ON RANDOM WALKS ON THE MAPPING CLASS GROUP

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ABSTRACT. We define an electrification of the curve graph of a surface S of finite type and use it to identify the Poisson boundary of a random walk on the mapping class group of S with some logarithmic moment condition as a stationary measure on the space of minimal and maximal geodesic laminations on S, equipped with the Hausdorff topology.

1. INTRODUCTION

The *mapping class group* $\mathcal{MCG}$ of a surface of finite type, that is, a closed surface S of genus $g \geq 0$ from which $m \geq 0$ points, so-called *punctures*, have been deleted, is the group of isotopy classes of diffeomorphisms of S. We assume that S is *non-exceptional*, that is, S is not a sphere with at most 4 punctures or a torus with at most 1 puncture. The mapping class group is finitely presented.

A *random walk* on the mapping class groups is generated by a probability measure μ on $\mathcal{MCG}$. The n-th convolution of μ is the measure defined inductively by $\mu^{(n)} = \sum_h \mu(h)\mu^{(n-1)}(h^{-1}g)$, with $\mu^{(1)} = \mu$. We then can consider the *Poisson boundary* of the random walk which is a maximal *stationary measure* ν. Here a measure ν on some measure space with a $\mathcal{MCG}$-action is called stationary if it satisfies the identity

$$\nu = \sum_h \mu(h)h_*\nu = \nu.$$

The Poisson boundary is a purely measure theoretic object, but we can ask whether it can be realized on a topological $\mathcal{MCG}$-space X. By this we mean that there exists a probability measure ν on X whose measure class is invariant and so that for the $\mathcal{MCG}$-action on X, the measure ν can be identified with the Poisson boundary, defined on the Borel σ-algebra on X.

Kaimanovich and Masur [KM96] discovered that the Poisson boundary of $\mathcal{MCG}$ can be realized on the space $\mathcal{PML}$ of *projective measured geodesic laminations* on S provided that the measure μ has finite entropy and finite first logarithmic moment with respect to the *Teichmüller distance* $d_{\mathcal{T}}$ on *Teichmüller space* $\mathcal{T}(S)$. This means that for any point $x \in \mathcal{T}(S)$, we have $\sum_h \mu(h)(\max\{0, \log d_{\mathcal{T}}(x, hx)\}) < \infty$.

Gadre and Maher [GM18] showed that under the stronger finite moment condition $\sum_h \mu(h)d(e,h) < \infty$ where d is a word metric on $\mathcal{MCG}$, almost every biinfinite

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path converges to a point in $\mathcal{PML}$, and the pair of forward and backward limit points define a Teichmüller geodesic in the *principal stratum of marked quadratic differentials*, consisting of differentials with only simple zeros. Moreover, the same holds true if the support of μ not necessarily generates $\mathrm{Mod}(S)$ but generates a semi-group H that is non-elementary and contains a pseudo-Anosov mapping class belonging to the principal stratum. By this we mean that a quadratic differential tangent to its invariant Teichmüller geodesic is contained in the principal stratum.

The main purpose of this note is to provide a different perspective on this result and show the following strengthening. Recall that the *curve graph* $\mathcal{CG}(S)$ of S is the graph whose vertices are isotopy classes of essential simple closed curves on S and where two such curves are connected by an edge of length one if they can be realized disjointly. The curve graph $\mathcal{CG}(S)$, equipped with the natural simplicial metric $d_{\mathcal{CG}(S)}$, is a Gromov hyperbolic geodesic metric graph [MM99], and the mapping class group acts on $\mathcal{CG}(S)$ as a group of simplicial automorphisms.

The Gromov boundary $\partial\mathcal{CG}(S)$ of the curve graph $\mathcal{CG}(S)$ can naturally be identified with the space of *minimal filling geodesic laminations*, equipped with the so-called *coarse Hausdorff topology.* Here a geodesic lamination is filling if it decomposes S into ideal polygons and once punctured ideal polygons. If these polygons are all ideal triangles and once punctured monogons, then the geodesic lamination is called *maximal.*

A probability measure μ on $\mathcal{MCG}$ has *finite logarithmic moment* for the action on $\mathcal{CG}(S)$ if $\sum_h \mu(h)(\max\{d_{\mathcal{CG}(S)}(c,hc)),0\}) < \infty$. Note that this condition is weaker than stating that the logarthmic moment for the action on $\mathcal{T}(S)$ is finite.

Theorem 1. *Let μ be a probability measure on $\mathcal{MCG}$ whose support generates a non-elementary semi-subgroup of $\mathcal{MCG}$ which contains at least one pseudo-Anosov element belonging to the principal stratum. Assume moreover that μ has finite entropy and finite logarithmic moment for the action on the curve graph. Then the Poisson boundary of the random walk on $\mathcal{MCG}$ generated by μ can be realized as a stationary measure on the space of minimal and maximal geodesic laminations on S, equipped with the Hausdorff topology.*

The proof of Theorem 1 uses a result of independent interest which we explain next.

Definition. The *principal curve graph* of a closed surface S of genus $g \geq 2$ is the graph whose vertices are simple closed curves and where two such curves c, d are connected by an edge of length one if $S \setminus (c \cup d)$ has a complementary component which neither is a fourgon nor a sixgon nor a once punctured bigon.

We show in Section 2 and Section 3.

