SMALL EIGENVALUES OF RANDOM 3-MANIFOLDS

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ABSTRACT. We show that for every $g \geq 2$ there exists a number $c = c(g) > 0$ such that the smallest positive eigenvalue of a random closed 3-manifold M of Heegaard genus g is at most $c(g)/\mathrm{vol}(M)^2$.

1. INTRODUCTION

By celebrated work of Perelman, any closed oriented aspherical atoroidal 3-manifold admits a hyperbolic metric, and such a metric is unique by Mostow rigidity. In recent years, there was considerable progress in the understanding of the relation between geometric and topological invariants of such a manifold. The program to construct an explicit combinatorial model which describes the geometry up to uniform quasi-isometry turned out to be particularly fruitful [40, 10, 11], but it is far from completed.

The main purpose of this article is obtain an understanding of geometric and topological invariants for *random* hyperbolic 3-manifolds in the sense of Dunfield and Thurston [16]. Namely, fix a genus $g \geq 2$. A closed 3-manifold of Heegaard genus at most g can be obtained by gluing two handlebodies of genus g along the boundary with a diffeomorphism ϕ. The resulting 3-manifold M only depends on the isotopy class of ϕ, and it is aspherical atoroidal and hence hyperbolic if ϕ is sufficiently complicated. Thus hyperbolic 3-manifolds of Heegaard genus g correspond to suitable elements of the mapping class group $\mathrm{Mod}(\Sigma)$ of the boundary surface Σ of a handlebody of genus g (in fact, they correspond to double cosets in this group, see [16]).

Now let us choose a symmetric probability measure on $\mathrm{Mod}(\Sigma)$ whose finite support generates $\mathrm{Mod}(\Sigma)$. This measure generates a random walk on $\mathrm{Mod}(\Sigma)$, and hence it induces a notion of a random 3-manifold, glued from two handlebodies with a random gluing map. A random 3-manifold is hyperbolic [16] and hence we can study the behavior of geometric invariants of such random hyperbolic 3-manifolds M.

Our main technical result (Theorem 6.7) constructs for a 3-manifold obtained from a gluing map with some additional properties a Riemannian metric of sectional curvature close to -1 everywhere and different from -1 only in geometrically controlled regions where the injectivity radius is

bounded from below by a universal constant. These constraints are fulfilled for random gluing maps.

We use this construction to obtain information on the spectrum of the Laplacian of a random hyperbolic 3-manifold M. List the eigenvalues as $0 = \lambda_0(M) < \lambda_1(M) \leq \lambda_2(M) \leq \lambda_3(M) \leq \ldots$, with each eigenvalue repeated according to its multiplicity. By [44] and [21], there exists a universal constant $\chi > 0$ such that

$$\lambda_1(M) \geq \frac{\chi}{\mathrm{vol}(M)^2} \quad \text{and } \lambda_{\mathrm{vol}(M)/\chi}(M) \geq \chi$$

for every closed hyperbolic 3-manifold M. Manifolds which fibre over the circle provide examples for which these estimates are essentially sharp. We refer to the introduction of [1] for a more comprehensive discussion.

On the other hand, it follows from the work of Buser [12] and Lackenby [26] that there exists a number $b(g) > 0$ such that for a hyperbolic 3-manifold M of Heegaard genus g, there is a bound

$$\lambda_1(M) \leq \frac{b(g)}{\mathrm{vol}(M)}.$$

Hyperbolic 3-manifolds constructed from expander graphs have arbitrarily large volume, yet their smallest positive eigenvalue is bounded from below by a universal constant. Hence in this estimate, the dependence of the constant $b(g)$ on the Heegaard genus g can not be avoided.

Under geometric constraints, one obtains better estimates. White [49] showed that there is a number $a(g) > 0$ such that $\lambda_1(M) \leq a(g)/\mathrm{vol}(M)^2$ if M is of Heegaard genus g and the injectivity radius of M is bounded from below by universal constant. The same holds true for random hyperbolic 3-manifolds which fibre over the circle, with fibre genus g [1].

Using the model metric for random hyperbolic 3-manifolds as our main tool we show.

Theorem 1. *For every $g \geq 2$ there exists a number $c(g) > 1$ such that*

$$\lambda_1(M) \leq \frac{c(g)}{\mathrm{vol}(M)^2} \quad and \ \lambda_{\mathrm{vol}(M)/c(g)}(M) \leq c(g)$$

for a random hyperbolic 3-manifold of Heegaard genus g.

Here the upper bound for $\lambda_{\mathrm{vol}(M)/c(g)}(M)$ is a straightforward consequence of domain monotonicity with Dirichlet boundary conditions. For the upper bound for $\lambda_1(M)$, we expect that the dependence of the constant $c(g)$ on g can not be avoided.

Strategy of the proof. As mentioned above, our main technical result is Theorem 6.7 which provides of an explicit Riemannian metric of curvature close to -1 on a 3-manifold of Heegaard genus g with some constraints on the gluing map. Constructions of geometrically controlled model metrics appear frequently in the literature, for example as a main tool in [42] and

in [41]. For doubly degenerate hyperbolic 3-manifolds whose fundamental group is isomorphic to the fundamental group of a closed surface, there is a completely explicit combinatorial model for the geometry [40, 10]. More recently these results were used to describe explicitly the geometry of hyperbolic 3-manifolds with a lower bound on the injectivity radius and some topological constraints [11].

We can not apply the constructions in [11] as there are no lower bounds for the injectivity radius of a random hyperbolic 3-manifold M. Instead we use properties of the random walk to locate regions in a random 3-manifold which are diffeomorphic to a trivial I-bundle over a closed surface and such that a combinatorial model would predict a uniform lower bound on the injectivity radius in those regions. This is the constraint on the gluing map required in Theorem 6.7. The model metric is then constructed by cutting M open at two such regions and by using information on suitable model metrics for the pieces.

For random hyperbolic 3-manifolds M, we find that the spectrum of the model metric fulfills the properties stated in Theorem 1.

The last step consists in comparing the model metric on M and the hyperbolic metric. A result of Tian [47] implies that the model metric is C^2-close to the hyperbolic metric. As this work is neither published nor available in electronic form, we prove a weak substitute which is sufficient for the proof of Theorem 1. Our argument is based on the methods introduced in [4].

Organization of the article. In Section 2 we collect some properties of the pointed geometric topology for 3-dimensional Riemannian manifolds which are used later on.

In Section 3 we introduce a relative version of bounded combinatorics and set up sufficient conditions for the construction of a model metric. This construction depends on the existence of *large thick collars*, a property which is introduced in Section 4.

Sections 5 and 6 are devoted to the proof of Theorem 6.7 which provides a model metric for a hyperbolic 3-manifold of fixed Heegaard genus with some additional properties. In Section 7 we show that random hyperbolic 3-manifolds have the properties required in Section 6, and in Section 8 we relate the model metric to the hyperbolic metric using tools from [4]. The information on the hyperbolic metric we obtain then leads to Theorem 1.

2. Hyperbolic structures on handlebodies

The goal of this section is to collect some results from the deformation theory of convex cocompact hyperbolic metrics on handlebodies in the form used later on. We also introduce some notations which are used throughout the article.

We begin with making precise what we understand by looking at a convex cocompact hyperbolic handlebody from the point of view of the boundary of the convex core. We give a quantitative description of the notion of a large collar with bounded geometry (a *large-thick collar*). As a preparation for Section 4, we describe some basic general compactness properties of the geometric topology.

Fix, once and for all, a genus $g \geq 2$. Let $\mathcal{H}$ be a handlebody of genus g, with boundary surface $\Sigma := \partial\mathcal{H}$. We fix on $\mathcal{H}$ an orientation, and we coherently orient Σ as the boundary of $\mathcal{H}$.

A *marked hyperbolic structure* on the handlebody $\mathcal{H}$ is a quotient $N = \mathbb{H}^3/\Gamma$ of hyperbolic 3$-$space by a discrete free subgroup $\Gamma < \mathrm{PSL}_2(\mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3)$, together with a homeomorphism (the marking) $\phi : \mathrm{int}(\mathcal{H}) \longrightarrow N$. We say that the marked structures $\phi : \mathrm{int}(\mathcal{H}) \longrightarrow N$ and $\psi : \mathrm{int}(\mathcal{H}) \longrightarrow N'$ are *equivalent* if there exists an isometry $f : N \longrightarrow N'$ such that $f \circ \phi$ is isotopic to ψ.

2.1. **Parametrization of marked convex cocompact structures.** We denote by $\mathcal{T} = \mathcal{T}(\Sigma)$ the *Teichmüller space* of marked hyperbolic metrics on Σ, and by $\mathcal{M} = \mathcal{M}(\Sigma) = \mathrm{Mod}(\Sigma)\backslash\mathcal{T}(\Sigma)$ the *moduli space* of hyperbolic metrics on Σ.

By classical results due to Bers, Kra, Maskit, Sullivan and others, so-called *convex cocompact* hyperbolic structures on the handlebody $\mathcal{H}$ are parametrized by a parameter that lies in the Teichmüller space $\mathcal{T}$ of the boundary surface.

Namely, let $N = \mathbb{H}^3/\Gamma$ be a hyperbolic structure on $\mathcal{H}$. Associated to Γ we have the *limit set* $\Lambda \subset \partial\mathbb{H}^3$ which consists of the points at infinity of a $\Gamma-$orbit closure, and the *domain of discontinuity* $\Omega_\Gamma = \partial\mathbb{H}^3 - \Lambda$, the complement of the limit set. The group Γ, isomorphic to a free group $\mathbb{F}_g$ of rank g, acts freely and properly discontinuously both on the convex hull of the limit set $\mathcal{CH}(\Lambda) \subset \mathbb{H}^3$ and on the domain of discontinuity $\Omega_\Gamma \subset \partial\mathbb{H}^3$.

The quotient

$$\mathcal{CC}(N) := \mathcal{CH}(\Lambda)/\Gamma$$

is the *convex core* of N. It is a convex topological submanifold of N, possibly with boundary. The manifold N is called *convex cocompact* if $\mathcal{CC}(N)$ is compact. The complement $N - \mathcal{CC}(N)$ is naturally homeomorphic to $\partial\mathcal{CC}(N) \times (0, \infty)$. In the remainder of this section we always assume that all hyperbolic structures on $\mathcal{H}$ are convex cocompact. In particular, we always have $\Omega_\Gamma \neq \emptyset$.

The quotient $\partial_c N := \Omega_\Gamma/\Gamma$ is the (unmarked) *conformal boundary* of N (we can think of it as a point in moduli space). As N is convex cocompact, $\partial_c N$ is homeomorphic to the closed surface $\Sigma = \partial\mathcal{H}$. The conformal boundary is equipped with a natural conformal structure and hence a hyperbolic metric (which we refer to as the Poincaré metric) coming from the fact that

Γ acts via Möbius transformations on $\partial\mathbb{H}^3$. The quotient

$$\overline{N} = \mathbb{H}^3 \cup \Omega_\Gamma/\Gamma = N \cup \partial_c N$$

gives a natural compactification of N.

Using a marking $\phi : \text{int}\,(\mathcal{H}) \longrightarrow N$, the isotopy class of the inclusion of the boundary $\Sigma := \partial\mathcal{H} \hookrightarrow \mathcal{H}$ determines an isotopy class of an embedding $\Sigma \hookrightarrow N$. We use this isotopy class to give a marking to the conformal boundary $\partial_c N$ and to the boundary of the convex core $\partial\mathcal{CC}(N)$.

In this terminology, Bers parametrization can be stated as follows: Equivalence classes of marked convex cocompact structures are parametrized by the marked conformal boundary. Given a marked conformal boundary $X \in \mathcal{T}$, we denote by $\mathcal{H}(X)$ the corresponding marked convex cocompact hyperbolic handlebody.

2.2. The boundary of the convex core.

As before, let $N = \mathbb{H}/\Gamma$ be a convex cocompact hyperbolic structure on $\mathcal{H}$. Then the boundary $\partial\mathcal{CC}(N) \subset N$ of the convex core is an embedded *pleated surface*.

DEFINITION (Pleated Surface, Thurston [46]). Let M be a hyperbolic 3-manifold and let us fix a homotopy class of maps $j : \Sigma \to M$. A pleated surface in the homotopy class of j consists of the following data:

- A hyperbolic metric σ on Σ.
- A path-isometry $f : (\Sigma, \sigma) \to M$ homotopic to j such that every point $x \in \Sigma$ is contained in a geodesic segment which is mapped to a geodesic in M.

Associated to every pleated map $f : (\Sigma, \sigma) \to M$ there is a geodesic lamination $\lambda \subset \Sigma$, called the pleating locus with the following property. Every leaf of λ is mapped to a geodesic by f, and the restriction of f to every component of $\Sigma - \lambda$, called a *flat piece*, is a locally isometric immersion. We say that f realizes $\lambda \subset \Sigma$ in M within the homotopy class j. For more on laminations and pleated surfaces we refer the reader to Chapter I.5 of [15].

There is a natural *nearest point retraction* (see Chapter II.1.3 of [15]) from the conformal boundary to the boundary of the convex core $r : \partial_c N \longrightarrow \partial\mathcal{CC}(N)$. With respect to the induced markings on the conformal boundary and on the boundary of the convex core, r lies in the homotopy class of the identity.

The following result of Bridgeman and Canary provides control of the boundary of the convex core when we have a good understanding of the geometry of the conformal boundary:

THEOREM 2.1 (Bridgeman-Canary, [8]). *There are maps $J, G : (0, \infty) \to (1, \infty)$ such that the following holds: Let $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ be a finitely generated, non-elementary, torsion free Kleinian group. Suppose that the length,*

measured with respect to the Poincaré metric, of every curve in the conformal boundary Ω_Γ/Γ which is compressible in the 3-manifold $\left(\mathbb{H}^3 \cup \Omega_\Gamma\right)/\Gamma$ is bounded from below by $\rho > 0$. Then the nearest point retraction from the conformal boundary to the boundary of the convex core is $J(\rho)-$Lipschitz and admits a $G(\rho)-$Lipschitz homotopy inverse.

2.3. **Limits of hyperbolic manifolds.** Let us choose for every (marked) convex cocompact structure on the handlebody $\mathcal{H}$ a basepoint $x \in \partial\mathcal{CC}(N)$ on the boundary of the convex core. We then can talk about a marked pointed convex cocompact handlebody.

DEFINITION (Geometric Convergence). A sequence $\{(M_n, m_n)\}_{n \in \mathbb{N}}$ of pointed hyperbolic 3-manifolds is said to *converge in the pointed geometric topology* to a pointed hyperbolic 3-manifold (M_∞, m_∞) if the following conditions are satisfied. For every $R > 0, \xi > 0$ there are numbers $n(R, \xi) > 0$, and for every $n \geq n(R, \xi)$ there exists a map (the *approximating map*) $k : U_n \subset M_\infty \to M_n$ such that

- k is defined on the ball $B_{M_\infty}(m_\infty, R)$ of radius R centered at the basepoint m_∞ of M_∞ and sends this basepoint to the base point of M_n.
- the restriction of k to the ball $B_{M_\infty}(m_\infty, \frac{R}{2})$ is ξ-close to an isometry in the $\mathcal{C}^2$-topology: The metric tensor ρ_∞ of M_∞ and the pullback $k^*\rho_n$ by k of the metric tensor ρ_n of M_n are ξ-close in the $\mathcal{C}^2$-norm on 2$-$tensors on the ball $B_{M_\infty}\left(m_\infty, \frac{R}{2}\right)$.

To be more precise, define

$$\|k^*\rho_n - \rho_\infty\|_{B\left(m_\infty, \frac{R}{2}\right)}$$
$$= \sup_{m \in B(m_\infty, R/2)} \sup_{x,y \in T^1_m M_\infty} \{|k^*\rho_n(x,y)_m - \rho_\infty(x,y)_m|\}.$$

Let Rm be the Riemannian curvature tensor and define

$$\|k^*\mathrm{Rm}_n - \mathrm{Rm}_\infty\|_{B\left(m_\infty, \frac{R}{2}\right)}$$
$$= \sup_{m \in B(m_\infty, R/2)} \sup_{x,y,z,w \in T^1_m M_\infty} \{|k^*\mathrm{Rm}_n(x,y,z,w)_m - \mathrm{Rm}_\infty(x,y,z,w)_m|\}.$$

Then under the condition in the above Definition, we have

$$\|k^*\rho_n - \rho_\infty\|_{B(m_\infty, R/2)} + \|k^*\mathrm{Rm}_n - \mathrm{Rm}_\infty\|_{B(m_\infty, R/2)} < \xi.$$

We then say that the restriction of k to $B(m_\infty, R/2)$ is $\xi-$*almost isometric.*

Note that this definition of geometric convergences is slightly more restrictive than what is found in the literature (see e.g. Chapter E of [2]).

We shall make use of the following compactness result for geometric convergence (Theorem E.1.10 of [2]).

THEOREM 2.2. *Suppose that $\{(M_n, m_n)\}_{n \in \mathbb{N}}$ is a sequence of pointed hyperbolic* 3$-$*manifolds such that there is a uniform positive lower bound $\eta > 0$ on the injectivity radius at the base points m_n. Then there exists a subsequence*

that converges in the geometric topology to a pointed hyperbolic 3-manifold (M_∞, m_∞).

We observe next that in combination with *Margulis' Lemma* (see e.g. [2]), Theorem 2.1 implies that if the conformal boundary of a convex-cocompact hyperbolic structure N on a handlebody $\mathcal{H}$ is ϵ-thick in the Poincaré metric (i.e. its injectivity radius is at least ϵ), then there is a uniform lower bound, only depending on $\epsilon > 0$ and $g = g(\Sigma)$, on the injectivity radius of N at points that are close to the boundary of the convex core $\mathcal{CC}(N)$. This enables us to take geometric limits.

For the formulation of this fact, for sufficiently small $\epsilon > 0$ we denote by $\mathcal{T}_\epsilon \subset \mathcal{T}$ the subset of Teichmüller space of all marked hyperbolic metrics on Σ with injectivity radius at least ϵ, and we let $\mathcal{M}_\epsilon = \mathrm{Mod}(\Sigma)\backslash \mathcal{T}_\epsilon$ be the ϵ-thick part of moduli space. Furthermore, let $\mathrm{inj}_x(N)$ be the injectivity radius of a hyperbolic manifold N at the point x, and let $\mathrm{inj}(N) = \inf_x\{\mathrm{inj}_x(N) \mid x \in N\}$ the global injectivity radius of N.

LEMMA 2.3. *For every $\epsilon > 0$ and $g \geq 2$ there exists $\eta = \eta(\epsilon, g) > 0$ such that the following holds: Let N be a convex cocompact hyperbolic structure on $\mathcal{H}$. If $\partial_c N \in \mathcal{M}_\epsilon$ then $\inf_{x \in \partial \mathcal{CC}(N)}\{\mathrm{inj}(x, N)\} \geq \eta$.*

In the proof and in the sequel we use the following notations:

> **Notation**. If X is a hyperbolic surface and $\gamma : S^1 \to X$ is a smooth closed curve, then we denote by $L(\gamma)$ the length of γ, and by $L_X(\gamma)$ the length of the geodesic representative of γ on X. For a curve γ in a hyperbolic $3-$manifold M we use the notation $l(\gamma)$ and $l_M(\gamma)$ for the analogous quantities.

Proof. By Theorem 2.1, we have that every simple closed curve on the boundary of the convex core $\gamma \subset \partial \mathcal{CC}(N)$ has length (with respect to the induced hyperbolic metric)

$$L_{\partial \mathcal{CC}(N)}(\gamma) \geq \frac{1}{G(\epsilon)} L_{\partial_c N}(\gamma) \geq \frac{2\epsilon}{G(\epsilon)}$$

as $\mathrm{inj}(\partial_c N) \geq \epsilon$.

Hyperbolic trigonometry shows that there exists a uniform upper bound for the diameter of $\partial \mathcal{CC}(N)$ in the intrinsic metric, say diam $\partial \mathcal{CC}(N) \leq D$ where $D = D(\epsilon, g)$ only depends on $\epsilon > 0$ and $g \geq 2$. Since the inclusion $\partial \mathcal{CC}(N) \subset N$ is $1-$Lipschitz by definition of the intrinsic path-metric, we have the same control on the diameter when we compute distances in N.

Let $\epsilon_3 > 0$ be a Margulis constant for hyperbolic manifolds in dimension 2 and 3. By Margulis' Lemma, for some small number $\rho < \epsilon_3$ the $\rho-$thin part $N_{(0,\rho]} := N - \{x \in N \mid \mathrm{inj}_x N > \rho\}$ of N is a disjoint union of Margulis tubes, i.e. metric tubular neighbourhoods of simple closed geodesics of length smaller than ρ.

