STABILITY OF EINSTEIN METRICS AND EFFECTIVE HYPERBOLIZATION IN LARGE HEMPEL DISTANCE

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Abstract. Extending earlier work of Tian, we show that if a manifold admits a metric that is almost hyperbolic in a suitable sense, then there exists an Einstein metric that is close to the given metric in the $C^{2,\alpha}$-topology. In dimension 3 the original manifold only needs to have finite volume, and the volume can be arbitrarily large. Applications include a new proof of the hyperbolization of 3-manifolds of large Hempel distance yielding some new geometric control on the hyperbolic metric, and an analytic proof of Dehn filling and drilling that allows the filling and drilling of arbitrary many cusps and tubes.

Contents

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1. Introduction

1.1. **Statement of the main results.** The search for *Einstein metrics* on a closed manifold M has a long and fruitful history. Such metrics can be found using the Ricci flow, perhaps with surgery. This approach was used by Perelman to prove the so-called geometrization conjecture for 3-manifolds, embarking from the easy fact that in dimensions 2 and 3, Einstein metrics have constant curvature.

An older method for the construction of Einstein metrics consists in starting from a metric $\bar{g}$ which is almost Einstein in a suitable sense, and construct a nearby Einstein metric as a perturbation of the given metric. The perturbation can be done using once again the Ricci flow, as for example in [MO90]. The cross curvature flow is another tool for evolving metrics on 3-manifolds towards an Einstein metric [KY09]. One may also use compactness properties for Riemannian manifolds with a uniform upper bound on the diameter, a uniform lower bound on the volume and a suitable curvature control, like an L^p-bound on the norm of the curvature tensor, to establish the existence of Einstein metrics which are close to a metric $\bar{g}$ with these properties and for which in addition the L^p-norm of $\mathrm{Ric}(\bar{g}) - \lambda\bar{g}$ is sufficiently small (see [PW97, Corollary 1.6] and also [And06] and [Bam12]).

Much more recently, Fine and Premoselli [FP20] constructed Einstein metrics using a gluing method which can be described as follows. Starting from a manifold M which is the union of two open submanifolds U, V admitting each an Einstein metric whose restrictions to $U \cap V$ are close to each other in a controlled way, one can glue these metrics on $U \cap V$ and try to use an implicit function theorem for the so-called *Einstein operator* at the glued metric to find a nearby Einstein metric. This method depends on stability of Einstein metrics near the given metric, which means that locally, if there is an Einstein metric in an a priori chosen neighborhood of the glued metric, then this metric is unique up to scaling and pull-back by diffeomorphisms. No a priori volume bound is necessary.

In the setting we are interested in, stability is guaranteed by curvature control. Namely, a classical result states that on compact manifolds of dimension $n \geq 3$, Einstein metrics with negative sectional curvature are isolated in the moduli space of Riemannian structures (see [Bes08, Corollary 12.73], [Koi78, Theorem 3.3]). For manifolds of infinite volume, this is no longer true. We refer to [Biq00] for more information and for references. We also note that for positive sectional curvature there are much stronger rigidity results. For example, Berger showed that if an Einstein metric has positive strictly $\frac{n-2}{n-1}$-pinched sectional curvature, then the sectional curvature is constant (see [Ber66]).

The main goal of this article is to develop a systematic approach for the construction of Einstein metrics by a perturbation of metrics whose sectional curvature is close to -1. The first result we prove is a general existence result for Einstein metrics in this setting. It requires a uniform lower bound on the injectivity radius, but no volume bounds. Its formulation is similar to the main result of an unpublished preprint of Tian [Tia], and the proof we give follows his outline. Section 5 contains a stronger but also more technical version of this result.

Theorem 1 (Stability of Einstein metrics with a lower injectivity radius bound)**.** *For any $n \geq 3$, $\alpha \in (0,1)$, $\Lambda \geq 0$, and $\delta \in (0, 2\sqrt{n-2})$ there exist constants $\varepsilon_0 = \varepsilon_0(n,\alpha,\Lambda,\delta) > 0$ and $C = C(n,\alpha,\Lambda,\delta) > 0$ with the following property. Let M be a closed n-manifold that admits a Riemannian metric $\bar{g}$ satisfying the following conditions for some $\varepsilon \leq \varepsilon_0$:*

i) $-1-\varepsilon \leq \sec_{(M,\bar{g})} \leq -1+\varepsilon$;
ii) $\operatorname{inj}(M,\bar{g}) \geq 1$;
iii) $\|\nabla \operatorname{Ric}(\bar{g})\|_{C^0(M,\bar{g})} \leq \Lambda$;
iv) It holds

$$\int_M e^{-(2\sqrt{n-2}-\delta) r_x(y)} |\operatorname{Ric}(\bar{g}) + (n-1)\bar{g}|^2_{\bar{g}}(y)\, d\mathrm{vol}_{\bar{g}}(y) \leq \varepsilon^2$$

for all $x \in M$, where $r_x(y) = d(x,y)$.

Then there exists an Einstein metric g_0 on M so that $\operatorname{Ric}(g_0) = -(n-1)g_0$ *and*

$$\|g_0 - \bar{g}\|_{C^{2,\alpha}(M,\bar{g})} \leq C\varepsilon^{1-\alpha}.$$

Moreover, if additionally $\operatorname{Ric}(\bar{g}) = -(n-1)\bar{g}$ *outside a region U, and if*

$$\int_U |\operatorname{Ric}(\bar{g}) + (n-1)\bar{g}|^2_{\bar{g}}\, d\mathrm{vol}_{\bar{g}} \leq \varepsilon^2,$$

then

$$|g_0 - \bar{g}|_{C^{2,\alpha}}(x) \leq C\varepsilon^{1-\alpha} e^{-(\sqrt{n-2}-\frac{1}{2}\delta)\operatorname{dist}_{\bar{g}}(x,U)}$$

for all $x \in M$.

We refer to Section 2.3 for a detailed explanation of the notion of Hölder norm used here.

As in [FP20], [And06] and [Bam12], the proof of Theorem 1 is based on an application of the implicit function theorem to the Einstein operator Φ (see Section 2.2 for the definition of Φ). The novelty of our approach consists in the use of Banach spaces for tensor fields whose construction is adapted to the specific geometric situation. These

Banach spaces are defined by *hybrid norms* which are a combination of Hölder- and weighted Sobolev norms.

For the main applications we have in mind, Theorem 1 is not strong enough due to the assumption of a uniform positive lower bound on the injectivity radius. But Theorem 1 does not extend in a straightforward way to finite volume manifolds without such a lower injectivity radius bound. We illustrate this in Section 6 by constructing for any $L > 1$ a metric on a closed 3-manifold which fulfills all assumptions of Theorem 1 with the exception of a uniform lower bound on the injectivity radius, with an arbitrarily small control constant ε, which is not L-bilipschitz equivalent to an Einstein metric

Finding the correct assumptions for 3-dimensional manifolds of finite volume for which an extension of Theorem 1 without the hypothesis of a uniform positive lower bound on the injectivity radius holds true is the main technical result of this article. For its formulation, recall that a complete manifold of bounded negative sectional curvature admits a decomposition into its *thick* part M_{thick} consisting of points where the injectivity radius is bigger than a fixed *Margulis constant* for the curvature bounds, and its complement M_{thin}.

Theorem 2 (Stability of Einstein metrics in dimension 3). *For all $\alpha \in (0,1)$, $\Lambda \geq 0$, $\delta \in (0,2)$, $b > 1$ and $\eta > 2$ there exist $\varepsilon_0 = \varepsilon_0(\alpha, \Lambda, \delta, b, \eta) > 0$ and $C = C(\alpha, \Lambda, \delta, b, \eta) > 0$ with the following property. Let M be a 3-manifold that admits a complete Riemannian metric $\bar{g}$ satisfying the following conditions for some $\varepsilon \leq \varepsilon_0$:*

i) $\mathrm{vol}(M, \bar{g}) < \infty$;

ii) $-1 - \varepsilon \leq \sec_{(M,\bar{g})} \leq -1 + \varepsilon$;

iii) It holds

$$\max_{\pi \subseteq T_x M} |\sec(\pi) + 1|,\ |\nabla R|(x),\ |\nabla^2 R|(x) \leq \varepsilon e^{-\eta d(x, M_{\text{thick}})}$$

for all $x \in M_{\text{thin}}$;

iv) $\|\nabla \mathrm{Ric}(\bar{g})\|_{C^0(M,\bar{g})} \leq \Lambda$;

v) It holds

$$e^{b d(x, M_{\text{thick}})} \int_M e^{-(2-\delta) r_x(y)} |\mathrm{Ric}(\bar{g}) + 2\bar{g}|^2_{\bar{g}}(y)\, d\mathrm{vol}_{\bar{g}}(y) \leq \varepsilon^2$$

for all $x \in M$, where $r_x(y) = d(x, y)$.

Then there exists a hyperbolic metric g_{hyp} on M so that

$$\|g_{\text{hyp}} - \bar{g}\|_{C^{2,\alpha}(M,\bar{g})} \leq C\varepsilon^{1-\alpha}.$$

Moreover, if additionally $\bar{g}$ is already hyperbolic outside a region $U \subseteq M$, and if

$$\int_U |\mathrm{Ric}(\bar{g}) + 2\bar{g}|^2_{\bar{g}}\, d\mathrm{vol}_{\bar{g}} \leq \varepsilon^2,$$

then for all $x \in M_{\text{thick}}$ it holds

$$|g_{\text{hyp}} - \bar{g}|_{C^{2,\alpha}}(x) \leq C\varepsilon^{1-\alpha} e^{-(1-\frac{1}{2}\delta)\mathrm{dist}_{\bar{g}}(x, U \cup \partial M_{\text{thick}})}.$$

Here R denotes the Riemann curvature endomorphism. Section 10 contains a slightly stronger but also more technical version of this result.

We believe that there is a version of Theorem 2 with similar assumptions which holds true in all dimensions. However, it turns out that the well known difference between geometric properties of the thin parts of negatively curved 3-manifolds and the thin parts of negatively curved manifolds in higher dimensions require a modification of the strategy we use, and we do not attempt to establish such an extension of Theorem 2 in this article.

In view of a good understanding of large scale geometric properties of asymptotically hyperbolic manifolds of infinite volume as considered for example in [Biq00] and [HQS12] we also expect that there are extensions of Theorem 1 to negatively curved manifolds of infinite volume with finitely generated fundamental group and asymptotically hyperbolic infinite volume ends.

Theorem 2 makes it possible to glue finite volume hyperbolic metrics which are defined on open submanifolds U, V of a given 3-manifold M along the intersection $U \cap V$ and deform the glued metrics to a hyperbolic metric. As a fairly immediate application, we obtain an analytic approach to hyperbolic Dehn filling and Dehn drilling in dimension 3, without the use of the deformation theory of hyperbolic cone manifolds. An earlier analytic proof of hyperbolic Dehn filling under a uniform upper bound for the volume which also is based on an implicit function theorem is due to Anderson [And06] and Bamler [Bam12]. We refer to [FPS19a], [HK08] for an overview on what is known to date about Dehn filling and Dehn drilling.

For the statement of our drilling result, recall that a *Margulis tube* in a negatively curved manifold M is a tubular neighborhood of a closed geodesic β which is a connected component of the thin part of M. The *radius* of the Margulis tube is the distance between the core geodesic of the tube and its boundary.

Theorem 3 (The drilling theorem). *For any $\varepsilon > 0$, $\kappa \in (0,1)$ and $m > 0$ there exists a number $R = R(\varepsilon, \kappa, m) > 0$ with the following property. Let M be a finite volume hyperbolic 3-manifold, and let $T_1, \dots, T_k$ be a family of Margulis tubes in M. Let $R_i > 0$ be the radius of the tube T_i, and let β_i be its core geodesic. If for each $r > 0$ and each $x \in M$ we have $\#\{i \mid \mathrm{dist}(x, T_i) \leq r\} \leq me^{\kappa r}$ and if $R_i \geq R$ for all i, then the manifold obtained from M by drilling each of the geodesics β_i admits a complete hyperbolic metric of finite volume, and the restriction of this hyperbolic metric to the complement of the cusps obtained from the drilling is ε-close in the C^2-topology to the restriction of the metric on M.*

The same argument which allows for drilling closed geodesics in finite volume hyperbolic 3-manifolds can also be used to Dehn fill cusps. This is formulated in our next result. Recall that the *meridian* of a solid torus T is a simple closed curve on the boundary torus ∂T of T which is homotopic to zero in T. A *torus cusp* in a hyperbolic 3-manifold is a cusp diffeomorphic to $T^2 \times [0, \infty)$ where T^2 denotes the 2-torus. Any cusp in a finite volume orientable hyperbolic 3-manifold is a torus cusp.

Theorem 4 (The filling theorem). *For any $\varepsilon > 0$, $\kappa \in (0,1)$ and $m > 0$ there exists a number $L = L(\varepsilon, \kappa, m) > 0$ with the following property. Let M be a finite volume hyperbolic 3-manifold, $C_1, \dots, C_k \subseteq M$ be a finite collection of torus cusps, and assume that for each $r > 0$ and each $x \in M$ we have $\#\{i \mid \mathrm{dist}(x, C_i) \leq r\} \leq me^{\kappa r}$. For each $i \leq k$ let α_i be a* *flat simple closed geodesic in ∂C_i of length $L_i \geq L$. Then the manifold obtained from M by filling the cusps C_i, with meridian α_i, is hyperbolic, and the restriction of its metric to the complement of the Margulis tubes obtained from the filling is ε-close to the metric on $M - \cup_i C_i$.*

Although unlike [FPS19a], we do not obtain effective constants for Dehn filling and drilling, Theorem 3 and Theorem 4 allows to drill or fill an arbitrary number of tubes or cusps with fixed meridional length as long as the boundaries of the tubes or cusps are sufficiently sparsely distributed in the manifold. We also obtain a more precise geometric control on the complement of the drilled or filled tubes or cusps which we discuss in Section 11.

The proofs of Theorem 3 and Theorem 4 use Theorem 2 to deform a metric glued from the metric on the given manifold and hyperbolic metrics on tubes and cups to a hyperbolic metric on the drilled or filled manifold. This strategy can also be used to construct hyperbolic metrics on manifolds glued in a controlled way from hyperbolic pieces as long as the gluing regions are sufficiently sparsely distributed. An example of such a construction can be found in the article [BMNS16]. We do however not discuss such a potential application of our main result here.

Apart from a new approach to drilling and filling, we also obtain results towards what sometimes is called *effective Mostow rigidity*, a program which lead among others to the solution of the so-called *ending lamination conjecture* (see [Min10], [BCM12]). The idea is as follows. Due to the groundbreaking work of Thurston and Perelman, a closed aspherical atoroidal 3-manifold admits a hyperbolic metric, which is moreover unique up to isotopy by Mostow rigidity. Thus topological information gives rise to geometric invariants, and some of these invariants, like for example the injectivity radius or the volume, should be recoverable from suitably chosen topological data. Even more ambitious, it may be possible to construct a bi-Lipschitz model for the hyperbolic metric from topological information as in [BCM12].

To implement this program, one may try to decompose a closed 3-manifold M into pieces which are equipped with hyperbolic metrics constructed from the knowledge of the pieces and knowledge on how these pieces glue together to M. This program is well suited for an application of Theorem 2.

For the formulation of our last main result, recall that a *handlebody* of genus $g \geq 1$ is a compact 3-manifold with boundary which is diffeomorphic to the connected sum of g solid tori. The boundary ∂H of such a handlebody H is a closed oriented surface ∂H of genus g. Any closed 3-manifold can be realized as the gluing $M_f = H_1 \cup_f H_2$ of two handlebodies H_1, H_2 of the same genus $g \geq 1$ along a diffeomorphism $f : \Sigma = \partial H_1 \to \partial H_2$ of the boundaries. The manifold M_f only depends on a double coset of the mapping class of Σ defined by f. The boundaries $\partial H_1, \partial H_2$ of H_1, H_2 contain collections $\mathcal{D}_1, \mathcal{D}_2$ of curves, the simple closed curves in $\partial H_1, \partial H_2$ which bound disks in H_1, H_2. We call these curves the *disk sets* of H_1, H_2. Using the identification of ∂H_1 and ∂H_2 via f, these disk sets define subsets in the *curve graph* $\mathcal{CG}(\Sigma)$ of the boundary surface Σ of H_1, H_2. The vertices of this graph are isotopy classes of simple closed curves on Σ, and two such curves are connected by an edge of length one if they can be realized disjointly.

The *Hempel distance* of the manifold M_f is the distance in $\mathcal{CG}(\Sigma)$ between the image of the disk set $\mathcal{D}_1$ of H_1 under the map f and the disk set $\mathcal{D}_2$ of H_2. Hempel [Hem01] showed that if the Hempel distance of M_f is at least 3, then M_f is aspherical atoroidal and hence hyperbolic by the work of Perelman.

The following result can be viewed as a first step towards an effective geometrization of 3-manifolds. In its formulation, we denote by $d_{\mathcal{CG}}$ the distance in the curve graph of the boundary surface Σ.

Theorem 5 (Effective hyperbolization in large Hempel distance). *For every $g \geq 2$ there exist numbers $R = R(g) > 0$ and $C = C(g) > 0$ with the following property. Let M_f be a closed 3-manifold with Heegaard surface Σ of genus g and gluing map f, and assume that $d_{\mathcal{CG}}(\mathcal{D}_1, \mathcal{D}_2) \geq R$. Then M_f admits a hyperbolic metric, and the volume of M_f for this metric is at least $C d_{\mathcal{CG}}(\mathcal{D}_1, \mathcal{D}_2)$.*

This result does not rely on the work of Perelman and does not use the Ricci flow, that is, we give a new proof of hyperbolization under the assumption of large Hempel distance. The lower bound on the Hempel distance we need is not effective, however we obtain some explicit information on the hyperbolic metric. An earlier proof of hyperbolization of random 3-manifolds without the use of the Ricci flow can be found in [FSV19].

The first geometric information which is new is the lower volume bound in terms of the Hempel distance stated in Theorem 5. Note that up to a universal constant, the volume of a closed hyperbolic 3-manifold coincides with its simplicial volume. Work of Brock (see [Bro03] and [HV22] for more details) shows that this simplicial volume is bounded from above by a fixed multiple of the smallest distance in the so-called *pants graph* of Σ between a pants decomposition consisting of pants curves in $\mathcal{D}_1$, and a pants decomposition consisting of pants curves in $\mathcal{D}_2$. We conjecture that this upper estimate computes the volume up to a universal multiplicative constant. This conjecture holds true for random 3-manifolds [Via21]. Our lower volume bound is expected to be far from sharp.

The second geometric information we obtain applies to manifolds M_f for which there is a sufficiently long segment of a minimal geodesic in the curve graph of Σ connecting $\mathcal{D}_1$ to $\mathcal{D}_2$ which has *bounded combinatorics*. In this case we obtain that the hyperbolic metric is uniformly close to a metric obtained by gluing two convex cocompact hyperbolic metrics on handlebodies near the boundary.

Motivated by [Min00], in the absence of such a segment, and assuming that the manifold M_f is equipped with a hyperbolic metric, we prove an a priori length bound for closed geodesics in M_f which arise in the following way. For a proper essential subsurface Y of Σ, denote by $\mathrm{diam}_Y(\mathcal{D}_1, \mathcal{D}_2)$ the diameter in the curve graph of Y of the *subsurface projections* of the disk sets $\mathcal{D}_1, \mathcal{D}_2$ into Y. Furthermore, for a multicurve c in Σ, let $\ell_f(c)$ be the length of the geodesic representative of c in the manifold M_f (which may be zero if c is compressible).

Theorem 6 (A priori length bounds). *Given Σ, there exists a number $p = p(\Sigma) \geq 3$, and for every $\varepsilon > 0$, there exists a number $k = k(\Sigma, \varepsilon) > 0$ with the following property. Let M_f be a hyperbolic 3-manifold of Heegaard genus g and Hempel distance at least 4* *and let $Y \subset \Sigma$ be a proper essential subsurface of Σ such that $d_{\mathcal{CG}}(\partial Y, \mathcal{D}_1 \cup \mathcal{D}_2) \geq p$. If* $\mathrm{diam}_Y(\mathcal{D}_1 \cup \mathcal{D}_2) \geq k$*, then* $\ell_f(\partial Y) \leq \varepsilon$.

Note that a priori, the statement of Theorem 6 relates a property which only depends on the manifold M_f, namely the hyperbolic length of a system of essential closed curves, which is well defined by Mostow rigidity, to a property which depends on choices, namely the choice of a Heegaard splitting for M_f. However, as the diameter of the subsurface projection of the disk sets of M_f into Y is required to be large, it follows from [JMM10] that up to isotopy, Y appears as a subsurface of every Heegaard surface of sufficiently small genus.

1.2. **Organization of the article and outline of the proofs.** The article is roughly divided into four parts which can be read independently. The first part, contained in Sections 2-5, is devoted to the proof of Theorem 1, and it is the only part containing results on manifolds of dimension different from 3. Section 2 organizes the basic set-up and collects some technical results used later on. It also introduces the conventions and notations we are going to use. In particular, we introduce the Einstein operator Φ, and we formulate and prove a general statement which allows to obtain C^0-estimates for a solution of its linearization from suitable integral bounds. The results in this section are variations of results available in the literature, adjusted to our needs.

The goal is to use an implicit function theorem to construct solutions of the equation $\Phi(\bar{g} + h) = 0$ (see Lemma 2.4). To this end it is necessary to invert the linearization of Φ and obtain a good norm control on this inverse, acting on suitably chosen Banach spaces of sections of the bundle of symmetric $(0,2)$-tensors over the manifold M. This is carried out in Section 3. As the Einstein operator is closely related to the Laplacian, we begin with establishing a uniform L^2-Poincare inequality for the manifolds with small pinched negative curvature. We then introduce the weighted Sobolev spaces which are our primary tool. They enter in the definition of the *hybrid norms* in Section 4, which are combinations of Hölder and weighted Sobolev norms. These norms are used in an a priori estimate for the linearized Einstein equation leading to an invertibility statement for the linearized equation. In Section 5 we then show that this is sufficient for an application of the implicit function theorem which completes the proof of Theorem 1.

A crucial point in this proof is a uniform C^0-bound for solutions of the linearization of the Einstein equation, which depends on a uniform lower bound on the injectivity radius. In Section 6 we show that there is no straightforward way of dropping this assumption by exhibiting a family of metrics on closed 3-manifolds obtained by Dehn filling a finite volume hyperbolic 3-manifold with a single cusp and slowly changing the conformal structure of the level tori for the distance to the core geodesics of the filled cusp while keeping the monodromy of the core curve fixed. This construction does not alter the metric in the thick part of the manifold, and we show that it can be done in such a way that its curvature is arbitrarily close to to -1, the weighted L^2-norm of $\mathrm{Ric}(g) + 2g$ is arbitrarily small, while the length ratio of the core curves of the modified metric and the hyperbolic metric can be made arbitrarily large.

The second part of this article is devoted to overcoming this difficulty for finite volume 3-manifolds satisfying suitable curvature assumptions. We begin with analyzing the case when the injectivity radius may be arbitrarily small, but an extension of the strategy used in the proof of Theorem 1 is possible. Namely, in Section 7 we define the *small part* of a finite volume negatively curved 3-manifold to consist of points of small injectivity radius and such that moreover the diameter of the distance torus or horotorus containing the points is bounded from above by a universal constant. In the case of hyperbolic 3-manifolds, the diameter of a component of the small part may be arbitrarily large. Using a counting argument for preimages of points in the thin but not small part of a negatively curved 3-manifold which are contained in a fixed size ball in the universal covering, in Proposition 7.5 we extend the main C^0-estimate in the proof of Theorem 1 to the thin but not small part of the 3-manifold.

Motivated by work of Bamler [Bam12], to deal with the small part of the manifold we take advantage of the fact that on the small part of a hyperbolic Margulis tube or cusp, solutions of the linearized Einstein equations can be controlled with an ODE. The main task is then to use the geometric assumptions to construct a hyperbolic model metric for the small parts of tubes and cusps in Section 8 and to use the ODE for the hyperbolic model metric to analyze the solutions of the linearized Einstein equation. We construct Banach spaces adapted to our needs which control the growth of solutions in the small part, and we use these Banach spaces to invert the linearized Einstein operator with uniformly controlled norm in Section 9. This then leads to the proof of Theorem 2 in Section 10.

In Section 11 which contains the third part of the article, we apply Theorem 2 to Dehn filling and Dehn drilling as formulated in Theorem 3 and Theorem 4.

The last part of this article is devoted to the proof of Theorem 5. We begin in Section 12 with showing that Theorem 2 together with a gluing result taken from [HV22] can fairly immediately be used to construct a hyperbolic metric close to a model metric on a closed 3-manifold $M_f = H_1 \cup_f H_2$ which has the following property. The Hempel distance of M_f is large, and a minimal geodesic in the curve graph $\mathcal{CG}(\Sigma)$ of the boundary surface $\Sigma = \partial H_1 = \partial H_2$ connecting the disk set $\mathcal{D}_1$ to the disk set $\mathcal{D}_2$ contains a sufficiently long segment whose endpoints have *bounded combinatorics*.

This statement is not sufficient for the proof of Theorem 5 as it uses an assumption which is not be fulfilled for an arbitrary 3-manifold of large Hempel distance. To complete the proof of Theorem 5 we use instead an approach which has some resemblance to the work [FSV19].

Namely, we first establish Theorem 6 which gives a length bound on closed geodesics in a *hyperbolic* manifold M_f arising as boundary curves of proper essential subsurfaces of Σ with large subsurface projections of the disk sets. Then we use this a priori length bound for two distinct curves c_1, c_2 arising from two different such subsurfaces together with Dehn surgery and Thurston's hyperbolization result for pared acylindrical manifolds to find a hyperbolic metric on M_f. This construction is carried out in Section 14.

Counting Margulis tubes and segments of a geodesic in the curve graph of Σ connecting the disk sets with no large subsurface projection then yields the lower volume bound stated in Theorem 5.

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2. The basic set-up

In this section we introduce the notations we are going to use, and collect some technical tools needed later on.

2.1. Notation.

We start by stating our notational conventions of various operations on tensors. We use the sign convention

$$R(x,y)z := \nabla_x\nabla_y z - \nabla_y\nabla_x z - \nabla_{[x,y]}z$$

for the Riemannian curvature endomorphism. The Ricci tensor $\mathrm{Ric}(g)$ of a Riemannian metric g is the $(0,2)$-tensor given by

$$\mathrm{Ric}(g)(x,y) = \mathrm{tr}_g\big(z \mapsto R(z,x)y\big) = \sum_{i=1}^n \langle R(e_i,x)y, e_i\rangle,$$

where $(e_i)_{1\le i\le n}$ is a local orthonormal frame. The associated $(1,1)$-tensor is denoted by

$$\mathrm{Ric}_g(x) = \sum_{i=1}^n R(x,e_i)e_i.$$

The $(1,1)$-tensor Ric_g induces the *Weitzenböck curvature operator*, also denoted by Ric_g, that acts on $(0,2)$-tensors h by

$$\mathrm{Ric}_g(h)(x,y) = h\big(\mathrm{Ric}_g(x),y\big) + h\big(x,\mathrm{Ric}_g(y)\big) - 2\,\mathrm{tr}_g h\big(\cdot, R(\cdot,x)y\big).$$

For the covariant differentiation of tensors we use the last input as the direction of differentiation, that is,

$$(\nabla h)(y_1,...,y_k,x) = (\nabla_x h)(y_1,...,y_k).$$

The *adjoint of the covariant derivative* is

$$(\nabla^* h)(y_1,...,y_{k-1}) := -\sum_{i=1}^n (\nabla h)(y_1,...,y_{k-1},e_i,e_i),$$

where $(e_i)_{1\le i\le n}$ is a local orthonormal frame. The *Connection Laplacian* and *Lichnerowicz Laplacian* of a $(0,2)$-tensor h are

$$\Delta h := \nabla^*\nabla h \quad \text{and} \quad \Delta_L h := \Delta h + \mathrm{Ric}_g(h).$$

Similarly, we define the Laplacian of a function $u : M \to \mathbb{R}$ as

$$\Delta u := \nabla^* du.$$

The *divergence* δ_g of a $(0,2)$-tensor h is defined as

$$\delta_g h := -\sum_{i=1}^{n} (\nabla_{e_i} h)(\cdot, e_i)$$

Finally, the *Bianchi operator* β_g of a metric g acts on $(0,2)$-tensors h by

$$\beta_g(h) := \delta_g(h) + \frac{1}{2} d\mathrm{tr}_g(h).$$

We will often drop the metric g from the notation, and hope that this leads to no confusion with the notation of Weitzenböck cruvature operator and the Ricci tensor. More precisely, if g is a background metric, then for a generic tensor h we simply write $\mathrm{Ric}(h)$ for $\mathrm{Ric}_g(h)$, while if g' is another metric, then $\mathrm{Ric}(g')$ is the Ricci tensor of g', and not $\mathrm{Ric}_g(g')$.

Throughout the article we shall use the following convention regarding the use of constants appearing in analytic estimates.

Convention 2.1. In a chain of inequalities, constants denoted by the same symbol may change from line to line, and may depend on varying sets of parameters. In short, the letter C does *not* always refer to the same constant.

We also use the following convention for the O-notation. Here X shall be an arbitrary set.

Notation 2.2. For functions $u, \varphi_1, ..., \varphi_m : X \to \mathbb{R}$ we write $u = \sum_{k=1}^m O(\varphi_k)$ if there are *universal* constants c_k such that $|u(x)| \le \sum_{k=1}^m c_k \varphi_k(x)$ for all $x \in X$.

Moreover, we always assume the following.

Convention 2.3. Unless otherwise stated, all manifolds are assumed to be connected and orientable.

2.2. **The Einstein operator.** As mentioned in the introduction, we shall construct the Einstein metric by an application of the implicit function theorem for the so-called *Einstein operator* (see [Biq00, Section I.1.C], [And06, page 228] for more information). This operator is defined as follows.

Consider the operator $\Psi : g \to \mathrm{Ric}(g) + (n-1)g$. As the diffeomorphism group $\mathrm{Diff}(M)$ of the manifold M acts on metrics by pull-back and Ψ is equivariant for this action, the linearization of Ψ is not elliptic. To remedy this problem, for a given background metric $\bar{g}$ one defines the *Einstein operator* $\Phi_{\bar{g}}$ by

$$\Phi_{\bar{g}}(g) := \mathrm{Ric}(g) + (n-1)g + \frac{1}{2}\mathcal{L}_{(\beta_{\bar{g}}(g))^\sharp}(g), \tag{2.1}$$

where the musical isomorphism $\sharp$ is with respect to the metric g. Using the formula for the linearisation of Ric ([Top06, Proposition 2.3.7]), one shows that the linearisation of $\Phi_{\bar{g}}$ at $\bar{g}$ is

$$(D\Phi_{\bar{g}})_{\bar{g}}(h) = \frac{1}{2}\Delta_L h + (n-1)h.$$

Hence $(D\Phi_{\bar{g}})_{\bar{g}}$ is an elliptic operator. This opens up the possibility for an application of the implicit function theorem.

It has been observed many times in the literature that the Einstein operator can detect Einstein metrics. The following observation can for example be found in [And06] (Lemma 2.1).

Lemma 2.4. *Let $(M,\bar{g})$ be a complete Riemannian manifold, and let g be another metric on M so that*

$$\sup_{x\in M} |\beta_{\bar{g}}(g)|(x) < \infty \quad \textit{and} \quad \mathrm{Ric}(g) \le \lambda g \textit{ for some } \lambda < 0,$$

where $\beta_{\bar{g}}(\cdot) = \delta_{\bar{g}}(\cdot) + \frac{1}{2}d\mathrm{tr}_{\bar{g}}(\cdot)$ is the Bianchi operator of the background metric $\bar{g}$. Denote by $\Phi = \Phi_{\bar{g}}$ the Einstein operator defined in (2.1). Then

$$\Phi(g) = 0 \quad \textit{if and only if} \quad g \textit{ solves the system} \quad \begin{cases} \mathrm{Ric}(g) = -(n-1)g \\ \beta_{\bar{g}}(g) = 0 \end{cases} .$$

2.3. **Hölder norms.** To apply the implicit function theorem to the Einstein operator Φ, we have to study its linearization $(D\Phi)_{\bar{g}}$ at the initial metric $\bar{g}$, acting on a suitably chosen Banach space of sections of the symmetric tensor product $\mathrm{Sym}^2(T^*M) = \mathrm{Sym}(T^*M \otimes T^*M)$. The Banach norms we shall use are hybrids of two rather classical Banach norms: $C^{k,\alpha}$-norms, defined locally using charts, and weighted L^2-norms.

It is important for our main results that Hölder estimates arising from Schauder theory for the Einstein operator on the manifold $(M,\bar{g})$ only depend on local geometric information: The injectivity radius, and a bound on $\|\mathrm{Ric}(\bar{g})\|_{C^1(M,\bar{g})}$. Since we were unable to find a suitable reference in the literature, we summarize what we need in the following proposition. The existence of $C^{k,\alpha}$-norms with the stated properties is part of the claim and will be established below. Similar statements can for example be found in [And06] (page 230 for his definition of Hölder norms, and inequality (3.16) for the estimate).

Proposition 2.5 (Schauder estimate for tensors)**.** *Let (M,g) be an n-dimensional Riemannian manifold satisfying*

$$\|\mathrm{Ric}(g)\|_{C^1(M,g)} \le \Lambda \quad \textit{and} \quad \mathrm{inj}(M) \ge i_0.$$

Let $\mathcal{S} \in \mathrm{End}\big(T^{(0,2)}M\big)$ and let $\mathcal{R} \in \mathrm{Hom}\big(T^{(0,3)}M, T^{(0,2)}M\big)$ be such that

$$\|\mathcal{R}\|_{C^{0,\alpha}(M)}, \quad \|\mathcal{S}\|_{C^{0,\alpha}(M)} \le \lambda.$$

For $f \in C^{0,\alpha}\big(T^{(0,2)}M\big)$ let $h \in C^{2,\alpha}\big(T^{(0,2)}M\big)$ be a solution of the equation

$$\Delta h + \mathcal{R}(\nabla h) + \mathcal{S}(h) = f.$$

Then it holds

$$\|h\|_{C^{2,\alpha}(M)} \le C\left(\|f\|_{C^{0,\alpha}(M)} + \|h\|_{C^0(M)}\right)$$

and

$$\|h\|_{C^{1,\alpha}(M)} \le C\left(\|f\|_{C^0(M)} + \|h\|_{C^0(M)}\right)$$

for some $C > 0$ only depending on $n, \alpha, \lambda, \Lambda, i_0$.

The remainder of this subsection is devoted to the construction of the Hölder norms and a sketch of the proof of Proposition 2.5.

Proof of Proposition 2.5. In local coordinates, the Connection Laplacian Δ has the form

$$(\Delta h)_{ij} = -g^{kl}\partial^2_{kl}(h_{ij}) + \text{Lower Order Terms}$$

and the coefficients of the lower order terms can involve up to two derivatives of g_{ij}. Therefore, to import the Schauder estimates from $\mathbb{R}^n$ by writing the equation in local coordinates, we need coordinates φ that have the following properties:

- The matrix (g^{ij}_φ) is uniformly elliptic;
- $\|g^\varphi_{ij}\|_{C^{2,\alpha}}$ is bounded by a universal constant;
- The coordinates are defined on a metric ball of a priori size.

