INVARIANT RADON MEASURES ON MEASURED LAMINATION SPACE

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Abstract. Let S be an oriented surface of genus $g \geq 0$ with $m \geq 0$ punctures and $3g-3+m \geq 2$. We classify all Radon measures on the space of measured geodesic laminations which are invariant under the action of the mapping class group of S.

1. Introduction

Let S be an oriented surface of finite type, i.e. S is a closed surface of genus $g \geq 0$ from which $m \geq 0$ points, so-called *punctures*, have been deleted. We assume that $3g-3+m \geq 1$, i.e. that S is not a sphere with at most 3 punctures or a torus without puncture. In particular, the Euler characteristic of S is negative. Then the *Teichmüller space* $\mathcal{T}(S)$ of S is the quotient of the space of all complete hyperbolic metrics of finite volume on S under the action of the group of diffeomorphisms of S which are isotopic to the identity. The *mapping class group* $MCG(S)$ of all isotopy classes of orientation preserving diffeomorphisms of S acts properly discontinuously on $\mathcal{T}(S)$ with quotient the *moduli space* $\mathrm{Mod}(S)$.

A *geodesic lamination* for a fixed choice of a complete hyperbolic metric of finite volume on S is a *compact* subset of S foliated into *simple* geodesics. A *measured geodesic lamination* is a geodesic lamination together with a transverse translation invariant measure. The space $\mathcal{ML}$ of all measured geodesic laminations on S, equipped with the weak*-topology, is homeomorphic to $S^{6g-7+2m} \times (0,\infty)$ where $S^{6g-7+2m}$ is the $6g-7+2m$-dimensional sphere. The mapping class group $MCG(S)$ naturally acts on $\mathcal{ML}$ as a group of homeomorphisms preserving a Radon measure in the *Lebesgue measure class*. Up to scale, this measure is induced by a natural symplectic structure on $\mathcal{ML}$ (see [PH92] for this observation of Thurston), and it is ergodic under the action of $MCG(S)$. Moreover, the measure is *non-wandering*. By this we mean that $\mathcal{ML}$ does not admit an $MCG(S)$-invariant countable Borel partition into sets of positive measure.

If S is a once punctured torus or a sphere with 4 punctures, then the Teichmüller space of S has a natural identification with the upper half-plane $\mathbf{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. Up to passing to the quotient by the hyperelliptic involution if S is the once punctured torus, the mapping class group $MCG(S)$ is just the

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group $PSL(2,\mathbb{Z})$ acting on $\mathbf{H}^2$ by linear fractional transformations. The action of $MCG(S)$ on measured lamination space can in this case be identified with the quotient of the standard *linear* action of $SL(2,\mathbb{Z})$ on $\mathbb{R}^2$ under the reflection at the origin (see the book [BM00] of Bekka and Mayer for more and for references and compare the survey [H08a]).

Extending earlier work of Furstenberg, Dani completely classified all $SL(2,\mathbb{Z})$-invariant Radon measures on $\mathbb{R}^2$ [D79]. He showed that if such a measure η is ergodic under the action of $SL(2,\mathbb{Z})$ then either it is non-wandering and coincides with the usual Lebesgue measure λ up to scale, or it is *rational*, which means that it is supported on a single $SL(2,\mathbb{Z})$-orbit of points whose coordinates are dependent over $\mathbb{Q}$.

If the surface S is *non-exceptional*, i.e. if $3g-3+m\geq 2$, then the $MCG(S)$-invariant Radon measures on $\mathcal{ML}$ which naturally correspond to the rational measures for exceptional surfaces are defined as follows. A *weighted geodesic multi-curve* on S is a measured geodesic lamination whose *support* is a union of simple closed geodesics. The orbit of a weighted geodesic multi-curve under the action of $MCG(S)$ is a discrete subset of $\mathcal{ML}$ (see Section 5 for this easy and well known fact) and hence it supports a ray of invariant purely atomic Radon measures which we call *rational*. This definition coincides with the one given above for a once punctured torus or a forth punctured sphere.

For a non-exceptional surface S, there are additional $MCG(S)$-invariant Radon measures on $\mathcal{ML}$. Namely, a *proper bordered subsurface* S_0 of S is a union of connected components of the space which we obtain from S by cutting S open along a collection of disjoint simple closed geodesics. Then S_0 is a surface with non-empty geodesic boundary and of negative Euler characteristic. If two boundary components of S_0 correspond to the same closed geodesic γ in S then we require that $S-S_0$ contains a connected component which is an annulus with core curve γ. Let $\mathcal{ML}(S_0)\subset\mathcal{ML}$ be the space of all measured geodesic laminations on S which are contained in the interior of S_0. The space $\mathcal{ML}(S_0)$ can naturally be identified with the space of measured geodesic laminations on the surface $\hat{S}_0$ of finite type which we obtain from S_0 by collapsing each boundary circle to a puncture. The stabilizer in $MCG(S)$ of the subsurface S_0 is the direct product of the group of all elements which can be represented by diffeomorphisms leaving S_0 pointwise fixed and a group which is naturally isomorphic to a subgroup G of finite index of the mapping class group $MCG(\hat{S}_0)$ of $\hat{S}_0$.

Let c be a weighted geodesic multi-curve on S which is disjoint from the interior of S_0. Then for every $\zeta\in\mathcal{ML}(S_0)$ the union $c\cup\zeta$ is a measured geodesic lamination on S which we denote by $c\times\zeta$. Let $\mu(S_0)$ be an $G<MCG(\hat{S}_0)$-invariant Radon measure on $\mathcal{ML}(S_0)$ which is contained in the Lebesgue measure class. The measure $\mu(S_0)$ can be viewed as a Radon measure on $\mathcal{ML}$ which gives full measure to the laminations of the form $c\times\zeta$ ($\zeta\in\mathcal{ML}(S_0)$) and which is invariant and ergodic under the stabilizer of $c\cup S_0$ in $MCG(S)$. The translates of this measure under the action of $MCG(S)$ define an $MCG(S)$-invariant ergodic wandering measure on $\mathcal{ML}$ which we call a *standard subsurface measure*. We observe in Section 5 that if

the weighted geodesic multi-curve c contains the boundary of S_0 then the standard subsurface measure defined by $\mu(S_0)$ and c is a Radon measure on $\mathcal{ML}$.

The goal of this note is to show that every $MCG(S)$-invariant ergodic Radon measure on $\mathcal{ML}$ is of the form described above.

Theorem. (1) *An invariant ergodic non-wandering Radon measure for the action of $MCG(S)$ on $\mathcal{ML}$ coincides with the Lebesgue measure up to scale.*
(2) *An invariant ergodic wandering Radon measure for the action of $MCG(S)$ on $\mathcal{ML}$ is either rational or a standard subsurface measure.*

The organization of the paper is as follows. In Section 2 we discuss some properties of geodesic laminations, quadratic differentials and the curve graph needed in the sequel. In Section 3 we introduce conformal densities for the mapping class group. These conformal densities are families of finite Borel measures on the projectivization $\mathcal{PML}$ of $\mathcal{ML}$, parametrized by the points in Teichmüller space. They are defined in analogy to the conformal densities for discrete subgroups of the isometry group of a hyperbolic space. Up to scale, there is a unique conformal density in the Lebesgue measure class. We show that this is the only conformal density which gives full measure to the $MCG(S)$-invariant subset of $\mathcal{PML}$ of all projective measured geodesic laminations whose support is minimal and fills up S.

Every conformal density gives rise to an $MCG(S)$-invariant Radon measure on $\mathcal{ML}$. The investigation of invariant Radon measure which are not of this form relies on the structural results of Sarig [S04]. To apply his results we use train tracks to construct partitions of measured lamination space which have properties similar to Markov partitions. Section 4 summarizes those facts about train tracks which are needed for this purpose. The proof of the theorem is completed in Section 5. In the appendix we present a result of Minsky and Weiss [MW02] in the form needed for the proof of our theorem.

At the time this paper was posted on the arXiv I received the preprint [LM07] of Lindenstrauss and Mirzakhani which contains another proof of the above theorem.

2. Quadratic differentials and the curve graph

In this introductory section we summarize some properties of (measured) geodesic laminations, quadratic differentials and the curve graph which are needed later on. We also introduce some notations which will be used throughout the paper.

In the sequel we always denote by S an oriented surface of genus $g \geq 0$ with $m \geq 0$ punctures and where $3g - 3 + m \geq 1$.

2.1. **Geodesic laminations.** A *geodesic lamination* for a complete hyperbolic structure of finite volume on the surface S is a *compact* subset of S which is foliated into simple geodesics. A geodesic lamination λ is called *minimal* if each of its half-leaves is dense in λ. Thus a simple closed geodesic is a minimal geodesic lamination. A minimal geodesic lamination with more than one leaf has uncountably many leaves and is called *minimal arational*. Every geodesic lamination λ consists of a disjoint union of finitely many minimal components and a finite number of isolated leaves. Each of the isolated leaves of λ either is an isolated closed geodesic and hence a minimal component, or it *spirals* about one or two minimal components [CEG87]. A geodesic lamination is *maximal* if its complementary regions are all ideal triangles or once punctured monogons. A geodesic lamination *fills up* S if its complementary regions are all topological discs or once punctured topological discs.

A *measured geodesic lamination* is a geodesic lamination λ together with a translation invariant transverse measure. Such a measure assigns a positive weight to each compact arc in S which intersects λ nontrivially and transversely and whose endpoints are contained in the complementary regions of λ. The geodesic lamination λ is called the *support* of the measured geodesic lamination; it consists of a disjoint union of minimal components [CEG87]. The space $\mathcal{ML}$ of all measured geodesic laminations on S equipped with the weak*-topology is homeomorphic to $S^{6g-7+2m} \times (0,\infty) \sim \mathbb{R}^{6g-6+2m} - \{0\}$. Its projectivization is the space $\mathcal{PML}$ of all *projective measured geodesic laminations*. The measured geodesic lamination $\mu \in \mathcal{ML}$ *fills up* S if its support fills up S. The projectivization of a measured geodesic lamination which fills up S is also said to fill up S. A measured geodesic lamination is called *uniquely ergodic* if its support admits a single transverse measure up to scale. There is a continuous symmetric pairing $i : \mathcal{ML} \times \mathcal{ML} \to [0,\infty)$, the so-called *intersection form*, which extends the geometric intersection number between two simple closed curves.

2.2. **Quadratic differentials.** The fibre bundle $\mathcal{Q}^1(S)$ of all *holomorphic quadratic differentials of area one* over the Teichmüller space $\mathcal{T}(S)$ of the surface S can naturally be viewed as the unit cotangent bundle of $\mathcal{T}(S)$ for the *Teichmüller metric*. The *Teichmüller geodesic flow* Φ^t on $\mathcal{Q}^1(S)$ commutes with the action of the mapping class group $MCG(S)$ of all isotopy classes of orientation preserving diffeomorphisms of S. Thus this flow descends to a flow on the quotient $\mathcal{Q}(S) = \mathcal{Q}^1(S)/MCG(S)$, again denoted by Φ^t.

A measured geodesic lamination can be viewed as an equivalence class of *measured foliations* on S [L83]. Therefore every holomorphic quadratic differential q on S defines a pair $(\mu,\nu) \in \mathcal{ML} \times \mathcal{ML}$ where the *horizontal measured geodesic lamination* μ corresponds to the *horizontal measured foliation* of q which is expanded under the Teichmüller flow, and where the *vertical measured geodesic lamination* ν corresponds to the vertical measured foliation of q which is contracted under the Teichmüller flow. The area of the quadratic differential is just the intersection number $i(\mu,\nu)$. Note that the transverse measure of the *vertical* measured geodesic lamination is expanded under the Teichmüller geodesic flow, i.e. with the identification of q with the pair (μ,ν) the Teichmüller flow acts by $\Phi^t(\mu,\nu) = (e^{-t}\mu, e^t\nu)$.

For a quadratic differential $q \in \mathcal{Q}^1(S)$ define the *strong unstable manifold* $W^{su}(q)$ to be the set of all quadratic differentials $z \in \mathcal{Q}^1(S)$ whose horizontal measured geodesic lamination equals the horizontal measured geodesic lamination for q. Similarly, define the *strong stable manifold* $W^{ss}(q)$ to be the set of all quadratic differentials $z \in \mathcal{Q}^1(S)$ whose vertical measured geodesic lamination coincides with the vertical measured geodesic lamination of q. The *stable manifold* $W^s(q) = \cup_{t\in\mathbb{R}}\Phi^t W^{ss}(q)$ and the *unstable manifold* $W^u(q) = \cup_{t\in\mathbb{R}}\Phi^t W^{su}(q)$ are submanifolds of $\mathcal{Q}^1(S)$. The canonical projection

$$P : \mathcal{Q}^1(S) \to \mathcal{T}(S)$$

maps each stable and each unstable manifold onto $\mathcal{T}(S)$ [HM79].

The sets $W^s(q)$ (or $W^{ss}(q), W^{su}(q), W^u(q)$) ($q \in \mathcal{Q}^1(S)$) define a foliation of $\mathcal{Q}^1(S)$ which is invariant under the mapping class group and hence projects to a singular foliation on $\mathcal{Q}(S)$ which we call the *stable foliation* (or the *strong stable, strong unstable, unstable foliation*). There is a distinguished family of Lebesgue measures λ^s on the leaves of the stable foliation which are conditional measures of a Φ^t-invariant Borel probability measure λ on $\mathcal{Q}(S)$ in the Lebesgue measure class. The measure λ is ergodic and mixing under the Teichmüller geodesic flow (see [M82] and also [V82, V86]).

Every area one quadratic differential $q \in \mathcal{Q}^1(S)$ defines a singular euclidean metric on S of area one together with two orthogonal foliations by straight lines, with singularities of cone angle $k\pi$ for some $k \geq 3$. This metric is given by a distinguished family of isometric charts $\varphi : U \subset S \to \varphi(U) \subset \mathbb{C}$ on the complement of the zeros (or poles at the punctures) of q which map the distinguished foliations to the foliation of $\mathbb{C}$ into the *horizontal* lines parallel to the real axis and into the *vertical lines* parallel to the imaginary axis.

The group $SL(2,\mathbb{R})$ acts on the bundle $\mathcal{Q}^1(S)$ by replacing for $q \in \mathcal{Q}^1(S)$ and $M \in SL(2,\mathbb{R})$ each isometric chart φ for q by $M \circ \varphi$ where M acts linearly on $\mathbb{R}^2 = \mathbb{C}$. This preserves the compatibility condition for charts. The Teichmüller geodesic flow Φ^t then is the action of the diagonal group

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \quad (t \in \mathbb{R}).$$

The so-called *horocycle flow* h_t is given by the action of the unipotent subgroup

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \quad (t \in \mathbb{R}).$$

2.3. **The curve graph.** The *curve graph* $\mathcal{C}(S)$ of the surface S is a metric graph whose vertices are the free homotopy classes of *essential* simple closed curves on S, i.e. curves which are neither contractible nor freely homotopic into a puncture of S. In the sequel we often do not distinguish between an essential simple closed curve and its free homotopy class whenever no confusion is possible. If the surface S is non-exceptional, i.e. if $3g - 3 + m \geq 2$, then two such curves are connected in $\mathcal{C}(S)$ by an edge of length one if and only if they can be realized disjointly. If S is a once punctured torus then two simple closed curves on S are connected in $\mathcal{C}(S)$ by an edge of length one if and only if they intersect in precisely one point. If S is a forth punctured sphere then two simple closed curves on S are connected in $\mathcal{C}(S)$ by an edge of length one if and only if they intersect in precisely two points. The curve graph $\mathcal{C}(S)$ is connected. Any two elements $c, d \in \mathcal{C}(S)$ of distance at least 3 *jointly fill up* S, i.e. they decompose S into topological discs and once punctured topological discs. The mapping class group naturally acts on $\mathcal{C}(S)$ as a group of simplicial isometries.

By Bers' theorem, there is a number $\chi_0 > 0$ depending on S such that for every complete hyperbolic metric h on S of finite volume there is a *pants decomposition* of S consisting of $3g - 3 + m$ pairwise disjoint simple closed geodesics of length at most χ_0. On the other hand, the number of essential simple closed curves α on S whose hyperbolic length $\ell_h(\alpha)$ (i.e. the length of a geodesic representative of its free homotopy class) does not exceed $2\chi_0$ is bounded from above by a constant not depending on h, and the diameter of the subset of $\mathcal{C}(S)$ containing these curves is uniformly bounded as well (see [MM99, Bw06, H07, H08d] for a more detailed discussion).

Define a map

$$\Upsilon_{\mathcal{T}} : \mathcal{T}(S) \to \mathcal{C}(S)$$

by associating to a complete hyperbolic metric h on S of finite volume a curve $\Upsilon_{\mathcal{T}}(h)$ whose h-length is at most χ_0. If Υ' is any other choice of such a map then $d(\Upsilon_{\mathcal{T}}(h), \Upsilon'_{\mathcal{T}}(h) \leq \text{const}$. By [MM99] there is a constant $L > 1$ depending on S such that

$$d(\Upsilon_{\mathcal{T}}(g), \Upsilon_{\mathcal{T}}(h)) \leq L d(g, h) + L \quad \text{for all} \quad g, h \in \mathcal{T}(S) \tag{1}$$

where by abuse, we use the same symbol d to denote the distance on $\mathcal{T}(S)$ defined by the Teichmüller metric and the distance on the curve graph $\mathcal{C}(S)$ (see also the discussion in [H06] and Lemma 2.1 of [H08d]).

For a quadratic differential $q \in \mathcal{Q}^1(S)$ define the *q-length* of an essential closed curve α on S to be the infimum of the lengths of a representative of the free homotopy class of α with respect to the singular euclidean metric defined by q. We have.

Lemma 2.1. *For every $\chi > 0$ there is a number $a(\chi) > 0$ with the following property. For any quadratic differential $q \in \mathcal{Q}^1(S)$ the diameter in $\mathcal{C}(S)$ of the set of all simple closed curves on S of q-length at most χ does not exceed $a(\chi)$.*

Proof. By Lemma 5.1 of [MM99] (see also Lemma 5.1 of [Bw06] for an alternative proof) there is a number $b > 0$ and for every singular euclidean metric on S defined by a quadratic differential q of area one there is an embedded annulus $A \subset S$ of width at least b. This means that the q-distance between the two boundary components of A is at least b. If we denote by γ the core curve of A, then the q-length of every simple closed curve c on S is at least $i(c, \gamma)b$. As a consequence, for every $\chi > 0$ the intersection number with γ of every simple closed curve c on S whose q-length is at most χ is bounded from above by χ/b. This implies that the set of such curves is of diameter at most $\chi/b + 1$ in $\mathcal{C}(S)$ [MM99, Bw06]. □

By possibly enlarging the constant $\chi_0 > 0$ as above we may assume that for every $q \in \mathcal{Q}^1(S)$ there is an essential simple closed curve on S of q-length at most χ_0 (see [Bw06, R07] and the proof of Lemma 2.2 for a justification of this well known fact). Thus we can define a map

$$\Upsilon_{\mathcal{Q}} : \mathcal{Q}^1(S) \to \mathcal{C}(S)$$

by associating to a quadratic differential q a simple closed curve $\Upsilon_{\mathcal{Q}}(q)$ whose q-length is at most χ_0. By Lemma 2.1, if $\Upsilon'_{\mathcal{Q}}$ is any other choice of such a map then we have $d(\Upsilon_{\mathcal{Q}}(q), \Upsilon'_{\mathcal{Q}}(q)) \leq a(\chi_0)$ for all $q \in \mathcal{Q}^1(S)$ where as before, d is the distance function on $\mathcal{C}(S)$.

The map $\Upsilon_{\mathcal{T}} : \mathcal{T}(S) \to \mathcal{C}(S)$ associates to a complete hyperbolic metric h on S of finite volume a simple closed curve c of h-length at most χ_0. Let again $P : \mathcal{Q}^1(S) \to \mathcal{T}(S)$ be the canonical projection. The following simple observation is related to recent work of Rafi [R05]. We include its short proof for completeness.

Lemma 2.2. *There is a number $\chi_1 > 0$ such that $d(\Upsilon_{\mathcal{Q}}(q), \Upsilon_{\mathcal{T}}(Pq)) \leq \chi_1$ for all $q \in \mathcal{Q}^1(S)$.*

Proof. By Lemma 2.1, it is enough to show that for every $q \in \mathcal{Q}^1(S)$ and every simple closed curve α on S whose length with respect to the hyperbolic metric Pq is bounded from above by χ_0, the q-length of α is uniformly bounded.

For this observe that by the collar lemma of hyperbolic geometry, a simple closed geodesic α on a hyperbolic surface whose length is bounded from above by $\chi_0 > 0$ is the core curve of an embedded annulus A whose *modulus* is bounded from below by a universal constant $b > 0$. Then the *extremal length* of the core curve of A is bounded from above by a universal constant $c > 0$. Now the area of q equals one and therefore the q-length of the core curve α of A does not exceed $\sqrt{c}$ by the definition of extremal length (see e.g. [Mi96]). This shows the lemma. □

Choose a smooth function $\sigma : [0, \infty) \to [0, 1]$ with $\sigma[0, \chi_0] \equiv 1$ and $\sigma[2\chi_0, \infty) \equiv 0$ where as before, $\chi_0 > 0$ is a Bers constant for S. For every $h \in \mathcal{T}(S)$ we obtain a finite Borel measure μ_h on the curve graph $\mathcal{C}(S)$ by defining

$$\mu_h = \sum_{c \in \mathcal{C}(S)} \sigma(\ell_h(c))\delta_c$$

where δ_c denotes the Dirac mass at c. The total mass of μ_h is bounded from above and below by a positive constant only depending on S, and the diameter of the support of μ_h in $\mathcal{C}(S)$ is uniformly bounded as well. Note that $\Upsilon_{\mathcal{T}}(h)$ is contained in the support of μ_h, and the weight of $\Upsilon_{\mathcal{T}}(h)$ for the measure μ_h equals one. The measures μ_h depend continuously on $h \in \mathcal{T}(S)$ in the weak*-topology. This means that for every bounded function $f : \mathcal{C}(S) \to \mathbb{R}$ the function $h \to \int f d\mu_h$ is continuous.

