STABILITY OF QUASI-GEODESICS IN TEICHMÜLLER SPACE

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Abstract. Let S be a surface S of genus $g \geq 0$ with $m \geq 0$ punctures and $3g-3+m \geq 2$. We show that a Teichmüller quasi-geodesic in the thick part of Teichmüller space for S is contained in a bounded neighborhood of a geodesic if and only if it induces a quasi-geodesic in the curve graph.

1. Introduction

Let S be an oriented surface of finite type, i.e. S is a closed surface of genus $g \geq 0$ from which $m \geq 0$ points, so-called *punctures*, have been deleted. We assume that $3g - 3 + m \geq 2$, i.e. that S is not a sphere with at most four punctures or a torus with at most one puncture. We then call the surface S *nonexceptional.*

Since the Euler characteristic of S is negative, the *Teichmüller space* $\mathcal{T}(S)$ of S is the quotient of the space of all complete hyperbolic metrics on S of finite volume under the action of the group of diffeomorphisms of S which are isotopic to the identity. The *mapping class group* $\mathrm{Mod}(S)$ of all isotopy classes of orientation preserving diffeomorphisms of S acts properly discontinuously on $\mathcal{T}(S)$. For every $\epsilon > 0$, this action preserves the subset $\mathcal{T}(S)_\epsilon$ of $\mathcal{T}(S)$ of all hyperbolic metrics whose *systole*, i.e. the shortest length of a closed geodesic, is at least ϵ. Moreover, the action of $\mathrm{Mod}(S)$ on $\mathcal{T}(S)_\epsilon$ is cocompact.

The *Teichmüller metric* $d_{\mathcal{T}}$ is a complete length metric on $\mathcal{T}(S)$ with the property that any two points in $\mathcal{T}(S)$ can be connected by a unique geodesic. For a number $L > 1$, an *L-quasi-geodesic* in $\mathcal{T}(S)$ is a map $\gamma : J \to \mathcal{T}(S)$ such that

$$|s-t|/L - L \leq d_{\mathcal{T}}(\gamma(s), \gamma(t)) \leq L|s-t| + L \text{ for all } s, t \in J$$

where $J \subset \mathbb{R}$ is a closed connected set. The goal of this note is to shed some light on large scale properties of quasi-geodesics in $\mathcal{T}(S)$.

A geodesic metric space (X, d) is called *hyperbolic in the sense of Gromov* if there is a number $\delta > 0$ with the following property. For any geodesic triangle with sides a, b, c, the side a is contained in the δ-neighborhood of $b \cup c$. In a hyperbolic geodesic metric space, quasi-geodesics are stable: Any quasi-geodesic is contained in a uniformly bounded neighborhood of a geodesic. The Teichmüller metric, however, is not hyperbolic. Minsky [Mi96] gave a precise description of the Teichmüller metric in the thin part of Teichmüller space and obtained as a consequence that on the

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large scale, it is far from being similar to hyperbolic geodesic metric. In particular, quasi-geodesics are in general not stable. Masur and Minsky [MM00b] pointed out that quasi-geodesics in the thick part of Teichmüller space are in general not stable either: There is a number $L > 1$, a number $\epsilon > 0$ and a biinfinite L-quasi-geodesic in $\mathcal{T}(S)_\epsilon$ which is not contained in any bounded neighborhood of any geodesic.

It turns out that stability of quasi-geodesics in the thick part of Teichmüller space can be described using the relation between the geometry of Teichmüller space and the geometry of the *curve graph* of S. The curve graph is a geodesic metric graph $(\mathcal{CG}(S), d_{\mathcal{C}})$ whose vertex set is the set $\mathcal{C}(S)$ of all free homotopy classes of *essential* simple closed (non-oriented) curves on S (i.e. simple closed curves which are neither contractible nor homotopic into a puncture) and where two such curves are connected by an edge of length one if and only if they can be realized disjointly.

By a result of Bers, there is a constant $\chi_0 = \chi_0(S) > 0$ such that for every complete hyperbolic metric h on S of finite volume there is a *pants decomposition* for S consisting of simple closed geodesics of h-length at most χ_0. We define a map $\Upsilon_{\mathcal{T}} : \mathcal{T}(S) \to \mathcal{C}(S)$ by associating to a marked hyperbolic metric $h \in \mathcal{T}(S)$ a simple closed curve $\Upsilon_{\mathcal{T}}(h)$ whose h-length does not exceed χ_0. Note that such a map is by no means unique, but we observe in Section 2 that it is coarsely equivariant with respect to the action of the mapping class group. The purpose of this note is to characterize stable quasi-geodesics in the thick part of Teichmüller space as those quasi-geodesics which are mapped by $\Upsilon_{\mathcal{T}}$ to quasi-geodesics in $\mathcal{CG}(S)$. For the formulation of this result, define the *Hausdorff distance* $d_H(A, B) \in [0, \infty]$ between two subsets A, B of $\mathcal{T}(S)$ as the infimum of all numbers $r > 0$ such that A is contained in the r-neighborhood of B and B is contained in the r-neighborhood of A.

Theorem.
1. *For every $L > 1$ there is a constant $\epsilon = \epsilon(L) > 0$ with the following property. Let $J \subset \mathbb{R}$ be a closed connected set of length at least $1/\epsilon$ and let $\gamma : J \to \mathcal{T}(S)$ be an L-quasi-geodesic. If $\Upsilon_{\mathcal{T}} \circ \gamma$ is an L-quasi-geodesic in $\mathcal{CG}(S)$ then there is a Teichmüller geodesic $\xi : J' \to \mathcal{T}(S)_\epsilon$ such that $d_H(\gamma(J), \xi(J')) \leq 1/\epsilon$.*
2. *For every $\epsilon > 0$ there is a constant $L(\epsilon) > 1$ with the following property. Let $J \subset \mathbb{R}$ be a closed connected set and let $\gamma : J \to \mathcal{T}(S)$ be a $1/\epsilon$-quasi-geodesic. If there is a Teichmüller geodesic arc $\xi : J' \to \mathcal{T}(S)_\epsilon$ with $d_H(\gamma(J), \xi(J')) \leq 1/\epsilon$ then $\Upsilon_{\mathcal{T}} \circ \gamma$ is an $L(\epsilon)$-quasi-geodesic in $\mathcal{CG}(S)$.*

The second part of Theorem 1 is implicitly contained in the work of Masur and Minsky [MM99]. For completeness we nevertheless include a proof in Section 2.

A subset B of a geodesic metric space (X, d) is called *c-quasi-convex* for a constant $c > 0$ if any geodesic arc with both endpoints in B is contained in the c-neighborhood of B. We say that B is *quasi-convex* if it is c-quasi-convex for some number $c > 0$. A quasi-convex subset B of (X, d) is *hyperbolic* if there is a number $\delta > 0$ such that for every geodesic triangle in X with vertices in B and sides a, b, c, the side a is contained in the δ-neighborhood of $b \cup c$. As an immediate corollary of the above discussion we obtain the following statement (which is probably known to the experts but not explicitly available in the literature).

Corollary. *For $\epsilon > 0$, a set $B \subset \mathcal{T}(S)_\epsilon$ which is quasi-convex for the Teichmüller metric is hyperbolic.*

Further applications of the above theorem include a more elementary approach to Minsky's proof of the ending lamination conjecture for hyperbolic three-manifolds with a positive lower bound for the injectivity radius [HLO09] as well as an Anosov closing lemma for Teichmüller geodesics in the thick part of Teichmüller space [H07b].

The proof of the first part of Theorem 1 relies on an idea of Mosher [Mo03]. The task is to set up a framework to which this idea can be applied. The main difficulty is the non-local-compactness of the curve graph. We overcome this difficulty in Section 3 by working directly in Teichmüller space and controlling distances in the curve graph via a continuous distance-like symmetric model function on $\mathcal{T}(S) \times \mathcal{T}(S)$.

2. Cobounded Teichmüller geodesics

Let S be an oriented surface of genus $g \geq 0$ with $m \geq 0$ punctures and $3g-3+m \geq 2$. Let $\mathcal{C}(S)$ be the set of all free homotopy classes of unoriented *essential* simple closed curves on S, i.e. simple closed curves which are neither contractible nor freely homotopic into a puncture. The *curve graph* $\mathcal{CG}(S)$ of S is the graph with vertex set $\mathcal{C}(S)$ and where two vertices are joined by an edge if and only if the corresponding free homotopy classes can be realized disjointly. Since $3g-3+m \geq 2$ by assumption, $\mathcal{CG}(S)$ is connected (see [MM99] and the references given there). In the sequel we often do not distinguish between an essential simple closed curve α on S and the vertex of the curve graph defined by α.

Providing each edge in $\mathcal{CG}(S)$ with the standard euclidean metric of diameter 1 equips the curve graph with a geodesic metric $d_{\mathcal{C}}$. However, $\mathcal{CG}(S)$ is not locally finite and therefore the metric space $(\mathcal{CG}(S), d_{\mathcal{C}})$ is not locally compact. Masur and Minsky [MM99] showed that nevertheless its geometry can be understood quite explicitly. Namely, $\mathcal{CG}(S)$ is hyperbolic of infinite diameter (see also [Bw06, H07a] for alternative shorter proofs). The mapping class group naturally acts on $\mathcal{CG}(S)$ as a group of simplicial isometries.