Having bounded diameter and carrying all the information about the fundamental group of $\mathcal{H}$, the surface $\partial\mathcal{CC}(N)$ cannot penetrate deeply into a Margulis tube. Namely, let $\gamma \subset N$ be the core geodesic of the ρ-Margulis tube containing $x \in \partial\mathcal{CC}(N)$. Standard hyperbolic geometry yields that the distance between the boundary of the $\rho-$Margulis tube and the boundary of the ϵ_3-Margulis tube of γ grows to ∞ as ρ approaches 0. In particular, if ρ is sufficiently small then by the diameter bound for $\partial\mathcal{CC}(N)$, this surface is entirely contained in the ϵ_3-Margulis tube. This contradicts the fact that the inclusion $\partial\mathcal{CC}(N) \hookrightarrow N$ is π_1-surjective. Hence the injectivity radius of N at points in $\partial\mathcal{CC}(N)$ is bounded from below by a universal positive constant.   □

By Lemma 2.3 and compactness of pleated surfaces (see section I.5.2 of [15], in particular Theorem I.5.2.2), given a sequence of triples

$$\{(N_n, j_n : X_n \subset \mathcal{M} \to \partial\mathcal{CC}(N_n), x_n)\}_{n\in\mathbb{N}}$$

consisting of convex-cocompact hyperbolic structures, corresponding (pleated surface parametrizations of the) boundaries of the convex cores and basepoints such that the conformal boundary is ϵ-thick, we can always extract a subsequence (say the whole sequence) that converges in the geometric topology to a triple $(N_\infty, j_\infty : X_\infty \to N_\infty, x_\infty)$, consisting of a hyperbolic $3-$manifold, a pleated surface and a common basepoint, in the following sense:

- The sequence of pointed 3-manifolds (N_n, x_n) converges to (N_∞, x_∞).
- The sequence of pointed hyperbolic surfaces $(\partial\mathcal{CC}(N_n), x_n)$ converges to the pointed hyperbolic surface $(X_\infty, \overline{x}_\infty)$. Observe that, by Theorem 2.1, $\partial\mathcal{CC}(N_n)$ has a uniform lower bound on the injectivity radius and a uniform upper bound on the diameter. In particular, the surfaces X_n are contained in a compact subset of moduli space, and the surface $X_\infty \in \mathcal{M}$ is an accumulation point of the sequence (X_n). In particular, it shares the same uniform bounds on the injectivity radius and the diameter.
- The pleated surface embeddings $\partial\mathcal{CC}(N_n) \hookrightarrow N_n$, which we denote by j_n, converge to a pleated surface $j_\infty : X_\infty \to N_\infty$. The diagram where all the maps respect the basepoints, and the vertical arrows are the approximating maps provided by the geometric convergence,

$$
\begin{array}{ccc}
\partial\mathcal{CC}(N_n) & \xrightarrow{\;j_n\;} & N_n \\[4pt]
{\scriptstyle \phi_n}\uparrow & & \uparrow{\scriptstyle k_n} \\[4pt]
X_\infty & \xrightarrow[\;j_\infty\;]{} & N_\infty
\end{array}
$$

commutes up to local (pointed) homotopies, i.e. those homotopies that respect the base points and take place in small neighbourhoods of the images of the pleated surfaces.

- Lastly, we also recall that $\partial\mathcal{CC}(N_n)$ and X_∞ come together with a marking (as in the definition of pleated surface), i.e. they are isometrically identified with $\left(\Sigma, \sigma_{\partial\mathcal{CC}(N_n)}\right)$ and (Σ, σ_∞). These markings consist of collections of curves whose lengths are uniformly bounded for the induced hyperbolic metrics. We can assume that the composition of the identification $\left(\Sigma, \sigma_{\partial\mathcal{CC}(N_n)}\right) \simeq \partial\mathcal{CC}(N_n)$ with the inclusion in N_n is isotopic to the marking of N_n, and that the marked (and hence parametrized) pleated surfaces converge.

3. Relative bounded combinatorics

Bounded combinatorics is a combinatorial condition which translates into explicit geometric control of the hyperbolic metric on the convex cocompact handlebody near the boundary of its convex core.

Relative versions of bounded combinatorics were introduced in [41], [42] and [11]. We are interested in bounded combinatorics relative to a decorated handlebody $\mathcal{H}$, where the decoration is either a marking μ on $\partial\mathcal{H}$ or a point $X \in \mathcal{T}$, which we can think of as a "fixed convex cocompact structure" on $\mathcal{H}$. Such a decorated handlebody will be denoted by a pair $(\mathcal{H}, \mu)$ (if μ is a marking) or by $(\mathcal{H}, X)$ if $X \in \mathcal{T}$. If $X \in \mathcal{T}$ then the geometric realization $\mathcal{H}(X)$ is the convex cocompact handlebody whose marked conformal boundary equals X.

Following ideas of Minsky [40] and of Brock-Canary-Minsky [10], our goal is to construct for a random 3-manifold M a model metric which is close to the hyperbolic metric and such that on a submanifold M_0 of M whose volume is bigger than a definite proportion of the volume of M, this metric can explicitly be described.

Our construction is a variation of earlier constructions of [41], [42] and [11]. Namely, we glue the model metric from hyperbolic metrics on pieces with large overlap on which the metrics have large injectivity radius and are very close. The condition we need to successfully glue these metrics on the overlap to a metric which is close to a hyperbolic metric can be described as "bounded combinatorics and large height".

Our setup, however, is different from the setup in these earlier work as we can not assume any global control of the injectivity radius, i.e. unbounded geometry appears. The purpose of this section is to provide the control needed in the sequel.

We begin with collecting the essential facts about coarse geometry and Gromov hyperbolic spaces, in particular about the geometry of the *curve graph* and the *disk graph*.

3.1. **Coarse Geometry.** A map $f : (X, d_X) \to (Y, d_Y)$ between metric spaces is an $(L, C)-$*quasi-isometric embedding* if for every $x, x' \in X$

$$\frac{1}{L} d_X(x, x') - C \le d_Y(f(x), f(x')) \le L d_X(x, x') + C.$$

If moreover it is *coarsely surjective*, i.e. there exists $D \ge 0$ such that for every $y \in Y$ there exists $x \in X$ such that $d_Y(y, f(x)) \le D$, then f is said to be a *quasi-isometry*.

A parametrized $(L, C)-$*quasi-geodesic* segment, ray or line is an (L, C)-quasi-isometric embedding of an interval, a half line or the entire real line $\mathbb{R}$. Later on we will have to deal also with *unparametrized* $(L, C)-$quasi-geodesic segments, rays and lines which are maps $f : I \subset \mathbb{R} \longrightarrow (X, d_X)$ such that there exists a homeomorphism ϕ from an interval $I' \subset \mathbb{R}$ onto the interval I with the property that the composition $f \circ \phi : I' \longrightarrow (X, d_X)$ is a $(L, C)-$quasi-isometric embedding (the intervals I, I' can be finite or infinite).

3.2. **Curve and Disk Graphs.** Masur and Minsky proved in [32] that the curve graph $\mathcal{C} := \mathcal{C}(\Sigma)$ of the closed surface Σ of genus $g \ge 2$ is a *Gromov hyperbolic* space of infinite diameter, and Klarreich [24] identified the *Gromov boundary* $\partial_\infty \mathcal{C} = \partial_\infty \mathcal{C}(\Sigma)$ with the space $\mathcal{EL} = \mathcal{EL}(\Sigma)$ of minimal filling unmeasured laminations (see also [19] for a different approach).

DEFINITION (Gromov Product and Convergence). Given $\alpha, \beta, \gamma \in \mathcal{C}$, the quantity

$$(\alpha|\beta)_\gamma := \frac{1}{2} [d(\alpha, \gamma) + d(\beta, \gamma) - d(\alpha, \beta)]$$

is the Gromov product of α, β based at γ. A sequence $\{\alpha_n\}_{n \in \mathbb{N}} \subset \mathcal{C}$ *converges at infinity* to a point in $\partial_\infty \mathcal{C}$ if and only if for some base point γ (and hence for any) we have $\liminf_{n,m \to \infty} (\alpha_n|\alpha_m)_\gamma \to \infty$. If $\{\alpha_n\}_{n \in \mathbb{N}}$ converges at infinity and $\{\beta_n\}_{n \in \mathbb{N}}$ satisfies $\liminf_{n,m \to \infty} (\alpha_n|\beta_m) = \infty$, then $\{\beta_n\}_{n \in \mathbb{N}}$ converges to the same point in $\partial_\infty \mathcal{C}$.

For material on Gromov products and Gromov boundaries we refer the reader to Section 3 of Chapter III.H of [9].

The geometry of the curve graph is coarsely tied to the geometry of Teichmüller space. There is a (coarsely well-defined) $\mathrm{Mod}(\Sigma)-$equivariant map

$$\Upsilon : \mathcal{T} \to \mathcal{C},$$

called the *systole map*, that associates to evey marked hyperbolic structure $X \in \mathcal{T}$ a shortest geodesic $\Upsilon(X)$. It follows from Masur-Minsky [32] that there exist constants $L = L(\Sigma) \ge 1, C = C(\Sigma) \ge 0$ such that for every Teichmüller geodesic $l : I \to \mathcal{T}$ (here I can be an interval, a half-line or the whole real line) the composition $\Upsilon \circ l : I \to \mathcal{C}$ is an *unparametrized* $(L, C)-$quasi-geodesic.

Moreover, if we restrict our attention to the ϵ-thick part $\mathcal{T}_\epsilon$ of Teichmüller space, then the situation improves: In [20] it is shown that for every $\epsilon > 0$ there exist $L_\epsilon \geq 1, C_\epsilon \geq 0$ such that if l is parametrized by arc length on an interval of length $l(I) \geq L_\epsilon$ and if $l(I) \subset \mathcal{T}_\epsilon$ then $\Upsilon \circ l$ is a *parametrized* $(L_\epsilon, C_\epsilon)-$quasi-geodesic.

The *disk graph* $\mathcal{D}$ associated to the identification $\Sigma = \partial \mathcal{H}$ is the subgraph of $\mathcal{C}$ spanned by disk-bounding curves. Masur and Minsky showed in [34] that the disk graph $\mathcal{D}$ is a *quasi-convex* subset of the curve graph $\mathcal{C}$. Being quasi-convex, by hyperbolicity of $\mathcal{C}$, there is a coarsely defined nearest point projection $\pi_\mathcal{D} : \mathcal{C} \longrightarrow \mathcal{D}$.

3.3. **Subsurface projection and Bounded Combinatorics.**

An essential tool for describing the geometry of the curve graph is the notion of *subsurface projection* introduced by Masur-Minsky in [33]: For every proper essential subsurface $W \subset \Sigma$ (with some care for annuli and pairs of pants) and every point $\alpha \in \mathcal{C} \cup (\partial_\infty \mathcal{C} = \mathcal{EL})$, there is a subsurface projection $\pi_W(\alpha) \subset \mathcal{C}(W)$ which consists of the (possibly empty) subset of $\mathcal{C}(W)$ of all the possible essential surgeries of $\alpha \cap W$ (see [33] for the details).

We can also define subsurface projections for markings (taking the projection of the marking as a subset of $\mathcal{C}$). All the markings μ on Σ we consider are *complete*, i.e. they are given, as a subset of $\mathcal{C}$, by a *pants decomposition*, called the *base* of the marking, and for every curve α in the base a *transversal* t_α, that is, a simple closed curve which intersects α essentially in the least possible number of points (either one or two points) and does not intersect the other curves in the base. The total geometric intersection number between all curves in a marking is required to be uniformly bounded.

For every $X \in \mathcal{T}$ we denote by μ_X a short marking on the hyperbolic surface X. A short marking μ on X is a marking which is shortest among all markings. If we denote for $X \in \mathcal{T}$ by $L_X(\mu)$ the sum of the geodesic lengths of all curves in the marking μ, then for every $\epsilon > 0$ there exists $B_\epsilon > 0$ such that if $X \in \mathcal{T}_\epsilon$, then $L_X(\mu_X) \leq B_\epsilon$.

Bounded combinatorics means no large subsurface projections:

DEFINITION (Bounded Combinatorics, [39]). Let $R > 0$ be a positive number. Let α, β be either complete markings or unmeasured minimal filling laminations on Σ. The pair α, β has $R-$*bounded combinatorics* if for every proper essential subsurface $W \subset \Sigma$ we have

$$d_W(\alpha, \beta) := \mathrm{diam}_{\mathcal{C}(W)}(\pi_W(\alpha) \cup \pi_W(\beta)) \leq R.$$

For some $\epsilon > 0$, two points $X, Y \in \mathcal{T}_\epsilon$ have R-*bounded combinatorics* if this holds true for short markings μ_X, μ_Y for X, Y.

Just like ϵ-thickness, R-bounded combinatorics implies nice compactness properties: The following can be found as Proposition 6.2 in [41]

LEMMA 3.1. *Let $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathcal{C}$ be a sequence of markings that have R-bounded combinatorics with respect to a marking β. Then either there exists*

a constant subsequence, or there exists a subsequnce that converges to an unmeasured minimal filling lamination in the Gromov boundary $\lambda \in \partial_\infty \mathcal{C}$. *Moreover,* λ *has* $(R+1)$-*bounded combinatorics with respect to* β.

3.4. **Bounded combinatorics and Heegaard splittings.** When considering $\partial \mathcal{H} = \Sigma$ one has to take into account the compressibility of the boundary. Denote as before by $\mathcal{D}$ the disk set of $\mathcal{H}$, viewed as a quasi-convex subset of the curve graph $\mathcal{C}$.

Motivated by a construction of Namazi [41], we shall use the following relative version of bounded combinatorics for convex cocompact handlebodies.

DEFINITION (Relative Bounded Combinatorics). We say that an ordered pair (μ, ν) of markings of $\partial \mathcal{H}$ has *relative R-bounded combinatorics* with respect to the handlebody $\mathcal{H}$ if the pair μ, ν has $R-$bounded combinatorics and the following holds:

$$(1) \qquad\qquad d_\mathcal{C}(\mathcal{D}, \mu) + d_\mathcal{C}(\mu, \nu) \le d_\mathcal{C}(\mathcal{D}, \nu) + R.$$

The *height of the pair* (μ, ν) is $d_\mathcal{C}(\mu, \nu)$.

For a fixed thickness threshold $\epsilon > 0$, we say that an ordered pair $(Y, X) \in \mathcal{T} \times \mathcal{T}$ has relative $R-$bounded combinatorics with respect to $\mathcal{H}$ if $Y, X \in \mathcal{T}_\epsilon$ and the pair (μ_Y, μ_X) satisfies the above conditions. The height in this case is $d_\mathcal{T}(Y, X)$.

In the definition "μ (or Y) lies between $\mathcal{D}$ and ν (or X)".

The next lemma, which is analogous to Lemma 3.1, provides some compactness in our setting:

LEMMA 3.2. *Fix* $g \ge 2$ *and* $R > 0$. *Let* $\mathcal{H}$ *be a handlebody of genus* g. *Let* $\{(\alpha_n, \beta_n)\}_{n \in \mathbb{N}}$ *be a sequence of ordered pairs of markings on* $\Sigma := \partial \mathcal{H}$. *Suppose that:*

- *The pair* (α_n, β_n) *has relative* $R-$*bounded combinatorics.*
- *The sequence of heights diverges, i.e.* $H_n = d_\mathcal{C}(\alpha_n, \beta_n) \longrightarrow \infty$.

Then we have

$$(\delta | \alpha_n)_{\beta_n} \longrightarrow \infty$$

uniformly in $\delta \in \mathcal{D}$. *If we renormalize the configuration by translating* β_n *to a fixed base-point* $\beta \in \mathcal{C}$ *with* $\tau_n \in \mathrm{Mod}(\Sigma)$, *then the sequence* $\{\tau_n \mathcal{D}\}_{n \in \mathbb{N}}$ *converges, up to possibly passing to a subsequence, to a point* $\lambda \in \partial_\infty \mathcal{C}$. *Moreover* λ *has* $(R+1)-$*bounded combinatorics with respect to* β.

The meaning of the term "uniformly" in the statement is the following: For every $M > 0$ there exists $n_0 > 0$ such that for every $n \ge n_0$ and $\delta \in \mathcal{D}$ we have $(\delta | \alpha_n)_{\beta_n} \ge M$. Informally, Lemma 3.2 says that the disk set $\mathcal{D}$ disappears if we look at it from the point of view of β_n.

Proof. The proof is an easy application of Lemma 3.1 and basic properties of hyperbolic spaces. By definition, the Gromov product is computed by the

following formula

$$(\delta|\alpha_n)_{\beta_n} := \frac{1}{2}\left[d_{\mathcal{C}}(\beta_n, \delta) + d_{\mathcal{C}}(\beta_n, \alpha_n) - d_{\mathcal{C}}(\delta, \alpha_n)\right].$$

The Gromov product measures the fellow-traveling of the segments $[\beta_n, \delta]$ and $[\beta_n, \alpha_n]$.

Fix $M > 0$. Consider a disk $\delta \in \mathcal{D}$. We claim that $(\delta|\alpha_n)_{\beta_n} \geq M$ for large n. To show the claim it suffices to analyze the geodesic segment $[\beta_n, \delta]$. Namely, the quasi-convexity of $\mathcal{D}$ and the Condition (1) imply together that $[\beta_n, \alpha_n]$ uniformly fellow-travels $[\beta_n, \delta]$.

By quasi-convexity of $\mathcal{D}$, the segment $[\beta_n, \delta]$ passes uniformly close to the nearest-point projection of β_n to $\mathcal{D}$, which we denote by $\overline{\beta}_n = \pi_{\mathcal{D}}(\beta_n)$. Hence we have:

$$d_{\mathcal{C}}(\beta_n, \delta) \approx d_{\mathcal{C}}(\beta_n, \overline{\beta}_n) + d_{\mathcal{C}}(\overline{\beta}_n, \delta).$$

Here the symbol $\approx$ means "equal up to a uniform additive constant". The same holds for α_n: If we denote by $\overline{\alpha}_n := \pi_{\mathcal{D}}(\alpha_n)$ the projection to the disk set, then we have $d_{\mathcal{C}}(\alpha_n, \delta) \approx d_{\mathcal{C}}(\alpha_n, \overline{\alpha}_n) + d_{\mathcal{C}}(\overline{\alpha}_n, \delta)$.

The conclusion $(\delta|\alpha_n)_{\beta_n} \approx d(\alpha_n, \beta_n) = H_n$ would follow directly from the formula of the Gromov product if we knew that α_n, β_n have coarsely the same projection to $\mathcal{D}$, i.e. $\overline{\alpha}_n \approx \overline{\beta}_n$, and the segment $\left[\beta_n, \overline{\beta}_n\right]$ passes uniformly close to α_n, i.e. $d_{\mathcal{C}}(\beta_n, \overline{\beta}_n) \approx d_{\mathcal{C}}(\beta_n, \alpha_n) + d_{\mathcal{C}}(\alpha_n, \overline{\beta}_n)$. These properties are a consequence of Condition (1) and can be derived from the fact that equality holds in the following chain of inequalities:

$$d_{\mathcal{C}}(\beta_n, \overline{\beta}_n) = d_{\mathcal{C}}(\beta_n, \mathcal{D}) \approx d_{\mathcal{C}}(\beta_n, \alpha_n) + d_{\mathcal{C}}(\alpha_n, \mathcal{D})$$
$$= d_{\mathcal{C}}(\beta_n, \alpha_n) + d_{\mathcal{C}}(\alpha_n, \overline{\alpha}_n) \geq d_{\mathcal{C}}(\beta_n, \overline{\alpha}_n) \geq d_{\mathcal{C}}(\beta_n, \overline{\beta}_n).$$

Now we normalize the situation by translating β_n to a fixed point $\beta \in \mathcal{C}$ and prove the convergence statement. Let us denote by $\hat{\alpha}_n$ also the translate of α_n and by $\mathcal{D}_n$ the translated disk sets. Since having bounded combinatorics is invariant under $\mathrm{Mod}(\Sigma)$, every $\hat{\alpha}_n$ has $R-$bounded combinatorics with respect to β.

By Lemma 3.1 the sequence $\{\hat{\alpha}_n\}_{n \in \mathbb{N}}$, not having any constant subsequence as $d_{\mathcal{C}}(\beta, \hat{\alpha}_n) = H_n \to \infty$, admits a subsequence, say the whole sequence, converging to a minimal filling lamination in the Gromov boundary $\lambda \in \partial_\infty \mathcal{C}$ which furthermore has $(R+1)$-bounded combinatorics with respect to β. We show that $\mathcal{D}_n \to \lambda$ as well.