Theorem 2. *The principal curve graph is a quasi-tree of infinite diameter. Its Gromov boundary equals the space of minimal and maximal geodesic laminations, equipped with the Hausdorff topology.*

Theorem 1 is obtained from Theorem 2 and the work of Maher and Tiozzo [MT18]. The principal curve graph and its variants mentioned in Remark 3.5 is also very useful for the understanding of the geometry of the curve graph and the mapping class group, but we defer this discussion to forthcoming work.

The main part of the article is divided into three sections. In Section 2 we introduce the principal curve graph and show that it is hyperbolic, of infinite diameter. In Section 3 we show that it is a quasi-tree whose Gromov boundary can be identified with the space of minimal and maximal geodesic laminations, equipped with the Hausdorff topology. In Section 4 we prove Theorem 1.

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2. The principal curve graph

The vertices of the *curve graph* $\mathcal{CG}(S)$ of a non-exceptional surface S of finite type are isotopy classes of essential simple closed curves on S, that is, simple closed curves which are not contractible or homotopic into a puncture, and where two such vertices are connected by an edge of length one if they can be realized disjointly.

Let c, d be two simple closed curves on S in minimal position, that is, c, d intersect in the minimal number of points. In the sequel we always assume that this is the case. Then $S \setminus (c \cup d)$ is a union of complementary regions whose boundaries consist of subarcs of c and d in alternating order. In particular, a complementary component which is simply connected is a polygon with an even number of sides, and a complementary component which is a punctured disc is a punctured polygon with an even number of sides.

The curves c, d *bind* S if each component of $S \setminus (c \cup d)$ is a disc or a once punctured disc. This is equivalent to stating that there is no component of $S \setminus (c \cup d)$ which contains an essential simple closed curve.

Let us assume in the sequel that c, d bind S. By reasons of Euler characteristic, there is at least one component of $S \setminus (c \cup d)$ which is a polygon with at least 6 sides or a once punctured polygon with at least 4 sides. As in the introduction, we define the principal curve graph $\mathcal{PC}(S)$ of S as the graph whose vertices are isotopy classes of essential simple closed curves on S and where two such curves c, d are connected by an edge of length one if there is a component of $S \setminus (c \cup d)$ which either contains an essential simple closed curve of S, or is a polygon with at least 8 sides or a once punctured polygon with at least 6 sides. Clearly the mapping class group acts on $\mathcal{PC}(S)$ as a group of simplicial automorphisms.

The goal of this section is to show.

Proposition 2.1. *The principal curve graph is a hyperbolic geodesic metric space of infinite diameter.*

The principal curve graph $\mathcal{PC}(S)$ contains a $\mathcal{MCG}$-invariant subgraph $\mathcal{CG}_0(S)$, with the same set of vertices, that is, vertices are simple closed curves, and where two such vertices c, d are connected by an edge of length one if there is a component of $S \setminus (c \cup d)$ which contains an essential simple closed curve of S. Then this curve is disjoint from both c, d and therefore the distance in the curve graph $\mathcal{CG}(S)$ of S between c, d is at most two. Vice versa, if c, d are connected in $\mathcal{CG}(S)$ by an edge, then they are disjoint. If S is not a five punctured sphere or a twice punctured torus, then $S \setminus (c \cup d)$ contains a component which is not simply connected and c, d are connected in $\mathcal{CG}_0(S)$ by an edge. If S is a five punctured sphere or a twice punctured torus then it is easy to see that there is a simple closed curve e which is connected to both c, d by an edge in $\mathcal{CG}_0(S)$. Thus we have shown.

Lemma 2.2. *$\mathcal{CG}_0(S)$ is two-quasi isometric to the curve graph $\mathcal{CG}(S)$ of S.*

As $\mathcal{PC}(S)$ is obtained from $\mathcal{CG}_0(S)$ by adding some edges, the graph $\mathcal{PC}(S)$ can be thought of as an electrification of $\mathcal{CG}(S)$. If $d_{\mathcal{PC}(S)}(c, d) \geq 2$ then we say that c, d *completely fill* S.

A *train track* on S is an embedded 1-complex $\tau \subset S$ whose edges (called *branches*) are smooth arcs with well-defined tangent vectors at the endpoints. At any vertex (called a *switch*) the incident edges are mutually tangent. Through each switch there is a path of class C^1 which is embedded in τ and contains the switch in its interior. In particular, the half-branches which are incident on a fixed switch are divided into two classes according to the orientation of an inward pointing tangent at the switch. Each closed curve component of τ has a unique bivalent switch, and all other switches are at least trivalent. The complementary regions of the train track have negative Euler characteristic, which means that they are different from discs with $0, 1$ or 2 cusps at the boundary and different from annuli and once-punctured discs with no cusps at the boundary. A train track is called *maximal* if each of its complementary components either is a trigon, that is, a topological disc with three cusps at the boundary, or a once punctured monogon, that is, a once punctured disc with one cusp at the boundary. We always identify train tracks which are isotopic. The book [PH92] contains a comprehensive treatment of train tracks which we refer to throughout the article.

A train track is called *generic* if all switches are at most trivalent. The train track τ is called *transversely recurrent* if every branch b of τ is intersected by an embedded simple closed curve $c = c(b) \subset S$ of class C^1 which intersects τ transversely and is such that $S \setminus (\tau \cup c)$ does not contain an embedded *bigon*, that is, a disc with two corners at the boundary.

A *transverse measure* on a train track τ is a nonnegative weight function μ on the branches of τ satisfying the *switch condition*: for every switch s of τ, the half-branches incident on s are divided into two classes, and the sums of the weights over all half-branches in each of the two classes coincide. The train track is called *recurrent* if it admits a transverse measure which is positive on every branch. We denote by $\mathcal{TT}$ the set of all recurrent and transversely recurrent (marked) train tracks on S. The mapping class group of S acts on $\mathcal{TT}$ as a group of transformations.