The curve graph $\mathcal{C}(S)$ is a hyperbolic geodesic metric space [MM99] and hence it admits a *Gromov boundary* $\partial\mathcal{C}(S)$. For every $c \in \mathcal{C}(S)$ there is a complete distance function δ_c on $\partial\mathcal{C}(S)$ of uniformly bounded diameter and there is a number $\beta > 0$ such that

$$\delta_c \leq e^{\beta d(c,a)}\delta_a \text{ for all } c, a \in \mathcal{C}(S). \tag{2}$$

The distances δ_c are equivariant with respect to the action of $MCG(S)$ on $\mathcal{C}(S)$ and on $\partial\mathcal{C}(S)$.

For $h \in \mathcal{T}(S)$ define a distance δ_h on $\partial\mathcal{C}(S)$ by

$$\delta_h(\xi, \zeta) = \int \delta_c(\xi, \zeta) d\mu_h(c).$$

Clearly the metrics δ_h are equivariant with respect to the action of $MCG(S)$ on $\mathcal{T}(S)$ and $\partial\mathcal{C}(S)$. Moreover, there is a constant $\kappa > 0$ such that

$$\delta_h \leq e^{\kappa d(h,z)} \delta_z \text{ for all } h, z \in \mathcal{T}(S) \tag{3}$$

(here as before, d denotes the Teichmüller metric). Namely, the function σ is smooth, with uniformly bounded differential. Moreover, for every simple closed curve $c \in \mathcal{C}(S)$, the function $h \to \log \ell_h(c)$ on $\mathcal{T}(S)$ is smooth, with uniformly bounded differential with respect to the norm induced by the Teichmüller metric (Lemma 3.1 of [W79]). Since σ is supported in $[0, 2\chi_0]$, this implies that for each $c \in \mathcal{C}(S)$ the function $h \to \sigma(\ell_h(c))$ on $\mathcal{T}(S)$ is smooth, with uniformly bounded differential. As a consequence, for all $\xi \neq \eta \in \partial\mathcal{C}(S)$ the function $h \to \delta_h(\xi, \eta)$ is smooth, and the differential of its logarithm is uniformly bounded with respect to the Teichmüller norm, independent of ξ, η. From this and the definitions, the estimate (3) above is immediate. By the inequalities (2,3) and the definitions, via enlarging the constant κ we may also assume that

$$\kappa^{-1}\delta_h \leq \delta_{\Upsilon_{\mathcal{T}}(h)} \leq \kappa \delta_h \text{ for every } h \in \mathcal{T}(S). \tag{4}$$

3. Conformal densities

In this section we study *conformal densities* on the space $\mathcal{PML}$ of projective measured geodesic laminations on S. Recall that $\mathcal{PML}$ equipped with the weak* topology is homeomorphic to the sphere $S^{6g-7+2m}$, and the mapping class group $MCG(S)$ naturally acts on $\mathcal{PML}$ as a group of homeomorphisms. By the Hubbard Masur theorem [HM79], for every $x \in \mathcal{T}(S)$ and every $\lambda \in \mathcal{PML}$ there is a unique holomorphic quadratic differential $q(x, \lambda) \in \mathcal{Q}^1(S)_x$ of area one on x whose vertical measured geodesic lamination $q_v(x, \lambda)$ is contained in the class λ. For all $x, y \in \mathcal{T}(S)$ there is a number $\Psi(x, y, \lambda) \in \mathbb{R}$ such that $q_v(y, \lambda) = e^{\Psi(x,y,\lambda)} q_v(x, \lambda)$. The function $\Psi : \mathcal{T}(S) \times \mathcal{T}(S) \times \mathcal{PML} \to \mathbb{R}$ is continuous, moreover it satisfies the *cocycle identity*

$$\Psi(x, y, \lambda) + \Psi(y, z, \lambda) = \Psi(x, z, \lambda) \tag{5}$$

for all $x, y, z \in \mathcal{T}(S)$ and all $\lambda \in \mathcal{PML}$.

Definition. A *conformal density of dimension* $\alpha \geq 0$ on $\mathcal{PML}$ is an $MCG(S)$-equivariant family $\{\nu^y\}$ $(y \in \mathcal{T}(S))$ of finite Borel measures on $\mathcal{PML}$ which are absolutely continuous and satisfy $d\nu^z / d\nu^y = e^{\alpha\Psi(y,z,\cdot)}$ almost everywhere.

The conformal density $\{\nu^y\}$ is *ergodic* if the $MCG(S)$-invariant measure class it defines on $\mathcal{PML}$ is ergodic. There is an ergodic conformal density $\{\lambda^x\}$ of dimension $\alpha = 6g - 6 + 2m$ in the Lebesgue measure class induced by the symplectic form on the space $\mathcal{ML}$ of all measured geodesic laminations on S, see [M82]. Note that since the action of $MCG(S)$ on $\mathcal{PML}$ is minimal, the measure class of a conformal density is always of full support.

In Section 5 we will see that every conformal density gives rise to an $MCG(S)$-invariant Radon measure on $\mathcal{ML}$, so the classification of conformal densities is essential for the classification of invariant Radon measures on $\mathcal{ML}$.

Following [Su79], we construct from a conformal density $\{\nu^x\}$ of dimension α an $MCG(S)$-invariant family $\{\nu^{su}\}$ of locally finite Borel measures on strong unstable manifolds $W^{su}(q)$ $(q \in \mathcal{Q}^1(S))$ which transform under the Teichmüller geodesic flow Φ^t via $\nu^{su} \circ \Phi^t = e^{\alpha t}\nu^{su}$. For this let

$$\pi : \mathcal{Q}^1(S) \to \mathcal{PML} \tag{6}$$

be the natural projection which maps a quadratic differential $q \in \mathcal{Q}^1(S)$ to its vertical projective measured geodesic lamination. Let $P : \mathcal{Q}^1(S) \to \mathcal{T}(S)$ be the canonical projection. The restriction of the projection $\pi : \mathcal{Q}^1(S) \to \mathcal{PML}$ to the strong unstable manifold $W^{su}(q)$ is a homeomorphism onto its image and hence the measure ν^{Pq} on $\mathcal{PML}$ induces a Borel measure $\tilde{\nu}^{su}$ on $W^{su}(q)$. The measure ν^{su} on $W^{su}(q)$ defined by $d\nu^{su}(u) = e^{\alpha\Psi(Pq,Pu,\pi(u))}d\tilde{\nu}^{su}(u)$ is locally finite and does not depend on the choice of q. The measures ν^{su} on strong unstable manifolds transform under the Teichmüller flow as required.

The *flip* $\mathcal{F} : q \to -q$ maps strong stable manifolds homeomorphically onto strong unstable manifolds and therefore we obtain a family ν^{ss} of locally finite Borel measures on strong stable manifolds by defining $\nu^{ss} = \nu^{su} \circ \mathcal{F}$. Let dt be the usual Lebesgue measure on the flow lines of the Teichmüller flow. The locally finite Borel measure $\tilde{\nu}$ on $\mathcal{Q}^1(S)$ defined by $d\tilde{\nu} = d\nu^{ss} \times d\nu^{su} \times dt$ is invariant under the Teichmüller geodesic flow Φ^t and the action of the mapping class group. If we denote by Δ the diagonal in $\mathcal{PML} \times \mathcal{PML}$ then the desintegration $\hat{\nu}$ of $\tilde{\nu}$ along the flow lines of the Teichmüller flow is an $MCG(S)$-invariant locally finite Borel measure on $\mathcal{PML} \times \mathcal{PML} - \Delta$. Let ν be the Φ^t-invariant locally finite Borel measure on $\mathcal{Q}(S)$ which is the projection of the restriction of $\tilde{\nu}$ to a Borel fundamental domain for the action of $MCG(S)$. For the conformal density $\{\lambda^x\}$ in the Lebesgue measure class the resulting Φ^t-invariant measure λ on $\mathcal{Q}(S)$ is finite.

Call a quadratic differential $q \in \mathcal{Q}(S)$ *forward returning* if there is a compact subset K of $\mathcal{Q}(S)$ and for every $k > 0$ there is some $t > k$ with $\Phi^t q \in K$. The set of all forward returning points $q \in \mathcal{Q}(S)$ is a G_δ-subset of $\mathcal{Q}(S)$. To see that this is the case, choose a countable family $\{U_i\}$ of open subsets of $\mathcal{Q}(S)$ with compact closure and such that $U_i \subset U_{i+1}$, $\cup_i U_i = \mathcal{Q}(S)$. If $q \in \mathcal{Q}(S)$ is *not* forward returning, then for each $i > 0$ there is an integer $m(i) > 0$ such that $q \in A_{i,m(i)} = \{z \mid \Phi^t z \notin U_i$ for all $t \geq m(i)\}$. Now the sets $A_{i,m(i)} \subset \mathcal{Q}(S)$ are clearly closed and hence the set of all points $q \in \mathcal{Q}(S)$ such that $\{t > 0 \mid \Phi^t q \in U_i\}$ is unbounded is a G_δ-set. Then the set of all forward returing point is a G_δ-subset of $\mathcal{Q}(S)$ as well.

Call a projective measured geodesic lamination $\xi \in \mathcal{PML}$ *returning* if there is a quadratic differential $q \in \mathcal{Q}^1(S)$ whose vertical measured geodesic lamination is contained in the class ξ and whose projection to $\mathcal{Q}(S)$ is forward returning. The set of returning projective measured geodesic laminations is a Borel subset of $\mathcal{PML}$ which is invariant under the action of the mapping class group and is contained in the set of all uniquely ergodic projective measured geodesic laminations which fill up S [M82]. Note however that Cheung and Masur [CM06] constructed an example of a uniquely ergodic projective measured geodesic lamination which fills up S and is not returning.

Call a quadratic differential $q \in \mathcal{Q}(S)$ *forward recurrent* if q is contained in the ω-limit set of its own orbit under Φ^t. Since $\mathcal{Q}(S)$ is second countable, an argument along the line of the above discussion shows that the set of all forward recurrent points is a Borel subset of $\mathcal{Q}(S)$. Call a projective measured geodesic lamination $\xi \in \mathcal{PML}$ *recurrent* if there is a quadratic differential $q \in \mathcal{Q}^1(S)$ whose vertical measured geodesic lamination is contained in the class ξ and whose projection to $\mathcal{Q}(S)$ is forward returning and contains a forward recurrent point $q_0 \in \mathcal{Q}(S)$ in its ω-limit set. The set $\mathcal{RML}$ of all recurrent projective measured geodesic laminations is an $MCG(S)$-invariant Borel subset of $\mathcal{PML}$ which has full Lebesgue measure. We have.

Lemma 3.1. *For an ergodic conformal density $\{\nu^x\}$ of dimension α the following are equivalent.*

(1) *$\{\nu^x\}$ gives full mass to the returning projective measured geodesic laminations.*
(2) *$\{\nu^x\}$ gives full mass to the set $\mathcal{RML}$ of recurrent projective measured geodesic laminations.*
(3) *The measure $\hat\nu$ on $\mathcal{PML} \times \mathcal{PML} - \Delta$ is ergodic under the diagonal action of $MCG(S)$.*
(4) *The Teichmüller geodesic flow Φ^t is conservative for the measure ν on $\mathcal{Q}(S)$.*
(5) *The Teichmüller geodesic flow Φ^t is ergodic for ν.*

Proof. We follow Sullivan [Su79] closely. Namely, for $\epsilon > 0$ let $\mathrm{int}\mathcal{T}(S)_\epsilon \subset \mathcal{T}(S)$ be the open $MCG(S)$-invariant subset of all complete hyperbolic structures on S of finite volume whose systole (i.e. the length of a shortest closed geodesic) is bigger than ϵ and define $\mathcal{Q}^1(\epsilon) = \{q \in \mathcal{Q}^1(S) \mid Pq \in \mathrm{int}\mathcal{T}(S)_\epsilon\}$. Then the closure $\overline{\mathcal{Q}}^1(\epsilon)$ of $\mathcal{Q}^1(\epsilon)$ projects to a compact subset $\overline{\mathcal{Q}}(\epsilon)$ of $\mathcal{Q}(S)$. For $\delta < \epsilon$ we have $\overline{\mathcal{Q}}(\delta) \supset \overline{\mathcal{Q}}(\epsilon)$ and $\cup_{\epsilon>0}\overline{\mathcal{Q}}(\epsilon) = \mathcal{Q}(S)$. For $\epsilon > 0$ define $\mathcal{B}_\epsilon \subset \mathcal{PML}$ to be the set of all projective measured geodesic laminations ξ such that there is a quadratic differential $q \in \mathcal{Q}^1(S)$ with $\pi(q) = \xi$ and a sequence $\{t_i\} \to \infty$ with $\Phi^{t_i}q \in \mathcal{Q}^1(\epsilon)$ for all $i > 0$. The set $\mathcal{B}_\epsilon$ is invariant under the action of $MCG(S)$, and $\mathcal{B} = \cup_{\epsilon>0}\mathcal{B}_\epsilon$ is the set of all returning projective measured geodesic laminations.

Let $\{\nu^x\}$ be an ergodic conformal density of dimension α which gives full mass to the set $\mathcal{B}$ of returning projective measured geodesic laminations. By invariance and ergodicity, there is a number $\epsilon > 0$ such that ν^x gives full mass to $\mathcal{B}_\epsilon$. Let ν be the Φ^t-invariant Radon measure on $\mathcal{Q}(S)$ defined by $\{\nu^x\}$. By the results of [M80, M82], for every forward returning quadratic differential $q \in \mathcal{Q}(S)$ and every $z \in W^{ss}(q)$ we have $d(\Phi^t q, \Phi^t z) \to 0$ $(t \to \infty)$ where here d is any distance on $\mathcal{Q}(S)$ defining the usual topology. Thus for ν-almost every $q \in \mathcal{Q}(S)$ the Φ^t-orbit of q enters the compact set $\overline{\mathcal{Q}}(\epsilon/2)$ for arbitrarily large times.

By Lemma 3.1 of [W79], for $\delta > 0$ such that $\log\delta = \log(\epsilon/2) - 1$ and for ν-almost every $q \in \overline{\mathcal{Q}}(\delta)$ there are infinitely many integers $m > 0$ with $\Phi^m q \in \overline{\mathcal{Q}}(\delta)$. As a consequence, the first return map to $\overline{\mathcal{Q}}(\delta)$ of the homeomorphism Φ^1 of $\mathcal{Q}(S)$ defines a measurable map $G : \overline{\mathcal{Q}}(\delta) \to \overline{\mathcal{Q}}(\delta)$ which preserves the restriction $\nu_0 = \nu|\overline{\mathcal{Q}}(\delta)$ of ν. Since ν is a Radon measure, ν_0 is finite and hence the system $(\overline{\mathcal{Q}}(\delta), \nu_0, G)$ is conservative. But then the measure ν is conservative for the time-one map Φ^1 of the Teichmüller flow. Moreover, by the Poincaré recurrence theorem, applied to the measure preserving map $G : \overline{\mathcal{Q}}(\delta) \to \overline{\mathcal{Q}}(\delta)$, we obtain that ν-almost every $q \in \mathcal{Q}(S)$ is forward recurrent. Thus 1) above implies 2) and 4).

Using the usual Hopf argument [Su79] we conclude that the measure ν is ergodic. Namely, choose a continuous positive function $\rho : \mathcal{Q}(S) \to (0,\infty)$ such that $\int \rho d\nu = 1$; such a function ρ exists since the measure ν is locally finite by assumption. By the above, if $q, q' \in \mathcal{Q}(S)$ are typical for ν and contained in the same strong stable manifold then the orbits of Φ^t through q, q' are forward asymptotic [M80, M82]. By the Birkhoff ergodic theorem, for every continuous function f with compact support the limit $\lim_{t\to\infty} \int_0^T f(\Phi^t q)dt / \int_0^T \rho(\Phi^t q)dt = f_\rho(q)$ exists almost everywhere, and the Hopf argument shows that the function f_ρ is constant along ν-almost all strong stable and strong unstable manifold. From this ergodicity follows as in [Su79]. In particular, 1) above implies 5).

The remaining implications in the statement of the lemma are either trivial or standard. Since they will not be used in the sequel, we omit the proof. □

Every ergodic conformal density either gives full measure or zero measure to the $\mathcal{ML}$-invariant Borel subset $\mathcal{RML}$ of recurrent projective measured geodesic laminations. The main goal of this section is to show that a conformal density $\{\nu^x\}$ which gives full measure to $\mathcal{RML}$ is contained Lebesgue measure class. For this we adapt ideas of Sullivan [Su79] to our situation. Namely, the set $\mathcal{RML}$ can be viewed as the radial limit set for the action of $MCG(S)$ on $\mathcal{T}(S)$, where $\mathcal{PML}$ is identified with the Thurston boundary of $\mathcal{T}(S)$. This means that every point in $\mathcal{RML}$ is contained in a nested sequence of neighborhoods which are images of a fixed set under elements of the mapping class group. The mass of these sets with respect to the measures ν^x, λ^x (where as before, $\{\lambda^x\}$ is a conformal density in the Lebesgue measure class) are controlled, and this allows a comparison of measures.

To carry out this approach, we have to construct such nested sequences of neighborhoods of points in $\mathcal{RML}$ explicitly. We use the Gromov distances on the boundary $\partial\mathcal{C}(S)$ of the curve graph $\mathcal{C}(S)$ for this purpose. This boundary consists of all unmeasured minimal geodesic laminations which fill up S, equipped with a coarse Hausdorff topology.

Denote by $\mathcal{FML} \subset \mathcal{PML}$ the set of all projective measured geodesic laminations whose support is a geodesic lamination which is minimal and fills up S. Then $\mathcal{FML}$ is a G_δ-subset of $\mathcal{PML}$. Namely, identify $\mathcal{PML}$ with a section Σ of the projection $\mathcal{ML} \to \mathcal{PML}$. A measured geodesic lamination $\mu \in \Sigma$ is not contained in $\mathcal{FML}$ if and only if there is a simple closed curve c on S with $i(\mu, c) = 0$. Since the intersection form i is continuous, the set $\{\mu \in \Sigma \mid i(\mu, c) \neq 0\}$ is closed. But there are only countably many free homotopy classes of simple closed curves on S and hence $\mathcal{FML}$ is indeed a G_δ-set.

The natural forgetful map

$$F : \mathcal{FML} \to \partial\mathcal{C}(S)$$

which assigns to a projective measured geodesic lamination in $\mathcal{FML}$ its support is a continuous $MCG(S)$-equivariant surjection [Kl99, H06]. The restriction of the projection map F to the Borel set $\mathcal{RML} \subset \mathcal{FML}$ is injective [M82].

Let again $P : \mathcal{Q}^1(S) \to \mathcal{T}(S)$ be the canonical projection. For $q \in \mathcal{Q}^1(S)$ and $r > 0$ define

$$B(q, r) = \{u \in W^{su}(q) \mid d(Pq, Pu) \le r\}. \tag{7}$$

Then $B(q, r)$ is a compact neighborhood of q in $W^{su}(q)$ with dense interior which depends continuously on q in the following sense. If $q_i \to q$ in $\mathcal{Q}^1(S)$ then $B(q_i, r) \to B(q, r)$ in the Hausdorff topology for compact subsets of $\mathcal{Q}^1(S)$. We have.

Lemma 3.2. (1) *The map $F : \mathcal{FML} \to \partial\mathcal{C}(S)$ is continuous and closed.*
(2) *If the vertical measured geodesic lamination of the quadratic differential $q \in \mathcal{Q}^1(S)$ is uniquely ergodic then the sets $F(\pi(B(q, r)) \cap \mathcal{FML})$ $(r > 0)$ form a neighborhood basis for $F(\pi(q))$ in $\partial\mathcal{C}(S)$.*

Proof. The first part of the lemma is immediate from the description of the Gromov boundary of $\mathcal{C}(S)$ in [Kl99, H06].

To show the second part, note first that for every $q \in \mathcal{Q}^1(S)$ the restriction of the projection $\pi : W^{su}(q) \to \mathcal{PML}$ is a homeomorphism of $W^{su}(q)$ onto an *open* subset of $\mathcal{PML}$. To see this, identify $\mathcal{PML}$ with a section Σ of the fibration $\mathcal{ML} \to \mathcal{PML}$. Then $\xi \in \Sigma$ is contained in $\pi W^{su}(q)$ if and only if the function $\zeta \to i(\xi, \zeta) + i(\pi(-q), \zeta)$ on Σ is *positive.* Since the intersection form i on $\mathcal{ML}$ is continuous and since Σ is compact, this is an open condition for points $\xi \in \Sigma$.

Now if the vertical measured geodesic lamination of $q \in \mathcal{Q}^1(S)$ is uniquely ergodic then by [Kl99] we have $F^{-1}(F(\pi(q))) = \{\pi(q)\}$ and hence if $r > 0$ is arbitrary then $F(\mathcal{FML} - \pi(\text{int}B(q, r)))$ is a closed subset of $\partial\mathcal{C}(S)$ which does not contain $F(\pi(q))$. In particular,

$$F(\pi(B(q, r)) \cap \mathcal{FML})$$

is a neighborhood of $F(\pi(q))$ in $\partial\mathcal{C}(S)$. From this and continuity of F the second part of the lemma follows. ☐

Define

$$\mathcal{A} = \pi^{-1}(\mathcal{FML}) \subset \mathcal{Q}^1(S).$$

Then $\mathcal{A}$ is a G_δ-subset of $\mathcal{Q}^1(S)$. The map $F \circ \pi : \mathcal{A} \to \partial\mathcal{C}(S)$ is continuous.