The goal of this section is to relate the geometry of the thick part of Teichmüller space to the geometry of the curve graph in a quantitative way. To achieve this goal, we use a map from the Teichmüller space into the curve graph as defined in the introduction. Namely, for every marked hyperbolic metric $h \in \mathcal{T}(S)$, every essential free homotopy class α on S can be represented by a closed h-geodesic which is unique up to parametrization. This geodesic is simple if the free homotopy class admits a simple representative. The *h-length* $\ell_h(\alpha)$ of the class α is defined to be the length of its geodesic representative. Equivalently, $\ell_h(\alpha)$ equals the minimum of the h-lengths of all closed curves representing the class α.

A *pants decomposition* for S is a collection of $3g-3+m$ pairwise disjoint simple closed essential curves on S which decompose S into $2g-2+m$ *pairs of pants.* Here by a pair of pants we mean a surface which is homeomorphic to a three-holed sphere. By a classical result of Bers (see [B92]), there is a number $\chi_0 > 0$ only depending on the topological type of S such that for every complete hyperbolic metric h on S of finite volume there is a pants decomposition for S consisting of simple closed curves of h-length at most χ_0. A number $\chi_0 > 0$ with this property is called a *Bers constant* for S.

Define a map
$$\Upsilon_{\mathcal{T}} : \mathcal{T}(S) \to \mathcal{C}(S)$$
by associating to a complete hyperbolic metric h on S of finite volume an essential simple closed curve $\Upsilon_{\mathcal{T}}(h) \in \mathcal{C}(S)$ whose h-length is at most χ_0. Note that such a map is not unique. However, the following lemma due to Masur and Minsky [MM99] shows that the ambiguity in its definition is uniformly controlled. For its proof and for later use, define the *intersection number* $i(\alpha, \beta)$ between two simple closed curves α, β on S to be the minimum of the number of intersection points of any two simple closed curves which are freely homotopic to α, β.

Lemma 2.1. *For every number $\chi > 0$ there is a number $a(\chi) > 0$ with the following property. Let $h \in \mathcal{T}(S)$ and let α, β be two simple closed curves of h-length at most χ. Then $d_{\mathcal{C}}(\alpha, \beta) \leq a(\chi)$.*

Proof. By the collar lemma for hyperbolic surfaces (see [B92]), for any metric $h \in \mathcal{T}(S)$ and any number $\chi > 0$ there is a number $p(\chi) > 0$ only depending on χ such that every simple closed h-geodesic α of length at most χ is the core curve of an embedded annulus in S whose *width*, i.e. the distance between its two boundary circles, is at least $p(\chi)$. As a consequence, every essential intersection between α and a simple closed curve β on S contributes at least $p(\chi)$ to the length of β. Thus if the length of β is at most χ, then $i(\alpha, \beta) \leq \chi/p(\chi)$.

On the other hand, the distance in $\mathcal{CG}(S)$ between two curves $\alpha, \beta \in \mathcal{C}(S)$ is bounded from above by $i(\alpha, \beta) + 1$ [MM99, Bw06]. Therefore the diameter in $\mathcal{CG}(S)$ of the set of all simple closed curves of h-length at most χ is bounded from above by a universal constant $a(\chi) = \chi/p(\chi) + 1$ not depending on h. □

The next lemma due to Masur and Minsky [MM99] shows that the map $\Upsilon_{\mathcal{T}}$ is coarsely Lipschitz with respect to the Teichmüller distance $d_{\mathcal{T}}$ on $\mathcal{T}(S)$ and the distance $d_{\mathcal{C}}$ on the curve graph. We include its easy proof for completeness of exposition.

Lemma 2.2. *There is a number $L > 1$ such that*

(1) *$d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(g), \Upsilon_{\mathcal{T}}(h)) \leq L d_{\mathcal{T}}(g, h) + L$ for all $g, h \in \mathcal{T}(S)$.*
(2) *$d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(\varphi g), \varphi \Upsilon_{\mathcal{T}}(g)) \leq L$ for all $g \in \mathcal{T}(S), \varphi \in \mathrm{Mod}(S)$.*

Proof. By a result of Wolpert [W79], for every essential simple closed curve α on S we have

$$d_{\mathcal{T}}(g, h) \geq |\log \ell_g(\alpha) - \log \ell_h(\alpha)| \text{ for all } g, h \in \mathcal{T}(S). \tag{1}$$

This implies that there is a constant $b > 1$ (in fact we can take $b = e$) such that for all $g, h \in \mathcal{T}(S)$ with $d_{\mathcal{T}}(g, h) \leq 1$ and every $\alpha \in \mathcal{C}(S)$ with $\ell_g(\alpha) \leq \chi_0$ we have $\ell_h(\alpha) \leq b\chi_0$. Lemma 2.1 then shows that $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(g), \Upsilon_{\mathcal{T}}(h)) \leq a(b\chi_0)$ whenever $d_{\mathcal{T}}(g, h) \leq 1$.

Now the Teichmüller metric is a geodesic metric, i.e. any two points in $\mathcal{T}(S)$ can be connected by a minimal geodesic. Thus if $d_{\mathcal{T}}(g,h) \in (m-1,m]$ for some integer $m \geq 1$ then there are points $u_0 = g, u_1, \dots, u_m = h$ with $d_{\mathcal{T}}(u_{i-1},u_i) \leq 1$ for all i. The estimate in the previous paragraph together with the triangle inequality for $d_{\mathcal{C}}$ then yields that

$$d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(g), \Upsilon_{\mathcal{T}}(h)) \leq a(b\chi_0)m \leq a(b\chi_0)d_{\mathcal{T}}(g,h) + a(b\chi_0).$$

The first part of the lemma follows.

The second part of the lemma is derived in the same way. Namely, by the properties of the action of $\mathrm{Mod}(S)$ on $\mathcal{T}(S)$ and by the definition of the map $\Upsilon_{\mathcal{T}}$, for every $g \in \mathcal{T}(S)$ and every $\varphi \in \mathrm{Mod}(S)$ the φg-lengths of the curves $\Upsilon_{\mathcal{T}}(\varphi g)$ and $\varphi \Upsilon_{\mathcal{T}}(g)$ are at most χ_0. Thus by Lemma 2.1, the distance in $\mathcal{CG}(S)$ between these curves does not exceed $a(\chi_0)$. □

Using these basic properties of the map $\Upsilon_{\mathcal{T}}$, we can begin to relate the geometry of Teichmüller space equipped with the Teichmüller metric to the geometry of the curve graph. For this let $J \subset \mathbb{R}$ be a closed connected subset, i.e. either J is a closed interval or a closed ray or the whole line. For some $p > 1$, a map $\gamma : J \to \mathcal{CG}(S)$ is called a *p-quasi-geodesic* if for all $s, t \in J$ we have

$$d_{\mathcal{C}}(\gamma(s), \gamma(t))/p - p \leq |s-t| \leq p d_{\mathcal{C}}(\gamma(s), \gamma(t)) + p.$$

A map $\gamma : J \to \mathcal{CG}(S)$ is called an *unparametrized p-quasi-geodesic* if there is a closed connected set $I \subset \mathbb{R}$ and a homeomorphism $\zeta : I \to J$ such that $\gamma \circ \zeta : I \to \mathcal{CG}(S)$ is a p-quasi-geodesic.

The following result of Masur and Minsky (Theorem 2.6 and Theorem 2.3 of [MM99]; the precise quantitative version which we will use is Theorem 4.1 of [H07a]) is crucial for the proof of the Theorem from the introduction. For its formulation, we call a geodesic in $\mathcal{T}(S)$ for the Teichmüller metric a *Teichmüller geodesic.* A Teichmüller geodesic is always parametrized by arc length on a closed connected subset of the real line.

Theorem 2.3. *There is a number $p > 1$ only depending on the topological type of S such that the image under $\Upsilon_{\mathcal{T}}$ of every Teichmüller geodesic in $\mathcal{T}(S)$ is an unparametrized p-quasi-geodesic in $\mathcal{CG}(S)$.*

In general, the image under $\Upsilon_{\mathcal{T}}$ of a Teichmüller geodesic is not a quasi-geodesic with its proper parametrization. More concretely, an unparametrized quasi-geodesic in $\mathcal{CG}(S)$ which is the image under the map $\Upsilon_{\mathcal{T}}$ of an infinite Teichmüller geodesic can be bounded or unbounded. We are interested in the case when the diameter of the image is infinite, and for this we recall some standard facts about Teichmüller geodesics as well as a result of Klarreich [Kl99].

A *geodesic lamination* for a complete hyperbolic structure on S of finite volume is a *compact* subset of S which is foliated into simple geodesics. A geodesic lamination λ on S is called *minimal* if each of its half-leaves is dense in λ. Thus a simple closed geodesic is a minimal geodesic lamination. A minimal geodesic lamination with more than one leaf has uncountably many leaves and is called *minimal arational.* A geodesic lamination λ is said to *fill up* S if every simple closed geodesic on S

intersects λ transversely. This is equivalent to stating that the complementary components of λ are all topological discs or once punctured topological discs.