Consider a sequence of disks $\{\delta_n \in \mathcal{D}_n\}_{n \in \mathbb{N}}$. We identify the Gromov boundary $\partial_\infty \mathcal{C}$ with the space of equivalence classes of diverging sequences $\{x_n\}_{n \in \mathbb{N}}$. The claim follows by showing that $\{\delta_n\}_{n \in \mathbb{N}}$ and $\{\hat{\alpha}_n\}_{n \in \mathbb{N}}$ are equivalent, i.e. $(\delta_m|\hat{\alpha}_n)_\beta \to \infty$ $(n, m \to \infty)$.

Fix $M > 0$. By the first part of the proof, there exists $N > 0$ such that for every $n, m \geq N$ we have $(\delta_n|\hat{\alpha}_n)_\beta \geq M$. The claim follows from basic

properties of Gromov products:

$$(\delta_m | \hat\alpha_n)_\beta \gtrapprox \min \left\{ (\delta_m | \hat\alpha_m)_\beta \,, (\delta_n | \hat\alpha_n)_\beta \right\} \geq M$$

for every $n, m \geq N$. Here $\gtrapprox$ means "greater or equal up to a uniform additive constant".   □

3.5. **Gluings with relative bounded combinatorics.** The next definition describes the class of gluings for which we can relate the glued metric to the hyperbolic metric on the glued manifold.

DEFINITION (Gluings with Relative Bounded Combinatorics). Given a gluing map $f \in \mathrm{Mod}\,(\Sigma)$, a quadruple of markings (μ, ν, ν_f, μ_f) *has relative* (f, R)*-bounded combinatorics* with respect to $\mathcal{H}$ if:

- (μ, ν) has relative R-bounded combinatorics with respect to $\mathcal{D}$.
- (ν_f, μ_f) has relative R-bounded combinatorics with respect to $f\mathcal{D}$.

The *height* of the pair is $\min \{d_\mathcal{C}(\mu, \nu), d_\mathcal{C}(\mu_f, \nu_f)\}$.

As in the definition of pairs with relative bounded combinatorics, for a fixed a thickness threshold $\epsilon > 0$, there is also a version for quadruples of points $\left(Y, X, \overline{X}, \overline{Y}\right) \in$ in Teichmüller space $\mathcal{T}_\epsilon$.

REMARK 3.3. We make the following observations:

(i) The point Y lies between the disk set $\mathcal{D}$ and X. On the other side $\overline{Y}$ lies between $f\mathcal{D}$ and $\overline{X}$.
(ii) The pair $\left(\overline{Y}, \overline{X}\right)$ has relative $R-$bounded combinatorics with respect to the handlebody defined by the disk set $f\mathcal{D}$, i.e. by declaring that the curves in $f\mathcal{D} \subset \mathcal{C}(\Sigma)$ are exactly the compressible ones.
(iii) If $\left(Y, X, \overline{X}, \overline{Y}\right)$ is a quadruple with (f, R)-relative bounded combinatorics, then also every quadruple $\left(Y_0, X_0, \overline{X}_0, \overline{Y}_0\right)$, where the segments $[Y_0, X_0], \left[\overline{Y}_0, \overline{X}_0\right] \subset \mathcal{T}_\epsilon$ are, respectively, subsegments of $[Y, X]$ and $\left[\overline{Y}, \overline{X}\right]$, satisfies the (f, R)-bounded combinatorics condition.

4. LARGE-THICK COLLARS

The goal of this section is to prove the following Proposition. Recall that for $X \in \mathcal{T}$ we denote by $\mathcal{H}(X)$ a convex cocompact handlebody with conformal boundary X.

PROPOSITION 4.1. *Let* $g \geq 2$ *be fixed. For all* $R, L, \epsilon > 0$ *there exists* $H = H(R, L, \epsilon) > 0$ *such that the following holds: If the pair* $(Y, X) \in \mathcal{T}_\epsilon^2$ *has* R-*relative bounded combinatorics with respect to* $\mathcal{H}$ *and height at least* H, *then the boundary of the convex core of* $\mathcal{H}(X)$ *has a collar of width at least* L *and injectivity radius at least* $\eta > 0$ *where* η *only depends on* g *and* R.

The strategy is easy to state: We argue by contradiction. Suppose we have a sequence of counterexamples $\mathcal{H}(X_n)$ with relative R-bounded combinatorics and diverging heights, but no large-thick collar. Using the results

from Section 2, we can take a geometric limit N_∞ by looking at $\mathcal{H}(X_n)$ from the boundary of the convex core. The main result of this section states that N_∞ is a singly degenerate structure on $\Sigma \times \mathbb{R}$ with bounded geometry. Once we know this we are done because we can pull-back a large thick collar of arbitrary size via the approximating maps, thus obtaining a contradiction.

We begin with collecting some structural facts used in the proof.

4.1. **Ends of hyperbolic 3-manifolds.** Let M be a hyperbolic 3-manifold. For a fixed $\epsilon > 0$ let us denote by $M_{[\epsilon,\infty)}$ the $\epsilon-$*thick part* of M, the set of points in M where the injectivity radius is greater or equal to ϵ, and by $M_{(0,\epsilon]}$ the $\epsilon-$*thin part* of M, the closure of the complement of $M_{[\epsilon,\infty)}$. There exists a universal constant $\epsilon_3 > 0$, called a *Margulis constant*, such that for every $\epsilon \leq \epsilon_3$, every connected component of the $\epsilon-$thin part of the *thick-thin decomposition* $M = M_{(0,\epsilon]} \cup M_{[\epsilon,\infty)}$ is of one of the following two types: Margulis tubes, i.e. metric tubular neighbourhood of a closed geodesic γ (the *core* of the tube) of length $l_M(\gamma) \leq 2\epsilon$, rank two cusps (which will not be relevant for us) or rank one cusps, isometric to a quotient of a horoball $\mathcal{O} \subset \mathbb{H}^3$ by an infinite cyclic group of parabolic isometries. A simple closed curve generating the fundamental group of the cusp is called the *core* of the cusp (see Chapter D of [2] for more information).

From now on let us assume that M is homeomorphic to $\Sigma \times \mathbb{R}$. As Σ is closed, $\pi_1(M) \simeq \pi_1(\Sigma)$ does not contain subgroups isomorphic to $\mathbb{Z}^2$ and hence M does not have rank two cusps. Every element $\gamma \in \pi_1(\Sigma)$ is hyperbolic on Σ, but might act as a parabolic motion on the universal covering $\mathbb{H}^3$ of M. In this case we call γ an *accidental parabolic*. Such an element generates the fundamental group of a rank one cusp in M.

Fix a Margulis constant $\epsilon \leq \epsilon_3$ and consider the *non-cuspidal part* of M defined as $M_0 = M - M_{(0,\epsilon]}^{\text{cusp}}$ where $M_{(0,\epsilon]}^{\text{cusp}}$, the *cuspidal part*, is the union of the interiors of the (rank one) cusps in $M_{(0,\epsilon]}$. Scott proved in [45] that there exists a compact submanifold $\mathcal{SC} \subset M$, called a *Scott core*, homeomorphic to $\Sigma \times [0,1]$, with the following properties.

- The inclusion of $\mathcal{SC}$ into M is a homotopy equivalence.
- The intersection of $\mathcal{SC}$ with the closure of the cuspidal part $M_{(0,\epsilon]}^{\text{cusp}}$ consists of a disjoint union of annuli on $\partial \mathcal{SC}$ whose cores represent the cores of the corresponding rank one cusps.
- The topological *ends* of M_0 (which are *relative ends* for M) are in bijective correspondence with the connected components of $M_0 - \mathcal{SC}$. The closure of every connected component $E \subset M_0 - \mathcal{SC}$ is homeomorphic to $Y \times [0,\infty)$ where Y is the (connected) subsurface of $\partial \mathcal{SC}$ obtained as the intersection $\overline{E} \cap \mathcal{SC}$.

This description of the (relative) compact core uses results of McCullogh [36] and Kulkarni-Shalen [25].

By work of Thurston [46], Bonahon [7], Canary [13] and others, to each relative end $E \simeq Y \times [0, \infty)$ we can associate an *end invariant* which either is a finite type hyperbolic structure (conformal boundary) or a minimal filling lamination (the *ending lamination*) on Y. In the first case we say that the end is *geometrically finite*, while in the second case it is *simply degenerate*. Following Thurston [46], Bonahon [7] and Canary [13], this dichotomy can be characterized as follows:

- The end E is geometrically finite if there is a compact set $K \subset M$ such that $E - K$ does not contain any closed geodesic.
- The end E is simply degenerate if there exists a sequence of simple closed curves $\{\gamma_n \subset Y\}$ with geodesic representatives $\{\gamma_n^* \subset E\}_{n \in \mathbb{N}}$ (equivalently, a sequence of pleated surfaces $\{f_n : (Y, \sigma_n) \to E\}_{n \in \mathbb{N}}$ in the homotopy class of the inclusion $Y \subset E$), that *exit the end*, i.e. γ_n (or $f_n(Y)$) is eventually contained in $E - K$ for any compact set $K \subset M$. The curves γ_n (or the curves $\Upsilon(Y, \sigma_n)$, see Section 3) converge in $\mathcal{C}(Y)$ to the ending lamination $\lambda_E \in \partial_\infty \mathcal{C}(Y)$.

4.2. **Marked hyperbolic structures on I-bundles.** A marked hyperbolic structure on $\Sigma \times \mathbb{R}$, is a hyperbolic 3-manifold $Q = \mathbb{H}^3/\Gamma$ homeomorphic to $\Sigma \times \mathbb{R}$ together with a homotopy equivalence $\phi : \Sigma \to Q$, the *marking*. We always assume that there are no *accidental parabolics*, i.e. every element of $\Gamma < \mathrm{Isom}^+(\mathbb{H}^3)$ is hyperbolic. Equivalently, there are no cusps in Q.

Every marked hyperbolic structure Q on $\Sigma \times \mathbb{R}$ without accidental parabolics has exactly two relative ends homeomorphic to $\Sigma \times [0, \infty)$. To them we can associate a pair of end invariants (μ^+, μ^-), each of which is either a marked hyperbolic structure or a minimal filling lamination on Σ.

- If both μ^+ and μ^- are marked hyperbolic surfaces we call the manifold *quasi-fuchsian*. Its convex core $\mathcal{CC}(Q)$ is compact and homeomorphic to $\Sigma \times [0, 1]$ (except in the *fuchsian* case $\mu^+ = \mu^-$ where $\mathcal{CC}(Q)$ is a totally geodesic embedded surface). The boundary components are pleated surfaces.
- If μ_+ and μ_- are distinct minimal filling laminations, then we call the manifold Q *doubly degenerate*. In this case the convex core coincides with the whole manifold $\mathcal{CC}(Q) = Q$.
- If one end invariant is a marked hyperbolic structure and the other is a filling lamination, then the manifold is *singly degenerate*. The convex core is homeomorphic to $\mathcal{CC}(Q) = \Sigma \times [0, \infty)$.

By Bers Simultaneous Uniformization [3], marked quasi-fuchsian structures Q on $\Sigma \times \mathbb{R}$ are parametrized by $\mathcal{T} \times \mathcal{T}$ via the map that associates to Q the conformal boundary $\partial_c Q = (\mu^+, \mu^-) \in \mathcal{T} \times \mathcal{T}$. Given a pair $(Y, X) \in \mathcal{T} \times \mathcal{T}$ of conformal structures at infinity, we denote by $Q(Y, X)$ the unique quasi-fuchsian manifold that realizes those boundary data.

The solution of the Ending Lamination Conjecture by Minsky [40] and Brock-Canary-Minsky [10] implies that, as a marked hyperbolic structure, Q is uniquely determined by its end invariants (μ^+, μ^-). We recall that the manifold Q has *bounded geometry* if there is a positive lower bound on the injectivity radius.

The mapping class group $\mathrm{Mod}\,(\Sigma)$ acts on marked hyperbolic structures Q on $\Sigma \times \mathbb{R}$ by precomposition of marking. On the quasi-fuchsian subspace, the action coincides with the diagonal action on $\mathcal{T} \times \mathcal{T}$.

4.3. **Convergence to singly degenerate.**

We now prove what we stated at the beginning of the section. Namely, given a sequence of marked convex-cocompact handlebodies $\{N_n = \mathcal{H}(X_n), x_n \in \partial\mathcal{CC}(N_n)\}_{n \in \mathbb{N}}$ whose conformal boundaries satisfy the condition of relative $R-$bounded combinatorics and diverging heights, we can extract a subsequence that converges geometrically to a singly degenerate structure on $\Sigma \times \mathbb{R}$ with uniformly bounded geometry.

PROPOSITION 4.2. *Let $\left\{\left(Y_n, X_n\right) \in \mathcal{T}_\epsilon^2\right\}_{n \in \mathbb{N}}$ be a sequence where every pair has relative $R-$bounded combinatorics, and the heights H_n diverge. Then up to passing to a subsequence, the sequence $\{(N_n := \mathcal{H}(X_n), x_n)\}_{n \in \mathbb{N}}$ of pointed convex cocompact handlebodies converges geometrically to a singly degenerate hyperbolic structure N_∞ on $\Sigma \times \mathbb{R}$ with $\mathrm{inj}(N_\infty) \geq \eta$. Here η only depends on g and R.*

For convenience, we divide the proof of Proposition 4.2 into several small steps: Lemma 4.3, Lemma 4.4, Lemma 4.5 and its Corollaries.

To begin with, note that by Theorem 2.2 and Lemma 2.3, from the sequence of triples $\{(N_n, j_n : \partial\mathcal{CC}(N_n) \hookrightarrow N_n, x_n)\}_{n \in \mathbb{N}}$ we can extract a subsequence that converges to $(N_\infty, j_\infty : X_\infty \hookrightarrow N_\infty, x_\infty)$ in the sense described in Section 2. We then have a diagram provided by geometric convergence

$$
\begin{array}{ccc}
\partial\mathcal{CC}(N_n) & \xrightarrow{\;j_n\;} & N_n \\
\phi_n \uparrow & & \uparrow k_n \\
X_\infty & \xrightarrow[\;j_\infty\;]{} & N_\infty
\end{array}
$$

where the vertical arrows are the approximating maps, where $j_n : \partial\mathcal{CC}(N_n) = \left(\Sigma, \sigma_{\partial\mathcal{CC}(N_n)}\right) \to N_n$ is a pleated surface parametrization of $\partial\mathcal{CC}(N_n)$ in the isotopy class of the marking of N_n, and $X_\infty = (\Sigma, \sigma_\infty) \to N_\infty$ is the "limit" pleated surface in N_∞.

LEMMA 4.3. *The map j_∞ is incompressible.*

Proof. For large n, the maps $k_n^{-1} j_n \phi_n$ are defined and they are homotopic to j_∞ within a neighbourhood of $j_\infty(X_\infty)$ of uniformly bounded diameter. Thus j_∞ is compressible if and only if $k_n^{-1} j_n \phi_n$ is compressible. As the diameters of the pleated surfaces $j_n \partial\mathcal{CC}(N_n)$ are uniformly bounded, there

exists n_0 such that the map $k_n^{-1} j_n \phi_n$ is an embedding for every $n \geq n_0$. Suppose that j_∞ is compressible. By the Loop Theorem, there exists a simple closed curve $\gamma \subset X_\infty$ such that $k_{n_0}^{-1} j_{n_0} \phi_{n_0}(\gamma)$ bounds an embedded disk D^2 in N_∞.

For large enough $n \geq n_0$ the map k_n is defined on the disk D^2. Since the maps $k_n j_\infty$ and $j_n \phi_n$ are (locally) homotopic we observe that the simple closed curve $\phi_n(\gamma) \subset \partial\mathcal{CC}(N_n)$ whose image under the inclusion j_n is freely homotopic to $k_n j_\infty(\gamma)$, is compressible. As N_n is a handlebody, this means that $\phi_n(\gamma) \in \mathcal{D}$, where, as before, $\mathcal{D} \subset \mathcal{C}$ is the set of diskbounding curves. The length $L_{\partial\mathcal{CC}(N_n)}(\phi_n(\gamma))$ of $\phi_n(\gamma)$ is bounded from above by $\leq 2 L_{X_\infty}(\gamma)$ for n large enough.

To obtain a contradiction, it suffices to show that the $\partial\mathcal{CC}(N_n)-$lengths of any sequence of simple closed diskbounding curves $\zeta_n \subset \mathcal{D}$ blow up along the sequence. To this end let as before μ_{X_n} be a short marking for the conformal boundary X_n of N_n, with base the pants decomposition P_n. Denote by $i(\xi, \zeta)$ the geometric intersection number between two simple closed curves ξ, ζ in the boundary surface Σ. Let $B = B(\Sigma, \epsilon)$ be an upper bound for the length of a short marking for a surface $X \in \mathcal{T}_\epsilon$. Then there is a number $C = C(\Sigma, \epsilon) > 0$ such that

$$ L_{X_n}(\zeta_n) \geq C \cdot i\left(\zeta_n, P_n\right) \geq \frac{C}{B} 2^{(d_\mathcal{C}(\zeta_n, P_n)-2)/2} \xrightarrow[n \to \infty]{} \infty. $$

Namely, the first inequality follows from the fact that $L_{X_n}(P_n) \leq B$ and standard hyperbolic geometry as explicitly described in Lemma 4.2 in [17]. The second inequality is a consequence of $d_\mathcal{C}(\xi, \zeta) \leq 2 \log_2 i(\xi, \zeta) + 2$ for all $\xi, \zeta \in \mathcal{C}$. $\qquad \square$

Let us consider now the $\pi_1(j_\infty(X_\infty), x_0)-$covering of N_∞ which we denote by $p : \overline{N}_\infty \to N_\infty$. By covering theory, the map $j_\infty : X_\infty \to N_\infty$ lifts to $\overline{N}_\infty$, and any lift $\overline{j}_\infty : X_\infty \to \overline{N}_\infty$ is a homotopy equivalence. We fix once and for all such a lift and denote by $\overline{X}_\infty = \overline{j}_\infty(X_\infty)$ the image of X_∞ under this lift.

By work of Thurston [46] and Bonahon [7], we know that, in this setting, $\overline{N}_\infty$ is homeomorphic to $\Sigma \times \mathbb{R}$.

The next step of the proof consists in showing that $\overline{N}_\infty$ has a "visible" geometrically finite end homeomorphic to $\Sigma \times [0, \infty)$.

LEMMA 4.4. $\overline{N}_\infty$ *has a geometrically finite end homeomorphic to* $\Sigma \times [0, \infty)$.

Proof. For large n consider the nearest point retraction to the convex core $r_n : N_n \to \mathcal{CC}(N_n) \subset N_n$. It is a (base point preserving) 1-Lipschitz projection map, i.e. $r_n^2 = r_n$, with the property that $r_n j_n = j_n$. It also fits into a one-parameter family of projections that gives a deformation of r_n to the identity Id_{N_n}. Namely, for $t \geq 0$ define r_n^t to be the nearest point retraction

to the t-neighbourhood of the convex core which is a convex subset $\mathcal{CC}_t(N_n)$. Notice that $r_n^t \overset{t\to\infty}{\longrightarrow} \mathrm{Id}_{N_n}$, uniformly on compact subsets.

The map $h_n : N_n \times [0,\infty) \to N_n$ defined by

$$h_n(x,t) = r_n^t(x)$$

is 1-Lipschitz with respect to the product metric. Hence, by geometric convergence, there is an induced 1-Lipschitz family of projections $h_\infty : N_\infty \times [0,\infty) \to N_\infty$. Let us denote by $r_\infty : N_\infty \to N_\infty$ the map induced by the r_n's, i.e. $r_\infty(\cdot) = h_\infty(\cdot, 0)$. It has the properties $r_\infty^2 = r_\infty$ and $r_\infty j_\infty = j_\infty$. The last property implies, in particular, that h_∞ lifts to a map $\bar{h}_\infty : \overline{N}_\infty \times [0,\infty) \to \overline{N}_\infty$ with $\bar{h}_\infty(\cdot, 0) = \bar{r}_\infty$, the lift of r_∞ satisfying $\bar{r}_\infty \bar{j}_\infty = \bar{j}_\infty$.

Since $h_\infty(\cdot, t) \to \mathrm{Id}_{N_\infty}$ $(t \to \infty)$ we also get $\bar{h}_\infty(\cdot, t) \to \mathrm{Id}_{\overline{N}_\infty}$ $(t \to \infty)$. From the existence of such a map we can immediately conclude that the image $\bar{r}_\infty(\overline{N}_\infty)$ is a convex subset of $\overline{N}_\infty$ containing the convex core $\mathcal{CC}(\overline{N}_\infty)$. To this end let $x, y \in \bar{r}_\infty(\overline{N}_\infty)$ be a pair of points. Let α be a geodesic joining them. Then $\bar{r}_\infty(\alpha)$ is homotopic, relative to the endpoints, to α and has length $l(\bar{r}_\infty(\alpha)) \leq l(\alpha)$. Since α is the unique length minimizer in its homotopy class with fixed endpoints we conclude that $\bar{r}_\infty(\alpha) = \alpha$. To summarize, $\bar{r}_\infty(\overline{N}_\infty)$ is convex.

