_i \to \gamma$ locally uniformly, for sufficiently large i the ball K is disjoint from γ_i. Let $u_i = \pi_{\gamma_i(J_i)}(x), z_i = \pi_{\gamma_i(J_i)}(y)$. If $i > 0$ is sufficiently large that K is disjoint from γ_i then we have $d(u_i, z_i) \leq B$ since γ_i is B-contracting. Moreover, the distance between u_i and x and between z_i and y is bounded independently of i. Thus up to passing to a subsequence we may assume that $u_i \to u, z_i \to z$. Then $u, z \in \gamma(J)$ and $d(u,z) \leq B$ by continuity and therefore up to possibly exchanging x and y we may assume that $u \neq \pi_{\gamma(J)}(x)$. Since the shortest distance projection of x into $\gamma(J)$ is unique, we have $d(u,x) > d(\pi_{\gamma(J)}(x), x)$. But $\gamma_i \to \gamma$ locally uniformly and $d(u_i, x) \to d(u,x)$ and therefore for sufficiently large i the point $\pi_{\gamma_i(J_i)}(\pi_{\gamma(J)}(x)) \in \gamma_i(J_i)$ is closer to x than u_i. This contradicts the choice of u_i and shows the lemma. □

4. Rank-one isometries

As before, let X be a proper CAT(0)-space. For an isometry g of X define the *displacement function* d_g of g to be the function $x \to d_g(x) = d(x, gx)$.

Definition 4.1. An isometry g of X is called *semisimple* if d_g achieves its minimum in X. If g is semisimple and $\min d_g = 0$ then g is called *elliptic*. A semisimple isometry g with $\min d_g > 0$ is called *axial*.

By the above definition, an isometry is elliptic if and only if it fixes at least one point in X. Any isometry of X which admits a bounded orbit in X is elliptic [**6**]. By Proposition II.3.3 of [**1**], an isometry g of X is axial if and only if there is a geodesic $\gamma : \mathbb{R} \to X$ such that $g\gamma(t) = \gamma(t+\tau)$ for every $t \in \mathbb{R}$ where $\tau = \min d_g > 0$. Such a geodesic is called an *oriented axis* for g. Note that the geodesic $t \to \gamma(-t)$ is an oriented axis for g^{-1}. The endpoint $\gamma(\infty)$ of γ is a fixed point for the action of g on ∂X which is called the *attracting fixed point*. The closed convex set $A \subset X$ of all points for which the displacement function of g is minimal is isometric to $C \times \mathbb{R}$ where $C \subset A$ is closed and convex. For each $x \in C$ the set $\{x\} \times \mathbb{R}$ is an axis of g.

The following definition is due to Bestvina and Fujiwara (Definition 5.1 of [**5**]).

Definition 4.2. An isometry $g \in \mathrm{Iso}(X)$ is called *B-rank-one* for some $B > 0$ if g is axial and admits a B-contracting axis.

We call an isometry g *rank-one* if g is B-rank-one for some $B > 0$.

The following statement is Theorem 5.4 of [**5**].

Proposition 4.3. *An axial isometry of X with axis γ is rank-one if and only if γ does not bound a flat half-plane.*

A homeomorphism g of a compact space K is said to act with *north-south dynamics* if there are two fixed points $a \neq b \in K$ for the action of g such that for every neighborhood U of a, V of b there is some $k > 0$ such that $g^k(K - V) \subset U$ and $g^{-k}(K - U) \subset V$. The point a is called the *attracting fixed point* for g, and b is the *repelling fixed point*. A rank-one isometry acts with north-south dynamics on ∂X (see Lemma 3.3.3 of [**1**]). For completeness of exposition, we provide a proof of this fact.

Lemma 4.4. *An axial isometry g of X is rank-one if and only if g acts with north-south dynamics on ∂X.*

Proof. Let g be a B-rank-one isometry of X and let γ be a B-contracting axis of g. Let $A \subset \gamma$ be a compact connected fundamental domain for the action of g on γ. A Busemann function $b_\xi(x, \cdot)$ depends continuously on $\xi \in \partial X$, and $b_{\gamma(\infty)}(\gamma(0), \gamma(t)) = -t$ and $b_{\gamma(-\infty)}(\gamma(0), \gamma(t)) = t$ for all $t \in \mathbb{R}$. By Lemma 3.3, for every $\xi \in \partial X - \{\gamma(\infty), \gamma(-\infty)\}$ the set $\pi_{\gamma(\mathbb{R})}(\xi)$ is a compact subset of $\gamma(\mathbb{R})$ of diameter at most $6B + 4$. Therefore the set $K = \{\xi \in \partial X - \{\gamma(\infty), \gamma(-\infty)\} \mid \pi_{\gamma(\mathbb{R})}(\xi) \cap A \neq \emptyset\}$ is closed and does not contain $\gamma(\infty), \gamma(-\infty)$.

Using again Lemma 3.3, for every $\xi \in \partial X - \{\gamma(\infty), \gamma(-\infty)\}$ there is some $k \in \mathbb{Z}$ with $\pi_{\gamma(\mathbb{R})}(\xi) \cap g^k A \neq \emptyset$. By equivariance under the action of the infinite cyclic subgroup of $\mathrm{Iso}(X)$ generated by g, this means that $\xi \in g^k K$ and hence $\cup_k g^k(K) = \partial X - \{\gamma(-\infty), \gamma(\infty)\}$.