A *geodesic lamination* for a complete hyperbolic structure on S of finite volume is a *compact* subset of S which is foliated into simple geodesics. A geodesic lamination λ is *minimal* if each of its half-leaves is dense in λ. A geodesic lamination is *maximal* if its complementary regions are all ideal triangles or once punctured monogons (note that a minimal geodesic lamination can also be maximal). The space of geodesic laminations on S equipped with the *Hausdorff topology* is a compact metrizable space.

A geodesic lamination λ is called *complete* if λ is maximal and can be approximated in the Hausdorff topology by simple closed geodesics. The space $\mathcal{CL}$ of all complete geodesic laminations equipped with the Hausdorff topology is compact. The mapping class group $\mathcal{MCG}(S)$ naturally acts on $\mathcal{CL}$ as a group of homeomorphisms. Every geodesic lamination λ which is a disjoint union of finitely many minimal components is a *sublamination* of a complete geodesic lamination, that is, there is a complete geodesic lamination which contains λ as a closed subset (Lemma 2.2 of [H09]).

A train track or a geodesic lamination σ is *carried* by a transversely recurrent train track τ if there is a map $\varphi : S \to S$ of class C^1 which is homotopic to the identity and maps σ into τ in such a way that the restriction of the differential of φ to the tangent space of σ vanishes nowhere; note that this makes sense since a train track has a tangent line everywhere. We call the restriction of φ to σ a *carrying map* for σ. Every geodesic lamination λ which is carried by σ is also carried by τ. A train track τ is called *complete* if it is generic and transversely recurrent and if it carries a complete geodesic lamination. The space of complete geodesic laminations carried by a complete train track τ is open and closed in $\mathcal{CL}$ (Lemma 2.3 of [H09]). In particular, the space $\mathcal{CL}$ is totally disconnected.

The following lemma is based on a construction due to Masur and Minsky (Section 4 of [MM04]).

Lemma 2.3. *Let c, d be any two simple closed curves on S which bind S. Then there exist a recurrent one-switch train track $\eta(c, d)$ which carries d and intersects c in a single point. This train track is maximal only if c, d completely fill S.*

Proof. Using the notations from the lemma, choose a component I of $c \setminus d$ contained in the boundary of a polygonal component of $S \setminus (c \cup d)$ with at least 6 sides or a once punctured polygonal component of $S \setminus (c \cup d)$ with at least four sides. Such a component C always exists by reasons of Euler characteristic. If the distance between c, d in $\mathcal{PC}(S)$ equals one, then we may assume that C is a polygonal component with at least 8 sides or a once punctured polygonal component with at least 4 sides. Contract $c \setminus I$ to a point. The graph G obtained from $c \cup d$ in this way has a single vertex p which is the image of $c \setminus I$. Furthermore, it intersects c in a single point. Impose a switch structure at the vertex p as follows.

Declare all half-edges of G which are subarcs of d and leave c to a fixed side to be incoming, and declare the half-edges of G which are subarcs of d and leave c to the opposite side to be outgoing. The result of this construction is a *bigon track* $\hat{\sigma}$, that is, a graph which has all properties of a train track except that it may contain bigons. These bigons can be collapsed to yield a train track σ which carries d and intersects c in a single point. As σ carries the simple closed curve d and is filled by d, that is, a carrying map $d \to \sigma$ to surjective, the train track σ is recurrent.

Let us inspect the complementary regions of $\hat{\sigma}$. If E is a component of $S \setminus (c \cup d)$ with 2ℓ sides not containing I, then each side e of E contained in c is contracted to a point in the construction of $\hat{\sigma}$, and the two sides of E adjacent to e meet at a cusp of the complementary region of $\hat{\sigma}$ which is the collapse of E. Thus any complementary polygon (or once punctured complementary polygon) of $S \setminus (c \cup d)$ with 2ℓ sides not containing I gives rise to a complementary component of $\hat{\sigma}$ which is a topological disk (or once punctured disk) with ℓ cusps in the boundary. Complementary quadrangles collapse to complementary bigons.

The side I of the component C is removed, and the component C merges with the second component C' of $S \setminus (c \cup d)$ which contains I in its boundary to a complementary component D of $\hat{\sigma}$. To analyze this component, we distinguish three cases.

If $C' = C$ then D contains an essential simple closed curve. Namely, in this case there is a simple closed curve in $C \cup I$ which intersects c in a single point contained in I, and this simple closed curve is contained in the complementary region D of $\hat{\sigma}$.

Now let us assume that C is a polygon without puncture and $2\ell \geq 6$ sides and that $C' \neq C$. If C' is a polygon with $2k \geq 4$ sides, then D is disk with $\ell + k - 2 \geq \ell$ cusps in the boundary. Similarly, if C' is a once punctured polygon with $2k \geq 2$ sides, then D is a once punctured disk with $\ell + k - 2 \geq \ell - 1$ cusps in the boundary.

If both C, C' are once punctured polygons, then the component D contains two punctures and hence it contains an essential simple closed curve on S surrounding the punctures.