Recall that there is a number $\kappa > 0$ and there is an $MCG(S)$-equivariant family of distance functions δ_h $(h \in \mathcal{T}(S))$ on $\partial\mathcal{C}(S)$ such that $\delta_h \le e^{\kappa d(h,z)}\delta_z$ for all $h, z \in \mathcal{T}(S)$ (inequality (3) in Section 2). For $q \in \mathcal{A}$ and $\chi > 0$ define $D(q, \chi) \subset \partial\mathcal{C}(S)$ to be the closed δ_{Pq}-ball of radius χ about $F\pi(q) \in \partial\mathcal{C}(S)$. Note that we have $gD(q, \chi) = D(gq, \chi)$ for all $q \in \mathcal{Q}^1(S)$ and all $g \in MCG(S)$. The following lemma is a translation of hyperbolicity of the curve graph into properties of the distance functions δ_h.

Lemma 3.3. (1) *For every $\beta > 0$ there is a number $\rho = \rho(\beta) > 0$ such that*

$$D(\Phi^t q, \rho) \subset D(q, \beta)$$

for every $q \in \mathcal{A}$ and every $t \geq 0$.

(2) *There is a number $\beta_0 > 0$ with the following property. For every $q \in \mathcal{A}$ and every $\epsilon > 0$ there is a number $T(q, \epsilon) > 0$ such that $D(\Phi^t q, \beta_0) \subset D(q, \epsilon)$ for every $t \geq T(q, \epsilon)$.*

Proof. By the results of [MM99] (see Theorem 4.1 of [H07] for an explicit statement), there is a number $L > 0$ such that the image under $\Upsilon_{\mathcal{T}}$ of every Teichmüller geodesic is an *unparametrized L-quasi-geodesic in $\mathcal{C}(S)$*. This means that for every $q \in \mathcal{Q}^1(S)$ there is an increasing homeomorphism $\sigma_q : \mathbb{R} \to \sigma_q(\mathbb{R}) \subset \mathbb{R}$ such that the curve $t \to \Upsilon_{\mathcal{T}}(P\Phi^{\sigma_q(t)} q)$ is an L-quasi-geodesic in $\mathcal{C}(S)$.

If $q \in \mathcal{A}$ then we have $\sigma_q(t) \to \infty$ $(t \to \infty)$ and the unparametrized L-quasi-geodesic $t \to \Upsilon_{\mathcal{T}}(P\Phi^t q)$ converges as $t \to \infty$ in $\mathcal{C}(S) \cup \partial\mathcal{C}(S)$ to the point $F(\pi(q)) \in \partial\mathcal{C}(S)$ (see [Kl99, H06, H08d]). In particular, for $q \in \mathcal{A}$ and every $T > 0$ there is a number $\tau = \tau(q, T) > 0$ such that $d(\Upsilon_{\mathcal{T}}(P\Phi^t q), \Upsilon_{\mathcal{T}}(Pq)) \geq T$ for all $t \geq \tau$.

Since $\mathcal{C}(S)$ is a hyperbolic geodesic metric space, any finite subarc of an L-quasi-geodesic is contained in a tubular neighborhood of a geodesic in $\mathcal{C}(S)$ of uniformly bounded radius. This implies that there is no backtracking along an unparametrized L-quasi-geodesic: There is a constant $b > 0$ only depending on L and the hyperbolicity constant of $\mathcal{C}(S)$ such that $d(\gamma(t), \gamma(0)) \geq d(\gamma(s), \gamma(0)) - b$ for all $t \geq s \geq 0$ and every L-quasi-geodesic $\gamma : [0, \infty) \to \mathcal{C}(S)$ (see also Lemma 2.4 of [H08d]). From the definition of the Gromov distances δ_c $(c \in \mathcal{C}(S))$ (and property (2) in Section 2) we obtain the existence of a number $\alpha > 0$ such that for every L-quasi-geodesic ray $\gamma : [0, \infty) \to \mathcal{C}(S)$ with endpoint $\gamma(\infty) \in \partial\mathcal{C}(S)$ and every $t > 0$ the Gromov distances $\delta_{\gamma(t)}$ on $\partial\mathcal{C}(S)$ satisfy

$$\delta_{\gamma(t)} \geq \alpha e^{\alpha d(\gamma(t), \gamma(0))} \delta_{\gamma(0)}$$

on the $\delta_{\gamma(t)}$-ball of radius α about $\gamma(\infty)$. Let $\kappa > 0$ be as in inequality (4) from Section 2 and define $\beta_0 = \alpha/\kappa^2$.

By inequality (4) from Section 2, for $q \in \mathcal{A}$ and $t \geq 0$ we have

$$\delta_{P\Phi^t q} \geq \kappa^{-2} \alpha e^{\alpha d(\Upsilon_{\mathcal{T}}(P\Phi^t q), \Upsilon_{\mathcal{T}}(Pq))} \delta_{Pq} \tag{8}$$

on the $\delta_{P\Phi^t q}$-ball $D(\Phi^t q, \beta_0)$. Thus if for $\epsilon > 0$ we choose $T_1 > 0$ sufficiently large that $\epsilon \alpha e^{\alpha T_1} \geq \kappa^2 \beta_0$ then for $q \in \mathcal{A}$, for $T = \tau(q, T_1) > 0$ and for $t > T$ we have $D(\Phi^t q, \beta_0) \subset D(q, \epsilon)$ which shows the second part of the lemma.

To show the first part of the lemma, for $\beta < \beta_0$ define $\rho(\beta) = \alpha\beta/\kappa^2$. Then the estimate (8) above shows that $D(\Phi^t q, \rho) \subset D(q, \beta)$ for every $q \in \mathcal{A}$ and every $t \geq 0$. This completes the proof of the lemma. □

For a forward recurrent point $q_0 \in \mathcal{Q}(S)$ let $\mathcal{RML}(q_0) \subset \mathcal{RML}$ be the Borel subset of all recurrent projective measured geodesic laminations $\xi \in \mathcal{RML}$ such that there is some $q \in \pi^{-1}(\xi)$ with the following property. The projection to $\mathcal{Q}(S)$ of the orbit of q under the Teichmüller geodesic flow contains q_0 in its ω-limit set. By definition, the set $\mathcal{RML}(q_0)$ is invariant under the action of $MCG(S)$. Moreover, every recurrent point $\xi \in \mathcal{RML}$ is contained in one of the sets $\mathcal{RML}(q_0)$ for some forward recurrent point $q_0 \in \mathcal{Q}(S)$. Note moreover that an orbit of Φ^t in $\mathcal{Q}(S)$ which is typical for the Φ^t-invariant Lebesgue measure on $\mathcal{Q}(S)$ is dense and hence for every forward recurrent point $q_0 \in \mathcal{Q}(S)$ the set $\mathcal{RML}(q_0)$ has full Lebesgue measure. Write

$$C(q_0) = F(\mathcal{RML}(q_0)) \subset \partial\mathcal{C}(S) \quad \text{and} \quad A(q_0) = \pi^{-1}\mathcal{RML}(q_0) \subset \mathcal{Q}^1(S).$$

Following p. 151 of [F69], a *Borel covering relation* for a Borel subset C of a metric space (X, d) is a family $\mathcal{V}$ of pairs (x, V) where $V \subset X$ is a Borel set, where $x \in V$ and such that

$$C \subset \bigcup \{V \mid (z, V) \in \mathcal{V} \text{ for some } z \in C\}.$$

The covering relation $\mathcal{V}$ is called *fine* at every point of C if for every $x \in C$ and every $\alpha > 0$ there is some $(y, V) \in \mathcal{V}$ with $x \in V \subset U(x, \alpha)$ where $U(x, \alpha)$ denotes the open ball of radius α about x.

For $\chi > 0$ and the forward recurrent point $q_0 \in \mathcal{Q}(S)$ with lift $q_1 \in \mathcal{Q}^1(S)$ there is by continuity of $F \circ \pi : \mathcal{A} \to \partial\mathcal{C}(S)$ a compact neighborhood K of q_1 in $\mathcal{Q}^1(S)$ such that $F \circ \pi(K \cap \mathcal{A}) \subset D(q_1, \chi)$. We call K a χ-*admissible* neighborhood of q_1. For a number $\chi > 0$ and such a χ-admissible neighborhood K of q_1 define

$$\begin{aligned} \mathcal{V}_{q_0,\chi,K} = &\{(F\pi(q), gD(q_1, \chi)) \mid \\ &q \in W^{su}(q_1) \cap A(q_0), g \in MCG(S), gK \cap \cup_{t>0}\Phi^t q \neq \emptyset.\} \end{aligned}$$

We sometimes identify a pair $(\xi, gD(q_1, \chi)) \in \mathcal{V}_{q_0,\chi,K}$ with the set $gD(q_1, \chi)$ whenever the point ξ has no importance. Let $\beta_0 > 0$ be as in Lemma 3.3. We have.

Lemma 3.4. *Let $q_0 \in \mathcal{Q}(S)$ be a forward recurrent point and let $q_1 \in \mathcal{Q}^1(S)$ be a lift of q_0. Then for every $\chi < \beta_0/4$ and every χ-admissible compact neighborhood K of q_1 the family $\mathcal{V}_{q_0,\chi,K}$ is a Borel covering relation for $C(q_0) \subset (\partial\mathcal{C}(S), \delta_{Pq_1})$ which is fine at every point of $C(q_0)$.*

Proof. Using the above notations, let $q_0 \in \mathcal{Q}(S)$ be a forward recurrent point and let q_1 be a lift of q_0 to $\mathcal{Q}^1(S)$. Let $\chi < \beta_0/4$ where $\beta_0 > 0$ is as in Lemma 3.3. It clearly suffices to show the lemma for the covering relations $\mathcal{V}_{q_0,\chi,K}$ where K is a sufficiently small χ-admissible neighborhood of q_1.

From relation (3) in Section 2 we infer that for every sufficiently small χ-admissible neighborhood K of q_1 we have

$$\delta_{Pq_1}/2 \le \delta_{Pq} \le 2\delta_{Pq_1} \text{ for every } q \in K.$$

In particular, for $q \in K \cap \mathcal{A}$ the set $D(q_1, \chi)$ contains $F\pi(q)$ and is contained in $D(q, 4\chi)$.

By the construction of the distances δ_h $(h \in \mathcal{T}(S))$ on $\partial\mathcal{C}(S)$ it suffices to show that for every $q \in A(q_0)$ and every $\epsilon > 0$ there is some $g \in MCG(S)$ such that the set $gD(q_1, \chi)$ contains $F(\pi(q))$ and is contained in the open δ_{Pq}-ball of radius ϵ about $F(\pi(q))$.

For $q \in A(q_0)$ and $\epsilon > 0$ let $T(q, \epsilon) > 0$ be as in the second part of Lemma 3.3. Choose some $t > T(q, \epsilon)$ such that $\Phi^t q \in \tilde{K} = \cup_{g \in MCG(S)} K$; such a number exists by the definition of the set $A(q_0)$ and by [M80]. By Lemma 3.3 we have $D(\Phi^t q, 4\chi) \subset D(q,\epsilon)$. Now if $g \in MCG(S)$ is such that $\Phi^t q \in gK$ then we obtain from χ-admissibility of the set K and equivariance under the action of the mapping class group that

$$F\pi(q) = F\pi(\Phi^t q) \in gD(q_1,\chi) \subset D(\Phi^t q, 4\chi) \subset D(q,\epsilon).$$

Since $\epsilon > 0$ was arbitrary, this shows the lemma. □

The next proposition is the main technical result of this section. For its formulation, we refer to p. 151 of [F69] for the definition of a *Vitali relation* for a finite Borel measure on the Borel subset $C(q_0)$ of $\mathcal{C}(S)$. We show.

Proposition 3.5. *Let $q_0 \in \mathcal{Q}(S)$ be a forward recurrent point for the Teichmüller geodesic flow. Then for every sufficiently small $\chi > 0$, every sufficiently small χ-admissible compact neighborhood K of q_1 and for every conformal density $\{\nu^x\}$ on $\mathcal{PML}$ which gives full measure to the set $\mathcal{RML}(q_0)$, the covering relation $\mathcal{V}_{q_0,\chi,K}$ for $C(q_0)$ is a Vitali relation for the measure $F_*\nu^x$ on $\partial\mathcal{C}(S)$.*

Proof. The strategy of proof is to use the properties of the balls $D(q,\epsilon)$ established in Lemma 3.3 to gain enough control on ν^x-volumes for a conformal density $\{\nu^x\}$ on $\mathcal{PML}$ that Theorem 2.8.17 of [F69] can be applied.

Let $q_0 \in \mathcal{Q}(S)$ be a forward recurrent point for Φ^t and let $q_1 \in \mathcal{Q}^1(S)$ be a lift of q_0. Since no torsion element of $MCG(S)$ fixes pointwise the Teichmüller geodesic defined by q_1 we may assume that the point $Pq_1 \in \mathcal{T}(S)$ is not fixed by any nontrivial element of the mapping class group.

By Lemma 3.4 and using the notations from this lemma, for every $\chi < \beta_0/4$ and every sufficiently small χ-admissible compact neighborhood K of q_1 the covering relation $\mathcal{V}_{q_0,\chi,K}$ for $C = F(\mathcal{RML}(q_0) - \pi(-q_1)) \subset \partial\mathcal{C}(S)$ is fine for the metric δ_{Pq_1} at every point of C.

We first establish some geometric control on the covering relation $\mathcal{V}_{q_0,\chi,K}$ for some particularly chosen small $\chi < \beta_0/4$ and a suitably chosen χ-admissible neighborhood K of q_1. For this let again d be the distance on $\mathcal{T}(S)$ defined by the Teichmüller metric. Choose a number $r > 0$ which is sufficiently small that the images under the action of the mapping class group of the closed d-ball $B(Pq_1, 5r)$ of radius $5r$ about Pq_1 are pairwise disjoint. By the estimate (3) for the family of distance functions δ_z $(z \in \mathcal{T}(S))$, via decreasing the size of the radius r we may assume that

$$\delta_x/2 \leq \delta_u \leq 2\delta_x \text{ for all } x,u \in B(Pq_1,5r). \tag{9}$$

Recall from (7) above the definition of the closed balls $B(q,r) \subset W^{su}(q)$ $(q \in \mathcal{Q}^1(S))$. By continuity of the projection π there is an open neighborhood U_1 of q_1 in $\mathcal{Q}^1(S)$ such that

$$\pi B(z,r) \subset \pi B(q,2r) \text{ for all } z,q \in U_1$$

and that moreover the projection PU_1 of U_1 to $\mathcal{T}(S)$ is contained in the open ball of radius r about Pq_1. This implies in particular that $gU_1 \cap U_1 = \emptyset$ for every nontrivial element $g \in MCG(S)$.

Since the vertical measured geodesic lamination of q_1 is uniquely ergodic, Lemma 3.2 shows that there is a number $\beta > 0$ such that

$$F(\pi B(q_1, r) \cap \mathcal{FML}) \supset D(q_1, 8\beta).$$

Since the map $F \circ \pi : \mathcal{A} \to \partial\mathcal{C}(S)$ is continuous, there is an open neighborhood $U_2 \subset U_1$ of q_1 such that $U_2 \cap \mathcal{A} \subset (F \circ \pi)^{-1} D(q_1, \beta)$. By the choice of U_1 we have $D(z, \beta) \subset D(q, 8\beta)$ for all $q, z \in U_2 \cap \mathcal{A}$. For all $q, z \in U_2 \cap \mathcal{A}$ we also have

$$D(z, \beta) \subset D(q_1, 4\beta) \subset F(\pi B(q_1, r) \cap \mathcal{FML}) \subset F(\pi B(q, 2r) \cap \mathcal{FML}). \tag{10}$$

By Lemma 3.3, there is a number $\sigma \leq \beta$ such that for every $t \geq 0$ we have

$$D(\Phi^t q, \sigma) \subset D(q, \beta).$$

Now U_2 is an open neighborhood of q_1 in $\mathcal{Q}^1(S)$ and therefore $U_2 \cap W^{su}(q_1)$ is an open neighborhood of q_1 in $W^{su}(q_1)$. In particular, there is a number $r_1 < r$ such that $B(q_1, r_1) \subset W^{su}(q_1) \cap U_2$. Thus by Lemma 3.2 there is a number $\chi \leq \sigma/16$ such that

$$F(\pi(W^{su}(q_1) \cap U_2) \cap \mathcal{FML}) \supset D(q_1, 16\chi). \tag{11}$$

Note that we have

$$D(\Phi^t q, 16\chi) \subset D(q, \beta) \text{ for all } q \in U_2 \cap \mathcal{A} \text{ and all } t > 0. \tag{12}$$

Using once more continuity of the map $F \circ \pi : \mathcal{A} \to \partial\mathcal{C}(S)$, there is a compact neighborhood $K \subset U_2$ of q_1 such that

$$K \cap \mathcal{A} \subset (F \circ \pi)^{-1} D(q_1, \chi).$$

In particular, K is χ-admissible. By inequality (9) for the dependence of the metrics δ_{Pq} on the points $q \in K \subset U_1$ we then have

$$F\pi(z) \in D(q_1, \chi) \subset D(z, 4\chi) \subset D(q_1, 16\chi) \text{ for all } z \in K \cap \mathcal{A}. \tag{13}$$

By (11) above and continuity of the strong unstable foliation and of the map π we may moreover assume that

$$F(\pi(W^{su}(q) \cap U_2) \cap \mathcal{FML}) \supset D(q_1, 16\chi) \text{ for every } q \in K. \tag{14}$$

Note that if $z \in K \cap \mathcal{A}$ then $D(z, 4\chi) \subset D(q_1, 16\chi)$ and hence if $u \in W^{su}(q_1) \cap A(q_0)$ is such that $F\pi(u) \in D(z, 4\chi)$ then $u \in U_2$. Namely, the projective measured geodesic lamination $\pi(u)$ of every $u \in A(q_0)$ is uniquely ergodic and therefore $(F \circ \pi)^{-1}(F(\pi(u)) \cap W^{su}(q_1)$ consists of a unique point. However, by (11) above, the set $U_2 \cap W^{su}(q_1)$ contains such a point. Consequently, inequality (9) above shows that $D(z, 4\chi) \subset D(u, 16\chi)$.

Define

$$\mathcal{V}_0 = \mathcal{V}_{q_0, \chi, K}.$$

By Lemma 3.4, $\mathcal{V}_0$ is a covering relation for the set $C \subset C(q_0) \subset \partial\mathcal{C}(S)$ which is fine at every point of C.

By the choice of the set $K \subset U_1$, if $q \in W^{su}(q_1) \cap A(q_0)$, if $g \in MCG(S)$ and if $t > 0$ are such that $\Phi^t q \in gK$ then $g \in MCG(S)$ is uniquely determined by $\Phi^t q$. For $(\xi, V) \in \mathcal{V}_0$ define

$$\begin{aligned}\rho(\xi, V) = &\max\{e^{-t} \mid q \in W^{su}(q_1) \cap A(q_0), t \geq 0,\\ &V = gD(q_1, \chi), \Phi^t q \in gK, \pi(q) = \xi\}.\end{aligned}$$

Following p. 144 of [F69], for $(\xi, V) \in \mathcal{V}_0$ define the *ρ-enlargement* of V by

$$\hat{V} = \bigcup\{W \mid (\zeta, W) \in \mathcal{V}_0, W \cap V \cap C(q_0) \neq \emptyset, \rho(\zeta, W) \leq e^r \rho(\xi, V)\} \tag{15}$$

where in this definition, the constant $r > 0$ is chosen as in the beginning of this proof.

Let $\{\nu^x\}$ be a conformal density of dimension $\alpha \geq 0$ which gives full measure to the set $\mathcal{RML}(q_0)$. We may assume that the density is ergodic, i.e. that the $MCG(S)$-invariant measure class it defines on $\mathcal{PML}$ is ergodic. The measure ν^x induces a Borel measure $F_*\nu^x$ on the set $C = F(\mathcal{RML}(q_0) - \pi(-q_1)) \subset C(q_0) \subset \partial\mathcal{C}(S)$.

Recall from the beginning of this section that the measures ν^y $(y \in \mathcal{T}(S))$ define a family of $MCG(S)$-invariant Radon measures ν^{su} on strong unstable manifolds in $\mathcal{Q}^1(S)$. These measures are invariant under holonomy along strong stable manifolds and they are quasi-invariant under the Teichmüller geodesic flow, with transformation $d\nu^{su} \circ \Phi^t = e^{\alpha t} d\nu^{su}$. For $q \in \mathcal{Q}^1(S)$ the measure ν^{su} on $W^{su}(q)$ projects to a Borel measure ν_q on C. For $q, z \in \mathcal{Q}^1(S)$ the measures ν_q, ν_z are absolutely continuous, with continuous Radon Nikodym derivative depending continuously on q, z. By invariance of the measures ν^{su} under holonomy along strong stable manifolds and by the choice of the point q_1 and the number $\chi > 0$ there is a number $a > 0$ such that $1/a \geq \nu_q D(q_1, \chi) \geq a$ for all $q \in K$.