A *measured geodesic lamination* is a geodesic lamination λ together with a translation invariant transverse measure. Such a measure assigns a positive weight to each compact arc in S which intersects λ nontrivially and transversely and whose endpoints are contained in complementary regions of λ. The geodesic lamination λ is called the *support* of the measured geodesic lamination; it consists of a disjoint union of minimal components. Vice versa, every minimal geodesic lamination is the support of a measured geodesic lamination.

The space $\mathcal{ML}$ of measured geodesic laminations on S can be equipped with the weak*-topology. Its projectivization $\mathcal{PML}$ is called the space of *projective measured geodesic laminations* and is homeomorphic to the sphere $S^{6g-7+2m}$. There is a continuous *length function* on $\mathcal{T}(S) \times \mathcal{ML} \to (0,\infty)$ which assigns to a hyperbolic metric $h \in \mathcal{T}(S)$ and a measured geodesic lamination μ the h-length $\ell_h(\mu)$ of μ. This length function satisfies $\ell_h(a\mu) = a\ell_h(\mu)$ for all $h \in \mathcal{T}(S), \mu \in \mathcal{ML}$ and every $a > 0$. For every fixed $h \in \mathcal{T}(S)$ the set of all measured geodesic laminations of h-length one is a section of the projection $\mathcal{ML} \to \mathcal{PML}$.

There is also a continuous symmetric pairing $\iota : \mathcal{ML} \times \mathcal{ML} \to (0,\infty)$, the so-called *intersection form*, which satisfies $\iota(a\xi, b\eta) = ab\iota(\xi,\eta)$ for all $a, b \geq 0$ and all $\xi, \eta \in \mathcal{ML}$. If α, β are simple closed geodesics, viewed as measured geodesic laminations (i.e. equipped with the transverse counting measure), then $i(\alpha, \beta)$ is just the number of intersection points between α and β. The measured geodesic lamination $\nu \in \mathcal{ML}$ is said to *fill up* S if its support fills up S. This is equivalent to stating that $i(\nu, c) > 0$ for every simple closed curve c.

Since $\mathcal{CG}(S)$ is a hyperbolic geodesic metric space, it admits a *Gromov boundary* $\partial\mathcal{CG}(S)$ which is a (non-compact) metrizable topological space equipped with an action of $\mathrm{Mod}(S)$ by homeomorphisms (see [BH99] for the definition of the Gromov boundary of a hyperbolic geodesic metric space and for references). Following Klarreich [Kl99] (see also [H06]), this boundary can naturally be identified with the space of all (unmeasured) minimal geodesic laminations which fill up S equipped with the topology which is induced from the weak*-topology on $\mathcal{PML}$ via the measure forgetting map.

Let $\mathcal{Q}^1(S)$ be the bundle of *holomorphic quadratic differentials* of area one over Teichmüller space. The cotangent vector at $\gamma(0)$ of a Teichmüller geodesic $\gamma : \mathbb{R} \to \mathcal{T}(S)$ (which we always assume to be parametrized by arc length without further mentioning) is a holomorphic quadratic differential $q \in \mathcal{Q}^1(S)$ of area one on the Riemann surface $\gamma(0)$. The quadratic differential q corresponds to a pair $(q_h, q_v) \in \mathcal{ML} \times \mathcal{ML}$ of measured geodesic laminations which satisfy $i(q_h, q_v) = 1$ and *jointly fill up* S. This means that $i(\mu, q_h) + i(\mu, q_v) > 0$ for every measured geodesic lamination $\mu \in \mathcal{ML}$. The measured geodesic lamination q_v is called the *vertical* measured geodesic lamination of q, and q_h is called *horizontal*. For every $t \in \mathbb{R}$ the unit cotangent vector of γ at $\gamma(t)$ is the quadratic differential $\Phi^t q$ defined by the pair $(e^t q_v, e^{-t} q_h)$.

If the vertical measured geodesic lamination q_v of q fills up S then the unparametrized p-quasi-geodesic $t \to \Upsilon_{\mathcal{T}}(\gamma(t))$ in $\mathcal{CG}(S)$ is of infinite diameter and converges to the support of q_v, viewed as a point in $\partial\mathcal{CG}(S)$ [Kl99, H06]. Namely, the length of $\Upsilon_{\mathcal{T}}(\gamma(t))$ with respect to the singular euclidean metric defined by $\Phi^t q$ (i.e. the minimal $\Phi^t q$-length of a simple closed curve representing the class $\Upsilon_{\mathcal{T}}(\gamma(t))$) is uniformly bounded (see [MM99, Bw06], and see [R07a] for a more precise statement). Since for every simple closed curve $\alpha \in \mathcal{C}(S)$ the $\Phi^t q$-length of α is bounded from below by $\max\{e^t i(\alpha, q_v), e^{-t} i(\alpha, q_h)\}$ (see e.g. [R05]), the intersection numbers $e^t i(\Upsilon_{\mathcal{T}}(\gamma(t)), q_v)$ are bounded from above by a universal constant. Let ν_t be the measured geodesic lamination of $\gamma(0)$-length one which is in the class of $\Upsilon_{\mathcal{T}}(\gamma(t))$. Since the shortest length of a simple closed curve for $\gamma(0)$ is bounded from below by a positive constant, we have $\nu_t = a_t \Upsilon_{\mathcal{T}}(\gamma(t))$ for a number $a_t > 0$ which is uniformly bounded. Up to passing to a subsequence the measured geodesic laminations ν_t converge as $t \to \infty$ to a measured geodesic lamination ν. By continuity of the intersection form, we have $i(\nu, q_v) = 0$. Since q_v fills up S, this implies that the support of ν equals the support of q_v. On the other hand, Theorem 1.1 of [H06] shows that a sequence $(c_i) \subset \mathcal{C}(S)$ converges in the space $\mathcal{CG}(S) \cup \partial\mathcal{CG}(S)$ to a point $\xi \in \partial\mathcal{CG}(S)$ if the curves c_i viewed as projective measured geodesic laminations converge in $\mathcal{PML}$ to a projective measured geodesic lamination with support ξ.

We begin the proof of the second part of Theorem from the introduction with an easy no-retraction lemma for quasi-geodesics in the hyperbolic geodesic metric space $\mathcal{CG}(S)$ which is well known but not so easy to track in the literature. For its formulation, define the *Hausdorff distance* between two subsets A, B of a metric space X to be the infimum of all numbers $r > 0$ such that A is contained in the r-neighborhood of B and B is contained in the r-neighborhood of A. It will be convenient to allow that the sets A, B are not necessarily closed and that the metric space is unbounded (i.e. the "Hausdorff distance" is by no means a distance).

Lemma 2.4. *For $p > 1$ there is a constant $c = c(p) > 0$ with the following property. Let $\gamma : J \to \mathcal{CG}(S)$ be any unparametrized p-quasi-geodesic; if $t_1 < t_2 < t_3 \in J$ then $d_{\mathcal{C}}(\gamma(t_1), \gamma(t_3)) \geq d_{\mathcal{C}}(\gamma(t_1), \gamma(t_2)) + d_{\mathcal{C}}(\gamma(t_2), \gamma(t_3)) - c$.*

Proof. Let $p > 1$; by the definition of an unparametrized p-quasi-geodesic, it is enough to show the existence of a number $c > 0$ such that for every (parametrized) p-quasi-geodesic $\gamma : [0, n] \to \mathcal{CG}(S)$ and all $0 < t < n$ we have $d_{\mathcal{C}}(\gamma(0), \gamma(n)) \geq d_{\mathcal{C}}(\gamma(0), \gamma(t)) + d_{\mathcal{C}}(\gamma(t), \gamma(n)) - c$.

Since $\mathcal{CG}(S)$ is a hyperbolic geodesic metric space, there is a constant $c > 0$ only depending on p with the following property (see [BH99]). Let $n > 0$, let $\gamma : [0, n] \to \mathcal{CG}(S)$ be any p-quasi-geodesic and let $\zeta : [0, m] \to \mathcal{CG}(S)$ be a geodesic connecting $\zeta(0) = \gamma(0)$ to $\zeta(m) = \gamma(n)$. Then the Hausdorff distance between $\gamma[0, n]$ and $\zeta[0, m]$ is smaller than $c/2$.

In particular, for every $t \in [0, n]$ there is a point $s \in [0, m]$ such that

$$d_{\mathcal{C}}(\gamma(t), \zeta(s)) \leq c/2.$$

Thus we have $d_{\mathcal{C}}(\gamma(0), \gamma(t)) + d_{\mathcal{C}}(\gamma(t), \gamma(n)) \leq d_{\mathcal{C}}(\zeta(0), \zeta(s)) + d_{\mathcal{C}}(\zeta(s), \zeta(m)) + c = d_{\mathcal{C}}(\gamma(0), \gamma(n)) + c$ which shows the lemma. □

For $\epsilon > 0$ let $\mathcal{T}(S)_\epsilon$ be the closed subset of $\mathcal{T}(S)$ of all hyperbolic metrics h for which the length of the shortest closed h-geodesic is at least ϵ. Informally we think of $\mathcal{T}(S)_\epsilon$ as the ϵ-thick part of Teichmüller space. The mapping class group preserves the set $\mathcal{T}(S)_\epsilon$ and acts on it cocompactly. Moreover, every Mod(S)-invariant subset of $\mathcal{T}(S)$ on which Mod(S) acts cocompactly is contained in $\mathcal{T}(S)_\epsilon$ for some $\epsilon > 0$.