The existence of such charts is guaranteed if we have a lower bound on the injectivity radius and an upper bound on $\|\mathrm{Ric}(g)\|_{C^1(M,g)}$. Namely, Anderson proved the following (see [JK82], [And90, Main Lemma 2.2], [And06, page 230] and [Biq00, Proposition I.3.2]):

For any $n \in \mathbb{N}$, $\alpha \in (0,1)$, $\Lambda \geq 0$, $i_0 > 0$ there exist $\rho = \rho(n,\alpha,\Lambda,i_0) > 0$ and $C = C(n,\alpha,\Lambda,i_0)$ with the following property. Let (M,g) be a Riemannian n-manifold satisfying

$$\|\mathrm{Ric}(g)\|_{C^1(M,g)} \leq \Lambda \quad \text{and} \quad \mathrm{inj}(M) \geq i_0.$$

Then for all $p \in M$ there exists a *harmonic* chart $\varphi : B(p,2\rho) \subseteq M \to \mathbb{R}^n$ centered at p so that

$$e^{-Q}|v|_g \leq |(D\varphi)(v)|_{\text{eucl.}} \leq e^{Q}|v|_g \tag{2.2}$$

for all $v \in TB(p,2\rho)$, and

$$\|g^\varphi_{ij}\|_{C^{2,\alpha}} \leq C \tag{2.3}$$

for all $i,j = 1,\dots,n$. Here $Q > 0$ is a very small fixed constant, and $\|\cdot\|_{C^{2,\alpha}}$ is the usual Hölder norm of the coefficient functions in $\varphi(B(p,2\rho)) \subseteq \mathbb{R}^n$.

Assume from now on that the Riemannian manifold (M,g) satisfies

$$\|\mathrm{Ric}(g)\|_{C^1(M,g)} \leq \Lambda \quad \text{and} \quad \mathrm{inj}(M) \geq i_0$$

for some $\Lambda \geq 0$ and $i_0 > 0$.

Remark 2.6. The C^1 bound on $\mathrm{Ric}(g)$ is only used at two places in this article. First, we need it for the construction of our notion of Hölder norm. Second, it is used to obtain an upper bound on $\|\mathrm{Ric}(\bar g) + (n-1)\bar g\|_{C^{0,\alpha}}$ (see the proof of Theorem 5.1).

We now state the definition of the Hölder norms. For $k = 1,2$, a C^k-tensor field T and $p \in M$ we define the $C^{k,\alpha}$-norm of T at p as

$$|T|_{C^{k,\alpha}}(p) := \max_{i,j} \|T^\varphi_{ij}\|_{C^{k,\alpha}},$$

where φ is some harmonic chart satisfying (2.2) and (2.3), and $\|\cdot\|_{C^{k,\alpha}}$ is the usual Hölder norm in $\varphi(B(p,\frac{\rho}{2})) \subseteq \mathbb{R}^n$. Similarly, we define the $C^{0,\alpha}$-norm of T at p to be

$$|T|_{C^{0,\alpha}}(p) := \max_{i,j} \|T^\varphi_{ij}\|_{C^{0,\alpha}},$$

where φ is some harmonic chart satisfying (2.2) and (2.3), and $\|\cdot\|_{C^{0,\alpha}}$ is the usual Hölder norm in $\varphi(B(p,\rho)) \subseteq \mathbb{R}^n$. For all $k = 0,1,2$ we also define

$$\|T\|_{C^{k,\alpha}(M)} := \sup_{p\in M} |T|_{C^{k,\alpha}}(p).$$

Note that the euclidean domain in which the Hölder norms $C^{0,\alpha}$ are computed is bigger than for the $C^{1,\alpha}$ and $C^{2,\alpha}$-norm.

By (2.2), the matrix (g_{φ}^{ij}) is uniformly elliptic, and we have

$$\operatorname{diam}\big(\varphi(B(p,2\rho))\big) \le 4e^{Q}\rho \text{ and } \operatorname{dist}_{\mathbb{R}^n}\big(\varphi(B(p,\frac{\rho}{2})),\partial\varphi(B(p,\rho))\big) \ge \frac{1}{2}e^{-Q}\rho.$$

Therefore, the classical interior Schauder estimates imply

$$|h|_{C^{2,\alpha}}(p) \le C\Big(|\mathcal{L}h|_{C^{0,\alpha}}(p) + \sup_{B(p,\rho)}|h|\Big) \tag{2.4}$$

and

$$|h|_{C^{1,\alpha}}(p) \le C\Big(\sup_{B(p,\rho)}|\mathcal{L}h| + \sup_{B(p,\rho)}|h|\Big) \tag{2.5}$$

for a constant $C = C(n,\alpha,\lambda,\Lambda,i_0)$, where $\mathcal{L}$ is the elliptic operator from Proposition 2.5. This immediately yields Proposition 2.5. □

Apart from Schauder estimates, there is one more basic property that an elliptic operator $\mathcal{L}$ should satisfy, namely the continuity property $\|\mathcal{L}h\|_{C^{0,\alpha}(M)} \le C\|h\|_{C^{2,\alpha}}$. In fact, an elliptic operator $\mathcal{L}$ as in Proposition 2.5 satisfies

$$|\mathcal{L}h|_{C^{0,\alpha}}(p) \le \sup_{q\in B(p,\rho)}|h|_{C^{2,\alpha}}(q) \tag{2.6}$$

for some $C = C(n,\alpha,\lambda,\Lambda,i_0)$.

Proof of (2.6). Fix $p \in M$ and let $\varphi : B(p,2\rho) \to \mathbb{R}^n$ be a harmonic chart satisyfing (2.2) and (2.3). Let $q \in B(p,\rho)$ be arbitrary and choose a harmonic chart $\psi : B(q,2\rho) \to \mathbb{R}^n$ satisfying (2.2) and (2.3). It suffices to show that the $C^{3,\alpha}$-norm of the coordinate change

$$\psi\circ\varphi^{-1} : B(\varphi(q),\frac{1}{4}e^{-Q}\rho) \subseteq \mathbb{R}^n \to \psi(B(q,\frac{1}{2}\rho)) \subseteq \mathbb{R}^n$$

is bounded by a universal constant. Note that this coordinate change is well-defined since φ is a e^{Q}-biLipschitz equivalence by (2.2). In fact, this coordinate change is even defined on $B(\varphi(q),\frac{1}{2}e^{-Q}\rho)$. Abbreviate $B_1 := B(\varphi(q),\frac{1}{4}e^{-Q}\rho)$, $B_2 := B(\varphi(q),\frac{1}{2}e^{-Q}\rho)$ and $F := \psi\circ\varphi^{-1}$. As ψ is a harmonic chart, we have $\Delta_g\psi^m = 0$ for every coordinate function ψ^m of ψ. Also $-\Delta_g u = g_{\varphi}^{ij}\frac{\partial^2 u}{\partial x^i\partial x^j}$ for any C^2 function u because φ is harmonic. Hence for every $m = 1,\dots,n$ we get

$$g_{\varphi}^{ij}\frac{\partial^2 F^m}{\partial x^i\partial x^j} = 0 \quad \text{in } B_2.$$

Invoking the classical interior Schauder estimates yields

$$\|F^m\|_{C^{3,\alpha}(B_1)} \le C\|F^m\|_{C^0(B_2)}$$

for a universal constant C. We may without loss of generality assume that $\psi(q) = 0 \in \mathbb{R}^n$. Then $\operatorname{Im}(\psi) \subseteq B(0,2e^{Q}\rho) \subseteq \mathbb{R}^n$. In particular, $\|F^m\|_{C^0(B_2)} \le 2e^{Q}\rho$. This completes the proof. □

This argument also shows that (up to equivalence) the choice of harmonic chart for the definition of the pointwise Hölder norm is irrelevant (as long as the chart satisfies (2.2) and (2.3)).

Remark 2.7. For Theorem 2, we have to consider manifolds without a lower injectivity radius bound. However, these manifolds are negatively curved, and hence their universal covers have infinite injectivity radius. So our notion of Hölder norms applies to the universal cover. We then define $|T|_{C^{m,\alpha}}(p) := |\tilde{T}|_{C^{m,\alpha}}(\tilde{p})$ where $\tilde{T}$ is the pull-back of T to the universal cover.

2.4. **C^0-estimates.** To obtain C^0-estimates for the linearization of the Einstein operator, we use once again a standard tool, the De Giorgi–Nash–Moser estimates on manifolds in the following form. In its formulation $\rho = \rho(n, \alpha, \Lambda, i_0) > 0$ shall denote the constant appearing in the definition of the $C^{k,\alpha}$-norms (see the proof of Proposition 2.5).

Lemma 2.8 (C^0-estimates)**.** *For $\alpha \in (0,1)$, $\Lambda \geq 0$, $i_0 > 0$, $q > n$, $\lambda > 0$, and $r \in (0, \rho)$ there exists a constant $C = C(n, \alpha, \Lambda, i_0, q, \lambda, r) > 0$ with the following property. Let (M, g) be a Riemannian n-manifold satisfying*

$$\|\mathrm{Ric}(g)\|_{C^1(M,g)} \leq \Lambda \quad \textit{and} \quad \mathrm{inj}(M) \geq i_0.$$

Let X be a continuous vector field, and let c be a continuous function on M so that $\|X\|_{C^0(M)}, \|c\|_{C^0(M)} \leq \lambda$. If $u \in C^2(M)$ and $f \in C^0(M)$ satisfy

$$-\Delta u + \langle X, \nabla u \rangle + cu \geq f,$$

then for all $x \in M$ it holds

$$u(x) \leq C\Big(\|u\|_{L^2(B(x,r))} + \|f\|_{L^{q/2}(B(x,r))}\Big). \tag{2.7}$$

Moreover, if $-\Delta u + \langle X, \nabla u \rangle + cu = f$, then the same upper bounds holds for $|u|(x)$.

Remark 2.9. It will be apparent from the proof that assuming $\|\mathrm{Ric}(g)\|_{C^0(M,g)} \leq \Lambda$ is sufficient for the statement of Lemma 2.8.

Note that when dealing with negatively curved manifolds without a lower injectivity radius bound, the terms on the right hand side of inequality (2.7) has to be replaced with the corresponding term in the universal cover.

Proof. Fix $x_0 \in M$ and pick a harmonic chart $\varphi : B(x_0, 2\rho) \to \mathbb{R}^n$ satisfying (2.2) and (2.3). Also fix $r \in (0, \rho)$, and abbreviate $\Omega := \varphi\big(B(x_0, r)\big) \subseteq \mathbb{R}^n$. In the local coordinates given by φ the differential inequality reads

$$g_\varphi^{ij} \partial_i \partial_j (u \circ \varphi^{-1}) + X^i \partial_i (u \circ \varphi^{-1}) + c(u \circ \varphi^{-1}) \geq (f \circ \varphi^{-1}) \quad \text{in } \Omega \subseteq \mathbb{R}^n.$$

By (2.2) (g_φ^{ij}) is uniformly elliptic. Note that as φ is an e^Q-bi-Lipschitz equivalence onto its image, it holds $B(\varphi(x_0), 2r') \subseteq \Omega$ for $r' = \frac{1}{2} e^{-Q} r$. The classical De Giorgi–Nash–Moser estimates (see [GT01, Theorem 8.17]) yield that there is $C = C(\lambda, q, r', n, \alpha, \Lambda, i_0)$ so that

$$\sup_{B(\varphi(x_0), r')} (u \circ \varphi^{-1}) \leq C\Big(\|u \circ \varphi^{-1}\|_{L^2(\Omega)} + \|f \circ \varphi^{-1}\|_{L^{q/2}(\Omega)}\Big).$$

Since $\varphi^{-1} : \Omega \to B(p, r)$ is an e^Q-biLipschitz equivalence, this completes the proof. □

3. Integral inequalities

Our goal is to invert the linearisation $(D\Phi)_{\bar{g}}(h) = \frac{1}{2}\Delta h + \frac{1}{2}\mathrm{Ric}(h) + (n-1)h$ of the Einstein operator Φ.

3.1. **Poincaré inequalities.** As a first step, we establish that the Laplacian Δ acting on the space of $(0,2)$-tensors has a spectral gap, that is, that it satisfies a *Poincaré inequality.* This is expressed by the following proposition, which is taken from [Tia] (Corollary 1 of Section 3).

Proposition 3.1. *For every $n \geq 2$ there exist numbers $\varepsilon(n) > 0$ and $c = c(n) > 0$ with the following property. Let M^n be a Riemannian manifold with $|\sec + 1| \leq \varepsilon \leq \varepsilon(n)$; then*

$$\|h\|^2_{L^2(M)} \leq \frac{1}{n - c\cdot\varepsilon}\|\nabla h\|^2_{L^2(M)}$$

for all $h \in C^2_c\big(\mathrm{Sym}^2(T^*M)\big)$ with $\mathrm{tr}(h) = 0$.

The proof of this proposition is divided into two steps. We begin with some purely algebraic control of the curvature tensor R of a Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$ whose sectional curvature is close to -1, summarized in Lemma 3.2 below. The second step consists in controlling L^2-norms of covariant derivatives.

For the algebraic control, recall that a Riemannian manifold (M has constant sectional curvature κ if and only if the curvature endomorphism satisfies $R(x,y)z = \kappa(\langle y,z\rangle x - \langle x,z\rangle y)$. Motivated by this, we define for $\kappa \in \mathbb{R}$ the tensors R^κ and Rm^κ by

$$R^\kappa(x,y)z := \kappa(\langle y,z\rangle x - \langle x,z\rangle y) \quad \text{and} \quad Rm^\kappa(x,y,z,w) := \langle R^\kappa(x,y)z, w\rangle. \tag{3.1}$$

More specifically, we denote $R^{hyp} := R^{-1}$ and $Rm^{hyp} := Rm^{-1}$, where hyp stands for *hyperbolic.* We also remark that bounding $|Rm - Rm^\kappa|$ is equivalent to bounding $|\sec - \kappa|$. More precisely, there is a constant $c(n) > 0$ such that

$$\sup_\sigma \big|\sec(\sigma) - \kappa\big| \leq |Rm - Rm^\kappa| \leq c(n)\cdot \sup_\sigma \big|\sec(\sigma) - \kappa\big|, \tag{3.2}$$

where the supremum is taken over all planes $\sigma \subseteq T_pM$ and $p \in M$ is arbitrary. This is clear because curvature operators are determined by their sectional curvatures through an explicit formula (see [Pet16, Exercise 3.4.29]).

With these notations we observe

Lemma 3.2. *Let M be a Riemannian n-manifold and $\kappa \in \mathbb{R}$. Then the pointwise estimate*

$$\left|\frac{1}{2}\langle \mathrm{Ric}(h), h\rangle - \kappa\big(n|h|^2 - \mathrm{tr}(h)^2\big)\right| \leq (1+\sqrt{n})|Rm - Rm^\kappa||h|^2$$

holds for all symmetric $(0,2)$-tensors h.

By the irreducible decomposition of the curvature tensor (see Section G of Chapter 1 in [Bes08]) Rm decomposes as

$$Rm = \frac{\mathrm{scal}}{2n(n-1)} g \owedge g + \frac{1}{n-2}\left(\mathrm{Ric}(g) - \frac{\mathrm{scal}}{n} g\right) \owedge g + W,$$

where $\owedge$ is the Kulkarni-Nomizu product, $\mathrm{scal} = \mathrm{tr}(\mathrm{Ric}(g))$ is the scalar curvature, and W is the Weyl tensor. Using the Kulkarni-Nomizu one can write $Rm^\kappa = \frac{\kappa}{2} g \owedge g$. Therefore, if g is almost Einstein in the sense that $|\mathrm{Ric}(g) - (n-1)\kappa g|$ is small, then $|Rm - Rm^\kappa|$ is small if and only if the norm of the Weyl tensor $|W|$ is small.

Proof. For this proof we introduce the following abbreviations (here $(e_i)_{1\le i\le n}$ is an orthonormal frame):

- $\rho(h)(x,y) = \frac{1}{2}\Big(h\big(\mathrm{Ric}(x), y\big) + h\big(x, \mathrm{Ric}(y)\big)\Big)$;
- $L(h)(x,y) \coloneqq \mathrm{tr} h\big(\cdot, R(\cdot,x)y\big)$;
- $\mathrm{Ric}^\kappa(x,y) = \mathrm{tr} Rm^\kappa(x,\cdot,\cdot,y)$ and $\mathrm{Ric}^\kappa(x) = \sum_i R^\kappa(x,e_i)e_i$;
- $\rho^\kappa(h)(x,y) = \frac{1}{2}\Big(h\big(\mathrm{Ric}^\kappa(x), y\big) + h\big(x, \mathrm{Ric}^\kappa(y)\big)\Big)$;
- $L^\kappa(h)(x,y) \coloneqq \mathrm{tr} h\big(\cdot, R^\kappa(\cdot,x)y\big)$.

Claim (1). *For every symmetric* $(0,2)$*-tensor* h*, it holds*

$$\big|\langle L(h) - L^\kappa(h), h\rangle\big| \le |Rm - Rm^\kappa||h|^2 \quad \textit{and} \quad \big|\langle \rho(h) - \rho^\kappa(h), h\rangle\big| \le \sqrt{n}\,|Rm - Rm^\kappa||h|^2.$$

Proof of Claim 1. The first inequality follows by writing the expression in an orthonormal basis and invoking the Cauchy-Schwarz inequality.

For the second inequality, choose an orthonormal basis $(e_i)_{1\le i\le n}$ so that $h(e_k,e_l) = 0$ for $k \ne l$. Writing the expression in this frame and using $|\mathrm{Ric}(x,y) - \mathrm{Ric}^\kappa(x,y)| \le \sqrt{n}|Rm - Rm^\kappa||x||y|$ (which holds since $|\mathrm{tr}(\cdot)| \le \sqrt{n}|\cdot|$) yields the second inequality. ☐

Claim (2). *For every symmetric* $(0,2)$*-tensor* h*, we have*

$$\langle L^\kappa(h), h\rangle = \kappa\big(\mathrm{tr}(h)^2 - |h|_g^2\big) \quad \textit{and} \quad \langle \rho^\kappa(h), h\rangle = \kappa(n-1)|h|_g^2.$$

Proof of Claim 2. Note that $\mathrm{Ric}^\kappa(x) = \kappa(n-1)x$. So the second equality is clear. Choose an orthonormal basis $(e_i)_{1\le i\le n}$ for the metric g so that $h(e_k,e_l) = 0$ for $k \ne l$. Then

$$L^\kappa(h)(e_l,e_l) = \sum_{i,j} h(e_i,e_j)\kappa(\langle e_i,e_j\rangle - \langle e_i,e_l\rangle\langle e_j,e_l\rangle) = \sum_i h_{ii}\kappa(1-\delta_{il}) = \kappa\sum_{i\ne l} h_{ii}$$

and thus

$$\langle L^\kappa(h), h\rangle = \sum_l L^\kappa(h)_{ll}h_{ll} = \kappa\sum_l\sum_{i\ne l} h_{ll}h_{ii} = \kappa\big((\sum_i h_{ii})^2 - \sum_i h_{ii}^2\big) = \kappa\big(\mathrm{tr}(h)^2 - |h|_g^2\big).$$

This finishes the proof of the second claim. ☐

As $\frac{1}{2}\langle \mathrm{Ric}(h), h\rangle = \langle \rho(h) - L(h), h\rangle$, the two claims immediately imply the desired result. ☐

The following L^2-identity is the second auxiliary result we need.

Lemma 3.3. *Let* M *be a Riemannian manifold. Then it holds*

$$0 \le \|\nabla h\|_{L^2(M)}^2 + \frac{1}{2}\big(\mathrm{Ric}(h), h\big)_{L^2(M)}$$

for all $h \in C_c^2\big(\mathrm{Sym}^2(T^*M)\big)$.

Proof. This follows immediately from the equation on the bottom of page 355 of [Bes08]. Indeed, this equation states

$$(\delta^\nabla d^\nabla + d^\nabla \delta^\nabla)h = \nabla^*\nabla h - \overset{\circ}{R}h + h \circ r.$$

By $\overset{\circ}{R}h$ Besse denotes $(\overset{\circ}{R}h)(x,y) = \mathrm{tr}h\big(\cdot, R(\cdot,x)y)\big)$ (see [Bes08, page 51] and note that [Bes08] uses the sign convention opposite to ours). Moreover, r is the $(0,2)$-Ricci tensor $\mathrm{Ric}(g)$, and $\circ$ denotes the symmetric product, that is, the pairing that under the isomorphism $\mathrm{Sym}^2(T^*M) \cong \mathrm{Sym}\big(\mathrm{End}(TM)\big)$ corresponds to the symmetric product $(L,L') \to \frac{1}{2}(L\circ L' + L'\circ L)$, where $L \circ L'$ is the composition of endomorphisms. So $(h\circ r)(x,y) = \frac{1}{2}\big(h(\mathrm{Ric}(x),y) + h(x,\mathrm{Ric}(y))\big)$, and hence the right hand side in the equation above is just $\nabla^*\nabla h + \frac{1}{2}\mathrm{Ric}(h)$.

Moreover, d^∇ is the exterior differential on T^*M-valued 1-forms induced by the Levi-Civita connection ∇, and δ^∇ denotes its dual (note that a symmetric $(0,2)$-tensor can be thought of as a T^*M-valued 1-form). Therefore,

$$\begin{aligned}
0 \leq &\|d^\nabla h\|^2_{L^2(M)} + \|\delta^\nabla h\|^2_{L^2(M)} \\
=&\big((\delta^\nabla d^\nabla + d^\nabla\delta^\nabla)h, h\big)_{L^2(M)} \\
=&(\nabla^*\nabla h, h)_{L^2(M)} + \frac{1}{2}\big(\mathrm{Ric}(h), h\big)_{L^2(M)} \\
=&\|\nabla h\|^2_{L^2(M)} + \frac{1}{2}\big(\mathrm{Ric}(h), h\big)_{L^2(M)}.
\end{aligned}$$

This completes the proof. □

We are now ready for the proof of Proposition 3.1.

Proof of Proposition 3.1. We copy the proof from [Tia, Corollary 1 in Section 3]. By Lemma 3.3, we have $0 \leq \|\nabla h\|^2_{L^2(M)} + \frac{1}{2}\big(\mathrm{Ric}(h),h\big)_{L^2(M)}$. As $|\sec + 1| \leq \varepsilon$, it holds $|Rm - Rm^{hyp}| \leq \varepsilon c_0$ for some constant $c_0 = c_0(n)$ due to (3.2). Hence Lemma 3.2 yields $-\frac{1}{2}\langle \mathrm{Ric}(h), h\rangle \geq \big(n - \varepsilon c_0(1+\sqrt{n})\big)|h|^2$. Combining these two inequalities completes the proof in case ε is small enough that $n - \varepsilon c_0(1+\sqrt{n}) > 0$. □

3.2. **Weighted L^2-norms.** The main tool for obtaining a priori C^0-estimates for the differential equation $(D\Phi)_{\bar{g}}(h) = f$ which are independent of $\mathrm{vol}(M)$ is the use of *hybrid norms* that are a mixture of Hölder norms and weighted Sobolev norms (see Section 4.1). The next proposition establishes the required a priori estimates for weighted L^2-norms. It follows Corollary 2 in Section 3 of [Tia].

Proposition 3.4. *Let M be a complete Riemannian n-manifold of finite volume, let $f \in C^0\big(\mathrm{Sym}^2(T^*M)\big) \cap L^2(M)$ and let $h \in C^2\big(\mathrm{Sym}^2(T^*M)\big) \cap L^2(M)$ be a solution of*

$$\frac{1}{2}\Delta_L h + (n-1)h = f.$$

Let $\varphi \in C^\infty(M)$ be so that φh, $\varphi f \in L^2(M)$. Then it holds

$$(n-2)\int_M \varphi^2|h|^2\,d\mathrm{vol} \le 2\int_M \varphi^2\langle h,f\rangle\,d\mathrm{vol} + \int_M |\nabla\varphi|^2|h|^2\,d\mathrm{vol} + (1+\sqrt{n})\int_M \varphi^2|Rm - Rm^{hyp}||h|^2\,d\mathrm{vol}.$$

The weights appearing in our hybrid norms will *not* be smooth. For convenience, we state the precise version of Proposition 3.4.

Corollary 3.5. *Let M, h, f be as in Proposition 3.4, and let $\rho : \mathbb{R} \to \mathbb{R}$ be a smooth function so that ρ and ρ' are bounded on $\mathbb{R}_{\ge 0}$. Then for every $x \in M$ we have*

$$(n-2)\int_M \rho(r_x)^2|h|^2\,d\mathrm{vol} \le 2\int_M \rho(r_x)^2\langle h,f\rangle\,d\mathrm{vol} + \int_M \big(\rho'(r_x)\big)^2|h|^2\,d\mathrm{vol} + (1+\sqrt{n})\int_M \rho(r_x)^2|Rm - Rm^{hyp}||h|^2\,d\mathrm{vol},$$

where $r_x = d_M(\cdot, x)$.

Proof. Theorem 1 in [AFLMR07] shows that for every $\epsilon > 0$ there is a Lipschitz function $r_\epsilon \in C^\infty(M)$ such that $\|r_\epsilon - r_x\|_{C^0(M)} < \epsilon$ and $\mathrm{Lip}(r_\epsilon) \le 1+\epsilon$. Consider $\varphi_\epsilon := \rho\circ r_\epsilon$, and note that $\varphi_\varepsilon h, \varphi_\varepsilon f \in L^2(M)$ as ρ is bounded. Applying Proposition 3.4 to $\varphi_\epsilon := \rho \circ r_\epsilon$ and taking $\epsilon \to 0$ implies the desired result. ☐

Proof of Proposition 3.4. We adapt the proof from [Tia, Corollary 2 in Section 3] and include further details.

We first consider the case that φ has compact support. Abbreviate $\tilde{h} = \varphi h$. A direct computation yields

$$\Delta\tilde{h} = (\Delta\varphi)h - 2\mathrm{tr}^{(1,4)}(\nabla\varphi\otimes\nabla h) + \varphi\Delta h.$$

This implies

$$\Delta\tilde{h} = (\Delta\varphi)h - 2\mathrm{tr}^{(1,4)}(\nabla\varphi\otimes\nabla h) + 2\varphi f - 2(n-1)\tilde{h} - \mathrm{Ric}(\tilde{h})$$

as $f = \frac{1}{2}\Delta h + \frac{1}{2}\mathrm{Ric}(h) + (n-1)h$. Consider the 1-form $\omega = \varphi|h|^2 d\varphi$. A straightforward calculation in a local orthonormal frame shows

$$-\nabla^*\omega = |\nabla\varphi|^2|h|^2 + 2\langle \mathrm{tr}^{(1,4)}(\nabla\varphi\otimes\nabla h), \tilde{h}\rangle - \langle(\Delta\varphi)h, \tilde{h}\rangle.$$

Note that $\int_M \nabla^*\omega\,d\mathrm{vol} = 0$. Indeed, $-\nabla^*\omega = \mathrm{div}(\omega^\sharp)$ because the musical isomorphisms commute with covariant differentiation, and $\int_M \mathrm{div}(\omega^\sharp)\,d\mathrm{vol} = 0$ by the divergence theorem since $\omega^\sharp$ is compactly supported.

Thus $\big((\Delta\varphi)h, \tilde{h}\big)_{L^2} - 2\big(\mathrm{tr}^{(1,4)}(\nabla\varphi\otimes\nabla h), \tilde{h}\big)_{L^2} = \|\,|\nabla\varphi|\,h\|^2_{L^2}$. Together with Lemma 3.3 this shows

$$\begin{aligned} 0 \le& \big(\Delta\tilde{h}, \tilde{h}\big)_{L^2} + \frac{1}{2}\big(\mathrm{Ric}(\tilde{h}), \tilde{h}\big)_{L^2} \\ =& \|\,|\nabla\varphi|\,h\|^2_{L^2} + 2\big(\varphi f, \tilde{h}\big)_{L^2} - 2(n-1)\|\tilde{h}\|_{L^2} - \frac{1}{2}\big(\mathrm{Ric}(\tilde{h}), \tilde{h}\big)_{L^2}. \end{aligned}$$

Lemma 3.2 implies

$$\begin{aligned}
-\frac{1}{2}\langle \mathrm{Ric}(\tilde{h}), \tilde{h}\rangle \leq& \big(n|\tilde{h}|^2 - \mathrm{tr}(\tilde{h})^2\big) + (1+\sqrt{n})|Rm - Rm^{hyp}||\tilde{h}|^2 \\
\leq& n|\tilde{h}|^2 + (1+\sqrt{n})|Rm - Rm^{hyp}||\tilde{h}|^2.
\end{aligned}$$

Hence

$$(n-2)\|\tilde{h}\|_{L^2}^2 \leq 2\big(\varphi f, \tilde{h}\big)_{L^2} + \|\,|\nabla\varphi|\,h\|_{L^2}^2 + (1+\sqrt{n})\int_M |Rm - Rm^{hyp}||\tilde{h}|^2\, d\mathrm{vol}.$$

Since $\tilde{h} = \varphi h$ this finishes the case that φ is compactly supported.

We now consider the general case. Choose a pointwise non-decreasing sequence of bump functions $(\psi_k)_{k\in\mathbb{N}} \subseteq C_c^\infty(M)$ so that $0 \leq \psi_k \leq 1$, $\|\nabla\psi_k\|_{C^0(M)} \leq \frac{1}{k}$, and $\psi_k \to 1$ pointwise. Since $(a+b)^2 \leq (1+k)a^2 + (1+\frac{1}{k})b^2$, we have

$$|\nabla(\psi_k\varphi)|^2 \leq (1+k)|\nabla\psi_k|^2\varphi^2 + \left(1+\frac{1}{k}\right)\psi_k^2|\nabla\varphi|^2 \leq \frac{1+k}{k^2}\varphi^2 + \left(1+\frac{1}{k}\right)|\nabla\varphi|^2.$$

Applying the result for the case of compact support to $\varphi_k := \psi_k\varphi$ gives

$$\begin{aligned}
(n-2)\int_M \psi_k^2\varphi^2|h|^2\, d\mathrm{vol} \leq& 2\int_M \psi_k^2\varphi^2\langle h, f\rangle\, d\mathrm{vol} \\
&+ \frac{1+k}{k^2}\|\varphi h\|_{L^2}^2 + \left(1+\frac{1}{k}\right)\int_M |\nabla\varphi|^2|h|^2\, d\mathrm{vol} \\
&+ (1+\sqrt{n})\int_M \psi_k^2\varphi^2|Rm - Rm^{hyp}||h|^2\, d\mathrm{vol}.
\end{aligned}$$

Taking $k\to\infty$ implies the desired result. Indeed, the second summand on the right hand side converges to 0 since $\varphi h \in L^2(M)$, and the first and fourth summand converge by dominated convergence because $\varphi h, \varphi f \in L^2(M)$. Also we may assume $|\nabla\varphi|h \in L^2(M)$ since otherwise the desired inequality trivially holds. So the third summand on the right hand side also converges. $\square$

4. Invertibility of $\mathcal{L}h = \frac{1}{2}\Delta_L h + (n-1)h$

In order to apply the implicit function theorem it is necessary to invert the linearisation of the Einstein operator Φ at the original metric $\bar{g}$. This linearisation is given by

$$(D\Phi)_{\bar{g}}(h) = \frac{1}{2}\Delta_L h + (n-1)h.$$

For simplicity of notation we abbreviate this operator by $\mathcal{L}$. It is of utmost importance that $\|\mathcal{L}^{-1}\|_{\mathrm{op}}$ is bounded by some constant that is independent of $\mathrm{vol}(M)$. To achieve this we consider special *hybrid norms* that are defined in Section 4.1. The a priori estimate is then proven in Section 4.2.

4.1. **The hybrid norm.** Bounding the operator norm of the inverse $\mathcal{L}^{-1}$ boils down to proving an a priori estimate $\|h\|_{\mathrm{source}} \leq C\|\mathcal{L}h\|_{\mathrm{target}}$. As $\mathcal{L}$ is an elliptic operator, it is natural to work with Hölder norms and use the Schauder estimates established in Proposition 2.5. To obtain constants that are independent of $\mathrm{vol}(M)$ we define norms that are a combination of Hölder and weighted Sobolev norms. The basic reason for this is that C^0-bounds can be deduced from L^2-bounds by De Giorgi–Nash–Moser estimates (Lemma 2.8).

Let M be a Riemannian manifold of dimension $n \geq 3$. For $\bar{\epsilon} > 0, \delta \in (0, 2\sqrt{n-2})$ and $r_0 \geq 1$ define

$$E := E(M; \bar{\epsilon}, \delta, r_0) := \left\{ x \in M \;\middle|\; \int_{B(x,2r_0) \setminus B(x,r_0)} e^{-(2\sqrt{n-2}-\delta) r_x(y)} \, \mathrm{dvol}(y) \leq \bar{\epsilon} \right\}, \tag{4.1}$$

where $r_x(y) = d_M(x,y)$. Inside E we will be able to bound $\|h\|_{C^0}$ in terms of $\|f\|_{C^0}$. Outside of E we will use Lemma 2.8 to bound $\|h\|_{C^0}$ in terms of $\|f\|_{L^2}$. The norms we define are supposed to capture L^2-information outside of E.

We now come to the precise definition, which we take from Section 5 of [Tia].

Definition 4.1. For $\alpha \in (0,1)$, $\bar{\epsilon} > 0$, $\delta \in (0, 2\sqrt{n-2})$ and $r_0 \geq 1$ the *hybrid norms* $\|\cdot\|_k$ on $C^{k,\alpha}(\mathrm{Sym}^2(T^*M))$ are defined as

$$\|h\|_2 := \max\left\{ \|h\|_{C^{2,\alpha}(M)}, \sup_{x \notin E} \left(\int_M e^{-(2\sqrt{n-2}-\delta) r_x(y)} \big(|h|^2 + |\nabla h|^2 + |\Delta h|^2 \big) \, \mathrm{dvol}(y) \right)^{\frac{1}{2}} \right\} \tag{4.2}$$

and

$$\|f\|_0 := \max\left\{ \|f\|_{C^{0,\alpha}(M)}, \sup_{x \notin E} \left(\int_M e^{-(2\sqrt{n-2}-\delta) r_x(y)} |f|^2 \, \mathrm{dvol}(y) \right)^{\frac{1}{2}} \right\}, \tag{4.3}$$

where $E = E(M; \bar{\epsilon}, \delta, r_0)$ is the set defined in (4.1).

Strictly speaking these norms also depend on a choice of constants $\Lambda \geq 0$ and $i_0 > 0$ for which it holds $\|\nabla \mathrm{Ric}(g)\|_{C^0(M,g)} \leq \Lambda$ and $\mathrm{inj}(M,g) \geq i_0$. This is because our notion of Hölder norm depends on this geometric information (see Proposition 2.5 and its proof).

The reason why we use weights of the form $e^{-(2\sqrt{n-2}-\delta) r_x}$ (and not $e^{-a r_x}$ for arbitrary big $a > 0$) is that in order to obtain weighted L^2-estimates we can only use weights $e^{2\omega}$ for functions ω satisfying $|\nabla \omega| < \sqrt{n-2}$. This is because in the estimate of Proposition 3.4 the factor $n-2$ on the left hand side needs to be able to absord the factor $|\nabla \omega|^2$ on the right hand side.

For later purposes it will be useful to mention the following equivalent version of the norm $\|\cdot\|_2$.