Write $\nu_1 = \nu_{q_1}$; we claim that there is a number $c > 0$ such that $\nu_1(\hat{V}) \leq c\nu_1(V)$ for all $(\xi, V) \in \mathcal{V}_0$ where $\hat{V}$ is the ρ-enlargement of V as defined in (15). For this let $(\xi, V) \in \mathcal{V}_0$ be arbitrary; then there is some $q \in W^{su}(q_1) \cap A(q_0)$ with $F\pi(q) = \xi$ and there is a number $t \geq 0$ and some $g \in MCG(S)$ such that $\Phi^t q \in gK$ and that $V = gD(q_1, \chi)$ and $\rho(\xi, V) = e^{-t}$. By equivariance under the action of the mapping class group and by the inclusion (10) above, we have

$$V = gD(q_1, \chi) \subset F(\pi B(\Phi^t q, 2r) \cap \mathcal{FML}).$$

Let $(\zeta, W) \in \mathcal{V}_0$ be such that

$$\rho(\xi, V) \leq \rho(\zeta, W) \leq e^r \rho(\xi, V)$$

and that $W \cap V \cap C \neq \emptyset$. Then there is a number $\epsilon \in [0, r]$, a point $z \in W^{su}(q_1)$ such that $F\pi(z) = \zeta$ and some $h \in MCG(S)$ such that $\Phi^{t-\epsilon} z \in hK$ and that $W = hD(q_1, \chi)$, $\rho(\zeta, W) = e^{\epsilon - t}$. By equivariance under the action of $MCG(S)$ and the inclusion (10) above, we have

$$W = hD(q_1, \chi) \subset F(\pi B(\Phi^{t-\epsilon} z, 2r) \cap \mathcal{FML})$$

and hence from the definition of the sets $B(q,R)$ and the definition of the strong unstable manifolds we conclude that

$$W \subset F(\pi B(\Phi^t z, 4r) \cap \mathcal{FML}).$$

Since the restriction of the map F to $\mathcal{RML}$ is injective and the restriction of the map π to $W^{su}(\Phi^t q)$ is injective and since $V \cap W \cap C \neq \emptyset$ by assumption we obtain that $B(\Phi^t z, 4r) \cap B(\Phi^t q, 2r) \neq \emptyset$. As a consequence, the distance in $\mathcal{T}(S)$ between the points $P(\Phi^t z)$ and $P(\Phi^t q)$ is at most $6r$ and hence the distance between $P(\Phi^{t-\epsilon} z) \in PhK = hPK$ and $P(\Phi^t q) \in PgK = gPK$ is at most $7r$. On the other hand, since $K \subset U_1$, for $u \neq v \in MCG(S)$ the distance in $\mathcal{T}(S)$ between uPK and vPK is not smaller than $8r$. Therefore we have $g = h$ and $V = W$. This shows that

$$\begin{aligned}\nu_1(\bigcup\{W \mid (\zeta, W) \in \mathcal{V}_0, \rho(\xi, V) \leq & \rho(\zeta, W) \\ \leq e^r \rho(\xi, V), W \cap V \cap C \neq \emptyset\}) = & \nu_1(V).\end{aligned}$$

On the other hand, if $z \in W^{su}(q_1) \cap A(q_0)$, if $s \geq 0$ and $h \in MCG(S)$ are such that $\Phi^s z \in hK$ and $hD(q_1, \chi) = W$ and if $(F\pi(z), W) \in \mathcal{V}_0$ is such that

$$e^{-s} = \rho(F\pi(z), W) \leq \rho(\xi, V)$$

and $V \cap W \cap C \neq \emptyset$ then $s \geq t$. By the choice of the set K, equivariance under the action of the mapping class group and the inclusion (13) above, we have $W \subset D(\Phi^s z, 4\chi)$ and $V \subset D(\Phi^t q, 4\chi)$ and hence $D(\Phi^t q, 4\chi) \cap D(\Phi^s z, 4\chi) \cap C \neq \emptyset$. In other words, there is some $u \in A(q_0) \cap W^{su}(\Phi^t q)$ with $F(\pi(u)) \in D(\Phi^t q, 4\chi) \cap D(\Phi^s z, 4\chi)$.

By the inclusions (13) and (14) and the following remark, since $u \in W^{su}(\Phi^t q) \cap A(q_0)$, $\Phi^t q \in gK$ and $F\pi(u) \in D(\Phi^t q, 4\chi) \subset gD(q_1, \sigma)$ we have $u \in gU_2 \cap A(q_0)$ and moreover

$$\Phi^{s-t} u \in W^{su}(\Phi^s z) \cap A(q_0) \quad \text{and} \quad W \subset D(\Phi^s z, 4\chi) \subset D(\Phi^{s-t} u, 16\chi).$$

From (12) above and invariance under the action of the mapping class group we obtain $D(\Phi^{s-t} u, 16\chi) \subset D(u, \beta)$. The inclusion (10) then yields that

$$W \subset D(\Phi^{s-t} u, 16\chi) \subset D(u, \beta) \subset F(\pi B(\Phi^t q, 2r) \cap \mathcal{FML}).$$

This shows that the ρ-enlargement $\hat{V}$ of V is contained in $F(\pi B(\Phi^t q, 2r) \cap \mathcal{FML})$.

Since $\Phi^t q \in \cup_{g \in MCG(S)} gK$ by assumption, by invariance under the action of the mapping class group the $\nu_{\Phi^t q}$-mass of $F(\pi B(\Phi^t q, 2r) \cap \mathcal{FML})$ is uniformly bounded. Therefore by the transformation rule for the measures ν_z under the action of the Teichmüller geodesic flow, the ν_1-mass of $\hat{V}$ is bounded from above by a fixed multiple of the ν_1-mass of V. Thus by Theorem 2.8.17 of [F69] and by Lemma 3.4, the covering relation $\mathcal{V}_{q_0,\chi,K}$ is indeed a Vitali relation for the measure $F_*\nu_1$ and hence it is a Vitali relation for the measure $F_*\nu^x$ as well. Note that the same is true for the covering relation $\mathcal{V}_{q_0,\epsilon,K'}$ for every $\epsilon < \chi$ and every sufficiently small ϵ-admissible neighborhood K' of q_1. □

Using Lemma 3.1, Lemma 3.4 and Proposition 3.5 we can now show.

Lemma 3.6. (1) *A conformal density on $\mathcal{PML}$ which gives full measure to $\mathcal{FML}$ is of dimension at least $6g-6+2m$, with equality if and only if it coincides with the Lebesgue measure up to scale.*
(2) *A conformal density which gives full measure to the set of returning projective measured geodesic laminations is of dimension $6g-6+2m$ and coincides with the Lebesgue measure up to scale.*

Proof. Let $\{\nu^x\}$ be a conformal density of dimension $\alpha \geq 0$ which gives full measure to the set $\mathcal{FML}$. We may assume that the density is ergodic, i.e. that the $MCG(S)$-invariant measure class it defines on $\mathcal{PML}$ is ergodic. Let moreover $\{\lambda^x\}$ be the conformal density of dimension $h = 6g-6+2m$ which defines the Φ^t-invariant probability measure on $\mathcal{Q}(S)$ in the Lebesgue measure class. We have to show that $\alpha \geq h$, with equality if and only if $\nu^x = \lambda^x$ up to scale. For this assume that $\alpha \leq h$.

The Lebesgue measure λ on $\mathcal{Q}(S)$ is of full support and ergodic under the Teichmüller flow and therefore the Φ^t-orbit of λ-almost every $q \in \mathcal{Q}(S)$ is dense. This implies that there is a recurrent point $q_0 \in \mathcal{Q}(S)$ with the property that the measure λ^x gives full mass to $\mathcal{RML}(q_0)$.

Recall that the conformal densities $\{\nu^x\}, \{\lambda^x\}$ define families ν^{su}, λ^{su} of Radon measures on the strong unstable manifolds. For $q \in \mathcal{Q}^1(S)$ denote by ν_q, λ_q the image under the map $F \circ \pi$ of the restriction of these measures to $W^{su}(q) \cap \mathcal{A}$. Since the conformal densities $\{\nu^x\}, \{\lambda^x\}$ give full measure to the set $\mathcal{FML}$, for every $q \in \mathcal{Q}^1(S)$ the measures λ_q, ν_q on $\partial\mathcal{C}(S)$ are of full support.

Let $q_1 \in \mathcal{Q}^1(S)$ be a lift of q_0 to $\mathcal{Q}^1(S)$. By Proposition 3.5, for sufficiently small $\chi > 0$ and for a sufficiently small compact neighborhood K of q_1 the covering relation $\mathcal{V}_{q_0,\chi,K}$ is a Vitali relation for the measure λ_{q_1} on $\partial\mathcal{C}(S)$. Using equivariance under the action of $MCG(S)$ and the fact that ν_{q_1} is of full support, if the measures ν_{q_1}, λ_{q_1} are singular then for λ^{su}-almost every $q \in W^{su}(q_1)$ there is a sequence $t_i \to \infty$ such that for every $i > 0$ the following holds.

(1) $\Phi^{t_i} q \in g_i K$ for some $g_i \in MCG(S)$.
(2) The $\nu_{\Phi^{t_i}q}$-mass and the $\lambda_{\Phi^{t_i}q}$-mass of $D(g_i q_1, \chi)$ is bounded from above and below by a universal constant.
(3) The limit $\lim_{i\to\infty} \nu_{q_1}(D(g_i q_1,\chi))/\lambda_{q_1}(D(g_i q_1,\chi))$ exists and equals zero.

In particular, for every $k > 0$ and for all sufficiently large i, say for all $i \geq i(k)$, we have $\lambda_{q_1} D(g_i q_1, \chi) \geq k\nu_{q_1} D(g_i q_1, \chi)$. On the other hand, we also have

$$\lambda_{q_1} D(g_i q_1, \chi) = e^{-ht_i} \lambda_{\Phi^{t_i}q} D(g_i q_1, \chi) \leq ce^{-ht_i} \tag{16}$$

for a universal constant $c > 0$ and $\nu_{q_1} D(g_i q_1, \chi) \geq de^{-\alpha t_i}$ for a universal constant $d > 0$. If $k > 0$ is sufficiently large that $kd \geq 2c$ then for $i \geq i(k)$ we obtain a contradiction.

In other words, if $\alpha \leq h$ then the measures $\{\nu^x\}$ and $\{\lambda^x\}$ are absolutely continuous. Moreover, they give full mass to the recurrent projective measured geodesic laminations. Then they define absolutely continuous Φ^t-invariant Radon measures ν, λ on $\mathcal{Q}(S)$ which are ergodic by Lemma 3.1. As a consequence, the measures ν, λ

coincide up to scale and hence the measures $\{\nu^x\}, \{\lambda^x\}$ coincide up to scale as well. This shows the first part of the lemma.

To show the second part of the lemma, assume that $\alpha \geq h$ and that the conformal density $\{\nu^x\}$ gives full measure to the subset of $\mathcal{PML}$ of returning points. By Lemma 3.1, $\{\nu^x\}$ gives full measure to the set $\mathcal{RML}$ of recurrent points. Since $\{\nu^x\}$ is ergodic, there is a forward recurrent quadratic differential $q_0 \in \mathcal{Q}(S)$ such that $\{\nu^x\}$ gives full measure to the set $\mathcal{RML}(q_0)$. This implies that we can exchange the roles of $\{\lambda^x\}$ and $\{\nu^x\}$ in the above argument and obtain that $\{\nu^x\}, \{\lambda^x\}$ coincide up to scale. □

The following proposition uses the results of Minsky and Weiss [MW02] to show that up to scale, the Lebesgue measure is the unique conformal density on $\mathcal{PML}$ which gives full measure to the set $\mathcal{FML}$ of filling projective measured geodesic laminations.

Proposition 3.7. *Let $\{\nu^x\}$ be a conformal density which gives full measure to $\mathcal{FML}$. Then $\{\nu^x\}$ is the Lebesgue measure up to scale.*

Proof. We argue by contradiction and we assume that there is a conformal density $\{\nu^x\}$ which gives full measure to the set $\mathcal{FML}$ of all projective measured geodesic laminations whose support is minimal and fills up S and which is singular to the Lebesgue measure. Without loss of generality we can assume that $\{\nu^x\}$ is ergodic. By Lemma 3.6, the dimension α of $\{\nu^x\}$ is strictly bigger than $h = 6h - 6 + 2m$ and the ν^x-measure of the set of returning points vanishes. The conformal density induces a family $\{\nu^{su}\}$ of locally finite Borel measures on the leaves of the strong unstable foliation as before.

Let ν be the locally finite Borel measure on $\mathcal{Q}(S)$ which can be written in the form $d\nu = d\nu^{su} \times d\lambda^s$ where λ^s is the family of Lebesgue measures on stable manifolds which transforms under the Teichmüller flow Φ^t via $d\lambda^s \circ \Phi^t = e^{-ht} d\lambda^s$. The measure ν is quasi-invariant under the Teichmüller geodesic flow and transforms via

$$\nu \circ \Phi^t = e^{(\alpha - h)t} \nu.$$

Since $\alpha > h$ this implies that the measure ν is infinite. Moreover, it gives full mass to quadratic differentials whose vertical measured geodesic lamination is minimal and fills up S.

The family λ^s of Lebesgue measures on stable manifolds is invariant under the horocycle flow h_t as defined in Section 2. By the explicit construction of the measures ν^{su}, this implies that the measure ν is invariant under h_t.

However, following the reasoning of Dani [D79] (see the proof of Corollary 2.6 of [MW02] for a discussion in our context), this implies that the measure ν is necessarily *finite*. Namely, by the Birkhoff ergodic theorem, applied to the horocycle flow h_t and the locally finite h_t-invariant measure ν (see Theorem 2.3 of [K85] for the version of the Birkhoff ergodic theorem for locally finite infinite measures needed here), for ν-almost every $q \in \mathcal{Q}(S)$ and for every continuous positive function f on $\mathcal{Q}(S)$ with $\int f d\nu < \infty$ the limit

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(h_t q)dt = F(q)$$

exists, and the resulting function F is h_t-invariant and ν-integrable. On the other hand, consider the family of Borel probability measures

$$\mu(q,T) = \frac{1}{T}\int_0^T \delta(h_t q)dt$$

on $\mathcal{Q}(S)$ where $\delta(x)$ denotes the Dirac mass at x. By Theorem H2 of [MW02] (more precisely, by the theorem in the appendix which is a slightly extended version of this result), for every $\epsilon > 0$ there is a compact set $K_\epsilon \subset \mathcal{Q}(S)$ such that for ν-almost every $q \in \mathcal{Q}(S)$, every weak limit $\mu(q,\infty)$ of the measures $\mu(q,T)$ as $T \to \infty$ satisfies $\mu(q,\infty)(K_\epsilon) \geq 1-\epsilon$. Since the function f is *positive*, we have $\inf\{f(z) \mid z \in K_{1/2}\} = 2c > 0$ and therefore $F(q) = \int f d\mu(q,\infty) \geq c$ for ν-almost every $q \in \mathcal{Q}(S)$. But this contradicts the fact that the measure ν is infinite and that F is ν-integrable and shows the proposition. □

4. Train tracks

In Section 3 we showed that conformal densities for the mapping class group which give full measure to the set of filling measured geodesic laminations coincide with the Lebesgue measure up to scale. Conformal densities induce $MCG(S)$-invariant Radon measures on measured lamination space (see the discussion in Section 5). There may be other invariant Radon measures on $\mathcal{ML}$ which give full measure to the filling measured geodesic laminations. Namely, there may be such measures which give full measure to a measurable section of the fibration $\mathcal{ML} \to \mathcal{PML}$. It follows from the work of Sarig [S04] (see the discussion in Section 5) that this is the only case we have to rule out.

For this we follow the guidelines of Ledrappier and Sarig [LS06]. The main idea is to obtain good control of the action of $MCG(S)$ on $\mathcal{PML}$ with the help of Markov partitions. However, for the action of the mapping class group on $\mathcal{PML}$, such Markov partitions have no obvious reason to exist (due to the lack of hyperbolicity). Instead, we use partitions defined by *train tracks* which can be controlled sufficiently well for our purpose. Such control is achieved by relating data associated to train tracks to Teichmüller distances in Teichmüller space. We begin with summarizing those properties of train tracks which are needed to carry out this idea.

A *maximal generic train track* on the surface S is an embedded 1-complex $\tau \subset S$ whose edges (called *branches*) are smooth arcs with well-defined tangent vectors at the endpoints. At any vertex (called a *switch*) the incident edges are mutually tangent. Every switch is trivalent. Through each switch pass two germs of open C^1-arcs of S^1 contained in τ. In particular, the half-branches which are incident on a fixed switch are divided into two classes which are "incoming" or "outgoing" according to a given orientation of the line tangent to τ at the switch. The complementary regions of a maximal generic train track are trigons, i.e. discs with three cusps at the boundary, or once punctured monogons, i.e. once punctured discs with one cusp at the boundary. We always identify train tracks which are isotopic. We refer to [PH92] for a comprehensive account on train tracks.

A maximal generic train track or a geodesic lamination σ is *carried* by a maximal generic train track τ if there is a map $F : S \to S$ of class C^1 which is homotopic to the identity and maps σ into τ in such a way that the restriction of the differential of F to the tangent space of σ vanishes nowhere; note that this makes sense since a train track has a tangent line everywhere. We call the restriction of F to σ a *carrying map* for σ. Write $\sigma \prec \tau$ if the maximal generic train track or the geodesic lamination σ is carried by the maximal generic train track τ. If σ is carried by τ with carrying map F and if τ is carried by η with carrying map G, then σ is carried by η with carrying map $G \circ F$.

A half-branch $\hat{b}$ in a maximal generic train track τ incident on a switch v of τ is called *large* if every embedded arc of class C^1 containing v in its interior passes through $\hat{b}$. A half-branch which is not large is called *small*. A branch b in a maximal generic train track τ is called *large* if each of its two half-branches is large; in this case b is necessarily incident on two distinct switches, and it is large at both of them. A branch is called *small* if each of its two half-branches is small. A branch is called *mixed* if one of its half-branches is large and the other half-branch is small (for all this, see [PH92] p.118).

A *transverse measure* on a maximal generic train track τ is a nonnegative weight function μ on the branches of τ satisfying the *switch condition*: for every switch s of τ, the sum of the weights over all incoming half-branches at s is required to coincide with the sum of the weights over all outgoing half-branches at s. Thus if a, b, c are the half-branches of τ which are incident on s and if a is large, then we require that $\mu(a) = \mu(b) + \mu(c)$. The maximal generic train track is called *recurrent* if it admits a transverse measure which is positive on every branch. We call such a transverse measure μ *positive*, and we write $\mu > 0$. The space $\mathcal{V}(\tau)$ of all transverse measures on τ has the structure of a convex cone in a finite dimensional real vector space. Via a carrying map, a measured geodesic lamination carried by τ defines a transverse measure on τ, and every transverse measure arises in this way (via Construction 1.7.7 of [PH92]). Thus $\mathcal{V}(\tau)$ can naturally be identified with a subset of $\mathcal{ML}$ which is invariant under scaling. A maximal generic train track τ is recurrent if and only if the subset $\mathcal{V}(\tau)$ of $\mathcal{ML}$ has nonempty interior.

A *tangential measure* β for a maximal generic train track τ associates to every branch b of τ a nonnegative weight $\beta(b)$ such that for every complementary trigon with sides c_1, c_2, c_3 we have $\beta(c_i) \leq \beta(c_{i+1}) + \beta(c_{i+2})$. Here $\beta(c_i)$ is the sum over all branches $b \subset c_i$ of $\beta(b)$, counted with multiplicities, and indices are taken modulo three. (The complementary once-punctured monogons define no constraint on tangential measures.) The space $\mathcal{V}^*(\tau)$ of all tangential measures on τ has the structure of a convex cone in a finite dimensional real vector space. The maximal generic train track τ is called *transversely recurrent* if it admits a tangential measure β which is positive on every branch [PH92].

For a maximal generic train track τ, every tangential measure on τ defines uniquely a measured geodesic lamination which *hits τ efficiently* (we refer to [PH92] for an explanation of this terminology), and every measured geodesic lamination which hits τ efficiently can be obtained in this way. However, in general there are many tangential measures which correspond to a fixed measured geodesic lamination λ which hits τ efficiently. Namely, let s be a switch of τ and let a, b, c be the half-branches of τ incident on s and such that the half-branch a is large. If β is a tangential measure on τ which determines the measured geodesic lamination λ then it may be possible to drag the switch s across some of the leaves of λ and modify the tangential measure β on τ to a tangential measure $\nu \neq \beta$. Then $\beta - \nu$ is a multiple of a vector of the form $\delta_a - \delta_b - \delta_c$ where δ_w denotes the characteristic function of the branch w.

In this way the space of all measured geodesic laminations which hit τ efficiently can be identified with the quotient of the convex linear cone $\mathcal{V}^*(\tau)$ by the subspace H spanned by all vectors of the form $\delta_a - \delta_b - \delta_c$ arising from the different switches of τ as described above. In other words, there is a bijection between the set of all measured geodesic laminations μ which hit τ efficiently and classes of tangential measures for τ in the quotient cone $\mathcal{V}^*(\tau)/H$. With this identification, the pairing $\mathcal{V}(\tau) \times \mathcal{V}^*(\tau) \to [0, \infty)$ defined by

$$(\mu, \beta) \to \sum_b \mu(b)\beta(b) \tag{17}$$

descends to the intersection form on $\mathcal{ML}$ [PH92].

A maximal generic train track τ is called *complete* if it is recurrent and transversely recurrent. Note that in a slight deviation from the terminology introduced in [PH92], a complete train track is always assumed to be generic. In the sequel we denote by $\mathcal{TT}$ the set of isotopy classes of all complete train tracks on S.

Intersection numbers between measured geodesic laminations which are carried by a complete train track τ can also be controlled. The following observation is a slight extension of Corollary 2.3 of [H06]. For its formulation, define the *total weight* of a transverse measure on a complete train track τ to be the sum of the weights over all branches of τ.

Lemma 4.1. *Let μ, ν be measured geodesic laminations which are carried by a complete train track τ. Then $i(\mu, \nu) \leq \mu(\tau)\nu(\tau)$ where $\mu(\tau), \nu(\tau)$ is the total weight of the transverse measure on τ defined by μ, ν.*

Proof. If the measured geodesic laminations μ, ν define transverse measures on τ with rational weights then μ, ν are weighted geodesic multi-curves [PH92].

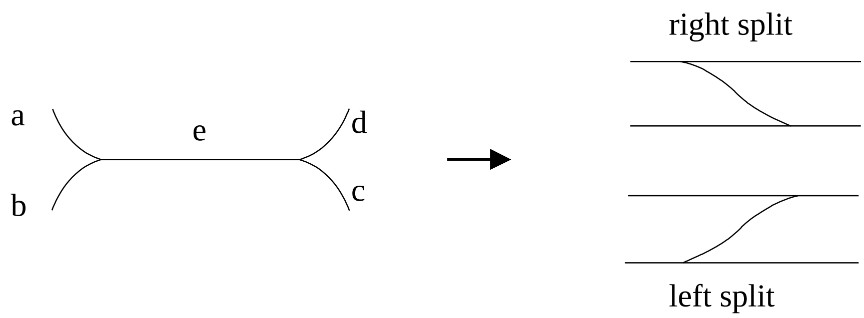
On the other hand, transverse measures with rational weights are dense in the space of all transverse measures. Moreover, the assignment which associates to a transverse measure on τ the corresponding measured geodesic lamination which is carried by τ is continuous. Since the intersection form on $\mathcal{ML}$ is continuous as well, it suffices to show the lemma for weighted geodesic multi-curves which are carried by τ.