Now the boundary $\overline{X}_\infty$ of $\bar{r}_\infty(\overline{N}_\infty)$ is a closed embedded (by convexity) incompressible pleated surface, and such a surface is contained in the convex core of $\overline{N}_\infty$. To be more precise, the preimage $\tilde{X}_\infty$ of $\overline{X}_\infty$ in $\mathbb{H}^3$ is a pleated surface which bounds a convex $\pi_1(\overline{N}_\infty)$- half-space V. Furthermore, the group $\pi_1(\overline{N}_\infty)$ acts properly and cocompactly on $\tilde{X}_\infty$, equipped with the intrinsic path metric. If $\tilde{\lambda}$ is a pleating lamination for $\tilde{X}_\infty$, then every leaf of $\tilde{\lambda}$ connects two points in the limit set of $\pi_1(\overline{N}_\infty)$ by invariance under the action of $\pi_1(\overline{N}_\infty)$. But $\tilde{X}_\infty$ is contained in the convex hull of $\tilde{\lambda}$, whence $X_\infty \subset \partial\mathcal{CC}(\overline{N}_\infty)$. This shows that $\overline{X}_\infty$ bounds a geometrically finite end in $\overline{N}_\infty$. $\qquad\square$

Up to now we have not fully used the condition of relative bounded combinatorics. We do it now by observing the following

LEMMA 4.5. *There exists a minimal filling lamination $\lambda \in \partial_\infty \mathcal{C}(X_\infty)$ such that for any sequence of diskbounding curves $\{\delta_n \in \mathcal{C}(\partial\mathcal{CC}(N_n))\}_{n\in\mathbb{N}}$, the sequence of simple closed curves $\{\phi_n^{-1}\delta_n\}_{n\in\mathbb{N}} \subset \mathcal{C}(X_\infty)$ converges (up to passing to subsequences) to λ.*

Proof. Let μ_{X_∞} be a short marking on X_∞. Since ϕ_n is almost an isometry, the marking $\phi_n \mu_{X_\infty}$ is short also for $\partial\mathcal{CC}(N_n)$, i.e. we can choose $\mu_{\partial\mathcal{CC}(N_n)} = \phi_n \mu_{X_\infty}$. By assumption, we also find markings ν_n on $\partial\mathcal{CC}(N_n)$ that satisfy the relative bounded combinatorics conditions.

By Lemma 3.2, we have $(\phi_n^{-1}\delta_n | \phi_n^{-1}\nu_n)_{\mu_{X_\infty}} = (\delta_n | \nu_n)_{\phi_n \mu_{X_\infty}} \to \infty$. Moreover, μ_{X_∞} and $\phi_n^{-1}\nu_n$ have $R-$bounded combinatorics and their distances in $\mathcal{C}(X_\infty)$, being equal to the distances between $\mu_{\partial\mathcal{CC}(N_n)}$ and ν_n, diverge.

In particular $\phi_n^{-1}\nu_n$ converges (up to passing to subsequences) to a minimal filling lamination $\lambda \in \partial_\infty \mathcal{C}(X_\infty)$ and so does $\phi_n^{-1}\delta_n$ (see the proof of Lemma 3.2). $\hfill\square$

As an easy consequence, we obtain

COROLLARY 4.6. *The lamination $\bar{j}_\infty(\lambda) \subset \overline{X}_\infty$ is not realized in $\overline{N}_\infty$.*

Proof. Suppose we can realize $\bar{j}_\infty(\lambda)$. Then by composition with the covering projection, the lamination $\lambda \subset X_\infty$ can be realized in N_∞. Let $Y_\lambda \subset N_\infty$ be a pleated surface realizing λ.

Lemma 4.5, combined with the "long-branches-small-switch-angles" train-track argument due to Bonahon [7] (see the proof of Proposition 3.2 in [42] for a nice exposition), tells us that for sufficiently large n and a diskbounding simple closed curve δ_n on $X_n \subset N_n$, we could also realize $\phi_n^{-1}\delta_n$ as a closed geodesic, in a bounded neighbourhood of Y_λ in N_∞.

By geometric convergence, this implies that for large n, we can represent the curve $j_n\delta_n$ in N_n as a curve with very small geodesic curvature. Such a curve is not nullhomotopic in N_n, But this is absurd as δ_n is, by definition, compressible in N_n. $\hfill\square$

The next corollary is certainly well known. As we were not able to locate it in the literature in the form we need, we include a proof in the Appendix. The main issue here is the possible presence of accidental parabolics.

COROLLARY 4.7. $\overline{N}_\infty$ *has a simply degenerate end homeomorphic to $\Sigma \times [0, \infty)$ with ending lamination λ.*

To conclude, we found that $\overline{N}_\infty$ is a hyperbolic structure on $\Sigma \times \mathbb{R}$ for which one of the end invariants is a minimal filling lamination, and the second is a marked conformal structure on Σ. Since there is no room for other ends, we see that $\overline{N}_\infty$ is singly degenerate. Our final step consists in showing that the covering $p : \overline{N}_\infty \to N_\infty$ is trivial. This step concludes the proof of Proposition 4.2 with the exception of the bounded geometry condition.

LEMMA 4.8. *The covering $p : \overline{N}_\infty \to N_\infty$ is trivial.*

Proof. By Canary's Covering Theorem [14], $p : \overline{N}_\infty \to N_\infty$ is finite-to-one, N_∞ is homeomorphic to $\Omega \times \mathbb{R}$ where Ω is a closed surface, and the covering is induced by the inclusion $\pi_1(\Sigma) < \pi_1(\Omega)$. However, by Lemma 4.3, we also have an incompressible embedding of Σ into $N_\infty \simeq \Omega \times \mathbb{R}$. We conclude by evoking the standard fact of 3-manifold topology that the closed incompressible surfaces in $\Omega \times \mathbb{R}$ are all isotopic to the standard embedding $\Omega \hookrightarrow \Omega \times \{0\}$ (see Proposition 9.3.18 of [30]). $\hfill\square$

We are left with the observation that the injectivity radius of $\overline{N}_\infty$ is bounded from below by a universal positive constant. As there are no accidential parabolics and the ending lamination λ has $\overline{R}-$bounded combinatorics with respect to X_∞ ($\overline{R}$ might be bigger than the initial R, but it is still uniformly bounded), this is an immediate consequence of Minsky's theorem on Bounded Geometry for Kleinian Surface Groups [39].

5. Cut and Glue construction

Recall that our goal is to produce a model metric on a Heegaard splitting $H_1 \cup_f H_2$ and that we want to achieve it by gluing together some simple building blocks which are pieces of hyperbolic manifolds we understand better. In our case the building blocks are two convex cocompact handlebodies interpolated by a quasi-fuchsian manifold.

This section describes how the gluing works. We use a standard procedure, summarized in Lemma 5.1. Suppose we are given the following data:

(1) A pair $(N, \partial N)$, $(M, \partial M)$ of Riemannian $3-$manifolds with boundary, with metrics ρ_N, ρ_M.
(2) A pair of collars U, V of $\partial N, \partial M$ with a smooth diffeomorphism $k : U \to V$ between them.
(3) A smooth bump function $\theta : U \to [0, 1]$ which takes value 1 on ∂N and 0 on the other boundary $\partial U \setminus \partial N$.

Then we can form the Riemannian $3-$manifold $N \cup_{k:U\to V} M$ where the metrics ρ_N and ρ_M are replaced on U by the convex combination $(1 - \theta)\rho_N + \theta k^* \rho_M$. A crucial feature that can be implemented is a control on the *sectional curvatures*: If the diffeomorphism is almost isometric and the bump function has uniformly bounded derivatives, then the sectional curvatures of the gluing will be comparable to those of the pieces.

Let us now describe the case of a convex cocompact handlebody N and quasi-fuchsian manifold Q. The first item of the list is provided by small smooth neighbourhoods of their convex cores $\mathcal{CC}(N)$ and $\mathcal{CC}(Q)$.

The second item requires us to control a pair of collars of $\partial\mathcal{CC}(N)$ and $\partial\mathcal{CC}(Q)$, and this is provided by Proposition 4.1. We also have to produce a nice diffeomorphism between them, but this will be the object of the next section (Proposition 6.1). We anticipate here that, in general, it will not be possible to use exactly the collars of $\partial\mathcal{CC}(N)$ and $\partial\mathcal{CC}(Q)$, but one has to allow a more flexible notion of *product region* which we define below.

As for the last item, Lemma 5.2 will produce for us uniformly controlled bump functions on product regions with bounded geometry.

Now we describe the details. Let us start with a definition:

Definition (Product Region). Let $(M, \partial M, j : \Sigma \to \partial_0 M)$ be a compact oriented $3-$manifold with boundary ∂M and a distinguished parametrized (or simply, marked) boundary component $j : \Sigma \to \partial_0 M \subset \partial M$. A *product*

region $U \subset M$ relative to $\partial_0 M \subset \partial M$ is a codimension 0 submanifold homeomorphic to $\Sigma \times [-1, 1]$ which is isotopic to a collar of $\partial_0 M$.

Using the product structure of U we can define a *top boundary* $\partial_+ U$, the one that faces $\partial_0 M$, and a *bottom boundary* $\partial_- U$. We denote by M_- and M_+ the parts of M that lie *below* $\partial_+ U$ and *above* $\partial_- U$ respectively. In particular, U is a collar of a boundary component of both M_-, M_+.

We will be interested in essentially three parameters of a product region: The injectivity radius inj $U := \inf \{\mathrm{inj}_x M \,|\, x \in U\}$, the diameter diam U and the *width* width $U := \inf \{d(x, y) \,|\, x \in \partial_+ U, y \in \partial_- U\}$. When the injectivity radius, the width and the diameter are uniformly bounded we say that the product region has *bounded geometry*.

Finally, we also observe that the marking $j : \Sigma \to \partial_0 M$ can be isotoped into U producing a *marking* of U.

One can cut and glue 3$-$manifolds along product regions. Namely, suppose we have a pair $(M, \partial M, j : \Sigma \to \partial_0 M)$ and $(N, \partial N, i : \Sigma \to \partial_0 N)$ of compact 3$-$manifolds with boundary together with distinguished parametrized boundary components. Let $U \subset M$ and $V \subset N$ be product regions relative to $\partial_0 M, \partial_0 N$ respectively. Let $k : U \to V$ be an orientation preserving diffeomorphism between them. Then we can form the 3-manifold

$$\mathbb{X} := M_- \cup_{k:U \to V} N_+.$$

Up to homeomorphism, the result only depends on the homotopy class of k, which we are going to define. Denote by $j_U, j_V : \Sigma \to U, V$ the induced markings on the product regions. The composition $f := j_V^{-1} \circ k \circ j_U$ is well-defined up to isotopy, that is, it does not depend on the choice of induced markings. It is called the *homotopy class of k with respect to the markings j, i*. The manifold $\mathbb{X}$ is homeomorphic to

$$M \cup_{\phi : \partial_0 M \to \partial_0 N} N$$

where $\phi := i \circ f \circ j^{-1}$.

Now we turn to the Riemannian part of the construction. The following observation, which we state as a lemma, is the main conclusion of the cut and glue construction. The proof is straightforward, and we omit it.

LEMMA 5.1. *Let $(M, \partial M, j : \Sigma \to \partial_0 M)$, $(N, \partial N, i : \Sigma \to \partial_0 N)$ be marked hyperbolic structures on M and N with distinguished boundary components $\partial_0 M, \partial_0 N$. Denote by ρ_M, ρ_N the Riemannian metrics of M, N. Suppose we have product regions $U \subset M, V \subset N$ relative to $\partial_0 M, \partial_0 N$ and an orientation preserving diffeomorphism $k : U \to V$ between them. Suppose also that $\theta : U \to [0, 1]$ is a smooth function with $\theta|_{\partial_- U, \partial_+ U} \equiv 0, 1$. Then we can form the 3$-$manifold*

$$\mathbb{X} = M_- \cup_{k:U \to V} N_+$$

and endow it with the Riemannian metric

$$\rho := \begin{cases} \rho_M & on\ M_- \setminus U \\ (1-\theta)\rho_M + \theta k^*\rho_N & on\ U \\ \rho_N & on\ N_+ \setminus V. \end{cases}$$

On $U \subset \mathbb{X}$ we have the following sectional curvature bound

$$|1 + \sec_{\mathbb{X}}| \leq ||\theta||_{\mathcal{C}^2} \cdot \left(||\mathrm{Rm}_M - k^*\mathrm{Rm}_N||_{\mathcal{C}^2} + ||\rho_M - k^*\rho_N||_{\mathcal{C}^2} \right).$$

The compositions of the inclusions $M_- \subset \mathbb{X}, N_+ \subset \mathbb{X}$ with the (induced) markings $j_U, i_V : \Sigma \to M, N$ are homotopic.

Once we fix the size of a product region we can produce a uniform bump function $\theta : U \to [0,1]$ on it.

LEMMA 5.2. *For all $\eta, D > 0$ there exists $B > 0$ such that the following holds: Let $U \simeq \Sigma \times [0,1]$ be a product region with bounded geometry, namely* inj $U \geq \eta$, diam $U \leq 2D$, width $U \geq D$. *Then there exists a smooth function $\theta : U \to [0,1]$ with the following properties:*

- *Near the boundaries it is constant: $\theta|_{\partial_- U} \equiv 0$ and $\theta|_{\partial_+ U} \equiv 1$.*
- *Uniformly bounded $\mathcal{C}^2-$norm: $||\theta||_{\mathcal{C}^2} \leq B$.*

Proof. For every $D, \eta > 0$, the space of pointed hyperbolic $3-$manifolds

$$\mathcal{S}(\eta, 2D, D) = \left\{ (M, \star \in U) \ \middle| \ \begin{array}{c} U \simeq \Sigma \times [-1,1] \text{ is a product,} \\ \text{inj } U > \eta, \text{diam } U < 2D, \text{width } U > D \end{array} \right\}$$

is relatively compact in the geometric topology. For any $\delta > 0$, the accumulation points not in the space are still contained in $\mathcal{S}(\eta - \delta, 2D + \delta, D - \delta)$. $\square$

6. ALMOST-ISOMETRIC EMBEDDINGS AND GLUING

If we want to apply the cut-and-glue construction, we have to understand when we can find almost isometric product regions in a convex cocompact handlebody and in a quasi-fuchsian manifold. The following technical proposition provides the control we need.

PROPOSITION 6.1. *Fix bounded combinatorics parameters $R, \epsilon > 0$ and an almost-isometry parameter $\xi > 0$. There exists $L_0 = L_0(R, \epsilon) > 0$ such that for every $L \geq L_0$ there exists a height $H = H(L, R, \epsilon, \xi) > 0$ such that the following holds: Let $(Y, X) \in \mathcal{T}_\epsilon \times \mathcal{T}_\epsilon$ be a pair with relative $R-$bounded combinatorics and height at least H. Let $Z \in \mathcal{T}_\epsilon$ be any other point such that the Teichmüller geodesic $[Y, Z]$ contains $[Y, X]$ as a subsegment. Consider the convex cocompact handlebody $N = \mathcal{H}(X)$ and the quasi-fuchsian manifold $Q = Q(Y, Z)$. Then there exist product regions $U \subset N$ and $V \subset Q$ and an orientation preserving diffeomorphism $k : U \to V$ such that*

- (1) *Bounded geometry:* inj $U \geq \eta = \eta(R, \epsilon) > 0$, diam $U \leq 2L$ *and* width $U \geq L$.
- (2) *Almost isometry: $||\rho_N - k^*\rho_Q||_{\mathcal{C}^2} + ||\mathrm{Rm}_N - k^*\mathrm{Rm}_Q||_{\mathcal{C}^2} < \xi$.*

(3) *Homotopy class: k lies in the isotopy class of the identity with respect to the markings.*

Moreover, U contains the geodesic representative of $\alpha \in \mathcal{C}$, a curve which has moderate length for both N and some hyperbolic surface $T \in [Y, X]$, i.e. $l_N(\alpha), L_T(\alpha) \leq B = B(g, \epsilon)$.

We call the point $Z \in \mathcal{T}_\epsilon$ the *free boundary* of $Q(Y, Z)$.

Conditions (1)-(3) guarantee that we can uniformly glue $\mathcal{H}(X)$ to $Q(Y, Z)$ using the cut and glue construction. The application to Heegaard splitting is given in Theorem 6.7 at the end of the section.

A few words on the proof: We have seen that the boundary of the convex core of a convex cocompact handlebody with relative bounded combinatorics and large height has a large-thick collar. This means that, to some extent, we can treat it as if it was a hyperbolic structure on $\Sigma \times \mathbb{R}$.

Then the strategy is to reduce the problem to the following statement, which solves the analogue question of finding almost isometric embeddings of product regions in hyperbolic manifolds homeomorphic to $\Sigma \times \mathbb{R}$

PROPOSITION 6.2. *For every $\epsilon, \xi, \delta, L > 0$ there exists $H = H(\epsilon, \xi, \delta, L) > 0$ such that the following holds: Let Q_1, Q_2 be marked hyperbolic structures on $\Sigma \times \mathbb{R}$ without accidental parabolics with associated Teichmüller geodesics $l_i : I_i \subseteq \mathbb{R} \to \mathcal{T}$ with $i = 1, 2$. Suppose that l_1, l_2 $\delta-$fellow travel on a subsegment J of length at least $20H$ and entirely contained in the $\epsilon-$thick part $\mathcal{T}_\epsilon$. Then there exist product regions $U_i \subset Q_i$ with $\mathrm{diam}(U_i) \leq 2L$, $\mathrm{width}(U_i) \geq L$ and a $\xi-$almost isometric embedding $k : U_1 \to U_2$ in the homotopy class of the identity with respect to the markings. Moreover, U_i contains the geodesic representative of $\alpha \in \mathcal{C}$, a curve which has moderate length for both Q_i and $T \in J$ the midpoint of the segment, i.e. $l_{Q_i}(\alpha), L_T(\alpha) \leq B = B(\epsilon, g)$.*

6.1. **Lipschitz model.** Proposition 6.2 is a direct consequence of the Lipschitz Model by Minsky [40]. We use the following statement:

THEOREM 6.3 (Minsky [40]). *Fix $\epsilon > 0$. Let Q be a marked hyperbolic structure on $\Sigma \times \mathbb{R}$ without accidental parabolics. Let $l : I \to \mathcal{T}$ be the corresponding Teichmüller geodesics. There exists $H_0 = H_0(g, \epsilon) > 0$ and $B = B(g, \epsilon) > 0$ such that the following holds: Suppose that I contains a subsegment $[a - H_0, b + H_0]$ such that $b - a \geq 20H_0$ and $l[a - H_0, b + H_0] \subset \mathcal{T}_\epsilon$. Denote by J the subsegment $[a, b]$. Then for every $X \in J$ and any curve $\alpha \in \mathcal{C}$ with $L_X(\alpha) \leq B$ there exists a pleated surface $i_X : \hat{X} \to Q$ realizing α and such that $d_\mathcal{T}(X, \hat{X}) \leq H_0$. In particular $l_Q(\alpha) \leq Be^{2H_0}$.*

We remark the following crucial consequence of the Margulis Lemma:

LEMMA 6.4. *There exists $\eta = \eta(g, \epsilon) > 0$ such that $\mathrm{inj}_x(Q) > \eta$ for every $x \in i_X(\hat{X})$ and every $X \in J$.*

Proof. The surface $i_X(\hat{X})$ is π_1−surjective and has uniformly bounded diameter (1−Lipschitz image of $\hat{X} \in \mathcal{T}_{\epsilon'(\epsilon,H)}$). Such a surface cannot enter any very thin part of Q (see Lemma 2.3). $\qquad\square$

We prove Proposition 6.2.

Proof of Proposition 6.2. We argue by contradiction. Suppose we have a sequence of structures Q_1^n, Q_2^n that satisfy the assumptions, but do not satisfy the conclusions. Denote by J_n the δ−fellow-traveling ϵ−thick segment for their Teichmüller geodesics $l_i^n : I_i^n \to \mathcal{T}$ with $i = 1, 2$. Let $c_n \in J_n$ the midpoint. Up to the action of the mapping class group we can assume that c_n lies in a fixed compact set of $\mathcal{T}$. Let us parametrize J_n by $j_n : [-a_n, a_n] \to J_n$ so that $j_n(0) = c_n$. After the renormalization, we can extract a subsequence that converges uniformly on compact sets to a bi-infinite Teichmüller geodesic $j_\infty : \mathbb{R} \to J_\infty$ entirely contained in $\mathcal{T}_\epsilon$ with distinct uniquely ergodic minimal filling endpoints $\lambda^-, \lambda^+ \in \mathcal{PML}$.