By comparison, for every neighborhood V of $\gamma(-\infty)$ there is a number $m > 0$ such that V contains the endpoints of all geodesic rays $\zeta : [0, \infty) \to \partial X$ issuing from $\gamma(0) = \zeta(0)$ which pass through the $3B + 1$-neighborhood of $\gamma(-t)$ for some $t \geq m$. Lemma 3.2 and Lemma 2.2 show that if $\xi \in \partial X$ is such that there is a point $z \in \pi_{\gamma(\mathbb{R})}(\xi) \cap \gamma(-\infty, -m]$ then the geodesic ray connecting $\gamma(0)$ to ξ intersects the $3B + 1$-neighborhood of z and hence $\xi \in V$. This means that if $k \geq 0$ is sufficiently large that $g^{-k} A \subset \gamma(-\infty, -m]$ then $\cup_{j \leq -k} g^j K \subset V$ and hence $\partial X - V \subset \cup_{j > -k} g^j K \cup \{\gamma(\infty)\}$. Similarly, for every neighborhood U of a there is some $\ell > 0$ such that $\cup_{j \geq \ell} g^j K \subset U$. Then by equivariance, we have $g^{k+\ell}(\partial X - V) \subset U$. This shows that g acts with north-south dynamics on ∂X.

On the other hand, let g be an axial isometry of X which acts with north-south dynamics on ∂X, with attracting and repelling fixed point $a, b \in \partial X$, respectively. If g is not rank-one then there is an oriented axis γ for g which bounds a flat half-plane $F \subset X$. For $k \in \mathbb{Z}$ the image $g^k F$ of F under the isometry g^k is again a flat half-plane with boundary γ.

By the definition of the topology on ∂X, there is a neighborhood U of a in ∂X with the following property. Let $\xi : [0, \infty) \to X$ be a geodesic ray with $\xi(0) = \gamma(0)$ which is contained in a flat half-plane G bounded by γ. If ξ encloses an angle with the oriented ray $\gamma[0, \infty)$ in G which is bigger than $\pi/4$ then $\xi(\infty) \notin U$.

Let $\xi : [0, \infty) \to F$ be the geodesic ray with $\xi(0) = \gamma(0)$ which meets $\gamma(\mathbb{R})$ perpendicularly at $\gamma(0)$. Since by assumption g acts with north-south dynamics on ∂X, there is some $k > 0$ such that $g^k \xi(\infty) \in U$. On the other hand, $g^k \xi$ is a geodesic ray in the flat half-plane $g^k F$ which is perpendicular to γ. Then the angle in $g^k F$ between γ and the ray in $g^k F$ which connects x to $g^k \xi(\infty)$ equals $\pi/2$ as well. However, this means that $g^k \xi(\infty) \notin U$ which is a contradiction. Therefore g is rank-one. The lemma is proven. ☐

Remark: The proof of the statement that a rank-one isometry of X acts with north-south dynamics on ∂X is valid without the assumption that X is proper.

We conclude this section with a characterization of rank-one isometries which is easier to verify.

LEMMA 4.5. *Let $g \in \mathrm{Iso}(X)$ and assume that there are non-trivial open subsets V_1, V_2 of ∂X with the following properties.*

1. *The closures $\overline{V_1}, \overline{V_2}$ of V_1, V_2 are disjoint.*
2. *There is a number $B > 0$ and there is a B-contracting biinfinite geodesic with both endpoints in $\partial X - \overline{V_1} - \overline{V_2}$.*
3. *The distance between any biinfinite B-contracting geodesic with both endpoints in V_1 and any biinfinite B-contracting geodesic with both endpoints in V_2 is bounded from below by a universal positive constant.*
4. *$g(\partial X - V_2) \subset V_1$ and $g^{-1}(\partial X - V_1) \subset V_2$.*

Then g is rank-one, with attracting fixed point in V_1 and repelling fixed point in V_2.

PROOF. Let $g \in \mathrm{Iso}(X)$ be any isometry with the properties stated in the lemma for open subsets V_1, V_2 of ∂X with disjoint closure $\overline{V_1}, \overline{V_2}$. We have to show that g is rank-one. Note that g maps a nontrivial open neighborhood of $\overline{V_1}$ into $\overline{V_1}$ and hence the order of g is infinite. We show first that g is not elliptic.

For this assume to the contrary that g is elliptic. Then every orbit in X of the infinite cyclic subgroup G of $\mathrm{Iso}(X)$ generated by g is bounded. Let γ be a biinfinite B-contracting geodesic whose endpoints $\gamma(\infty), \gamma(-\infty)$ are contained in $\partial X - \overline{V_1} - \overline{V_2}$. Such a geodesic exists by the second requirement for V_1, V_2 stated in the lemma. For every $k \in \mathbb{Z}$ the geodesic $g^k \gamma$ is B-contracting.

Since the G-orbit of every point in X is bounded, the geodesics $g^k \gamma$ all pass through a fixed compact subset of X. By the Arzela Ascoli theorem, for every sequence $k_i \to \infty$ there is a subsequence (k_{i_j}) such that the geodesics $g^{k_{i_j}} \gamma$ converge locally uniformly to a biinfinite geodesic in X. Lemma 3.6 shows that such a limiting geodesic ζ is B-contracting. Moreover, by the properties of g the endpoints of ζ in ∂X are contained in $\overline{V_1}$. The collection $\mathcal{G}_+$ of all such limiting geodesics is closed for the compact open topology, moreover $\mathcal{G}_+$ is g-invariant.