As a consequence, if C is a polygon with at least 8 sides or a once punctured polygon with at least 4 sides, then the bigon track $\hat{\sigma}$ and hence the train track σ contains a complementary component which either contains an essential simple closed curve, or is a polygon with at least 8 sides, or is a once punctured polygon with at least 4 sides. In particular, σ is not maximal. This completes the proof of the lemma. $\square$

Denote by
$$\Psi : \mathcal{CG}(S) \to \mathcal{PC}(S)$$
the map induced by the vertex inclusion. This map is one-Lipschitz if S is distinct from a five punctured sphere or a twice punctured torus and is two-Lipschitz otherwise. An *unparameterized L-quasi-geodesic* in a geodesic metric space X is a map $\psi : [a, b] \to X$ with the property that there exists an increasing homeomorphism $\rho : [0, c] \to [a, b]$ such that $\psi \circ \rho : [0, c] \to X$ is an L-quasi-geodesic.

A *vertex cycle* of a recurrent train track τ is an immersed simple closed curve in τ which is an extreme point for the cone of all transverse measures for τ. Such a vertex cycle either is an embedded simple closed curve in τ or a dumbbell. In particular, a closed trainpath defined by a vertex cycle passes through any branch at most twice. As a consequence, the geometric intersection number between any two vertex cycles on τ is uniformly bounded, and there is a coarsely well defined map

$$\Upsilon : \mathcal{TT} \to \mathcal{CG}(S) \tag{1}$$

which associates to a train track one of its vertex cycles. Here coarsely well defined means that Υ depends on choices, but any two choices give rise to maps which map a given point to images of uniformly bounded distance. We refer to [H06] for more information on this construction. We use Lemma 2.3 to show.

Proposition 2.4. *The principal curve graph $\mathcal{PC}(S)$ is hyperbolic. Geodesics in $\mathcal{CG}(S)$ map by Ψ to uniform unparameterized quasi-geodesics in $\mathcal{PC}(S)$.*

Proof. The proposition follows from the work of Kapovich and Rafi [KR14] if we can verify that the conditions in Corollary 2.4 of [KR14] are fulfilled. For this it suffices to show the existence of a number $L > 1$ with the following property. Whenever c, d are curves whose distance in the principal curve graph is one, then there exists an L-quasi-geodesic $\rho : [0, m] \to \mathcal{CG}(S)$ connecting $\rho(0) = c$ to $\rho(m) = d$ such that the diameter of $\Psi(\rho[0, m]) \subset \mathcal{PC}(S)$ is at most L.

This is obvious if c, d do not bind S as in this case, their distance in the curve graph is at most two. If c, d bind S then we construct such a quasi-geodesic using the fact that so-called *splitting sequences* of train tracks on S are mapped by the map Υ to uniform unparameterized quasi-geodesics in $\mathcal{CG}(S)$ [H06].

Let σ be a one-switch train track constructed from c, d as in Lemma 2.3. Then σ has a complementary component different from a trigon or a once punctured monogon. In particular, any two simple closed curves carried by σ have distance at most one in $\mathcal{PC}(S)$. Furthermore, the distance in $\mathcal{PC}(S)$ between a vertex cycle of σ and the curve c is uniformly bounded since this holds true for their distance in the curve graph.

By Theorem 2.4.1 of [PH92], there is a *splitting and collision* sequence σ_i issuing from σ which consists of train tracks which carry d and which connects σ to the train track which consists of the single curve d. Each of the train tracks σ_i is carried by σ. Associating to each train track σ_i one of its vertex cycles c_i defines a uniform unparameterized quasi-geodesic in $\mathcal{CG}(S)$ [H06] which connects a vertex cycle c_0 of $\sigma = \sigma_0$, that is, a simple closed curve in a uniformly bounded neighborhood of c in $\mathcal{CG}(S)$, to the curve d. Since each of the curves c_i is carried by σ, this quasi-geodesic consists of curves whose distance to d in the principal curve graph equals at most one.

Hyperbolicity of the principal curve graph now follows from Corollary 2.4 of [KR14]. This result also shows that geodesics in $\mathcal{CG}(S)$ map to uniform unparameterized quasi-geodesic in $\mathcal{PC}(S)$. This is what we wanted to show. □

Proposition 2.4 does not imply that the principal curve graph has infinite diameter. We next show that this is indeed the case.

A *measured geodesic lamination* is a geodesic lamination equipped with a transverse invariant measure. A projective measured geodesic lamination is an equivalence class of measured laminations which are obtained from each other by scaling. The space $\mathcal{PML}$ of projective measured geodesic laminations, equipped with the weak*-topology, is compact. A simple closed curve admits a unique transverse measure up to scale and hence can be viewed as a projective measured geodesic lamination.

By a construction due to Thurston and Veech, simple closed curves c, d which bind S determine a line of *area one holomorphic quadratic differentials* on S with horizontal and vertical measured geodesic lamination supported in c, d, respectively. These differentials define the cotangent line of a *Teichmüller geodesic*. If c, d are connected by an edge in the principal curve graph, then these quadratic differentials either have a zero of order at least two, or one of the marked points (punctures) is a regular point or a zero of the differential and not a simple pole. We refer to [H24] for a more comprehensive discussion.

Using compactness of the space $\mathcal{PML}$ of projective measured geodesic laminations we observe.

Lemma 2.5. *Let (c_i, d_i) be a sequence of pairs of simple closed curves such that $d_{\mathcal{PC}(S)}(c_i, d_i) \leq 1$ for all i. Assume that the sequence c_i converges as $i \to \infty$ in $\mathcal{PML}$ to a projective measured geodesic lamination whose support μ is both minimal and maximal. Then up to passing to a subsequence, the sequence d_i converges to a projective measured geodesic lamination with support μ.*

Proof. Let $\rho \in \mathcal{PML}$ be the limit of the sequence c_i. The support of ρ equals μ. Using the notations from the lemma, by compactness and passing to a subsequence of the sequence d_i, we may assume that d_i converges in $\mathcal{PML}$ to a projective measured geodesic lamination ξ.