By the Double Limit Theorem and the Ending Lamination Theorem, the geodesic j_∞ defines a unique doubly degenerate structure on $\Sigma \times \mathbb{R}$ whose ending laminations are λ^+, λ^-. We show that we can choose basepoints $x_j^n \in Q_j^n$ so that the sequence of pointed manifolds (Q_j^n, x_j^n) converges geometrically to Q_∞ and derive a contradiction.

Let us focus on Q_1^n. Parametrize l_1^n so that when we restrict it to $[-a_n + \delta, a_n - \delta]$ it δ−fellow travels $j_n : [-a_n + \delta, a_n - \delta] \to J_n$. Observe that the geodesic l_1^n converges uniformly on compact sets to some bi-infinite geodesic J_1^∞ which is entirely contained in $\mathcal{T}_\epsilon$ and has uniquely ergodic minimal filling endpoints λ_1^+, λ_1^-. Since l_1^n and j_n are δ−fellow travelers, we have $\lambda_1^\pm = \lambda_\infty^\pm$. Therefore $J_1^\infty = J^\infty$, as cobounded geodesics in Teichmüller space are uniquely determined by their endpoints.

The points $X_1^n := l_1^n(0)$ converge to $X_\infty = j_\infty(0)$, hence, for n large enough, by Theorem 6.3, we can find pleated surfaces $i_{X_1^n} : \hat{X}_1^n \to Q_1^n$ that realize any short curve $\gamma \in \Upsilon(X_\infty)$ for X_∞. We choose a basepoint x_1^n on $i_{X_1^n}(\hat{X}_1^n)$. Lemma 6.4 tells us that we can take a geometric limit of the sequence triples $(Q_1^n, x_1^n, i_{X_1^n} : \hat{X}_1^n \to Q_1^n)$. The limit is a triple $(Q_1^\infty, x_1^\infty, i_{X_\infty} : \hat{X}_1^\infty \to Q_1^\infty)$ where i_{X_∞} is a pleated surface realizing γ, the short curve on $\Upsilon(X_\infty)$ as above, in Q_1^∞.

The proof can now proceed with the standard arguments of Section 4 with less complications. We only give a sketch. The map i_{X_∞} is incompressible and determines a covering of Q_1^∞ homeomorphic to $\Sigma \times \mathbb{R}$. We claim that it is doubly degenerate. It suffices to check that λ^+ and λ^- are not realized. Choose a diverging sequence of times $t_n \in [0, a_n]$ such that $l_1^n(t_n)$ is very close to J_∞. Theorem 6.3 implies that a short curve α_n^+ for $l_1^n(t_n)$ has moderate length for Q_1^n. As $t_n \uparrow \infty$ we have $\alpha_n^+ \to \lambda^+$. Suppose that λ^+ is realized. The "long-branches-small-switch-angles" train track argument implies that we can also realize α_n^+ in a fixed compact set and hence $l_{Q_1^\infty}(\alpha_n^+)$ must

necessarily diverge. By geometric convergence, the same must happen in Q_1^n, but α_n^+ has always moderate length in Q_1^n, a contradiction. The argument for λ^- is the same. By the Ending Lamination Theorem, the i_{X_∞}−covering is isometric to Q_∞. The Covering Theorem [14] implies that the covering is trivial. The uniform bound on the injectivity radius is guaranteed by the fact that λ^+, λ^- have uniformly bounded combinatorics [39].

In conclusion, for any fixed size L and almost isometric parameter ξ, we can pull-back a product region of that size from Q_∞ to Q_1^n in a ξ−almost isometric fashion. Moreover, we can also assume that the product region lies uniformly close to the basepoint and that, if the size L is sufficiently large, it also contains the geodesic representative of a short curve γ on $l_1^n(0)$, the midpoint of J_n. This contradicts the initial assumptions. $\square$

We are now ready to prove Proposition 6.1

6.2. **Proof of Proposition 6.1.** We argue again by contradiction.

Suppose we have a sequence of examples $\{(Y_n, X_n, Z_n)\}_{n \in \mathbb{N}}$ with relative R−bounded combinatorics and diverging heights, but $N_n := \mathcal{H}(X_n)$ and $Q_n := Q(Y_n, Z_n)$ do not satisfy the conclusion of the proposition. As a first step, we reduce the problem to the realm of hyperbolic structures on $\Sigma \times \mathbb{R}$. If we fix basepoints $x_n \in \partial\mathcal{CC}(N_n)$, we know that we can take, up to passing to subsequences, a geometric limit of the sequence

$$\{(N_n, x_n, j_n : (\Sigma, \sigma_n) \to \partial\mathcal{CC}(N_n))\}_{n \in \mathbb{N}}.$$

The limit is a triple $(N_\infty, x_\infty, j_\infty : (\Sigma, \sigma_\infty) \to N_\infty)$ where N_∞ is a singly degenerate hyperbolic structure on $\Sigma \times \mathbb{R}$ with injectivity radius bounded from below by $\eta = \eta(R, \epsilon) > 0$ and j_∞ is a pleated surface marking. The approximating maps $k_n : N_\infty \to N_n$ and the change of marking $\phi_n : \Sigma \to \Sigma$ provided by the pleated surface convergence fit into the following diagram that commutes up to local homotopies

$$
\begin{array}{ccc}
(\Sigma, \sigma_n) & \xrightarrow{\;\;j_n\;\;} & N_n \\
{\scriptstyle\phi_n}\big\uparrow & & \big\uparrow{\scriptstyle k_n} \\
(\Sigma, \sigma_\infty) & \xrightarrow[\;\;j_\infty\;\;]{} & N_\infty.
\end{array}
$$

The ending lamination λ is the support of a unique projective measured lamination that can be characterized as the limit in $\mathcal{T} \cup \mathcal{PML}$ of the sequence of remarked surfaces $Y_n' := \phi_n^{-1} Y_n$, i.e. $\lambda = \lim_{n \to \infty} Y_n' \in \mathcal{PML}$ (see [24]).

We fully renormalize the picture by introducing $X_n' := \phi_n^{-1} X_n$, $Z_n' := \phi_n^{-1} Z_n$ and the quasi-fuchsian manifolds $Q_n' := Q(Y_n', Z_n')$. Observe that the points $X_n' \in [Y_n', Z_n']$ lie in a fixed compact set. In fact, by the main theorem of [8], X_n' lies uniformly close to $\phi_n^{-1}(\Sigma, \sigma_n)$ which is converging to (Σ, σ_∞), the hyperbolic structure of the limit pleated surface. Hence, up to
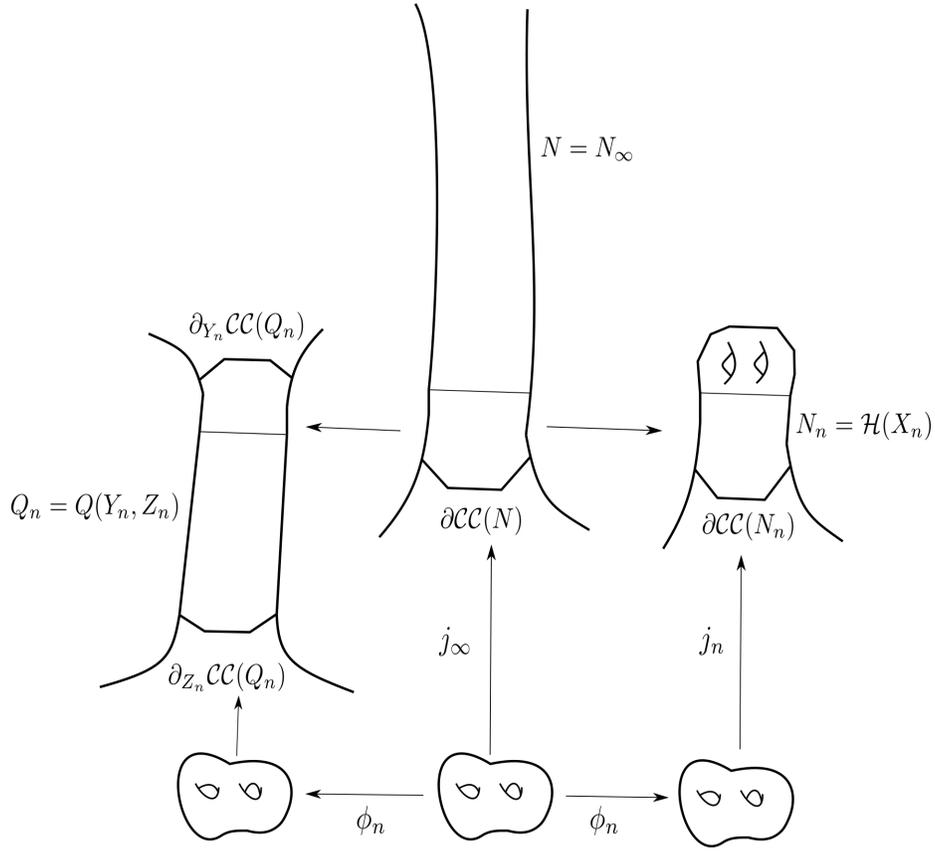
FIGURE 1. Almost-isometric embeddings of product regions.

subsequences, the segments $[Y'_n, X'_n]$ converge uniformly on compact sets to the Teichmüller geodesic ray $(\lambda, X_\infty]$.

The next lemma is used to determine a product region of N_∞ which can be embedded in all N_n and Q'_n with n large via Proposition 6.2 (see Figure 1). Let $l : (-\infty, 0] \to \mathcal{T}$ the Teichmüller ray corresponding to N_∞ with $l(-\infty) = \lambda$. As N_∞ has bounded geometry, it is entirely contained in a ϵ'−thick part (by [39], [43]) for some ϵ' only depending on R, ϵ. Moreover, by Theorem A of [37], it passes uniformly close to (Σ, σ_∞) and hence to X_∞.

LEMMA 6.5. *There exists $\delta, b > 0$ such that for every $H > 0$, if n is sufficiently large, then there is a point $W'_n \in [Y'_n, X'_n]$ such that $[W'_n, X'_n]$, parametrized in this order, δ−fellow travels the restriction of $l : (-\infty, 0] \to \mathcal{T}$ to $J_H = [-b - 200H, -b]$.*

Proof. The geodesics $(\lambda, X_\infty]$ and l have the same endpoint at infinity, the uniquely ergodic projective measured lamination λ. By Masur [31], these geodesics stay at a uniformly bounded distance δ. As $[Y'_n, X'_n]$ converges

uniformly on compact sets to $(\lambda, X_\infty]$, if n is large enough, we find $W'_n \in [Y'_n, X'_n]$ which is δ-close to l and with $d_{\mathcal{T}}(W'_n, X'_n)$ as large as we want. $\quad\square$

By Lemma 6.5, for any fixed H, we can apply Proposition 6.2 to the $\delta-$fellow traveling $\epsilon'-$thick geodesics $[Z'_n, Y'_n]$ and l along $l(J_H)$ if n is sufficiently large. We choose H to be larger than $H(\epsilon', \xi, \delta, L)$. For n large enough, we get a product region $U_n \subset N_\infty$ of L-bounded geometry and a $\xi-$almost isometric embedding $h_n : U_n \to Q'_n$. Moreover the product region U_n contains a geodesic α_n^* of uniformly bounded length that represents a short curve α_n for the midpoint T_n of $[W'_n, X'_n]$. The curve α_n has also moderate length for the midpoint T of $l(J_H)$ as $L_T(\alpha_n) \leq L_{T_n}(\alpha_n)e^{2\delta}$. As there is only a finite number of curves of moderate length on T, we can assume that $\alpha_n = \alpha$ is fixed.

Consider a sufficiently large collar U of $\partial\mathcal{CC}(N_\infty)$ containing the $2L$-neighbourhood of the geodesic representative of $\alpha \in \Upsilon(T)$. If n is sufficiently large, the approximating map $N_\infty \to N_n$ is defined and $\xi-$almost isometric on U. The product region U_n, containing α^* and having size comparable with L, is contained in U. This is a contradiction.

The proof of Proposition 6.1 is now complete.

6.3. **Position of the product regions.** As we have already pointed out, Proposition 6.1 guarantees that we can uniformly glue $\mathcal{H}(X)$ to $Q(Y, Z)$ using the cut and glue construction. However, for the model metric on $H_1 \cup_f H_2$, we need a more quantitative control: If we want to glue a pair of convex cocompact handlebodies to a single quasi-fuchsian manifold on top and on bottom, we have to make sure that the gluing regions appear in the *right order* along the quasi-fuchsian manifold. We control the order using the distance from the boundaries of the convex core: In the notations of Proposition 6.1 we have

LEMMA 6.6. *There exists some function $A : (0, \infty) \to (0, \infty)$ such that*

$$d_N (U, \partial\mathcal{CC}(N)) \leq A(d_{\mathcal{T}}(X, Y)),$$
$$d_Q (k(U), \partial_Y\mathcal{CC}(Q)) \leq A(d_{\mathcal{T}}(X, Y)).$$

Here $\partial_Y\mathcal{CC}(Q = Q(Y, Z))$ denotes the boundary component of the convex core that faces the conformal boundary Y.

Proof. Let α be a the curve which is of moderate length for some $T \in [Y, X]$ and whose geodesic representative has length $l_Q(\alpha) \in [\eta, B]$ and lies in U as in Proposition 6.1. We have

$$d_N (U, \partial\mathcal{CC}(N)) \leq d_N (\alpha^*, \partial\mathcal{CC}(N)),$$
$$d_Q (k(U), \partial_Y\mathcal{CC}(Q)) \leq d_Q (k(\alpha^*), \partial_Y\mathcal{CC}(Q)).$$

Since k is $\xi-$almost isometric, the curve $k(\alpha^*)$ has uniformly bounded geodesic curvature, hence its lift to $\mathbb{H}^3$ is a uniform quasi-geodesic and lies uniformly close to its geodesic representative by the Morse Lemma. The

length of the geodesic representative for $k(\alpha^*)$ is uniformly comparable with the one of α^*, in particular it is uniformly bounded away from 0 and ∞. By basic hyperbolic geometry

$$\cosh\left(d_Q\left(\alpha^*, \partial \mathcal{CC}(N)\right)\right) \leq L_{\partial \mathcal{CC}(N)}(\alpha)/l_N(\alpha),$$
$$\cosh\left(d_Q\left(k(\alpha)^*, \partial_Y \mathcal{CC}(Q)\right)\right) \leq L_{\partial_Y \mathcal{CC}(Q)}(\alpha)/l_Q(\alpha).$$

Thus, it is enough to show that the numerators are uniformly bounded:

$$L_{\partial \mathcal{CC}(N)}(\alpha) \simeq L_X(\alpha) \leq L_T(\alpha)e^{2d_\mathcal{T}(X,T)} \leq Be^{2d_\mathcal{T}(X,Y)},$$
$$L_{\partial_Y \mathcal{CC}(Q)}(\alpha) \simeq L_Y(\alpha) \leq L_T(\alpha)e^{2d_\mathcal{T}(Y,T)} \leq Be^{2d_\mathcal{T}(X,Y)}.$$

The inequalities are applications of Theorem 2.1 and Wolpert's inequality $L_R(\alpha) \leq L_S(\alpha)e^{2d_\mathcal{T}(R,S)}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6.4. The gluing.

The following theorem is the main technical result of this article. Recall that we denote by M_f the closed 3-manifold obtained by gluing two handlebodies with boundary Σ with a map $f \in \mathrm{Mod}(\Sigma)$.

THEOREM 6.7. *Let* $R, \epsilon, \xi > 0$ *be fixed. There exists* $H_{\mathrm{gluing}}(R, \epsilon, \xi) > 0$ *such that for every* $H \geq H_{\mathrm{gluing}}$ *the following holds: Let* $f \in \mathrm{Mod}\,(\Sigma)$ *be a gluing map. Suppose that* $\left(Y, X, \overline{X}, \overline{Y}\right) \in \mathcal{T}_\epsilon^4$ *is a quadruple with relative* $(f, R)-$*bounded combinatorics and height in* $[H, 2H]$. *Then there exists a metric* g *on* $M_f = H_1 \cup_f H_2$ *with the following properties.*

(1) *The sectional curvature of the metric is contained in the interval* $\sec \in (-1 - \xi, -1 + \xi)$.
(2) *The curvature of* g *is constant outside the union* Ω *of two disjoint regions of uniformly bounded diameter and uniform lower bound on the injectivity radius diffeomorphic to* $\Sigma \times [0, 1]$.
(3) $M_f - \Omega = H_1 \cup H_2 \cup Q$ *where* Q *is isometric to the complement in* $Q(Y, \overline{Y})$ *of a collar neighborhood of* $\partial \mathcal{CC}(Q(Y, \overline{Y})$ *of uniformly bounded radius (depending on* H), *and where* H_1, H_2 *are isometric to the complement in* $\mathcal{H}(X_0), \mathcal{H}(\overline{X}_0)$ *of a collar neighborhood of* $\partial \mathcal{CC}(\mathcal{H}(X_0)), \partial \mathcal{CC}(\mathcal{H}(\overline{X}_0))$ *of uniformly bounded diameter (where* $X_0, \overline{X}_0$ *are points on* $[Y, X], \left[\overline{X}, \overline{Y}\right]$).

Proof. Let L_0 be as in Proposition 6.1. Let $B_0 := B_0(L_0, \eta)$ be the $\mathcal{C}^2-$bound produced by Lemma 5.2. Consider the height $H_0 := H_0(2L_0, R, \epsilon, \xi/B_0)$ provided by Proposition 6.1.

We choose $X_0 \in [Y, X]$ (resp. $\overline{X}_0 \in \left[\overline{X}, \overline{Y}\right]$) so that $d_\mathcal{T}(Y, X_0) = H_0$ (resp. $d_\mathcal{T}(\overline{X}_0, \overline{Y}) = H_0$). This is possible if the height is sufficiently large. By Remark 3.3 the $(f, R)-$relative bounded combinatorics condition is still satisfied by $(Y, X_0, \overline{X}_0, \overline{Y})$. We use Proposition 6.1 twice in order to produce product regions:

- When applied to the pair (Y, X_0) using $\overline{Y}$ as a free boundary: A ξ/B_0-almost-isometric embedding in the homotopy class of the identity of a product region (with $2L_0-$bounded geometry and injectivity radius bounded by η) $k_U : U \subset \mathcal{H}(X_0) \to Q(Y, \overline{Y})$.
- When applied to the pair $(\overline{Y}, \overline{X}_0)$ using Y as a free boundary: A ξ/B_0-almost-isometric embedding in the homotopy class of f of a product region with $(2L_0-$bounded geometry and injectivity radius bounded by η) $k_V : V \subset \mathcal{H}(\overline{X}_0) \to Q(\overline{Y}, Y)$.
- Observe that the manifolds $Q := Q(Y, \overline{Y})$ and $Q(\overline{Y}, Y)$ are isometric via an orientation reversing isometry.

By Lemma 6.6 we have

$$d_Q\left(k_U(U), \partial_Y \mathcal{CC}(Q)\right) \leq A(2H_0) \text{ and } d_Q\left(k_V(V), \partial_{\overline{Y}} \mathcal{CC}(Q)\right) \leq A(2H_0).$$

In particular, if $d_Q\left(\partial_Y \mathcal{CC}(Q), \partial_{\overline{Y}} \mathcal{CC}(Q)\right)$ is much bigger than $A(2H_0)$ then $k_U(U), k_V(V)$ are disjoint and appear in the correct order along Q. Proposition 4.1 implies that there exists $H_{\text{gluing}} \geq H_0$ such that this condition is satisfied (we can choose H_{gluing} to be the height that implies the presence of a large-thick collar of $\partial_Y \mathcal{CC}(Q)$ of width at least $20A(2H_0)$).

Lemma 5.2 gives us uniform bump functions $\theta_U, \theta_V : U, V \to [0, 1]$ on U, V whose $\mathcal{C}^2-$norm is bounded by B_0. Finally, we apply Lemma 5.1 twice and glue $\mathcal{H}(X_0), Q, \mathcal{H}(\overline{X}_0)$ along $k_U : U \to Q$ and $k_V : V \to Q$ using uniform bump functions θ_U, θ_V. The sectional curvatures of the resulting manifold satisfy $|\sec + 1| \leq B_0 \cdot \xi/B_0 = \xi$. The requirements (2) and (3) follow from the cut and glue construction and Lemma 6.6. $\square$

7. Random Heegaard splittings

The goal of this section is to establish some geometric control on random 3-manifolds. We begin with defining the type of control we need.

DEFINITION. For $\delta \in (0, 1/2), b > 1$ and $g \geq 2$, a (b, δ)-*product region of genus* g in a Riemannian 3-manifold M is a closed subset V of M with the following properties.