Let $x \in X$ be a fixed point of g and let $K_+ = \{\pi_{\zeta(\mathbb{R})}(x) \mid \zeta \in \mathcal{G}_+\}$. Since $\mathcal{G}_+$ is closed and g-invariant and consists of B-contracting geodesics passing through a fixed compact subset of X, the set K_+ is compact and g-invariant. Moreover, by the third requirement in the statement of the lemma, there is a number $c > 0$ such that the distance in X of K_+ to any biinfinite B-contracting geodesic with both endpoints in $\overline{V_2}$ is at least $2c$. On the other hand, by definition of the set K_+, for $y \in K_+$ there is some $z \in X$ with $d(y, z) < c$ and there is some $\ell > 0$ such that $g^{-\ell}(z) \in \gamma$. Then for $m > \ell$, the point $g^{-m}(z)$ lies on a B-contracting geodesic connecting two points in $\overline{V_2}$. In particular, its distance to K_+ is at least $2c$. Since K_+ is g-invariant, this implies that $d(g^{-m}(y), g^{-m}(z)) \geq 2c$, on the other hand also $d(g^{-m}(y), g^{-m}(z)) = d(y, z) < c$. This is a contradiction and shows that indeed g is not elliptic.

Next assume to the contrary that g is not semisimple. By the classification of isometries of proper CAT(0)-spaces (Proposition 3.4 of [**1**]), in this case g fixes a point $\xi \in \partial X$ and it preserves every Busemann function at ξ. Let $x \in X$ and let $H = b_\xi(x, \cdot)^{-1}(-\infty, 0]$ be a closed horoball at ξ. Then H is a closed g-invariant convex subset of X whose closure $\overline{H}$ in $X \cup \partial X$ intersects ∂X in a closed subset ∂H.

We claim that for $i = 1, 2$ the intersection $\partial H \cap V_i$ contains a fixed point for g. For this let $\eta \in \partial X - \overline{V_1} - \overline{V_2} - \{\xi\}$ be the endpoint of a contracting geodesic ray. Such a point exists by the second assumption in the lemma. Define the shortest distance projection $\pi_H(\eta) \subset \overline{H}$ of η into H as follows. If an arbitrarily fixed Busemann function $b_\eta(x, \cdot)$ at η assumes a minimum on H then let $\pi_H(\eta)$ be the closure in $\overline{H}$ of the set of minima of $b_\eta(x, \cdot)$. Otherwise let $\pi_H(\eta) \subset \partial H$ be the set of accumulation points of sequences $(x_i) \subset H$ so that $b_\eta(x, x_i)$ converges to the infimum of $b_\eta(x, \cdot)$ on H as $i \to \infty$. Then $\pi_H(\eta)$ is a closed convex subset of $\overline{H}$.

By Lemma 3.5, η is a visibility point and hence there is a geodesic ρ connecting η to ξ. This geodesic satisfies $b_\eta(\rho(0), \rho(t)) = t = -b_\xi(\rho(0), \rho(t))$ for all t. In particular, if ρ is parametrized in such a way that $b_\xi(x, \rho(0)) = 0$ then since Busemann functions are convex and one-Lipschitz we have $\rho(0) \in \pi_H(\eta)$ (compare the simple argument in the proof of Lemma 2.2). If $z \in \pi_H(\eta) \cap H$ is another point then $z \in b_\eta(\rho(0), \cdot)^{-1}(0) \cap b_\xi(x, \cdot)^{-1}(-\infty, 0]$, and there is a geodesic ρ' connecting η to ξ which passes through $\rho'(0) = z$. The function $t \to d(\rho(t), \rho'(t))$ is convex and bounded and hence it is constant. By comparison, ρ and ρ' bound a totally geodesic embedded flat strip.

Now η is the endpoint of a contracting geodesic ray and hence by Lemma 3.2 and Lemma 3.8 of [**5**], the geodesic ray $\rho(-\infty, 0]$ is C-contracting for some $C > 0$. In particular, Lemma 3.2 shows that there is a subray of $\rho'(-\infty, 0]$ which is contained in a uniformly bounded neighborhood of $\rho(-\infty, 0]$. Therefore the width of the flat strip bounded by ρ, ρ' is uniformly bounded. On the other hand, we have $b_\eta(\rho(0), \rho(t)) = b_\eta(\rho(0), \rho'(t)) = -t$ for all t and consequently the distance between $\rho(0), \rho'(0)$ is uniformly bounded as well. This shows that $\pi_H(\eta) \cap H$ is a bounded and hence compact subset of H. By convexity of $\pi_H(\eta)$ we conclude that $\pi_H(\eta)$ is contained in H.

The horoball H is invariant under g and hence $\pi_H(g^k\eta) = g^k\pi_H(\eta)$ for all $k \in \mathbb{Z}$. Thus if there is a compact subset of X which intersects each of the sets $\pi_H(g^k\eta)$ $(k > 0)$ whose diameter is uniformly bounded then g has a bounded orbit and hence g is elliptic. It follows that there is a sequence $k_i \to \infty$ such that $g^{k_i}\pi_H(\eta) \subset H$

leave any compact subset of X. Then up to passing to a subsequence, the sequence $(g^{k_i}\rho(0))$ converges as $i\to\infty$ to a point in ∂H.

The horoball H is closed and convex, and $g^k\rho(0)\in\pi_H(g^k\eta)$ for all k. As a consequence, for each $k\neq 0$ the *Alexandrov angle* at $g^k\rho(0)$ between $\rho(0)$ and $g^k\eta$ is not smaller than $\pi/2$ (see [**6**] for a comprehensive treatment of Alexandrov angles). This implies that $\pi_{g^k\rho(-\infty,0]}(\rho(0))=g^k\rho(0)$ and hence since $g^k\rho(-\infty,0]$ is C-contracting for all k, the geodesic connecting $\rho(0)$ to $g^k\eta$ passes through a uniformly bounded neighborhood of $g^k\rho(0)$. By comparison, this implies that the sequence $(g^{k_i}\eta)\subset\partial X$ converges as $i\to\infty$ to the limit point in ∂H of the sequence $(g^{k_i}\rho(0))$. On the other hand, for every k the distance between $g^k\rho(0)$ and $g^{k+1}\rho(0)$ is uniformly bounded and hence using again comparison, the limit point is a fixed point for g. By the fourth assumption in the lemma, this fixed point is contained in $\partial H\cap V_1$. The same argument, applied to the sets $g^{-k}\pi_H(\eta)$ for $k>0$ shows the existence of a fixed point in $\partial H\cap V_2$.