The transverse measure on τ defined by two disjoint simple closed geodesics c_1, c_2 carried by τ is the sum of the transverse measures defined by c_1 and c_2. Moreover, we have $i(ac_1 + bc_2, \xi) = ai(c_1, \xi) + bi(c_2, \xi)$ for all $a, b > 0, \xi \in \mathcal{ML}$. Consequently it is enough to show the lemma for two simple closed geodesics carried by τ.

However, this follows as in the proof of Corollary 2.3 of [H06]. Namely, let μ be a simple closed curve which is carried by τ and let $m = \mu(\tau)$. Then there is a *trainpath of length* m, i.e. a C^1-immersion $\rho[0, m] \to \tau$ which maps each interval $[i, i+1]$ onto a branch of τ and which parametrizes the image of μ under a carrying map $\mu \to \tau$. Deform μ with a homotopy to a closed curve $\rho' : [0, m] \to S$ which is mapped to ρ by a carrying map and is such that for $i \leq n$, $\rho'[i, i + 1]$ intersects τ in at most one point, contained in the interior of the branch $\rho[i, i + 1]$.

Now if ν is the transverse measure on τ defined by a second simple closed geodesic carried by τ, then the number of intersection points between $\rho'[i, i + 1]$ and a trainpath on τ defined by this geodesic is not bigger than $\nu(\rho[i, i + 1]) \leq \nu(\tau)$. Together we obtain that $i(\mu, \nu) \leq \mu(\tau)\nu(\tau)$ as claimed. □

There is a simple way to modify a complete train track τ to another complete train track. Namely, if e is a large branch of τ then we can perform a right or left *split* of τ at e as shown in the figure below. Note that a right split at e is uniquely determined by the orientation of S and does not depend on an orientation of e. Using the labels in the figure, in the case of a right split we call the branches a and c *winners* of the split, and the branches b, d are *losers* of the split. If we perform a left split, then the branches b, d are winners of the split, and the branches a, c are losers of the split.



A (right or left) split τ' of a maximal generic train track τ is carried by τ, and up to isotopy, there is a natural choice of a carrying map which maps the switches of τ' to the switches of τ. Since carrying is a transitive relation between maximal generic train tracks, a train track which can be obtained from τ by a *splitting sequence*, i.e. by successive modifications by splits beginning with the train track τ, is carried by τ. There is a natural bijection of the set of branches of τ onto the set of branches of τ' which maps the branch e to the diagonal e' of the split. The split of a complete train track is maximal, transversely recurrent and generic, but it may not be recurrent (Lemma 1.3.3.b and Lemma 2.1.3 of [PH92]).

If τ' is a complete train track which can be obtained from τ by a single split then a carrying map $\tau' \to \tau$ induces a linear push-forward map $L : \mathcal{V}(\tau') \to \mathcal{V}(\tau)$ which maps the transverse measure μ on τ' defined by a measured geodesic lamination λ carried by τ' to the transverse measure on τ defined by the same measured geodesic lamination λ. There is a dual map $L^* : \mathcal{V}^*(\tau) \to \mathcal{V}^*(\tau')$ which is determined by the equation

$$(L\mu, \beta) = (\mu, L^*(\beta))$$

where $(L\mu, \beta)$ is defined as in (17). The image of a tangential measure on τ which defines a measured geodesic lamination ζ which hits τ efficiently is a tangential measure on τ' which defines the same measured geodesic lamination ζ (Proposition 3.4.2 and Lemma 3.4.3 of [PH92]). In particular, a measured geodesic lamination which hits τ efficiently also hits τ' efficiently.

For a complete train track $\tau \in \mathcal{TT}$ denote by $\mathcal{V}_0(\tau) \subset \mathcal{V}(\tau)$ the convex set of all transverse measures on τ whose total weight equals one. Let moreover

$$\mathcal{Q}(\tau) \subset \mathcal{Q}^1(S)$$

be the set of all area one quadratic differentials whose vertical measured geodesic lamination is carried by τ and determines a transverse measure in $\mathcal{V}_0(\tau)$, and whose horizontal measured geodesic lamination hits τ efficiently. Then $\mathcal{Q}(\tau)$ is a closed subset of $\mathcal{Q}^1(S)$ which however may not be compact in general.

As in [MM99], define a *vertex cycle* for τ to be a transverse measure on τ which spans an extreme ray in the convex cone $\mathcal{V}(\tau)$. Up to rescaling, such a vertex cycle is the counting measure of a simple closed curve which is carried by τ (see p. 115 of [MM99] for this fact). In the sequel we identify a vertex cycle for τ with this simple closed curve on S. With this convention, the transverse measure on τ defined by a vertex cycle is integral.

Define a map

$$\Psi : \mathcal{TT} \to \mathcal{C}(S)$$

by associating to a complete train track τ one of its vertex cycles. Recall from Section 2 the definition of the map $\Upsilon_{\mathcal{Q}} : \mathcal{Q}^1(S) \to \mathcal{C}(S)$. We have:

Lemma 4.2. *There is a number $\chi_2 = \chi_2(S) > 0$ such that $d(\Upsilon_{\mathcal{Q}}(q), \Psi(\tau)) \leq \chi_2$ for every complete train track $\tau \in \mathcal{TT}$ and every quadratic differential $q \in \mathcal{Q}(\tau)$.*

Proof. Let $\tau \in \mathcal{TT}$ and let $q \in \mathcal{Q}(\tau)$. Then the transverse measure $\lambda \in \mathcal{V}_0(\tau)$ of the vertical measured geodesic lamination of q can be written in the form $\lambda = \sum_i a_i \alpha_i$ where α_i is the transverse measure on τ defined by a vertex cycle and where $a_i \geq 0$. Now the number of vertex cycles for τ is bounded from above by a universal constant only depending on the topological type of the surface S. The weight disposed on a branch of τ by the counting measure of a vertex cycle does not exceed two (Lemma 2.2 of [H06]). This means that there is a number $a > 0$ only depending on the topological type of S and there is some $i > 0$ such that $a_i \geq a$.

The horizontal measured geodesic lamination μ of q defines a class of tangential measures on τ. If $\beta \in \mathcal{V}^*(\tau)$ is any representative of this class, then we have $\sum_b \lambda(b)\beta(b) = i(\lambda, \mu) = 1$. This implies that the vertex cycle $\alpha = \alpha_i$ satisfies $i(\alpha, \mu) = \sum_b \alpha(b)\beta(b) \leq i(\lambda, \mu)/a = 1/a$. On the other hand, by Lemma 4.1 and by Lemma 2.2 of [H06] (as mentioned in the previous paragraph), the intersection number $i(\alpha, \lambda)$ does not exceed $2p$ where $p > 0$ is the number of branches of a complete train track on S. Moreover, the q-length of the simple closed curve α is bounded from above by $i(\alpha, \lambda) + i(\alpha, \mu)$. (This fact is well known to the experts but a bit difficult to find explicitly in the literature. In Section 5 of [Bw06], Bowditch discusses in detail the case when the supports of the vertical and the horizontal measured geodesic laminations of q are simple closed multi-curves, i.e. disjoint unions of simple closed geodesics. The general case follows from continuity of the intersection form and the fact that quadratic differentials whose horizontal and vertical measured geodesic laminations are supported in simple closed multi-curves are dense in the space of all quadratic differentials [St84]). Hence the q-length of α is uniformly bounded and therefore Lemma 2.1 implies that the distance between $\Upsilon_{\mathcal{Q}}(q)$ and α is bounded from above by a constant only depending on S.

On the other hand, the number of vertex cycles on a complete train track is uniformly bounded, and up to the action of $MCG(S)$, there are only finitely many isotopy classes of complete train tracks on S. Therefore the distance in $\mathcal{C}(S)$ between any two vertex cycles of a complete train track on S is uniformly bounded. The lemma follows. □

A *framing* [H08b] (or a *complete clean marking* in the terminology of [MM00]) for the surface S consists of a pants decomposition P for S and a collection of $3g-3+m$ essential simple closed *spanning curves*. For each pants curve $c \in P$ there is a unique such spanning curve which is disjoint from $P-c$, which is not freely homotopic to any pants curve of P and which has the minimal number (one or two) of intersection points with c among all simple closed curves with these properties. Any two such curves differ by a multiple Dehn twist about c.

There is a number $\hat{\chi}_0 > 0$ such that for every framing F there exists a hyperbolic metric $h \in \mathcal{T}(S)$ with the property that the h-length of each curve from F is at most $\hat{\chi}_0$. We call such a hyperbolic metric *short for F*. Such a metric can be constructed as follows. Equip each pair of pants defined by P with a hyperbolic metric such that the length of each boundary geodesic equals, say, one; then glue these pairs of pants in such a way that the spanning curves have the smallest possible length. By standard hyperbolic trigonometry, there is a number $\delta > 0$ such that every hyperbolic metric which is short for some framing F of S is contained in the set $\mathcal{T}(S)_\delta$ of all metrics whose *systole*, i.e. the shortest length of a closed geodesic, is at least δ. Moreover, using Fenchel-Nielsen coordinates based on the pants decomposition P for a fixed framing F, it is straightforward that the set of all hyperbolic metrics which are short for F is compact.

The mapping class group $MCG(S)$ naturally acts on the collection of all framings of S. There are only finitely many orbits for this action. Thus by invariance under the action of the mapping class group, the diameter in $\mathcal{T}(S)$ of all hyperbolic metrics which are short for a fixed framing is bounded from above by a constant only depending on S.

Also, there is a number $k > 0$ and for every complete train track $\tau \in \mathcal{TT}$ there is a framing F of S which consists of simple closed curves carried by τ and such that the total weight of the counting measures on τ defined by these curves does not exceed k. We call such a framing *short for τ*. Namely, let c be a simple closed curve which is carried by τ and such that the minimum weight c puts on any branch of τ is not smaller than three. Such a simple closed curve exists since τ is recurrent.

By Lemma 4.5 of [MM99] and completeness, τ carries every simple closed curve α with $i(c,\alpha) \leq 2$. However, a framing of S with a pants decomposition containing c as a component consists of simple closed curves α with $i(c,\alpha) \leq 2$. This implies that τ carries a framing for S. Since up to the action of $MCG(S)$ there are only finitely many isotopy classes of complete train tracks on S, we can find a number $k > 0$ as required.

By Lemma 4.1, the intersection number $i(c,c')$ between any two simple closed curves c, c' which are carried by some $\tau \in \mathcal{TT}$ and which define counting measures on τ of total weight at most k is bounded from above by k^2. In particular, for any two short framings F, F' for τ the distance in $\mathcal{T}(S)$ between any two hyperbolic metrics which are short for F, F' is uniformly bounded (see Lemma 4.7 of [Mi93] and [MM99, Bw06]).

Define a map $\Lambda : \mathcal{TT} \to \mathcal{T}(S)$ by associating to a complete train track τ a hyperbolic metric $\Lambda(\tau) \in \mathcal{T}(S)$ which is short for a short framing for τ. By our above discussion, there is a number $\chi_3 > 0$ only depending on the topological type of S such that if Λ' is another choice of such a map then we have $d(\Lambda(\tau), \Lambda'(\tau)) \leq \chi_3$ for every $\tau \in \mathcal{TT}$. In particular, the map Λ is *coarsely $MCG(S)$-equivariant*: For every $\tau \in \mathcal{TT}$ and every $g \in MCG(S)$ we have $d(\Lambda(g\tau), g\Lambda(\tau)) \leq \chi_3$.

In Section 5, the sets of all projective measured geodesic laminations which are carried by a complete train track τ are used as substitutes for a Markov partition for the action of the mapping class group on $\mathcal{PML}$. These sets are invariant under the action of the mapping class group, and there is some control on their intersections. For example, if $\tau \in \mathcal{TT}$, if e is a large branch of τ and if $\tau_\ell, \tau_r \in \mathcal{TT}$ are obtained from τ by a left and right split at e, respectively, then $\mathcal{V}(\tau) = \mathcal{V}(\tau_r) \cup \mathcal{V}(\tau_\ell)$, and $\mathcal{V}(\tau_r) \cap \mathcal{V}(\tau_\ell)$ is the intersection of $\mathcal{V}(\tau)$ with a hyperplane in $\mathcal{PML}$ (see [MM99] for a discussion). However, for arbitrary train tracks $\tau, \sigma \in \mathcal{TT}$ the intersection $\mathcal{V}(\tau) \cap \mathcal{V}(\sigma)$ is difficult to control. Lemma 4.3 and Lemma 4.4 give some quantitative information on nesting of these sets which is sufficient for our purpose.

We begin with a technical result which relates nesting of such sets to geometric properties of Teichmüller geodesics which are easier to control. For its formulation, let again $P : \mathcal{Q}^1(S) \to \mathcal{T}(S)$ be the canonical projection.

Lemma 4.3. *There is a number $\ell > 0$ and for every $\epsilon > 0$ there is a number $m(\epsilon) > 0$ with the following property. Let $\sigma, \tau \in \mathcal{TT}$ and assume that σ is carried by τ and that the distance in $\mathcal{C}(S)$ between any vertex cycle of σ and any vertex cycle of τ is at least ℓ. Let $q \in \mathcal{Q}(\tau)$ be a quadratic differential whose vertical measured geodesic lamination q_v is carried by σ. If the total weight of the transverse measure on σ defined by q_v is not smaller than ϵ, then $d(\Lambda(\tau), Pq) \leq m(\epsilon)$.*

Proof. Let $\chi_0 > 0$ be as in the definition of the map $\Upsilon_{\mathcal{T}} : \mathcal{T}(S) \to \mathcal{C}(S)$ in Section 2. Let $\chi_1 > 0$ be as in Lemma 2.2 and let $\kappa_1 > 0$ be such that for every $x \in \mathcal{T}(S)$ the diameter in $\mathcal{C}(S)$ of the set of simple closed curves whose x-length is at most χ_0 is bounded from above by κ_1. Let $\chi_2 > 0$ be as in Lemma 4.2 and let $\ell > 2\kappa_1 + 2\chi_1 + 2\chi_2 + 3$.

Let $\sigma \prec \tau \in \mathcal{TT}$ be such that the distance in $\mathcal{C}(S)$ between any vertex cycle of τ and any vertex cycle of σ is at least ℓ. Let $\epsilon > 0$ and let $q \in \mathcal{Q}(\tau)$ be such that the vertical measured geodesic lamination q_v of q is carried by σ and defines on σ a transverse measure of total weight at least ϵ. We call such a quadratic differential *ϵ-big for* (σ, τ). In particular, if $t > 0$ is such that $\Phi^t q \in \mathcal{Q}(\sigma)$ then $t \leq -\log \epsilon$. Note that such a number $t \in \mathbb{R}$ exists since by the discussion preceding Lemma 4.2 and its obvious extensions to train track which are related by carrying (as discussed in Proposition 3.4.2 and Lemma 3.4.3 of [PH92]), the horizontal measured geodesic lamination of q hits σ efficiently.

We claim that the Pq-length of any essential simple closed curve on S is at least $\epsilon\chi_0$. Namely, if there is a simple closed curve c on S whose Pq-length is smaller than $\epsilon\chi_0$ then the $P\Phi^t q$-length of c is at most χ_0 (Lemma 3.1 of [W79]). By the choice of the constant κ_1, this means that $d(\Upsilon_{\mathcal{T}}(Pq), \Upsilon_{\mathcal{T}}(P\Phi^t q)) \leq 2\kappa_1$ and hence Lemma 2.2 shows that $d(\Upsilon_{\mathcal{Q}}(q), \Upsilon_{\mathcal{Q}}(\Phi^t q)) \leq 2\kappa_1 + 2\chi_1$. From Lemma 4.2 we conclude that $d(\Psi(\sigma), \Psi(\tau)) \leq 2\kappa_1 + 2\chi_1 + 2\chi_2 < \ell$ which contradicts our choice of σ and τ. Thus we have $Pq \in \mathcal{T}(S)_{\epsilon\chi_0}$.

Now by Lemma 3.3 of [Mi94] (which is an immediate consequence of invariance under the action of the mapping class group and cocompactness), there is a constant $L > 0$ depending on ϵ such that for every $q \in \mathcal{Q}^1(S)$ with $Pq \in \mathcal{T}(S)_{\epsilon\chi_0}$ the singular euclidean metric defined by q is L-bilipschitz equivalent to the hyperbolic metric Pq. Moreover, if a given framing is short for two hyperbolic metrics (here $\Lambda(\tau)$ and Pq), then these are close in $\mathcal{T}(S)$. Thus for the proof of the lemma it suffices to show that for a quadratic differential $q \in \mathcal{Q}(\tau)$ which is ϵ-big for (σ, τ) the q-length of some short framing for τ is uniformly bounded.

Let q_h be the horizontal measured geodesic lamination of q. Then q_h defines a class of tangential measures on τ. Let $\beta \in \mathcal{V}^*(\tau)$ be a representative of this class. We claim that the weights $\beta(b)$ where b varies through the branches of τ are bounded from above by a constant only depending on ϵ (and the topological type of S), but not on the quadratic differential q which is ϵ-big for (σ, τ) or on the choice of β. For this note that the transverse measure on the train track σ defined by the vertical measured geodesic lamination q_v of q can be represented in the form $q_v = \sum_i a_i \alpha_i$ where the α_i are the vertex cycles for σ. By the discussion in the first paragraph of the proof of Lemma 4.2, there is a number $a > 0$ only depending on S such that $a_i \geq \epsilon a$ for at least one index i.

By the choice of σ and τ, the distance in $\mathcal{C}(S)$ between any vertex cycle of σ and any vertex cycle for τ is at least $\ell > 3$. Thus by Lemma 4.9 of [MM99], the image of α_i under a carrying map $\alpha_i \to \tau$ is a *large subtrack* ω of τ. This means that this image is a train track on S which is a subset of τ and whose complementary components are topological discs or once punctured topological discs. Each such complementary component is a polygon or a once punctured polygon which is a union of complementary components of τ.

By definition of the linear push-forward map $\mathcal{V}(\sigma) \to \mathcal{V}(\tau)$ induced by the carrying map $\sigma \to \tau$, since $q_v = \sum_i a_i \alpha_i$, the q_v-weight of every branch of ω is at least ϵa. Thus we obtain from the identity $i(q_h, q_v) = 1$ that the weight of every branch of ω with respect to the tangential measure β is bounded from above by $1/\epsilon a$. From the definition of a tangential measure for τ and the fact that ω is large we deduce that the total weight that β puts on the branches of τ is bounded from above by a constant only depending on ϵ and the topological type of S. Namely, let D be a complementary polygon of ω with more than 3 sides. Then the branches of τ which are contained in D decompose D into trigons. There is at least one such trigon T with two sides contained in the boundary of D, i.e. with two sides contained in ω. By the definition of a tangential measure, the total weight put by β on the third side of T which crosses the interior of D is bounded from above by the sum of the total weights on the sides of T contained in ω. Thus by the above consideration, the total weight put by β on the boundary of T is uniformly bounded. Since the number of complementary components of τ only depends on the topological type of S, with a uniformly bounded number of such steps we obtain inductively a universal bound on the total β-weight of $D \cap \tau$. A similar argument is also valid for once punctured complementary polygons of ω. Namely, a once punctured complementary polygon D in ω which is not a complementary component of τ has at least two sides, and the branches of τ which are contained in D decompose D into trigons and one once punctured monogon. Then we can fill in successively the sides of the complementary trigons of τ which are contained in D as before and obtain a control on the total weight of β.

Since the total weight of the tangential measure β on τ representing q_h is uniformly bounded, the intersection number $i(c, q_h)$ for every curve from a short framing for τ is uniformly bounded as well. Moreover, by Lemma 4.1, for $\tau \in \mathcal{TT}$, for a simple closed curve c contained in a short framing for τ and for every $\nu \in \mathcal{V}_0(\tau)$ we have $i(\nu, c) \leq k$ where $k > 0$ is as in the definition of a short framing for τ. Apply this estimate with $\nu = q_v$. Now for every quadratic differential $z \in \mathcal{Q}^1(S)$ with horizontal and vertical measured geodesic lamination z_h, z_v the z-length of a simple closed curve c is bounded from above by $i(z_h, c) + i(z_v, c)$ (compare the remark in the proof of Lemma 4.2). This shows that the q-length of every simple closed curve from a short framing of τ is uniformly bounded and completes the proof of the lemma. □

Every projective measured geodesic lamination $\xi \in \mathcal{PML}$ determines an unstable manifold $W^u(\xi) \subset \mathcal{Q}^1(S)$ consisting of all area one quadratic differentials whose horizontal measured geodesic lamination is contained in the class ξ. This unstable manifold can naturally be identified with the set of all measured geodesic laminations ζ such that ζ and ξ *jointly fill up* S. This means that for any representative $\nu \in \mathcal{ML}$ of the class β and *any* measured geodesic lamination $\alpha \in \mathcal{ML}$ we have $i(\zeta, \alpha) + i(\nu, \alpha) > 0$. Note that this does not depend on the choice of the representative ν. By the Hubbard-Masur theorem, for every $\xi \in \mathcal{PML}$ the restriction of the canonical projection $P : \mathcal{Q}^1(S) \to \mathcal{T}(S)$ to the unstable manifold $W^u(\xi)$ is a homeomorphism. Hence the Teichmüller metric defines a distance function d on $W^u(\xi)$.

For a train track $\tau \in \mathcal{TT}$ let $\mathcal{PE}(\tau)$ be the projectivization of the set of all measured geodesic laminations which hit τ efficiently (as defined implicitly via tangential measures for τ). Recall from the above discussion that if $\sigma \prec \tau$ then $\mathcal{PE}(\tau) \subset \mathcal{PE}(\sigma)$. For $\xi \in \mathcal{PE}(\tau)$ and a number $R > 0$ the train track τ is called *R−ξ-tight* if the diameter of $\mathcal{Q}(\tau) \cap W^u(\xi)$ with respect to the lift of the Teichmüller metric is at most R. We use Lemma 4.3 to derive the following result which is the main technical ingredient for the proof in Section 5 that invariant Radon measures which give full mass to the filling measured geodesic laminations are defined by conformal densities.