Remark 4.2. Let M be a closed Riemannian n-manifold with $|\mathrm{sec}| \leq 2$. The norm $\|\cdot\|_2$ is equivalent to the norm

$$\|h\|_2' := \max\left\{ \|h\|_{C^{2,\alpha}(M)}, \sup_{x \notin E} \left(\int_M e^{-(2\sqrt{n-2}-\delta) r_x(y)} \big(|h|^2 + |\nabla h|^2 + |\nabla^2 h|^2 \big) \, \mathrm{dvol}(y) \right)^{\frac{1}{2}} \right\}.$$

and the equivalence constants can be chosen to depend only on n.

Sketch of proof. By approximation it suffices to show that for any smooth Lipschitz function $\varphi : M \to \mathbb{R}$ and any smooth $(0,2)$-tensor h it holds

$$\int_M e^{\varphi} |\nabla^2 h|^2 \, \mathrm{dvol} \leq c \int_M e^{\varphi} \big(|h|^2 + |\nabla h|^2 + |\Delta h|^2 \big) \, \mathrm{dvol}$$

for a constant c depending only on n and $\mathrm{Lip}(\varphi)$ (see [AFLMR07, Theorem 1] for the approximation of Lipschitz functions). It is a well-known fact that for $u \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ it holds $\int_{\mathbb{R}^n} |\nabla^2 u|\, dx = \int_{\mathbb{R}^n} |\Delta u|^2\, dx$ (see (3) on page 326 of [Eva10]). In fact, this only needs integration by parts and the fact that second order partial derivatives commute. A similar calculation applies in the present context. The only difference is that now second order partial derivatives only commute up to a term involving the curvature tensor. But since $|\sec| \le 2$ we have uniform control on any term involving the curvature. □

4.2. **The a priori estimate.** After having introduced the hybrid norms $\|\cdot\|_2$ and $\|\cdot\|_0$ in the previous subsection, we now prove that

$$\mathcal{L} : \Big(C^{2,\alpha}\big(\mathrm{Sym}^2(T^*M)\big), \|\cdot\|_2\Big) \longrightarrow \Big(C^{0,\alpha}\big(\mathrm{Sym}^2(T^*M)\big), \|\cdot\|_0\Big)$$

satisfies an a priori estimate with a constant independent of $\mathrm{vol}(M)$. This is Proposition 5.1 in [Tia].

Proposition 4.3. *For all $n \ge 3$, $\alpha \in (0,1)$, $\Lambda \ge 0$, $\delta \in (0, 2\sqrt{n-2})$ and $r_0 \ge 1$ there exist constants ε_0, $\bar{\epsilon}_0$ and $C > 0$ with the following property. Let M be a closed Riemannian n-manifold with*

$$|\sec + 1| \le \varepsilon_0, \quad \mathrm{inj}(M) \ge 1 \quad \textit{and} \quad \|\nabla \mathrm{Ric}\|_{C^0(M)} \le \Lambda.$$

Then for all $h \in C^{2,\alpha}\big(\mathrm{Sym}^2(T^*M)\big)$ it holds

$$\|h\|_2 \le C \|\mathcal{L}h\|_0,$$

where $\|\cdot\|_2$ and $\|\cdot\|_0$ are the norms defined in (4.2) and (4.3) with respect to any $\bar{\epsilon} \le \bar{\epsilon}_0$.

Proof. Our proof follows that in [Tia]. We add further details and at times give alternative arguments. Abbreviate $f := \mathcal{L}h$.

Step 1 (Integral estimate): Let $c(n,\delta) := n - 2 - (\sqrt{n-2} - \delta/2)^2$ and $\epsilon_0 \le \frac{c(n,\delta)}{2(1+\sqrt{n})}$. Assume that $\|Rm - Rm^{hyp}\|_{C^0(M)} \le \varepsilon_0$. An application of Corollary 3.5 with $\rho(t) = e^{-(\sqrt{n-2}-\delta/2)t}$ gives

$$c(n,\delta) \int_M e^{-(2\sqrt{n-2}-\delta)r_x} |h|^2 \,\mathrm{dvol} \le 2 \int_M e^{-(2\sqrt{n-2}-\delta)r_x} \langle f, h\rangle \,\mathrm{dvol} + (1+\sqrt{n})\varepsilon_0 \int_M e^{-(2\sqrt{n-2}-\delta)r_x} |h|^2 \,\mathrm{dvol},$$

and consequently

$$\frac{c(n,\delta)}{2} \int_M e^{-(2\sqrt{n-2}-\delta)r_x} |h|^2 \,\mathrm{dvol} \le \int_M e^{-(2\sqrt{n-2}-\delta)r_x} \left(\frac{c(n,\delta)}{4} |h|^2 + \frac{4}{c(n,\delta)} |f|^2 \right) \mathrm{dvol}$$

by the Cauchy-Schwarz inequality and the inequality between the arithmetic and the geometric mean. Hence

$$\int_M e^{-(2\sqrt{n-2}-\delta)r_x} |h|^2 \,\mathrm{dvol} \le \frac{16}{c(n,\delta)^2} \int_M e^{-(2\sqrt{n-2}-\delta)r_x} |f|^2 \,\mathrm{dvol}. \tag{4.4}$$

Note that $|\mathrm{Ric}(h)| \le c(n)|h|$ because $|\mathrm{sec}| \le 2$. Testing the equation $f - \frac{1}{2}\mathrm{Ric}(h) - (n-1)h = \frac{1}{2}\Delta h$ with $e^{-(2\sqrt{n-2}-\delta)r_x}\Delta h$ and using the inequality between the weighted arithmetic and the weighted geometric mean implies

$$\int_M e^{-(2\sqrt{n-2}-\delta)r_x}|\Delta h|^2\,d\mathrm{vol} \le C(n)\int_M e^{-(2\sqrt{n-2}-\delta)r_x}\big(|h|^2 + |f|^2\big)\,d\mathrm{vol}. \tag{4.5}$$

Set $\varphi := (2\sqrt{n-2} - \delta)r_x$. We may act as if φ were smooth. Indeed, by Theorem 1 in [AFLMR07] there exists a sequence $(\varphi_\varepsilon)_{\varepsilon>0} \subseteq C^\infty(M)$ so that $\lim_{\varepsilon\to 0}\|\varphi - \varphi_\varepsilon\|_{C^0(M)} = 0$ and $\mathrm{Lip}(\varphi_\varepsilon) \le 2\sqrt{n-2}$. Then all the arguments below apply to φ_ε, so that (4.8) will hold for φ_ε instead of φ. But then taking $\varepsilon \to 0$ will yield (4.8) for φ.

Using $\frac{1}{2}\Delta(|h|^2) = \langle \Delta h, h\rangle - |\nabla h|^2 \le \frac{1}{2}|\Delta h|^2 + \frac{1}{2}|h|^2 - |\nabla h|^2$ we obtain

$$\begin{aligned}\int_M e^{-\varphi}|\nabla h|^2\,d\mathrm{vol} \le &\frac{1}{2}\int_M e^{-\varphi}|h|^2\,d\mathrm{vol} + \frac{1}{2}\int_M e^{-\varphi}|\Delta h|^2\,d\mathrm{vol}\\ &- \frac{1}{2}\int_M e^{-\varphi}\Delta(|h|^2)\,d\mathrm{vol}.\end{aligned} \tag{4.6}$$

Integration by parts shows that

$$-\frac{1}{2}\int_M e^{-\varphi}\Delta(|h|^2)\,d\mathrm{vol} = -\frac{1}{2}\int_M \langle \nabla(|h|^2), \nabla(e^{-\varphi})\rangle\,d\mathrm{vol},$$

moreover $\left|\frac{1}{2}\langle\nabla(|h|^2), \nabla\varphi\rangle\right| \le |h||\nabla h||\nabla\varphi| \le \frac{1}{2}|\nabla\varphi|^2|h|^2 + \frac{1}{2}|\nabla h|^2$. Absorbing $\frac{1}{2}e^{-\varphi}|\nabla h|^2$ to the left hand side of inequality (4.6) and using $|\nabla\varphi| \le 2\sqrt{n-2}$ yields

$$\begin{aligned}\int_M e^{-\varphi}|\nabla h|^2\,d\mathrm{vol} \le &\int_M e^{-\varphi}|h|^2\,d\mathrm{vol} + \int_M e^{-\varphi}|\Delta h|^2\,d\mathrm{vol}\\ &+ 4(n-2)\int_M e^{-\varphi}|h|^2\,d\mathrm{vol}.\end{aligned} \tag{4.7}$$

Therefore

$$\int_M e^{-(2\sqrt{n-2}-\delta)r_x}\big(|h|^2 + |\nabla h|^2 + |\Delta h|^2\big)\,d\mathrm{vol} \le C(n,\delta)\int_M e^{-(2\sqrt{n-2}-\delta)r_x}|f|^2\,d\mathrm{vol} \tag{4.8}$$

by combining (4.4), (4.5) and (4.7). This completes the integral estimates.

Step 2 (C^0-estimate): It remains to estimate $\|h\|_{C^{2,\alpha}(M)}$. By Proposition 2.5, it suffices to bound $\|h\|_{C^0(M)}$. We reduce the C^0-estimate to an L^2-estimate. Namely, we show that there is a constant $C = C(n,\alpha,\Lambda)$ so that for each $x \in M$ it holds

$$|h|(x) \le C\big(\|h\|_{L^2(B(x,\rho))} + \|f\|_{C^0(B(x,\rho))}\big), \tag{4.9}$$

where ρ is the constant appearing in the definition of the Hölder norms. This will follow from the De Giorgi–Nash–Moser estimates of Lemma 2.8. The problem is that De Giorgi–Nash–Moser estimates only hold for scalar equations, but not for systems. For this reason we can not directly apply Lemma 2.8 to $\mathcal{L}h = f$. To remedy this, we show that $|h|$ satisfies an elliptic partial differential inequality.

Recall $f = \frac{1}{2}\Delta h + \frac{1}{2}\mathrm{Ric}(h) + (n-1)h$. Using $\frac{1}{2}\Delta(|h|^2) = \langle \Delta h, h\rangle - |\nabla h|^2$ and the estimate on $\frac{1}{2}\langle \mathrm{Ric}(h), h\rangle$ from Lemma 3.2 we get (assuming $\varepsilon_0 \le \frac{1}{1+\sqrt{n}}$)

$$\begin{aligned}
-\frac{1}{2}\Delta(|h|^2) &= -2\langle f, h\rangle + \langle \mathrm{Ric}(h), h\rangle + 2(n-1)|h|^2 + |\nabla h|^2 \\
&\ge -2|f||h| - 2(n + \varepsilon_0(1+\sqrt{n}))|h|^2 + 2\mathrm{tr}(h)^2 + 2(n-1)|h|^2 + |\nabla h|^2 \\
&\ge -2|f||h| - 4|h|^2 + |\nabla h|^2. \qquad (4.10)
\end{aligned}$$

Suppose for the moment that $h \neq 0$ everywhere. Then $|h|$ is a nowhere vanishing C^2 function. Observe

$$|\nabla(|h|)| \le |\nabla h| \quad \text{and} \quad -\frac{1}{2}\Delta(|h|^2) = -|h|\Delta(|h|) + |\nabla(|h|)|^2.$$

Combining this with inequality (4.10) and dividing by $|h|$ shows

$$-\Delta(|h|) \ge -2|f| - 4|h|. \qquad (4.11)$$

Applying Lemma 2.8 to (4.11) yields (4.9).

Recall that we assumed $h \neq 0$ everywhere. We will now show that this assumption can be dropped. Namely, (4.9) is stable under C^2-convergence, that is, if (4.9) holds for a sequence of h_i and if $h_i \to h$ in the C^2-topology, then (4.9) also holds for h. Therefore, it suffices to construct a sequence h_i converging to h in the C^2-topology so that $h_i \neq 0$ everywhere.

Let $h \in C^{2,\alpha}\big(\mathrm{Sym}^2(T^*M)\big)$ be arbitrary. Then h can be approximated in the C^2-topology by symmetric $(0,2)$-tensors h_i $(i \ge 1)$ which are transverse to the zero-section of $\mathrm{Sym}^2(T^*M)$. For reasons of dimension, such a section is disjoint from the zero-section, in other words, the tensors h_i vanish nowhere. Therefore, the estimate (4.9) holds for all $h \in C^{2,\alpha}\big(\mathrm{Sym}^2(T^*M)\big)$ and $x \in M$.

Fix $h \in C^{2,\alpha}\big(\mathrm{Sym}^2(T^*M)\big)$. Choose $x \in M$ so that $|h|(x) \ge \frac{1}{2}\|h\|_{C^0(M)}$. Then (4.9) implies

$$\frac{1}{2}\|h\|_{C^0(M)} \le C\big(\|h\|_{L^2(B(x,\rho))} + \|f\|_{C^0(M)}\big) \qquad (4.12)$$

for some $C = C(n, \alpha, \Lambda)$. We can without loss of generality assume that the ρ from the definition of Hölder norms is at most 1. So it suffices to bound $\|h\|_{L^2(B(x,r_0))}$ as $r_0 \ge 1 \ge \rho$. To this end we distinguish two cases.

We first consider the case that $x \notin E$. By (4.4)

$$\begin{aligned}
\int_{B(x,r_0)} |h|^2 \, d\mathrm{vol} &\le e^{2\sqrt{n-2}r_0} \int_M e^{-(2\sqrt{n-2}-\delta)r_x} |h|^2 \, d\mathrm{vol} \\
&\le e^{2\sqrt{n-2}r_0} C(n,\delta) \int_M e^{-(2\sqrt{n-2}-\delta)r_x} |f|^2 \, d\mathrm{vol} \\
&\le C(n,\delta,r_0)\|f\|_0^2,
\end{aligned}$$

where in the last line we used (4.3) and $x \notin E$. This finishes the case $x \notin E$.

Now consider the case $x \in E$. Choose $\eta : \mathbb{R} \to \mathbb{R}$ smooth such that $\eta = 1$ on $(-\infty, 1]$, $\eta = 0$ on $[2, \infty)$ and $-\frac{3}{2} \le \eta' \le 0$. Let $\rho(t) = \eta(\frac{t}{r_0})e^{-(\sqrt{n-2}-\delta/2)t}$. Compute

$$\rho'(t) = e^{-(\sqrt{n-2}-\delta/2)t}\left(\frac{1}{r_0}\eta'(t/r_0) - \eta(t/r_0)(\sqrt{n-2}-\delta/2)\right).$$

Abbreviate $\sigma := n-2-(\sqrt{n-2}-\delta/2)^2 - (1+\sqrt{n})\varepsilon_0$. As $\|Rm - Rm^{hyp}\| \le \varepsilon_0$, Corollary 3.5 implies

$$\begin{aligned}
\int_M & e^{-(2\sqrt{n-2}-\delta)r_x}\eta^2\big(\sigma|h|^2 - 2\langle h, f\rangle\big)\,\mathrm{dvol} \\
&\le \int_M \left(\eta(-\eta')\frac{2\sqrt{n-2}-\delta}{r_0} + (\frac{\eta'}{r_0})^2\right)|h|^2 e^{-(2\sqrt{n-2}-\delta)r_x}\,\mathrm{dvol} \\
&\le \int_{B(x,2r_0)\setminus B(x,r_0)} \left(\frac{3\sqrt{n-2}}{r_0} + \frac{9}{4r_0^2}\right)|h|^2 e^{-(2\sqrt{n-2}-\delta)r_x}\,\mathrm{dvol} \\
&\le C(n,r_0)\|h\|^2_{C^0(M)}\int_{B(x,2r_0)\setminus B(x,r_0)} e^{-(2\sqrt{n-2}-\delta)r_x}\,\mathrm{dvol} \\
&\le \bar{\epsilon}C(n,r_0)\|h\|^2_{C^0(M)},
\end{aligned} \tag{4.13}$$

where we used that $\eta'(\frac{r_x}{r_0}) = 0$ outside $B(x,2r_0)\setminus B(x,r_0)$ in the second inequality, and $x \in E$ together with the definition (4.1) of E in the last inequality. Moreover,

$$\int_M e^{-(2\sqrt{n-2}-\delta)r_x}\eta^2|f|^2\,\mathrm{dvol} \le e^{4\sqrt{n-2}r_0}\int_{B(x,2r_0)} e^{-(2\sqrt{n-2}-\delta)r_x}|f|^2\,\mathrm{dvol} \le c(n,r_0)\|f\|_0^2$$

since $\eta(\frac{r_x}{r_0}) = 0$ outside $B(x,2r_0)$. Combining this with (4.13) yields

$$\sigma\int_M e^{-(2\sqrt{n-2}-\delta)r_x}\eta^2\big|h - \frac{1}{\sigma}f\big|^2\,\mathrm{dvol} \le \bar{\epsilon}C(n,r_0)\|h\|^2_{C^0(M)} + \frac{c(n,r_0)}{\sigma}\|f\|_0^2.$$

Note $\sigma = c(n,\delta) - (1+\sqrt{n})\varepsilon_0$. Assume $\varepsilon_0 \le \frac{c(n,\delta)}{2(1+\sqrt{n})}$. Then $\sigma \ge \frac{c(n,\delta)}{2}$. Hence

$$\begin{aligned}
\int_{B(x,r_0)}\big|h - \frac{1}{\sigma}f\big|^2\,\mathrm{dvol} &\le e^{2\sqrt{n-2}r_0}\int_M e^{-(2\sqrt{n-2}-\delta)r_x}\eta^2\big|h-\frac{1}{\sigma}f\big|^2\,\mathrm{dvol} \\
&\le C(n,\delta,r_0)\big(\bar{\epsilon}\|h\|^2_{C^0(M)} + \|f\|_0^2\big).
\end{aligned}$$

Using the triangle inequality we get

$$\|h\|_{L^2(B(x,r_0))} \le C(n,\delta,r_0)\big(\bar{\epsilon}^{\frac{1}{2}}\|h\|_{C^0(M)} + \|f\|_0\big).$$

Combining this with (4.12) yields

$$\frac{1}{2}\|h\|_{C^0(M)} \le C\big(\bar{\epsilon}^{\frac{1}{2}}\|h\|_{C^0(M)} + \|f\|_0\big)$$

for some $C = C(n,\alpha,\delta,r_0,\Lambda)$. Thus for $\bar{\epsilon} \le \frac{1}{16C^2}$

$$\frac{1}{2}\|h\|_{C^0(M)} \le \frac{1}{4}\|h\|_{C^0(M)} + C\|f\|_0.$$

This implies the desired C^0-estimate. □

We now make some further remarks concerning this proof. First, we point out the following estimate that can be extracted from the proof. In fact, it follows immediately from the estimates (4.4) and (4.9). This estimate will be the key ingredient to obtain the exponential decay estimate in Theorem 1.

Remark 4.4. Let $f \in C^{0,\alpha}\big(\mathrm{Sym}^2(T^*M)\big)$ and let $h \in C^{2,\alpha}\big(\mathrm{Sym}^2(T^*M)\big)$ be a solution of $\mathcal{L}h = f$. Then there is a constant $C = C(n,\alpha,\delta,\Lambda)$ so that

$$|h|(x) \leq C\left(\|f\|_{C^0(B(x,\rho))} + \left(\int_M e^{-(2\sqrt{n-2}-\delta)r_x}|f|^2\,d\mathrm{vol}\right)^{\frac{1}{2}}\right)$$

for all $x \in M$. Here $\rho > 0$ is the constant appearing in the definition of the Hölder norms.

For the proof of Theorem 2 we have to deal with manifolds that may no longer be compact (but have finite volume), and do *not* have a positive lower bound on the injectivity radius. The next two remarks explain to what extend the arguments from the proof of Proposition 4.3 are still valid in that situation.

Remark 4.5. Let M be a finite volume manifold that satisfies all the assumptions from Proposition 4.3 except the compactness assumption and the lower bound on the injectivity radius. If $h \in C^2\big(\mathrm{Sym}^2(T^*M)\big) \cap H^2(M)$ and if $\mathcal{L}(h) = f$, then the inequality (4.8) is still valid. Here $h \in C^2\big(\mathrm{Sym}^2(T^*M)\big) \cap H^2(M)$ just means that h is C^2 and that $\int_M \big(|h|^2 + |\nabla h|^2 + |\nabla^2 h|^2\big)\,d\mathrm{vol} < \infty$.

Proof. The proof of inequality (4.8) carries over without change provided we can verify the equality

$$\int_M e^{-\varphi}\Delta(|h|^2)\,d\mathrm{vol} = \int_M \langle\nabla(e^{-\varphi}), \nabla(|h|^2)\rangle\,d\mathrm{vol}, \tag{4.14}$$

which involved an integration by parts. Here $\varphi = (2\sqrt{n-2} - \delta)r_x$. As in the proof of Proposition 4.3 we may act as if φ were smooth.

Consider the vector field $X := e^{-\varphi}\nabla(|h|^2)$. As $h \in H^2(M)$ and because φ is Lipschitz and bounded from below, it is easy to see that $X \in L^1(M)$ and $\mathrm{div}(X) = \langle\nabla(e^{-\varphi}), \nabla(|h|^2)\rangle - e^{-\varphi}\Delta(|h|^2) \in L^1(M)$. Therefore, the main result of [Gaf54] shows $\int_M \mathrm{div}(X)\,d\mathrm{vol} = 0$. ☐

Remark 4.6. Let M be a finite volume manifold that satisfies all the assumptions from Proposition 4.3 except the compactness assumption and the lower bound on the injectivity radius. Then there exist $\bar{\epsilon}_0 = \bar{\epsilon}_0(n,\alpha,\Lambda,\delta,r_0) > 0$ and $C = C(n,\alpha,\Lambda,\delta,r_0)$ with the following property. Let $h \in C^2\big(\mathrm{Sym}^2(T^*M)\big) \cap H^2(M)$, and assume there is $x_0 \in M_{\text{thick}}$ with $|h|(x_0) \geq \frac{1}{2}\|h\|_{C^0(M)}$. Then it holds

$$\|h\|_{C^0} \leq C\|\mathcal{L}h\|_0,$$

where $\|\cdot\|_0$ is the norm defined in (4.3) with respect to any $\bar{\epsilon} \leq \bar{\epsilon}_0$.

Proof. A priori (4.9) only holds in the universal cover. But for $x \in M_{\text{thick}}$, the norm $\|\cdot\|_{L^2(B(x,\rho))}$ is the same in the universal cover and in the base manifold (here we assume without loss of generality that the universal radius ρ used to define Hölder norms is smaller than a chosen Margulis constant). So (4.9) holds for $x \in M_{\text{thick}}$, and the rest of the argument of Proposition 4.3 applies without change. $\square$

We finish this section by showing that the operator $\mathcal{L}$ is invertible. This follows from Proposition 4.3 by standard techniques.

Proposition 4.7. *For all $n \geq 3$, $\alpha \in (0,1)$, $\Lambda \geq 0$, $\delta \in (0, \sqrt{n-2})$ and $r_0 \geq 1$ there exist constants ε_0, $\bar{\epsilon}_0$ and $C > 0$ with the following property. Let M be a closed Riemannian n-manifold with*

$$|\sec + 1| \leq \varepsilon_0, \quad \operatorname{inj}(M) \geq 1 \quad \textit{and} \quad \|\nabla \operatorname{Ric}\|_{C^0(M)} \leq \Lambda.$$

Then the operator

$$\mathcal{L} : \left(C^{2,\alpha}\left(\operatorname{Sym}^2(T^*M)\right), \|\cdot\|_2\right) \longrightarrow \left(C^{0,\alpha}\left(\operatorname{Sym}^2(T^*M)\right), \|\cdot\|_0\right)$$

is invertible, and

$$\|\mathcal{L}\|_{\text{op}}, \|\mathcal{L}^{-1}\|_{\text{op}} \leq C,$$

where $\|\cdot\|_2$ and $\|\cdot\|_0$ are the norms defined in (4.2) and (4.3) with respect to any $\bar{\epsilon} \leq \bar{\epsilon}_0$.

Proof. By Proposition 4.3, it remains to show that $\mathcal{L}$ is surjective. We split up the equation into its trace and its trace-free part. Namely, note that any $(0,2)$-tensor f can be written as $f = f^\circ + \varphi g$, where f° has vanishing trace, φ is a function, and g is the given Riemannian metric of M.

Note that $\mathcal{L}(ug) = \left(\frac{1}{2}\Delta u + (n-1)u\right)g$. The bilinear form $a_0 : H^1(M) \times H^1(M) \to \mathbb{R}$ associated to the equation $\frac{1}{2}\Delta u + (n-1)u = \varphi$ is given by

$$a_0(u,v) = \int_M \left(\frac{1}{2}\langle \nabla u, \nabla v\rangle + (n-1)uv\right) d\text{vol}.$$

This is clearly bounded and coercive. Thus by Lax-Milgram and the Weyl Lemma, for any $\varphi \in C^\infty(M)$ there is $u \in C^\infty(M)$ so that $\mathcal{L}(ug) = \varphi g$.

Let $E \to M$ be the vector bundle of symmetric $(0,2)$-tensors with vanishing trace. The bilinear form $a : H^1(E) \times H^1(E) \to \mathbb{R}$ associated to $\mathcal{L}$ is given by

$$a(h,h') = \int_M \left(\frac{1}{2}\langle \nabla h, \nabla h'\rangle + \frac{1}{2}\langle \operatorname{Ric}(h), h'\rangle + (n-1)\langle h, h'\rangle\right) d\text{vol}.$$

By Proposition 3.1 the Poincaré inequality holds for tensors with vanishing trace. Together with the estimate from Lemma 3.2 we get

$$a(h,h) \geq \frac{1}{2}\|\nabla h\|^2_{L^2(M)} - (1 + (1+\sqrt{n})\varepsilon_0)\|h\|^2_{L^2(M)} \geq \left(\frac{1}{2} - \frac{1 + (1+\sqrt{n})\varepsilon_0}{n - c(n)\varepsilon_0}\right)\|\nabla h\|^2_{L^2(M)}$$

for all $h \in H^1(E)$, so that for $\varepsilon_0 > 0$ small enough, the form a is coercive on E. So again by Lax-Milgram and the Weyl Lemma, for any $f^\circ \in C^\infty(E)$ there is $h \in C^\infty(E)$ so that $\mathcal{L}h = f^\circ$.

Therefore, splitting any f up into its trace part φg and its trace-free part f°, we obtain that for any $f \in C^\infty\left(\operatorname{Sym}^2(T^*M)\right)$ there is $h \in C^\infty\left(\operatorname{Sym}^2(T^*M)\right)$ so that $\mathcal{L}h = f$.

Recall the well-known fact that for any $u \in C^{0,\alpha}(\mathbb{R}^n)$ there is a sequence $(u_\varepsilon)_{\varepsilon > 0} \subseteq C^\infty(\mathbb{R}^n)$ so that $\lim_{\varepsilon \to 0} \|u_\varepsilon - u\|_{C^{0,\beta}(\mathbb{R}^n)} = 0$ for any $\beta \in (0,\alpha)$. Moreover, if u has compact support in some open set $\Omega \subseteq \mathbb{R}^n$, then u_ε can be assumed to have compact support in Ω too. Now let $f \in C^{0,\alpha}\big(\mathrm{Sym}^2(T^*M)\big)$ be arbitrary. Applying this approximation result locally, we obtain a sequence $(f_i)_{i\in\mathbb{N}}$ in $C^\infty\big(\mathrm{Sym}^2(T^*M)\big)$ converging to f with respect to the $C^{0,\frac{\alpha}{2}}$-norm. Let h_i be the solutions of $\mathcal{L}h_i = f_i$. Note that the norms $\|\cdot\|_{C^{0,\frac{\alpha}{2}}}$ and $\|\cdot\|_0$ are equivalent on $C^{0,\frac{\alpha}{2}}\big(\mathrm{Sym}^2(T^*M)\big)$ (but with a non-universal constant). It follows from Proposition 4.3 (applied with $\frac{\alpha}{2}$)

$$\|h_i - h_j\|_{C^2} \le C\|f_i - f_j\|_{C^{0,\frac{\alpha}{2}}} \to 0 \quad \text{as } i,j \to \infty$$

for a (non-universal) constant C. So $(h_i)_{i\in\mathbb{N}} \subseteq C^2\big(\mathrm{Sym}^2(T^*M)\big)$ is a Cauchy sequence. Denote the limit tensor field by h. Clearly h solves $\mathcal{L}h = f$. Finally, $h \in C^{2,\alpha}\big(\mathrm{Sym}^2(T^*M)\big)$ by elliptic regularity theory. Therefore, $\mathcal{L}$ is bijective. The bound on $\|\mathcal{L}^{-1}\|_{\mathrm{op}}$ follows from Proposition 4.3, and the one for $\|\mathcal{L}\|_{\mathrm{op}}$ is obvious. □

Recall that by Convention 2.3 we assume all manifolds to be orientable. Nonetheless, we have the following.

Remark 4.8. Proposition 4.7 also holds when M is not orientable.

Proof. Proposition 4.7 holds for the orientation cover $\hat{M}$ of M. Moreover, since the non-trivial decktransformation $\tau : \hat{M} \to \hat{M}$ is an isometry, Proposition 4.7 shows that the elliptic operator $\mathcal{L}$ on $\hat{M}$ restricts to an isomorphism between subbundles of τ-invariant Hölder sections of symmetric $(0,2)$-tensors. But τ-invariant Hölder sections on $\hat{M}$ are nothing else than Hölder sections on M. □

5. Proof of the pinching theorem with lower injectvity radius bound

We start by stating a more precise formulation of Theorem 1.

Theorem 5.1. *For any $n \ge 3$, $\alpha \in (0,1)$, $\Lambda \ge 0$, $\delta \in (0, 2\sqrt{n-2})$ and $r_0 \ge 1$ there exist constants ε_0 and $C > 0$ with the following property. Let M be a closed n-manifold that admits a Riemannian metric $\bar{g}$ satisfying the following conditions for some $\varepsilon \le \varepsilon_0$:*

i) $-1-\varepsilon \le \sec_{(M,\bar{g})} \le -1+\varepsilon$;
ii) $\mathrm{inj}(M,\bar{g}) \ge 1$;
iii) $\|\nabla \mathrm{Ric}(\bar{g})\|_{C^0(M,\bar{g})} \le \Lambda$;
iv) It holds

$$\int_M e^{-(2\sqrt{n-2}-\delta)r_x(y)} |\mathrm{Ric}(\bar{g}) + (n-1)\bar{g}|_{\bar{g}}^2(y)\, d\mathrm{vol}_{\bar{g}}(y) \le \varepsilon^2$$

for all $x \in M$ with

$$\int_{B(x,2r_0)\smallsetminus B(x,r_0)} e^{-(2\sqrt{n-2}-\delta)r_x(y)}\, d\mathrm{vol}_{\bar{g}}(y) > \varepsilon_0,$$

where $r_x(y) = d_{\bar{g}}(x,y)$.

Then there exists an Einstein metric g_0 on M so that $\mathrm{Ric}(g_0) = -(n-1)g_0$ *and*

$$\|g_0 - \bar{g}\|_2 \le C\varepsilon^{1-\alpha},$$

where $\|\cdot\|_2$ *is the norm defined in (4.2) with respect to the metric* $\bar{g}$ *and the constants* ϵ_0, δ, r_0.

Moreover, if for some $\beta \le 2\sqrt{n-2} - \delta$ *and* $U \subseteq M$ *it holds*

$$\int_M e^{-(2\sqrt{n-2}-\delta)r_x(y)} |\mathrm{Ric}(\bar{g}) + (n-1)\bar{g}|^2(y)\, d\mathrm{vol}(y) \le \varepsilon^{2(1-\alpha)} e^{-2\beta\, \mathrm{dist}_{\bar{g}}(x,U)} \quad \text{for all } x \in M,$$

then

$$|g_0 - \bar{g}|_{C^{2,\alpha}}(x) \le C\varepsilon^{1-\alpha} e^{-\beta\, \mathrm{dist}_{\bar{g}}(x,U)} \quad \text{for all } x \in M.$$

In particular, if $\mathrm{Ric}(\bar{g}) = -(n-1)\bar{g}$ *outside a region* U, *and if*

$$\int_U |\mathrm{Ric}(\bar{g}) + (n-1)\bar{g}|^2\, d\mathrm{vol}_{\bar{g}} \le \varepsilon^2,$$

then

$$|g_0 - \bar{g}|_{C^{2,\alpha}}(x) \le C\varepsilon^{1-\alpha} e^{-(\sqrt{n-2}-\frac{1}{2}\delta)\, \mathrm{dist}_{\bar{g}}(x,U)} \quad \text{for all } x \in M.$$

As mentioned previously, we will prove this using the implicit function theorem. The linearisation $(D\Phi)_{\bar{g}}$ of the Einstein operator Φ at the initial metric was studied in Section 4, and we showed that it is invertible, with controlled norm of its inverse. To control the size of a neighborhood of $\Phi(\bar{g})$ in which Φ is invertible requires an estimate of the Lipschitz constant of the mapping $g \mapsto (D\Phi)_g$. This will follow from the next lemma.

Lemma 5.2. *For all* $n \ge 2$, $\alpha \in (0,1)$, $\Lambda \ge 0$, $i_0 > 0$ *there exist* $\varepsilon = \varepsilon(n, \alpha, \Lambda, i_0) > 0$ *and* $C = C(n, \alpha, \Lambda, i_0)$ *with the following property. Let* $(M, \bar{g})$ *be a Riemannian* n*-manifold with*

$$\|\mathrm{Ric}(\bar{g})\|_{C^1(M,\bar{g})} \le \Lambda \quad \text{and} \quad \mathrm{inj}(M, \bar{g}) \ge i_0$$

and let $g \in C^{2,\alpha}(\mathrm{Sym}^2(T^*M))$ *be another Riemannian metric so that* $\|g - \bar{g}\|_{C^{2,\alpha}(M,\bar{g})} \le \varepsilon$. *Then the linearization of the Einstein operator* $\Phi = \Phi_{\bar{g}}$ *defined in (2.1) satisfies the pointwise estimates*

$$|(D\Phi)_g(h) - (D\Phi)_{\bar{g}}(h)|_{C^{0,\alpha}}(x) \le C \max_{y \in B(x,\rho)} |g - \bar{g}|_{C^{2,\alpha}}(y) |h|_{C^{2,\alpha}}(y)$$

and

$$|(D\Phi)_g(h) - (D\Phi)_{\bar{g}}(h)|_{C^0}(x) \le C|g - \bar{g}|_{C^2}(x)|h|_{C^2}(x)$$

for all $h \in C^{2,\alpha}(\mathrm{Sym}^2(T^*M))$, *where all norms are taken w.r.t. the background metric* $\bar{g}$.

The term $\max_{y \in B(x,\rho)}$ comes from the fact that $C^{0,\alpha}$-norms are defined in bigger local charts than $C^{2,\alpha}$-norms (also see inequality (2.6)). Also note that since all norms are taken w.r.t. $\bar{g}$, we do *not* need an upper bound on $\|\nabla \mathrm{Ric}(g)\|_{C^0}$.