We argue by contradiction and assume that the support of ξ is distinct from μ. Since μ is minimal and maximal, this implies that ξ together with the projective measured geodesic lamination ρ determines a Teichmüller geodesic whose cotangent line γ consists of area one quadratic differentials with vertical and horizontal projective measured geodesic laminations ρ, ξ, respectively.

Namely, two projective measured geodesic laminations ρ, ξ determine a Teichmüller geodesic if and only if for every measured geodesic lamination ζ, the intersection between ζ and one of the two laminations ρ, ξ is positive. Note that this property is invariant under scaling and hence makes sense for projective measured geodesic laminations. As this is an open condition (for example, by continuity of the intersection form), we conclude that for sufficiently large i the pair (c_i, d_i) binds S and hence defines the cotangent line γ_i of a Teichmüller geodesic, that is, an orbit of the *Teichmüller flow* on the bundle of area one quadratic differentials. Furthermore, by continuity, the cotangent lines γ_i of these Teichmüller geodesics converge locally uniformly to γ in the bundle over Teichmüller space whose fiber over a Riemann surface X is the sphere of area one quadratic differentials on X.

On the other hand, as the distance in the principal curve graph between c_i, d_i equals one, the cotangent line γ_i is *not* contained in the *principal stratum* of quadratic differentials with only simple zeros and simple poles at the marked points (here we view an abelian differential as a quadratic differential with all zeros of even order). As the complement of the principal stratum in the Teichmüller space of quadratic differentials is closed and invariant under the Teichmüller flow, the cotangent line of the limiting Teichmüller geodesic is contained in the complement of the principal stratum as well. But any quadratic differential whose horizontal measured lamination is supported in a minimal complete geodesic lamination is contained in the principal stratum. This is a contradiction which shows the lemma. □

We are now ready to show that the diameter of $\mathcal{PC}(S)$ is infinite and complete the proof of Proposition 2.1.

Proposition 2.6. *The diameter of $\mathcal{PC}(S)$ is infinite.*

Proof. The proof of this proposition is a variation of an argument of Luo as explained in Section 4.3 of [MM99].

Let λ be a minimal complete geodesic lamination. Then λ determines a point in the Gromov boundary $\partial\mathcal{CG}(S)$ of the curve graph $\mathcal{CG}(S)$. There exists $p > 1$ and an edge path $\gamma : [0, \infty) \to \mathcal{CG}(S)$ which is a p-quasi-geodesic and connects a fixed simple closed curve c_0 to λ.

By Proposition 2.4, the assignment $i \to c_i = \gamma(i)$ defines a uniform unparameterized quasi-geodesic in the principal curve graph. Let us assume to the contrary that this quasi-geodesic is of finite diameter, that is, that $d_{\mathcal{PC}(S)}(c_0, c_i)$ is uniformly bounded. Then by passing to a subsequence, we may assume that there is a number $k > 0$ so that $d_{\mathcal{PC}(S)}(c_0, c_i) = k$ for all i. Let $c_i^1 \in \mathcal{PC}(S)$ be a vertex so that $d_{\mathcal{PC}(S)}(c_0, c_i^1) = k - 1$ and $d_{\mathcal{PC}(S)}(c_i^1, c_i) = 1$ for all sufficiently large i.

By Lemma 2.5, we know that $c_i^1 \to \lambda$ in the Hausdorff topology. Repeat this argument with the sequence c_i^1. After k such steps we conclude that $c_0 \to \lambda$ in the Hausdorff topology, which is a contradiction. □

3. Gromov boundary and the action of the mapping class group

Since the graph $\mathcal{PC}(S)$ is hyperbolic of infinite diameter, its *Gromov boundary* $\partial\mathcal{PC}(S)$ is nontrivial. Furthermore, since $\mathcal{MCG}$ acts on $\mathcal{PC}(S)$ as a group of simplicial isometries, it acts on $\partial\mathcal{PC}(S)$ as a group of transformations.

Proposition 3.1. *The Gromov boundary of the principal curve graph is the space of minimal complete geodesic laminations, equipped with the Hausdorff topology.*

Proof. A point in the Gromov boundary $\partial\mathcal{CG}(S)$ of $\mathcal{CG}(S)$ can be viewed as an equivalence class of uniform quasi-geodesic rays in $\mathcal{CG}(S)$ where two such quasi-geodesic rays are equivalent if their Hausdorff distance in $\mathcal{CG}(S)$ in finite (this is true in the situation at hand in spite of the fact that the curve graph is not locally finite).

Since $\mathcal{CG}(S)$ and $\mathcal{PC}(S)$ have the same vertices and, by Proposition 2.4, any two points in $\mathcal{PC}(S)$ can be connected by a uniform quasi-geodesic which is a reparameterization of a uniform quasi-geodesic in $\mathcal{CG}(S)$ with the same endpoints, we conclude that the Gromov boundary of the principal curve graph is a quotient of the subspace of the Gromov boundary of $\mathcal{CG}(S)$ consisting of equivalence classes of those quasi-geodesic rays in $\mathcal{CG}(S)$ whose diameter in $\mathcal{PC}(S)$ are infinite. The quotient is taken by the equivalence relation which identifies two points if they correspond to quasi-geodesic rays whose Hausdorff distance in $\mathcal{PC}(S)$ is finite.