As mentioned above, the following lemma is implicitly contained in the work of Masur and Minsky [MM00a] (see also the more recent work of Rafi [R07b]).

Lemma 2.5. *For every $\epsilon > 0$ there is a number $\nu_0 = \nu_0(\epsilon) > 0$ with the following property. Let $\gamma : J \to \mathcal{T}(S)_\epsilon$ be a Teichmüller geodesic; then the curve $\Upsilon_{\mathcal{T}} \circ \gamma : J \to \mathcal{CG}(S)$ is a ν_0-quasi-geodesic.*

Proof. As in Theorem 2.3, let $p > 1$ be such that the image under $\Upsilon_{\mathcal{T}}$ of every Teichmüller geodesic is an unparametrized p-quasi-geodesic in $\mathcal{CG}(S)$. Let $c = c(p) > 0$ be as in Lemma 2.4.

We claim that for every $\epsilon > 0$ there is a constant $k_0 = k_0(\epsilon) > 0$ with the following property. Let $k \geq k_0$ and let $\gamma : [0, k] \to \mathcal{T}(S)_\epsilon$ be a Teichmüller geodesic arc of length at least k_0; then $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(0), \Upsilon_{\mathcal{T}}\gamma(k)) \geq 2c$.

To see that this is the case, we argue by contradiction and we assume otherwise. Then there is a number $\epsilon > 0$ and there is a sequence $k_i \to \infty$ and for every $i > 0$ there is a Teichmüller geodesic arc $\gamma_i : [0, k_i] \to \mathcal{T}(S)_\epsilon$ such that

$$d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma_i(0), \Upsilon_{\mathcal{T}}\gamma_i(k_i)) \leq 2c.$$

Lemma 2.4 implies that in fact

$$d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma_i(0), \Upsilon_{\mathcal{T}}\gamma_i(t)) \leq 3c$$

for every $t \in [0, k_i]$.

Let $L > 0$ be as in Lemma 2.2. The action of Mod(S) on $\mathcal{T}(S)_\epsilon$ is isometric and cocompact. Let K_0 be a compact fundamental domain for the action of Mod(S) on $\mathcal{T}(S)_\epsilon$. By the second part of Lemma 2.2, up to replacing γ_i by its image under a suitably chosen element of Mod(S) and up to replacing the constant $3c$ by $3c + 2L$, we may assume that $\gamma_i(0) \in K_0$ for every $i > 0$. Then the unit cotangent vectors $q_i \in \mathcal{Q}^1(S)$ of the geodesics γ_i at $\gamma_i(0)$ are contained in the compact subset K of $\mathcal{Q}^1(S)$ of all area one quadratic differentials with foot-point in K_0.

Consequently, by passing to a subsequence we may assume that the quadratic differentials q_i converge as $i \to \infty$ to a quadratic differential $q \in K$. Then the Teichmüller geodesic arcs γ_i converge locally uniformly as $i \to \infty$ to the Teichmüller geodesic ray $\gamma : [0, \infty) \to \mathcal{T}(S)$ with unit cotangent vector q at $\gamma(0)$. Since $\mathcal{T}(S)_\epsilon \subset \mathcal{T}(S)$ is closed and $\gamma_i[0, k_i] \subset \mathcal{T}(S)_\epsilon$ for all i, we have $\gamma[0, \infty) \subset \mathcal{T}(S)_\epsilon$. Moreover, since

$$d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma_i(0), \Upsilon_{\mathcal{T}}\gamma_i(s)) \leq 3c + 2L$$

for all $s > 0$ such that $k_i > s$, the first part of Lemma 2.2 shows that

$$d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(s), \Upsilon_{\mathcal{T}}(\gamma(0)) \leq 3c + 4L \text{ for all } s \geq 0.$$

Let $q_v \in \mathcal{ML}$ be the vertical measured geodesic lamination of the quadratic differential q which defines the Teichmüller geodesic ray γ. Since

$$\gamma[0,\infty) \subset \mathcal{T}(S)_\epsilon,$$

the Teichmüller geodesic γ projects into a compact subset of the moduli space $\mathcal{T}(S)/\mathrm{Mod}(S)$. Thus by a result of Masur [M82], the measured geodesic lamination q_v fills up S. This implies that the curve $\Upsilon_{\mathcal{T}} \circ \gamma$ is an unparametrized quasi-geodesic in $\mathcal{CG}(S)$ of *infinite* diameter (see [Kl99, H06] and the comments before the statement of the lemma). This is a contradiction and shows for every $\epsilon > 0$ the existence of a constant $k_0 = k_0(\epsilon) > 0$ as claimed.

Let $\epsilon > 0$, let $k_0 = k_0(\epsilon)$, let $n > 0$ and let $\gamma : [0, k_0 n] \to \mathcal{T}(S)_\epsilon$ be any Teichmüller geodesic. The image under $\Upsilon_{\mathcal{T}}$ of every Teichmüller geodesic in $\mathcal{T}(S)$ is an unparametrized p-quasi-geodesic. By the choice of c, for all $0 \le s \le t \le u \le k_0 n$ we have

$$d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(s), \Upsilon_{\mathcal{T}}\gamma(u)) \ge d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(s), \Upsilon_{\mathcal{T}}\gamma(t)) + d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(t), \Upsilon_{\mathcal{T}}\gamma(u)) - c. \tag{2}$$

On the other hand, for every integer $\ell < n$ we have $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(\ell k_0), \Upsilon_{\mathcal{T}}\gamma((\ell+1)k_0)) \ge 2c$ by the choice of k_0, and therefore Lemma 2.4 and the estimate (2) imply that

$$d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma((\ell+1)k_0), \Upsilon_{\mathcal{T}}\gamma(s)) \ge d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(\ell k_0), \Upsilon_{\mathcal{T}}\gamma(s)) + c \text{ for all } s \le \ell k_0.$$

Inductively we deduce that $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(\ell k_0), \Upsilon_{\mathcal{T}}\gamma(m k_0)) \ge c|\ell - m|$ for all integers $\ell, m \le n$.

By the first part of Lemma 2.2, the map $\Upsilon_{\mathcal{T}} : \mathcal{T}(S) \to \mathcal{CG}(S)$ is coarsely Lipschitz. It follows that $\Upsilon_{\mathcal{T}} \circ \gamma$ is a ν_0-quasi-geodesic for a constant $\nu_0 > 0$ only depending on ϵ. More precisely, if $L > 1$ is as in Lemma 2.2 then we have

$$c|s-t|/k_0 - k_0 L - L \le d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(s), \Upsilon_{\mathcal{T}}\gamma(t)) \le L|s-t| + L \text{ for all } s,t \in [0, k_0 n].$$

This shows the lemma. ☐

As in the introduction, denote by d_H the Hausdorff distance with respect to the Teichmüller metric for (not necessarily closed or bounded) subsets of $\mathcal{T}(S)$.

The following corollary shows the second part of the theorem from the introduction.

Corollary 2.6. *For every $\epsilon > 0$ there is a number $\nu = \nu(\epsilon) > 1$ with the following property. Let $\gamma : J \to (\mathcal{T}(S)_\epsilon, d_{\mathcal{T}})$ be a $1/\epsilon$-quasi-geodesic. Assume that there is a Teichmüller geodesic $\zeta : I \to \mathcal{T}(S)$ such that $d_H(\gamma(J), \zeta(I)) < 1/\epsilon$. Then $\Upsilon_{\mathcal{T}} \circ \gamma : J \to \mathcal{CG}(S)$ is a ν-quasi-geodesic.*

Proof. Let $\epsilon > 0$ and let $\gamma : J \to \mathcal{T}(S)_\epsilon$ be a $1/\epsilon$-quasi-geodesic for the Teichmüller metric. Assume that there is a Teichmüller geodesic $\zeta : I \to \mathcal{T}(S)$ such that $d_H(\gamma(J), \zeta(I)) < 1/\epsilon$. This means that there is a map $\rho : J \to I$ such that

$$d_{\mathcal{T}}(\gamma(t), \zeta(\rho(t))) \le 1/\epsilon \text{ for all } t. \tag{3}$$

Moreover, we have $\zeta(I) \subset \mathcal{T}(S)_\delta$ for a number $\delta > 0$ only depending on ϵ by Wolpert's estimate (1) above.

By assumption, the map γ is a $1/\epsilon$-quasi-geodesic in $\mathcal{T}(S)_\epsilon$, moreover ζ realizes the distance between any of its points. Thus by the triangle inequality,

$$\epsilon|s-t| - 3/\epsilon \leq d_{\mathcal{T}}(\zeta(\rho(s)), \zeta(\rho(t))) = |\rho(s)-\rho(t)| \leq |s-t|/\epsilon + 3/\epsilon$$

which just means that the map ρ is a b-*quasi-isometry* for $b = 3/\epsilon > 1$.

By Lemma 2.5, the curve $\Upsilon_{\mathcal{T}} \circ \zeta : I \to \mathcal{CG}(S)$ is a ν_0-quasi-geodesic for a constant $\nu_0 > 0$ only depending on ϵ. Then the map $t \in J \to \Upsilon_{\mathcal{T}}\zeta(\rho(t)) \in \mathcal{CG}(S)$ is the composition of a b-quasi-isometry with a ν_0-quasi-isometric embedding and hence it is a $\tilde{b}$-quasi-geodesic for a constant $\tilde{b} > 1$ only depending on ϵ.