Proof. The linearisation of the operator $g \to \mathrm{Ric}(g)$ is given by (see [Top06, Proposition 2.3.7])

$$(D\mathrm{Ric})_g(h) = \frac{1}{2}\Delta_L h - \frac{1}{2}\mathcal{L}_{(\beta_g(h))^{\sharp_g}}(g),$$

where $\beta_g = \delta_g(\cdot) + \frac{1}{2}d\mathrm{tr}_g(\cdot)$ is the Bianchi operator of g, $\sharp_g : T^*M \to TM$ is the musical isomorphism associated to g, and $\mathcal{L}_X(\cdot)$ is the Lie derivative in direction X. Therefore, it follows from the definition (2.1) of Φ and the product rule that

$$(D\Phi)_g(h) = \Delta_L h + (n-1)h + \frac{1}{2}\mathcal{L}_{(\beta_{\bar{g}}(g))^{\sharp_g}}(h) + \frac{1}{2}\mathcal{L}_{(\beta_{\bar{g}}(h)-\beta_g(h))^{\sharp_g}}(g) + \frac{1}{2}\mathcal{L}_{(D_h\sharp)_g(\beta_{\bar{g}}(g))}(g),$$

where $(D_h\sharp)_g : T^*M \to TM$ is the linearisation of $g \to \sharp_g$ in direction h. Denote by $\flat_g : TM \to T^*M, v \mapsto g(v,\cdot)$ the inverse of $\sharp_g$. Differentiating the identity $\mathrm{id}_{TM} = \sharp_g \circ \flat_g$ in direction h, and applying $\flat_g$ yields $(D_h\sharp)_g = -\sharp_g \circ \flat_h \circ \sharp_g$, where $\flat_h(v) = h(v,\cdot)$. In local coordinates this reads $\big((D_h\sharp)_g(\omega)\big)^m = -g^{mk}h_{kj}g^{ji}\omega_i$. Therefore, Lemma 5.2 can be checked by a straightforward (albeit tedious) calculation in local coordinates. We will not carry this out in more detail. $\square$

Lemma 5.2 and Remark 4.2 immediately imply the following corollary.

Corollary 5.3. *For any $n \geq 3$, $\alpha \in (0,1)$, $\Lambda \geq 0$ there exist $\varepsilon = \varepsilon(n,\alpha,\Lambda) > 0$ and $C = C(n,\alpha,\Lambda)$ with the following property. Let $(M,\bar{g})$ be a closed Riemannian n-manifold with*

$$|\sec| \leq 2, \quad \mathrm{inj}(M,\bar{g}) \geq 1 \quad \textit{and} \quad \|\nabla \mathrm{Ric}(\bar{g})\|_{C^0(M,\bar{g})} \leq \Lambda$$

and let $g \in C^{2,\alpha}\big(\mathrm{Sym}^2(T^*M)\big)$ be another Riemannian metric so that $\|g-\bar{g}\|_{C^{2,\alpha}(M,\bar{g})} \leq \varepsilon$. Then the operator $\Phi = \Phi_{\bar{g}}$ defined in (2.1) satisfies

$$\|(D\Phi)_g(h) - (D\Phi)_{\bar{g}}(h)\|_0 \leq C\|g-\bar{g}\|_2\|h\|_2$$

for all $h \in C^{2,\alpha}\big(\mathrm{Sym}^2(T^*M)\big)$, where $\|\cdot\|_2$ and $\|\cdot\|_0$ are the norms defined in (4.2) and (4.3) with respect to the metric $\bar{g}$ and any $\bar{\epsilon},\delta,r_0$.

We now come to the proof of Theorem 5.1.

Proof of Theorem 5.1. In this proof, $\|\cdot\|_2$ resp. $\|\cdot\|_0$ shall denote the norms defined in (4.2) and (4.3) with respect to the metric $\bar{g}$ and the constants ε_0,δ,r_0, and $C^{2,\alpha}\big(\mathrm{Sym}^2(T^*M)\big)$ is understood to be equipped with $\|\cdot\|_2$. Metric balls $B(h,R)$ of radius R about a section h are taken with respect to that norm.

Define the operator

$$\Psi : B(0,1) \subseteq C^{2,\alpha}\big(\mathrm{Sym}^2(T^*M)\big) \to C^{2,\alpha}\big(\mathrm{Sym}^2(T^*M)\big)$$

by

$$\Psi(h) := h - \mathcal{L}^{-1}\big(\Phi(\bar{g}+h)\big).$$

Here $\Phi = \Phi_{\bar{g}}$ is the Einstein operator defined in (2.1), and $\mathcal{L} = (D\Phi)_{\bar{g}}$.

By Proposition 4.3 and Corollary 5.3, there is a constant $C = C(n,\alpha,\delta,r_0,\Lambda)$ such that $\|\mathcal{L}^{-1}\|_{\mathrm{op}} \leq C$ and $\mathrm{Lip}\big((D\Phi)_\bullet\big) \leq C$. Thus it follows from $(D\Psi)_h = \mathcal{L}^{-1} \circ \big(\mathcal{L} - (D\Phi)_{\bar{g}+h}\big)$ that for $R = R(n,\alpha,\delta,\Lambda) > 0$ small enough, the restriction of Ψ to the closed ball $\bar{B}(0,R)$ is $\frac{1}{2}$-Lipschitz. Moreover, since $\|\cdot\|_{C^{0,\alpha}} \leq C\|\cdot\|_{C^0}^{1-\alpha}\|\cdot\|_{C^1}^{\alpha}$ and $\Phi(\bar{g}) = \mathrm{Ric}(\bar{g}) + (n-1)\bar{g}$, the assumptions $i)$ and $iii)$ imply that $\|\Phi(\bar{g})\|_{C^{0,\alpha}} \leq C\varepsilon^{1-\alpha}$. Together with condition $iv)$, this shows

$$\|\Phi(\bar{g})\|_0 \leq C\varepsilon^{1-\alpha}$$

due to the definition of the norm $\|\cdot\|_0$. As a consequence, for $\varepsilon_0 = \varepsilon_0(n,\alpha,\delta,\Lambda) > 0$ small enough, we have $\|\Psi(0)\|_2 \le \frac{R}{2}$ and hence Ψ restricts to a map $\bar{B}(0,R) \to \bar{B}(0,R)$. By the Banach fixed point theorem there exists a fixed point h_0 of Ψ. By definition of Ψ this means $\Phi(\bar{g}+h_0) = 0$, and hence $g_0 = \bar{g} + h_0$ is an Einstein metric due to Lemma 2.4. Moreover, as Ψ is $\frac{1}{2}$-Lipschitz, it holds

$$\|h_0\|_2 = \|\Psi(h_0)\|_2 \le \|\Psi(h_0) - \Psi(0)\|_2 + \|\Psi(0)\|_2 \le \frac{1}{2}\|h_0\|_2 + C\varepsilon^{1-\alpha}.$$

This implies $\|h_0\|_2 \le C\varepsilon^{1-\alpha}$.

It remains to show the improved estimate on $|g_0 - \bar{g}|_{C^{2,\alpha}}(x)$. Let $\beta \le 2\sqrt{n-2} - \delta$ and $U \subseteq M$ be as in the statement of Theorem 5.1.

Recall that the pointwise Hölder norm $|\cdot|_{C^{0,\alpha}}(x)$ is computed in a local chart defined on the ball $B(x,\rho)$, where $\rho = \rho(n,\alpha,\Lambda) > 0$ is a universal constant (see the proof of Proposition 2.5 for more details). In particular, $\|\cdot\|_{C^0(B(x,\rho))} \le C|\cdot|_{C^{0,\alpha}}(x)$. So Remark 4.4 and the pointwise Schauder estimate (2.4) show that there is a universal constant C_0 so that for all $h \in C^{2,\alpha}\big(\mathrm{Sym}^2(T^*M)\big)$ it holds

$$|h|_{C^{2,\alpha}}(x) \le C_0\left(|\mathcal{L}h|_{C^{0,\alpha}}(x) + \left(\int_M e^{-(2\sqrt{n-2}-\delta)r_x(y)}|\mathcal{L}h|^2(y)\,d\mathrm{vol}_{\bar{g}}\right)^{\frac{1}{2}}\right). \tag{5.1}$$

Choose C_0 large enough so that the a priori estimate from Proposition 4.3, and the weighted integral estimates from Step 1 of the proof of Proposition 4.3 hold, that is,

$$\|h\|_2 \le C_0\|\mathcal{L}h\|_0 \quad\text{and}\quad \|h\|_{H^2(M;\omega_x)} \le C_0\|\mathcal{L}h\|_{L^2(M;\omega_x)} \ \text{ for all } x \in M, \tag{5.2}$$

where $\|\cdot\|_{H^2(M,\omega_x)} := \left(\int_M e^{-(2\sqrt{n-2}-\frac{1}{2}\delta)r_x(y)}|\cdot|_{C^2}(y)\,d\mathrm{vol}_{\bar{g}}(y)\right)^{\frac{1}{2}}$ is the weighted H^2-norm, and analogously $\|\cdot\|_{L^2(M;\omega_x)}$ shall denote the weighted L^2-norm. Moreover, we assume that C_0 is large enough so that $\|\mathrm{Ric}(\bar{g}) + (n-1)\bar{g}\|_{C^{0,\alpha}(M)} \le C_0\varepsilon^{1-\alpha}$.

Define $C_1 := 2C_0^2 e^{\rho\sqrt{n-2}} + 2C_0$, and consider the set

$$\mathcal{U} := \Big\{h \in \mathrm{Dom}(\Psi) \,|\, h \text{ satisfies the inequalities (5.4), (5.5) for all } x \in M\Big\}, \tag{5.3}$$

where the inequalities (5.4) and (5.5) appearing in the definition of $\mathcal{U}$ are

$$|h|_{C^{2,\alpha}}(x) \le C_1\varepsilon^{1-\alpha}e^{-\beta\,\mathrm{dist}_{\bar{g}}(x,U)} \tag{5.4}$$

and

$$\|h\|_{H^2(M;\omega_x)} \le C_1\varepsilon^{1-\alpha}e^{-\beta\,\mathrm{dist}_{\bar{g}}(x,U)}. \tag{5.5}$$

We will show that $\Psi(\mathcal{U}) \subseteq \mathcal{U}$. This implies the desired estimate, because the fixed point h_0 is then necessarily contained in $\mathcal{U}$.

To prove $\Psi(\mathcal{U}) \subseteq \mathcal{U}$ we first observe that for all $h \in \mathrm{Dom}(\Psi)$ it holds

$$\Psi(h) - \Psi(0) = \int_0^1 \mathcal{L}^{-1}\big(\mathcal{L}h - (D\Phi)_{\bar{g}+th}h\big)\,dt \tag{5.6}$$

by the Fundamental Theorem of Calculus. Denote by $C_2 \coloneqq \mathrm{Lip}\big((D\Phi)_\bullet\big)$ the universal continuity constant given by Lemma 5.2, so that it holds

$$|\mathcal{L}h - (D\Phi)_{\bar{g}+th}h|_{C^{0,\alpha}}(x) \le C_2 \sup_{y \in B(x,\rho)} |h|^2_{C^{2,\alpha}}(y) \tag{5.7}$$

and

$$|\mathcal{L}h - (D\Phi)_{\bar{g}+th}h|_{C^0}(x) \le C_2 |h|^2_{C^2}(x). \tag{5.8}$$

Now let $h \in \mathcal{U}$ be arbitrary. We start by showing that $\Psi(h)$ satisfies (5.5). Combining the Jensen-inequality, (5.2), (5.6), and (5.8) yields

$$\begin{aligned}
\|\Psi(h) - \Psi(0)\|^2_{H^2(M;\omega_x)} &\overset{(5.6)}{\le} \int_0^1 \big\|\mathcal{L}^{-1}\big(\mathcal{L}h - (D\Phi)_{\bar{g}+th}h\big)\big\|^2_{H^2(M;\omega_x)}\, dt \\
&\overset{(5.2)}{\le} C_0^2 \int_0^1 \big\|\mathcal{L}h - (D\Phi)_{\bar{g}+th}h\big\|^2_{L^2(M;\omega_x)}\, dt \\
&\overset{(5.8)}{\le} C_0^2 C_2^2 \int_M e^{-(2\sqrt{n-2}-\delta)r_x(y)} |h|^4_{C^2}(y)\, d\mathrm{vol}_{\bar{g}}(y).
\end{aligned} \tag{5.9}$$

Note $\|h\|_{C^2(M)} \le C_1 \varepsilon^{1-\alpha}$ by (5.4) and since $h \in \mathcal{U}$. Together with (5.5) and (5.9) this implies

$$\begin{aligned}
\|\Psi(h) - \Psi(0)\|^2_{H^2(M;\omega_x)} &\overset{(5.9)}{\le} C_0^2 C_2^2 \int_M e^{-(2\sqrt{n-2}-\delta)r_x(y)} |h|^4_{C^2}(y)\, d\mathrm{vol}(y) \\
&\le C_0^2 C_2^2 C_1^2 \varepsilon^{2(1-\alpha)} \int_M e^{-(2\sqrt{n-2}-\delta)r_x(y)} |h|^2_{C^2}(y)\, d\mathrm{vol}(y) \\
&\overset{(5.5)}{\le} C_0^2 C_2^2 C_1^2 \varepsilon^{2(1-\alpha)} C_1^2 \varepsilon^{2(1-\alpha)} e^{-2\beta \operatorname{dist}_{\bar{g}}(x,U)}.
\end{aligned} \tag{5.10}$$

Note $\Phi(\bar{g}) = \mathrm{Ric}(\bar{g}) + (n-1)\bar{g}$. Hence $\|\Phi(\bar{g})\|_{L^2(M;\omega_x)} \le \varepsilon^{1-\alpha} e^{-\beta \operatorname{dist}_{\bar{g}}(x,U)}$ by assumption, so that $\|\Psi(0)\|_{H^2(M;\omega_x)} \le C_0 \varepsilon^{1-\alpha} e^{-\beta \operatorname{dist}_{\bar{g}}(x,U)}$ by (5.2). Applying (5.10) and the triangle inequality yields

$$\|\Psi(h)\|_{H^2(M;\omega_x)} \le \Big(C_0 + C_0 C_2 C_1^2 \varepsilon^{(1-\alpha)}\Big) \varepsilon^{(1-\alpha)} e^{-\beta \operatorname{dist}_{\bar{g}}(x,U)}.$$

As $C_0 + C_0 C_2 C_1^2 \varepsilon^{(1-\alpha)} \le C_1$ for $\varepsilon > 0$ small enough, we conclude that $\Psi(h)$ satisfies (5.5) for all $x \in M$.

It remains to show that $\Psi(h)$ satisfies (5.4) for all $x\in M$, because then $\Psi(h)\in\mathcal{U}$ by the definition (5.3) of $\mathcal{U}$. Combining (5.1), (5.6), (5.7), and (5.8) shows

$$\begin{aligned}
|\Psi(h)-\Psi(0)|_{C^{2,\alpha}}(x) &\overset{(5.6)}{\leq} \int_0^1 \left|\mathcal{L}^{-1}\big(\mathcal{L}h-(D\Phi)_{\bar g+th}h\big)\right|_{C^{2,\alpha}}(x)\,dt\\
&\overset{(5.1)}{\leq} \int_0^1 C_0|\mathcal{L}h-(D\Phi)_{\bar g+th}h|_{C^{0,\alpha}}(x)\,dt\\
&\quad +C_0\int_0^1 \|\mathcal{L}h-(D\Phi)_{\bar g+th}h\|_{L^2(M;\omega_x)}\,dt\\
&\overset{(5.7),(5.8)}{\leq} C_0C_2\sup_{y\in B(x,\rho)}|h|_{C^{2,\alpha}}(y)\\
&\quad +C_0C_2\left(\int_M e^{-(2\sqrt{n-2}-\delta)r_x(y)}|h|^4_{C^2}(y)\,d\mathrm{vol}_{\bar g}(y)\right)^{\frac12}. \qquad (5.11)
\end{aligned}$$

Note that the last summand is estimated in (5.10). Using (5.4) to estimate $|h|_{C^{2,\alpha}}(y)$, and remembering $\beta\leq\sqrt{n-2}$ we get

$$\begin{aligned}
|\Psi(h)-\Psi(0)|_{C^{0,\alpha}}(x) &\overset{(5.11)}{\leq} C_0C_2\sup_{y\in B(x,\rho)}|h|^2_{C^{2,\alpha}}(y)\\
&\quad + C_0C_2\left(\int_M e^{-(2\sqrt{n-2}-\delta)r_x(y)}|h|^4_{C^2}(y)\,d\mathrm{vol}_{\bar g}(y)\right)^{\frac12}\\
&\overset{(5.4),(5.10)}{\leq} C_0C_2C_1^2\varepsilon^{2(1-\alpha)}\sup_{y\in B(x,\rho)} e^{-2\beta\,\mathrm{dist}_{\bar g}(y,U)}\\
&\quad + C_0C_2C_1^2\varepsilon^{2(1-\alpha)}e^{-\beta\,\mathrm{dist}_{\bar g}(x,U)}\\
&\leq \left(2C_0C_2C_1^2e^{2\rho\sqrt{n-2}}\varepsilon^{(1-\alpha)}\right)\varepsilon^{(1-\alpha)}e^{-\beta\,\mathrm{dist}_{\bar g}(x,U)}.
\end{aligned}$$

Recall $\|\Phi(\bar g)\|_{L^2(M;\omega_x)}\leq\varepsilon^{1-\alpha}e^{-\beta\,\mathrm{dist}_{\bar g}(x,U)}$ and $\Psi(0)=-\mathcal{L}^{-1}\Phi(\bar g)$. Thus applying (5.1) shows $|\Psi(0)|_{C^{2,\alpha}}(x)\leq C_0\big(|\Phi(\bar g)|_{C^{0,\alpha}}(x)+\varepsilon^{1-\alpha}e^{-\beta\,\mathrm{dist}_{\bar g}(x,U)}\big)$. Since $|\cdot|_{C^{0,\alpha}}(x)$ is computed in a local chart defined on $B(x,\rho)$, it holds $|\Phi(\bar g)|_{C^{0,\alpha}}(x)=0$ if $\mathrm{dist}_{\bar g}(x,U)>\rho$. For $x\in M$ with $\mathrm{dist}_{\bar g}(x,U)\leq\rho$ it holds $|\Phi(\bar g)|_{C^{0,\alpha}}(x)\leq\|\Phi(\bar g)\|_{C^{0,\alpha}(M)}\leq C_0\varepsilon^{1-\alpha}\leq C_0e^{\beta\rho}\varepsilon^{1-\alpha}e^{-\beta\,\mathrm{dist}_{\bar g}(x,U)}$. All in all we conclude (again remembering $\beta\leq\sqrt{n-2}$)

$$|\Psi(h)|_{C^{2,\alpha}}(x)\leq\left(2C_0C_2C_1^2e^{2\rho\sqrt{n-2}}\varepsilon^{(1-\alpha)}+C_0+C_0^2e^{\rho\sqrt{n-2}}\right)\varepsilon^{1-\alpha}e^{-\beta\,\mathrm{dist}_{\bar g}(x,U)}$$

for all $x\in M$. Recall that $C_1=2C_0^2e^{\rho\sqrt{n-2}}+2C_0$. As $2C_0C_2C_1^2e^{2\beta\rho}\varepsilon^{(1-\alpha)}\leq C_0$ for $\varepsilon>0$ small enough, we conclude that $\Psi(h)$ satisfies (5.4) for all $x\in M$. Therefore, $\Psi(h)\in\mathcal{U}$. Since $h\in\mathcal{U}$ was arbitrary, we obtain $\Psi(\mathcal{U})\subseteq\mathcal{U}$. This completes the proof. □

Since the proof of Theorem 5.1 is merely an application of the Banach fixed point theorem based on Proposition 4.7, the next remark is immediate due to Remark 4.8.

Remark 5.4. Theorem 5.1 also holds when M is non-orientable.

6. Counterexamples

For our most important applications, we need a version of Theorem 5.1 which does not require a lower injectivity radius bound. The purpose of this section is to show that at least in dimension 3, such a result can not be obtained as a straightforward extension of Theorem 5.1 by providing examples which show that such straightforward extensions do not hold true. The mechanism behind these examples lies in the fact that hyperbolic metrics on *Margulis tubes* or *cusps* admit nontrivial hyperbolic deformations, and such deformations can be used to construct families of metrics on closed hyperbolic 3-manfolds violating the a priori stability estimates which are essential for an application of the implicit function theorem. The geometric feactures of these examples motivate our approach towards our second main result Theorem 2 which is valid for 3-dimensional manifolds without the assumption of a lower injectivity radius bound.

Proposition 6.1. *For any $\varepsilon > 0$, $\lambda \in (0,2)$ and any $C > 0$ there exists a closed 3-manifold M_ϵ and a Riemannian metric g on M_ε with the following properties.*

i) *The sectional curvature of g is contained in the interval $[-1-\varepsilon, -1+\varepsilon]$.*
ii) *For each component A of the thin part of (M_ε, g), we have*
$$\int_A \frac{1}{(\mathrm{inj})^{2-\lambda}} |\mathrm{Ric}(g) + 2g|_g^2 d\mathrm{vol} \le \varepsilon^2.$$
iii) *The volume of (M_ε, g) is bounded from above by a constant independent of ε.*
iv) *There is no constant curvature metric g_{const} on M_ϵ with the property that the identity $\mathrm{id}_M : (M_\epsilon, g) \to (M_\epsilon, g_{\mathrm{const}})$ is a C-bilipschitz equivalence.*

Remark 6.2. It will be apparent from the proof that the geometric properties we use for the construction of the examples are special to dimension 3 and, by a result of Gromov [Gro78], do not extend to dimension at least 4. Although we expect a result similar to our second main theorem to hold true in all dimensions, such an extension may require a new strategy for the proof.

For the construction of the manifolds M_ε we start with an orientable hyperbolic 3-manifold M of finite volume, with a single cusp. For example, the figure 8 knot complement will do. The cusp has a neighborhood $B = \Gamma\backslash H$ which is the quotient of a horoball H in $\mathbb{H}^3$ by an abelian subgroup $\Gamma = \mathbb{Z}^2$ of parabolic isometries. The group Γ preserves each of the horospheres which foliate H, and the quotient of each horosphere under Γ is a flat two-torus T^2.

Let us write the cusp neighborhood B and its hyperbolic metric g in the form
$$B = T^2 \times [0,\infty), \quad g = e^{-2t} g_0 + dt^2$$
where g_0 is a fixed flat metric on T^2. In other words, we have $(T^2, g_0) = \Gamma_0\backslash\mathbb{R}^2$ where Γ_0 is a group of translations of $\mathbb{R}^2$ isomorphic to $\mathbb{Z}^2$.

We now look at deformations of the metric g_0 of the following form. Let (e_1, e_2) be any orthonormal basis of $\mathbb{R}^2$. For $s \in \mathbb{R}$ consider the matrix
$$A(s) = \begin{pmatrix} 1 & 0 \\ 0 & e^s \end{pmatrix}.$$

It acts as a linear isomorphism on $\mathbb{R}^2$ not preserving the euclidean metric. Let g_s be the pull-back of the euclidean metric by $A(s)$. This pullback metric preserves orthogonality of the vectors e_1, e_2, preserves the length of e_1 and scales the length of e_2 by the factor e^s.

For numbers $\delta > 0, R > 0$ let $f_{\delta,R} : [0,\infty) \to [0,\infty)$ be a smooth function with the following properties.

(1) $f_{\delta,R}$ is supported in $[0, \frac{R}{\delta} + 4]$.
(2) $|f_{\delta,R}| \le \delta$.
(3) $|f'_{\delta,R}| \le \delta$.
(4) $|f''_{\delta,R}| \le 1$.
(5) $\int_0^\infty f_{\delta,R}(s)ds = R$.

Let $g_s = g(s, \delta, R, e_1, e_2)$ be the pull-back of the euclidean metric on the torus T^2 by the linear isomorphism $A(\int_0^s f_{\delta,R}(u)du)$. Then for each s, the metric g_s is a flat metric on T which is the pullback of the standard metric by an affine automorphism and as such determined by the property that the vectors $e_1, e_2/e^{\int_0^s f_{\delta,R}(u)du}$ are orthonormal.

Since the curvature of a Riemannian metric is computed by second derivatives of the metric, and since furthermore for any of the flat metrics g_s on T^2, the metric $e^{-2t}g_s + dt^2$ on B is hyperbolic, that is, of constant curvature -1, the following is a consequence of the construction.

Lemma 6.3. *For any $\varepsilon > 0$ there exists a number $\delta(\varepsilon) > 0$ such that for any $\delta < \delta(\varepsilon)$, $m > 0$ the curvature of each of the metrics*

$$e^{-2t}g(t-m, \delta, R, e_1, e_2) + dt^2$$

on $B = T^2 \times [0,\infty)$ is contained in the interval $[-1-\varepsilon, -1+\varepsilon]$.

Proof. As curvature computation is local, we can carry this out in the universal covering. Thus for a fixed point $y \in T^2 \times \{t\} \subset B$, we compute in the universal covering $\mathbb{R}^2 \times [0,\infty)$, which we can identify with the horoball $H = \{x_3 \ge c\}$ for some $c > 0$ in hyperbolic 3-space $\mathbb{H}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$. We also may assume that the horosphere $S = \{x_3 = 1\} \subset H$ contains a preimage $\tilde{y}$ of y. Furthermore, we consider the standard hyperbolic metric h on H, where the normalization is such that the standard flat metric on $\{x_3 = 1\}$ is the preimage of the flat metric on the slice $T^2 \times \{t\}$ containing y.

For simplicity of notation, write $u = t - s$. With this description, the hyperbolic metric near $\tilde{y}$ determined by the flat metric on $T^2 \times \{s\}$ is the warped product metric $h = e^{-2u}h_0 + du^2$ on H where $u = \log x_3$, and the lift $\hat{g}$ of the metric $e^{-2u}g(s-m, \delta, R, e_1, e_2)$ near $\tilde{y}$ is of the form

$$\hat{g} = e^{-2u}A(\beta(u))^* h_0 + du^2$$

where $\beta(u)$ is a smooth function on an interval containing 0 which satisfies $\beta(0) = 0$ and whose first and second derivatives near 0 are smaller than 2δ in absolute value.

Thus in standard coordinates, the Christoffel symbols for the metric $\hat{g}$ and their first derivatives are uniformly near the Christoffel symbols and their first derivatives for the hyperbolic metric. This implies that for any $\varepsilon > 0$ we can find a number $\delta(\varepsilon) > 0$ so that the statement of the lemma holds true for this ϵ. ☐

We now use this construction as follows. Consider as before a finite volume hyperbolic 3-manifold M with a single cusp B. We shall Dehn-fill the cusp and use a deformation of the Dehn filled metric to achieve our goal. The geometric control we are looking for is obtained from geometric information of Margulis tube of the Dehn filled manifold corresponding to the cusp of M.

To set up this construction, note that a Margulis tube in a hyperbolic 3-manifold is given as the quotient of a tubular neighborhood $N(\tilde{\gamma}, R)$ of radius $R > 0$ of a geodesic $\tilde{\gamma} \subseteq \mathbb{H}^3$ by an infinite cyclic group $\langle \phi \rangle \subseteq SO(3,1)$ of isometries, generated by an element ϕ which preserves $\tilde{\gamma}$ and acts on $\tilde{\gamma}$ as a translation. Then ϕ can be represented as a product of a transvection preserving $\tilde{\gamma}$ and an isometry ψ which fixes $\tilde{\gamma}$ pointwise and acts as a rotation on the orthogonal complement of $\tilde{\gamma}' \subseteq T_{\tilde{\gamma}}\mathbb{H}^3$.

Parameterize $\tilde{\gamma}$ by arc length and write $x_0 = \tilde{\gamma}(0)$. There is a totally geodesic hyperbolic plane $\mathbb{H}^2 \subseteq \mathbb{H}^3$ orthogonal to $\tilde{\gamma}'$ which passes through x_0. The quotient by ϕ of the tubular neighborhood $N(\tilde{\gamma}, R)$ intersects $\mathbb{H}^2$ in a hyperbolic disk whose boundary is an embedded circle in the two-torus $T^2 = \partial N(\tilde{\gamma}, R)/\phi$ of length $2\pi \sinh(R)$. This circle is the meridian of the solid torus $B = N(\tilde{\gamma}, R)/\phi$.

Choose a totally geodesic hyperbolic plane $H_0 \subseteq \mathbb{H}^3$ which contains $\tilde{\gamma}$. If $\tau > 0$ is the translation length of ϕ, then H_0 intersects the fundamental domain $\{\exp Y \mid Y \in T_{\tilde{\gamma}(u)}\mathbb{H}^3, 0 \le u \le \tau, Y \perp \tilde{\gamma}'\}$ for ϕ in a strip bounded by two geodesics which are orthogonal to $\tilde{\gamma}$, and the intersection of this strip with $\partial N(\tilde{\gamma}, R)$ contains an arc which descends to a straight line segment on the boundary torus T^2 of length $\tau \cosh(R)$. In particular, the translation length τ of ϕ, which equals the length of the closed geodesic in the free homotopy class defined by ϕ in the quotient manifold $\mathbb{H}^3/\langle \phi \rangle$, can explicitly be computed from the length of the meridian on T^2 and the length of a straight line segment orthogonal to the meridian which connects two points on the meridian and does not contain an intersection point with the meridian in its interior.

A Dehn filling of the finite volume hyperbolic manifold M is determined by the choice of a simple closed geodesic ζ on the boundary T^2 of the cusp, which is a flat torus. The Dehn filling along ζ is obtained from M by removal of the cusp and gluing a solid torus along the boundary whose meridian is glued to ζ. If ζ is sufficiently long in the flat metric on T^2, then the filled manifold is hyperbolic (see for example [HK08] or Section 11 of this article which does not depend on this section). Furthermore, as the lengths of such simple closed curves tend to infinity, the Dehn filled manifolds, equipped with their unique hyperbolic metrics, will be almost isometric to M on larger and larger neighborhoods of the complement of the cusp (see for example [BP92] or Section 11). As a consequence, for each $\ell > 0$ we can find such a Dehn filling with the property that the hyperbolic manifold obtained by this Dehn filling contains a copy of the ℓ-neighborhood of T^2 in the cusp $B \subset M$ up to a change of the metric in the C^2-topology which is as close to zero as we wish.

Fix a number $\lambda \in (0,2)$. The modification of the metric on such a Dehn filling of M will be carried out in a region of the form $T^2 \times [m, m + \frac{R}{\delta} + 4]$ in standard coordinates on the cusp where $\delta < \delta(\varepsilon)$ (for $\delta(\varepsilon)$ given by Lemma 6.3), and $m > 0$ is a number which is sufficiently large that

$$D_0^2 \int_m^\infty e^{-\frac{\lambda}{2}t} dt \le 1,$$

where $D_0 > 0$ is the (intrinsic) diamater of the boundary T^2 of the cusp (see Lemma 6.5).

After choosing m with this property, the length of the meridian for the Dehn filling is chosen large enough so that the Dehn filled metric is arbitrarily near the metric of the cusp on the neighbhood of radius $\ell = m + \frac{R}{\delta} + 4$ of the thick part of M. The deformation is chosen so that it stretches the direction orthogonal to the meridian ζ. For an arbitrarily chosen constant $C > 0$, the deformed metric g on the Dehn filled hyperbolic manifold (M_ζ, g_0) has the following properties.

(1) The metric coincides with the hyperbolic metric g_0 on a neighborhood of radius m of the thick part of the hyperbolic metric.
(2) The ratio of the lengths of the closed geodesics for the metrics g and g_0 in the filled manifold which are freely homotopic to the core curve of the tube is at least C.

Lemma 6.4. *For all sufficiently large m there is no constant curvature metric g_{const} so that $\mathrm{id}_M : (M_\zeta, g) \to (M_\zeta, g_{\mathrm{const}})$ is a $\sqrt{C}/2$-bilipschitz equivalence.*

Proof. Choose m sufficiently large that there exists a closed geodesic β in the union of the thick part of M with the m-neighborhood of the boundary torus of the cusp. The length of this geodesic in the Dehn filled manifold M_ζ, equipped with the hyperbolic metric g_0, is almost identical to the length of β. Furthermore, β is a closed geodesic for the deformed metric g since g coincides with the hyperbolic metric near β.

Assume there is a constant curvature metric g_{const} so that $\mathrm{id}_{M_\zeta} : (M_\zeta, g) \to (M_\zeta, g_{\mathrm{const}})$ is a $\sqrt{C}/2$-bilipschitz equivalence. By Mostow Rigidity, there is then a number $c > 0$ and a diffeomorphism ϕ homotopic to the identity so that $\phi^* g_{\mathrm{const}} = c^2 g_0$. Define $\hat{g} := \phi^* g$. Then $\mathrm{id}_{M_\zeta} : (M_\zeta, \hat{g}) \to (M_\zeta, c^2 g_0)$ is a $\sqrt{C}/2$-bilipschitz equivalence. Denote by γ_0 the core geodesic of the distinguished Margulis tube of the Dehn filled hyperbolic manifold (M_ζ, g_0). Note that γ_0 also is the core geodesic for (M_ζ, g). As a consequence, $\gamma := \phi^{-1}(\gamma_0)$ is the unique $\hat{g}$-geodesic in its free homotopy class. Moreover, γ and γ_0 are freely homotopic since $\phi \simeq \mathrm{id}$. Therefore,

$$\begin{aligned}
C\ell_{g_0}(\gamma_0) &\le \ell_g(\gamma_0) && \text{(by the construction of } g) \\
&= \ell_{\hat{g}}(\gamma) && \text{(by the definition of } \hat{g} \text{ and } \gamma) \\
&\le \ell_{\hat{g}}(\gamma_0) && \text{(because } \gamma \text{ is a } \hat{g}\text{-geodesic and } \gamma \simeq \gamma_0) \\
&\le \frac{1}{2}\sqrt{C}\ell_{c^2 g_0}(\gamma_0) && \text{(by the bilipschitz equivalence)} \\
&= \frac{1}{2} c\sqrt{C}\ell_{g_0}(\gamma_0).
\end{aligned}$$

Hence $c \geq 2\sqrt{C}$. Similarly, we have

$$\begin{aligned}
\ell_{g_0}(\beta) &= \ell_g(\beta) && \text{(by the first paragraph)}\\
&= \ell_{\hat{g}}(\phi^{-1}(\beta)) && \text{(by the definition of } \hat{g})\\
&\geq \frac{2}{\sqrt{C}}\ell_{c^2 g_0}(\phi^{-1}(\beta)) && \text{(by the bilipschitz equivalence)}\\
&\geq \frac{2c}{\sqrt{C}}\ell_{g_0}(\beta). && \text{(because } \beta \text{ is a } g_0\text{-geodesic and } \phi^{-1}(\beta) \simeq \beta)
\end{aligned}$$

Thus $c \leq \frac{1}{2}\sqrt{C}$. This is a contradiction. □

It follows from the following lemma that the constructed manifolds satisfy the curvature assumption $ii)$ of Proposition 6.1.

Lemma 6.5. *Let M be a closed 3-manifold, T a Margulis tube of M with core geodesic γ, and $\lambda \in (0,2)$. Assume that $-1-\varepsilon \leq \sec(M) \leq -1+\varepsilon$ for some $\varepsilon \leq \frac{1}{8}\lambda$, and that for some $m > 0$ the metric is hyperbolic outside the region*

$$\left\{ y \in T \mid m \leq \operatorname{dist}(y, M_{\text{thick}}) \leq \operatorname{Rad} - 1 \right\},$$

where Rad *is the Radius of T. Then for some universal constant $c > 0$ it holds*

$$\int_M \frac{1}{\operatorname{inj}(y)^{2-\lambda}} |\operatorname{Ric}(g) + 2g|^2(y)\, d\operatorname{vol}(y) \leq c D_0^2 \varepsilon^2 \int_m^{\operatorname{Rad}-1} e^{-\frac{\lambda}{2}r}\, dr,$$

where $D_0 := \operatorname{diam}(\partial T)$ is the (intrinsic) diameter of ∂T.