By Proposition 2.6 and its proof, a uniform quasigeodesic in $\mathcal{CG}(S)$ connecting a basepoint c to a minimal complete geodesic lamination λ is an unparameterized uniform quasi-geodesic in $\mathcal{PC}(S)$ of infinite diameter. Thus λ defines a point in the Gromov boundary of $\mathcal{PC}(S)$.

We claim that two distinct such minimal complete geodesic laminations define non-equivalent quasi-geodesic rays in $\mathcal{PC}(S)$ and hence distinct points in the Gromov boundary of $\mathcal{PC}(S)$. Namely, such a pair of distinct points can be connected by a biinfinite uniform quasi-geodesic in the curve graph and hence by Proposition 2.4, by a biinfinite unparameterized quasi-geodesic in $\mathcal{PC}(S)$. By Proposition 2.6, its two half-rays have infinite diameter in $\mathcal{PC}(S)$, which is only possible if these laminations defined disctinct points in the boundary of $\mathcal{PC}(S)$. As a consequence, the subspace of $\partial\mathcal{CG}(S)$ of all minimal complete geodesic laminations embeds (as a set) into the Gromov boundary of $\mathcal{PC}(S)$.

We show next that a minimal filling geodesic lamination which is not maximal defines an equivalence class of uniform quasi-geodesic rays in $\mathcal{CG}(S)$ which have finite diameter in $\mathcal{PC}(S)$.

Thus let λ be a minimal geodesic lamination which fills S but which is not maximal, that is, which is not complete. By the results in Section 3 of [H24], there exists a recurrent train track ξ of the same *combinatorial type* as λ (that is, with complementary regions whose topological types coincide with the topological types of the complementary regions of λ) which carries λ. In particular, at least one of the complementary components of ξ is a polygon with more than three sides or a once punctured polygon with at least two sides.

Let (ξ_i) be a λ-splitting sequence beginning at $\xi_0 = \xi$, that is, a splitting sequence so that each ξ_i carries λ. For each i let c_i be a vertex cycle of ξ_i. As both c_0, c_i are carried by ξ, there is at least one component of $S - (c_0 \cup c_i)$ which is a polygon with at least 8 sides or a once punctured polygon with at least four sides. This yields that the distance in $\mathcal{PC}(S)$ between c_0 and c_i is at most one for all i. Since $i \to c_i$ is an unparameterized quasi-geodesic in the curve graph of S which converges in $\mathcal{CG}(S) \cup \partial\mathcal{CG}(S)$ to $\lambda \in \partial\mathcal{CG}(S)$ [H06], this implies that λ does not define a point in the Gromov boundary of $\mathcal{PC}(S)$.

To summarize, as a set, the Gromov boundary of $\mathcal{PC}(S)$ can be identified with the subspace $\partial\mathcal{PC}(S)$ of $\partial\mathcal{CG}(S)$ of all minimal complete geodesic laminations. That this inclusion is a homeomorphism onto its image can be seen with the same arguments used before. Namely, as the inclusion $\mathcal{CG}(S) \to \mathcal{PC}(S)$ is one-Lipschitz, the topology on $\partial\mathcal{PC}(S)$ as a subspace of $\partial\mathcal{CG}(S)$ is finer than the topology on $\partial\mathcal{PC}(S)$ as the Gromov boundary of $\mathcal{PC}(S)$. Thus it suffices to show that the inclusion map $\partial\mathcal{PC}(S) \to \partial\mathcal{CG}(S)$ is continuous where $\partial\mathcal{PC}(S)$ is equipped with the topology as the Gromov boundary of $\mathcal{PC}(S)$, and this is equivalent to stating that if $\lambda_i \to \lambda$ in $\partial\mathcal{PC}(S)$, then λ_i converges to λ in the Hausdorff topology.

To establish this claim we argue by contradiction and we assume that there exists a sequence $\lambda_i \subset \partial\mathcal{PC}(S)$ which converges to $\lambda \in \partial\mathcal{PC}(S)$ but does not converge to λ in $\partial\mathcal{CG}(S)$. By compactness of the space of geodesic laminations equipped with the Hausdorff topology, up to passing to a subsequence, we may assume that $\lambda_i \to \mu$ in the Hausdorff topology where $\mu \neq \lambda$. Assume without loss of generality that $\lambda_i \neq \lambda$ for all i. Equip λ_i, λ with projective transverse measures α_i, β. Then for each i, the pair (α_i, β) of projective measured geodesic laminations determines a Teichmüller geodesic γ_i.

By passing to another subsequence, we may assume that the Teichmüller geodesics γ_i converge locally uniformly to a Teichmüller geodesic γ defined by a pair (α, β) of projective measured geodesic laminations, where α is supported in $\mu \neq \lambda$. Now by [MM99], associating to a point X on the Teichmüller geodesic γ a simple closed curve in $\mathcal{CG}(S)$ whose length for the hyperbolic metric X is uniformly bounded defines a uniform unparameterized quasi-geodesic in $\mathcal{CG}(S)$. As the Teichmüller geodesics γ_i, γ all pass through a fixed compact subset of Teichmüller space, together with Proposition 2.4, this implies that there are uniform quasi-geodesics in $\mathcal{PC}(S)$ connecting λ to λ_i which pass through a fixed subset of $\mathcal{PC}(S)$ of uniformly bounded diameter, violating the assumption that $\lambda_i \to \lambda$ in $\partial\mathcal{PC}(S)$. This completes the proof of the proposition. □

Since the space of complete geodesic laminations, equipped with the Hausdorff topology, is totally disconnected, we obtain.