On the other hand, the first part of Lemma 2.2 together with inequality (3) shows that

$$d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(\gamma(t)), \Upsilon_{\mathcal{T}}(\zeta \circ \rho(t))) \leq L/\epsilon + L \text{ for all } t.$$

Since $t \to \Upsilon_{\mathcal{T}}\zeta(\rho(t))$ is a $\tilde{b}$-quasi-geodesic, this shows that $\Upsilon_{\mathcal{T}} \circ \gamma$ is a ν-quasi-geodesic for a constant $\nu > 0$ only depending on ϵ. □

Corollary 2.6 easily implies the Corollary from the introduction. For its formulation, call a subset B of $(\mathcal{T}(S), d_{\mathcal{T}})$ p-*quasi-convex* for a number $p > 0$ if any Teichmüller geodesic arc with both endpoints in B is contained in the p-neighborhood of B. A subset B of $\mathcal{T}(S)$ is called *quasi-convex* if B is p-quasi-convex for some $p > 0$.

Call B *hyperbolic* if there is a number $\delta > 0$ such that for every geodesic triangle in $(\mathcal{T}(S), d_{\mathcal{T}})$ with vertices in B and sides a, b, c, the side a is contained in the δ-neighborhood of $b \cup c$.

Corollary 2.7. *For every $\epsilon > 0$, a quasi-convex subset B of $(\mathcal{T}(S)_\epsilon, d_{\mathcal{T}})$ is hyperbolic.*

Proof. Let $B \subset \mathcal{T}(S)_\epsilon$ be a p-quasi-convex set. Let $\delta > 0$ be sufficiently small that the p-neighborhood of $\mathcal{T}(S)\epsilon$ is contained in $\mathcal{T}(S)_\delta$. Such a number exists by invariance under the action of the mapping class group and the fact that the Teichmüller metric is complete. Then for $x, y \in B$, the Teichmüller geodesic ζ connecting x to y is contained in $\mathcal{T}(S)_\delta$. Corollary 2.6 shows that there is a number $L > 1$ only depending on δ such that

$$d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(x), \Upsilon_{\mathcal{T}}(y))/L - L \leq d_{\mathcal{T}}(x, y) \leq L d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(x), \Upsilon_{\mathcal{T}}(x)) + L.$$

Thus the restriction of $\Upsilon_{\mathcal{T}}$ to $(B, d_{\mathcal{T}})$ is a quasi-isometric embedding of B into $\mathcal{CG}(S)$.

Now let a, b, c be the sides of a geodesic triangle in $\mathcal{T}(S)$ with vertices in B. Then there are uniform quasi-geodesics $\tilde{a}, \tilde{b}, \tilde{c} \subset B$ with the same endpoints whose Hausdorff distance to a, b, c is at most p. These quasi-geodesics are obtained by associating to a point x on a, b, c a point $\tilde{x} \in B$ whose distance to x is minimal and hence at most p. By the above, the quasi-geodesics $\tilde{a}, \tilde{b}, \tilde{c}$ are mapped by $\Upsilon_{\mathcal{T}}$ to $\tilde{L}$-quasi-geodesics in $\mathcal{CG}(S)$ for a universal number $\tilde{L} > 0$.

By hyperbolicity of $\mathcal{CG}(S)$, there is a constant $r > 0$ such that if $\hat{a}, \hat{b}, \hat{c}$ is a triangle in $\mathcal{CG}(S)$ with $\tilde{L}$-quasi-geodesic sides then $\hat{a}$ is contained in the r-neighborhood of

$\hat{b} \cup \hat{c}$. Since the restriction of $\Upsilon_{\mathcal{T}}$ to B is a quasi-isometric embedding, we conclude that there is a universal constant $R > 0$ such that the quasi-geodesic $\tilde{a}$ in $(B, d_{\mathcal{T}})$ is contained in the R-neighborhood of $\tilde{b} \cup \tilde{c}$. Together with the choice of $\tilde{a}, \tilde{b}, \tilde{c}$ this implies that a is contained in the $R + 2p$-neighborhood of $b \cup c$. Thus B is indeed hyperbolic. □

3. Quasi-convex curves for the Teichmüller metric

Using the assumptions and notations from Section 2, the goal of this section is to show the first part of the Theorem from the introduction.

We begin again with a simple observation which shows that a quasi-geodesic in $\mathcal{T}(S)$ with respect to the Teichmüller metric $d_{\mathcal{T}}$ whose image under the map $\Upsilon_{\mathcal{T}}$ is a uniform quasi-geodesic in $\mathcal{CG}(S)$ is contained in the thick part of Teichmüller space.

Lemma 3.1. *For every $\nu > 1$ there is a number $\epsilon_0 = \epsilon_0(\nu) > 0$ with the following properties. Let $\gamma : [0, n] \to \mathcal{T}(S)$ be a ν-quasi-geodesic whose image $\Upsilon_{\mathcal{T}} \circ \gamma$ in $\mathcal{CG}(S)$ is a ν-quasi-geodesic. If $n \geq 1/\epsilon_0$ then $\gamma[0, n] \subset \mathcal{T}(S)_{\epsilon_0}$.*

Proof. The simple idea for the proof is as follows. If a quasi-geodesic in $\mathcal{T}(S)$ for the Teichmüller metric enters deeply into the thin part of Teichmüller space then it needs a long time to exit the set of metrics for which a fixed simple closed curve is short. Hence its image under the map $\Upsilon_{\mathcal{T}}$ remains for a long time in a set of uniformly bounded diameter.

For a quantitative statement, let $n > 0, \nu > 1$ and let $\gamma : [0, n] \to (\mathcal{T}(S), d_{\mathcal{T}})$ be a ν-quasi-geodesic such that $\Upsilon_{\mathcal{T}} \circ \gamma$ is a ν-quasi-geodesic in $\mathcal{CG}(S)$. Then we have

$$d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(t), \Upsilon_{\mathcal{T}}\gamma(s)) \geq |s - t|/\nu - \nu \text{ for all } s, t \in [0, n]. \tag{4}$$

Let as before $\chi_0 > 0$ be a Bers constant for S and as in Lemma 2.1, let $a(\chi_0) > 0$ be an upper bound for the diameter in $\mathcal{CG}(S)$ of the collection of all simple closed curves on S whose h-length is at most χ_0 for an arbitrary but fixed $h \in \mathcal{T}(S)$. Let $[a, b] \subset [0, n]$ be an interval for which there is a simple closed curve $\alpha \in \mathcal{C}(S)$ so that $\ell_{\gamma(t)}(\alpha) \leq \chi_0$ for all $t \in [a, b]$ (here as before, $\ell_{\gamma(t)}(\alpha)$ is the $\gamma(t)$-length of α). Then we have $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(\gamma(a)), \alpha) \leq a(\chi_0), d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(\gamma(b)), \alpha) \leq a(\chi_0)$ and therefore inequality (4) shows that

$$|b - a| \leq 2\nu a(\chi_0) + \nu^2. \tag{5}$$

Now by Wolpert's result (1), for all $\alpha \in \mathcal{C}(S)$ and all $h, h' \in \mathcal{T}(S)$ the Teichmüller distance between h and h' is at least $|\log \ell_h(\alpha) - \log \ell_{h'}(\alpha)|$. Thus if the ν-quasi-geodesic $\gamma : [0, n] \to (\mathcal{T}(S), d_{\mathcal{T}})$ is such that there is a point $t \in [0, n]$ and some $\alpha \in \mathcal{C}(S)$ with

$$\log(\ell_{\gamma(t)}(\alpha)) \leq \log(\chi_0) - 2\nu a(\chi_0) - 2\nu^2$$

then $\ell_{\gamma(s)}(\alpha) \leq \chi_0$ for every $s \in [0, n]$ with $d_{\mathcal{T}}(\gamma(s), \gamma(t)) \leq 2\nu a(\chi_0) + 2\nu^2$ and hence for every $s \in [0, n]$ with $|s - t| \leq 2\nu^2 a(\chi_0) + 2\nu^3 + \nu^2$. Consequently inequality (5) shows that $\Upsilon_{\mathcal{T}} \circ \gamma$ is *not* a ν-quasi-geodesic provided that $n \geq 4\nu^2 a(\chi_0) + 4\nu^3 + 2\nu^2$. □

The strategy for the proof of the first part of the Theorem from the introduction is to lift the metric $d_{\mathcal{C}}$ on the curve graph with the map $\Upsilon_{\mathcal{T}}$ to a symmetric function on $\mathcal{T}(S) \times \mathcal{T}(S)$. By Lemma 2.1, this function is locally uniformly bounded. This enables us to modify the function with an averaging procedure to a continuous symmetric Mod(S)-invariant function ρ on $\mathcal{T}(S) \times \mathcal{T}(S)$ whose large-scale properties coincide with the large-scale properties of $d_{\mathcal{C}} \circ (\Upsilon_{\mathcal{T}} \times \Upsilon_{\mathcal{T}})$. We then can use continuity, local compactness and equivariance under the action of the mapping class group to derive the theorem using an idea from Section 3.9 of [Mo03].