Proof. For $r \geq 0$ denote by $T(r)$ the torus in the Margulis tube T all whose points have distance r to ∂T. It follows from standard Jacobi field estimates that for some universal constant $c > 0$ it holds

$$\operatorname{area}\big(T(r)\big) \leq c e^{-2(1-\varepsilon)r} \operatorname{area}(\partial T) \leq c D_0^2 e^{-2(1-\varepsilon)r}$$

for all $r \in [0, \operatorname{Rad}-1]$. Similarly, a comparison argument shows that for all $y \in T(r)$ with $r \in [0, \operatorname{Rad}-1]$ it holds

$$\frac{1}{\operatorname{inj}(y)} \leq c e^{(1+\varepsilon)r}$$

for some universal constant $c > 0$ (see the proof of Corollary 7.7 for more details). Thus

$$\int_{T(r)} \frac{1}{\operatorname{inj}(y)^{2-\lambda}}\, d\operatorname{vol}_2(y) \leq c D_0^2 e^{((2-\lambda)(1+\varepsilon)-2(1-\varepsilon))r}$$

for all $r \in [0, \operatorname{Rad}-1]$. Note that $(2-\lambda)(1+\varepsilon) - 2(1-\varepsilon) = \varepsilon(4-\lambda) - \lambda \leq -\frac{1}{2}\lambda$ since by assumption $\varepsilon \leq \frac{1}{8}\lambda$. Therefore, the desired estimate follows from the fact that the curvature assumption $\sec(M) \in [-1-\varepsilon, -1+\varepsilon]$ implies $|\operatorname{Ric}(g) + 2g|^2 \leq 3(2\varepsilon)^2$. □

We quickly review the construction of the counterexamples constructed in this section and point out what should be taken away from these examples. We started with a hyperbolic metric. The new metric was defined by slowly changing the conformal structure on the horotori (quotients of horospheres). But this change only started deep in the thin part of the manifold. More precisely, the change of the conformal structure only starts at horotori that have distance at least m to M_{thick}, where for some arbitrary constant $\lambda \in (0,2)$, the number $m > 0$ was chosen so large that $D_0^2 \int_m^\infty e^{-\frac{\lambda}{2} r} dr \le 1$, where D_0 is the (intrinsic) diameter of the boundary of the filled Margulis tube T. In particular, the change of the conformal structure only occours on tori $T(r)$ whose diameter is bounded by some universal constant. Here $T(r) := \{y \in T \mid d(y, \partial T) = r\}$. Indeed, for $r \ge m$ it holds

$$
\begin{aligned}
\operatorname{diam}(T(r)) &\le \operatorname{diam}(T(m)) && \text{(monotonicity)}\\
&\le c D_0 e^{-m} && \text{(the metric is hyperbolic up to } T(m))\\
&\le c D_0^2 \int_m^\infty e^{-r}\, dr && (\operatorname{diam}(\partial T) \ge \operatorname{inj}(\partial T) = \mu)\\
&\le c D_0^2 \int_m^\infty e^{-\frac{1}{2}\lambda r}\, dr && (\lambda < 2)\\
&\le c && \text{(definition of } m),
\end{aligned}
$$

where $c > 0$ is a universal constant, and μ is a Margulis constant. With the terminology of Section 7 we can express this by saying that the change of the conformal structure only happens in the *small part* of the Margulis tube.

These geometric facts (which however are inherent to dimension 3, see [Gro78]) prevent the establishment of the C^0-estimate required in the proof of Theorem 5.1, and this problem can not be resolved by enriching the hybrid norms $\|\cdot\|_2$ and $\|\cdot\|_0$ with weighted L^2-norms whose weight involves $\operatorname{inj}(y)$.

Our second main result Theorem 2 overcomes this difficulty by constructing new hybrid Banach spaces and imposing stronger geometric control in the thin part of a negatively curved 3-manifold M. This is done in two steps. In a first step, carried out in Section 7, we locate the region in the thin part of M for which we can obtain C^0-estimates sufficient for our goal with a direct adaptation of the proof of Theorem 5.1. In a second step, we control its complement, which we call the *small part* of the manifold, with an ODE-Ansatz motivated by the work [Bam12].

7. The thin but not small part of a negatively curved 3-manifold

In the proof of Theorem 5.1, the assumption on a lower bound for the injectivity radius was used to establish a C^0-estimate for a symmetric $(0,2)$-tensor field h from knowledge of $\mathcal{L}(h)$. The examples in Section 6 show that without such a bound, we can not expect that such an a priori estimate holds true.

In the remainder of this section, M denotes a finite volume Riemannian 3-manifold of sectional curvature in $[-4, -1/4]$, with universal cover $\tilde{M}$. We know that each cusp is diffeomorphic to $T^2 \times [0, \infty)$ where T^2 is the 2-torus. This is true since by Convention 2.3 we assume that M is orientable. In Section 7.1 we introduce a region M_{small} in M, called *small part* of M, and we establish some of its basic properties. We show in Section 7.2 that if M satisfies the hypothesis on the curvature stated in Theorem 5.1, then in $M \setminus M_{\text{small}}$, a modified version of such a C^0-estimate holds true in spite of the fact that the injectivity radius may be arbitrarily small. The proof rests on a counting result for preimages in $\tilde{M}$ of points in $M - M_{\text{small}}$ which is proved in Section 7.3.

7.1. **The small part of M.** Choose once and for all a Margulis constant $\mu \leq 1$ for 3-manifolds M with curvature in the interval $[-4, -1/4]$. This constant determines the thin part

$$M_{\text{thin}} = \{x \in M \mid \text{inj}(x) < \mu\}$$

of M. For a number $\chi \leq \mu$ let $M^{<\chi} \subset M_{\text{thin}}$ be the set of all points x with $\text{inj}(x) < \chi$. The set M_{thin} is a disjoint union of *cusps* and *Margulis tubes*, where a Margulis tube is a tubular neighborhood of a closed geodesic of length smaller than 2μ.

From now on, we shall only consider Margulis tubes of radius $r \geq 3$, where the radius means the distance between the core curve of the tube and the boundary, determined by the constant μ. Comparison shows that this only excludes tubes whose core curves have length bounded from below by a fixed positive constant. In other words, tubes of radius at most three can be thought of belonging to the thick part of M for a properly adjusted Margulis constant.

Consider a Margulis tube T of M and let γ be its core geodesic. For $r > 0$ denote by

$$T(r) := \{x \in M \,|\, d(x, \gamma) = r\}$$

the torus of distance r to γ. The *small part* of the Margulis tube T is

$$T_{\text{small}} := \Big\{x \in T \,|\, d(x, \gamma) \leq 2 \text{ or } \text{diam}\big(T(r_\gamma(x))\big) \leq D\Big\},$$

where $r_\gamma = d(\cdot, \gamma)$, the diameter is with respect to the intrinsic metric on the torus, and $D > 0$ is a universal constant which will be determined later.

The *small part* C_{small} of a rank 2 cusp C is defined similarly. The only difference is that we have to consider Busemann functions (instead of $r_\gamma(\cdot)$) to define the level tori $T(r)$. Fix a rank 2 cusp C, and let $\xi \in \partial_\infty \tilde{M}$ be a point corresponding to C. Choose a Busemann function $b_\xi : \tilde{M} \to \mathbb{R}$ associated to ξ (see [BGS85] for more information on Busemann functions). This induces a Busemann function $\bar{b}_\xi : C \to \mathbb{R}$. For $r \in \mathbb{R}$ we define $T(r) := \{x \in C \,|\, \bar{b}_\xi(x) = r\}$. As in the case of a tube, we define the *small part* of the cusp C to be

$$C_{\text{small}} := \bigcup_r \Big\{T(r) \mid \text{diam}(T(r)) \leq D\Big\},$$

where D is the same universal constant as before that will be determined later, and the diameter is the intrinsic diameter.

Finally, the *small part* M_{small} of the manifold M is the union

$$M_{\text{small}} := \bigcup_T T_{\text{small}} \cup \bigcup_C C_{\text{small}},$$

where T ranges over all Margulis tubes T of radius at least 3 and C ranges over all rank 2 cusps of M.

Remark 7.1. Although the definition of the small part of a negatively curved manifold makes sense in all dimensions, it follows from [Gro78] that the small part of a closed negatively curved manifold M of dimension $n \geq 4$ is the union of tubular neighborhoods of short geodesics of uniformly bounded radius. The fact that this is not true in dimension 3 (see Section 6) is the main reason for introducing the small part of a negatively curved 3-manifold.

Remark 7.2. Since we use intrinsic diameters for the definition of the small part of M, standard Jacobi field estimates show that the small part of a Margulis tube or cusp is a connected subset of the tube or cusp.

Remark 7.3. If M is non-orientable, we define $M_{\mathrm{small}} := \pi(\hat{M}_{\mathrm{small}})$, where $\pi : \hat{M} \to M$ is the orientation cover of M.

Choose

$$D < \min\{\mu, 1/4\}$$

sufficiently small so that the one-neighborhood of the D-thin part M^D is contained in the $\mu/2$-thin part $M^{<\mu/2}$ of M. Using comparison of Jacobi fields, one observes that such a constant D only depends on the choice of μ and the curvature bounds.

Lemma 7.4. *$M_{\mathrm{small}} \subset M^{<D}$ and hence if $x \in M_{\mathrm{small}}$, then the torus $T(r)$ containing x is a C^1-submanifold of M contained in $\mathrm{int}(M_{\mathrm{thin}})$ whose distance to M_{thick} is at least $\mu/2$.*

Proof. Let $x \in M_{\mathrm{small}}$ be arbitrary. We only present the case that x is contained in a Margulis tube T, the case of a cusp being similar.

Let γ denote the core geodesic of T. We first consider the case that $r_\gamma(x) = d(x, \gamma) \leq 2$. As we only consider Margulis tubes of radius at least three, the distance of x to M_{thick} is at least $1 > \mu/2$.

So we may assume $r := r_\gamma(x) > 2$. By the definition of M_{small}, the diameter of the distance torus $T(r)$ containing x is at most D. Let $\alpha \subseteq T(r)$ be a shortest closed geodesic for the induced metric on $T(r)$ which is not contractible as a curve in $T(r)$. The length of α is at most $2D$. If α is contractible in M, then α is a meridian in $T(r)$. Thus comparison of Jacobi fields shows that the length of α is at least $4\pi \sinh(r/2)$. As $r > 2$ and $D \leq \frac{1}{4}$, this length is at least $4\pi \sinh(1) > 1 > 2D$, which is a contradiction.

As a consequence, α is not contractible in M and hence defines an essential loop in M of length at most $2D$. But then x is contained in the D-thin part of M and hence the D-neighborhood of x (which contains $T(r)$) is contained in the $\mu/2$-thin part of M by the choice of D. In particular, $T(r)$ is the projection to M of the level set of a function in the universal covering $\tilde{M}$ of M which either is the distance function to a geodesic line or a Busemann function. Such functions are known to be of class C^1 and non-singular away from their minimum [BGS85]. This completes the proof. □

7.2. **The C^0-estimate.** The main reason for introducing the small part of M is that for points in $M_{\mathrm{thin}} \setminus M_{\mathrm{small}}$, we can prove a C^0-estimate which is weaker than the estimate established in the proof of Theorem 5.1 but sufficient for an analogous conclusion. To obtain C^0-estimates for points in M_{small}, we shall use an ODE-Ansatz inspired by [Bam12] (see Section 9.3 for more details). As before, $\mathcal{L}$ denotes the elliptic differential operator given by $\mathcal{L}h = \frac{1}{2}\Delta_L h + 2h$.

Proposition 7.5. *For all $\alpha \in (0,1)$, $\Lambda \geq 0$, $\delta \in (0,2)$, and $b > 1$ there exist $\varepsilon_0 = \varepsilon_0(\delta, b) > 0$ and $C = C(\alpha, \Lambda, \delta, b) > 0$ with the following property. Let M be a Riemannian 3-manifold of finite volume so that*

$$|\sec +1| \leq \varepsilon_0 \quad \text{and} \quad \|\nabla \mathrm{Ric}\|_{C^0(M)} \leq \Lambda.$$

Then for all $h \in C^2(\mathrm{Sym}^2(T^*M)) \cap H^2(M)$ and $x \in M_{\mathrm{thin}} \setminus M_{\mathrm{small}}$ it holds

$$|h|(x) \leq C\left(\|\mathcal{L}h\|_{C^0(M)} + e^{\frac{b}{2}d(x, M_{\mathrm{thick}})}\left(\int_M e^{-(2-\delta)r_x(y)}|\mathcal{L}h|^2(y)\,d\mathrm{vol}(y)\right)^{\frac{1}{2}}\right).$$

The estimate in Proposition 7.5 motivates the definition of the norm $\|\cdot\|_{0,\lambda}$ that will be introduced in Section 9.2. The proof of Proposition 7.5 is based on a counting result for the number of preimages of points in $M \setminus M_{\mathrm{small}}$ in the universal covering $\tilde{M}$ of M contained in a ball in $\tilde{M}$ of fixed size which is the main result of Section 7.3. We denote by $N_r(M \setminus M_{\mathrm{small}})$ $(r > 0)$ the r-neighborhood of $M \setminus M_{\mathrm{small}}$.

Proposition 7.6 (Counting preimages)**.** *There is a constant $C > 0$ so that for every $x \in M^{<D} \cap N_{1/4}(M \setminus M_{\mathrm{small}})$ and every lift $\tilde{x} \in \tilde{M}$ it holds*

$$\#\left(\pi^{-1}(x) \cap B(\tilde{x}, D)\right) \leq C\frac{1}{\mathrm{inj}(x)},$$

where $\pi : \tilde{M} \to M$ is the universal covering projection.

The complete proof of Proposition 7.6 is is a bit technical. For this reason we postpone it to Section 7.3. However, in the special case that $\sec \equiv -1$ in M_{thin}, the proof is very simple and it already contains the core ideas for the general case. Moreover, this special case is sufficient for our applications to drilling and filling, and effective hyperbolization.

Proof of Proposition 7.6 when the thin part is hyperbolic. Fix some $x_0 \in M^{<\mu'} \cap N_{1/4}(M \setminus M_{\mathrm{small}})$, and denote by $T := T(r(x_0))$ the distance torus or horotorus containing x_0. Since the intrinsic geometry of T is uniformly bilipschitz to its extrinsic geometry (see Proposition 7.8 for a detailed formulation), it suffices to show

$$\#\left(\pi_T^{-1}(x_0) \cap B(\tilde{x}_0, D)\right) \leq C\frac{1}{\mathrm{inj}(T)},$$

where $\pi_T : \mathbb{R}^2 \to T$ is the universal covering projection of T, and $B(\tilde{x}_0, D) \subseteq \mathbb{R}^2$.

Since radial projections are uniformly Lipschitz (see Proposition 7.8), it follows from the definition of the small part of M that $\mathrm{diam}(T) \geq D'$ for some universal constant D'. Note that T is a flat torus since $\sec \equiv -1$ in M_{thin}. Thus by a simple volume counting argument for balls in $\mathbb{R}^2$, it suffices to assume that $\mathrm{inj}(T) \leq \frac{1}{2}D'$.

By the results of Section 2.24 in [GHL04], there is a fundamental region $\{tv_1 + sv_2 \mid s, t \in [0,1]\} \subseteq \mathbb{R}^2$ for T with

$$|v_1| = 2\mathrm{inj}(T) \quad \text{and} \quad \theta := \measuredangle(v_1, v_2) \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right].$$

Clearly $D' \leq \operatorname{diam}(T) \leq \frac{1}{2}\big(|v_1| + |v_2|\big)$, and thus $|v_2| \geq D'$ since we assume $\operatorname{inj}(T) \leq \frac{1}{2}D'$. For any $\tilde{x} \in \pi_T^{-1}(x_0)$ consider

$$\mathcal{R}_{\tilde{x}} := \tilde{x} + \left\{ tv_1 + s\frac{v_2}{|v_2|} \,\Big|\, t \in [0,1/2], s \in [0, D'/2] \right\},$$

and note that they are pairwise disjoint. All of them have area

$$\operatorname{area}(\mathcal{R}_{\tilde{x}}) = \sin(\theta)\frac{|v_1|}{2}\frac{D'}{2} \geq \frac{\sqrt{3}D'}{4}\operatorname{inj}(T).$$

Also $\bigcup_{\tilde{x}\in B(\tilde{x}_0,D)} \mathcal{R}_{\tilde{x}} \subseteq B(\tilde{x}_0, D + 2D')$ since $\operatorname{diam}(\mathcal{R}_{\tilde{x}}) \leq \frac{1}{2}\big(|v_1| + D'\big) < 2D'$. As the sets $\mathcal{R}_{\tilde{x}}$ are pairwise disjoint, volume counting implies

$$\#\big(\pi_T^{-1}(x_0) \cap B(\tilde{x}_0, D)\big) \leq \frac{\operatorname{area}\big(B(\tilde{x}_0, D + 2D')\big)}{\operatorname{area}(\mathcal{R})} \leq \frac{C}{\operatorname{inj}(T)}.$$

This completes the proof of the special case. □

In order to deduce Proposition 7.5 from Proposition 7.6 we have to replace $\frac{1}{\operatorname{inj}(x)}$ by a function that is easier to control.

Corollary 7.7. *There is a universal constant $C > 0$ with the following property. Let M be a Riemannian 3-manifold with*

$$-b^2 \leq \sec_M \leq -1/4$$

for some $1 \leq b \leq 2$. Then for all $x \in N_{1/4}(M \smallsetminus M_{\text{small}})$ and every lift $\tilde{x}$ of x to $\tilde{M}$, it holds

$$\#\big(\pi^{-1}(x) \cap B(\tilde{x}, D)\big) \leq Ce^{bd(x, M_{\text{thick}})}.$$

The proof of Corollary 7.7 is also contained in Section 7.3. We now show how Corollary 7.7 can be used to prove Proposition 7.5.

Proof of Proposition 7.5. Abbreviate $f := \mathcal{L}h$. It holds $\mathcal{L}\tilde{h} = \tilde{f}$ in the universal cover. By the argument which led to (4.9), we have

$$|\tilde{h}|(\tilde{x}) \leq C\Big(\|\tilde{h}\|_{L^2(B(\tilde{x},D/2))} + \|\tilde{f}\|_{C^0(\tilde{M})}\Big) \tag{7.1}$$

for a constant $C = C(n, \alpha, \Lambda)$. Therefore, it suffices to bound $\|\tilde{h}\|_{L^2(B(\tilde{x},D/2))}$. To this end, we invoke the following basic claim, which states that an integral in the universal cover can be estimated by a weighted integral in the manifold when the weight is an upper bound for the number of preimages.

Claim. *Let $x \in M$ and $\rho : M \to \mathbb{R}$ be a function so that*

$$\#\left(\pi^{-1}(y) \cap B(\tilde{y}, D)\right) \leq \rho(y)$$

holds for all $y \in B(x, D/2) \subseteq M$. Let $u : M \to \mathbb{R}_{\geq 0}$ be a non-negative integrable function and denote by $\tilde{u} := u \circ \pi$ its lift to the universal cover. Then we have

$$\int_{B(\tilde{x},D/2)} \tilde{u}(\tilde{y})\, d\mathrm{vol}_{\tilde{g}}(\tilde{y}) \leq \int_{B(x,D/2)} \rho(y)u(y)\, d\mathrm{vol}_g(y).$$

Proof of the claim. By the triangle inequality, if $\tilde{y} \in B(\tilde{x}, D/2)$ then $B(\tilde{x}, D/2) \subseteq B(\tilde{y}, D)$. Thus by assumption, a point $y \in B(x, D/2)$ has at most $\rho(y)$ preimages in $B(\tilde{x}, D/2)$. Hence the claim holds true for the indicator function $u = \chi_U$ of a small open subset $U \subseteq B(x, D/2)$. By linearity and monotonicity the result follows for all non-negative simple functions. A standard approximation argument completes the proof. □

For $\varepsilon_0 = \varepsilon_0(b) > 0$ small enough, $|\sec + 1| \le \varepsilon_0$ implies $-b^2 \le \sec \le -1/4$. Hence Corollary 7.7 shows that for any $x \in M \setminus M_{\mathrm{small}}$ the function $\rho(y) = Ce^{bd(y, M_{\mathrm{thick}})}$ satisfies the assumption of the claim. Thus

$$\int_{B(\tilde{x}, D/2)} |\tilde{h}|^2(\tilde{y})\, d\mathrm{vol}_{\tilde{g}}(\tilde{y}) \le C \int_{B(x, D/2)} e^{bd(y, M_{\mathrm{thick}})} |h|^2(y)\, d\mathrm{vol}_g(y).$$

As $d(y, M_{\mathrm{thick}}) \le d(x, M_{\mathrm{thick}}) + D/2$ for $y \in B(x, D/2)$,

$$\int_{B(\tilde{x}, D/2)} |\tilde{h}|^2(\tilde{y})\, d\mathrm{vol}_{\tilde{g}}(\tilde{y}) \le Ce^{bD/2} e^{bd(x, M_{\mathrm{thick}})} \int_{B(x, D/2)} |h|^2(y)\, d\mathrm{vol}_g(y). \tag{7.2}$$

Moreover,

$$\int_{B(x, D/2)} |h|^2(y)\, d\mathrm{vol}_g(y) \le e^D \int_{B(x, D/2)} e^{-(2-\delta) r_x(y)} |h|^2(y)\, d\mathrm{vol}_g(y). \tag{7.3}$$

By Remark 4.5 the integral estimate (4.8) from the proof of Proposition 4.3 is still valid. In particular, for $\varepsilon_0 = \varepsilon_0(\delta) > 0$ small enough it holds

$$\int_M e^{-(2-\delta) r_x(y)} |h|^2(y)\, d\mathrm{vol}_g(y) \le C \int_M e^{-(2-\delta) r_x(y)} |f|^2(y)\, d\mathrm{vol}_g(y) \tag{7.4}$$

for a constant $C = C(\delta)$. Combining (7.1)-(7.4) yields the desired estimate. □

7.3. **Counting preimages.** This subsection is concerned with the proof of Proposition 7.6. Before we come to the more technical details, we begin with a short overview of the proof. Let $x \in M^{<\mu'} \cap N_{1/4}(M \setminus M_{\mathrm{small}})$ and let as before $\mathrm{inj}(x)$ be the injectivity radius of M at x. There is a geodesic loop of length at most $2\mathrm{inj}(x)$ based at x. This loop can be homotoped with fixed endpoints to a loop c_1 lying entirely in the distance torus containing x of controlled comparable length. Using the assumption that $x \in N_{1/4}(M \setminus M_{\mathrm{small}})$, we then show that any closed curve on the torus whose homotopy class is not a multiple of the class of c_1 has length at least ℓ where $\ell > 0$ is a fixed constant. Namely, we show that otherwise the torus has a small diameter, contradicting that x is contained in a small neighbourhood of $M \setminus M_{\mathrm{small}}$. Therefore, in the universal cover of the torus, a preimage $\tilde{x}$ of x either lies on the lift $\tilde{c_1}$ of c_1 through $\tilde{x}$, or it has distance at least ℓ from $\tilde{x}$. A volume counting argument then completes the proof.

The main step in the implementation of this argument lies in obtaining sufficient geometric control on the tori so that the volume counting argument used in the case when the thin part is hyperbolic can be applied. The following proposition summarizes geometric properties of distance tubes and horospheres in simply connected manifolds of pinched negative curvature which are used in the proof of Proposition 7.6. Note that although a priori Busemann functions are only of class C^2, the Gauß equations show that their sectional curvature is defined and continuous.

Proposition 7.8. *For every $n \ge 2$ there exists numbers $A = A(n) \ge 1$ and $B = B(n) > 0$ such that for any simply connected complete n-manifold $\tilde{M}$ of curvature $\sec_{\tilde{M}} \subseteq [-4, -1/4]$ the following holds true. Let $\tilde{\gamma} \subseteq \tilde{M}$ be a geodesic.*

i) For $r \ge 3/4$, the sectional curvature of the level sets

$$\{d(\tilde{\gamma}, \cdot) = r\}$$

with respect to the induced metric is contained in $[-A^2, A^2]$, and the injectivity radius is at least B.

ii) For $r \ge 3/4$, the radial projection

$$\{r - 1/4 \le d(\tilde{\gamma}, \cdot) \le r + 1/4\} \to \{d(\tilde{\gamma}, \cdot) = r\}$$

is A-Lipschitz.

iii) For points x, y on distance level sets $\{d(\tilde{\gamma}, \cdot) = r\}$ $(r \ge 3/4)$ in $\tilde{M}$, the distance between x, y with respect to the intrinsic metric on the level set is at most $Ad_{\tilde{M}}(x, y)$ provided that $d_{\tilde{M}}(x, y) \le 1/4$.

Analogous properties also hold true for horospheres in $\tilde{M}$, with the same constants $A > 1, B > 0$.

Proof of Proposition 7.8. We sketch an argument for the first part of the proposition and refer to [Esc87] and [HIH77] for more information about the remaining parts.

In simply connected manifolds of constant sectional curvature $\kappa < 0$, the distance cylinders of distance r about geodesics have principal curvatures $\sqrt{-\kappa}\tanh(\sqrt{-\kappa}r)$ and $\sqrt{-\kappa}\coth(\sqrt{-\kappa}r)$. By standard comparison results for solutions of the Riccati equation, the principal curvatures λ of the distance tori in M are bounded by the maximal resp. minimal principal curvatures of the distance tori in the spaces of constant curvature -4 resp. $-1/4$. Thus $\frac{1}{2}\tanh\left(\frac{1}{2}r\right) \le \lambda \le 2\coth(2r)$.

There is a constant $c > 1$ so that $\frac{1}{c} \le \frac{1}{2}\tanh\left(\frac{1}{2}r\right), 2\coth(2r) \le c$ for all $r \ge 1$. So the principal curvatures λ of the distance tori in M are contained in $[1/c, c]$. Thus there are uniform bounds for the shape operator of the level sets, and since the curvature of the ambient manifold is contained in $[-4, -1/4]$ by assumption, the curvature of the level sets is uniformly bounded by the Gauß equations (see Chapter 6 of [dC92]). This completes the proof of the curvature control stated in the proposition.

To establish a uniform lower bound on the injectivity radius of the level sets $Z = \{d(\tilde{\gamma}, \cdot) = r\}$ $(r \ge 3/4)$, note first that for $\delta \le 1/4$, the ball $B^Z(\tilde{x}, \delta)$ of radius δ about a point $\tilde{x} \in Z$ for the intrinsic metric contains the radial projection of the intersection with Z of the ball $B(\tilde{x}, \delta/A)$ of radius δ/A in $\tilde{M}$ about $\tilde{x}$ (this uses *ii*)). This implies that $B(\tilde{x}, \delta/A)$ is contained in the preimage of $B^Z(\tilde{x}, \delta)$ under the restriction of the radial projection to the $1/4$-neighborhood of Z in $\tilde{M}$.

Since the radial projections are uniformly Lipschitz continuous, Fubini's theorem implies that the volume of $B^Z(\tilde{x}, \delta)$ is bounded from below by $C_0\mathrm{vol}(B(\tilde{x}, \delta/A))$, and the latter is bounded from below by $C_1(\delta/A)^n$ where C_0, C_1 only depend on the curvature bounds of $\tilde{M}$.

As a consequence, for $\delta = 1/4$ fixed, the volume of $B^Z(\tilde{x}, 1/4)$ is bounded from below by a universal constant not depending on $\tilde{x}$ or Z. Since the sectional curvature of the distance hypersurface Z is bounded from above by a universal constant, this implies that its injectivity radius is bounded from below by a universal constant $B > 0$ by a result of Cheeger, Gromov and Taylor (see [CGT82, Theorem 4.7]).

All remaining statements follow in a similar way, and their proofs will be omitted. □

Proof of Proposition 7.6. **Step 1 (Large injectivity radius):** Due to the definition of $D \leq 1/2$, the ball of radius 1 about a point in $M^{\leq \mu'}$ is contained in M_{thin}. Therefore, it suffices to consider the covering $\hat{M} = \tilde{M}/\Gamma$ of M where Γ is the fundamental group of the component of M_{thin} containing x.

Let $A = A(3) > 1$ and $B = B(3) > 0$ be as in Proposition 7.8. Assume without loss of generality that $B < D/2A$ where $D > 0$ is as in the definition of M_{small}. If the injectivity radius $\text{inj}(x)$ of M at x is at least $B/8A$, then the ball of radius $B/8A$ about x is diffeomorphic to a ball of the same radius about a preimage $\tilde{x}$ of x in $\tilde{M}$. By the curvature bounds, the volume of this ball is bounded from below by a universal positive constant $c_0 > 0$. Similarly, the volume of the ball $B(\tilde{x}, 2D)$ is bounded from above by a universal constant $c_1 > 0$. As the balls of radius $B/8A < D$ about the preimages of x in $B(\tilde{x}, D)$ are pairwise disjoint and contained in $B(\tilde{x}, 2D)$, the number of preimages of x contained in $B(\tilde{x}, \mu')$ is at most c_1/c_0. Thus in the sequel we may always assume that $\text{inj}(x) < B/8A$.

Let $x \in M^{<B/8A} \cap N_{1/4}(M \smallsetminus M_{\text{small}})$, choose a lift $\hat{x}$ of x to $\hat{M}$ and let $\tilde{x} \in \tilde{M}$ be a lift of $\hat{x}$. Let T be the distance torus of $\hat{M}$ containing $\hat{x}$. There exists a geodesic loop σ based at $\hat{x}$ with $\ell(\sigma) = 2\text{inj}(x)$.

By the definition of M_{small}, the distance of $\hat{x}$ to the core geodesic is at least $3/4$. Thus by Proposition 7.8, there is a curve c_1 lying entirely in T that is homotopic to σ relative endpoints and that satisfies $\ell(c_1) \leq A\ell(\sigma) < B/4$. It follows from the definition of σ that the curve c_1 is not contractible in T.

Assume without loss of generality that $c_1 \subseteq T$ is the shortest essential based loop at $\hat{x}$. Then c_1 is simple, that is, c_1 does not have self-intersections, and it is a geodesic with at most one breakpoint at $\hat{x}$. Cut T open along c_1 and let Z be the resulting metric cylinder. Let $\partial^0 Z, \partial^1 Z$ be the two distinct boundary components of Z.

Step 2 (Loops independent from c_1 are long): The distance $d := d_Z(\partial^0 Z, \partial^1 Z)$ can be realized by an embedded arc $c_2 \subseteq Z$ connecting $\partial^0 Z$ to $\partial^1 Z$. Concatenation of c_2 with a subarc of c_1 gives a closed essential curve $c_{2'} \subseteq T$ of length $\ell(c_{2'}) \leq d + \ell(c_1)/2 < d + B/4$.

By Proposition 7.8, since $T \subseteq N_{1/4}(M - M_{\text{small}})$, the diameter of T with respect to the intrinsic metric is at least $D/A > 2B$. We use this to show that $d \geq B/4$. To this end we argue by contradiction and we assume otherwise. Let $\gamma_1 \subseteq T$ be the closed geodesic of minimal length in the free homotopy class of c_1. Its length $\ell(\gamma_1)$ is at most $\ell(c_1) < B/4$.

Cut T along γ_1 and denote the resulting cylinder by Z'. The connected components of $c_{2'} \smallsetminus (\gamma_1 \cap c_{2'})$ lift to arcs in Z' whose endpoints lie on the one of the boundary components $\partial^0 Z'$ or $\partial^1 Z'$ of Z'. At least one of these lifts must connect $\partial^0 Z'$ and $\partial^1 Z'$. Namely, otherwise $c_{2'}$ is freely homotopic to a multiple of γ_1, which contradicts that $c_1, c_{2'}$ intersect in a single point. This implies that $d_{Z'}(\partial^0 Z', \partial^1 Z') \leq \ell(c_{2'}) < B/2$.

Let τ be a minimal geodesic in Z' from $\partial^0 Z'$ to $\partial^1 Z'$. Since γ_1 is a periodic geodesic, τ intersects $\partial^0 Z'$ and $\partial^1 Z'$ perpendicularly. Cutting Z' open along τ, we see that T has a rectangular fundamental region $\mathcal{R}$ in the universal covering $\tilde{T}$ of T with geodesic sides of side lengths $\ell(\gamma_1) < B/4$ and $\ell(\tau) < B/2$, intersecting each other perpendicularly.

If $v \in \tilde{T}$ is a vertex of $\mathcal{R}$, then any point in the boundary $\partial\mathcal{R}$ of $\mathcal{R}$ is of distance smaller than $3B/4$ to v. As a consequence, $\partial\mathcal{R}$ is a Jordan curve embedded in the ball of radius B about v. Since the injectivity radius of $\tilde{T}$ is at least B, this ball is diffeomorphic to a disk in $\mathbb{R}^2$. Then $\partial\mathcal{R}$ encloses a compact disk embedded in this ball. On the other hand, since $\tilde{T}$ is diffeomorphic to $\mathbb{R}^2$, the disk $\mathcal{R}$ is the unique disk in $\tilde{T}$ bounded by $\partial\mathcal{R}$. Hence $\mathcal{R}$ is contained in the open disk of radius B about v which yields that the diameter of $\mathcal{R}$ is smaller than $2B$. Consequently the diameter of T is smaller than $2B < D/A$ which is a contradiction to the assumption that $x \in N_{1/4}(M - M_{\text{small}})$.

Step 3 (Counting argument): The main idea in this step is the following. By the result of Step 2, all preimages of $\hat{x}$ in the universal covering $\pi_T : \tilde{T} \to T$ of the distance torus T that are contained in a ball of radius $r \le B/4$ come from the action of $[c_1] \in \pi_1(T, \hat{x})$, and thus a volume counting argument (similar to the proof of the special case) should complete the proof.

We now make this more precise. Since the deck group of $\tilde{T}$ is isomorphic to $\mathbb{Z}^2$ and acts freely and isometrically, the union of all lifts of the simple geodesic loop c_1 from Step 2 above form a $\pi_1(T, \hat{x})$-invariant countable collection $\mathcal{L}$ of disjoint piecewise geodesic lines in $\tilde{T} = \mathbb{R}^2$. By Step 2, the distance for the metric on $\tilde{T}$ between any two of these lines is at least $B/4$. Furthermore, these lines contain all preimages of $\hat{x}$ in $\tilde{T}$.

Now if c_1 is *smooth*, that is, if c_1 does not have a breakpoint at $\hat{x}$, then the lines in $\mathcal{L}$ are biinfinite geodesics. Since the injectivity radius of $\tilde{T}$ is at least B, this implies that the number of preimages of $\hat{x}$ which are contained in the ball of radius $B/4$ about a fixed preimage is at most $B/4\ell(c_1)$. As $\ell(c_1) \ge 2\mathrm{inj}(x)$, we conclude that this number is at most $B/4\,\mathrm{inj}(x)$, completing the proof of the proposition in this case (note that by the same argument as in Step 1, bounds on the number of preimages in a ball of radius $B/4$ implies bounds on the number of preimages in a ball of radius D).