Corollary 3.2. *The Gromov boundary of $\mathcal{PC}(S)$ is totally disconnected.*

It seems to be well known that a Gromov hyperbolic space with totally disconnected boundary and satisfying some additional assumptions like extendibility of uniform quasi-geodesics is a quasi-tree, that is, it is quasi-isometric to a tree. As we were not able to find references which can be applied to the principal curve graph we provided a complete proof of the following statement which completes the proof of Theorem 2.

Proposition 3.3. *The principal curve graph is a quasi-tree.*

Proof. By [Man05], we have to verify the so-called *bottleneck property*: If $\gamma : [0, k] \to \mathcal{PC}(S)$ is a geodesic edge path of length $k \geq 0$, then any edge path connecting $\gamma(0)$ to $\gamma(k)$ passes through a uniformly bounded neighborhood of $\gamma(k/2) = e$.

To see that this holds true we first establish the following

Claim: Let $c \neq d$ be simple closed curves and let τ be a complete train track which carries c but does not carry d. Then any edge path in $\mathcal{PC}(S)$ connecting c to d passes through the one-neighborhood of some vertex cycle of τ.

Proof of the claim: We proceed by induction on the length of an edge path α in the curve graph $\mathcal{CG}(S)$ connecting some simple closed curve c to some simple closed curve d. That this is legitimate follows from the fact that $\mathcal{CG}(S)$ and $\mathcal{PC}(S)$ have the same set of vertices.

Assume first that this length equals one. If c does not completely fill τ then the image of c under a carrying map $c \to \tau$ is not surjective. Then this image is a proper subtrack σ of τ, in particular, σ is not maximal. As a consequence, the distance in $\mathcal{PC}(S)$ between c and a vertex cycle of σ, which also is a vertex cycle of τ, is at most one. On the other hand, if c completely fills τ, then as d is disjoint from c, it follows from Lemma 4.5 of [MM99] that d is carried by τ, a contradiction. This shows the claim for curves c, d of distance one in $\mathcal{CG}(S)$.

Suppose the claim is true for all edge paths $\alpha : [0, m] \to \mathcal{CG}(S)$ of length at most $m-1$ for some $m \geq 2$. Let $\alpha : [0, m] \to \mathcal{CG}(S)$ be an edge path of length m and let τ be a complete train track carrying $\alpha(0)$ but not $\alpha(m)$. As before, if $\alpha(0)$ does not completely fill the train track τ, then the distance between $\alpha(0)$ and a vertex cycle of τ equals one and we are done. Otherwise $\alpha(1)$ is carried by τ by Lemma 4.5 of [MM99]. We then can apply the induction hypothesis to the path $\alpha[1, m]$ to complete the proof of the claim. ■

Let γ, e be as in the first paragraph of this proof. Assume without loss of generality that e is a vertex of $\mathcal{PC}(S)$, that is, e is a simple closed curve on S. Choose a pants decomposition P of S containing e, and choose moreover a system of *spanning curves* for P. These data define a *marking* of S. The marking in turn determines a finite collection of *train tracks in standard form* for the marking. Each of these train tracks either carries e or carries a curve which intersects e in at most two points and is of distance at most two to e in the curve graph $\mathcal{CG}(S)$. Any geodesic lamination λ on S is carried by one of these trains tracks, and if λ is complete, then it is carried by a unique such train track [PH92, H09]. Let τ_1, τ_2 be the two of these train tracks which carry c, d. If $\tau_1 \neq \tau_2$ then a path γ connecting c to d passes through the three-neighborhood of e by the above claim and the choice of τ_1, τ_2.

In the case that $\tau_1 = \tau_2 = \tau$ we modify τ with a splitting sequence of maximal length so that the resulting train track η still carries both c, d. Then no nontrivial split of η carries both c, d. We then can modify η with a single split so that the split track η' carries c but not d. As the image under Υ of a splitting sequence is a uniform unparameterized quasi-geodesic in $\mathcal{CG}(S)$ and $\mathcal{PC}(S)$, the image under the map Υ of a splitting sequence connecting τ to η' is a subarc of a uniform quasi-geodesic connecting e to both c, d. Since $\mathcal{PC}(S)$ is hyperbolic and e is contained in a geodesic connecting c to d, this implies that the distance in $\mathcal{PC}(S)$ between e and a vertex cycle of η' is uniformly bounded. Thus the proposition follows from the above claim, applied to c, d and η'. □

Recall that any pseudo-Anosov mapping class $\varphi \in \mathcal{MCG}$ has a unique *axis* in Teichmüller space $\mathcal{T}(S)$ of S, that is, an invariant Teichüller geodesic on which φ acts as a nontrivial translation.

Corollary 3.4. *Let $\varphi \in \mathcal{MCG}$ be a pseudo-Anosov mapping class. Then φ acts as a hyperbolic isometry on the principal curve graph if and only if the cotangent line of its axis is contained in the principal stratum of quadratic differentials with only simple zeros.*

Proof. A point $X \in \mathcal{T}(S)$ is a marked complete finite volume hyperbolic metric on S. Define $\Upsilon_0(X) \in \mathcal{CG}(S)$ to be a simple closed curve of uniformly bounded X-length. Such a curve exists by the Gauss Bonnet theorem which shows that the area of a hyperbolic metric on S is a topological invariant and hence a metric disc of fixed radius has to contain an essential curve as it can not lift to a diffeomorphic disc in the hyperbolic plane. If γ is any Teichmüller geodesic, then the assignment $t \to \Upsilon_0(\gamma(t))$ is a uniform unparameterized quasi-geodesic in $\mathcal{CG}(S)$ [MM99].