By property 4) in the statement of the lemma, this implies that the fixed point set $\mathrm{Fix}(g)$ of g is disconnected. However, Corollary 3.3 of [**10**] (see also Theorem 1.1 of [**11**] for a related result) shows that $\mathrm{Fix}(g)$ is connected in the topology on ∂X induced by the Tits metric d_T. Since the identity $(\partial X,d_T)\to\partial X$ is continuous [**6**], $\mathrm{Fix}(g)$ is connected in the visual (cone) topology as well. This is a contradiction. Together we conclude that g is necessarily axial.

Let ζ be an oriented axis of g and let again $\eta\in\partial X-\overline{V_1}-\overline{V_2}-\{\xi\}$ be the endpoint of a contracting geodesic. Then η can be connected to $\zeta(\infty)$ by a geodesic ρ. This geodesic satisfies $b_\eta(\rho(0),\rho(t))=t$ for all t. Now $t\to d(\rho(t),\zeta(t))$ is decreasing and hence by the definition of the shortest distance projection, we conclude that $A=\pi_{\zeta(\mathbb{R})}(\eta)$ is bounded and hence compact. Equivariance under the action of g implies that $\pi_{\zeta(\mathbb{R})}(g^k\eta)=g^kA$ for all $k\in\mathbb{R}$ and therefore $g^k\eta\to\zeta(\infty)$ $(k\to\infty)$. On the other hand, by the fourth assumption in the lemma we have $g^k\eta\in V_1$ for all $k\geq 1$ and hence $\zeta(\infty)\in V_1$. The same argument also shows that $\zeta(-\infty)\in V_2$.

We are left with showing that g is rank-one. Namely, we saw in the previous paragraph that g admits an axis with endpoints $a\in V_1, b\in V_2\subset\partial X$. If g is not rank-one then ζ bounds a flat half-plane F whose ideal boundary is an arc ∂F connecting $a\in V_1$ to $b\in V_2$. Then ∂F intersects the open set $\partial X-\overline{V_1}-\overline{V_2}$. For every $k>0$, the set g^kF is a flat half-plane bounded by ζ whose closure in $\overline{X}$ intersects ∂X in the arc $g^k\partial F=\partial g^kF$ connecting a to b.

By the Arzela-Ascoli theorem, there is a sequence $k_i\to\infty$ such that the sequence $(g^{k_i}F)$ of flat half-planes bounded by ζ converges uniformly on compact sets to a flat half-plane G. The ideal boundary ∂G of G intersects $\partial X-\overline{V_1}-\overline{V_2}$ nontrivially. Let $z\in\partial G-\overline{V_1}-\overline{V_2}$ be such an intersection point, let $x\in\zeta$ be any point and let $\beta>0$ be the angle between ζ and the geodesic ray ξ connecting x to $z=\xi(\infty)$. For sufficiently large i the endpoint z_i in $\partial g^{k_i}F$ of the geodesic ray which issues from x, which is contained in $g^{k_i}F$ and which encloses an angle β with ζ is contained in $\partial X-\overline{V_1}-\overline{V_2}$. The reasoning in the proof of Lemma 4.4 shows that for $j>i$ the point $g^{k_j-k_i}z_i$ is the endpoint of the geodesic ray in $g^{k_j}F$ which issues from x and encloses the angle β with ζ. In particular, for $j>i$ we have $g^{k_j-k_i}z_i\in\partial X-\overline{V_1}$ which contradicts the assumption 4) in the statement of the lemma. The lemma is proven. □

5. Non-elementary groups of isometries

As in the previous sections, let X be a proper CAT(0)-space. The isometry group $\mathrm{Iso}(X)$ of X can be equipped with a natural locally compact σ-compact metrizable topology, the so-called *compact open topology*. With respect to this topology, a sequence $(g_i) \subset \mathrm{Iso}(X)$ converges to some isometry g if and only if $g_i \to g$ uniformly on compact subsets of X. A closed subset $A \subset \mathrm{Iso}(X)$ is compact if and only if there is a compact subset K of X such that $gK \cap K \neq \emptyset$ for every $g \in A$. In particular, the action of $\mathrm{Iso}(X)$ on X is proper.

Let $G < \mathrm{Iso}(X)$ be a subgroup of the isometry group of X. The *limit set* Λ of G is the set of accumulation points in ∂X of one (and hence every) orbit of the action of G on X. If the closure of G is non-compact then its limit set is a compact non-empty G-invariant subset of ∂X. If $g \in G$ is axial with axis γ, then $\gamma(\infty), \gamma(-\infty) \in \Lambda$. In particular, the two fixed points for the action on ∂X of a rank-one element are contained in Λ.

A compact space is *perfect* if it does not have isolated points. We first observe

Lemma 5.1. *Let $G < \mathrm{Iso}(X)$ be a subgroup which contains a rank-one element g. Then the limit set Λ of G is the closure in ∂X of the set of fixed points of conjugates of g in G. If Λ contains at least three points then Λ is perfect.*

Proof. Let $G < \mathrm{Iso}(X)$ be a subgroup which contains a rank-one element $g \in G$. Let Λ be the limit set of G. We claim that Λ is contained in the closure of the G-orbit of the two fixed points of g. For this let $\xi \in \Lambda$, let γ be a B-contracting axis of g for some $B > 0$ and let $(g_i) \subset G$ be a sequence such that $(g_i\gamma(0))$ converges to ξ. There are two cases possible.