In general, we can not hope that c_1 is smooth. We use instead a volume counting argument. Namely, the lines in the family $\mathcal{L}$ divide $\tilde{T}$ in a union of disjoint strips with boundary in $\mathcal{L}$. Let α be a minimal geodesic connecting two adjacent lines L_1, L_2 from $\mathcal{L}$. This is an embedded geodesic arc embedded in one of the strips, say the strip S, with endpoints on the two distinct boundary lines L_1, L_2. The infinite cyclic subgroup of $\pi_1(T, \hat{x})$ which is generated by the class φ of c_1 preserves the lines in $\mathcal{L}$ and the strip S, and its maps α to a geodesic $\varphi(\alpha)$ disjoint from α. Namely, if they did intersect, they intersect transversely, and thus an elementary variational argument yields that one can find a curve connecting L_1 to L_2 of strictly shorter length than α. This contradicts the minimality of α. As a consequence, the subsegments a_i of L_i connecting the endpoints of α and $\varphi(\alpha)$ bound together with α and $\varphi(\alpha)$ a rectangular region $\mathcal{R}$ in $\tilde{T}$ which is a fundamental domain for the action of the deck group of T.

As we assume that L_1, L_2 are not smooth, each of the lines L_1, L_2 contains countably many breakpoints of the same breaking angle. For an orientation of $\tilde{T}$ and the induced orientation of L_1, L_2 as oriented boundary of S, for one of the boundary lines, say the line L_1, all internal angles at the breakpoints are strictly bigger than π. In other words, L_1 is a locally concave boundary component of S. Since α and $\varphi(\alpha)$ are minimal geodesics connecting L_1 to L_2, either they meet L_1 at a singular point of L_1 and the arc a_1 does not contain a singular point in its interior, or they meet L_1 orthogonally at a smooth point, and a_1 contains a unique singular point in its interior. In case α and $\varphi(\alpha)$ meet L_1 at a singular point, the angle they form with the smooth subsegments of a_1 exiting the breakpoint is at least $\pi/2$.

We shall show that the fundamental region $\mathcal{R}$ contains embedded rectangles of width at least $\ell(c_1)/2$ and height $B/4$. We only consider the case that α and $\varphi(\alpha)$ meet L_1 at a smooth point, the other case being similar (even a bit easier). Throughout we use the fact that since the injectivity radius of $\tilde{T}$ is at least B (and since $B < D/2A < \pi/2A$) the convexity radius is at least $B/2$ (see [CE08, Theorem 5.14]). In particular, this implies the following. Let $p \in \tilde{T}$, β_1, β_2 be two geodesics segments emanating from p whose endpoints are connected by a geodesic segment c. If $\beta_1 \cup \beta_2 \cup c \subset B(p, B/2)$ and if c meets β_1 orthogonally, then the interior angle at the endpoint of β_2 of the triangle with sides β_1, β_2, c is strictly smaller than $\pi/2$.

Recall from Step 1 that $\ell(a_1) = \ell(c_1) < B/4$. Let $\hat{a}_1 : [0, \delta] \to \tilde{T}$ be a subsegment of a_1 of length at least $\ell(c_1)/2$ which connects the cone point $\hat{a}_1(0) \in a_1 \cap \pi_T^{-1}(\hat{x})$ to the endpoint $\hat{a}_1(\delta) = \alpha \cap L_1$ of a_1. Let $t \to \nu(t)$ be the unit normal field along $\hat{a}_1$ pointing inside of the strip S. We claim that the restriction of the normal exponential map to the set $\{s\nu(t) \mid 0 \le s \le B/4, 0 \le t \le \delta\}$ is an embedding into $\mathcal{R}$.

First, observe that this is an embedding into the strip S. Indeed, if two distinct orthogonal segments $s \to \exp(s\nu(t_1))$ and $s \to \exp(s\nu(t_2))$ $(0 \le t_1 < t_2 \le \delta)$ intersect in a point p, then they are sides of a triangle with edge opposite to p is the arc $\hat{a}_1|_{[t_1,t_2]}$. This triangle is contained in $B(p, B/2)$ and has two right angles, contradicting convexity.

If the image intersects $S - \mathcal{R}$, then since the arc $\{\exp(s\nu(\delta)) \mid 0 \le s \le B/4\}$ is contained in the side α of $\mathcal{R}$, the arc $\beta := \{\exp(s\nu(0)) \mid 0 \le s \le B/4\}$ has to intersect the geodesic segment $\varphi(\alpha)$ in some point p (perhaps after replacing φ with φ^{-1}). Thus we obtain a triangle whose sides are the subarc of β connecting $\hat{a}_1(0)$ to p, the subarc of $\phi(\alpha)$ connecting $\phi(\alpha) \cap L_1$ to p and the subarc of a_1 connecting $\hat{a}_1(0)$ to $\phi(\alpha) \cap L_1$ (see Figure 1). This triangle is contained in $B(p, B/2)$, and it has a right angle at $\phi(\alpha) \cap L_1$ and an angle $\ge \pi/2$ at $\tilde{x}$ since the interior angle at the breakpoint $\tilde{x}$ is strictly bigger than π. As before, this violates convexity. This finishes the proof that restriction of the normal exponential map to the set $\{s\nu(t) \mid 0 \le s \le B/4, 0 \le t \le \delta\}$ is an embedding into $\mathcal{R}$.

Using once more the lower bound on the injectivity radius of $\tilde{T}$ and the upper bound on the Gauß curvature, we conclude from $\ell(\hat{a}_1) \ge \ell(c_1)/2$ that

$$\operatorname{area}(\exp\{s\nu(t) \mid 0 \le s \le B/4, 0 \le t \le \delta\}) \ge \kappa \ell(c_1),$$

where $\kappa > 0$ is a universal constant.

Since the images of the rectangle $\mathcal{R}$ under the action of the deck group of T have pairwise disjoint interiors, we conclude that the area of the $B/4$-neighborhood of the ball

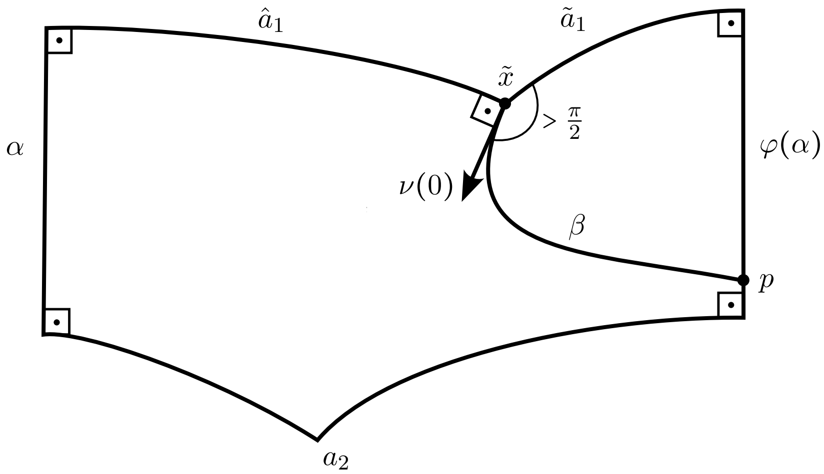

FIGURE 1. The argument by contradiction showing that the image of the normal exponential map is contained in $\mathcal{R}$.

of radius B about any point $z \in \pi_T^{-1}(\hat{x})$ is at least

$$\#\big(\pi_T^{-1}(\hat{x}) \cap B^{\tilde{T}}(z,B)\big) \cdot \kappa\ell(c_1)$$

(here the notation $B^{\tilde{T}}(\tilde{x},B)$ makes is precise that we take a ball in $\tilde{T}$). On the other hand, this area is bounded from above by a universal constant. Since moreover for any $\tilde{x} \in \pi_{\hat{M}}^{-1}(\hat{x})$ we have $\pi_M^{-1}(\hat{x}) \cap B^{\tilde{M}}(\tilde{x},B/A) \subseteq \pi_T^{-1}(\hat{x}) \cap B^{\tilde{T}}(\tilde{x},B)$ (note that this is an abuse of notation since $\pi_{\hat{M}}^{-1}(T)$ is an infinite cylinder if Γ is an infinite cyclic group of hyperbolic isometries, that is, if the component of M_{thin} with fundamental group Γ is a Margulis tube), and since $2\ell(c_1) \geq \text{inj}(x)$, this shows that

$$\#\big(\pi_M^{-1}(\hat{x}) \cap B^{\tilde{M}}(\tilde{x},B/A)\big) \leq \kappa'/\text{inj}(x).$$

Again, as in Step 1, estimates on the number of preimages in a ball of radius B/A implies bounds on the number of preimages in a ball of radius D. This completes the proof. □

Corollary 7.7 is now an easy consequence of Proposition 7.6.

Proof of Corollary 7.7. It suffices to prove the estimate for those $x \in N_{1/4}(M \smallsetminus M_{\text{small}})$ with $\text{inj}_M(x) \leq D$. By Proposition 7.6 this follows if for those x it holds $\text{inj}(x) \geq Ce^{-bd(x,M_{\text{thick}})}$ for a universal constant $C > 0$.

Thus let $x \in N_{1/4}(M \smallsetminus M_{\text{small}})$ and let x^* be the first point on the radial geodesic through x that lies in $\partial M_{\text{thick}}$. Abbreviate $R = d(x,x^*) = d(x,M_{\text{thick}})$.

Let Γ be the fundamental group of the component of M_{thin} containing x. Consider the intermediate cover $\hat{M} := M/\Gamma$, and choose lifts $\hat{x}$ and $\hat{x}^*$ of x and x^* with $d(\hat{x},\hat{x}^*) = R$. It holds $\text{inj}_{\hat{M}}(\hat{x}) = \text{inj}_M(x)$ and $\text{inj}_{\hat{M}}(\hat{x}^*) = \text{inj}_M(x^*)$. Let $\sigma \subset \hat{M}$ be an essential based loop at $\hat{x}$ of minimal length. Radially project σ to a curve σ^* based at $\hat{x}^*$. Note that if x is contained in a Margulis tube, then x has distance at least $3/4$ from the core geodesic due to the definition of M_{small}. Hence Jacobi field comparison shows $\ell(\sigma^*) \le Ce^{bR}\ell(\sigma)$ for a universal constant C. As $\text{inj}_{\hat{M}}(\hat{x}^*) = \text{inj}_M(x^*) = \mu$, we have $\ell(\sigma^*) \ge 2\mu$, where μ is the chosen Margulis constant for manifolds with sectional curvature contained in $[-4, -1/4]$. Thus $Ce^{bR}\ell(\sigma) \ge 2\mu$. The definition of σ, and the fact that $\text{inj}_{\hat{M}}(\hat{x}) = \text{inj}_M(x)$, imply $Ce^{bR}\text{inj}_M(x) \ge \mu$. Since $R = d(x, M_{\text{thick}})$, this completes the proof. □

7.4. **Generalisations.** In Section 9.4 we will prove a global C^0-estimate in terms of a new hybrid norm. For this we will need a slightly more general version of Proposition 7.5. This is based on the following more general version of Proposition 7.6. Recall that for $r \ge 0$ we denote by $N_r(M \smallsetminus M_{\text{small}})$ the r-neighbourhood of $M \smallsetminus M_{\text{small}}$.

Lemma 7.9. *For all $\bar{R} \ge 0$ there exists a constant $C(\bar{R}) > 0$ with the following property. Let $x \in M^{\le \mu'} \cap N_{\bar{R}+1/4}(M \smallsetminus M_{\text{small}})$, and if x is contained in a Margulis tube, assume in addition that $N_{\bar{R}}(M \smallsetminus M_{\text{small}})$ is disjoint from the one-neighbourhood of the core geodesic. Then it holds*

$$\#\big(\pi^{-1}(x) \cap B(\tilde{x}, D)\big) \le C(\bar{R}) \frac{1}{\text{inj}(x)},$$

where $\pi : \tilde{M} \to M$ is the universal covering projection.

Proof. We quickly review the proof of Proposition 7.6. We choose the constants A and B from Proposition 7.8, and we assumed without loss of generality that $B < \frac{D}{2A}$. The reason for chosing $\frac{D}{A}$ is the following. For all $x \in N_{1/4}(M \smallsetminus M_{\text{small}})$ the level torus T containing x satisfies $\text{diam}(T) > D/A$ (see Step 2).

We now explain how to adjust the argument from the proof of Proposition 7.6. Standard Jacobi field estimates show that for any $\bar{R} \ge 0$ there exists $\bar{D}(\bar{R}) > 0$ with the following property. For any x as stated in Lemma 7.9 it holds $\text{diam}(T) > \bar{D}(\bar{R})$ for the level torus T containing x. Choose some $B(\bar{R}) < \min\{B(3), \frac{1}{2}\bar{D}(R)\}$. The proof of Proposition 7.6 goes through without change when replacing B by $B(\bar{R})$ (and $A = A(3)$ still given by Proposition 7.8). □

The next result is the generalisation of Proposition 7.5 that we need for the global C^0-estimate in Section 9.4. It follows from Lemma 7.9 analogous to how Proposition 7.5 followed from Proposition 7.6. We omit the details.

Lemma 7.10. *For all $\alpha \in (0,1)$, $\Lambda \ge 0$, $\delta \in (0,2)$, $b > 1$, and $\bar{R} \ge 0$ there exist $\varepsilon_0 = \varepsilon_0(\delta, b) > 0$ and $C(\bar{R}) = C(\bar{R}, \alpha, \Lambda, \delta, b) > 0$ with the following property. Let M be a Riemannian 3-manifold of finite volume so that*

$$|\sec +1| \le \varepsilon_0 \quad \textit{and} \quad \|\nabla \text{Ric}\|_{C^0(M)} \le \Lambda.$$

Let $x \in N_{\bar{R}}(M \smallsetminus M_{\text{small}})$, and if x is contained in a Margulis tube, assume in addition that $N_{\bar{R}}(M \smallsetminus M_{\text{small}})$ is disjoint from the one-neighbourhood of the core geodesic. Then for all $h \in C^2\big(\text{Sym}^2(T^*M)\big) \cap H^2(M)$ it holds

$$|h|(x) \le C(\bar{R}) \left(\|\mathcal{L}h\|_{C^0(M)} + e^{\frac{b}{2} d(x, M_{\text{thick}})} \left(\int_M e^{-(2-\delta) r_x(y)} |\mathcal{L}h|^2(y) \, d\text{vol}(y) \right)^{\frac{1}{2}} \right).$$

8. Model metrics in tubes and cusps

The examples of Section 6 show that Theorem 1 no longer holds true without the assumption of a uniform lower bound on the injectivity radius. At the end of Section 6 we pointed out that the counterexamples model a deformation of hyperbolic structures on a fixed Margulis tube, obtained by slowly changing the conformal structure of the tori $T(r)$ contained in M_{small}. The goal of this section is to formulate a geometric condition for the tubes and cusps, controlled asymptotic hyperbolicity, which rules out such examples. This section can be skipped by readers who are mainly interested in the applications to drilling, filling and hyperbolization. For these applications, it suffices to consider metrics which have constant curvature in the thin parts of the manifold.

Asymptotically hyperbolic metrics on non-compact manifolds have been widely studied in the literature, however mainly in the context of manifolds with flaring ends. We refer to [HQS12] for an overview of some related results.

In the sequel, $\eta > 1$ is a constant fixed once and for all. Let M be a complete Riemannian 3-manifold of finite volume that satisfies the following curvature decay condition:

$$\max_{\pi\subseteq T_xM} |\sec(\pi)+1|,\ |\nabla R|(x),\ |\nabla^2 R|(x) \le \varepsilon_0 e^{-\eta d(x,\partial M_{\text{small}})} \quad \text{for all } x\in M_{\text{small}}. \tag{8.1}$$

Here R denotes the Riemann curvature endomorphism.

As before, we know that all cusps are diffeomorphic to $T^2\times[0,\infty)$ since by Convention 2.3 we assume that M is orientable. We construct in this section a hyperbolic model metric in the small part of cusps and the complements of the 1-neigbhborhood of the core curves of the small part of tubes. These auxiliary metrics are used in Section 9.2 to construct Banach spaces geared at controlling solutions of the equation $\mathcal{L}(h)=f$ in the small part of M.

Let as before T^2 be a two-torus. Call a metric g on $T^2\times I$ (where I is an interval) a *cusp metric* if it is of the form

$$g = e^{-2r} g_{Flat} + dr^2,$$

where g_{Flat} is some flat metric on T^2 and r is the I-coordinate. Let T be a Margulis tube and C a rank 2 cusp of M. Note that $C_{\text{small}} \cong T^2\times[0,\infty)$ and $T_{\text{small}}\smallsetminus N_1(\gamma)\cong T^2\times[0,R-1]$, where R is the *radius* of T_{small}, that is, the distance of the boundary of T_{small} to the core curve of T, and $\cong$ stands for diffeomorphic. The given metric on M will in general not be a cusp metric on these sets.

The following two statements are the main results of this section.

Proposition 8.1. *For any $\eta>1$ there exists $\varepsilon_0=\varepsilon_0(\eta)>0$ with the following property. Let M be an Riemannian 3-manifold satisfying the curvature decay condition (8.1) and let T be a Margulis tube of M with core geodesic γ. Then there exists a cusp metric g_{cusp} on $T_{\text{small}}\smallsetminus N_1(\gamma)$ so that for all $x\in T_{\text{small}}\smallsetminus N_1(\gamma)$ it holds*

$$|g-g_{cusp}|_{C^2}(x) = O\big(e^{-2r_\gamma(x)}+\varepsilon_0 e^{-\eta r_{\partial T}(x)}\big),$$

where $r_{\partial T}(x)=d(x,\partial T_{\text{small}})$, and $r_\gamma(x)=d(x,\gamma)$.

See Notation 2.2 for our convention of the O-notation. For cusps we have a slightly better estimate.

Proposition 8.2. *For any $\eta > 1$ there exists $\varepsilon_0 = \varepsilon_0(\eta) > 0$ with the following property. Let M be a Riemannian 3-manifold satisfying the curvature decay condition (8.1) and let C be a rank 2 cusp of M. Then there exists a cusp metric g_{cusp} on C_{small} so that for all $x \in C_{\text{small}}$ it holds*

$$|g - g_{cusp}|_{C^2}(x) = O(\varepsilon_0 e^{-\eta r(x)}),$$

where $r(x) = d(x, \partial C_{\text{small}})$.

The remainder of this section is devoted to the proof for Proposition 8.1 and Proposition 8.2. The main idea for the proof is to compare the Jacobi equation in M with the one in the comparison space $\bar{M} = \mathbb{H}^3$. To do so we require the following stability estimate for linear ODEs.

Lemma 8.3. *Let $A, \bar{A} : [0,T] \to \operatorname{End}(\mathbb{R}^n)$ and $b, \bar{b} : [0,T] \to \mathbb{R}^n$ be continuous, and assume that the following conditions are satisfied:*

i) $\|A(t)\|_{\text{op}} \leq a$ and $\|\bar{A}(t)\|_{\text{op}} \leq \bar{a}$ for all $t \in [0,T]$;
ii) $\|A(t) - \bar{A}(t)\|_{\text{op}} = O\big(\varepsilon e^{\eta(t-T)}\big)$ for some $\eta > a - \bar{a}$;
iii) $|\bar{b}(t)| = O\big(\bar{\beta} e^{\bar{\mu} t}\big)$ for some $\bar{\mu} > \max\{a, \bar{a}\}$ and $\bar{\beta} \geq 0$;
iv) $|b(t) - \bar{b}(t)| = O\big(\beta e^{\mu t}\big)$ for some $\mu > a$ and $\beta \geq 0$.

Then the solutions $y, \bar{y} : [0,T] \to \mathbb{R}^n$ of the ODEs

$$y'(t) = A(t)y(t) + b(t) \quad \textit{and} \quad \bar{y}'(t) = \bar{A}(t)\bar{y}(t) + \bar{b}(t)$$

with initial conditions $y(0) = y_0$ and $\bar{y}(0) = \bar{y}_0$ satisfy

$$|\bar{y}(t) - y(t)| = O\Big(|\bar{y}_0 - y_0| e^{at} + \varepsilon |\bar{y}_0| e^{\bar{a}t} e^{\eta(t-T)} + \varepsilon \bar{\beta} e^{\bar{\mu} t} e^{\eta(t-T)} + \beta e^{\mu t}\Big).$$

The same estimates hold for second order linear ODEs $v''(t) = R(t)v(t)$ if $\bar{y}_0$ is replaced by $|\bar{v}(0)| + |\bar{v}'(0)|$ (similarly for $|\bar{y}_0 - y_0|$), and a is replaced by $\max\{1, \max_t \|R(t)\|_{\text{op}}\}$ (similarly for $\bar{a}$). Indeed, substituting $y = (v, v')$ the second order ODE is equivalent to

$$y'(t) = \begin{pmatrix} 0 & \mathrm{id}_{\mathbb{R}^n} \\ R(t) & 0 \end{pmatrix} y(t) \quad \text{and} \quad y(0) = (v(0), v'(0)),$$

and it holds

$$\left\| \begin{pmatrix} 0 & \mathrm{id}_{\mathbb{R}^n} \\ R(t) & 0 \end{pmatrix} \right\|_{\text{op}} = \max\{1, \|R(t)\|_{\text{op}}\}.$$

Proof. Consider a linear ODE

$$\chi'(t) = \Sigma(t)\chi(t) + \xi(t),$$

and assume

$$\max_t \|\Sigma(t)\|_{\text{op}} \leq \sigma \quad \text{and} \quad |\xi(t)| \leq \sum_i \kappa_i e^{\lambda_i t} \text{ for some } \lambda_i > \sigma \text{ and } \kappa_i \geq 0.$$

Then it holds

$$|\chi(t)| \leq |\chi(0)| e^{\sigma t} + \sum_i (\lambda_i - \sigma)^{-1} \kappa_i e^{\lambda_i t}. \tag{8.2}$$

Indeed, this is a straightforward consequence of inequality (4.9) on page 56 of [Har82].

Compute
$$(\bar{y}-y)'(t) = A(t)(\bar{y}-y)(t) + (\bar{A}-A)(t)\bar{y}(t) + (\bar{b}-b)(t).$$
As $\bar{\mu} > \bar{a}$ we may apply (8.2) to $\bar{y}' = \bar{A}\bar{y} + \bar{b}$ to obtain $|\bar{y}|(t) = \big(|\bar{y}_0|e^{\bar{a}t} + \bar{\beta}e^{\bar{\mu}t}\big)$. Hence $|(\bar{A}-A)\bar{y}|(t) = O\big(\varepsilon|\bar{y}_0|e^{\bar{a}t}e^{\eta(t-T)} + \varepsilon\bar{\beta}e^{\bar{\mu}t}e^{\eta(t-T)}\big)$ due to condition $ii)$. Since $\bar{a}+\eta, \bar{\mu}+\eta, \mu > a$ we can apply (8.2) to the ODE satisfied by $\bar{y}-y$. Therefore,
$$|\bar{y}-y|(t) = O\Big(|\bar{y}_0 - y_0|e^{at} + \varepsilon|\bar{y}_0|e^{\bar{a}t}e^{\eta(t-T)} + \varepsilon\bar{\beta}e^{\bar{\mu}t}e^{\eta(t-T)} + \beta e^{\mu t}\Big).$$
This completes the proof. □

We now come to the construction of g_{cusp} on $T_{\mathrm{small}} \setminus N_1(\gamma)$. As an intermediate step we first construct another metric g_{tube} on T_{small}.

Let γ be the core curve of T and let $\tilde{\gamma} \subseteq \tilde{M}$ be a lift of γ. Denote by $\varphi : \tilde{M} \to \tilde{M}$ the deck transformation corresponding to $[\gamma] \in \pi_1(M)$ which preserves $\tilde{\gamma}$ and acts on it as a translation. To the element φ we associate its *translation length* which is the length of γ, and the *rotation angle*, defined by parallel transport of the orthogonal complement of γ' in $TM|\gamma$. Let $\beta \subseteq \mathbb{H}^3$ be a geodesic, and let $\psi : \mathbb{H}^3 \to \mathbb{H}^3$ be an orientation preserving loxodromic isometry, with axis β and the same translation length and the same rotation angle as φ.

Define $\hat{M} := \tilde{M}/\langle\varphi\rangle$ and $\hat{\mathbb{H}}^3 := \mathbb{H}^3/\langle\psi\rangle$. Using the normal exponential maps for $\tilde{\gamma}$ in $\tilde{M}$ and for β in $\mathbb{H}^3$, we see that there is a diffeomorphism $\hat{M} \supseteq N_R(\hat{\gamma}) \xrightarrow{\cong} N_R(\hat{\beta}) \subseteq \mathbb{H}^3/\langle\psi\rangle$ of the full distance tori. The projection $\hat{M} \to M$ also induces a diffeomorphism $\hat{M} \supseteq N_R(\hat{\gamma}) \xrightarrow{\cong} N_R(\gamma) \subseteq M$ when R is the radius of T_{small} because $T_{\mathrm{small}} \subseteq M_{\mathrm{thin}}$ by Lemma 7.4. Note that $N_R(\gamma) = T_{\mathrm{small}}$ by the definition of the radius R of T_{small}. The *tube metric* g_{tube} on T_{small} is the pullback of the hyperbolic metric on $N_R(\hat{\beta})$ via the diffeomorphism $T_{\mathrm{small}} \xrightarrow{\cong} N_R(\hat{\beta}) \subseteq \hat{\mathbb{H}}^3$.

The *cusp metric* g_{cusp} on $T_{\mathrm{small}} \setminus N_1(\gamma) \cong \partial T_{\mathrm{small}} \times [0, R-1]$ is the metric
$$g_{cusp} = e^{-2r}g_{Flat} + dr^2,$$
where g_{Flat} is the flat metric on $\partial T_{\mathrm{small}}$ induced by the tube metric g_{tube}. One can easily check by explicit calculations that
$$|g_{tube} - g_{cusp}|_{C^2}(x) = O\big(e^{-2r_\gamma(x)}\big) \tag{8.3}$$
for all $x \in T_{\mathrm{small}} \setminus N_1(\gamma)$.

We next verify that g_{cusp} has the properties stated in Proposition 8.1.

Proof of Proposition 8.1. By (8.3), it suffices to show that
$$|g - g_{tube}|_{C^2}(x) = O\big(\varepsilon_0 e^{-\eta r_{\partial T}(x)}\big) \tag{8.4}$$
for all $x \in T_{\mathrm{small}} \setminus N_1(\gamma)$. Let γ be the core geodesic of the tube T. It suffices to prove the estimates in the universal cover. Let $\tilde{\gamma} \subseteq \tilde{M}$ be a lift of γ. Choose parallel unit vector fields ν_1, ν_2 along $\tilde{\gamma}$ so that $\tilde{\gamma}', \nu_1, \nu_2$ is a positively oriented orthonormal frame along $\tilde{\gamma}$. Define a map $\varphi : \mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R} \to \tilde{M}$ by
$$\varphi(s,t,\theta) := \exp_{\tilde{\gamma}(s)}\big(t\nu_\theta(s)\big),$$

where $\nu_\theta(s) := \cos(\theta)\nu_1(s) + \sin(\theta)\nu_2(s)$. We think of φ as a 2-dimensional variation of geodesics. The main idea is that the variational fields of φ solve Jacobi equations, and so one can use Lemma 8.3 to compare the situation with the comparison space $\mathbb{H}^3$.

We fix some notation. Let V be a vector field along φ, i.e., a map $V : \mathbb{R}\times\mathbb{R}_{\geq 0}\times\mathbb{R} \to T\tilde{M}$ so that $\pi \circ V = \varphi$. Denote by $D_t V$ the vector field whose value at (s_0, t_0, θ_0) equals the covariant derivative of V along the curve $t \to \varphi(s_0, t, \theta_0)$ at $t = t_0$. The vector fields $D_s V$ and $D_\theta V$ are defined analogously. For example, for $t = 0$ the vector $D_\theta V(s_0, 0, \theta_0)$ is just the usual derivative of the curve $\theta \to V(s_0, 0, \theta)$ in $T_{\tilde{\gamma}(s_0)}\tilde{M}$. We will also write $()'$ for D_t.

Fix s_0 and θ_0, and consider the geodesic $\sigma(t) = \varphi(s_0, t, \theta_0)$ with $\sigma(0) = \tilde{\gamma}(s_0)$ and $\sigma'(0) = \nu_{\theta_0}(s_0)$. Let E_1, E_2, E_3 be a parallel orthonormal frame along σ so that $E_1(0) = \tilde{\gamma}'(s_0)$, $E_2(t) = \sigma'(t)$, and $E_3(0) = \nu_{\theta_0}^{\perp}(s_0) := -\sin(\theta_0)\nu_1(s_0) + \cos(\theta_0)\nu_2(s_0)$.

Let i be either the s- or the θ-coordinate. The restriction of the variational field $J_i := \partial_i \varphi$ to σ is a Jacobi field, that is, it solves the Jacobi equation

$$J_i''(t) + R(t)J_i(t) = 0,$$

where $R(t) = R(\cdot, \sigma'(t))\sigma'(t)$ and $J_i'' = D_t D_t J_i$.

We do the same set-up in the comparison space $\bar{M} = \mathbb{H}^3$. Using the orthonormal frames $(E_i)_{i=1}^3$ we can think of J_i resp. R as a curve resp. a family of symmetric matrices in $\mathbb{R}^3$. Similarly, $(\bar{E}_i)_{i=1}^3$ can be used to think of $\bar{J}_i$ resp. $\bar{R}$ as a curve resp. a family of symmetric matrices in $\mathbb{R}^3$. Hence it makes sense to write $J_i(t) - \bar{J}_i(t)$ and $R(t) - \bar{R}(t)$. The curvature decay condition (8.1) translates to (for $t \in [0, R]$)

$$\|R(t) - \bar{R}(t)\|_{\mathrm{op}} \leq \varepsilon_0 e^{\eta(t-R)},$$

where R is the radius of T_{small}, i.e., the distance of the core geodesic γ to $\partial T_{\mathrm{small}}$. Observe that $\bar{a} := \max\{1, \max_t \|\bar{R}(t)\|_{\mathrm{op}}\} = 1$ and $a := \max\{1, \max_t \|R(t)\|_{\mathrm{op}}\} \leq 1+\varepsilon_0$. So condition *ii)* of Lemma 8.3 is satisfied if $\varepsilon_0 < \eta$.

Note that J_i and $\bar{J}_i$ have the same initial conditions. Therefore, invoking Lemma 8.3 with $\bar{\beta} = \beta = 0$ yields (for $t \in [0, R]$)

$$|J_i(t) - \bar{J}_i(t)|, |J_i'(t) - \bar{J}_i'(t)| = O\big(\varepsilon_0 e^t e^{\eta(t-R)}\big). \tag{8.5}$$

Note $|\bar{J}_i|(t), |\bar{J}_i'|(t) = O(e^t)$ due to (8.2), and by (8.5) the same holds for J_i.

Let j be either the s- or the θ-coordinate. Consider the variational fields $D_j J_i$ restricted to σ. A straightforward calculation shows that $D_j J_i$ solves a inhomogeneous Jacobi equation, that is,

$$(D_j J_i)'' + R(t) D_j J_i(t) = b(t)$$

for a vector field b along σ. In fact, one can show

$$\begin{aligned} b(t) = &- (\nabla R)(J_j, \sigma', J_i, \sigma') - (\nabla R)(J_i, \sigma', \sigma', J_j) \\ &- R(J_j', \sigma')J_i - 2R(J_j, \sigma')J_i' - R(J_i, J_j')\sigma' - R(J_i, \sigma')J_j'. \end{aligned}$$

The analogous statements hold in the comparison space. We again use the parallel orthonormal frames $(E_i)_{i=1}^3$ and $(\bar{E}_i)_{i=1}^3$ to view $D_j J_i$, $\bar{D}_j \bar{J}_i$, b, and $\bar{b}$ as curves in $\mathbb{R}^3$. Then (8.1), (8.5) and the growth estimates $|J_i^{(\prime)}|(t), |\bar{J}_i^{(\prime)}|(t) = O(e^t)$ can be used to show

$|\bar{b}(t)| = O(e^{2t})$ and $|b(t) - \bar{b}(t)| = O\big(\varepsilon_0 e^{2t} e^{\eta(t-R)}\big)$. One also computes $D_j J_i(0) = 0 = \bar{D}_j \bar{J}_i(0)$, and $|(D_j J_i)'(0) - (\bar{D}_j \bar{J}_i)'(0)| = O(e^{-\eta R})$ by using the curvature decay condition (8.1). Then one can again use Lemma 8.3 to obtain

$$|D_j J_i(t) - \bar{D}_j \bar{J}_i(t)|, |(D_j J_i)'(t) - (\bar{D}_j \bar{J}_i)'(t)| = O\big(\varepsilon_0 e^{2t} e^{\eta(t-R)}\big). \tag{8.6}$$

Again note that $|\bar{D}_j \bar{J}_i|(t), |(\bar{D}_j \bar{J}_i)'|(t) = O(e^{2t})$ due to (8.2) and $|\bar{b}|(t) = O(e^{2t})$. By (8.6) the same estimate holds for $D_j J_i$.

Finally, let k be either the s- or θ-coordinate. Using arguments similar as for $D_j J_i$ (that is, inhomogeneous Jacobi equation, and Lemma 8.3) one can show

$$|D_k D_j J_i(t) - \bar{D}_k \bar{D}_j \bar{J}_i(t)|, |(D_k D_j J_i)'(t) - (\bar{D}_k \bar{D}_j \bar{J}_i)'(t)| = O\big(\varepsilon_0 e^{3t} e^{\eta(t-R)}\big). \tag{8.7}$$

The estimates (8.5), (8.6), and (8.7) imply the desired estimate on $|g - g_{tube}|_{C^2}$ in (8.4). Indeed, for $m > 0$ define $c : \{s, t, \theta\}^m \to \mathbb{N}$ as $c(i_1, ..., i_m) := \#\{u \,|\, i_u \neq t\}$. Then it follows from (8.5), (8.6), and (8.7) that

$$|g_{ij} - \bar{g}_{ij}| = O\big(\varepsilon_0 e^{c(i,j)t} e^{\eta(t-R)}\big) \tag{8.8}$$

$$|\partial_k g_{ij} - \partial_k \bar{g}_{ij}| = O\big(\varepsilon_0 e^{c(i,j,k)t} e^{\eta(t-R)}\big) \tag{8.9}$$

$$|\partial_l \partial_k g_{ij} - \partial_l \partial_k \bar{g}_{ij}| = O\big(\varepsilon_0 e^{c(i,j,k,l)t} e^{\eta(t-R)}\big) \tag{8.10}$$

for all $i, j, k, l \in \{s, t, \theta\}$. Note $\bar{g} = \cosh^2(t) ds^2 + dt^2 + \sinh^2(t) d\theta^2$, and so in particular the matrix $(\bar{g}_{ij})$ is diagonalised. So for any $(0, m)$-tensor T

$$|T|^2_{\bar{g}} = \sum_{i_1, ..., i_m} (T_{i_1 ... i_m})^2 (\bar{g}_{i_1 i_1})^{-1} \cdot ... \cdot (\bar{g}_{i_m i_m})^{-1}.$$

For $t \geq 1$, $\cosh(t)$ and $\sinh(t)$ agree with e^t up to a uniform multiplicative constant. So $(\bar{g}_{i_1 i_1})^{-1} \cdot ... \cdot (\bar{g}_{i_m i_m})^{-1} = O\big(e^{-2c(i_1, ..., i_m)t}\big)$, and thus (8.8), (8.9), and (8.10) imply (8.4). ☐

We now come to the sketch of proof for Proposition 8.2. This is a bit more involved than the case of a tube. The main idea is to pull back the conformal structure of the distance tori at infinity for the construction of the cusp metric and use the fact that up to scale, a flat metric on T^2 is determined by the conformal structure it defines. To produce a cusp metric that is close to the given metric, we establish an effective version of the Uniformization Theorem.