Let $\gamma \subset \mathcal{T}(S)$ be the axis of a pseudo-Anosov element φ and assume that the cotangent line of γ consists of quadratic differentials in the principal stratum. Then the horizontal and vertical measured geodesic laminations are minimal and complete as there can not be any horizontal or vertical saddle connections on a Teichmüller geodesic which projects to a closed curve in moduli space.

By Proposition 2.4 and Proposition 2.6, the image under Υ_0 of the line γ is an unparameterized quasi-geodesic in $\mathcal{PC}(S)$ of infinite diameter. As φ acts on this quasi-geodesic as a nontrivial translation, this quasi-geodesic is a quasi-axis for φ acting on $\mathcal{PC}(S)$ and hence φ acts as a hyperbolic isometric on $\mathcal{PC}(S)$.

Vice versa, if the axis of φ is not contained in the principal stratum, then the support of the horizontal and vertical measured geodesic laminations defined by this axis is not complete. By Proposition 3.1, this implies that the image of the axis under the map Υ_0 has finite diameter in $\mathcal{PC}(S)$. The corollary follows. □

Remark 3.5. The construction of the principal curve graph, which can be thought of as an electrification of the usual curve graph, can be used to find a system of intermediate graphs which are obtained from the curve graph by adding edges and are obtained from the principal curve graph by deleting some edges. An example is the graph whose vertices are simple closed curves and where two such vertices c, d are connected by an edge if there is at least one component of $S \setminus \{c, d\}$ which either is not simply connected or which is a polygon with at least 10 sides or if there are at least two components which are different from quadrangles or six-gons. It can be shown as in Proposition 2.4 that this graph $\mathcal{E}$ is hyperbolic, and geodesics in $\mathcal{CG}(S)$ map by the vertex inclusion to uniform unparameterized quasi-geodesics in $\mathcal{E}$. It can also fairly easily be seen that the graph $\mathcal{E}$ is not a quasi-tree but admits a quasi-isometric embedding into a product of quasi-trees. As applications of this construction are not immediate, we postpone this discussion to forthcoming work.

4. Random walks

We use Proposition 3.1 and Corollary 3.4 to show Theorem 1 from the introduction. For its formulation, call a subgroup Γ of $\mathcal{MCG}$ *non-elementary* if it contains at least two independent pseudo-Anosov elements.

The *entropy* of a probability measure μ on the group $\mathcal{MCG}$ is defined as

$$H(\mu) = -\sum_{g\in\mathcal{MCG}} \mu(g)\log\mu(g).$$

The measure μ is said to have *finite logarithmic moment* for the action of $\mathcal{MCG}$ on the curve graph $\mathcal{CG}(S)$ if

$$\sum_{g\in\mathcal{MCG}} \mu(g)(\max\{0, d_{\mathcal{CG}(S)}(c, gc)\}) < \infty.$$

The following is a more precise version of Theorem 1.

Theorem 4.1. *Let Ω be a random walk on $\mathcal{MCG}$ generated by a probability measure μ on $\mathcal{MCG}$ with the following properties.*

(1) *The support of μ generates a non-elementary semi-subgroup H of $\mathcal{MCG}$ which contains at least one pseudo-Anosov element whose attracting fixed point is a minimal complete geodesic lamination.*
(2) *μ has finite entropy and finite logarithmic moment for the action of $\mathcal{MCG}$ on the curve graph.*

Then the Poisson boundary of the walk can be realized as a $\mathcal{MCG}$-invariant measure class on the space $\partial\mathcal{PC}(S)$ of minimal complete geodesic laminations.

Proof. Since by assumption and Corollary 3.4, the semigroup H generated by the support of μ is non-elementary and contains at least one pseudo-Anosov element φ which acts as a hyperbolic isometry on $\mathcal{PC}(S)$, the semigroup H is non-elementary as a group of isometries acting on $\mathcal{PC}(S)$ (apply Proposition 3.1 and Corollary 3.4 to conjugates of φ).

Theorem 1 of [MT18] now shows that for any vertex $x \in \mathcal{PC}(S)$, almost every sample path $(\omega_n x) \subset \mathcal{PC}(S)$ converges to a point $\omega_+ \in \partial\mathcal{PC}(S)$. The resulting hitting measure ν is non-atomic, and it is the unique μ-stationary measure on $\partial\mathcal{PC}(S)$.

The same construction also applies for the action of the random walk on $\mathcal{CG}(S)$. As $\partial\mathcal{PC}(S) \subset \partial\mathcal{CG}(S)$ by Proposition 3.1, the μ-stationary measure ν on $\partial\mathcal{PC}(S)$ also can be viewed as a μ-stationary measure on $\partial\mathcal{CG}(S)$. By uniqueness, ν equals the hitting measure of the random walk on $\mathcal{CG}(S)$.

Now the action of $\mathcal{MCG}$ on $\mathcal{CG}(S)$ is acylindrical [Bw08] and therefore by Theorem 1.5 of [MT18], the Poisson boundary of $(\mathcal{MCG}, \mu)$ equals the Gromov boundary $\partial\mathcal{CG}(S)$ of $\mathcal{CG}(S)$, equipped with the hitting measure ν. But this hitting measure is just the unique stationary measure for the action of $\mathcal{MCG}$ on the boundary of the principal curve graph, which completes the proof of the theorem. $\square$

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