In the first case, up to passing to a subsequence, the geodesics $g_i\gamma$ eventually leave every compact set. Let $x_0 = \gamma(0)$ and for $i \geq 1$ let $x_i = \pi_{g_i\gamma(\mathbb{R})}(\gamma(0))$. Then $d(x_0, x_i) \to \infty$ $(i \to \infty)$. On the other hand, $g_i\gamma$ is B-contracting. Hence by Lemma 3.2, a geodesic ζ_i connecting x_0 to g_ix_0 passes through the $3B+1$-neighborhood of x_i, and the same is true for a geodesic η_i connecting x_0 to $g_i\gamma(\infty)$. Since $d(x_0, x_i) \to \infty$ $(i \to \infty)$, by convexity, by the description of the topology on ∂X as the topology of uniform convergence on compact sets for geodesic rays issuing from x_0, by CAT(0)-comparison and compactness, we conclude the following. After passing to a subsequence, the sequences (x_i) and (g_ix_0) and $(g_i\gamma(\infty))$ converge as $i \to \infty$ to the same point in ∂X. But $g_ix_0 \to \xi$ and therefore $g_i\gamma(\infty) \to \xi$. However, $g_i\gamma(\infty)$ is a fixed point of the conjugate $g_igg_i^{-1}$ of g. This shows that indeed ξ is contained in the closure of the fixed points of all conjugates of g.

In the second case there is a compact subset K of X such that $g_i\gamma \cap K \neq \emptyset$ for all i. Since X is proper by assumption, up to passing to a subsequence we may assume that the B-contracting geodesics $g_i\gamma$ converge locally uniformly to a geodesic ζ. On the other hand, we have $d(g_ix_0, x_0) \to \infty$ $(i \to \infty)$ and hence up to passing to a subsequence the geodesic arcs connecting x_0 to g_ix_0 converge as $i \to \infty$ to a geodesic ray which connects x_0 to one of the endpoints of ζ in ∂X. Then the limit ξ of the sequence (g_ix_0) is an endpoint of ζ and hence once again, ξ is contained in the closure of the fixed points of conjugates of g as claimed.

Now assume that the limit set Λ of G contains at least 3 points. Let g be any rank-one element of G. Then Λ contains at least one point ξ which is not a fixed point of g. Since by Lemma 4.4 g acts with north-south dynamics on ∂X, the sequence $(g^k\xi)$ consists of pairwise distinct points which converge as $k \to \infty$ to the attracting fixed point of g. Similarly, the sequence $(g^{-k}\xi)$ converges as $k \to \infty$ to the repelling fixed point of g. Moreover, by the above, a point $\xi \in \Lambda$ which is not a fixed point of a rank-one element of G is a limit of fixed points of rank-one elements. This shows that Λ is perfect and completes the proof of the lemma. $\square$

A subgroup of $\mathrm{Iso}(X)$ which contains a rank-one element and whose limit set contains at least three points may fix globally a point in ∂X. An easy example is the upper triangular subgroup of $SL(2,\mathbb{R})$ acting simply transitively on the hyperbolic plane $\mathbf{H}^2$ and fixing one point on the boundary of $\mathbf{H}^2$. Therefore we define a subgroup G of $\mathrm{Iso}(X)$ to be *non-elementary* if its limit set contains at least 3 points and if moreover G does not fix globally a point in ∂X.

The action of a group G on a topological space Y is called *minimal* if every G-orbit is dense. The following lemma completes the proof of the first part of the theorem from the introduction.

Lemma 5.2. *Let $G < \mathrm{Iso}(X)$ be a non-elementary group with limit set Λ which contains a rank-one element $g \in G$ with fixed points $a \neq b \in \Lambda$. Then for every non-empty open set $V \subset \Lambda$ there is some $u \in G$ with $u\{a,b\} \subset V$. Moreover, the action of G on Λ is minimal.*

Proof. Let $G < \mathrm{Iso}(X)$ be a non-elementary subgroup with limit set Λ and let $g \in G$ be a rank-one isometry with attracting and repelling fixed points $a, b \in \Lambda$, respectively. Let $V \subset \Lambda$ be a non-empty open set. By Lemma 5.1, the limit set Λ is perfect and up to replacing g by g^{-1} (and exchanging a and b) there is an element $v \in G$ which maps a to $v(a) \in V - \{a,b\}$. Then $h = vgv^{-1}$ is a rank-one element with fixed points $v(a) \in V, v(b) \in \Lambda$. In particular, by Lemma 4.4, h acts with north-south dynamics on Λ and hence if $v(b) \neq b$ then we have $h^k\{a,b\} \subset V$ for sufficiently large k.

If $v(b) = b$ then let $\rho \in G$ be an element with $\rho(b) \neq b$. Such an element exists since by assumption, G does not fix globally the point b. Since $v(a) \neq a$ the orbit of a under the action of the infinite cyclic subgroup H of G generated by $h = vgv^{-1}$ is infinite and hence we can find some $w \in H$ such that $w(a) \neq \rho^{-1}(b)$. Then $u = \rho \circ w$ maps $\{a,b\}$ to $\Lambda - \{b\}$, and for sufficiently large k, the isometry $h^k u$ maps $\{a,b\}$ into V.