We begin with constructing the function ρ. For this let again $\chi_0 > 0$ be a Bers constant for S and choose a smooth function $\sigma : [0, \infty) \to [0, 1]$ with $\sigma[0, \chi_0] \equiv 1$ and $\sigma[2\chi_0, \infty) \equiv 0$. For every $h \in \mathcal{T}(S)$ we obtain a finite Borel measure μ_h on $\mathcal{C}(S)$ by defining

$$\mu_h = \sum_\beta \sigma(\ell_h(\beta))\delta_\beta$$

where δ_β denotes the Dirac mass at β. By the collar lemma, the number of simple closed geodesics on (S, h) of length at most $2\chi_0$ is bounded from above independent of h (see also the proof of Lemma 2.1 where this fact is discussed). Thus the total mass of μ_h is bounded from above and below by a universal positive constant, and by Lemma 2.1 the diameter of the support of μ_h in $\mathcal{CG}(S)$ is uniformly bounded as well.

Since $\mathcal{C}(S)$ is countable, a finite measure on $\mathcal{C}(S)$ is just a summable nonnegative function on $\mathcal{C}(S)$. Thus the space of finite measures on $\mathcal{C}(S)$ can be equipped with the norm topology as a convex subset of the Banach space $\ell^1(\mathcal{C}(S))$ of summable functions on $\mathcal{C}(S)$.

The mapping class group acts on $\mathcal{C}(S)$ and hence it acts on the Banach space $\ell^1(\mathcal{C}(S))$. We have.

Lemma 3.2. *The assignment $h \to \mu_h$ is equivariant with respect to the action of the mapping class group on $\mathcal{T}(S)$ and $\ell^1(\mathcal{C}(S))$ and continuous.*

Proof. Equivariance of the map $h \to \mu_h$ under the action of the mapping class group is immediate from the definition.

To show continuity, let $h \in \mathcal{T}(S)$ and let U be a small neighborhood of h in $\mathcal{T}(S)$ contained in the ball of radius one with respect to the Teichmüller metric. By the collar lemma and by Wolpert's estimate (1), the number of simple closed curves $\alpha_1, \dots, \alpha_s \in \mathcal{C}(S)$ whose geodesic representative with respect to one of the metrics $u \in U$ has length at most $2\chi_0$ is bounded from above by a universal constant not depending on h and U. Thus each of the measures μ_u $(u \in U)$ is supported in $\cup_i \alpha_i$.

On the other hand, using again Wolpert's result, for each i the function $u \in \mathcal{T}(S) \to \ell_u(\alpha_i) \in (0, \infty)$ is continuous and hence the same is true for the function $u \to \sigma(\ell_u(\alpha_i))$. Thus the weight of α_i for the measure μ_u depends continuously on u. From this continuity of the map $h \to \mu_h$ is immediate. ☐

Lemma 3.2 implies in particular that the function $h \to \mu_h(\mathcal{C}(S))$ is continuous.

Define a symmetric non-negative function ρ on $\mathcal{T}(S) \times \mathcal{T}(S)$ by

$$\rho(h, h') = \int_{\mathcal{C}(S)\times\mathcal{C}(S)} d_{\mathcal{C}}(\cdot, \cdot) d\mu_h \times d\mu'_h / \mu_h(\mathcal{C}(S))\mu_{h'}(\mathcal{C}(S)).$$

Lemma 3.3. *The function ρ is continuous and invariant under the diagonal action of* $\mathrm{Mod}(S)$*. Moreover, there is a universal constant $a > 0$ such that*

$$\rho(h, h') - a \le d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(h), \Upsilon_{\mathcal{T}}(h')) \le \rho(h, h') + a \textit{ for all } h, h' \in \mathcal{T}(S).$$

Proof. We saw in the proof of Lemma 3.2 that for all $h, h' \in \mathcal{T}(S)$ there are neighborhoods U, U' of h, h' and there is a finite collections $\alpha_1, \dots, \alpha_s$ of points in $\mathcal{C}(S)$ such that for every $u \in U \cup U'$, the measure μ_u is supported in $A = \{\alpha_1, \dots, \alpha_s\}$. Since the diameter of A in $\mathcal{CG}(S)$ is finite, continuity of the function ρ follows from continuity of the map $h \to \mu_h$ with respect to the norm topology.

Invariance of ρ under the diagonal action of $\mathrm{Mod}(S)$ is immediate. To show the estimate in the lemma, simply observe that the total mass of μ_h and the diameter in $\mathcal{CG}(S)$ of the support of μ_h are uniformly bounded independent of h and that moreover $\sigma(\Upsilon_{\mathcal{T}}(h)) = 1$ for all h. □

Now we are ready to show the first part of the theorem from the introduction.

Proposition 3.4. *For every $\nu > 1$ there is a constant $\epsilon = \epsilon(\nu) > 0$ with the following property. Let $J \subset \mathbb{R}$ be a closed connected set of diameter at least $1/\epsilon$ and let $\gamma : J \to (\mathcal{T}(S), d_{\mathcal{T}})$ be a ν-quasi-geodesic (here as before, J denotes a closed connected subset of $\mathbb{R}$). If $\Upsilon_{\mathcal{T}} \circ \gamma$ is a ν-quasi-geodesic in $\mathcal{CG}(S)$ then there is a Teichmüller geodesic $\xi : J' \to \mathcal{T}(S)_\epsilon$ such that $d_H(\gamma(J), \xi(J')) \le 1/\epsilon$.*

Proof. For $\kappa > 1$ define a *κ-Lipschitz curve* in $\mathcal{T}(S)$ to be a κ-Lipschitz map $\gamma : J \to \mathcal{T}(S)$ with respect to the standard metric on $\mathbb{R}$ and the distance $d_{\mathcal{T}}$ on $\mathcal{T}(S)$ induced by the Teichmüller metric. Since $\mathcal{T}(S)$ is a smooth manifold and the distance $d_{\mathcal{T}}$ is geodesic, every ν-quasi-geodesic $\gamma : J \to (\mathcal{T}(S), d_{\mathcal{T}})$ can be replaced by a piecewise geodesic $\zeta : J \to \mathcal{T}(S)$ which is a 2ν-Lipschitz curve and which satisfies $d(\gamma(t), \zeta(t)) \le 4\nu$ for all $t \in J$. For this simply replace for every integer m the restriction of γ to $[m-1, m] \subset J$ by a Teichmüller geodesic segment with the same endpoints parametrized proportional to arc length on $[m-1, m]$. Let ζ be the resulting curve. Since $d_{\mathcal{T}}(\gamma(m-1), \gamma(m)) \le 2\nu$ by the definition of a quasi-geodesic, ζ is a 2ν-Lipschitz curve (this is valid for $J = \mathbb{R}$, and we leave it to the reader to make the necessary adjustment of this construction in the case that $J \ne \mathbb{R}$). Moreover, for every $t \in [m-1, m]$ we have $d_{\mathcal{T}}(\gamma(t), \gamma(m)) \le 2\nu, d_{\mathcal{T}}(\zeta(t), \zeta(m)) \le 2\nu$ and hence $d_{\mathcal{T}}(\zeta(t), \xi(t)) \le 4\nu$. Thus by the first part of Lemma 2.2 (and a change of notation for the constants used), it is enough to show the statement of the proposition for ν-Lipschitz curves $\gamma : J \to \mathcal{T}(S)$ which are ν-quasi-geodesics for $d_{\mathcal{T}}$ and such that $\Upsilon_{\mathcal{T}} \circ \gamma$ is a ν-quasi-geodesic in $\mathcal{CG}(S)$.

Let $\epsilon_0 = \epsilon_0(\nu)$ be as in Lemma 3.1 and let $a(\chi_0) > 0$ be as in Lemma 2.1. In the sequel we always assume that the diameter $|J|$ of the set J is bigger than $\max\{1/\epsilon_0, \nu(\nu + 2a(\chi_0) + 3)\}$; then $\gamma(J) \subset \mathcal{T}(S)_{\epsilon_0}$, and the distance in $\mathcal{CG}(S)$

between any two curves $\alpha, \beta \in \mathcal{C}(S)$ whose length at the two endpoints of γ is at most χ_0 is not smaller than 3.

The curve graph $\mathcal{CG}(S)$ is hyperbolic, and $\Upsilon_{\mathcal{T}} \circ \gamma : J \to \mathcal{CG}(S)$ is a ν-quasi-geodesic by assumption. Thus if J is not bounded from above (or below) then the points $\Upsilon_{\mathcal{T}}(\gamma(t))$ converge as $t \to \infty$ (or as $t \to -\infty$) to a point in the Gromov boundary $\partial\mathcal{CG}(S)$ of $\mathcal{CG}(S)$. We call this point the *right endpoint* (or the *left endpoint*) of $\Upsilon_{\mathcal{T}} \circ \gamma$ (see [BH99]). Recall that $\partial\mathcal{CG}(S)$ consists of minimal geodesic laminations which fill up S.