(1) V is diffeomorphic to $\Sigma \times [0, 1]$ where Σ is a closed surface of genus g, and V *separates* M, i.e. $M - \text{int}(V)$ consists of two connected components with boundary $\Sigma \times \{0\}, \Sigma \times \{1\}$, respectively.
(2) The injectivity radius of M at points in V is contained in the interval $[\delta, 1/\delta]$, and the diameters of the surfaces $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$ are at most $1/\delta$.
(3) The restriction of the metric of M to V is of constant curvature -1.
(4) The distance between the boundary components $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$ equals at least b.

Note that as $b > 1$, the volume of an (b, δ)-product region is bounded from below by a universal constant which can be chosen to be the volume of a ball of radius δ in hyperbolic 3-space, and up to a universal additive

constant, its diameter is bounded from above by the distance between the boundary surfaces.

EXAMPLE. Let M be a doubly degenerate hyperbolic 3-manifold which is homeomorphic to $\Sigma \times \mathbb{R}$ for a closed surface Σ of genus $g \geq 2$ and whose injectivity radius is at least δ. Then the injectivity radius is also bounded from above by a universal constant (see [40] for details), and for any $b > 1$, any sufficiently large metric ball in M contains a (b, δ)-product region of genus g.

By definition, a (b, δ)-product region $V \subset M$ separates M. In particular, if $V' \subset M$ is another such region which is disjoint from V, then it is contained in one of the two components of $M - V$. Thus if $\mathcal{V} \subset M$ is a disjoint union of $k \geq 1$ (b, δ)-product regions in M, then the dual graph whose vertices are the components of $M - \mathcal{V}$ and where two such components are connected by an edge if their closures intersect the same component of $\mathcal{V}$ is a tree. We say that the components of $\mathcal{V}$ are *linearly aligned* if this tree is just a line segment.

We shall show that for there exists a number $C_1 > 0$, and for given numbers $b > 1, \delta > 0$ there exist a number $C_2 = C_2(b, \delta) > 0$ such that for any $\epsilon > 0$, a random 3$-$manifold admits a negatively curved metric as described in Theorem 6.7 with the following additional properties.

(a) The gluing control parameter ξ is smaller than ϵ.
(b) $\mathrm{vol}(H_1 \cup H_2 \cup \Omega) \leq \epsilon n$ where n is the step of the walk.
(c) $\mathrm{vol}(Q) \geq C_1 n$ where n is the step of the walk.
(d) The set Q contains a subset Q' which is a disjoint union of linearly aligned (h, δ)-product regions of genus g and cardinality at least $C_2 n$.

We call such a gluing a *gluing with (ϵ, b, δ)-controlled geometry*. The point here is that $C_1 > 0$ is a universal constant, and the constant $C_2(b, \delta)$ only depends on b, δ and, in particular, is independent of ϵ.

We begin with describing the basic setup of random 3-manifolds.

DEFINITION (Random Walk). Let us fix a symmetric probability measure μ on $\mathrm{Mod}(\Sigma)$ whose support is a finite generating set S. Let $\{S_j\}_{j \in \mathbb{N}}$ be a sequence of independent, $\mu-$distributed random variables with values in $\mathrm{Mod}(\Sigma)$. The $n-$th step of the random walk is the random variable $\omega_n := S_1 \cdots S_n$ (with $\omega_0 := \mathrm{Id}_\Sigma$). The random walk is the discrete process $(\omega_n)_{n \in \mathbb{N}}$.

Let $\mathcal{P}$ be a property of mapping classes or 3-manifolds. We say that $\mathcal{P}$ holds for a *random mapping class* (resp. for a *random 3-manifold*) if

$$\mathbb{P}_n \left[f \in \mathrm{Mod}(\Sigma) \,|\, f \text{ (resp. } M_f) \text{ has } \mathcal{P} \right] \overset{n \to \infty}{\longrightarrow} 1$$

where $\mathbb{P}_n$ is the distribution of the $n-$th step of the random walk ω_n and coincides with the n-th convolution of μ with itself.

The following is the main result of this section. The constants $C_1 > 0, C_2(b, \delta) > 0$ appearing implicitly in its statement depend on the probability measure μ and will be determined in the course of the proof.

PROPOSITION 7.1. *Let $g \geq 2$ and $\epsilon > 0, b > 0, \delta > 0$ be fixed. Let μ be a symmetric probability measure on $\mathrm{Mod}\,(\Sigma)$ whose support is a finite symmetric generating set. We have*

$$\mathbb{P}_n\left[f \in \mathrm{Mod}\,(\Sigma) \,|\, M_f \text{ has gluing with } (\epsilon, b, \delta)\text{-controlled geometry }\right] \underset{n \to \infty}{\longrightarrow} 1.$$

We first recall some facts about random walks on $\mathrm{Mod}\,(\Sigma)$.

7.1. Random walks on the mapping class group.
Much of the material we present here is also contained in higher generality and with more details in Section 6 of [1].

In the sequel we always consider a symmetric probability measure μ on $\mathrm{Mod}\,(\Sigma)$ whose support S is a finite generating set. Associated to the random walk generated by μ is a *space of sample paths* $(\Omega, \mathcal{E}, \theta)$ where $\Omega = \mathrm{Mod}\,(\Sigma)^{\mathbb{N}}$ is endowed with the product topology, $\mathcal{E}$ is the σ-algebra of Borel sets and $\mathbb{P}$ is the push-forward of the product measure $\mu^{\otimes \mathbb{N}}$ under the measurable map

$$T : \Omega \to \Omega, \quad \text{defined by } (T(s_i)_j) = s_1 \cdots s_j = \omega_j.$$

We have

THEOREM 7.2 (Maher [28]).

$$\mathbb{P}_n\left[f \in \mathrm{Mod}(\Sigma) \,|\, f \text{ is pseudo-Anosov }\right] \overset{n \to \infty}{\longrightarrow} 1.$$

We will use a geometric statement for the action of random mapping classes on Teichmüller space. The following result is due to Tiozzo.

THEOREM 7.3 (Tiozzo, Theorem 1 of [48]). *Fix some X in the Teichmüller space $\mathcal{T}$ of Σ. Then there exists $L_T > 0$ such that for almost all sample paths (ω_n) there exists a Teichmüller geodesic ray $\gamma : [0, \infty) \to \mathcal{T}$ with $\gamma(0) = X$ and such that*

$$\lim_{n \to \infty} \frac{d_{\mathcal{T}}(\omega_n X, \gamma(L_T n))}{n} \to 0.$$

In particular, the drift for the action of the random walk on Teichmüller space with the Teichmüller metric is positive.

There also is a statement concerning the action of the random walk on the curve graph $(\mathcal{C}, d_{\mathcal{C}})$ of Σ which is due to Maher and Tiozzo [29].

THEOREM 7.4 (Maher-Tiozzo, Theorem 1.2 and Theorem 1.3 of [29]). *Let $\alpha \in \mathcal{C}$ be a basepoint. Then there exists a constant $L_{\mathcal{C}} > 0$ such that for almost every sample path (ω_n) we have*

$$\lim_{n \to \infty} \frac{d_{\mathcal{C}}(\alpha, \omega_n \alpha)}{n} = L_{\mathcal{C}} > 0.$$

Moreover, there is a uniform quasigeodesic ray γ which tracks the sample path sublinearly, i.e.

$$\lim_{n \to \infty} \frac{d_{\mathcal{C}}(\omega_n \alpha, \gamma)}{n} = 0 \text{ almost surely .}$$

As an application of Theorem 7.3 and Theorem 7.4, we obtain the following result which was first shown by Kaimanovich and Masur [22]. For its formulation, recall that a point in $\partial_\infty \mathcal{C}$ is an unmeasured filling geodesic lamination on Σ.

THEOREM 7.5 (Kaimanovich-Masur [22], Maher-Tiozzo [29]). *For $\mathbb{P}$-almost every sample path $\omega = (\omega_n)_{n \in \mathbb{N}} \in \mathrm{Mod}\,(\Sigma)^{\mathbb{N}}$, the following holds true.*

(1) *For every base-point $\alpha \in \mathcal{C}$, the sequence $\{\omega_n \alpha\}_{n \in \mathbb{N}} \subset \mathcal{C}$ converges to a point $\mathrm{bnd}(\omega) \in \partial_\infty \mathcal{C}$ in the Gromov boundary which is independent of α.*

(2) *The point $\mathrm{bnd}(\omega)$ supports a unique transverse invariant measure up to scale, and the Teichmüller ray $\tau_{X,\mathrm{bnd}(\omega)}$ issuing from a fixed base-point $X \in \mathcal{T}$ which determined by $\mathrm{bnd}(\omega)$, equipped with this transverse invariant measure, has the sublinear tracking property from Theorem 7.3-*

Furthermore, the map $\mathrm{bnd} : \mathrm{Mod}\,(\Sigma)^{\mathbb{N}} \to \partial_\infty \mathcal{C}$ is measurable with respect to the σ-algebra of Borel subsets of $\partial_\infty \mathcal{C}$.

By Theorem 7.5, we may view the map bnd as both a map with values in the boundary $\partial_\infty \mathcal{C}$ of the curve graph as well as a map with values in the space $\mathcal{PML}$ of projective measured laminations. We will not distinguish between the two viewpoints in the sequel to keep the notations simple.

Theorem 7.5 leads to the next definition.

DEFINITION (Harmonic Measure). The measure $\nu := (\mathrm{bnd})_* \mathbb{P}$ on $\partial_\infty \mathcal{C}$ (or on $\mathcal{PML}$) is called the *harmonic measure* associated to the random walk (or to the distribution μ).

The next statement is Proposition 6.10 of [1]. It can be viewed as a statement about the harmonic measure on $\mathcal{PML}$.

PROPOSITION 7.6. *Let $W \subset \mathcal{T}$ be a $\mathrm{Mod}(\Sigma)$-invariant open subset that contains an axis of a pseudo-Anosov mapping class. Then for all $H > 0$ there exists a $\hat{c} = \hat{c}(W, H) > 0$ such that for almost every sample path ω, we have*

$$\liminf \frac{1}{T} |\{t \in [0, T] \mid \tau_{X,\mathrm{bnd}(\omega)}[t - H, t + H] \subset W\}| > \hat{c}.$$

The $\mathrm{Mod}(\Sigma)$-invariants sets W we are going to use in the sequel are the sets $\mathcal{T}_\delta$ for some suitably chosen numbers $\delta > 0$.

7.2. **Random handlebodies.** From now on we fix a handlebody $\mathcal{H}$ and a marking of the boundary surface Σ. The disk set $\mathcal{D}$ of $\mathcal{H}$ defines a subset $\partial \mathcal{D}$ of $\partial_\infty \mathcal{C}$ by taking its closure in $\mathcal{C} \cup \partial_\infty \mathcal{C}$ and intersecting with the boundary,

i.e. $\partial_\infty \mathcal{D} := \overline{\mathcal{D}} \cap \partial_\infty \mathcal{C}$. Maher, exploiting work of Kerckhoff [23] (Kerckhoff's proof contained a small gap that has been settled by Gadre in [18]), showed that $\partial_\infty \mathcal{D}$ has measure zero with respect to the harmonic measure ν.

THEOREM 7.7 (Maher, [27]). *The harmonic measure of the boundary of the disk set vanishes, i.e. $\nu (\partial_\infty \mathcal{D}) = 0$. Moreover, the Hempel distance increases linearly along the random walk, i.e. there exists a constant $K > 1$ such that*

$$\mathbb{P}_n \left[(\omega_n) \in \Omega \mid d_\mathcal{C} (\mathcal{D}, \omega_n \mathcal{D}) \in \left[\frac{1}{K} n, K n \right] \right] \overset{n \to \infty}{\longrightarrow} 1.$$

Maher's theorem has a few immediate consequences. First of all, for a random mapping class f, the 3-manifold M_f is hyperbolic (see Dunfield and Thurston [16]). Furthermore, let us choose once and for all a basepoint $X \in \mathcal{T}_\epsilon$ contained in the ϵ-thick part of Teichmüller space for a suitably chosen number $\epsilon > 0$. We select X so that it admits a short marking whose base is a pants decomposition made of diskbounding curves for $\mathcal{H}$. By Theorem 7.4, the distance in the curve graph between $\Upsilon(X)$ and $\Upsilon(\omega_n X)$ makes linear progress in n, and by Theorem 7.7, it makes linear progress away from the diskbounding curves. Here as before, $\Upsilon : \mathcal{T} \to \mathcal{C}$ denotes the systole map.

This property, however, is not sufficient to conclude that for a random element $f \in \mathrm{Mod}(\Sigma)$, the manifold M_f satisfies the assumptions in Proposition 7.1. As additional properties, we have to control the transition of the Teichmüller geodesic segment $\tau_{X, \omega_n X}$ connecting X to $\omega_n X$ through the thick part of Teichmüller space while controlling the rate of divergence of its trace from the disk set. We next establish this control.

Thus let $f \in \mathrm{Mod}(\Sigma)$ be a random mapping class. By Theorem 7.2 we know that f is p-A (pseudo-Anosov).

Quantitatively, a possible measure for the fellow-travelling of the disk set is given by the size of the nearest point projection of the disk set $\mathcal{D}$ to the uniformly quasi-convex subset

$$G_f := \Upsilon(\tau_{X, fX})$$

of the curve graph. We denote this nearest point projection by π_{G_f}. As G_f is a uniform unparametrized quasi-geodesic in the curve graph, hyperbolicity of $\mathcal{C}$ yields that the projection $\pi_{G_f}(\mathcal{D})$ is a quasi-convex subset of G_f. Let $|\pi_{G_f}(\mathcal{D})|$ be its diameter. Our next goal is to prove that as the step length tends to infinity, this diameter is arbitrarily small compared to the diameter $|G_f|$ of G_f.

PROPOSITION 7.8. *Let $g \geq 2$ and $\epsilon > 0$ be fixed. Let μ be a symmetric probability measure on $\mathrm{Mod}(\Sigma)$ whose support is a finite generating set. We have*

$$\mathbb{P}_n \left[f \in \mathrm{Mod}(\Sigma) \, \middle| f \text{ is p-A}, \left| \pi_{G_f}(\mathcal{D}) \right| / |G_f| \leq \epsilon \right] \overset{n \to \infty}{\longrightarrow} 1.$$

Proof. Let $\epsilon > 0$ be arbitrary. Let $K > 0$ be the constant from Theorem 7.7, let $L_{\mathcal{C}}$ be the constant from Theorem 7.4 and assume without loss of generality that $L_{\mathcal{C}}\epsilon < 1/2K$.

Let $\alpha = \Upsilon(X) \in \mathcal{C}$. We may assume that α is diskbounding in the handlebody $\mathcal{H}$.

For $n_0 > 0$ let $\Omega_{n_0} \subset \Omega$ be the set of all sample paths $\omega = (\omega_n)$ such that for all $n \geq n_0$ the following properties are fulfilled.

(1) $L_{\mathcal{C}}(1 - \epsilon/2)n \leq d_{\mathcal{C}}(\alpha, \omega_n\alpha) \leq L_{\mathcal{C}}(1 + \epsilon/2)n$.
(2) Let γ be a uniform quasigeodesic ray in $\mathcal{C}$ connecting $\gamma(0) = \alpha$ to $\gamma(\infty) = \text{bnd}(\omega)$; then $d_{\mathcal{C}}(\gamma, \omega_n(\alpha)) \leq L_{\mathcal{C}}\epsilon n/2$.
(3) $d_{\mathcal{C}}(\mathcal{D}, \omega_n\mathcal{D}) \geq n/2K$.

Note that we have $\Omega_{n_1} \supset \Omega_{n_0}$ for all $n_1 \geq n_0$. By Theorem 7.4 and Theorem 7.7, for every $\rho > 0$ there exists a number $n_0 = n_0(\rho) > 0$ so that $\mathbb{P}_n(\Omega_{n_0}) \geq 1 - \rho$.

The disk set $\mathcal{D} \subset \mathcal{C}$ is quasi-convex. Thus by hyperbolicity of $\mathcal{C}$, there exists a number $A > 0$ with the following property. Let $\zeta : [0, \infty) \to \mathcal{C}$ be a uniform quasi-geodesic ray beginning at $\zeta(0) = \alpha \in \mathcal{D}$; if $t > 0$ is such that $d_{\mathcal{C}}(\zeta(t), \mathcal{D}) > A$ and if $\beta \in \mathcal{C}$ is such that $\zeta(t)$ equals a shortest distance projection of β into ζ , then a shortest geodesic connecting β to $\mathcal{D}$ passes through a uniformly bounded neighborhood of $\zeta(t)$. In particular, up to increasing A, we have $\zeta(t) \notin \pi_{G_f}(\mathcal{D})$.

Assume from now on that $n_0/4K > A$. Let $(\omega_n) \in \Omega_{n_0}$ and let $n \geq n_0$. Denote by γ the quasi-geodesic ray in $\mathcal{C}$ as in property (2) above. Then on the one hand, we have

$$L_{\mathcal{C}}(1 - \epsilon/2)n \leq d_{\mathcal{C}}(\alpha, \omega_n(\alpha)) \leq L_{\mathcal{C}}(1 + \epsilon/2)n,$$

on the other hand also $d_{\mathcal{C}}(\gamma, \omega_n(\alpha)) \leq L_{\mathcal{C}}\epsilon n/2$. In particular, by property (3) above, the nearest point projection q_n of $\omega_n(\alpha)$ into γ is of distance at least $n/2K - L_{\mathcal{C}}\epsilon n/2 \geq n/4K > A$ from $\mathcal{D}$. This implies that a geodesic in $\mathcal{C}$ which connects $\omega_n(\alpha)$ to a shortest distance projection into $\mathcal{D}$ passes through a uniformly bounded neighborhood of q_n. Using again uniform quasi-convexity of $\mathcal{D}$ and the fact that $\alpha \in \mathcal{D}$ we conclude that the diameter of the shortest distance projection of $\mathcal{D}$ into the geodesic $\tau_{X,\omega_n X}$ does not exceed the distance between α and q_{n_0} which is at most $L_{\mathcal{C}}(1 + \epsilon)n_0$, independent of $n \geq n_0$ and $\omega \in \Omega_{n_0}$.

Let now $n_1 > 0$ be sufficiently large that $L_{\mathcal{C}}(1 + \epsilon)n_0 \leq \epsilon L_{\mathcal{C}}(1 - \epsilon)n_1$. Then for $\omega \in \Omega_{n_0}$ and for $n \geq n_1$, the distance between $\omega_n(\alpha)$ and α is at least $L_{\mathcal{C}}(1 - \epsilon/2)n$, while the diameter of the projection of $\mathcal{D}$ into $\tau_{X,\omega_n X}$ does not exceed $L_{\mathcal{C}}(1 + \epsilon)n_0$. By the choice of n_1, this means that the properties required in the proposition are fulfilled for this n_1, i.e. we have $|\pi_{G_{\omega_n}}(\mathcal{D})| \leq \epsilon|G_{\omega_n}|$ as claimed.

As $\rho > 0$ was arbitrary, the proposition follows. $\qquad\square$

7.3. **Good gluing regions.** The goal of this subsection is to show Proposition 7.1. The argument is very similar to the argument in the proof of Proposition 7.8. We begin with a volume control for convex cocompact hyperbolic structures on handlebodies. To this end choose as before once and for all a marking η for the boundary Σ of the handlebody $\mathcal{H}$ so that the base pants decomposition consists of diskbounding curves. The following proposition is well known in various settings. As we did not find a directly quotable statement in the literature, we sketch a proof.

PROPOSITION 7.9. *Let $\epsilon > 0$ be a fixed number and let ν be any marking on Σ of Hempel distance at least three to η. Suppose that $\mathcal{H}$ is equipped with a convex cocompact hyperbolic structure $\mathcal{H}(X)$ with conformal boundary $X \in \mathcal{T}_\epsilon$ such that ν is short for X. Then the volume of the convex core of $\mathcal{H}(X)$ is bounded from above by a fixed multiple of the distance between η, ν in the marking graph.*

Proof. The volume of any simplex with totally geodesic sides in a hyperbolic 3-manifold M is bounded from above by a universal constant. Furthermore, any abstract simplex, i.e. an embedded subset of M which is the image of an embedding $\Delta \to M$ where Δ is the standard 3-simplex, can be *straightened* in a unique way to a simplex with the same vertex set and with totally geodesic sides [2], and this construction is compatible with the side relation.

On the other hand, as by Theorem 2.1 the diameter of the boundary of the convex core of the handlebody $\mathcal{H}(X)$ is uniformly bounded, the volume of a uniformly bounded neighborhood of this boundary is uniformly bounded as well. Thus for the purpose of the proposition, it suffices to show that the complement in $\mathcal{CC}(\mathcal{H}(X))$ of a neighborhood of the boundary of uniformly bounded radius admits a triangulation by simplices with totally geodesic sides whose number does not exceed a fixed multiple of the distance between η and ν in the marking graph.