By Lemma 4.4, a rank-one isometry of X acts on ∂X with north-south dynamics and hence every non-empty closed G-invariant subset A of ∂X contains every fixed point of every rank-one element. Namely, if $a \neq b$ are the two fixed points of a rank-one element of G and if there is some $\xi \in A - \{a,b\}$ then also $\{a,b\} \subset A$ since A is closed. On the other hand, if $a \in A$ then the above consideration shows that there is some $h \in G$ with $h(a) \in \Lambda - \{a,b\}$ and once again, we conclude by invariance that $b \in A$ as well. Now the set of all fixed points of rank-one elements of G is G-invariant and hence the smallest non-empty closed G-invariant subset of ∂X is the closure of the set of fixed points of rank-one elements. This set contains the limit set Λ by Lemma 5.1 and hence it coincides with Λ. In other words, the action of G on Λ is minimal. The lemma is proven. $\square$

Corollary 5.3. *Let $G < \mathrm{Iso}(X)$ be a non-elementary subgroup which contains a rank-one element. Then every element $u \in G$ which acts on ∂X with north-south dynamics is rank-one.*

Proof. Let $G < \mathrm{Iso}(X)$ be a non-elementary subgroup with limit set Λ which contains a B-rank-one element for some $B > 0$. Assume that $u \in G$ acts on ∂X with north-south dynamics, with fixed points $a \neq b$. It suffices to verify that there is some $k > 0$ and there are neighborhoods V_1 of a, V_2 of b which satisfy the assumptions in Lemma 4.5 for u^k.

For this note first that since Λ is perfect, there is some $k > 0$ and there are open neighborhoods V_1, V_2 of a, b in ∂X with disjoint closures $\overline{V_1}, \overline{V_2}$ such that $\Lambda - \overline{V_1} - \overline{V_2} \neq \emptyset$ and that $u^k(\partial X - V_2) \subset V_1$ and that $u^{-k}(\partial X - V_1) \subset V_2$. By Lemma 5.2, there is then a biinfinite B-contracting geodesic with both endpoints in $\Lambda - \overline{V_1} - \overline{V_2}$. This B-contracting geodesic is the image of an axis of a B-rank-one element of G.

We are left with showing that via perhaps decreasing the neighborhoods V_1, V_2 of a, b we can guarantee that the distance in X between any B-contracting geodesic line with both endpoints in V_1 and any B-contracting geodesic line with both endpoints in V_2 is bounded from below by a universal positive constant.

For this fix a point $x \in X$ and let $\zeta_1, \zeta_2 : [0, \infty) \to X$ be geodesic rays connecting x to a, b, respectively. By the definition of the topology on ∂X, if U is a sufficiently small neighborhood of a and if γ is a B-contracting geodesic line with both endpoints in U then the geodesic connecting x to $\pi_{\gamma(\mathbb{R})}(x)$ longes ζ_1 for a long initial segment only depending on U. Similarly, for a sufficiently small neighborhood V of b and any B-contracting geodesic with endpoints in V, a geodesic connecting x to $\pi_{\eta(\mathbb{R})}(x)$ longes ζ_2 for a long initial segment only depending on V. From this the existence of neighborhoods V_1, V_2 of a, b with the properties stated in Lemma 4.5 is immediate. □

A free group with two generators is hyperbolic in the sense of Gromov [**12**]. In particular, it admits a Gromov boundary which can be viewed as a compactification of the group. As another immediate consequence of Lemma 4.5 we obtain the fourth part of the theorem from the introduction.

Corollary 5.4. *Let $G < \mathrm{Iso}(X)$ be a non-elementary group which contains a rank-one element. Let $\Lambda \subset \partial X$ be the limit set of G. Then G contains a free subgroup Γ with two generators and the following properties.*

1. *Every element $e \neq g \in \Gamma$ is rank-one.*
2. *There is a Γ-equivariant embedding of the Gromov boundary of Γ into Λ.*

Proof. Let $G < \mathrm{Iso}(X)$ be a non-elementary subgroup which contains a rank-one element g. By Lemma 5.2, there are two rank-one elements g, h whose fixed point sets are disjoint.

Let a, b be the attracting and repelling fixed point of g, respectively, and let x, y be the attracting and repelling fixed point of h. By Lemma 4.4, g, h act with north-south dynamics on ∂X. Then up to replacing g, h by some power we can find small open neighborhoods U_1, U_2, U_3, U_4 of a, b, x, y in ∂X with pairwise disjoint closure such that the pair U_1, U_2 satisfies the requirements stated in Lemma 4.5 for g, and that the same holds true for the pair U_3, U_4 and h. Since g, h act on ∂X with north-south dynamics there are numbers $k > 0, \ell > 0$ such that $g^{mk}(\overline{U_3} \cup \overline{U_4}) \subset U_1 \cup U_2$ and $h^{m\ell}(\overline{U_1} \cup \overline{U_2}) \subset U_3 \cup U_4$ for every $m \in \mathbb{Z}$ (here $\overline{U_i}$ is the closure of U_i). By the usual ping-pong argument (see e.g. p.136-138 of [**16**]), the isometries $g^{mk}, h^{m\ell}$ generate a free subgroup Γ of G. By construction and Lemma 4.5, the elements of

Γ are rank-one, and there is a Γ-equivariant embedding of the Gromov boundary of the free group into Λ. The corollary is proven. □

Lemma 4.5 is also used to show the second and third part of the theorem from the introduction.

LEMMA 5.5. *Let $G < \mathrm{Iso}(X)$ be a non-elementary subgroup which contains a rank-one element.*

(1) *The pairs of fixed points of rank-one elements of G are dense in $\Lambda\times\Lambda-\Delta$.*
(2) *The action of G on $\Lambda\times\Lambda-\Delta$ has a dense orbit.*

PROOF. Let $G < \mathrm{Iso}(X)$ be a closed non-elementary subgroup with limit set Λ which contains a B-rank one element g with attracting fixed point $a\in\Lambda$ and repelling fixed point $b\in\Lambda$.