Lemma 4.4. *Let $\ell > 0$ be as in Lemma 4.3. Then for every $\epsilon > 0$ there is a number $R = R(\epsilon) > 0$ with the following property. Let $\eta \prec \sigma \prec \tau$ and assume that the distance in $\mathcal{C}(S)$ between any vertex cycle of σ and any vertex cycle of η as well as any vertex cycle of τ is at least ℓ. Suppose there exists a quadratic differential $q \in \mathcal{Q}(\tau)$ whose vertical measured geodesic lamination is carried by η and defines a transverse measure on η of total weight at least ϵ. Let $\xi \in \mathcal{PE}(\tau)$ be the projective class of the horizontal measured geodesic lamination of q. Then σ is $R - \xi$-tight.*

Proof. Let $\delta > 0$ be such that $\Lambda(\tau) \in \mathcal{T}(S)_\delta$ for every $\tau \in \mathcal{TT}$. By possibly decreasing δ we may assume that $\mathcal{T}(S)_\delta$ is connected. If we equip $\mathcal{T}(S)_\delta$ with the length metric induced by the Finsler structure defining the Teichmüller metric then $\mathcal{T}(S)_\delta$ is a proper geodesic metric space on which the mapping class group $MCG(S)$ acts properly discontinuously and cocompactly as a group of isometries. The set $\mathcal{TT}$ is the set of vertices of the *train track complex* which is a connected metric graph on which the mapping class group acts properly and cocompactly as a group of isometries (Lemma 3.2 and Lemma 3.3 of [H08b]). Edges in the train track complex are of length one and connect two train train tracks τ, τ' if τ' can be obtained from τ by a single split. Since the map Λ is coarsely $MCG(S)$-equivariant this means that there is some $L > 1$ such that $\Lambda : \mathcal{TT} \to \mathcal{T}(S)_\delta$ is an L-quasi-isometry.

Let $\eta \prec \sigma \prec \tau$ be as in the statement of the lemma, let $0 < \epsilon < 1$ and assume that there is some $q \in \mathcal{Q}(\tau)$ such that the vertical measured geodesic lamination q_v of q is carried by η and that the total weight of η defined by q_v is bounded from below by ϵ. Then the total weight c on σ defined by q_v is contained in the interval $[\epsilon, 1]$. By Lemma 4.3, applied both to the train tracks $\sigma \prec \tau$ and to the train tracks $\eta \prec \sigma$ (with the quadratic differential $\Phi^s q$ for $s = -\log c$), the distance between $\Lambda(\sigma)$ and $\Lambda(\tau)$ is bounded from above by a number $\kappa_0 > 0$ only depending on ϵ. Namely, we have $d(\Lambda(\tau), Pq) \le m(\epsilon), d(\Lambda(\sigma), P\Phi^s q) \le m(\epsilon)$ and $d(Pq, P\Phi^s q) = s \le -\log \epsilon$.

We claim that the distance between $\Lambda(\sigma)$ and $\Lambda(\tau)$ in $\mathcal{T}(S)_\delta$ is bounded from above by a constant $\kappa_1 > 0$ only depending on ϵ. For this note that by Lemma 3.1 of [W79], a Teichmüller geodesic of length at most κ_0 connecting two points $\Lambda(\tau), \Lambda(\sigma)$ in $\mathcal{T}(S)_\delta$ entirely remains in $\mathcal{T}(S)_\nu$ where $\log \delta - \log \nu = \kappa_0$. As a consequence, the distance between $\Lambda(\sigma)$ and $\Lambda(\tau)$ in $\mathcal{T}(S)_\nu$ equipped with the length metric induced by the Finsler structure defining the Teichmüller metric is at most κ_0. However, since the mapping class group $MCG(S)$ acts properly and cocompactly on both length spaces $\mathcal{T}(S)_\delta$ and $\mathcal{T}(S)_\nu$, the ($MCG(S)$-equivariant) inclusion $\mathcal{T}(S)_\delta \to \mathcal{T}(S)_\nu$ is an L_1-quasi-isometry for a number $L_1 > 1$ only depending on δ, ν. This shows the claim.

As a consequence of this discussion, the distance between σ, τ in the train track complex is bounded from above by a constant $\kappa_2 > 0$ only depending on ϵ. However, there are only finitely many orbits under the action of the mapping class group of pairs $\sigma \prec \tau$ whose distance in the train track complex is at most κ_2. Thus by invariance under the mapping class group, there is a number $\kappa_3 > 0$ only depending on ϵ (and the topological type of S) with the following property: if α is any vertex cycle of σ, then the total weight defined by α on τ is at most κ_3. Then the carrying map $\sigma \to \tau$ maps $\mathcal{V}_0(\sigma)$ to a subset of $\mathcal{V}(\tau)$ consisting of transverse measures whose total weight is bounded from above by a constant $n(\epsilon) > 0$ depending only on ϵ and S.

Recall the definition of ξ, the horizontal projective measured geodesic lamination of q. Since $q \in \mathcal{Q}(\tau)$, (a representative of) the projective measured geodesic lamination ξ hits τ efficiently. Let us now show that σ is $R-\xi$-tight.

If $z \in \mathcal{Q}(\sigma) \cap W^u(\xi)$, let $c \in [1, n(\epsilon)]$ be the total weight of τ defined by the vertical measured geodesic lamination z_v of z (which is carried by σ and hence by τ), and define $z' = \Phi^{-\log c} z$. Then we have $z' \in \mathcal{Q}(\tau)$. Indeed, the total weight defined by the vertical measured geodesic lamination z'_v of z' on τ is 1 (i.e. $z'_v \in \mathcal{V}_0(\tau)$) by the choice of c, and the horizontal projective measured geodesic lamination of z' is ξ which hits τ efficiently.

But the total weight of σ defined by z'_v is $1/c \geq 1/n(\epsilon)$. By Lemma 4.3, there exists a constant $m = m(1/n(\epsilon)) > 0$ such that $d(Pz', \Lambda(\tau)) \leq m$. But $d(Pz, Pz') = \log c \leq \log n(\epsilon)$. Therefore Pz is within a bounded distance R (depending only on ϵ and S) of $\Lambda(\tau)$. This proves the lemma. □

5. Invariant Radon measures on $\mathcal{ML}$

In this section we complete the proof of the theorem from the introduction. We continue to use the assumptions and notations from Section 3. Recall first that for every point $x \in \mathcal{T}(S)$ and every $\xi \in \mathcal{PML}$ there is a unique quadratic differential $q(x, \xi) \in \mathcal{Q}^1(S)_x$ of area one on the Riemann surface x whose vertical measured geodesic lamination $q_v(x, \xi)$ is contained in the class ξ (this is the Hubbard-Masur theorem). The assignment $\xi \to q_v(x, \xi)$ determines a homeomorphism $\mathcal{PML} \times \mathbb{R} \to \mathcal{ML}$ by assigning to $(\xi, t) \in \mathcal{PML} \times \mathbb{R}$ the measured geodesic lamination $e^t q_v(x, \xi)$.

A conformal density $\{\nu^y\}$ on $\mathcal{PML}$ of dimension α defines a Radon measure Θ_ν on $\mathcal{ML}$ via $d\Theta_\nu(\xi, t) = d\nu^x(\xi) \times e^{\alpha t} dt$ where $\xi \in \mathcal{PML}, t \in \mathbb{R}$. By construction, this measure is quasi-invariant under the one-parameter group of translations T^s on $\mathcal{ML} = \mathcal{PML} \times \mathbb{R}$ given by $T^s(\xi, t) = (\xi, s+t)$. More precisely, we have $\frac{d\nu \circ T^s}{d\nu} = e^{\alpha s}$.

The measure Θ_ν is moreover invariant under the action of the mapping class group $MCG(S)$. Namely, for $\xi \in \mathcal{PML}$ and $g \in MCG(S)$ the measured geodesic lamination $g(q_v(x, \xi)) = q_v(g(x), g(\xi))$ equals $e^{\Psi(x, g(x), g(\xi))} q_v(x, g(\xi))$ where Ψ is the cocycle defined in the beginning of Section 3. On the other hand, we have

$$d(\Theta_\nu \circ g)(\xi, t) = d\nu^{g(x)}(g(\xi)) \times e^{\alpha t} dt.$$

By the definition of a conformal density, for ν^x-almost every $\xi \in \mathcal{PML}$ the Radon Nikodym derivative of the measure $\nu^{g(x)}$ with respect to ν^x at the point $g(\xi)$ equals $e^{\alpha \Psi(x, g(x), g(\xi))}$ and therefore the measure Θ_ν is indeed invariant under the action of

$MCG(S)$. As a consequence, every conformal density on $\mathcal{PML}$ induces a $MCG(S)$-invariant Radon measure on $\mathcal{ML} = \mathcal{PML} \times \mathbb{R}$ which is quasi-invariant under the one-parameter group of translations T^s.

Now let η be any ergodic $MCG(S)$-invariant Radon measure on $\mathcal{ML}$. Then

$$H_\eta = \{a \in \mathbb{R} \mid \eta \circ T^a \sim \eta\}$$

is a closed subgroup of $\mathbb{R}$ [ANSS02]. The next lemma is an easy consequence of Proposition 3.7 and Sarig's cocycle reduction theorem (Theorem 2 of [S04]). For its formulation, recall from Section 3 the definition of the space $\mathcal{FML}$ of all projective measured geodesic laminations which fill up S. We call a measured geodesic lamination $\lambda \in \mathcal{ML}$ *filling* if its projectivization is contained in $\mathcal{FML}$. The set of all filling measured geodesic laminations is invariant under the action of the mapping class group.

Lemma 5.1. *Let η be an ergodic $MCG(S)$-invariant Radon measure on $\mathcal{ML}$ which gives full mass to the filling measured geodesic laminations. If $H_\eta \neq \{0\}$ then η coincides with the Lebesgue measure up to scale.*

Proof. Define a measurable countable equivalence relation $\mathcal{R}$ on $\mathcal{PML}$ by $\chi\mathcal{R}\xi$ if and only if χ and ξ are contained in the same orbit for the action of the mapping class group. Recall the definition of the cocycle $\Psi : \mathcal{T}(S) \times \mathcal{T}(S) \times \mathcal{PML} \to \mathbb{R}$. For a fixed point $x \in \mathcal{T}(S)$ we obtain a real-valued cocycle for the action of $MCG(S)$ on $\mathcal{PML}$, again denoted by Ψ, via $\Psi(\lambda, g) = \Psi(x, g^{-1}x, \lambda)$ ($\lambda \in \mathcal{PML}$ and $g \in MCG(S)$). By the cocycle identity (5) for Ψ we have $\Psi(\lambda, hg) = \Psi(\lambda, g) + \Psi(g\lambda, h)$, i.e. Ψ is indeed a cocycle which can be viewed as a cocycle on $\mathcal{R}$. We also write $\Psi(\lambda, \xi)$ instead of $\Psi(\lambda, g)$ whenever $\xi = g\lambda$; note that this is only well defined if λ is not fixed by any element of $MCG(S)$, however this ambiguity will be of no importance in the sequel.

Recall that the choice of a point $x \in \mathcal{T}(S)$ determines a homeomorphism $\mathcal{ML} \to \mathcal{PML} \times \mathbb{R}$. The cocycle Ψ then defines an equivalence relation $\mathcal{R}_\Psi$ on $\mathcal{ML} = \mathcal{PML} \times \mathbb{R}$ by

$$\mathcal{R}_\Psi = \{((\lambda, t), (\xi, s)) \in (\mathcal{PML} \times \mathbb{R})^2 \mid (\lambda, \xi) \in \mathcal{R} \text{ and } s - t = \Psi(\lambda, \xi)\}.$$

Let η be an ergodic $MCG(S)$-invariant Radon measure on $\mathcal{ML}$ which gives full mass to the filling measured geodesic laminations. By the results in [ANSS02] (compare also the clear discussion on p.521 of [S04]), if $H_\eta = \mathbb{R}$ then η is induced by a conformal density as described in the beginning of this section. In particular, by Proposition 3.7, in this case the measure η equals the Lebesgue measure up to scale. Thus for the proof of our lemma we are left with the case that $H_\eta = c\mathbb{Z}$ for a number $c > 0$.

By the cocycle reduction theorem of Sarig (Theorem 2 of [S04]), in this case there is a Borel function $u : \mathcal{PML} \to \mathbb{R}$ such that $\Psi_u(x, y) = \Psi(x, y) + u(y) - u(x) \in H_\eta$ holds η-almost everywhere in $\mathcal{R}_\Psi$. Since $c > 0$ we may assume without loss of generality that the function u is *bounded*. Following [S04], for $a \in \mathbb{R}$ define $\theta_a(x, t) = (x, t - u(x) - a)$. By Lemma 2 of [S04], for a suitable choice of the number a the measure $\eta \circ \theta_a^{-1}$ is an $\mathcal{R}_{\Psi_u}$-invariant ergodic Radon measure supported on $\mathcal{PML} \times c\mathbb{Z}$.

We now follow Ledrappier and Sarig [LS06] (see also [Ld06]). Namely, since η is invariant and ergodic under the action of $MCG(S)$ and since the $\mathbb{R}$-action on $\mathcal{ML}$ commutes with the $MCG(S)$-action, for every $t \in \mathbb{R}$ the measure $\eta \circ T^t$ is also $MCG(S)$-invariant and ergodic. Thus either $\eta \circ T^t$ and η are singular or they coincide up to scale. As a consequence, there is some number $\alpha \in \mathbb{R}$ such that $\eta \circ T^c = e^{\alpha c}\eta$. Since $\theta = \theta_a$ and T^t commute, we also have $\eta \circ (\theta^{-1} \circ T^c) = e^{\alpha c}\eta \circ \theta^{-1}$. Consequently the measure $e^{-\alpha t}\eta \circ \theta^{-1}$ is invariant under the translation T^c. Since moreover $e^{-\alpha t}\eta \circ \theta^{-1}$ is supported in $\mathcal{PML} \times c\mathbb{Z}$, it follows that $e^{-\alpha t}\eta \circ \theta^{-1} = \nu \times m_{H_\eta}$ with some measure ν on $\mathcal{PML}$.

The measure ν is finite since $\eta \circ \theta^{-1}$ is Radon and the function u is bounded. The measure η is $MCG(S)$-invariant and therefore $\eta \circ \theta^{-1}$ is invariant under $\theta \circ MCG(S) \circ \theta^{-1}$. In particular, the measure class of ν is invariant under the action of $MCG(S)$. More precisely, we have

$$\frac{d\nu \circ g}{d\nu}(\xi) = e^{\alpha \Psi(\xi, g)} \frac{e^{-\alpha u(\xi)}}{e^{-\alpha u(g(\xi))}}$$

for all $g \in MCG(S)$ and ν-almost every $\xi \in \mathcal{PML}$ (see [LS06]). As a consequence, if we define $d\nu^x(\xi) = e^{\alpha u(\xi)} d\nu(\xi)$ and $d\nu^y(\xi) = e^{\alpha \Psi(x,y,\xi)} d\nu^x(\xi)$ for $y \in \mathcal{T}(S)$ then $\{\nu^y\}$ defines a conformal density of dimension α on $\mathcal{PML}$. Note that the measure $\nu^x = e^{\alpha u}\nu$ is finite since the function u is bounded. By our assumption on the measure η, the conformal density $\{\nu^x\}$ gives full measure to the $MCG(S)$-invariant set $\mathcal{FML}$ of projective measured geodesic laminations which fill up S. Hence we conclude from Proposition 3.7 that η equals the Lebesgue measure λ up to scale. However, the Lebesgue measure is quasi-invariant under the translations $\{T^t\}$ which is a contradiction to the assumption that $H_\eta = c\mathbb{Z}$ for some $c > 0$. This shows the lemma. □

The investigation of $MCG(S)$-invariant ergodic measures η on $\mathcal{ML}$ which give full measure to the filling measured geodesic laminations and satisfy $H_\eta = \{0\}$ is more difficult. We begin with an observation which is similar to Proposition 3.7. For this call a measured geodesic lamination *weakly recurrent* if its projectivization is contained in the set $\mathcal{RML}$ of recurrent projective measured geodesic laminations as defined in Section 3.

Lemma 5.2. *An $MCG(S)$-invariant Radon measure η on $\mathcal{ML}$ which gives full measure to the filling measured geodesic laminations gives full measure to the weakly recurrent measured geodesic laminations.*

Proof. Let η be an $MCG(S)$-invariant ergodic Radon measure on $\mathcal{ML}$ which gives full measure to the filling measured geodesic laminations. We use the measure η to construct a locally finite Borel measure ν on the moduli space $\mathcal{Q}(S)$ of quadratic differential of area one which is invariant under the horocycle flow. Namely, for every quadratic differential $q \in \mathcal{Q}^1(S)$ the assignment which associates to a quadratic differential z contained in the unstable manifold $W^u(q)$ its vertical measured geodesic lamination is a homeomorphism of $W^u(q)$ onto an open subset of $\mathcal{ML}$. Thus by equivariance under the action of the mapping class group, the measure η lifts to an $MCG(S)$-invariant family $\{\eta^u\}$ of locally finite measures on unstable manifolds $W^u(q)$ ($q \in \mathcal{Q}^1(S)$). This family of measures then projects to a family

$\{\nu^u\}$ of locally finite Borel measures on the leaves of the unstable foliation on $\mathcal{Q}(S)$. The family $\{\nu^u\}$ is invariant under holonomy along strong stable manifolds.

Define $d\nu = d\nu^u \times d\lambda^{ss}$ where λ^{ss} is a standard family of Lebesgue measures on strong stable manifolds which is invariant under the horocycle flow h_t. Then ν is a locally finite h_t-invariant Borel measure on $\mathcal{Q}(S)$. For ν-almost every point $q \in \mathcal{Q}(S)$ the horizontal and the vertical measured geodesic laminations of q fill up S. As in the proof of Proposition 3.7, Dani's argument together with the theorem from the appendix implies that ν is *finite*. Moreover, for every $\epsilon > 0$ there is a compact subset K_ϵ of $\mathcal{Q}(S)$ not depending on ν such that $\nu(K_\epsilon)/\nu(\mathcal{Q}(S)) \geq 1 - \epsilon$.

Via replacing ν by $\nu/\nu(\mathcal{Q}(S))$ we may assume that ν is a probability measure. For $s > 0$ define

$$\nu(s) = \frac{1}{s}\int_0^s \Phi^t \nu dt.$$

Since $h_t \circ \Phi^s = \Phi^s \circ h_{e^s t}$ for all $s, t \in \mathbb{R}$, the Borel probability measure $\nu(s)$ is h_t-invariant and gives full measure to the points with filling vertical measured geodesic laminations. Therefore we have $\nu(s)(K_\epsilon) \geq 1-\epsilon$ for all $s > 0$, all $\epsilon > 0$. This implies that there is a sequence $s_i \to \infty$ such that the measures $\nu(s_i)$ converge as $i \to \infty$ weakly to a Borel probability measure $\nu(\infty)$ on $\mathcal{Q}(S)$ which is invariant under both the horocycle flow and the Teichmüller geodesic flow. By the Poincaré recurrence theorem for the Teichmüller flow, $\nu(\infty)$ gives full measure to the forward *recurrent* quadratic differentials.

As a consequence, ν-almost every $q \in \mathcal{Q}(S)$ contains a forward recurrent point in its ω-limit set for the action of the Teichmüller geodesic flow. Namely, since $\nu(\infty)$ is Borel regular, for $\epsilon > 0$ there is a compact subset B_ϵ of $\mathcal{Q}(S)$ which consists of forward recurrent points and such that $\nu(\infty)(B_\epsilon) > 1 - \epsilon$. Let $\{U_\ell\}$ be a family of open neighborhoods of B_ϵ such that $U_\ell \supset U_{\ell+1}$ for all ℓ and $\cap_\ell U_\ell = B_\epsilon$. Then for every $\ell > 0$ there is some $i(\ell) > 0$ such that $\nu(s_i)(U_\ell) \geq 1 - 2\epsilon$ for all $i \geq i(\ell)$.

For $\ell > 0$ define $C_\ell = \{q \mid \Phi^t q \in U_\ell$ for infinitely many $t > 0\}$; then $C_\ell \supset C_{\ell+1}$ for all ℓ. We claim that $\nu(C_\ell) \geq 1 - 5\epsilon$ for every ℓ. Namely, otherwise there is a number $T > 0$ and there is a Borel subset A of $\mathcal{Q}(S)$ with $\nu(A) \geq 4\epsilon$ and such that $\Phi^t z \notin U_\ell$ for every $z \in A$ and every $t \geq T$. Then necessarily $\nu(s_i)(U_\ell) \leq 1 - 3\epsilon$ for all sufficiently large i which is impossible. Since the neighborhood U_ℓ of B_ϵ was arbitrary and since ν is Borel regular we conclude that $\nu(\cap_\ell C_\ell) \geq 1 - 5\epsilon$. Thus the ν-mass of all points $q \in \mathcal{Q}(S)$ which contain a point $z \in B_\epsilon$ in its ω-limit set is at least $1 - 5\epsilon$. Since B_ϵ consists of recurrent points and since $\epsilon > 0$ was arbitrary we conclude that ν-almost every $q \in \mathcal{Q}(S)$ contains a forward recurrent point for the Teichmüller geodesic flow in its ω-limit set. This shows the lemma. □

For the investigation of $MCG(S)$-invariant Radon measures η on $\mathcal{ML}$ with $H_\eta = \{0\}$ we use a construction reminiscent of symbolic dynamics where the Markov shift is replaced by complete train tracks and their splits. We next establish some technical preparations to achieve this goal.

Define a geodesic lamination ξ on S to be *complete* if ξ is maximal and can be approximated in the Hausdorff topology by simple closed geodesics. The space $\mathcal{CL}$ of all complete geodesic laminations equipped with the Hausdorff topology is a compact totally disconnected metrizable $MCG(S)$-space. Every minimal geodesic lamination λ is a sublamination of a complete geodesic lamination. If λ fills up S then the number of complete geodesic laminations which contain λ as a sublamination is bounded from above by a universal constant (Section 2.1 of [H08b]).