The strategy now is to construct for each marking of Σ a triangulation of Σ and control these triangulations as we move through the marking graph. We begin with noting that a marking decomposes the surface into a uniformly bounded number of polygonal disks. This means that the intersection points between the curves from the marking determine a collection of marked points on the boundaries of these disks. Subdivide each disk into triangles in such a way that the marked points are precisely the vertices of these triangles. Note that this procedure is by no means unique, but there are only finitely many combinatorial possibilities.

If we apply this procedure to the marking η, then we can extend this (topological) triangulation of the boundary of $\mathcal{CC}(\mathcal{H}(X))$ to a topological triangulation with uniformly few simplices. This is true because the base of η consists of diskbounding curves, and the disks with boundary in the base of η decompose $\mathcal{H}$ into balls.

Now let us assume that η' is obtained from η by a Dehn twist about the pants curves (i.e the base) of η. Let T be a triangulation of Σ defined by η and let T' be its image under the Dehn twist. Then there exists a triangulation τ of $\Sigma \times [0,1]$ which restricts to T, T' on the boundary. As up to the action of the mapping class group there are only finitely many combinatorial possibilities for this situation, we can find such a triangulation of $\Sigma \times [0,1]$ with a uniformly bounded number of simplices.

The same argument holds true for the move which replaces a pants curve by a marking curve and clears intersections. In a number of such steps whose number does not exceed a fixed multiple of the distance between η and ν in the marking graph, we obtain a triangulation of $\mathcal{CC}(\mathcal{H}(X))$. By the diameter bound for the boundary of $\mathcal{CC}(\mathcal{H}(X))$ and the assumption that ν is short for X and hence by Theorem 2.1), ν is short for the boundary of the convex core, straightening this triangulation then yields a triangulation of a subset of $\mathcal{CC}(\mathcal{H}(X))$ whose complement is contained in a uniformly bounded neighborhood of the boundary and hence has uniformly bounded volume. This yields the proposition. $\qquad\square$

Using Proposition 7.9 we are now ready to complete the proof of Proposition 7.1.

Proof of Proposition 7.1. Let X be a point in the thick part of Teichmüller space for which a fixed marking η on Σ with pants curves consisting of diskbounding curves is short. The strategy is to isolated a region on the Teichmüller geodesic connecting X to its image under a random pseudo-Anosov mapping class which fulfills the assumptions in Theorem 6.7. Furthermore, this region should be contained in the initial subsegment of the geodesic of length at most ϵ times the total length. We also isolate a region with similar properties near the end of the segment.

Using Proposition 7.9 we then argue that the sum of the volumes of the convex cocompact handlebodies corresponding to this initial and terminal segment of the geodesic is small compared to the volume of the center piece and that the center piece contains linearly aligned product regions as predicted in the proposition.

Let as before $A > 0$ be sufficiently large that the following holds true. Let $\gamma : [0, \infty) \to \mathcal{C}$ be a uniform quasi-geodesic beginning at the diskbounding curve $\gamma(0) = \alpha$ (this should mean that we choose once and for all a quasi-geodesic constant so that any two distinct points in $\mathcal{C} \cup \partial_\infty \mathcal{C}$ can be connected by a quasi-geodesic for this constant). We require that whenever $\beta \in \mathcal{C}$ is such that a shortest distance projection $\gamma(t)$ of β into γ has distance at least A from $\mathcal{D}$, then a shortest geodesic connecting β to $\mathcal{D}$ passes through a uniformly bounded neighborhood of $\gamma(t)$.

Let μ be a finitely supported probability measure on $\mathrm{Mod}(\Sigma)$ which induces the probability measure $\mathbb{P}$ on Ω. Let $\epsilon > 0, H > 0, \delta > 0$ be arbitrarily

fixed. We require that $\delta > 0$ is small enough that the conditions in Proposition 7.6 are fulfilled for $W = \mathcal{T}_{2\delta}$.

Let $f \in \mathrm{Mod}(\Sigma)$ and consider as before the Teichmüller geodesic $\tau_{X,fX}$ connecting X to fX. We say that M_f admits a (H, δ, ϵ)-*good gluing region* if the following holds true. Let $\ell(\tau_{X,fX})$ be the length of the geodesic segment $\tau_{X,fX}$; then there exists an initial subsegment $\tau_{X,fX}[0, \rho]$ of length $\rho \le \epsilon\ell(\tau_{X,fX})$ such that the distance between $\Upsilon(\tau_{X,fX}[\rho - 2H, \rho])$ and $\mathcal{D}$ is at least A and that $\tau_{X,fX}[\rho - 2H, \rho] \subset \mathcal{T}_\delta$. We claim that

$$\mathbb{P}_n[f \in \mathrm{Mod}(\Sigma) \mid f \text{ is p.A. and } f \text{ has a } (H, \delta, \epsilon) \text{ good gluing region }] \to 1.$$

Note that a (H, δ, ϵ)-good gluing region is related but a priori different from a gluing with controlled geometry.

To show the claim let $\sigma > 0$. By Theorem 7.4 and Proposition 7.8, we can find a number $n_0 = n_0(\sigma) > 0$ with the following property.

Let $\Omega_{n_0} \subset \Omega$ be the set of all sample paths (ω_n) so that

$$d_{\mathcal{C}}(\alpha, w_n\alpha) \in [(1 - \epsilon)L_{\mathcal{C}}, (1 + \epsilon)L_{\mathcal{C}})]$$

for all $n \ge n_0$ and that furthermore $|\pi_{G_{\omega_n}}(\mathcal{D})/|G_{\omega_n}| \le \epsilon/2$ for all $n \ge n_0$; then $\mathbb{P}(\Omega_{n_0}) \ge 1 - \sigma$.

Note that by convexity and hyperbolicity, if $(\omega_n) \in \Omega_{n_0}$ then the diameter of the projection $\pi_{G_{\omega_n}}(\mathcal{D})$ is at most $n_0 q$ for some fixed number q, independent of n (see the proof of Proposition 7.8). Thus by Proposition 7.6, for a fixed number $A > 0$ there is a number $T > 0$ with the following property. Let $\Omega' \subset \Omega_{n_0}$ be the set of all $\omega = (\omega_n) \in \Omega_{n_0}$ such that the geodesic segment $\tau_{\mathrm{bnd}(\omega)}[0, T]$ contains a subsegment of length at least $2H + 2A$ entirely contained in $\mathcal{T}_{2\delta}$; then $\mathbb{P}_n(\Omega_{n_1}) \ge 1 - 2\sigma$ for all $n \ge n_1$.

By Theorem 7.3, the orbit of X under the random path ω tracks the geodesic ray $\tau_{\mathrm{bnd}(\omega)}$ sublinearly. This implies the following. For $n > 0$ let $s(n) \ge 0$ be such that $\tau_{X,\mathrm{bnd}(\omega)}(s(n))$ is the shortest distance projection of $\omega_n(X)$ into $\tau_{X,\mathrm{bnd}(\omega)}$; then $s(n) \to \infty$ as $n \to \infty$.

Minsky [38] showed that for a given number $\delta > 0$, there exists a number $A > 0$ so that the following holds true. Consider for the moment two Teichmüller geodesics $\gamma, \zeta : [0, \infty) \to \mathcal{T}$ with the same starting point $\gamma(0) = \zeta(0)$. Suppose that for some $T > A$ and some $H \ge 0$ the segment $\gamma[0, T]$ contains a subsegment β of length $H + 2A$ entirely contained in $\mathcal{T}_{2\delta}$. Suppose furthermore that for some large m the shortest distance projection of $\zeta(m)$ into γ is contained in $\gamma - \gamma[0, T]$; then ζ contains a subsegment of length at least H which is contained in a uniformly bounded neighborhood of β.

Together with the above discussion, this implies that there exists a number $n_1 > n_0$ such that the set $\Omega_{n_1} \subset \Omega_{n_0}$ of all $(\omega_n) \in \Omega_{n_0}$ with the property that $\tau_{0,\omega_n(o)}$ contains a subsegment of length $2H$ entirely contained in $\mathcal{T}_\delta$ satisfies $\mathbb{P}_n(\Omega_{n_1}) \ge 1 - 2\sigma$ for all $n \ge n_1$.

Now following the reasoning in the proof of Proposition 7.8, we conclude that there exists a number $n_2 > 0$ so that for $n \ge n_2$ the proportion of

the length of the smallest initial subsegment of the geodesic $\tau_{X,\omega_n X}$ which contains the above segment of length $2H$ with respect to the total length of $\tau_{X,\omega_n X}$ is at most ϵ. In particular, for $n > n_2$ we have $\mathbb{P}_n(\Omega_{n_1}) \geq 1 - 2\sigma$ and, furthermore, if $(\omega_n) \in \Omega_{n_1}$ and if $n \geq n_2$ then ω_n is p.A. and admits a (H, δ, ϵ)-good gluing region.

On the other hand, for a fixed (sufficiently small) number $\delta > 0$ and a given number $H > 0$, Theorem 7.6 shows that there exists a number $\hat{c} = \hat{c}(H, \delta)$ such that for almost every sample path ω, we have

$$\liminf \frac{1}{T} |\{t \in [0, T] \mid \tau_{X,\text{bnd}(\omega)}[t - 2h, t + 2h] \subset T_{2\delta}\}| > \hat{c}.$$

This implies that for this number $\hat{c}$ and for $\sigma > 0$ as before, there exists a number $n_3 > n_2$ such that for $n \geq n_3$ we have

$\mathbb{P}_n\{(\omega_n) \in \Omega_{n_0} \mid \tau_{X,\omega_n X}$ contains $\hat{c}(1 - \epsilon)/2H$ pairwise disjoint segments of length at least $2H$ and contained in $\mathcal{T}_{2\delta}\} > 1 - 3\rho$.

However, if $n \geq n_3$ and if we consider the quasifuchsian manifold defined by the (H, δ, ϵ)-good gluing region and the Teichmüller segment $\tau_{X,\omega_n X}$ then Minsky's model theorem shows that this quasifuchsian manifold satisfies property (4) in the definition of a gluing with (ϵ, b, δ)-controlled geometry where the constant $H > 0$ as above depends on the choice of the a priori prescribed number $b > 1$.

Now the direction of the walk can be reversed and hence we can transfer statements about initial segments of the walk to statements about terminal segments. As $\sigma > 0$ was arbitrary, together this then yields the proposition. $\square$

8. GEOMETRIC CONTROL OF RANDOM HYPERBOLIC 3-MANIFOLDS

In Section 7 we established that a random hyperbolic 3-manifold of Heegaard genus g admits a Riemannian metric of sectional curvature close to -1 with some specific geometric properties. Furthermore, for any given numbers $b > 1, \delta > 0$, a definitive proportion of the volume for this metric is contained in a union of pairwise disjoint linearly aligned (b, δ)-product regions. Here the proportionality constant depends on the numbers b, δ.

The main goal of this section is to show that this property carries over to the hyperbolic metric on a random 3-manifold. The following lemma shows that this suffices for the proof of Theorem 1 from the introduction.

LEMMA 8.1. *For fixed $g \geq 2, \delta > 0$ and sufficiently large $b > 1$, there exists a number $C = C(g, b, \delta) > 0$ with the following property. Let M be a hyperbolic 3-manifold, and suppose that M contains $n \geq 1$ pairwise disjoint linearly aligned (b, δ)-product regions of genus g; then $\lambda_1(M) \leq C/n^2$ and $\lambda_n(M) \leq 1/C$.*

Proof. Let M be as in the lemma. Denote by $\mathcal{V} \subset M$ the union of the n linearly aligned (b, δ)-product regions of genus g whose existence is assumed in the statement of the lemma.

For each component $A = \Sigma \times [0, 1]$ of $\mathcal{V}$ there is an L-Lipschitz function $\psi_A : A \to [0, 1]$ for some $L > 0$ only depending on b so that $\psi_A(\Sigma \times \{0\}) = 0$ and $\psi_A(\Sigma \times \{1\}) = 1$. Since the components of $\mathcal{V}$ are linearly aligned, functions of the form $\psi_A + b_A$ or of the form $1 - \psi_A + b_A$ for a constant b_A can be pasted together to a function on M which is constant on the components of $M - \mathcal{V}$ and whose Rayleigh quotient is bounded from above by a universal multiple of $1/n^2$. We refer to [1] for details.

This shows the upper bound for $\lambda_1(M)$, and the upper bound for $\lambda_n(M)$ follows from the fact that the first eigenvalue of a (b, δ)-product region with Dirichlet boundary conditions is bounded from above by a universal constant together with domain monotonicity of eigenvalues with vanishing Dirichlet data. $\qquad\square$

Theorem 1 from the introduction now follows from Proposition 7.1, Lemma 8.1 and the following statement which is the main result of this section. Recall that by hyperbolization, a closed 3-manifold M which admits a Riemannian metric of sectional curvature contained in $[-1 - \epsilon, -1 + \epsilon]$ for some $\epsilon < 1/2$ admits a hyperbolic metric, unique up to isometry by Mostow rigidity.

THEOREM 8.2. *For every $g \geq 2, a \in (0, 1), b > 4, \delta > 0$ there exist numbers $\epsilon = \epsilon(g, a, b, \delta) > 0, a' = a'(g, a, b, \delta) \in (0, 1)$ with the following property. Let M be a closed aspherical atoroidal 3-manifold of Heegaard genus g, and let ρ be a Riemannian metric on M of curvature contained in $(-1 - \epsilon, -1 + \epsilon)$. Assume that (M, ρ) contains a linearly aligned collection $\mathcal{V}$ of pairwise disjoint (b, δ)-product regions of genus g whose total volume is at least $a$vol(M, ρ). Let ρ_0 be the hyperbolic metric on M. Then (M, ρ_0) contains a linearly aligned collection $\mathcal{W}$ of pairwise disjoint $(b - 1, \delta/2)$-product regions of volume at least a'vol(M, ρ_0).*

By Proposition 7.1, for a fixed choice of a number $b > 4$ and sufficiently small $\delta > 0$, a random 3-manifold M of Heegaard genus g admits a Riemannian metric ρ which fulfills the assumption in Theorem 8.2 for some number $a \in (0, 1)$. Note that b, δ are independent of M, and the number $a \in (0, 1)$ depends on the random walk. Thus Theorem 1 is an immediate consequence of Theorem 8.2 and Lemma 8.1.

We are left with the proof of Theorem 8.2 which is carried out in the remainder of this section. We use a construction of [4], [5]. The following is a special case of the main result of [5].

THEOREM 8.3. *Let (M, ρ) and (M_0, ρ_0) be closed oriented Riemannian manifolds of dimension 3 and suppose that for some constant $b \geq 1$*

$$\mathrm{Ric}_\rho \geq -2, \quad and \quad -b^2 \leq K_{\rho_0} \leq -1.$$

If there exists a map $f : M \to M_0$ of degree one then

$$\mathrm{vol}(M, \rho) \geq \mathrm{vol}(M_0, \rho_0),$$

with equality if and only if $(M, \rho), (M_0, \rho_0)$ are isometric and hyperbolic.

Here Ric_ρ and K_{ρ_0} are the Ricci curvature and the sectional curvature of ρ and ρ_0.

COROLLARY 8.4. *For $\epsilon < 1/2$ let ρ be a Riemannian metric on the closed 3-manifold M of curvature contained in $(-1 - \epsilon, -1 + \epsilon)$ and let ρ_0 be the hyperbolic metric on M. Then*

$$\mathrm{vol}(M, \rho)/\mathrm{vol}(M, \rho_0) \in [(1 - \epsilon)^{3/2}, (1 + \epsilon)^{3/2}].$$

Proof. Rescaling the metric ρ with the factor $(1 - \epsilon)^{-1}$ yields a new metric on M whose volume is $(1 - \epsilon)^{-3/2}\mathrm{vol}(M, \rho)$ and whose sectional curvature is bounded from below by -1. In particular, the Ricci curvature of this metric is at least -2. An application of Theorem 8.3 then implies that $\mathrm{vol}(M, \rho) \geq (1 - \epsilon)^{3/2}\mathrm{vol}(M, \rho_0)$.

Similarly, rescaling the metric ρ on M with the factor $(1 + \epsilon)^{-1}$ yields a metric whose sectional curvature is bounded from above by -1 and whose volume equals $(1 + \epsilon)^{-3/2}\mathrm{vol}(M, \rho)$. Another application of Theorem 8.3, with the roles of (M, ρ) and (M, ρ_0) exchanged, shows that $\mathrm{vol}(M, \rho_0) \geq (1 + \epsilon)^{-3/2}\mathrm{vol}(M, \rho)$. Together the corollary follows. $\square$

The *volume entropy* $h(\rho)$ of a negatively curved metric ρ on M is the asymptotic growth rate of the volume of balls in its universal covering. The volume entropy of a hyperbolic metric equals 2, and the volume entropy of a metric whose sectional curvature is bounded from below by $-b^2$ for some $b > 0$ is at most $2b$.

For $c > h(\rho)$ there exists a smooth *natural map* $F_c : (M, \rho) \to (M, \rho_0)$ [4]. The following statement summarizes some of the results from Section 7 of [4]. Part of the statement is only implicitly contained in [4], but an explicit version can be found in Theorem 2.1 of [6]. We always assume that the constant ϵ which controls the curvature of M is smaller than $1/2$ and that the number $c > h(\rho)$ is bounded from above by 4 to make all constants uniform.

PROPOSITION 8.5. *Let $c > h(\rho)$ and let $F_c : (M, \rho) \to (M, \rho_0)$ be the natural map.*

(1) *F_c is of degree one, and its Jacobian satisfies*

$$|\mathrm{Jac}(F_c)| \leq \left(\frac{c}{2}\right)^3$$

pointwise.

(2) *There are $\kappa > 0, r \in (0, 1)$ and $L > 1$ not depending on (M, ρ) with the following property. If $x \in (M, \rho)$ is such that $|\mathrm{Jac}(F_c)(x)| \geq$*

$(1 - \kappa)(\frac{c}{2})^3$ then the restriction of the map F_c to the ball $B(x, r)$ of radius r about x in (M, ρ) is L-Lipschitz.

(3) For all $\theta > 0$ and $x \in M$ there exists $\beta > 0$ such that if $|\text{Jac}(F_c)(x)| \geq (1 - \beta)(\frac{c}{2})^3$ then

$$(1 - \theta)(\frac{c}{2})^3 < |d_x F_c(v)| < (1 + \theta)(\frac{c}{2})^3$$

for all unit tangent vectors $v \in T_x M$.

The strategy is now as follows. Given $a \in (0, 1)$ and $b > 4L$ where $L > 1$ is as in Proposition 8.5, for a manifold (M, ρ) which fulfills the assumption in Theorem 8.2 for sufficiently small $\epsilon > 0$, we find a union $\mathcal{W} \subset \mathcal{V}$ of components of the collection $\mathcal{V}$ of (b, δ)-product regions in (M, ρ) whose total measure is large and such that the restriction to this set of the natural map $F_c : (M, \rho) \to (M, \rho_0)$ for a suitably chosen $c > h(\rho)$ has large Jacobian outside of a subset which does not contain any ball of radius r where $r > 0$ is as in the second part of Proposition 8.5. Proposition 8.5 then yields that the map F_c is uniformly Lipschitz on $\mathcal{W}$. We then argue that the image under F_c of a (b, δ)-product region in $\mathcal{W}$ contains a (b', δ')-product region in (M, ρ_0) where b' is close to b and δ' is close to δ. The geometric control on the image of the map F_c is then used to show that suitably chosen sub-regions of these image product regions of controlled total volume are pairwise disjoint and linearly aligned.

The following lemma establishes a first volume control. In its formulation, the numbers $r > 0, L > 1$ are as in Proposition 8.5.

LEMMA 8.6. Let $a \in (0, 1), b > \max\{10r, 4\}, \delta > 0$ and $\xi > 0$. There exists a number $\epsilon_0 = \epsilon_0(a, b, \delta, \beta) > 0$ with the following property. Let (M, ρ) be as in Theorem 8.2, with sectional curvature contained in $(-1 - \epsilon_0, -1 + \epsilon_0)$. Then for $c > h(\rho)$ sufficiently close to $h(\rho)$, there is a subset $\mathcal{W} \subset \mathcal{V}$ with the following properties.

(1) $\mathcal{W}$ is a union of components of $\mathcal{V}$, and its total volume is at least $a\text{vol}(M, \rho)/2$.
(2) The restriction of F_c to each component of $\mathcal{W}$ is L-Lipschitz, and its image is contained in the σ-thick part of (M, ρ_0) for a universal constant $\sigma > 0$.
(3) If V is any component of $\mathcal{W}$ then $\text{vol}(F_c(V)) \geq (1 - \xi)\text{vol}(V)$, and there exists a subset A of V with $\text{vol}(A) \geq a\text{vol}(V)$ such that $F_c^{-1}(F_c(x)) \subset V$ for all $x \in A$.