The classical Uniformization Theorem states that for any Riemannian metric g on the two torus T^2, there exists a flat metric $\bar{g}$ on T^2 and a function $\rho : T^2 \to \mathbb{R}$ so that $g = e^\rho \bar{g}$. The flat metric is unique only up to a multiplicative constant, and hence ρ is only defined up to an additive constant. The following definition should be thought of as the choice of a canonical flat metric in the conformal class of g.

Definition 8.4. Let g be a metric on T^2. The *associated flat metric* g_{Flat} is the unique flat metric conformal to g such that the corresponding function ρ satisfies

$$\int_{T^2} \rho \, d\mathrm{vol}_g = 0.$$

We are now in a position to state the *effective Uniformization Theorem*. It says that if the given metric is "almost" a flat metric, then ρ is small. Here the function ρ is understood to define the associated flat metric.

Lemma 8.5. *There exist constants $\delta_0 > 0$ and $C > 0$ with the following property. If g is a metric on T^2 such that*

$$\mathrm{diam}(T^2, g) \leq 1 \quad \textit{and} \quad |\sec(g)| \leq \delta$$

for some $\delta \leq \delta_0$, then

$$|\rho(x)| \leq C\delta \quad \textit{for all } x \in T^2.$$

We postpone the proof of Lemma 8.5, and show next how to use Lemma 8.5 to obtain Proposition 8.2.

Proof of Proposition 8.2. For each $r \geq 0$ we denote by $T(r)$ the torus in C_{small} all of whose points have distance r to $\partial C_{\mathrm{small}}$. For the moment fix $r_0 > 0$. Note that by the curvature decay condition (9.1) it holds $|\sec + 1|, |\nabla R|, |\nabla^2 R| \leq \varepsilon_0 e^{-\eta r_0}$ for all points x that lie further inside C_{small} than $T(r_0)$. The results of [Shc83] show that under this condition, a Busemann function associated to the rank 2 cusp C is of class C^4, with controlled norm of the derivatives, and the shape operator $\mathcal{H}_{r_0}$ of the horotorus $T(r_0)$ satisfies $|\mathcal{H}_{r_0} - \mathrm{id}|_{C^2} = O\big(\varepsilon_0 e^{-\eta r_0}\big)$. This implies that $|K|_{C^2} = O\big(\varepsilon_0 e^{-\eta r_0}\big)$ for the Gauß curvature K of $T(r_0)$. As $T(r_0) \subseteq C_{\mathrm{small}}$ we can invoke Lemma 8.5 to conclude $|\rho| = O(\varepsilon_0 e^{-\eta r_0})$, where ρ on $T(r_0)$ is given by Definition 8.4. By Exercise 2 in Chapter 4.3 of [dC16] it holds $\Delta\rho = 2K$, where Δ is the Laplace operator with respect to the restriction of the metric g to $T(r_0)$. Therefore, Schauder estimates imply $\|\rho\|_{C^2} = O\big(\varepsilon_0 e^{-\eta r_0}\big)$.

Let $g_{Flat}^{(r_0)}$ be the flat metric on $T(r_0)$ defined by $e^{\rho} g_{Flat}^{(r_0)} = g|T(r_0)$ (see Definition 8.4). Let $\psi : \mathbb{R}^2 \to T(r_0)$ be the universal Riemannian covering map for $g_{Flat}^{(r_0)}$. Define $\varphi : \mathbb{R}^2 \times [0, r_0] \to C_{\mathrm{small}}$ by

$$\varphi(x,t) := \exp_{\psi(x)}(-t\partial_r),$$

where ∂_r is the radial vector field in C_{small} pointing to infinity. Fix $x_0 \in \mathbb{R}^2$, and consider the geodesic $\sigma(t) = \varphi(x_0, t)$.

As in the proof of Proposition 8.1 consider the variational fields $J_i := \partial_{x^i}\varphi$, $D_j J_i$, and $D_k D_j J_i$ along σ (for the notation see the proof of Proposition 8.1). These satisfy (in)homogeneous Jacobi equations. We claim that the initial conditions $J_i^{(\prime)}(0)$, $(D_j J_i)^{(\prime)}(0)$, $(D_k D_j J_i)^{(\prime)}(0)$ are $\varepsilon_0 e^{-\eta r_0}$-close to those in the comparison space $\bar{M} = \mathbb{H}^3$, where we write $V^{(\prime)}$ to denote V or V'. We show this for J_i, J_i' and $D_j J_i$, the other cases being similar.

As in the proof of Proposition 8.1 we use a parallel orthonormal frame $(E_i)_{i=1}^3$ along σ to view all vector fields as curves in $\mathbb{R}^3$, and all tensors as a family of tensors on $\mathbb{R}^3$. Choose the parallel orhonormal frame so that $E_1(0) = e^{-\rho/2}\frac{\partial}{\partial\psi^1}$, $E_2(0) = e^{-\rho/2}\frac{\partial}{\partial\psi^2}$ and $E_3(0) = \sigma'(0)$, so that $J_i(0) = e^{\rho/2}E_i(0)$ for $i = 1, 2$. In the comparison space $\bar{M} = \mathbb{H}^3$ it holds $\bar{J}_i(0) = \bar{E}_i(0)$. Using $|e^z - 1| \leq 2|z|$ for $|z|$ small, we obtain $|J_i - \bar{J}_i|(0) \leq 2|\rho| = O(e^{-\eta r_0})$. Moreover, $J_i'(0) = \nabla_{J_i(0)}\nu = \mathcal{H}_\nu\big(J_i(0)\big)$, where $\mathcal{H}_\nu$ is the shape operator of

$T(r_0)$ with respect to the unit normal $\nu := -\partial_r$. Now the shape operator $\bar{\mathcal{H}}$ of horospheres in $\mathbb{H}^3$ is the identity and therefore $|\mathcal{H} - \bar{\mathcal{H}}| = O(e^{-\eta r_0})$and $|J_i' - \bar{J}_i'|(0) = O(e^{-\eta r_0})$. To see that $D_j J_i(0)$ is close to $\bar{D}_j \bar{J}_i(0)$ observe that

$$D_j J_i(0) = \mathrm{II}_\nu\big(J_i(0), J_j(0)\big) + \nabla^{T(r_0)}_{J_i(0)} J_j(0),$$

where $\nabla^{T(r_0)}$ is the Levi-Civita connection of $T(r_0)$. Note that $J_i(0) = \partial_{\psi^i}$. The Christoffel symbol Γ^k_{ji} of $\nabla^{T(r_0)}$ is $\frac{1}{2}\big(\delta_{kj}\partial_i\rho + \delta_{ik}\partial_j\rho + \delta_{ij}\partial_k\rho\big)$ because $g = e^\rho \psi_* g_{\mathbb{R}^2}$ on $T(r_0)$. Therefore, the estimates on the shape operator and $\|\rho\|_{C^2}$ imply $|\bar{D}_j\bar{J}_i - D_j J_i|(0) = O(e^{-\eta r_0})$.

As in the proof of Proposition 8.1 the curvature decay condition (8.1) reads (for $t \in [0, r_0]$)

$$|R - \bar{R}|(t), |\nabla R - \bar{\nabla}\bar{R}|(t), |\nabla^2 R - \bar{\nabla}^2\bar{R}|(t) \le \varepsilon_0 e^{\eta(t-r_0)}.$$

Since all initial conditions are $e^{-\eta r_0}$-close to the ones in the comparison situation, one can, exactly as in the proof of Proposition 8.1, iteratively use Lemma 8.3 to conclude that the metric g is close to the metric $\bar{g}$ in the comparison situation (with error $O\big(\varepsilon_0 e^{\eta(t-r_0)}\big)$). Note that in $\mathbb{H}^3$ the hyperbolic metric is of the form $\bar{g} = e^{-2r} g_{Flat} + dr^2$ for a flat metric g_{Flat} on a reference horosphere. As $d(\sigma(t), \partial C_{\text{small}}) = r_0 - t$ we have

$$\big|g - g^{(r_0)}_{cusp}\big|_{C^2}(x) = O\big(\varepsilon_0 e^{-\eta r(x)}\big) \quad \text{whenever } \ r(x) \le r_0,$$

where $r(x) = d(x, \partial C_{\text{small}})$, and $g^{(r_0)}_{cusp} := e^{-2(r-r_0)} g^{(r_0)}_{Flat} + dr^2$.

Now choose a sequence $r_0 \to \infty$. After passing to a subsequence we may assume $g^{(r_0)}_{cusp} \to g_{cusp}$ in the pointed C^2-topology. This limit is the desired cusp metric. ☐

It remains to prove Lemma 8.5. For its proof we will need the fact that for metrics g on T^2 as stated in Lemma 8.5, the injectivity radius of the universal cover $(\tilde{T}^2, \tilde{g})$ is bounded from below by a universal constant. This follows from the next lemma.

Lemma 8.6. *Let $v > 0$, and let g be a Riemannian metric on T^2 so that*

$$\mathrm{vol}_g(T^2) \le v \quad \textit{and} \quad \sec(g) \le \delta$$

for some $0 < \delta \le \frac{2\pi}{v}$. Then $\mathrm{inj}(\tilde{T}^2, \tilde{g}) \ge \frac{\pi}{\sqrt{\delta}}$.

In fact, the same proof applies to all closed orientable surfaces.

Proof. Assume $\mathrm{inj}(\tilde{T}^2, \tilde{g}) < \frac{\pi}{\sqrt{\delta}}$. Choose $\tilde{x}_0 \in \tilde{T}^2$ so that $\mathrm{inj}_{\tilde{g}}(\tilde{x}_0) = \mathrm{inj}(\tilde{T}^2, \tilde{g})$. By the curvature assumption, the conjugate radius is not smaller than $\frac{\pi}{\sqrt{\delta}}$. Thus there exists a periodic geodesic $\tilde{\gamma}$ through $\tilde{x}_0$ with $\ell(\tilde{\gamma}) = 2\mathrm{inj}(\tilde{T}^2, \tilde{g})$ (see Proposition 2.12 in Chapter 13 of [dC92]). It suffices to prove the following claim.

Claim. The projected geodesic $\gamma := \pi \circ \tilde{\gamma}$ has no self-intersections, that is, the restriction of the universal covering projection $\pi : \tilde{T}^2 \to T^2$ to $\tilde{\gamma}$ is injective.

We first show how this claim can be used to finish the proof, and then we prove the claim. As $\tilde{\gamma}$ is a closed curve in the universal cover, γ is null-homotopic in T^2. Moreover, by the claim γ has no self-intersections. Therefore by the Jordan curve theorem, there is a closed disc $\Omega \subseteq T^2$ with $\partial\Omega = \gamma$. Using the local Gauß-Bonnet theorem, the curvature bound, and the volume bound we get

$$2\pi = 2\pi\chi(\Omega) = \int_\Omega \sec\, d\mathrm{vol}_g \leq \delta \mathrm{vol}_g(\Omega) < \delta \mathrm{vol}_g(T^2) \leq \delta v,$$

which contradicts $\delta \leq \frac{2\pi}{v}$. For the application of the local Gauß-Bonnet theorem observe that $\partial\Omega$ is a geodesic without corners.

So it remains to prove the claim. Arguing by contradiction, we assume that the claim is wrong. Then there exists a non-trivial deck transformation $\varphi : \tilde{T}^2 \to \tilde{T}^2$ so that for $\tilde{\gamma}^\varphi := \varphi(\tilde{\gamma})$ it holds

$$\tilde{\gamma} \cap \tilde{\gamma}^\varphi \neq \varnothing.$$

Observe that $\tilde{\gamma}$ and $\tilde{\gamma}^\varphi$ intersect each other transversely. Indeed, otherwise we have $\varphi(\tilde{\gamma}) = \tilde{\gamma}^\varphi = \tilde{\gamma}$ by uniqueness of geodesics. But then all powers of φ fix the compact set $\tilde{\gamma}$, which contradicts that the Deck group is torsion-free and acts properly on $\tilde{T}^2$.

It is well known that the mod 2 intersection number $i_{\mathbb{Z}/2\mathbb{Z}}(\cdot,\cdot)$ is homotopy-invariant. Hence $i_{\mathbb{Z}/2\mathbb{Z}}(\tilde{\gamma}, \tilde{\gamma}^\varphi) = 0$ because $\tilde{\gamma}$ and $\tilde{\gamma}^\varphi$ are both null-homotopic as $\tilde{T}^2$ is simply connected. So $\tilde{\gamma}$ and $\tilde{\gamma}^\varphi$ intersect in an even number of points. By assumption their intersection is non-trivial, and hence they intersect in at least two distinct points.

Choose two distinct points $\bar{x}, \bar{y} \in \tilde{\gamma} \cap \tilde{\gamma}^\varphi$. Choose subarcs c of $\tilde{\gamma}$ and c^φ of $\tilde{\gamma}^\varphi$ from $\bar{x}$ to $\bar{y}$ in such a way that $\ell(c), \ell(c^\varphi) \leq \frac{1}{2}\ell(\tilde{\gamma}) = \frac{1}{2}\ell(\tilde{\gamma}^\varphi)$. If c and c^φ are both strictly shorter than $\frac{1}{2}\ell(\tilde{\gamma})$, then $\exp_{\bar{x}}$ is *not* injective on $B\big(0, \frac{1}{2}\ell(\tilde{\gamma})\big) \subseteq T_{\bar{x}}\tilde{T}^2$, and so $\mathrm{inj}(\bar{x}) < \frac{1}{2}\ell(\tilde{\gamma})$. But this contradicts $\ell(\tilde{\gamma}) = 2\mathrm{inj}(\tilde{T}^2, \tilde{g})$. The argument in the general case is similar. By a variational argument we construct two geodesics with the same endpoints that have strictly smaller length than c and c^φ. The same reasoning will then lead to a contradiction.

Since c and c^φ intersect transversely at $\bar{y}$, there exists $v \in T_{\bar{y}}\tilde{T}^2$ with $\langle v, c' \rangle < 0$ and $\langle v, (c^\varphi)' \rangle < 0$. Choose a curve $c_v : (-\varepsilon, \varepsilon) \to \tilde{T}^2$ with $c_v(0) = \bar{y}$ and $c_v'(0) = v$. Since $\bar{y}$ is not conjugate to $\bar{x}$ along c or c^φ, there exist variations through geodesics $\Gamma, \Gamma^\varphi : (-\varepsilon, \varepsilon) \times [0,1]$ of c and c^φ with

$$\Gamma(s,0) = \bar{x} = \Gamma^\varphi(s,0) \quad \text{and} \quad \Gamma(s,1) = c_v(s) = \Gamma^\varphi(s,1)$$

for all $s \in (-\varepsilon, \varepsilon)$ (here we reparametrize c and c^φ to be defined on $[0,1]$). As c is a geodesic, the first variation formula shows

$$\frac{d}{ds}\bigg|_{s=0} \ell(\Gamma(s,\cdot)) = \langle \partial_s\Gamma(0,1), \partial_t\Gamma(0,1) \rangle = \langle v, c' \rangle < 0.$$

Similarly, $\frac{d}{ds}\big|_{s=0} \ell(\Gamma^\varphi(s,\cdot)) < 0$. So $\ell(\Gamma(s_0,\cdot)) < \ell(c)$ and $\ell(\Gamma^\varphi(s_0,\cdot)) < \ell(c^\varphi)$ for $s_0 > 0$ sufficiently small. Then $\Gamma(s_0,\cdot)$ and $\Gamma^\varphi(s_0,\cdot)$ are geodesics with the same starting point $\bar{x}$ and the same endpoint $c_v(s_0)$, and both are strictly shorter than $\frac{1}{2}\ell(\tilde{\gamma})$. Therefore, $\exp_{\bar{x}}$ is not injective on $B\big(0, \frac{1}{2}\ell(\tilde{\gamma})\big) \subseteq T_{\bar{x}}\tilde{T}^2$, and hence $\mathrm{inj}(\bar{x}) < \frac{1}{2}\ell(\tilde{\gamma}) = \mathrm{inj}(\tilde{T}^2, \tilde{g})$ which is a contradiction. This finishes the proof of the claim. □

We remainder of this section is devoted to the proof of the effective Uniformization Theorem.

Proof of Lemma 8.5. **Step 1 (averaged L^2-estimate):** Inequality $(**)$ on p. 520 of [GKS07] states the following. If M is a closed Riemannian n-manifold of diameter D satisfying $\mathrm{Ric} \geq -(n-1)$, then the smallest positive eigenvalue λ_1 of the Laplace operator satisfies $\lambda_1 \geq e^{-2(n-1)D}$.

By definition, we have $\mathrm{diam}(T^2, g) \leq 1$, $|\sec(g)| \leq \delta \leq 1$ and $\int_{T^2} \rho \, d\mathrm{vol}_g = 0$. So ρ is perpendicular to the space of constant functions in $H^1(T^2)$. Thus we have the Poincaré-inequality

$$\int_{T^2} \rho^2 d\mathrm{vol}_g \leq C \int_{T^2} |\nabla \rho|^2 \, d\mathrm{vol}_g$$

for $C = e^2$. It follows from Exercise 2 in Chapter 4.3 of [dC16] that

$$\Delta \rho = 2K,$$

where K is the Gauß curvature of g and Δ is the Laplace operator of g. Recall that by our sign convention $\Delta = -\mathrm{tr}(\nabla^2)$. We denote the average integral $\frac{1}{\mathrm{vol}_g(T^2)} \int_{T^2} d\mathrm{vol}_g$ by $⨍_{T^2} d\mathrm{vol}_g$. Testing $\Delta\rho = 2K$ with ρ gives

$$\begin{aligned} ⨍_{T^2} |\nabla \rho|^2 d\mathrm{vol}_g =& 2 ⨍_{T^2} \rho K d\mathrm{vol}_g \\ \leq& 2\delta ⨍_{T^2} |\rho| d\mathrm{vol}_g \\ \leq& 2\delta \left(⨍_{T^2} |\rho|^2 d\mathrm{vol}_g \right)^{\frac{1}{2}} \\ \leq& 2\delta C^{\frac{1}{2}} \left(⨍_{T^2} |\nabla\rho|^2 d\mathrm{vol}_g \right)^{\frac{1}{2}}, \end{aligned}$$

where we used the curvature assumption, the Cauchy-Schwarz and the Poincaré inequality. Thus

$$\left(⨍_{T^2} |\rho|^2 d\mathrm{vol}_g \right)^{\frac{1}{2}} \leq C^{\frac{1}{2}} \left(⨍_{T^2} |\nabla\rho|^2 d\mathrm{vol}_g \right)^{\frac{1}{2}} \leq 2\delta C. \tag{8.11}$$

Step 2 (C^0-estimate): Lifting $\Delta\rho = 2K$ to the universal cover we get $\Delta\tilde{\rho} = 2\tilde{K}$, where $\tilde{\rho}$ and $\tilde{K}$ are the lifts of ρ and K to $\tilde{T}^2$. Curvature bounds and diameter bounds imply upper volume bounds by the Bishop-Gromov volume comparison theorem. Hence it follows from Lemma 8.6 that for δ_0 small enough it holds $\mathrm{inj}(\tilde{T}^2, \tilde{g}) \geq i_0$ for a universal constant $i_0 > 0$. So we can apply Lemma 2.8, and conclude that for all $\tilde{x}_0 \in \tilde{T}^2$ it holds

$$|\tilde{\rho}|(\tilde{x}_0) \leq C\Big(\|\tilde{\rho}\|_{L^2(B(\tilde{x}_0,1))} + \|\tilde{K}\|_{C^0(\tilde{T}^2)} \Big) \tag{8.12}$$

for a universal constant C. Choose a partition of $\tilde{T}^2$ into fundamental regions for the action of the fundamental group of T^2 whose diameters do not exceed $2\mathrm{diam}(T^2, g)$. For each $\tilde{x} \in \tilde{T}^2$ denote by $\mathcal{F}_{\tilde{x}}$ the element of the partition containing $\tilde{x}$. Fix $\tilde{x}_0 \in \tilde{T}^2$. Define

$$\hat{\mathcal{F}} := \bigcup_{\tilde{x} \in B(\tilde{x}_0, 1)} \mathcal{F}_{\tilde{x}}.$$

Observe that $\hat{\mathcal{F}}$ is a finite union of fundamental regions, and thus

$$\fint_{\hat{\mathcal{F}}} \tilde{\rho}^2 \, d\mathrm{vol}_{\tilde{g}} = \fint_{T^2} \rho^2 \, d\mathrm{vol}_g, \tag{8.13}$$

where $\fint_U$ denotes the averaged integral $\frac{1}{\mathrm{vol}(U)} \int_U$. Since $\mathrm{diam}(\mathcal{F}_{\tilde{x}}) \leq 2\mathrm{diam}(T^2, g) \leq 2$ we also have $B(\tilde{x}_0, 1) \subseteq \hat{\mathcal{F}} \subseteq B(\tilde{x}_0, 3)$. From the lower bound i_0 for the injectivity radius of $\tilde{T}$ we deduce that $\mathrm{vol}(\hat{\mathcal{F}})$ is bounded from below and from above by a universal constant. Remembering $B(\tilde{x}_0, 1) \subseteq \hat{\mathcal{F}}$ this implies

$$\|\tilde{\rho}\|^2_{L^2(B(\tilde{x}_0,1))} \leq \|\tilde{\rho}\|^2_{L^2(\hat{\mathcal{F}})} = O\left(\fint_{\hat{\mathcal{F}}} \tilde{\rho}^2 \, d\mathrm{vol}_{\tilde{g}} \right). \tag{8.14}$$

Combining (8.11)-(8.14) and using the curvature assumption $|K| \leq \delta$ yields

$$|\tilde{\rho}|(\tilde{x}_0) \leq C\delta$$

for a universal constant C. This finishes the proof. □

9. Invertibility of $\mathcal{L}$ without a lower injectivity radius bound

9.1. **Statement and overview.** For the proof of Theorem 2 we need to analyze the invertibility of the elliptic operator $\mathcal{L} = \frac{1}{2}\Delta_L + 2\,\mathrm{id}$ in complete Riemannian 3-manifolds of finite volume that do *not* have a positive lower bound on the injectivity radius. The examples of Section 6 show that Proposition 4.3 can no longer hold in this more general situation. Also recall from the discussion at the end of Section 6 that the counterexamples were constructed by slowly changing the conformal structure of the level tori $T(r)$ contained in M_{small}. To exclude these examples, we shall use the geometric control of tubes and cusps established in Section 8 for manifolds M whose sectional curvature approaches constant curvature -1 exponentially fast in M_{small}.

We introduce new norms $\|\cdot\|_{2,\lambda;*}$ and $\|\cdot\|_{0,\lambda}$ for smooth sections of the bundle $\mathrm{Sym}^2(T^*M) = \mathrm{Sym}(T^*M \otimes T^*M)$ which are inspired by the work of Bamler [Bam12], and we use these norms to prove the following invertibility result. At this point we only mention that these norms depend on certain parameters $\alpha, \lambda, \delta, r_0, b, \bar{\epsilon}$. Recall that R denotes the Riemann curvature endomorphism.

Proposition 9.1. *For all $\alpha \in (0,1)$, $\Lambda \geq 0$, $\lambda \in (0,1)$, $\delta \in (0,2)$, $r_0 \geq 1$, $b > 1$ and $\eta \geq 2 + \lambda$ there exist constants ε_0, $\bar{\epsilon}_0$ and $C > 0$ with the following property. Let M be a Riemannian 3-manifold of finite volume that satisfies*

$$|\sec + 1| \leq \varepsilon_0, \quad \|\nabla \mathrm{Ric}\|_{C^0(M)} \leq \Lambda,$$

and

$$\max_{\pi \subseteq T_xM} |\sec(\pi) + 1|, \, |\nabla R|(x), \, |\nabla^2 R|(x) \leq \varepsilon_0 e^{-\eta d(x, \partial M_{\mathrm{small}})} \quad \textit{for all } x \in M_{\mathrm{small}}. \tag{9.1}$$

Then the operator

$$\mathcal{L} : \left(C^{2,\alpha}_{\lambda}(\mathrm{Sym}^2(T^*M)), \|\cdot\|_{2,\lambda;*} \right) \longrightarrow \left(C^{0,\alpha}_{\lambda}(\mathrm{Sym}^2(T^*M)), \|\cdot\|_{0,\lambda} \right)$$

is invertible and

$$\|\mathcal{L}\|_{\mathrm{op}}, \|\mathcal{L}^{-1}\|_{\mathrm{op}} \leq C,$$

where $\|\cdot\|_{2,\lambda;*}$ *and* $\|\cdot\|_{0,\lambda}$ *are the norms defined in (9.7) and (9.8) with respect to any* $\bar{\epsilon} \leq \bar{\epsilon}_0$*, and* $C^{k,\alpha}_{\lambda}(\mathrm{Sym}^2(T^*M))$ *is the corresponding Banach space of sections of* $\mathrm{Sym}^2(T^*M)$*.*

The proof of the existence of ε_0 and $\bar{\epsilon}_0$ is constructive. This is not the case for the constant C which involves an argument by contradiction.

Remark 9.2. Proposition 9.1 is also valid for all $\lambda \in (0,1)$ and $\eta > 1$. The proof we found is technically more involved, but it does not require any new insights. As we are not aware of additional applications, we restrict ourselves to the case $\eta \geq 2+\lambda$ and postpone the presentation of the stronger statement to forthcoming work.

Analogous to Remark 4.8 the following holds (also see Remark 7.3).

Remark 9.3. Proposition 9.1 also holds when M is non-orientable.

This section is structured as follows. The definition of the norms which appear in the statement of Proposition 9.1 is presented in Section 9.2. In Section 9.3 we prove that solutions h of the equation $\mathcal{L}h = f$ satisfy certain growth estimates in M_{small}. These will be used in Section 9.4 to show that $\mathcal{L}$ satisfies an a priori estimate $\|h\|_{2,\lambda;*} \leq C\|\mathcal{L}h\|_{0,\lambda}$. Finally, the surjectivity of $\mathcal{L}$ will be established in Section 9.5.

9.2. **Various norms.** In this subsection we present the definition of the norms $\|\cdot\|_{2,\lambda;*}$ and $\|\cdot\|_{0,\lambda}$ appearing in Proposition 9.1, and state some of their properties. To this end we first define *exponential norms* $\|\cdot\|_{C^?_\lambda}$. Second, adapting a construction in [Bam12], we present *decomposition norms* $\|\cdot\|_{C^?_\lambda;*}$. The *exponential hybrid norms* $\|\cdot\|_{2,\lambda;*}$ and $\|\cdot\|_{0,\lambda}$ are then a combination of the decomposition norms $\|\cdot\|_{C^?_\lambda;*}$ and the hybrid norms $\|\cdot\|_2$ and $\|\cdot\|_0$ defined in Section 4.1.

Throughout this section we assume that M satisfies the assumptions from Proposition 9.1, that is, M is a complete Riemannian 3-manifold of finite volume satisfying $|\sec+1| \leq \varepsilon_0$, $\|\nabla \mathrm{Ric}\|_{C^0(M)} \leq \Lambda$, and

$$\max_{\pi\subseteq T_xM} |\sec(\pi)+1|,\ |\nabla R|(x),\ |\nabla^2 R|(x) \leq \varepsilon_0 e^{-\eta d(x,\partial M_{\mathrm{small}})} \quad \text{for all } x \in M_{\mathrm{small}}$$

for some $\eta \geq 2+\lambda$. As before, $\mathcal{L}$ denotes the elliptic operator $\frac{1}{2}\Delta_L + 2\mathrm{id}$.

We begin with the definition of the *exponential norms.* Let $C_1,\dots,C_q$ be the cusps of M, and let $T_1,\dots,T_p$ be the Margulis tubes of M of radius at least 3. For any $k=1,\dots,p$ let R_k be the radius of $(T_k)_{\mathrm{small}}$, that is, the distance of the core geodesic γ_k to $\partial(T_k)_{\mathrm{small}}$. Denote by

$$r(x) \coloneqq d(x, M \smallsetminus M_{\mathrm{small}})$$

the distance of x to the complement of M_{small}. For $\lambda \in (0,1)$ define the *inverse weight function* $W_\lambda : M \to \mathbb{R}$ by

$$W_\lambda(x) \coloneqq \begin{cases} e^{-\lambda r(x)} & \text{if } x \in \bigcup_{i=1}^q (C_i)_{\mathrm{small}} \\ e^{-\lambda r(x)} + e^{\lambda(r(x)-R_k)} & \text{if } x \in (T_k)_{\mathrm{small}} \\ 1 & \text{otherwise} \end{cases}.$$

For any C^k or $C^{k,\alpha}$ norm $\|\cdot\|_{C^?}$ we define the corresponding *exponential* $C^?$ *norm* by

$$\|\cdot\|_{C^?_\lambda(M)} := \sup_{x\in M}\left(\frac{1}{W_\lambda(x)}|\cdot|_{C^?}(x)\right). \tag{9.2}$$

Observe that $\mathrm{Lip}\left(\log\left(\frac{1}{W_\lambda}\right)\right) \leq \lambda$ in M_{small}, and that outside M_{small} it holds $W_\lambda(x) = 1$. So the pointwise Schauder estimates (2.4) and (2.5) imply that there are Schauder estimates for the exponential norms, that is, there is a constant $C = C(\alpha,\Lambda,\lambda)$ so that

$$\|h\|_{C^{2,\alpha}_\lambda(M)} \leq C\Big(\|\mathcal{L}h\|_{C^{0,\alpha}_\lambda(M)} + \|h\|_{C^0_\lambda(M)}\Big) \tag{9.3}$$

and

$$\|h\|_{C^{1,\alpha}_\lambda(M)} \leq C\Big(\|\mathcal{L}h\|_{C^0_\lambda(M)} + \|h\|_{C^0_\lambda(M)}\Big). \tag{9.4}$$

Similarly, it follows from (2.6) that $\|\mathcal{L}h\|_{C^{0,\alpha}_\lambda(M)} \leq C\|h\|_{C^{2,\alpha}_\lambda(M)}$.

We continue with the definition of the *decomposition norms*. Following Section 3.2 of [Bam12], we first introduce *trivial Einstein variations*. In Definition 9.4 below, T^2 denotes a flat torus (with a fixed flat metric), and $I \subseteq \mathbb{R}$ is an interval. Even though T^2 is equipped with a metric, the product $T^2 \times I$ is only meant as a topological product. Moreover, in the definition we take coordinates (x^1, x^2, r) on $T^2 \times I$, where (x^1, x^2) are flat coordinates for T^2, and r is the standard coordinate for $I \subseteq \mathbb{R}$.

Definition 9.4. A $(0,2)$-tensor u on $T^2 \times I$ is called an *Einstein variation* if it is of the form

$$u = e^{-2r} u_{ij}\, dx^i dx^j$$

for some constants $u_{ij} \in \mathbb{R}$. Moreover, u is called a *trivial Einstein variation* if the trace of u with respect to the flat metric on T^2 vanishes everywhere, that is, if $\sum_i u_{ii} = 0$.

Here dx^i and dx^j are understood to be either dx^1 or dx^2, but not dr. Our definition looks slightly different than that in [Bam12], but this difference is only due to a change of coordinates.

Remark 9.5. The trace free condition guarantees that $\mathcal{L}_{cusp}u = 0$ for a trivial Einstein variation u (see (9.14)). Here $\mathcal{L}_{cusp}$ denotes the operator $\frac{1}{2}\Delta_L + 2\mathrm{id}$ with respect to the hyperbolic cusp metric $g_{cusp} = e^{-2r}g_{Flat} + dr^2$, where g_{Flat} is the given flat metric on T^2.

An Einstein variation should be thought of as an infinitesimal change in the conformal structure of the torus T^2, and a trivial Einstein variation is an infinitesimal change of the conformal structure that can *not* be detected by the operator $\mathcal{L}$. Therefore, to ensure that $\mathcal{L}$ is invertible, we have to work with a norm in the source space that isolates trivial Einstein variations. Also recall from the discussion at the end of Section 6, that the counterexamples of Proposition 6.1 are constructed by changing the conformal structure of the horotori that are contained in the small part of the manifold. This further justifies the use of a norm that is sensitive to changes in the conformal structures. A more precise explanation for the necessity of a norm that isolates trivial Einstein variations will be given in Remark 9.13.

For each $k = 1, ..., p$ let γ_k be the core geodesic of T_k. Choose cutoff functions ρ_k so that

$$\rho_k = 0 \text{ in } \big(M \smallsetminus (T_k)_{\text{small}}\big) \cup N_1(\gamma_k) \quad \text{and} \quad \rho_k = 1 \text{ in } N_{R_k - 1/4}(\gamma_k) \smallsetminus N_{5/4}(\gamma_k).$$

Similarly, choose cutoff functions ϱ_l so that

$$\varrho_l = 0 \text{ in } M \smallsetminus (C_l)_{\text{small}} \quad \text{and} \quad \varrho_l = 1 \text{ outside } N_1\big(M \smallsetminus (C_l)_{\text{small}}\big).$$

Here for any subset $X \subseteq M$, and any $r > 0$, $N_r(X)$ denotes the set of all points that have distance less than r to X. The cutoff functions are chosen in such a way that the Hölder norms $\|\rho_k\|_{C^{2,\alpha}}, \|\varrho_l\|_{C^{2,\alpha}}$ are bounded from above by a universal constant. Recall that by Lemma 7.4 the boundary $\partial(T_k)_{\text{small}}$ is a smooth torus, and $(T_k)_{\text{small}} \smallsetminus N_1(\gamma_k)$ is diffeomorphic to $\partial(T_k)_{\text{small}} \times [0, R_k - 1]$. By Proposition 8.1, there is a natural choice for a flat metric on $\partial(T_k)_{\text{small}}$, namely the flat metric induced by the model metric g_{cusp} of Proposition 8.1. Hence it makes sense to speak of trivial Einstein variations on $(T_k)_{\text{small}} \smallsetminus N_1(\gamma_k)$. Similarly, it makes sense to speak of trivial Einstein variations on $(C_l)_{\text{small}}$.

For a continuous symmetric $(0,2)$-tensor field h, consider decompositions

$$h = \bar{h} + \sum_{k=1}^{p} \rho_k u_k + \sum_{l=1}^{q} \varrho_l v_l, \tag{9.5}$$

where u_k is a trivial Einstein variation in $(T_k)_{\text{small}} \smallsetminus N_1(\gamma_k)$, and v_l is a trivial Einstein variation on $(C_l)_{\text{small}}$. Following p. 896 of [Bam12], for any C^k or $C^{k,\alpha}$ norm $\|\cdot\|_{C^?}$ define the corresponding *decomposition norm* by

$$\|h\|_{C^?_\lambda(M);*} := \inf\Big(\|\bar{h}\|_{C^?_\lambda(M)} + \max_{k=1,\dots,p} |u_k| + \max_{l=1,\dots,q} |v_l|\Big), \tag{9.6}$$

where the infimum is taken over all decompositions as in (9.5) and $|\cdot|$ is a norm on the finite dimensional space of trivial Einstein variations, e.g., $|\cdot| := \|\cdot\|_{C^0}$. We point out that our notation differs from that in [Bam12].

We now state some properties of the decomposition norm. We start with a basic inequality.