The set of all complete geodesic laminations which are carried by a complete train track τ is non-empty, open and closed in $\mathcal{CL}$ (Lemma 2.3 of [H08b]). This enables us to use complete train tracks for the construction of partitions of $\mathcal{CL}$ with properties similar to Markov partitions. To pass from complete geodesic laminations to projective measured geodesic laminations, let $\mathcal{H} \subset \mathcal{CL}$ be the set of all complete geodesic laminations $\lambda \in \mathcal{CL}$ which contain a uniquely ergodic minimal component which fills up S. It is not difficult to see that $\mathcal{H}$ is a Borel subset of $\mathcal{CL}$, however we do not need this fact in the sequel. We equip $\mathcal{H}$ with the subspace topology. The mapping class group $MCG(S)$ naturally acts on $\mathcal{H}$ as a group of transformations. There is a finite-to-one $MCG(S)$-equivariant map

$$E : \mathcal{H} \to \mathcal{FML} \tag{18}$$

which associates to every $\lambda \in \mathcal{H}$ the unique projective measured geodesic lamination which is supported in λ. The number of preimages in $\mathcal{H}$ of a point in $\mathcal{FML}$ is bounded from above by a universal constant.

Lemma 5.3. *The map E is continuous.*

Proof. Since the topology on $\mathcal{H}$ is metrizable, for the proof of the lemma it is enough to show the following. If $\{\lambda_i\} \subset \mathcal{H}$ is a sequence converging to some $\lambda \in \mathcal{H}$ then $E(\lambda_i) \to E(\lambda)$.

However, the space $\mathcal{PML}$ is compact and hence up to passing to a subsequence, we may assume that $E(\lambda_i) \to \mu \in \mathcal{PML}$. Then after passing to another subsequence, the supports of the projective measured geodesic lamination $E(\lambda_i)$ converge in the Hausdorff topology to a geodesic lamination containing the support of μ as a sublamination. But λ_i contains the support of $E(\lambda_i)$ as a closed subset and $\lambda_i \to \lambda$ in the Hausdorff topology and hence λ contains the support of μ as a sublamination. On the other hand, by definition of the set $\mathcal{H}$, the complete geodesic lamination λ contains a unique minimal component. This component fills up and is uniquely ergodic and hence $\mu = E(\lambda)$ is the unique projective measured geodesic lamination supported in λ. The lemma follows. □

As in Section 4, let $\mathcal{TT}$ be the set of isotopy classes of complete train tracks on S. A *full split* of a complete train track τ is a complete train track σ which can be obtained from τ by splitting τ at each large branch precisely once. A *full splitting sequence* is a sequence $\{\tau_i\} \subset \mathcal{TT}$ such that for each i, the train track τ_{i+1} can be obtained from τ_i by a full split. For a complete train track τ denote by $\mathcal{CL}(\tau)$ the set of all complete geodesic laminations which are carried by τ. Then $\mathcal{CL}(\tau)$ is a subset of $\mathcal{CL}$ which is both open and closed. If $\{\tau_i\}$ is an infinite full splitting sequence then $\cap_i \mathcal{CL}(\tau_i)$ consists of a unique point. For every complete train track τ and every complete geodesic lamination $\lambda \in \mathcal{CL}(\tau)$ there is a unique full splitting sequence $\{\tau_i(\lambda)\}$ issuing from $\tau_0(\lambda) = \tau$ such that $\cap_i \mathcal{CL}(\tau_i(\lambda)) = \{\lambda\}$. For every complete geodesic lamination $\lambda \in \mathcal{CL}$ there is a complete train track τ which carries λ (for all this, see [H08b]).

For $\tau \in \mathcal{TT}$ let as before $\mathcal{V}_0(\tau) \subset \mathcal{ML}$ be the set of all measured geodesic laminations ν whose support is carried by τ and such that the total weight of the transverse measure on τ defined by ν equals one. If the complete geodesic lamination $\lambda \in \mathcal{H}$ is carried by τ then there is a unique measured geodesic lamination $\nu(\lambda, \tau) \in \mathcal{V}_0(\tau)$ whose support is contained in λ. There is a number $a > 1$ only depending on the topological type of S such that for all $\lambda \in \mathcal{H} \cap \mathcal{CL}(\tau)$ and all $i \geq 0$ we have $\nu(\lambda, \tau_{i+1}(\lambda)) = e^s \nu(\lambda, \tau_i(\lambda))$ for some $s \in [0, a]$. Namely, $\tau_{i+1}(\lambda)$ is obtained from $\tau_i(\lambda)$ by a uniformly bounded number of splits. Moreover, if $\eta \in \mathcal{TT}$ is obtained from τ by a split at a large branch e, with losing branches b, d, and if ν is any transverse measure on η, then ν projects to a transverse measure $\hat{\nu}$ on τ with the following properties. Using the natural identification of the branches of τ with the branches of η, for every branch $h \neq e$ of τ, the $\hat{\nu}$-weight of h coincides with the ν-weight of the branch h in η. Moreover, we have $\hat{\nu}(e) = \nu(e) + \nu(b) + \nu(d)$.

As in Section 4, let $\mathcal{Q}(\tau) \subset \mathcal{Q}^1(S)$ be the set of all area one quadratic differentials whose vertical measured geodesic lamination is contained in $\mathcal{V}_0(\tau)$ and whose horizontal measured geodesic lamination hits τ efficiently. Let also $\mathcal{PE}(\tau)$ be the set of all projective measured geodesic laminations which hit τ efficiently. For a number $R > 0$ and some $\xi \in \mathcal{PE}(\tau)$, $\tau \in \mathcal{TT}$ is called $R - \xi$-tight if the diameter of $\mathcal{Q}(\tau) \cap W^u(\xi)$ with respect to the lift d of the Teichmüller metric is at most R.

Let $m > 0$ be the number of orbits of the action of $MCG(S)$ on $\mathcal{TT}$. The next technical observation is the main ingredient for the completion of the proof of the theorem from the introduction. It roughly says that controlled recurrence implies tightness. For its formulation, recall from Section 3 the definition of the set $\mathcal{RML}(q_0)$ for a forward recurrent $q_0 \in \mathcal{Q}(S)$.

Lemma 5.4. *Let $\tau \in \mathcal{TT}$ and let $q_1 \in \mathcal{Q}(\tau)$ be the lift of a forward recurrent point $q_0 \in \mathcal{Q}(S)$. Let ξ be the horizontal projective measured geodesic lamination of q_1. There are numbers $R > 2, k > 4R + 2ma$ depending on q_1 and for every complete geodesic lamination $\lambda \in \mathcal{CL}(\tau) \cap \mathcal{H}$ with $E(\lambda) \in \mathcal{RML}(q_0)$ there is a sequence $t_i \to \infty$ with the following properties.*

(1) *For each i there are numbers $j(i) > 0$, $\ell(i) > j(i)$ and there are numbers $s \in [t_i, t_i + ma], t \in [t_i + k, t_i + k + ma]$ such that $e^s \nu(\lambda, \tau) \in \mathcal{V}_0(\tau_{j(i)}(\lambda))$, $e^t \nu(\lambda, \tau) \in \mathcal{V}_0(\tau_{\ell(i)}(\lambda))$ and such that the train tracks $\tau_{j(i)}(\lambda), \tau_{\ell(i)}(\lambda)$ are $R - \xi$-tight.*
(2) *For every i there is some $g_i \in MCG(S)$ such that $g_i \tau_{j(i)}(\lambda) = \tau_{\ell(i)}(\lambda)$.*

Proof. By Lemma 2.2 and inequality (1) in Section 2, for all $q, z \in \mathcal{Q}^1(S)$ with $d(Pq, Pz) \leq a$ (where $a > 1$ is as above), the distance in $\mathcal{C}(S)$ between $\Upsilon_{\mathcal{Q}}(q)$ and $\Upsilon_{\mathcal{Q}}(z)$ is bounded from above by a universal constant $b > 0$. Let moreover $p > 0$ be the maximal distance in $\mathcal{C}(S)$ between any two vertex cycles of any complete train tracks η_1, η_2 on S so that either $\eta_1 = \eta_2$ or that η_2 can be obtained from η_1 by a full split. Let $\chi_2 > 0$ be as in Lemma 4.2 and let $\ell > 0$ be as in Lemma 4.3.

By assumption, the ω-limit set of the Φ^t-orbit of q_0 contains q_0. By Lemma 2.2 and the results of [Kl99, H06] we have $d(\Upsilon_{\mathcal{Q}}(\Phi^t q_1), \Upsilon_{\mathcal{Q}}(q_1)) \to \infty$ $(t \to \infty)$. Thus we can find small open relative compact neighborhoods $V \subset U$ of q_1 in $\mathcal{Q}^1(S)$, numbers $T_2 > T_1 > T_0 = 0$ and mapping classes $g_i \in MCG(S)$ such that for all $q \in V$ we have $\Phi^{T_i} q \in g_i U$ and

$$d(\Upsilon_{\mathcal{Q}}(\Phi^{T_i} q), \Upsilon_{\mathcal{Q}}(\Phi^{T_{i-1}} q)) \geq 2mp + \ell + 2\chi_2 + 2b \quad (i = 1, 2). \tag{19}$$

Let for the moment $q \in V$ be an arbitrary quadratic differential with vertical projective measured geodesic lamination contained in $E(\mathcal{H})$. Denote by q_h, q_v the horizontal measured geodesic lamination and the vertical measured geodesic lamination of q, respectively. Assume that there is a complete train track $\sigma \in \mathcal{TT}$ and a number $s_0 \in [0, a]$ with $\Phi^{s_0} q \in \mathcal{Q}(\sigma)$. Assume moreover that there is a complete geodesic lamination $\lambda \in \mathcal{CL}(\sigma)$ such that $E(\lambda)$ is the projective class of q_v. Then λ determines a full splitting sequence $\{\sigma_i(\lambda)\}$ issuing from $\sigma_0(\lambda) = \sigma$.

For $i = 1, 2$ and for $T_i > 0$ as above we can find some $j_2 > j_1 > j_0 = 0, s_i \in [0, a]$ such that $e^{T_i + s_i} q_v \in \mathcal{V}_0(\sigma_{j_i}(\lambda))$, which is equivalent to saying that $\Phi^{T_i + s_i} q \in \mathcal{Q}(\sigma_{j_i}(\lambda))$. By the choice of $b > 0$ we have

$$d(\Upsilon_{\mathcal{Q}}(\Phi^{T_i} q), \Upsilon_{\mathcal{Q}}(\Phi^{T_i + s_i} q)) \leq b$$

and therefore

$$d(\Upsilon_{\mathcal{Q}}(\Phi^{T_i + s_i} q), \Upsilon_{\mathcal{Q}}(\Phi^{T_{i-1} + s_{i-1}} q)) \geq 2mp + \ell + 2\chi_2 \quad (i = 1, 2)$$

by inequality (19).

Lemma 4.2 shows that $d(\Upsilon_{\mathcal{Q}}(\Phi^{T_i + s_i} q), \Psi(\sigma_{j_i}(\lambda))) \leq \chi_2$ and hence by the choice of p, for any $\alpha_i \in [j_i, j_i + m]$ the distance in $\mathcal{C}(S)$ between any vertex cycle of $\sigma_{\alpha_i}(\lambda)$ and any vertex cycle of $\sigma_{\alpha_{i-1}}(\lambda)$ is at least ℓ. Moreover, there is some $\tau_i \in [T_i, T_i + ma]$ such that $e^{\tau_i} q_v \in \mathcal{V}_0(\sigma_{\alpha_i}(\lambda))$ and hence the total weight of the complete train track $\sigma_{\alpha_2}(\lambda)$ defined by the measured geodesic lamination $e^{s_0} q_v$ which is carried by $\sigma_{\alpha_2}(\lambda)$ is bounded from below by $e^{-T_2 - ma}$.

Therefore Lemma 4.4, applied to the train tracks $\sigma_{\alpha_2}(\lambda) \prec \sigma_{\alpha_1}(\lambda) \prec \sigma$, shows the existence of a number $R > 0$ such that the train track $\sigma_{\alpha_1}(\lambda)$ is $R - [q_h]$-tight where $[q_h]$ is the projective class of the horizontal measured geodesic lamination of q. Note that the number $R > 0$ only depends on the point q_1 but not on $q \in V$ or the train track $\sigma \in \mathcal{TT}$ with $\Phi^{s_0} q \in \mathcal{Q}(\sigma)$ for some $s_0 \in [0, a]$.

For this number $R > 0$, choose a number $T_3 > T_2 + 4R + 2ma$ such that

$$d(\Upsilon_{\mathcal{Q}}(\Phi^{T_3} q_1), \Upsilon_{\mathcal{Q}}(\Phi^{T_2} q_1)) \geq 2mp + \ell + 2\chi_2 + b$$

and that $\Phi^{T_3} q_1 \in g_3 V$ for some $g_3 \in MCG(S)$; such a number exists since the projection q_0 of q_1 to $\mathcal{Q}(S)$ is forward recurrent. Choose an open neighborhood $W \subset V$ of q_1 such that $\Phi^{T_i} q \in g_i V$ for all $q \in W$ and $i = 1, 2, 3$. Write $T_4 = T_3 + T_1$ and $T_5 = T_3 + T_2$; then we have $\Phi^{T_j} q \in \cup_{h \in MCG(S)} hU$ for all $q \in W$ and every $j \in \{0, \dots, 5\}$. Define $k = T_4 - T_1 \geq 4R + 2ma$ and note that k only depends on q_1.

As above, let $\tau \in \mathcal{TT}$ be a train track which carries the vertical projective measured geodesic lamination of q_1 and such that the horizontal projective measured geodesic lamination ξ of q_1 hits τ efficiently. Let $\lambda \in \mathcal{CL}(\tau) \cap \mathcal{H}$ be such that $E(\lambda) \in \mathcal{RML}(q_0)$. Let $q \in \mathcal{Q}(\tau) \cap W^u(\xi)$ be such that the vertical projective measured geodesic lamination of q equals $E(\lambda)$. For the neighborhood $W \subset \mathcal{Q}^1(S)$ of q_1 as above, the set $\{t > 0 \mid \Phi^t q \in \cup_{h \in MCG(S)} hW\}$ is unbounded. Let $\{r_n\} \subset [0, \infty)$ be a sequence tending to infinity such that $\Phi^{r_n} q \in h_n W$ for some $h_n \in MCG(S)$ and every $n > 0$.

Let $\{\tau_i(\lambda)\}$ be the full splitting sequence determined by $\tau = \tau_0(\lambda)$ and λ. For $n > 0$ we have $\Phi^{r_n} q \in h_n W$. There is a number $s_0 = s_0(n) \in [0, a]$ and a number $j_0(n) > 0$ such that $\Phi^{r_n + s_0} q \in \mathcal{Q}(\tau_{j_0(n)}(\lambda))$. Using the above constants $T_i > 0$, there are numbers $j_5(n) > j_4(n) > j_3(n) > j_2(n) > j_1(n) > j_0(n)$ and numbers $s_i \in [0, a]$ such that $\Phi^{T_i + s_i + r_n} q \in \mathcal{Q}(\tau_{j_i(n)}(\lambda))$ $(i = 1, 2, 3, 4, 5)$. Since there are only m distinct orbits of complete train tracks under the action of the mapping class group, there are moreover numbers $j(n) \in [j_1(n), j_1(n) + m]$ and $\ell(n) \in [j_4(n), j_4(n) + m]$ and there is some $g \in MCG(S)$ such that $g\tau_{j(n)}(\lambda) = \tau_{\ell(n)}(\lambda)$. By the above consideration, the train tracks $\tau_{j(n)}(\lambda), \tau_{\ell(n)}(\lambda)$ are $R - \xi$-tight. Then the sequence $t_n = r_n + T_1$ and the numbers $j(n) > 0, \ell(n) > j(n)$ have the required properties stated in the lemma with $k = T_4 - T_1 > 4R + 2ma$. □

Now we are ready to complete the main step in the proof of the theorem from the introduction.

Proposition 5.5. *An $MCG(S)$-invariant Radon measure on $\mathcal{ML}$ which gives full mass to the filling measured geodesic laminations coincides with the Lebesgue measure up to scale.*

Proof. By Lemma 5.1 and Lemma 5.2, we only have to show that there is no $MCG(S)$-invariant ergodic Radon measure on $\mathcal{ML}$ with $H_\eta = \{0\}$ which gives full mass to the recurrent measured geodesic laminations.

For this we argue by contradiction and we assume that such a Radon measure η exists. Using once more the cocycle reduction theorem of Sarig (Theorem 2 of [S04]), there is a Borel function $u : \mathcal{PML} \to \mathbb{R}$ such that the measure η gives full mass to the graph $\{(x, u(x)) \mid x \in \mathcal{PML}\}$ of the function u. In particular, η projects to an $MCG(S)$-invariant ergodic measure class $\hat{\eta}$ on $\mathcal{PML}$ which gives full mass to the set $\mathcal{RML}$ of recurrent points. Using the notations from Proposition 3.5 and its proof, this means that there is a forward recurrent point $q_0 \in \mathcal{Q}(S)$ such that the measure class $\hat{\eta}$ on $\mathcal{PML}$ gives full measure to the set $\mathcal{RML}(q_0)$.

To derive a contradiction we adapt the arguments of Ledrappier and Sarig [LS06] to our situation. Denote again by $\mathcal{H} \subset \mathcal{CL}$ the set of all complete geodesic laminations containing a uniquely ergodic minimal component which fill up S and let $E : \mathcal{H} \to \mathcal{FML}$ be as in (18). Every finite Borel measure μ on $\mathcal{PML}$ which gives full mass to the returning projective measured geodesic laminations induces a finite Borel measure $\tilde{\mu}$ on $\mathcal{H}$. Namely, returning projective measured geodesic laminations are uniquely ergodic and hence for a Borel subset C of $\mathcal{H}$ we can define

$$\tilde{\mu}(C) = \int_{E(C)} \sharp(E^{-1}(z) \cap C) d\mu(z). \tag{20}$$

Let again $q_0 \in \mathcal{Q}(S)$ be a forward recurrent quadratic differential such that the measure class $\hat{\eta}$ gives full mass to $\mathcal{RML}(q_0)$. Let $q_1 \in \mathcal{Q}^1(S)$ be a lift of q_0. We may assume that the horizontal projective measured geodesic lamination ξ of q_1 is uniquely ergodic and fills up S. Since there is some complete train track σ which carries the vertical measured geodesic lamination of q_1 and such that the horizontal measured geodesic lamination of q_1 hits τ efficiently, Lemma 5.4 shows that by possibly replacing q_1 by $\Phi^t q_1$ for some (large) number $t > 0$ we may assume that there is a number $R_0 > 0$ and there is an $R_0 - \xi$-tight train track $\tau \in \mathcal{TT}$ with $q_1 \in \mathcal{Q}(\tau)$.

As before, the measure η on $\mathcal{ML}$ induces a locally finite Borel measure η^u on $W^u(q_1) = W(\xi)$. Let $\rho \in \mathbb{R}$ be such that $\cup_{\rho \leq s \leq \rho+1} \Phi^s W^{su}(q_1)$ contains some density point of the locally finite measure η^u on $W^u(q_1)$ whose horizontal measured geodesic lamination is carried by the train track τ. Such a number exists since by invariance under the mapping class group, the measure class $\hat{\eta}$ on $\mathcal{PML}$ is of full support and since moreover the set of all measured geodesic laminations carried by τ has non-empty interior.

Let $\hat{\mu}_0$ be the restriction of the measure η^u to the set $\cup_{\rho \leq s \leq \rho+1} \Phi^s W^{su}(q_1) \subset W^u(q_1)$. Since τ is $R_0 - \xi$-tight for some $R_0 > 0$, the set $\mathcal{Q}(\tau) \cap W^u(q_1)$ is relative compact and therefore the intersection of the support of $\hat{\mu}_0$ with $\cup_t \Phi^t \mathcal{Q}(\tau)$ is relative compact as well.

Denote by μ_0 the projection of $\hat{\mu}_0$ to $\mathcal{PML}$ via the map $\pi : \mathcal{Q}^1(S) \to \mathcal{PML}$ introduced in Section 3. Since η is a Radon measure on $\mathcal{ML}$ by assumption, the measure μ_0 is a locally finite Borel measure on $\pi W^u(q_1) \subset \mathcal{PML} - \xi$ which gives full mass to $\mathcal{RML}(q_0)$. Since η and hence $\hat{\mu}_0$ is a Radon measure, the total μ_0-mass of the set of all projective measured geodesic laminations which are carried by the train track τ is finite. By the definition in (20) above, μ_0 induces a finite nontrivial Borel measure $\tilde{\mu}_0$ on $\mathcal{CL}(\tau)$ which gives full mass to the set of complete geodesic laminations $\zeta \in \mathcal{H} \cap \mathcal{CL}(\tau)$ with $E(\zeta) \in \mathcal{RML}(q_0)$.

Similarly, for the constants $R > 2, m > 0, k > 4R + 2ma \geq 4R + ma + 1$ as in Lemma 5.4, let $\tilde{\mu}_1$ be the finite Borel measure on $\mathcal{CL}(\tau)$ which is induced in the above way from the restriction of η^u to $\cup_{\rho+k-2R-2ma \leq s \leq \rho+k+2R+2ma} \Phi^s W^{su}(q_1)$. Since $H_\eta = \{0\}$ by assumption, the measures $\tilde{\mu}_0, \tilde{\mu}_1$ are singular.

Define a *cylinder* in $\mathcal{CL}(\tau)$ to be a set of the form $\mathcal{CL}(\sigma)$ where $\sigma \in \mathcal{TT}$ is a complete train track which can be obtained from τ by a full splitting sequence. A cylinder is a subset of $\mathcal{CL}$ which is both open and closed [H08b]. The intersection of two cylinders is again a cylinder. Since every point in $\mathcal{CL}(\tau)$ is an intersection of countably many cylinders, the σ-algebra on $\mathcal{CL}(\tau)$ generated by cylinders is the usual Borel σ-algebra. Now $\tilde{\mu}_0, \tilde{\mu}_1$ are mutually singular Borel measures on $\mathcal{CL}(\tau)$ and hence there is a cylinder $\mathcal{CL}(\sigma) \subset \mathcal{CL}(\tau)$ such that $\tilde{\mu}_0(\mathcal{CL}(\sigma)) > 2\tilde{\mu}_1(\mathcal{CL}(\sigma))$.