Proof. Let $r > 0$ be as in the second part of Proposition 8.5. Assume without loss of generality that $r < 1$. For $x \in (M, \rho)$ let $B(x, r)$ be the open ball of radius r about x. Let $\mathcal{V}$ be a union of (b, δ)-product regions as in the statement of Theorem 8.2. Since the components of $\mathcal{V}$ are linearly aligned and $b > 4$, any ball $B(y, r)$ in (M, ρ) intersects at most two different components of $\mathcal{V}$.

Let us consider a point $x \in V$. The injectivity radius of (M, ρ) at x is at least δ. Therefore by comparison, the volume of the ball $B(x, r)$ is bounded from below by a universal constant $\alpha > 0$. On the other hand, as the diameters of the boundary surfaces of a component V of $\mathcal{V}$ are uniformly bounded, the volume of the r-neighborhood $N_r(V)$ of any component V of $\mathcal{V}$ is bounded from above by a universal constant $\beta > 0$. Thus if $x \in V$ then the ratio $\mathrm{vol}(B(x, r))/\mathrm{vol}(N_r(V))$ is bounded from below by a universal constant α/β.

Let $\xi > 0$. Define

$$Z = \{x \in M \mid |\mathrm{Jac}(F_c)(x)| \geq (1 - \xi)(\frac{c}{2})^3\}.$$

By Corollary 8.4 and the first part of Proposition 8.5, for sufficiently small $\epsilon > 0$ and for $c > h(\rho)$ sufficiently close to $h(\rho)$, the volume of the union $\mathcal{W}$ of all components V of $\mathcal{V}$ with the property that $N_r(V) - Z$ does not contain a ball of radius r centered at a point $x \in V$ is at least $3a\mathrm{vol}(M)/4$. Namely, if $V_1, \ldots, V_k$ are the components of $\mathcal{V} - \mathcal{W}$ and if $x_i \in V_i$ is such that $B(x_i, r) \subset M - Z$, then by the above discussion, any of the balls $B(x_i, r)$ intersects at most one other ball $B(x_j, r)$ for $j \neq 1$. In particular, at least $k/2$ of the balls $B(x_i, r)$ are pairwise disjoint and hence

$$\mathrm{vol}(\cup_i B(x_i, r)) \geq k\alpha/2.$$

Thus if $\mathrm{vol}(\mathcal{V} - \mathcal{W}) \geq a\mathrm{vol}(M, \rho)/4$ then $\mathrm{vol}(\cup_i B(x_i, r)) \geq \alpha a\mathrm{vol}(M, \rho)/8\beta$. But the restriction of F_c to $\cup_i B(x_i, r)$ decreases the volume by a definitive factor. For $\epsilon > 0$ sufficiently close to 0 and $c - h(\rho) > 0$ sufficiently small, this violates Corollary 8.4.

By the second part of Proposition 8.5, the restriction of F_c to any component V of $\mathcal{W}$ is L-Lipschitz where $L > 1$ is a universal constant. In particular, if γ is a closed loop entirely contained in V, then the length of its image $F_c(\gamma)$ is at most L times the length of γ.

By the definition of a (b, δ)-product region, for an arbitrary point $x \in V$ the subgroup of $\pi_1(M)$ generated by the homotopy classes of uniformly short loops at x which are entirely contained in V is not virtually abelian. But this implies that for any point $y \in F_c(V)$, there are closed loops of uniformly bounded length passing though y which generated a non-solvable subgroup of $\pi_1(M)$. As a consequence, the set $F_c(V)$ is contained in the σ-thick part of (M, ρ_0) for a universal constant $\sigma > 0$. Together this shows the first and second part of the lemma.

Now if V is a component of $\mathcal{W}$ and if $B = \{x \in V \mid |F_c^{-1}(F(x)) \not\subset V\}$ then the volume of (M, ρ_0) equals the volume of $F_c(M - B)$. Thus as $\epsilon \to 0$ and $c - h(\rho) \to 0$, by volume comparison the proportion of the volume of $\mathcal{W}$ contained in the union of those components of $\mathcal{W}$ which violate the conditions in the third part of the lemma has to tend to zero. This then implies the third part of the lemma. $\square$

For a number $\xi > 0$ we say that a map F between two metric spaces X, Y is a ξ-*coarse isometry* if $|d(Fx, Fy) - d(x, y)| \leq \xi$ for all x, y.

LEMMA 8.7. *For $b' < b, \delta' < \delta$ and $\xi > 0$ there exists a number $\epsilon_0 = \epsilon_0(b', \delta')$ with the following property. Let (M, ρ) be as in Lemma 8.6 and let V be a component of $\mathcal{W}$ where $\mathcal{W}$ is as in Lemma 8.6; then the restriction of F_c to V is a ξ-coarse isometry whose image contains a (b', δ')-product region of genus g.*

Proof. We argue by contradiction and we assume that a number $\epsilon_0 > 0$ as in the lemma does not exist. Then there exists a sequence of closed 3-manifolds (M_i, ρ) which fulfill the assumptions in Theorem 8.2 for a sequence $\epsilon_i \to 0$ and fixed numbers $g \geq 2, a > 0, b > 4, \delta > 0$ and such that for each i, there is a component V_i of the collection $\mathcal{W}_i$ as in Lemma 8.6 whose image under the natural map $F_i : (M_i, \rho) \to (M_i, \rho_0)$ does not contain a (b', δ') product region where $b' < b$ and $\delta' < \delta$ are fixed constants. Note that in contrast to similar statements in the literature, we do not assume the existence of a bound on the diameters of the manifolds (M_i, ρ). Let as before ρ_0 be the hyperbolic metric on the manifold M_i.

Let h_i be the volume entropy of M_i. We know that $h_i \to 2 \ (i \to \infty)$. Choose a sequence $\chi_i \to 0$ such that $h_i < 2 + \chi_i$. For each i consider the natural map $F_i : (M_i, \rho) \to (M_i, \rho_0)$ for the parameter $c_i = 2 + \chi_i$. By the choice of $\mathcal{W}_i$ and the second part of Lemma 8.6, we know that the restriction of F_i to V_i is L-Lipschitz where $L > 1$ does not depend on i. Furthermore, for each $\beta > 0$, the measure of the set of all points $z \in V_i$ so that $|\text{Jac}(F_i)(x)| \leq (1 - \beta)(\frac{c_i}{2})^3$ tends to zero as $i \to \infty$. By the third part of Proposition 8.5, as $i \to \infty$, on subset of the component V_i of $\mathcal{W}_i$ containing a larger and larger proportion of the volume of V_i, the differential of F_i is close to an isometry.

For each i let $x_i \in V_i$. The set $F_i(V_i)$ is contained in the σ-thick part of (M_i, ρ_0) where σ does not depend on i. Thus by passing to a subsequence, we may assume that the pointed manifolds (M_i, x_i, ρ) converge in the geometric topology to a pointed hyperbolic manifold (M, x) and that the pointed hyperbolic manifolds $(M_i, F_i(x_i), \rho_0)$ converge in the geometric topology to a pointed hyperbolic manifold (N, y).

Let (V, x) be the geometric limit of the pointed (b, δ)-product regions (V_i, x_i). Then V is a (b, δ)-product region in M containing the basepoint x. Furthermore, as the restriction of F_i to V_i is L-Lipschitz for a universal constant $L > 1$, up to passing to another subsequence we may assume that $F_i|V_i$ converges to an L-Lipschitz map $F : (V, x) \to (N, y)$.

By the definition of geometric convergence, for large enough i there exists a $(1 + \xi_i)$-bilipschitz homeomorphism ϕ_i of a neighborhood U of V in M onto a neighborhood U_i of V_i in M_i where $\xi_i \to 0 \ (i \to \infty)$. We use ϕ_i to identify U with U_i.

As $i \to \infty$ and by the choice of the sets V_i, the Jacobians of the restriction of F_i to V_i converge to one almost surely. We now follow the reasoning in the proof of Lemma 7.5 of [4]. Namely, using the map ϕ_i^{-1} we can think of U_i as a neighborhood of V in M. Egoroff's theorem then implies that for each n there exists a subset $K_n \subset V$ with $\mathrm{vol}(V - K_n) < 1/n$ and such that on K_n the differentials dF_i converge to an isometry uniformly. By Lemma 7.7 and Lemma 7.8 of [4], the map $F|V$ is one-Lipschitz. Its differential exists almost everywhere and is an isometry. It then follows from Appendix B that $F : V \to N$ is an isometric embedding. In particular, $F(V)$ is a (h, δ)-product region in N, and for sufficiently large i the map F_i is a ξ-coarse isometry.

Geometric convergence now implies that for large enough i, the image of V_i under F_i is a (b', δ')-product region in (M_i, ρ_0). This is a contradiction to the assumption on the sets V_i. $\qquad\square$

Proof of Theorem 8.2. We showed so far that for sufficiently small $\epsilon_0 > 0$, if (M, ρ) is as in Theorem 8.2, of sectional curvature contained in $(1-\epsilon_0, 1+\epsilon_0)$, then (M, ρ_0) contains a union of (b', δ')-product region for some b' close to b, δ' close to δ which cover a fixed proportion of the volume of (M, ρ). These product regions are the images under a suitably chosen natural map F_c of a subcollection $\mathcal{W} \subset \mathcal{V}$ of the family $\mathcal{V}$ of (b, δ)-product regions whose existence is assumed for (M, ρ). Furthermore, the volume of $\mathcal{W}$ is at least $a\mathrm{vol}(M, \rho)$ for some fixed number $a > 0$ (with a slight abuse of notation). The restriction of F_c to $\mathcal{W}$ is L-Lipschitz and a $1/4$-coarse isometry, and $\mathrm{vol}(F_c(\mathcal{W}))/\mathrm{vol}(\mathcal{W})$ is very close to one.

Let $\hat{b} < b - 2$ and $\hat{\delta} < \delta$ be such that each component V of $\mathcal{W}$ contains a $(\hat{b}, \hat{\delta})$-product region $\hat{V}$ in its interior whose one-neighborhood is entirely contained in V. The volume of $\hat{V}$ is at least $b\mathrm{vol}(V)$ for a universal constant $b > 0$.

Our goal is to show that there is a subcollection $\mathcal{Z}$ of $\mathcal{W}$ of volume at least $a\mathrm{vol}(M, \rho)/2$ with the additional property that whenever $V \neq W \in \mathcal{Z}$ then $\hat{V} \cap \hat{W} = \emptyset$.

To this end let us assume that for $V \neq W \in \mathcal{W}$ we have $F_c(\hat{V}) \cap F_c(\hat{W}) \neq \emptyset$. As the restriction of the map F_c is L-Lipschitz and a $1/4$-coarse isometry, this implies that there are balls $B_1 \subset V, B_2 \subset W$ of radius $1/2L$ such that $F_c(B_1) \subset F_c(W)$ and $F_c(B_2) \subset F_c(V)$. Namely, for all $z \in \hat{V}$ the ball of radius $1/2$ about $F_c(z)$ is contained in $F_c(V)$, furthermore F_c is L-Lipschitz.

Let $2\sigma > 0$ be a lower bound for the volume of a ball of radius $1/2L$ entirely contained in an (b, δ)-product region. Such a number exists since the injectivity radius in such a region is at least δ. Then the volume of $F_c(V \cup W)$ is at most $(\frac{c}{2})^3(\mathrm{vol}(V) + \mathrm{vol}(W) - 2\sigma)$. In particular, the contribution of $F_c(V)$ to the volume of $\mathcal{W}$ does not exceed $(\frac{c}{2})^3(\mathrm{vol}(V) - \sigma)$.

Since $\sigma > 0$ is independent of all choices and for c sufficiently close to 2 the restriction of the map F_c to $\mathcal{W}$ is very close to being volume preserving, we deduce that for c sufficiently close to 2 the union $\mathcal{Z}$ of all product regions $\hat{V}$ with $V \in \mathcal{W}$ and such that the sets from $\mathcal{Z}$ are mapped disjointly by F_c covers a fixed proportion of the volume of (M, ρ_0). Furthermore, the image of each of the components in $\mathcal{Z}$ contains a (b', δ')-product region for some fixed $b' < \hat{b}$ and some δ' close to $\hat{\delta}$. Thus we found a collection of pairwise disjoint product regions in (M, ρ_0) as claimed in the theorem.

We are left with showing that the regions $F_c(\hat{V})$ for $\hat{V} \in \mathcal{Z}$ are linearly aligned. However, F_c is a homotopy equivalence. If $\hat{V} \in \mathcal{Z}$ then as the restriction of F_c to $\hat{V}$ is a homeomorphism, for a fixed choice of an embedded surface $\Sigma \subset V$ which decomposes M into two handlebodies, the image surface $F_c(\Sigma)$ separates (M, ρ_0) into two components. The restriction of F_c to the closure of a component of $M - \Sigma$ is a generator of the relative homology group $H_3(M, M - F_c(\Sigma))$. But this homology group also is generated by the inclusion of a component of $M - F_c(\Sigma)$ and hence each component A of $M - \Sigma$ determines uniquely a component $\mathcal{F}(A)$ of $M - F_c(\Sigma)$ with the additional property that $F_c(A) \supset \mathcal{F}(A)$.

Now let $\hat{V} \neq \hat{W} \in \mathcal{Z}$; as the components of $\mathcal{Z}$ are pairwise disjoint, the component $\hat{W}$ is entirely contained in a component of $M - \hat{V}$, say the component A. Furthermore, as $F_c(\hat{V}), F_c(\hat{W})$ are disjoint, the component $F_c(\hat{W})$ is contained in a component Z of $M - F_c(\hat{V})$. We claim that $Z = \mathcal{F}(A)$.

Namely, let B be the component of $M - \hat{W}$ entirely contained in A. If $Z \neq \mathcal{F}(A)$ then we have $F_c(\hat{V}) \subset \mathcal{F}(B)$. But the restriction of F_c to B maps B to a subset that contains $\mathcal{F}(B)$. In particular, we have $F_c(\hat{V}) \subset F_c(M - \hat{V})$ which violates property (3) in Lemma 8.6.

But this just means that the components of $F_c(\mathcal{W})$ are linearly aligned. This completes the proof of the theorem. $\square$

Appendix A. Computation of end invariants

We give a proof, which is certainly well-known but hard to find in the literature, of Corollary 4.7.

LEMMA A.1. *Let M be a marked hyperbolic structure on $\Sigma \times \mathbb{R}$. Let $\lambda \in \mathcal{PML}$ be a minimal filling measured lamination that is not realized in M. Then λ is the ending lamination of a relative end of M.*

Proof. Let us fix a pleated surface $f_0 : (\Sigma, \sigma_0) \to M$ whose hyperbolic metric we will use to parametrize $\mathcal{ML}$ and $\mathcal{PML}$. Recall that the length function $L_{\sigma_0}(\bullet)$ and the intersection form $i(\bullet, \bullet)$ extends continuously to all measured laminations $\mathcal{ML}$.

Let us pick a representative $\lambda \in \mathcal{ML}$.

Consider a sequence of weights $a_n > $ such that the sequence of weighted curves $\{a_n \alpha_n\}_{n \in \mathbb{N}} \subset \mathcal{ML}$ converges to $\lambda \in \mathcal{ML}$. By continuity of the intersection form on $\mathcal{ML}$ and the fact that λ is filling, we have:

$$a_n i(\alpha_n, \gamma) \longrightarrow i(\lambda, \gamma) > 0$$

for every curve $\gamma \in \mathcal{C}(\Sigma)$.

Denote by $l_M : \mathcal{C}(\Sigma) \longrightarrow [0, \infty)$ the length function associated to the $3-$manifold M, i.e. $l_M(\alpha) = l_M(f_0(\alpha)^*)$ where $f_0(\alpha)^*$ is the geodesic representative of $f_0(\alpha)$. Since f_0 is a path-isometry, we have

$$L_{\sigma_0}(\alpha) = l(f_0(\alpha)) \geq l(f_0(\alpha)^*) = l_M(\alpha)$$

for every curve $\alpha \in \mathcal{C}(\Sigma)$.

Let $f : (\Sigma, \sigma) \longrightarrow M$ be a pleated surface realizing α in the homotopy class of f_0. By work of Thurston, there is a Margulis constant $\epsilon_1 > 0$ such that only the $\epsilon-$thin part of (Σ, σ) may enter the ϵ_1-thin part of M.

Let $\gamma \in \mathcal{C}(\Sigma)$ represent a cusp of M. Suppose that α intersects γ. By standard hyperbolic geometry, namely, by the Collar Lemma, we have

$$l_M(\alpha) = L_\sigma(\alpha) \geq i(\alpha, \gamma) \sinh^{-1}\left(\frac{1}{\sinh\left(\frac{L_\sigma(\gamma)}{2}\right)} \right).$$

In conclusion, putting together the previous observations, we get that the following holds: For the sequence of geodesics representatives of α_n realized in M by the pleated surfaces $f_n : (\Sigma, \sigma_n) \longrightarrow M$ we have

$$L_{\sigma_0}(\lambda) \simeq a_n L_{\sigma_0}(\alpha_n) \geq a_n l_M(\alpha_n) = a_n L_{\sigma_n}(\alpha_n)$$

$$\geq a_n i(\alpha_n, \gamma) \sinh^{-1}\left(\frac{1}{\sinh\left(\frac{L_{\sigma_n}(\gamma)}{2}\right)} \right) \simeq i(\lambda, \gamma) \sinh^{-1}\left(\frac{1}{\sinh\left(\frac{L_{\sigma_n}(\gamma)}{2}\right)} \right).$$

As a consequence $L_{\sigma_n}(\gamma)$ is bounded from below, say by $\eta_\gamma > 0$. Let η be much smaller than $\min \{\eta_\gamma \mid \gamma \text{ cusp } \}$ and Thurston constant ϵ. We have that $f_n(\Sigma) \cap M^{\mathrm{cusp}}_{(0,\eta]} = \emptyset$, i.e. $f_n(\Sigma) \subset M_0 = M \setminus M^{\mathrm{cusp}}_{(0,\eta)}$ in the notation of Section 4.

We are now able to prove that λ is an ending laminaiton of M. Observe that, in this setting, i.e. for sequences $f_n : (\Sigma, \sigma_n) \to M_0$, we have Compactness for Pleated Surfaces (see the proof of Lemma 6.13 in [35], note that, there, it is the hypothesis that the pleated surfaces are *type-preserving* that allows to conclude that $f_n(\Sigma) \subset M_0$).

The proof can be completed as in Corollary 6.14 in [35]. Since λ is not realized, the pleated surfaces f_n have to leave every compact set in M_0 and, in particular, they will miss the Scott core $\mathcal{SC} \subset M_0$ and exit at least one end for n sufficiently large. By the criterion for computing the end invariants of Section 4 we can conclude that λ is an ending lamination of a simply degenerate end of M. $\qquad \square$

Appendix B. Local control of one-Lipschitz maps

The goal of this appendix is to show (compare Appendix C of [4] for a different variation)

PROPOSITION B.1. *Let U be a domain in a hyperbolic 3-manifold and let $F : U \to N$ be a volume preserving one-Lipschitz map into a hyperbolic 3-manifold N. Then F is an isometric embedding.*

Proof. As F is volume preserving, all we need to show that F is a local isometry.

To this end let $x \in U$ and let $r_0 > 0$ be such that the closed balls $B(x, r_0)$, $B(F(x), r_0)$ of radius r_0 about x and $F(x)$ are isometric to the closed ball of the same radius in hyperbolic 3-space. Since F is one-Lipschitz we know that $F(B(x, r_0)) \subset B(F(x), r_0)$. Furthermore, as F is continuous and $B(x, r_0)$ is compact, $F(B(x, r_0))$ is a closed subset of $B(F(x), r_0)$ and hence coincides with $B(F(x), r_0)$ as F is volume preserving.

Using once more the fact that F is volume preserving, the differential of F exists almost everywhere and is an isometry. Furthermore, the set of all points $x \in U$ such that $F^{-1}(F(x)) = \{x\}$ has full measure.

Let x be such a point. We saw above that there is a closed subset A of the distance sphere of radius r_0 about x which is mapped by F onto the distance sphere of radius r_0 about $F(x)$. If $y \in A$ then using once more that F is a contraction, the geodesic γ_y connecting x to y is mapped by F to the geodesic γ_{Fy} connecting $F(x)$ to $F(y)$. As F is differentiable at x and $dF(x)$ is an isometry, we have $dF(\gamma_y'(0)) = \gamma_{Fy}'(0)$. In particular, if exp denotes the exponential map at x then $F(\exp(s \exp^{-1}(z))) = \exp(s dF(\exp^{-1}(z))$ for all $z \in A$. On the other hand, $F(A) = \partial B(F(y), r_0)$ and hence $A = \partial B(x, r_0)$ and the restriction of F to $B(x, r_0)$ is an isometry.

As x was a point from a subset of U of full measure, F is indeed a local isometry and hence an isometry. $\square$

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