Let $U\subset\Lambda\times\Lambda-\Delta$ be a non-empty open set. Our goal is to show that U contains a pair of fixed points of a rank-one element $g\in G$. For this we may assume that there are small open sets $V_i\subset\partial X-\{a,b\}$ with disjoint closure $\overline{V_i}$ and such that $U=V_1\times V_2\cap\Lambda\times\Lambda-\Delta$ and that $\Lambda-\overline{V_1}-\overline{V_2}\neq\emptyset$. We also may assume that the distance between any B-contracting geodesic with both endpoints in $\overline{V_1}$ and a B-contracting geodesic with both endpoints in $\overline{V_2}$ is bounded from below by a universal positive constant (compare the proof of Corollary 5.3). Moreover, by Lemma 5.2, there is some $q\in G$ with $q\{a,b\}\subset\Lambda-\overline{V_1}-\overline{V_2}$. The image under q of a B-contracting axis is a B-contracting geodesic with both endpoints in $\Lambda-\overline{V_1}-\overline{V_2}$.

Choose some $u\in G$ which maps $\{a,b\}$ into V_1. Such an element exists by Lemma 5.2. Then $v=ugu^{-1}$ is a B-rank-one isometry with fixed points $ua,ub\in V_1$. Similarly, there is a B-rank-one isometry w with both fixed points in V_2. Via replacing v,w by sufficiently high powers we may assume that $v(\partial X-V_1)\subset V_1, v^{-1}(\partial X-V_1)\subset V_1$ and that $w(\partial X-V_2)\subset V_2, w^{-1}(\partial X-V_2)\subset V_2$. Then we have $wv(\partial X-V_1)\subset V_2$ and $v^{-1}w^{-1}(\partial X-V_2)\subset V_1$ and hence by Lemma 4.5, wv is rank-one with fixed points in $V_1\times V_2$ and hence in U. The first part of the lemma is proven.

To show the second part of the lemma, we show first that for any non-empty open sets W_1,W_2 in $\Lambda\times\Lambda-\Delta$ there is some $h\in G$ with $hW_1\cap W_2\neq\emptyset$. For this assume without loss of generality that $W_1=U_1\times U_2, W_2=U_3\times U_4$ where U_1,U_2 and U_3,U_4 are non-empty open subsets of Λ with disjoint closure. Since Λ is perfect, by possibly replacing U_i be proper non-empty open subsets we may assume that the sets U_i are pairwise disjoint.

By the first part of the lemma, there is a rank-one element $u\in G$ with attracting fixed point in U_1 and repelling fixed point in U_4. Since u acts on ∂X with north-south dynamics, there is some $k>0$ and a small open neighborhood $U_5\subset U_1$ of the attracting fixed point of u such that $u^{-k}(U_5\times U_2)\subset U_1\times U_4$. Similarly, let $w\in G$ be a rank-one element with attracting fixed point in U_3 and repelling fixed point in $u^{-k}U_2\subset U_4$. Then we can find a number $\ell>0$ and an open subset U_6 of U_2 such that $w^{\ell}(u^{-k}(U_5\times U_6))\subset U_3\times U_4$.

The limit set Λ of G is compact and metrizable (see [**6**]) and hence $\Lambda\times\Lambda-\Delta$ is second countable. Choose a countable basis of open sets for $\Lambda\times\Lambda-\Delta$ of the form $U_i^1\times U_i^2$ where for each i the sets U_i^1,U_i^2 are non-empty, open and disjoint. We construct inductively a sequence of non-empty open sets $V_i^j\subset U_1^j$ $(j=1,2)$ such that for each $i\geq 1$ we have $V_{i+1}^j\subset V_i^j$ and that there is some $g_i\in G$ with $g_i(\overline{V_i^1}\times\overline{V_i^2})\subset U_i^1\times U_i^2$. Namely, write $V_1^j=U_1^j$ for $j=1,2$ and for some $i\geq 0$

assume that the sets V_i^j and the elements $g_i \in G$ have already been constructed. By the above, there is some $g_{i+1} \in G$ with $g_{i+1}(V_i^1 \times V_i^2) \cap U_{i+1}^1 \times U_{i+1}^2 \neq \emptyset$. Define $V_{i+1}^1 \times V_{i+1}^2$ to be any non-empty open product set whose closure is contained in $V_i^1 \times V_i^2 \cap g_{i+1}^{-1}(U_{i+1}^1 \times U_{i+1}^2)$. Then clearly the set $V_{i+1}^1 \times V_{i+1}^2$ and the map $g_{i+1} \in G$ satisfy the requirement of the inductive construction.

Since each of the sets $\overline{V_i^1} \times \overline{V_i^2}$ is compact and non-empty and since $V_{i+1}^1 \times V_{i+1}^2 \subset V_i^1 \times V_i^2$ for all i, there is some $z \in \cap_i(\overline{V_i^1} \times \overline{V_i^2})$. By construction, the G-orbit of z passes through each of the sets $U_i^1 \times U_i^2$ and hence the G-orbit of z is dense in $\Lambda \times \Lambda - \Delta$. This completes the proof of the lemma. □

We complete the discussion in this section by looking at groups of isometries of a proper CAT(0)-space X which are closed with respect to the compact open topology.

As before, let Δ be the diagonal of $\partial X \times \partial X$. We have (Lemma 6.1 of [**13**]).