By Lemma 5.4, for $\tilde{\mu}_0$-almost every $\lambda \in \mathcal{CL}(\sigma)$ there is a sequence $j(i) \to \infty$ and there is some $g(i, \lambda) \in MCG(S)$ with the following properties.

(1) The train tracks $\tau_{j(i)}(\lambda), g(i, \lambda)\tau_{j(i)}(\lambda)$ are $R - \xi$-tight.

(2) $g(i,\lambda)\tau_{j(i)}(\lambda)$ is carried by $\tau_{j(i)}(\lambda)$, in particular we have $g(i,\lambda)\mathcal{CL}(\tau_{j(i)}(\lambda)) \subset \mathcal{CL}(\tau_{j(i)}(\lambda))$.
(3) For $q \in \mathcal{Q}(g(i,\lambda)\tau_{j(i)}(\lambda)) \cap W^u(q_1)$ there is some $t \in [k - ma - 2R, k + ma + 2R]$ such that $e^{-t}q \in \mathcal{Q}(\tau_{j(i)}(\lambda)) \cap W^u(q_1)$.

To see the third property, note that by the construction of the sequence $j(i)$ in Lemma 5.4, for each i there is a point $q \in \mathcal{Q}(\tau) \cap W^u(q_1)$ and there are numbers $s > 0, t \in [s+k-ma, s+k+ma]$ such that $\Phi^s q \in \mathcal{Q}(\tau_{j(i)}(\lambda)), \Phi^t \in \mathcal{Q}(g(i,j)\tau_{j(i)}(\lambda))$. Together with properties 1) and 2), this yields property 3).

The above three properties imply the following. Let $q \in \cup_{\rho \leq t \leq \rho+1} \Phi^t W^{su}(q_1)$ be such that the vertical measured geodesic lamination q_v of q is carried by $\tau_{j(i)}(\lambda)$. Let $s \in \mathbb{R}$ be such that $\Phi^s q \in \mathcal{Q}(\tau_{j(i)}(\lambda))$ and let $z = W^{ss}(g(i,\lambda)\Phi^s q) \cap W^u(q_1)$; then the vertical measured geodesic lamination of z is carried by $\tau_{j(i)}(\lambda)$, and we have $z \in \Phi^{t+s} W^{su}(q_1)$ for some $t \in [k - 2R - ma, k + 2R + ma]$. Since $k > 4R + ma + 1$ it is now immediate from invariance of the measure η under the action of $MCG(S)$, from the definitions of the measures $\tilde{\mu}_0, \tilde{\mu}_1$ and the fact that the action of $MCG(S)$ on $\mathcal{ML}$ commutes with the action of the group of translations that $\tilde{\mu}_0(\mathcal{CL}(\tau_{j(i)}\lambda)) \leq \tilde{\mu}_1(\mathcal{CL}(\tau_{j(i)}\lambda))$.

On the other hand, there is a countable partition of a subset of $\mathcal{CL}(\sigma)$ of full $\tilde{\mu}_0$-mass into cylinders $\mathcal{CL}(\sigma_i)$ with train tracks $\sigma_i \in \mathcal{TT}$ $(i > 0)$ which can be obtained from σ by a full splitting sequence and which satisfy 1),2),3) above. This partition can inductively be constructed as follows. Beginning with the train track σ, there is a full splitting sequence of minimal length $n \geq 0$ issuing from σ which connects σ to some train track $\sigma_1 \in \mathcal{TT}$ with the above properties. Let $\eta_1, \dots, \eta_k$ be the collection of all train tracks which can be obtained from σ by a full splitting sequence of length n and assume after reordering that we have $\sigma_1 = \eta_1$. Repeat this construction simultaneously with the train tracks $\eta_2, \dots, \eta_k$. After countably many steps we obtain a partition of $\tilde{\mu}_0$-almost all of $\mathcal{CL}(\sigma)$ as required.

Together we conclude that necessarily $\tilde{\mu}_1(\mathcal{CL}(\sigma)) \geq \tilde{\mu}_0(\mathcal{CL}(\sigma))$ which contradicts our choice of σ. In other words, the case $H_\eta = \{0\}$ is impossible which completes the proof of the proposition. □

Remark: Let $q_0 \in \mathcal{Q}(S)$ be a forward recurrent point. The arguments in the proof of Proposition 5.5 can be used to construct a Vitali relation for the lift to $\mathcal{CL}$ of any Borel measure on $\mathcal{RML}(q_0)$. In other words, with some extra arguments, Proposition 3.5 can be deduced from Proposition 5.5 and its proof. However, we included Proposition 3.5 in the present form since its basic idea is simpler and more geometric, moreover it is used in [H08c].

Choose a complete hyperbolic metric on S of finite volume. Let S_0 be a proper bordered connected subsurface of S with geodesic boundary. Then S_0 has negative Euler characteristic. We do not require that S_0 is connected, and we allow that distinct boundary components of S_0 are defined by the same simple closed geodesic in S. Denote by $MCG(S_0)$ (or $MCG(S - S_0)$) the subgroup of $MCG(S)$ of all elements which can be represented by a diffeomorphism fixing $S - S_0$ (or S_0) pointwise (by convention, a Dehn twist about a boundary component of S_0 is not contained in

$MCG(S_0)$). The stabilizer $\mathrm{Stab}(S_0)$ of S_0 in $MCG(S)$ contains a subgroup of finite index of the form $MCG(S_0) \times MCG(S - S_0) \times \mathcal{D}(\partial S_0)$ where $\mathcal{D}(\partial S_0)$ is the free abelian group of Dehn twists about the geodesics in S which define the boundary of S_0.

Let $\widehat{S}_0$ (or $\widehat{S - S_0}$) be the surface of finite type which we obtain from S_0 (or $S - S_0$) by collapsing each boundary circle to a puncture. The space $\mathcal{ML}(S_0)$ of all measured geodesic laminations on $\widehat{S}_0$ can be identified with the space of all measured geodesic laminations on S whose support is contained in the interior of S_0. We say that a measured geodesic lamination $\nu \in \mathcal{ML}(S_0)$ *fills* S_0 if its support intersects every simple closed geodesic contained in the interior of S_0 transversely. There is a family of Lebesgue measures on $\mathcal{ML}(S_0)$ which are invariant under $\mathrm{Stab}(S_0)$. These measures are products of Lebesgue measures on the spaces of measured geodesic laminations which are supported on a connected component of S_0. Note that $\mathrm{Stab}(S_0)$ may contain elements which permutes some of the components of S_0, so perhaps not every product measure is $\mathrm{Stab}(S_0)$-invariant. For every $\varphi \in MCG(S) - \mathrm{Stab}(S_0)$, the image $\varphi(S_0)$ of S_0 under φ is a subsurface of S which is distinct from S_0. The image $\varphi(\zeta)$ under φ of a measured geodesic lamination ζ on S_0 which fills S_0 is a measured geodesic lamination which fills $\varphi(S_0)$ and hence this image is not contained in $\mathcal{ML}(S_0)$. Note that we have $\mathrm{Stab}(\varphi(S_0)) = \varphi \circ \mathrm{Stab}(S_0) \circ \varphi^{-1}$.

Now let c be any (possibly trivial) simple weighted geodesic multicurve on S which is disjoint from the interior of S_0. Then for every $\zeta \in \mathcal{ML}(S_0)$ the union $c \cup \zeta$ is a measured geodesic lamination on S in a natural way which we denote by $c \times \zeta$. Thus $c \times \mathcal{ML}(S_0)$ is naturally a closed subspace of $\mathcal{ML}$. This subspace can be equipped with a $\mathrm{Stab}(c \cup S_0) < \mathrm{Stab}(S_0)$-invariant ergodic Radon measure μ_{c,S_0} induced by a measure μ_0 from the cone of $\mathrm{Stab}(S_0)$-invariant Lebesgue measures on $\mathcal{ML}(S_0)$. By invariance, we obtain an $MCG(S)$-invariant ergodic wandering measure on $\mathcal{ML}$ by defining

$$\lambda_{c \times S_0} = \sum_{\varphi \in MCG(S)} \varphi_* \mu_{c,S_0}.$$

We call $\lambda_{c \times S_0}$ a *standard subsurface measure* of S_0. If the support of the weighted multi-curve c contains every boundary component of S_0 then we call the resulting $MCG(S)$-invariant measure $\lambda_{c \times S_0}$ on $\mathcal{ML}$ a *special standard subsurface measure* on $\mathcal{ML}$.

Recall from the introduction that a *rational* $MCG(S)$-invariant measure on $\mathcal{ML}$ is a sum of weigthed Dirac masses supported on the orbit of a simple weighted multi-curve. Such a rational measure is a special standard subsurface measure on $\mathcal{ML}$ (for the empty subsurface). Then rational measures are included in the following statement.

Lemma 5.6. *A special standard subsurface measure on $\mathcal{ML}$ is locally finite.*

Proof. Let g be any complete hyperbolic metric on S of finite volume. Then for every measured geodesic lamination μ on S the *g-length* $\ell_g(\mu)$ of μ is defined. By definition, this length is the total mass of the measure on S which is the product of the transverse measure for μ and the hyperbolic length element on the geodesics contained in the support of μ. For every *compact* subset K of $\mathcal{ML}$ there is a number $m > 0$ such that $K \subset K(m) = \{\mu \in \mathcal{ML} \mid \ell_g(\mu) \leq m\}$.

To show the lemma, observe first that for every weighted geodesic multi-curve c on S and every $m > 0$ the set $K(m)$ contains only finitely many images of c under the action of the mapping class group. Namely, let $a > 0$ be the minimal weight of a component of the support of c. Then for every $\varphi \in MCG(S)$ the g-length of the multi-curve $\varphi(c)$ is not smaller than a times the maximal length of any closed geodesic on the hyperbolic surface (S, g) which is contained in the free homotopy class defined by a component of $\varphi(c)$. However, there are only finitely many simple closed geodesics on S whose g-length is at most m/a and hence the intersection of $K(m)$ with the $MCG(S)$-orbit of c is indeed finite. In particular, a rational $MCG(S)$-invariant measure on $\mathcal{ML}$ is Radon.

Now let S_0 be a bordered subsurface of S of negative Euler characteristic and geodesic boundary. Let c be a simple weighted geodesic multi-curve which contains every boundary component of S_0. Then the stabilizer $\mathrm{Stab}(c)$ of c in $MCG(S)$ contains the stabilizer $\mathrm{Stab}(c \cup S_0)$ of $c \cup S_0$ as a subgroup of finite index. In particular, a $\mathrm{Stab}(S_0)$-invariant measure on $\mathcal{ML}(S_0)$ in the Lebesgue measure class induces an ergodic $\mathrm{Stab}(c)$-invariant Radon measure on the space of measured geodesic laminations on S containing c as a measured sublamination.

On the other hand, if $\varphi \in MCG(S)$ *does not* stabilize c then φ moves at least one component of c away from c. Therefore by the above consideration, there are only *finitely many* cosets in $MCG(S)/\mathrm{Stab}(c)$ containing some representative φ such that $\varphi(c \times \mathcal{ML}(S_0)) \cap K(m) \neq \emptyset$. This shows that a special subsurface measure on $\mathcal{ML}$ is Radon. $\square$

Finally we are able to complete the proof of the theorem from the introduction (note that in the spirit of Lemma 5.6, there is a slight redundancy in its statement).

Theorem 5.7. *Let η be an $MCG(S)$-invariant ergodic Radon measure on the space $\mathcal{ML}$ of all measured geodesic laminations on S.*

(1) *If η is non-wandering then η is the Lebesgue measure up to scale.*
(2) *If η is wandering then either η is rational or η is a standard subsurface measure.*

Proof. Let η be an ergodic $MCG(S)$-invariant Radon measure on $\mathcal{ML}$. By Proposition 5.5, if η gives full mass to the measured geodesic laminations which fill up S then η coincides with the Lebesgue measure up to scale. Thus by ergodicity and invariance we may assume that η gives full mass to the measured geodesic laminations which do not fill up S.

Let λ be a density point for η. The support of λ is a union of components $\lambda_1 \cup \cdots \cup \lambda_k$ for some $k \geq 1$. We assume that these components are ordered in such a way that there is some $\ell \leq k$ such that the components $\lambda_1, \ldots, \lambda_\ell$ are minimal arational and that the components $\lambda_{\ell+1}, \ldots, \lambda_k$ are simple closed curves. By ergodicity, for η-almost every $\mu \in \mathcal{ML}$ there is a decomposition of the support of μ of the same form.

If $\ell = 0$ then by ergodicity, η is rational and there is nothing to show. Thus assume that $\ell > 0$. Then $\mu = \lambda_1 \cup \cdots \cup \lambda_\ell$ *fills* a subsurface S_0 of S with ℓ connected components, each of which is of negative Euler characteristic.

Write $c = \lambda_{\ell+1} \cup \cdots \cup \lambda_k$. Since the $MCG(S)$-orbit of c is countable and discrete and since λ is a density point for η, the restriction of η to $c \times \mathcal{ML}(S_0)$ does not vanish. However, this restriction is a Stab$(c \cup S_0)$-invariant Radon measure on $MCG(S_0)$ which gives full mass to the measured geodesic laminations filling up S_0. Therefore this restriction is an interior point of the cone of Stab$(c \cup S_0)$-invariant Lebesgue measure on $c \times \mathcal{ML}(S_0)$. In other words, η is a standard subsurface measure. This completes the proof of the theorem. □

Remark: In [LM07], Lindenstrauss and Mirzakhani obtain a stronger result. They show that a locally finite standard subsurface measure on $\mathcal{ML}$ is special.

Appendix

The purpose of this appendix is to present some results from the paper [MW02] of Minsky and Weiss in the form needed in Section 3.

As in Section 2, denote by $\mathcal{Q}(S)$ the moduli space of area one holomorphic quadratic differentials on S. Every $q \in \mathcal{Q}(S)$ defines an isometry class of a singular euclidean metric on S. The set Σ of singular points for this metric coincides precisely with the set of zeros for q. We also assume that the differential has a simple pole at each of the punctures of S and hence it can be viewed as a meromorphic quadratic differential on the compactified surface $\hat{S}$ which we obtain by filling in the punctures in the standard way.

A *saddle connection* for q is a path $\delta : (0,1) \to S$ which does not contain any singular point, whose image under a distinguished isometric chart is an euclidean straight line and which extends continuously to a path $\bar{\delta} : [0,1] \to \hat{S}$ mapping the endpoints to singular points or punctures. A saddle connection is *horizontal* if it is mapped by a distinguished chart to a horizontal line segment.

A saddle connection does not have self-intersections. Two saddle connections δ_1, δ_2 are *disjoint* if $\delta_1(0,1) \cap \delta_2(0,1) = \emptyset$. The closure of any finite collection of pairwise disjoing saddle connections on S is an embedded graph in S. By Proposition 4.7 of [MW02] (see also [KMS86]), the number of pairwise disjoint saddle connections for a quadratic differential $q \in \mathcal{Q}(S)$ is bounded from above by a universal constant $M > 0$ only depending on the topology of S.

Recall that a *tree* is a graph without circuits. For $\epsilon > 0$ let $K(\epsilon) \subset \mathcal{Q}(S)$ be the set of all quadratic differentials q such that the collection of all saddle connections of q of length at most ϵ is a tree. We have.

Lemma. *For every $\epsilon > 0$ the set $K(\epsilon) \subset \mathcal{Q}(S)$ is compact.*

Proof. It is enough to show that for every $q \in K(\epsilon)$ the q-length of any simple closed curve on S is bounded from below by ϵ (see [R05, R07]).

Thus let c be any simple closed geodesic on S for the q-metric. Then up to replacing c by a freely homotopic simple closed curve of the same length we may assume that c consists of a sequence of saddle connections for q. Since the set of saddle connections of length at most ϵ does not contain a circuit, the curve c contains at least one saddle connection of length at least ϵ. But this just means that the q-length of c is at least ϵ as claimed. □

The following proposition is a modified version of Theorem 6.3 of [MW02]. We use the notations from [MW02]. Let $\mathcal{L}_q$ be the set of all saddle connections of the quadratic differential q. For $k \geq 1$ define

$$\mathcal{E}_k = \{E \subset \mathcal{L}_q \mid E \text{ consists of } k \text{ disjoint segments}\}.$$

Denote again by h_t the horocycle flow on $\mathcal{Q}(S)$. For $E \in \mathcal{E}_k$ and $t \in \mathbb{R}$ define $\ell_{q,E}(t) = \max_{\delta\in E} \ell_{q,\delta}(t)$ where $\ell_{q,\delta}(t)$ is the length of δ with respect to the singular euclidean metric defined by $h_t q$. For $k \geq 0$ let

$$\alpha_k(t) = \min_{E\in\mathcal{E}_k} \ell_{q,E}(t).$$

Proposition. *There are positive constants C, α, ρ_0 depending only on S with the following property. Let $q \in \mathcal{Q}(S)$, let $I \subset \mathbb{R}$ be an interval and let $0 < \rho' \leq \rho_0$. Define*

$$A = \{\delta \in \mathcal{L}_q \mid \ell_{q,\delta}(t) \leq \rho'\} \quad \textit{for all } t \in I.$$

If $\cup\{\overline{\delta} \mid \delta \in A\} \subset S$ is an embedded tree with $r \geq 0$ edges then for any $0 < \epsilon < \rho'$ we have:

$$|\{t \in I \mid \alpha_{r+1}(t) < \epsilon\}| \leq C\big(\frac{\epsilon}{\rho'}\big)^{\alpha}|I|.$$

Proof. Let $M > 0$ be such that for every $q \in \mathcal{Q}(S)$ the number of pairwise disjoint saddle connections of q is bounded from above by $M - 1$. By Proposition 6.1 of [MW02] there is a number $\rho_0 > 0$ with the following property. If $E \in \mathcal{E}_k$ is such that the closure $S(E)$ of the union of all simply connected components of $S - \cup_{\delta\in E}\overline{\delta}$ is all of S then $\ell_{q,E}(0) \geq \rho_0$.

Let $q \in \mathcal{Q}(S)$ and let $A \subset \mathcal{L}_q$ be a union of pairwise disjoint saddle connections whose closure is an embedded graph in S without circuits. Assume that A consists of $r \geq 0$ segments. We necessarily have $r < M$. For a number $C > 0$ to be determined later let $0 < \epsilon < C\rho'$ and let

$$V_\epsilon = \{t \in I \mid \alpha_{r+1}(t) < \epsilon\}.$$

For $k = 1, \dots, M - r - 1$ define

$$L_k = \epsilon\big(\frac{\rho'}{\epsilon}\big)^{\frac{k-1}{M-r-1}}.$$

We choose $C > 0$ in such a way that $L_k/L_{k+1} \leq C^{\frac{1}{M-r-1}}$. For $t \in V_\epsilon$ let

$$\kappa(t) = \max\{k \mid \alpha_k(t) < L_k\}$$

and let $V_k = \{t \in V_\epsilon \mid \kappa(t) = k\}$. Then $\kappa(t) \leq M - 1$ for all t and hence V_ϵ is the disjoint union of the measurable sets $V_{k+1}, \dots, V_{M-1}$. Thus there is some $k \in \{r+1, \dots, M-1\}$ for which

$$|V_k| \geq \frac{|V_\epsilon|}{M - r - 1}.$$

For this choice of k define $L = L_k$ and $U = L_{k+1}$. Note that we have

$$\alpha_{\kappa(t)}(t) < L_{\kappa(t)}, \quad \alpha_{\kappa(t)+1} \geq L_{\kappa(t)+1}.$$

Following [MW02], for $\delta \in \mathcal{L}_\delta - A$ let $H(\delta)$ be the set of $t \in I$ for which $\ell_{q,\delta}(t) < L$, and whenever $\delta \cap \delta' \neq \emptyset$ for $\delta \neq \delta' \in \mathcal{L}_q$ we have

$$\ell_{q,\delta'}(t) \geq \frac{U\sqrt{2}}{3}.$$

Following the argument in Section 6 of [MW02] we only have to verify that $V_k \subset \cup_{\delta \in \mathcal{L}_q - A} H(\delta)$. Namely, let $t \in V_k$ and let $E \in \mathcal{E}_k$ be such that $\ell_{q,E}(t) = \alpha_k(t) < L$. Denote by $S(E)$ the closure of the union of the simply connected components of $S - \cup_{\delta \in E} \overline{\delta}$. By Proposition 6.1 of [MW02] we have $S(E) \neq E$ and hence since $k > r$ and the graph defined by the saddle connections contained in A does not have circuits, the boundary of $S(E)$ contains at least one saddle connection δ which is *not* contained in A. But this just means that $t \in H(\delta)$ (see Claim 6.7 in [MW02]). This complete the proof of the proposition. □

As in [MW02] we use the lemma and the proposition to derive a recurrency property for the horocycle flow. For its formulation, denote by χ_C the characteristic function of the set $C \subset \mathcal{Q}(S)$.

Theorem. *For any $\epsilon > 0$ there is a compact set $K \subset \mathcal{Q}(S)$ such that for any $q \in \mathcal{Q}(S)$ with minimal vertical measured geodesic lamination which fills S we have*

$$\mathrm{Avg}_{t,q}(K) = \liminf_{t \to \infty} \frac{1}{t} \int_0^t \chi_K(h_t q) dt \geq 1 - \epsilon.$$

Proof. Let q be a quadratic differential with vertical measured geodesic lamination which fills up S. Then the vertical saddle connections of q form an embedded graph without circuits. Morever, the number of these saddle connections is bounded from above by a universal constant. Now if δ is any saddle connection whose length is constant along the horocycle flow then δ is vertical. But this just means that we can apply the above proposition as in the proof of Theorem H2 of [MW02] to obtain the theorem. □

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