A simple closed curve $\alpha \in \mathcal{C}(S)$ supports a unique projective measured geodesic lamination which we denote by $[\alpha]$. Similarly, for a measured geodesic lamination $\lambda \in \mathcal{ML}$ we denote by $[\lambda]$ the projective class of λ. Following Mosher [Mo03], we say that the projective measured geodesic lamination $[\alpha]$ defined by a simple closed curve $\alpha \in \mathcal{C}(S)$ is *realized* at some $t \in J$ if the length of α with respect to the metric $\gamma(t) \in \mathcal{T}(S)$ is at most χ_0. The number of projective measured geodesic laminations which are realized at a given point $t \in J$ is uniformly bounded, and $[\Upsilon_{\mathcal{T}}(\gamma(t))]$ is realized at $\gamma(t)$.

Similarly, we say that the projectivization $[\lambda]$ of a measured geodesic lamination λ is realized at an infinite right or left "endpoint" of J if the support of λ equals the corresponding right or left endpoint of the quasi-geodesic $\Upsilon_{\mathcal{T}} \circ \gamma : J \to \mathcal{CG}(S)$ in the Gromov boundary $\partial\mathcal{CG}(S)$ of $\mathcal{CG}(S)$, viewed as a minimal geodesic lamination. The set of projective measured geodesic laminations which are realized at a fixed infinite endpoint of J is a nonempty closed subset of $\mathcal{PML}$ [Kl99, H06]. We call a projective measured geodesic lamination which is realized at a (finite or infinite) endpoint of J an *endpoint lamination*.

The assignment $t \to \Upsilon_{\mathcal{T}}(\gamma(t))$ is a ν-quasi-geodesic in $\mathcal{CG}(S)$ by assumption, and the diameter in $\mathcal{CG}(S)$ of the set of all curves of length at most χ_0 with respect to some fixed hyperbolic metric $h \in \mathcal{T}(S)$ is at most $a(\chi_0)$ where $a(\chi_0) > 0$ is as in Lemma 2.1. The projective measured geodesic laminations $[\alpha], [\beta]$ defined by two curves $\alpha, \beta \in \mathcal{C}(S)$ with $d_{\mathcal{C}}(\alpha, \beta) \geq 3$ jointly fill up S, i.e. are such that $i([\alpha], \zeta) + i([\beta], \zeta) > 0$ for every measured geodesic lamination $\zeta \in \mathcal{ML}$ [MM99, Bw06] (this makes sense for projective measured geodesic laminations). Hence by the assumption on the lower bound for the diameter of the parameter interval J, any two projective measured geodesic laminations $[\alpha], [\beta]$ which are realized at the two distinct endpoints of J jointly fill up S. This is also valid if J is unbounded since in this case an endpoint lamination $[\lambda]$ at an infinite endpoint of J jointly fills up S with every projective measured geodesic lamination whose support does not coincide with the support of $[\lambda]$. Thus for every $\alpha \in \mathcal{C}(S)$ it jointly fills up S with $[\alpha]$, and it jointly fills up S with every projective measured geodesic lamination whose support is a point in $\partial\mathcal{CG}(S)$ distinct from the support of $[\lambda]$.

Any pair of distinct points $[\lambda] \neq [\mu] \in \mathcal{PML}$ which jointly fill up the surface S define up to parametrization a unique Teichmüller geodesic whose cotangent line consists of area one quadratic differentials with vertical measured geodesic lamination contained in the class $[\lambda]$ and with horizontal measured geodesic lamination contained in the class $[\mu]$. Therefore, for every ν-quasi-geodesic $\zeta : J \to (\mathcal{T}(S), d_{\mathcal{T}})$ with $|J| \geq \max\{1/\epsilon_0, \nu(\nu + 2a(\chi_0) + 3)\}$ such that $\Upsilon_{\mathcal{T}} \circ \zeta$ is a ν-quasi-geodesic in $\mathcal{CG}(S)$, any pair of projective measured geodesic laminations $[\lambda],[\mu]$ realized at the two (possibly infinite) endpoints of J defines up to parametrization a unique Teichmüller geodesic $\eta([\lambda],[\mu])$.

Let $\rho : \mathcal{T}(S)\times\mathcal{T}(S)\to[0,\infty)$ be as in Lemma 3.3. By Lemma 3.3, for every $\nu>1$ there is a constant $p=p(\nu)>1$ with the following property. If $\gamma : J\to\mathcal{T}(S)$ is such that $\Upsilon_{\mathcal{T}}\circ\gamma$ is a ν-quasi-geodesic in $\mathcal{CG}(S)$, then γ is a p-quasi-geodesic with respect to the function ρ. By this we mean that

$$\rho(\gamma(s),\gamma(t))/p-p\leq|s-t|\leq p\rho(\gamma(s),\gamma(t))+p \text{ for all } s,t\in J. \tag{6}$$

Vice versa, for every $p>1$ there is a constant $\nu=\nu(p)>1$ such that if $\gamma : J\to\mathcal{T}(S)$ is a p-quasi-geodesic with respect to ρ, then $\Upsilon_{\mathcal{T}}\circ\gamma$ is a ν-quasi-geodesic in $\mathcal{CG}(S)$.

As in Section 2, for $h\in\mathcal{T}(S)$ and $\mu\in\mathcal{ML}$ denote by $\ell_h(\mu)$ the *h-length* of μ; The function $(h,\mu)\in\mathcal{T}(S)\times\mathcal{ML}\to(0,\infty)$ is continuous.

Using an idea of Mosher [Mo03], for $p>1$ define Γ_p to be the set of all triples $(\gamma : J\to\mathcal{T}(S),\lambda_+,\lambda_-)$ with the following properties.

(1) $J\subset\mathbb{R}$ is a closed connected set of diameter at least $\max\{1/\epsilon_0,\nu(p)(\nu(p)+2a(\chi_0)+3)\}$ where $\epsilon_0=\epsilon_0(\nu(p))$ is as in Lemma 3.1, and $0\in J$.
(2) $\gamma : J\to\mathcal{T}(S)$ is a p-Lipschitz curve which is a p-quasi-geodesic with respect to the function ρ.
(3) $\lambda_+,\lambda_-\in\mathcal{ML}$ are measured geodesic laminations of $\gamma(0)$-length 1, and the projective measured geodesic lamination $[\lambda_+]$ is realized at the right endpoint of J, the projective measured geodesic lamination $[\lambda_-]$ is realized at the left endpoint of J.

We equip Γ_p with the product topology, using the weak*-topology on $\mathcal{ML}$ for the second and the third component of the triple and the compact-open topology for the arc $\gamma : J\to\mathcal{T}(S)$. Note that this topology is metrizable.

We follow Mosher (Proposition 3.17 of [Mo03]) and show that the action of $\mathrm{Mod}(S)$ on Γ_p is cocompact. For this note first that by Lemma 3.1, for every $(\gamma : J\to\mathcal{T}(S),\lambda_+,\lambda_-)\in\Gamma_p$ the image $\gamma(J)$ is contained in $\mathcal{T}(S)_{\epsilon_0}$ where $\epsilon_0=\epsilon_0(\nu(p))>0$ is as in the first part of the definition of Γ_p. Since $\mathrm{Mod}(S)$ acts isometrically and cocompactly on $\mathcal{T}(S)_{\epsilon_0}$ and since the function ρ is invariant under the diagonal action of $\mathrm{Mod}(S)$, it is enough to show that the subset of Γ_p consisting of triples with the additional property that $\gamma(0)$ is contained in a fixed compact subset A of $\mathcal{T}(S)_{\epsilon_0}$ is compact. Now the topology on Γ_p is metrizable and hence this follows if every sequence in Γ_p contained in the subset $\{(\gamma : J\to\mathcal{T}(S),\lambda_+,\lambda_-)\in\Gamma_p\mid\gamma(0)\in A\}$ has a convergent subsequence.

By the Arzela-Ascoli theorem, the set of p-Lipschitz maps $\gamma : J\to\mathcal{T}(S)_{\epsilon_0}$ where $J\subset\mathbb{R}$ is a closed connected subset containing 0 and such that $\gamma(0)\in A$ is compact with respect to the compact open topology. Moreover, the function ρ on $\mathcal{T}(S)\times\mathcal{T}(S)$ is continuous and hence if $\gamma_i : J_i\to\mathcal{T}(S)$ converges locally uniformly to $\gamma : J\to\mathcal{T}(S)$ and if γ_i is a p-quasi-geodesic with respect to ρ for all i then the same is true for γ. Since the function on $\mathcal{T}(S)\times\mathcal{ML}$ which assigns to a metric $h \in \mathcal{T}(S)$ and a measured geodesic lamination $\mu \in \mathcal{ML}$ the h-length of μ is continuous and since for every fixed $h \in \mathcal{T}(S)$ the set of measured geodesic laminations of h-length 1 is compact and a section for the map $\mathcal{ML} \to \mathcal{PML}$, the action of $\mathrm{Mod}(S)$ on Γ_p is indeed cocompact provided that the following holds: If $\gamma_i : J_i \to \mathcal{T}(S)_{\epsilon_0}$ $(i > 0)$ is a sequence of p-Lipschitz curves which converge locally uniformly to $\gamma : J \to \mathcal{T}(S)_{\epsilon_0}$, if for each i the projective measured geodesic lamination $[\lambda_i]$ is realized at the right endpoint of J_i and if $[\lambda_i] \to [\lambda]$ in $\mathcal{PML}$ $(i \to \infty)$ then $[\lambda]$ is realized at the right endpoint of J.