LEMMA 5.6. *Let $G < \text{Iso}(X)$ be a closed subgroup with limit set Λ. Let $(a, b) \in \Lambda \times \Lambda - \Delta$ be the pair of fixed points of a rank-one element. Then the G-orbit of (a, b) is a closed subset of $\Lambda \times \Lambda - \Delta$.*

Lemma 5.6 is used to show the following strengthening of Corollary 5.4 for closed non-elementary subgroups of $\text{Iso}(X)$ (compare [**4, 5**]).

PROPOSITION 5.7. *Let $G < \text{Iso}(X)$ be a closed non-elementary group which contains a rank-one element. Let $\Lambda \subset \partial X$ be the limit set of G. If G does not act transitively on $\Lambda \times \Lambda - \Delta$ then G contains a free subgroup Γ with two generators and the following properties.*

1. *Every element $e \neq g \in \Gamma$ is rank-one.*
2. *There is a Γ-equivariant embedding of the Gromov boundary of Γ into Λ.*
3. *There are infinitely many elements $u_i \in \Gamma$ $(i > 0)$ with fixed points a_i, b_i such that for all i the G-orbit of $(a_i, b_i) \in \Lambda \times \Lambda - \Delta$ is distinct from the orbit of $(b_j, a_j)(j > 0)$ or $(a_j, b_j)(j \neq i)$.*

PROOF. Let $G < \text{Iso}(X)$ be a closed non-elementary subgroup with limit set Λ which contains a rank-one element g with B-contracting axis γ. Assume that G does not act transitively on $\Lambda \times \Lambda - \Delta$. By Lemma 5.5, there are two rank-one elements $g, h \in G$ whose pairs of endpoints are contained in distinct orbits for the action of G on $\Lambda \times \Lambda - \Delta$. In particular, no positive powers of these elements are conjugate, and the elements g, h admit B-contracting axes for some $B > 0$. By Lemma 5.2, via replacing h by a conjugate we may assume that the fixed points of g, h are all distinct. Corollary 5.4 then shows that up to replacing g, h by suitably chosen powers we may assume that the subgroup Γ of G generated by g, h is free and consists of rank-one elements. Moreover, there is an equivariant embedding of the boundary of the free group with two generators into the limit set of Λ of G.

Now Proposition 2 of [**4**] implies that there are infinitely many elements in Γ which are pairwise not mutually conjugate in G and whose inverses are not conjugate. We give a version of this argument here which is in the spirit of the arguments used earlier. Namely, let a, b and x, y be the attracting and repelling fixed points of g, h, respectively. By Lemma 5.6, we may assume that there are open neighborhoods U_1, U_2 of a, b and U_3, U_4 of x, y such that the G-orbit of (a, b) does not intersect $U_3 \times U_4$ and that the G-orbit of (x, y) does not intersect $U_1 \times U_2$. By replacing g,h by suitable powers we may moreover assume that $g(\cup_{j\neq 2}\overline{U_j})\subset U_1$, $g^{-1}(\cup_{j\neq 1}\overline{U_j})\subset U_2$, $h(\cup_{j\neq 4}\overline{U_j})\subset U_3$ and $h^{-1}(\cup_{j\neq 3}\overline{U_j})\subset U_4$.

For numbers $n,m,k,\ell>2$ consider the isometry

$$f=f_{nmk\ell}=g^nh^mg^kh^{-\ell}\in\Gamma.$$

It satisfies $f(\overline{U_1})\subset U_1, f^{-1}(\overline{U_3})\subset U_3$ and hence the attracting fixed point of f is contained in U_1 and its repelling fixed point is contained in U_3.

Since $n>2$, its conjugate $f_1=g^{-1}fg$ satisfies $f_1(\overline{U_1})\subset U_1$ and $f_1^{-1}(\overline{U_2})\subset U_2$, i.e. its attracting fixed point is contained in U_1 and its repelling fixed point is contained in U_2. Furthermore, since $m>2$, its conjugate $f_2=h^{-1}g^{-n}fg^nh$ has its attracting fixed point in U_3 and its repelling fixed point in U_4, and its conjugate $f_3=h^{-1}fh$ has its attracting fixed point in U_4 and its repelling fixed point in U_3.

As a consequence, f is conjugate to both an element with fixed points in $U_1\times U_2$ as well as to an element with fixed points in $U_3\times U_4$. This implies that f is not conjugate to either g or h. Moreover, since g and h can not both be conjugate to h^{-1}, by eventually adjusting the size of U_3,U_4 we may assume that f is not conjugate to h^{-1}.

We claim that moreover via perhaps increasing the values of n,ℓ we can achieve that $f_{nmk\ell}$ is not conjugate to f^{-1}. Namely, as $n\to\infty$, the fixed points of the conjugate $g^{-n}f_{(2n)mkl}g^n=g^nh^mg^kh^{-\ell}g^n$ of $f_{(2n)mkl}$ converge to the fixed points of g. Similarly, the fixed points of the conjugate $h^{-\ell}f^{-1}_{nmk(2l)}h^\ell=h^\ell g^{-k}h^{-m}g^{-n}h^\ell$ of $f^{-1}_{nmk(2\ell)}$ converge as $\ell\to\infty$ to the fixed points of h. Thus after possibly conjugating with g,h, if $f_{nmk\ell}$ is conjugate in G to $f^{-1}_{nmk\ell}$ for all n,ℓ then there is a sequence of elements $g_i\in G$ which map a fixed compact subset K of X intersecting an axis for g into a fixed compact subset W of X intersecting an axis for h and such that $g_i(a,b)\to(x,y)$. However, G is a closed subgroup of $\mathrm{Iso}(X)$ and hence after passing to a subsequence we may assume that $g_i\to g\in G$. Then $g(a,b)=(x,y)$ which violates the choice of g,h.

Inductively we can construct in this way a sequence of elements of Γ with the properties stated in the proposition. □

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