To see that this is indeed the case, assume first that $J \cap [0, \infty) = [0, b]$ for some $b \in (0, \infty)$. Then for sufficiently large i we have $J_i \cap [0, \infty) = [0, b_i]$ with $b_i \in (0, \infty)$ and $b_i \to b$. Thus $\gamma_i(b_i) \to \gamma(b)$ $(i \to \infty)$ and therefore for sufficiently large i there is only a *finite* number of curves $\alpha \in \mathcal{C}(S)$ whose length with respect to one of the metrics $\gamma_j(b_j), \gamma(b)$ $(j \geq i)$ is at most χ_0. By passing to a subsequence we may assume that there is a simple closed curve $\alpha \in \mathcal{C}(S)$ with $[\lambda_j] = [\alpha]$ for all large j. Then $[\alpha] = [\lambda]$ is the limit of the sequence $([\lambda_j])$. The $\gamma_j(b_j)$-length of α is at most χ_0 for all sufficiently large j and hence the same is true for the $\gamma(b)$-length of α by continuity of the length function. As a consequence, the limit $[\lambda]$ of the sequence $([\lambda_j])$ is realized at the right endpoint $\gamma(b)$ of γ.

In the case that $[0, \infty) \subset J$ we argue as before. Assume first that $b_i < \infty$ for infinitely many i and that $b_i \to \infty$. Then for each i, there is a simple closed curve $\alpha_i \in \mathcal{C}(S)$ so that $[\lambda_i] = [\alpha_i]$, and by Lemma 2.1, α_i is contained in a ball about $\Upsilon_{\mathcal{T}}(\gamma_i(b_i))$ of radius $a(\chi_0) > 0$ independent of i. Since the curves $\gamma_i : J_i \to \mathcal{T}(S)_{\epsilon_0}$ converge uniformly on compact sets to the curve $\gamma : J \to \mathcal{T}(S =_{\epsilon_0}$, as $i \to \infty$ longer and longer subsegments of γ are uniformly fellow-traveled by the curves γ_i. Since the map $\Upsilon_{\mathcal{T}}$ is coarsely Lipschitz and since the maps $t \to \Upsilon_{\mathcal{T}}(\gamma_i(t))$ are $\nu(p)$-quasi-geodesics in $\mathcal{CG}(S)$, this implies that as $i \to \infty$, longer and longer subsegments of the quasi-geodesic $\Upsilon_{\mathcal{T}} \circ \gamma$ in $\mathcal{CG}(S)$ are uniformly fellow-traveled by the quasi-geodesics $\Upsilon_{\mathcal{T}} \circ \gamma_i$.

By hyperbolicity and the definition of the topology on the union of a hyperbolic geodesic metric space with its Gromov boundary, we conclude that as $i \to \infty$, the simple closed curves $\alpha_i \in \mathcal{C}(S)$ converge in $\mathcal{CG}(S) \cup \partial\mathcal{CG}(S)$ to the endpoint $\mu \in \partial\mathcal{CG}(S)$ of the $\nu(p)$-quasi-geodesic ray $\Upsilon_{\mathcal{T}} \circ \gamma[0, \infty)$. Namely, a neighborhood basis in $\mathcal{CG}(S) \cup \partial\mathcal{CG}(S)$ of a point $\mu \in \partial\mathcal{CG}(S)$ can be obtained as follows. Choose a fixed point $x \in \mathcal{CG}(S)$, any sufficiently large number $L > 1$ (depending on the hyperbolicity constant) and a $\nu(p)$-quasi-geodesic $\xi : [0, \infty) \to \mathcal{CG}(S)$ connecting x to μ. For every $m > 0$ let V_m be the set of all endpoints in $\mathcal{CG}(S) \cup \partial\mathcal{CG}(S)$ of finite or infinite $\nu(p)$-quasi-geodesics in $\mathcal{CG}(S)$ containing $\xi[0, m]$ in their L-neighborhood. Then the sets V_m $(m > 0)$ are a neighborhood basis of μ (see [BH99] for details).

Now by a result of Klarreich (Theorem 1.4 of [Kl99], see also [H06]), if $(\alpha_i) \subset \mathcal{C}(S)$ is any sequence which converges in $\mathcal{CG}(S) \cup \partial\mathcal{CG}(S)$ to a point $\mu \in \partial\mathcal{CG}(S)$, then up to passing to a subsequence, the projective measured geodesic laminations $[\alpha_i] \subset \mathcal{PML}$ converge to a projective measured geodesic lamination supported in μ. But $[\alpha_i] = [\lambda_i] \to [\lambda]$ in $\mathcal{PML}$ by assumption and hence the lamination $[\lambda]$ is supported in μ. In other words, $[\lambda]$ is realized at the right (infinite) endpoint of J.

In the case that $b_i = \infty$ for all but finitely many i, the same argument can also be applied. Namely, in this case the uniform quasi-geodesics $\Upsilon_{\mathcal{T}} \circ \gamma_i : [0,\infty) \to \mathcal{CG}(S)$ have right endpoints $\beta_i \in \partial\mathcal{CG}(S)$. Since $\gamma_i \to \gamma$ uniformly on compact sets, we conclude as before that the points β_i converge as $i \to \infty$ in $\partial\mathcal{CG}(S)$ to the endpoint β of $\Upsilon_{\mathcal{T}} \circ \gamma$. Now the projective measured geodesic lamination $[\lambda_i]$ is supported in β_i, and the sequence $[\lambda_i]$ converges in $\mathcal{PML}$ to $[\lambda]$. Since the topology of $\partial\mathcal{CG}(S)$, viewed as a space of geodesic laminations, is just the measure forgetting topology induced from the space $\mathcal{PML}$, we conclude that $[\lambda]$ is supported in β. But this just means that $[\lambda]$ is realized at the right (infinite) endpoint of J (see [H06]). As a consequence, the sufficient condition stated above for cocompactness of the action of $\mathrm{Mod}(S)$ on Γ_p is satisfied.

Now we follow Section 3.10 of [Mo03]. As discussed above, the requirement (1) in the definition of Γ_p implies that each point $(\gamma : J \to \mathcal{T}(S), \lambda_+, \lambda_-) \in \Gamma_p$ determines the geodesic $\eta([\lambda_+],[\lambda_-])$. The unit cotangent line of this geodesic is the set q_t of quadratic differentials with vertical and horizontal measured geodesic laminations

$$(e^t\lambda_+, e^{-t}\lambda_-/i(\lambda^+,\lambda^-))\,(t \in \mathbb{R}).$$

For $(\gamma : J \to \mathcal{T}(S), \lambda_+, \lambda_-) \in \Gamma_p$ let $\sigma(\gamma,\lambda_+,\lambda_-) \in \mathcal{T}(S)$ be the point on the geodesic $\eta([\lambda_+],[\lambda_-])$ which is the foot-point of the quadratic differential defined by the pair $(\lambda_+/\sqrt{i(\lambda^+,\lambda^-)}, \lambda_-/\sqrt{i(\lambda^+,\lambda^-)})$. Since the length function on $\mathcal{T}(S) \times \mathcal{ML}$ is continuous and the intersection form on $\mathcal{ML} \times \mathcal{ML}$ is continuous, the map taking $(\gamma : J \to \mathcal{T}(S), \lambda_+, \lambda_-)$ to $(\gamma(0), \sigma(\gamma,\lambda_+,\lambda_-)) \in \mathcal{T}(S) \times \mathcal{T}(S)$ is continuous. Moreover by construction, this map is equivariant with respect to the natural action of $\mathrm{Mod}(S)$ on Γ_p and on $\mathcal{T}(S) \times \mathcal{T}(S)$. Since the action of $\mathrm{Mod}(S)$ on Γ_p is cocompact, the same is true for the action of $\mathrm{Mod}(S)$ on the image of this map (see [Mo03] for a similar reasoning). Thus the distance between $\gamma(0)$ and $\sigma(\gamma,\lambda_+,\lambda_-)$ is bounded from above by a universal constant $b > 0$.

Let again $(\gamma : J \to \mathcal{T}(S), \lambda_+, \lambda_-) \in \Gamma_p$. For each $s \in J$ define

$$a_-(s) = \frac{1}{\ell_{\gamma(s)}(\lambda_-)}, \quad a_+(s) = \frac{1}{\ell_{\gamma(s)}(\lambda_+)}$$

where as before, $\ell_{\gamma(s)}(\lambda_\pm)$ is the $\gamma(s)$-length of $\lambda_\pm$. These are continuous functions of $s \in J$. For $s \in J$ define the shift $\gamma^s(t) = \gamma(t+s)$; then the ordered triple $(\gamma^s, a_+(s)\lambda_+, a_-(s)\lambda_-)$ lies in the $\mathrm{Mod}(S)$-cocompact set Γ_p and hence the distance between $\gamma(s)$ and a suitably chosen point on the geodesic $\eta([\lambda_+],[\lambda_-])$ is at most b. As a consequence, the arc γ is contained in the b-neighborhood of the geodesic $\eta([\lambda_+],[\lambda_-])$. Since the curve γ is p-Lipschitz, this implies that the Hausdorff distance between $\gamma(J)$ and a suitably chosen subarc of $\eta([\lambda_+],[\lambda_-])$ is uniformly bounded. The proposition is proven. □

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