Lemma 9.6. It holds

$$\|h\|_{C^0} \le 2\|h\|_{C^0_\lambda;*}.$$

Proof. Note $W_\lambda \le 2$, so that $\|\bar{h}\|_{C^0} \le 2\|\bar{h}\|_{C^0_\lambda}$. Now the desired inequality follows from the triangle inequality and the definition of $\|\cdot\|_{C^0_\lambda;*}$. □

Schauder estimates also hold for the decomposition norm (see [Bam12, Lemma 4.1]).

Lemma 9.7. *There is a universal constant C so that*

$$\|h\|_{C^{2,\alpha}_\lambda;*} \le C\Big(\|\mathcal{L}h\|_{C^{0,\alpha}_\lambda} + \|h\|_{C^0_\lambda;*}\Big)$$

for all $h \in C^{2,\alpha}\big(\mathrm{Sym}^2(T^*M)\big)$.

Proof. In this proof we use Notation 2.2. Let $\mathcal{L}_{cusp}$ denote the differential operator $\frac{1}{2}\Delta_L+2\mathrm{id}$ with respect to the model metrics g_{cusp} given by Proposition 8.1 and Proposition 8.2. By Remark 9.5, it holds $\mathcal{L}_{cusp}u_k = 0$ and $\mathcal{L}_{cusp}v_l = 0$. Hence $\|u_k\|_{C^{2,\alpha}} = O(|u_k|)$ and $\|v_l\|_{C^{2,\alpha}} = O(|v_l|)$ due to Schauder estimates.

The estimates on $|g-g_{cusp}|_{C^2}$ given by Proposition 8.2 and Proposition 8.1 yield that $|\mathcal{L}v_l|_{C^{0,\alpha}}(x) = O\big(\varepsilon_0 e^{-\eta r(x)}|v_l|\big)$ and $|\mathcal{L}u_k|_{C^{0,\alpha}}(x) = O\Big(\big(e^{2(r(x)-R_k)}+\varepsilon_0 e^{-\eta r(x)}\big)|u_k|\Big)$, where η is the decay rate in the curvature decay condition (9.1), and $r(x) = d(x, M\smallsetminus M_{\mathrm{small}})$. We then get $\|\mathcal{L}u_k\|_{C^{0,\alpha}_\lambda} = O(|u_k|)$ and $\|\mathcal{L}v_l\|_{C^{0,\alpha}_\lambda} = O(|v_l|)$ since $\lambda<1<\eta$. The rest of the argument carries over from [Bam12, Lemma 4.1] □

The next result and its proof is analogous to Lemma 4.2 in [Bam12]. It gives the canonical choice of a trivial Einstein variation in a Margulis tube. For rank 2 cusps the canonical choice of a trivial Einstein variation will be given by Proposition 9.10.

Lemma 9.8. *Let $h\in C^0\big(\mathrm{Sym}^2(T^*M)\big)$. Choose points $c_k\in T_k$ with $r(c_k)=\frac{R_k}{2}$, where $r=d(\cdot,M\smallsetminus M_{\mathrm{small}})$. For each k let u_k be the trivial Einstein variation in $(T_k)_{\mathrm{small}}\smallsetminus N_1(\gamma_k)$ such that $|h-u_k|(c_k)$ is minimal among all trivial Einstein variations in $(T_k)_{\mathrm{small}}\smallsetminus N_1(\gamma_k)$. Then for some universal constant C it holds*

$$\|h\|_{C^0_\lambda(M');*} \le \|\bar h\|_{C^0_\lambda(M')} + \max_k |u_k| \le C\|h\|_{C^0_\lambda(M');*},$$

where $\bar h := h-\sum_k \rho_k u_k$ and $M' = M\smallsetminus\bigcup_{l=1}^q (C_l)_{\mathrm{small}}$.

Note that on $(T_k)_{\mathrm{small}}$, the weight $\frac{1}{W_\lambda(r)}$ is maximal at $r=\frac{R_k}{2}$.

Proof. The proof of [Bam12, Lemma 4.2] goes through without modification. For later purpose we point out that $|u_k|(c_k)\le|h|(c_k)$, and $|\bar h|(c_k)\le|\bar h-u|(c_k)$ for any trivial Einstein variation u on $(T_k)_{\mathrm{small}}\smallsetminus N_1(\gamma_k)$. This is because by its definition, u_k is the image of the orthogonal projection from $\mathrm{Sym}^2(T^*_{c_k}M)$ to the space of trivial Einstein variations on $(T_k)_{\mathrm{small}}\smallsetminus N_1(\gamma_k)$. Also note that trivial Einstein variations have constant norm (if the norm is taken with respect to the cusp metric g_{cusp}). In particular, we have $|u_k|\le\|h\|_{C^0(M)}$. □

Finally, we come to the definition of the *exponential hybrid norms* $\|\cdot\|_{2,\lambda;*}$ and $\|\cdot\|_{0,\lambda}$ appearing in Proposition 9.1. Recall that for $k=0$ and $k=2$, in Definition 4.1 we defined the hybrid norms (when $n=3$) by

$$\|\cdot\|_k := \max\left\{\|\cdot\|_{C^{k,\alpha}(M)}, \sup_{x\notin E}\left(\int_M e^{-(2-\delta)r_x(y)}|\cdot|^2_{C^k}(y)\,d\mathrm{vol}(y)\right)^{\frac{1}{2}}\right\},$$

where $E\subseteq M$ is a subset defined by a volume growth condition and $|\cdot|_{C^k}(y)$ denotes the C^k-norm at the point y. We refer to Section 4.1 for more details. For ease of notation we abbreviate

$$\|h\|_{H^2(M;\omega_x)} := \left(\int_M e^{-(2-\delta)r_x(y)}|h|^2_{C^2}(y)\,d\mathrm{vol}(y)\right)^{\frac{1}{2}}$$

and

$$\|f\|_{L^2(M;\omega_x)} := \left(\int_M e^{-(2-\delta) r_x(y)} |f|^2_{C^0}(y)\, d\mathrm{vol}(y) \right)^{\frac{1}{2}}.$$

Here ω_x should indicate that there is a weight function involved that depends on $x \in M$.

Definition 9.9. For $\alpha \in (0,1)$, $\lambda \in (0,1)$, $b > 1$, $\bar{\epsilon} > 0$, $\delta \in (0,2)$ and $r_0 \geq 1$ the *exponential hybrid norms* $\|\cdot\|_{2,\lambda;*}$ and $\|\cdot\|_{0,\lambda}$ are defined by

$$\|h\|_{2,\lambda;*} := \max\left\{ \|h\|_{C^{2,\alpha}_\lambda(M);*}, \sup_{x \notin E} \|h\|_{H^2(M;\omega_x)}, \sup_{x \in M_{\mathrm{thin}} \smallsetminus M_{\mathrm{small}}} e^{\frac{b}{2} d(x, M_{\mathrm{thick}})} \|h\|_{H^2(M;\omega_x)} \right\} \tag{9.7}$$

and

$$\|f\|_{0,\lambda} := \max\left\{ \|f\|_{C^{0,\alpha}_\lambda(M)}, \sup_{x \notin E} \|f\|_{L^2(M;\omega_x)}, \sup_{x \in M_{\mathrm{thin}} \smallsetminus M_{\mathrm{small}}} e^{\frac{b}{2} d(x, M_{\mathrm{thick}})} \|f\|_{L^2(M;\omega_x)} \right\}, \tag{9.8}$$

where $E = E(M; \bar{\epsilon}, \delta, r_0)$ is the set defined in (4.1).

In the source space we use $\|\cdot\|_{C^{2,\alpha}_\lambda(M);*}$ instead of just $\|\cdot\|_{C^{2,\alpha}_\lambda(M)}$ so that the norm is sensitive to trivial Einstein variations. We refer to Remark 9.13 and the discussion after Remark 9.5 as to why this is necessary. The last integral terms in the definition of the norms are included so that we can employ the C^0-estimate from Proposition 7.5. As was the case with the previously defined hybrid norms $\|\cdot\|_2$ and $\|\cdot\|_0$, we suppress most of the constants in the notation for the norms $\|\cdot\|_{2,\lambda;*}$ and $\|\cdot\|_{0,\lambda}$.

Define the spaces $C^{0,\alpha}_\lambda(\mathrm{Sym}^2(T^*M))$ and $C^{2,\alpha}_\lambda(\mathrm{Sym}^2(T^*M))$ as

$$C^{0,\alpha}_\lambda(\mathrm{Sym}^2(T^*M)) := \left\{ f \in C^{0,\alpha}(\mathrm{Sym}^2(T^*M)) \;\middle|\; \|f\|_{C^{0,\alpha}_\lambda(M)} < \infty \right\} \tag{9.9}$$

and

$$C^{2,\alpha}_\lambda(\mathrm{Sym}^2(T^*M)) := \left\{ h \in C^{2,\alpha}(\mathrm{Sym}^2(T^*M)) \;\middle|\; \|h\|_{C^{2,\alpha}_\lambda(M);*} < \infty \right\}. \tag{9.10}$$

The latter is the space of sections h with the property that $\mathcal{L}h \in C^{0,\alpha}_\lambda(\mathrm{Sym}^2(T^*M))$ (see Lemma 9.21).

9.3. **Growth estimates.** In Section 9.4 we shall prove the a priori estimate of Proposition 9.1. An intermediate step towards this goal is Proposition 9.18, in which we prove a global C^0-estimate $\|h\|_{C^0(M)} \leq C\|\mathcal{L}h\|_{0,\lambda}$. For points in M_{thick} we can use the arguments from the proof of Proposition 4.3 to obtain such an estimate (see Remark 4.6). Moreover, Proposition 7.5 provides the desired estimate for points in $M_{\mathrm{thin}} \smallsetminus M_{\mathrm{small}}$. Therefore, it remains to obtain C^0-estimates in M_{small}. The main ingredient to obtain C^0-estimates in M_{small} are certain *growth estimates* that solutions of $\mathcal{L}h = f$ satisfy. These estimates are contained in the following Proposition 9.10 and Proposition 9.11 which are the main results of this section.

We begin with the growth estimate in a cusp. Besides the growth estimate, this result also states that there is a canonical choice of trivial Einstein variation inside a cusp (for tubes the canonical choice of trivial Einstein variation was given by Lemma 9.8).

Proposition 9.10 (Growth estimate in a cusp). *For all $\alpha \in (0,1)$, $\Lambda \geq 0$, $\lambda \in (0,1)$, $b > 1$, $\delta \in (0,2)$, and $\eta \geq 2 + \lambda$ there exists $\varepsilon_0 > 0$ with the following property.*

Let M be a finite volume 3-manifold that satisfies

$$|\sec + 1| \leq \varepsilon_0, \quad \|\mathrm{Ric}(g)\|_{C^0(M)} \leq \Lambda,$$

and

$$\max_{\pi \subseteq T_x M} |\sec(\pi) + 1|, |\nabla R|(x), |\nabla^2 R|(x) \leq \varepsilon_0 e^{-\eta d(x, \partial M_{\mathrm{small}})} \quad \text{for all } x \in M_{\mathrm{small}}.$$

Let $f \in C^{0,\alpha}_{\lambda}\big(\mathrm{Sym}^2(T^*M)\big)$, and let $h \in C^2\big(\mathrm{Sym}^2(T^*M)\big)$ with $\|h\|_{C^0(M)} < \infty$ be a solution of

$$\mathcal{L}h = f.$$

Fix a cusp C of M. Then there exists a unique trivial Einstein variation v in C_{small} satisfying

$$\|h - v\|_{C^0_\lambda(C_{\mathrm{small}})} < \infty,$$

and we have

$$|v| = O(\|f\|_{0,\lambda}).$$

Moreover, if $\|h\|_{C^0(M)}, \|f\|_{C^{0,\alpha}(M)} \leq 1$, then for all $x \in C_{\mathrm{small}}$ it holds

$$|h|(x) = O\Big(\|f\|_{0,\lambda} + e^{-r(x)}\Big) \tag{9.11}$$

and

$$e^{\lambda r(x)}|h - v|(x) = O\Big(\|f\|_{0,\lambda} + e^{-(1-\lambda) r(x)}\Big), \tag{9.12}$$

where $r(x) = d(x, \partial C_{\mathrm{small}})$.

We refer to Notation 2.2 for our convention of the O-notation. The component $\sup_{x \notin E} \|f\|_{L^2(M;\omega_x)}$ of the norm $\|f\|_{0,\lambda}$ is *not* needed for this estimate. For this reason we don't have to include constants $\bar{\epsilon} > 0$ and $r_0 \geq 1$ in the formulation of Proposition 9.10 as these only enter the definition of the set E.

The estimate in a tube is very similar, but it additionally involves the distance to the core geodesic.

Proposition 9.11 (Growth estimate in a tube). *Let all the constants and the manifold M be as in Proposition 9.10. Let $f \in C^{0,\alpha}_{\lambda}\big(\mathrm{Sym}^2(T^*M)\big)$ with $\|f\|_{0,\lambda} \leq 1$, and let $h \in C^{2,\alpha}\big(\mathrm{Sym}^2(T^*M)\big)$ with $\|h\|_{C^0(M)} \leq 1$ be a solution of*

$$\mathcal{L}h = f.$$

Fix a Margulis tube T of M, and denote its core geodesic by γ. For all $x \in T_{\mathrm{small}} \smallsetminus N_1(\gamma)$ it then holds

$$|h|(x) = O\left(\|f\|_{0,\lambda} + e^{-r_{\partial T}(x)} + e^{-\frac{3}{2} r_\gamma(x)}\right), \tag{9.13}$$

where $r_{\partial T}(x) = d(x, \partial T_{\mathrm{small}})$, and $r_\gamma(x) = d(x, \gamma)$.

The main idea to obtain these growth estimates is as follows. On M_{small} consider the model metric g_{cusp} obtained in Section 8. Following [Bam12, p. 901], we define with respect to this model metric an *averaging operator* that assigns to each tensor h another tensor $\hat{h}$ that only depends on $r = d(\cdot, M \smallsetminus M_{\mathrm{small}})$ (we will say in a moment what this means exactly). This averaging operator commutes with the differential operator $\mathcal{L}$ (if $\mathcal{L}$ is taken with respect to the model metric). So $\mathcal{L}\hat{h} = \hat{f}$. A key point is that as $\hat{h}$ and $\hat{f}$ both only depend on r, the equation $\mathcal{L}\hat{h} = \hat{f}$ actually just is a linear system of ODEs with constant coefficients whose fundamental solutions can be written down explicitly. Using standard ODE arguments we thus obtain growth estimates for $\hat{h}$. These will yield growth estimates for h since $|h - \hat{h}|(x)$ decays exponentially in $r(x)$.

We now explain these ideas in more detail. We start with some terminology. Recall from Section 8 that we call a metric g on $T^2 \times I$ (where I is an interval) a *cusp metric* if it is of the form

$$g = e^{-2r} g_{Flat} + dr^2,$$

where g_{Flat} is some flat metric on T^2, and r is the standard coordinate on $I \subseteq \mathbb{R}$. We call a covering $\varphi : \mathbb{R}^2 \times I \to T^2 \times I$ *cusp coordinates* if it is of the form $\varphi(x_1, x_2, r) = (\psi(x_1, x_2), r)$ for some local isometry $\psi : \mathbb{R}^2 \to (T^2, g_{Flat})$. We say that a tensor h on $T^2 \times I$ *only depends on* r if its coefficients h_{ij} in cusp coordinates only depend on r. This can also be stated without reference to local coordinates as follows. Note that there is an isometric $\mathbb{R}^2$-action on $(T^2 \times I, g)$ preserving the level tori $\{r = \mathrm{const}\}$. Then h only depends on r if it is invariant under this isometric $\mathbb{R}^2$-action.

Next we explain the averaging operator that assigns to each tensor h on $T^2 \times I$ another tensor $\hat{h}$ that only depends on r. The average $\hat{u}$ of a function $u : T^2 \times I \to \mathbb{R}$ is

$$\hat{u}(x) := \frac{1}{\mathrm{vol}_2(T(r))} \int_{T(r)} u(y) \, d\mathrm{vol}_2(y),$$

where $r = r(x)$, and $T(r) := T^2 \times \{r\}$. For a $(0,2)$-tensor h we define $\hat{h}$ componentwise, that is,

$$(\hat{h})_{ij}(x) := \widehat{h_{ij}}(x) = \frac{1}{\mathrm{vol}_2(T(r))} \int_{T(r)} h_{ij}(y) \, d\mathrm{vol}_2(y),$$

where the coefficients are with respect to cusp coordinates. It is clear that this definition is independent of the choice of cusp coordinates. The averaging for tensors of any type is defined in exactly the same way. We collect the properties of this averaging operation in the following lemma.

Lemma 9.12. *Let $T^2 \times I$ be equipped with a cusp metric. The averaging operation $\hat{\cdot}$ has the following properties:*

i) $\hat{h}$ only depends on r;
ii) There is a universal constant $c > 0$ so that

$$|\hat{h}|(x) \le c \frac{1}{\mathrm{vol}_2(T(r))} \int_{T(r)} |h|(y) \, d\mathrm{vol}_2(y).$$

In particular, $|\hat{h}|(x) \le c \max_{T(r(x))} |h|$ for a universal constant c;
iii) If h is of class C^1, then the same holds true for $\hat{h}$, and $\widehat{\nabla h} = \nabla \hat{h}$;

iv) $\hat{\cdot}$ *commutes with taking the trace, that is,* $\mathrm{tr}(\hat{h}) = \widehat{\mathrm{tr}(h)}$*;*
v) If h *is* C^1*, then*

$$|h - \hat{h}|(x) \leq cDe^{-r(x)} \max_{T(r(x))} |h|_{C^1},$$

where $D := \mathrm{diam}(T^2, g_{Flat})$ *and* c *is a universal constant.*

Property $v)$ is what allows us to deduce estimates on $|h|$ from estimates on $|\hat{h}|$.

Proof. These properties are all straighforward to check. The main fact to observe is that in cusp coordinates the g_{ij} (and hence the Christoffel symbols) only depend on r, and so one can take $g_{ij}(y)$ inside the integral $\int_{T(r)}$ out of the integral. □

Another crucial point is that if h and f are $(0,2)$-tensors that only depend on r, then the equation $\mathcal{L}h = f$ is a linear system of ODEs with constant coefficients (here $\mathcal{L}$ is taken with respect to the cusp metric). Namely, it holds

$$\begin{aligned}
-2(\mathcal{L}h)(\partial_3, \partial_3) &= (h_{33})'' - 2(h_{33})' - 4h_{33};\\
-2(\mathcal{L}h)(\partial_i, \partial_3) &= (h_{i3})'' - 4h_{i3};\\
-2(\mathcal{L}h)(\partial_i, \partial_j) &= (h_{ij})'' + 2(h_{ij})' - 2\delta_{ij} \sum_{k=1}^{2} h_{kk},
\end{aligned}$$

where $(\cdot)'$ denotes $\frac{d}{dr}$ and $\partial_3 = \frac{\partial}{\partial r}$. This can be checked by a straightforward calculation. Note that $|h|^2 = (h_{33})^2 + 2\sum_{i=1}^{2}(e^r h_{i3})^2 + \sum_{i,j=1}^{2}(e^{2r}h_{ij})^2$. Thus we are interested in equations for $h_{33}, e^r h_{i3}, e^{2r}h_{ij}$, rather than for h_{33}, h_{i3}, h_{ij}. Using the above, it is straightforward to check that if h, f only depend on r, then the equation $\mathcal{L}h = f$ is equivalent to

$$\begin{cases}
(h_{33})'' & - 2(h_{33})' & - 4h_{33} & = -2f_{33}\\
(e^r h_{i3})'' & - 2(e^r h_{i3})' & - 3e^r h_{i3} & = -2e^r f_{i3}\\
(e^{2r}h_{ij})'' & - 2(e^{2r}h_{ij})' & & = -2e^{2r} f_{ij} + 2\delta_{ij}(\mathrm{tr}(h) - h_{33}).
\end{cases} \tag{9.14}$$

The set of roots of the polynomials associated to these ODEs are $\{1-\sqrt{5}, 1+\sqrt{5}\}$, $\{-1, 3\}$ and $\{0, 2\}$. The exact form of this linear system of ODEs is *not* important. All what matters is that it is *some* system of ODEs whose fundamental solutions we can write down explicitly. Moreover, tracing the equation $\mathcal{L}h = f$ yields $\frac{1}{2}\Delta\mathrm{tr}(h) + 2\mathrm{tr}(h) = \mathrm{tr}(f)$. For a function u that only depends on r it holds $-\Delta u = u'' - 2u'$. Thus

$$\mathrm{tr}(h)'' - 2\mathrm{tr}(h)' - 4\mathrm{tr}(h) = -2\mathrm{tr}(f). \tag{9.15}$$

The roots of the polynomial $Q(X) = X^2 - 2X - 4$ associated to this ODE are $1 \pm \sqrt{5}$.

At this point we make another important comment as to why we work with the decomposition norm $\|\cdot\|_{C^{2,\alpha}_{\lambda};*}$ instead of just the exponential norm $\|\cdot\|_{C^{2,\alpha}_{\lambda}}$ (see Section 9.2 for the definition of these norms).

Remark 9.13. As mentioned previously, the counterexamples of Section 6 show that working with the hybrid norms of Section 4.1 will no longer be sufficient in the absence of a positive lower bound on the injectivity radius. A natural condition to rule out the counterexamples of Section 6 is to require that the sectional curvatures converge to -1

exponentially fast inside M_{small}, and thus it is natural to work with weighted Hölder norms. However, if we used $\|\cdot\|_{C^{2,\alpha}_\lambda}$ instead of $\|\cdot\|_{C^{2,\alpha}_\lambda;*}$ for definining the hybrid norm in the source space $C^{2,\alpha}_\lambda(\mathrm{Sym}^2(T^*M))$, then the operator $\mathcal{L} : C^{2,\alpha}_\lambda(\mathrm{Sym}^2(T^*M)) \to C^{0,\alpha}_\lambda(\mathrm{Sym}^2(T^*M))$ would *not* be invertible (with universal constants). This is because the system (9.14) has some constant fundamental solution (coming from the root 0). Therefore, in M_{small} there are bounded solutions of $\mathcal{L}h = 0$ that are *not* decaying. The fundamental solutions corresponding to the root 0 are the tensors with $e^{2r}h_{ij} = \text{const.}$, but these are exactly the trivial Einstein variations (see Definition 9.4). This explains why in the source space we have to work with a weighted norm that isolates trivial Einstein variations and only considers their unweighted C^0-norms.

Now we explain in detail how to use the averaging operator and the linear system of ODEs in (9.14) to obtain growth estimates in M_{small}. Let $f \in C^{0,\alpha}_\lambda(\mathrm{Sym}^2(T^*M))$ be arbitrary, and let h be a solution of

$$\mathcal{L}h = f,$$

where $\mathcal{L}$ is the elliptic operator given by $\mathcal{L}h = \frac{1}{2}\Delta_L h + 2h$ with respect to the given metric g of M. We start with considering a rank 2 cusp C of M. Note that $C_{\mathrm{small}} \cong T^2 \times [0,\infty)$. Let g_{cusp} be the model metric on C_{small} given by Proposition 8.2. This satisfies

$$|g - g_{cusp}|_{C^2}(x) = O(\varepsilon_0 e^{-\eta r(x)}),$$

where $r(x) = d(x, \partial C_{\mathrm{small}})$, and ε_0, η are the constants appearing in the curvature decay condition (9.1). Let $\mathcal{L}_{cusp}$ be the elliptic operator $\mathcal{L}_{cusp}h = \frac{1}{2}\Delta_L h + 2h$ with respect to the model metric g_{cusp}. Set $f_c := \mathcal{L}_{cusp}h$. Then $|f - f_c|(x) = O(\varepsilon_0 |h|_{C^2}(x)e^{-\eta r(x)})$ by the above estimate on $|g - g_{cusp}|_{C^2}$. Thus

$$\begin{aligned} |f_c|(x) \leq& |f|(x) + O\big(\varepsilon_0 |h|_{C^2}(x)e^{-\eta r(x)}\big) \\ =& O\big(\|f\|_{0,\lambda}e^{-\lambda r(x)} + \varepsilon_0 |h|_{C^2}(x)e^{-\eta r(x)}\big). \end{aligned}$$

Let $\hat{\cdot}$ be the averaging operator with respect to the model metric g_{cusp}. Using *ii)* of Lemma 9.12 we get

$$|\widehat{f_c}|(r) = O\left(\|f\|_{0,\lambda}e^{-\lambda r} + \varepsilon_0 e^{-(\eta-2)r}\int_{T(r)} |h|_{C^2}(y)\, d\mathrm{vol}_2(y)\right),$$

where we used that

$$\mathrm{vol}_2(T(r)) = e^{-2r}\mathrm{vol}_2(\partial C_{\mathrm{small}}) = O(e^{-2r})$$

since by definition $\mathrm{diam}(\partial C_{\mathrm{small}})$ is bounded by a universal constant. Define the function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ by

$$\psi(r) := \int_{T(r)} |h|_{C^2}(y)\, d\mathrm{vol}_2(y). \tag{9.16}$$

Hence

$$|\widehat{f_c}|(r) = O\left(\|f\|_{0,\lambda}e^{-\lambda r} + \varepsilon_0 \psi(r)e^{-\lambda r}\right) \tag{9.17}$$

since $\eta \geq 2 + \lambda$.

By *iii*) and *iv*) of Lemma 9.12 and the identity $f_c = \mathcal{L}_{cusp} h$, it holds

$$\mathcal{L}_{cusp}\hat{h} = \widehat{f_c}.$$

As $\hat{h}$ and $\widehat{f_c}$ only depend on r, $\mathcal{L}_{cusp}\hat{h} = \widehat{f_c}$ is the linear system of ODEs given by (9.14). Due to the growth estimate (9.17), we can invoke the following two basic ODE results to obtain a growth estimate for $\hat{h}$.

In the formulation of these ODE results, we denote by I either $\mathbb{R}_{\geq 0}$ or an interval of the form $[0, R-1]$ for some $R \geq 2$. Moreover, for a polynomial $Q = \sum_m a_m X^m$ we write $Q(\frac{d}{dr})$ for the differential operator $\sum_m a_m \frac{d^m}{dr^m}$.

Lemma 9.14. *Let $Q \in \mathbb{R}[X]$ be a quadratic polynomial with two distinct real roots λ_1, λ_2. Let $y : I \to \mathbb{R}$ be a solution of the ODE*

$$Q\left(\frac{d}{dr}\right)(y) = u,$$

where $u : I \to \mathbb{R}$ is a function satisyfing $u(r) = \sum_{k=1}^m O\big(\beta_k e^{\mu_k r}\big)$ for some $\beta_k \in \mathbb{R}_{\geq 0}$, and $\mu_k \in \mathbb{R} \setminus \{\lambda_1, \lambda_2\}$. Then

$$y(r) = A_1 e^{\lambda_1 r} + A_2 e^{\lambda_2 r} + \sum_{k=1}^m O\big(\beta_k e^{\mu_k r}\big)$$

for some constants $A_1, A_2 \in \mathbb{R}$.

Lemma 9.15. *Let $Q \in \mathbb{R}[X]$ be a quadratic polynomial with two distinct real roots λ_1, λ_2. Let $y : I \to \mathbb{R}$ be a solution of*

$$Q\left(\frac{d}{dr}\right)(y) = u,$$

where u satisfies $|u(r)| \leq e^{ar}\psi(r)$ for some $a \in \mathbb{R}$ and $\psi \in L^1(\mathbb{R}_{\geq 0})$. Then

$$y(r) = A_1 e^{\lambda_1 r} + A_2 e^{\lambda_2 r} + O\big(\|\psi\|_{L^1(\mathbb{R}_{\geq 0})} e^{ar}\big)$$

for some $A_1, A_2 \in \mathbb{R}$.

In Lemma 9.14 and Lemma 9.15, the universal constant absorbed by $O(...)$ is allowed to depend on λ_1, λ_2, and a, but *not* on R (in case $I = [0, R-1]$). We again refer to Notation 2.2 for our convention of the O-notation.

Proof of Lemma 9.14 and Lemma 9.15. Both of these lemmas follow easily from the explicit integral formulas for solutions of linear ODEs. □

In order to successfully apply Lemma 9.15, we need to control the L^1-norm of the function ψ defined in (9.16). This is expressed in the following lemma.

Lemma 9.16. *Let the constants and the manifold M be as in Proposition 9.10. Let $f \in C^{0,\alpha}\big(\mathrm{Sym}^2(T^*M)\big)$ with $\|f\|_{C^{0,\alpha}(M)} < \infty$, and let $h \in C^2(\mathrm{Sym}^2(T^*M)) \cap H_0^1(M)$ be a solution of*

$$\mathcal{L}h = f.$$

Fix a rank 2 cusp C of M, and let $\psi:\mathbb{R}_{\geq 0}\to\mathbb{R}$ be given by

$$\psi(r)=\int_{T(r)}|h|_{C^2}(y)\,d\mathrm{vol}_2(y),$$

where $T(r)\subseteq C_{\mathrm{small}}$ is the torus of all points of distance r to $\partial C_{\mathrm{small}}$. Then $\psi\in L^1(\mathbb{R}_{\geq 0})$, and

$$\|\psi\|_{L^1(\mathbb{R}_{\geq 0})}=O(\|f\|_{0,\lambda}),$$

where $\|\cdot\|_{0,\lambda}$ is the norm defined in (9.8).

Not all components of the hybrid norm $\|f\|_{0,\lambda}$ are needed for this estimate. In fact, we only need $\|f\|_{C^0}$ and $e^{\frac{b}{2}d(x,M_{\mathrm{thick}})}\|f\|_{L^2(M;\omega_x)}$ ($x\in M_{\mathrm{thin}}\smallsetminus M_{\mathrm{small}}$). Moreover, the curvature decay condition (9.1) is not needed for this estimate.

Proof. By the co-area formula it holds

$$\int_0^\infty\psi(r)\,dr=\int_{C_{\mathrm{small}}}|h|_{C^2}(x)\,d\mathrm{vol}(x).\tag{9.18}$$

Since by definition $\mathrm{diam}(\partial C_{\mathrm{small}})$ is bounded by a universal constant, it is easy to see that $\mathrm{vol}(C_{\mathrm{small}})$ is also bounded by a universal constant. Combining this with (9.18), and using the Cauchy-Schwarz inequality yields

$$\int_0^\infty\psi(r)\,dr\leq C\Big(\int_{C_{\mathrm{small}}}|h|^2_{C^2}(x)\,d\mathrm{vol}(x)\Big)^{\frac{1}{2}}\tag{9.19}$$

for some universal constant C. Therefore, it suffices to obtain H^2-estimates of h in C_{small}.

For $\varepsilon_0>0$ small enough it follows from Lemma 7.10 and Schauder estimates that

$$|h|_{C^2}(x)=O(\|f\|_{0,\lambda})\ \text{ for all }\ x\ \text{ with }\ r(x)\leq 1.\tag{9.20}$$

Choose a smooth bump function $\varphi:M\to[0,1]$ satisfying $\varphi=0$ on $M\smallsetminus C_{\mathrm{small}}$, $\varphi(x)=1$ for $x\in C_{\mathrm{small}}$ with $r(x)\geq 1$, and $\|\nabla\varphi\|_{C^0(M)}\leq 2$. Applying Proposition 3.4 with this φ, and using (9.20) we obtain (for $\varepsilon_0>0$ small enough)

$$\int_{C_{\mathrm{small}}}|h|^2\,d\mathrm{vol}\leq C\int_{C_{\mathrm{small}}}|f|^2\,d\mathrm{vol}+O(\|f\|^2_{0,\lambda}).$$

Analogous to Step 1 in the proof of Proposition 4.3 we get H^2-estimates for the solutions h of $\mathcal{L}h=f$ in terms of $\|h\|_{L^2}$ and $\|f\|_{L^2}$ by standard computations using integration by parts. Because of (9.20) we can control the boundary terms when invoking integration by parts on C_{small}. Therefore, for some universal constant $C>0$ it holds

$$\int_{C_{\mathrm{small}}}|h|^2_{C^2}\,d\mathrm{vol}\leq C\int_{C_{\mathrm{small}}}|f|^2\,d\mathrm{vol}+O(\|f\|^2_{0,\lambda}).\tag{9.21}$$

Note $\|f\|_{L^2(C_{\mathrm{small}})}=O(\|f\|_{C^0})$ since $\mathrm{vol}(C_{\mathrm{small}})$ is bounded by a universal constant. Thus the desired inequality follows from (9.19) and (9.21). ☐

We are now finally in a position to prove the growth estimate in a cusp.

Proof of Proposition 9.10. Later when we prove that $\mathcal{L}$ is surjective, it will be important to have an argument that only assumes $h \in C^2\big(\mathrm{Sym}^2(T^*M)\big) \cap H^1_0(M)$, but that does *not* assume $\|h\|_{C^0(M)} < \infty$. For this reason we will as long as possible only assume $h \in C^2\big(\mathrm{Sym}^2(T^*M)\big) \cap H^1_0(M)$, and point out from which point on we really need the assumption $\|h\|_{C^0} < \infty$.

Let g_{cusp} be the model metric in the rank 2 cusp C given by Proposition 8.2. Because of the estimate on $|g - g_{cusp}|_{C^2}$ in Proposition 8.2, it is irrelevant in C_{small} whether quantities such as the L^2-norm are taken w.r.t. g or g_{cusp}, and we will compute all quantities w.r.t. g_{cusp}. Denote by $\mathcal{L}_{cusp}$ the elliptic operator $\frac{1}{2}\Delta_L + 2\mathrm{id}$ with respect to the model metric g_{cusp}. We define $f_c := \mathcal{L}_{cusp} h$. Moreover, let $\hat{\cdot}$ denote the averaging operator with respect to the model metric g_{cusp} (see Lemma 9.12). In the discussion after Lemma 9.12 we showed (see (9.17))

$$\mathcal{L}_{cusp}\hat{h} = \widehat{f_c} \quad \text{and} \quad |\widehat{f_c}|(r) = O(\|f\|_{0,\lambda} e^{-\lambda r} + \varepsilon_0 \psi(r) e^{-\lambda r}),$$

where ψ was defined in (9.16). Moreover, as $\hat{h}$, $\widehat{f_c}$ only depend on r, $\mathcal{L}_{cusp}\hat{h} = \widehat{f_c}$ is the linear system of ODEs given by (9.14). Namely, by (9.15), and the first two equations in (9.14) we have

$$\begin{cases} Q_1(\frac{d}{dr})(\mathrm{tr}(\hat{h})) & = -2\mathrm{tr}(\widehat{f_c}) \\ Q_1(\frac{d}{dr})(\hat{h}_{33}) & = -2(\widehat{f_c})_{33} \\ Q_2(\frac{d}{dr})(e^r \hat{h}_{i3}) & = -2e^r(\widehat{f_c})_{i3} \end{cases}$$

for some quadratic polynomials Q_1 and Q_2 with roots $\{1-\sqrt{5}, 1+\sqrt{5}\}$ and $\{-1, 3\}$. As $|\widehat{f_c}|$ satisfies the above growth estimate, and since $-\lambda \notin \{1 \pm \sqrt{5}, -1, 3\}$ we can apply Lemma 9.14 and Lemma 9.15.

We know $\|\psi\|_{L^1(\mathbb{R}_{\geq 0})} = O(\|f\|_{0,\lambda})$ from Lemma 9.16 (note that Lemma 9.16 does *not* assume $\|h\|_{C^0} < \infty$). Thus we get from Lemma 9.14 and Lemma 9.15

$$\begin{cases} \mathrm{tr}(\hat{h})(r) & = a_1 e^{(1-\sqrt{5})r} + a_2 e^{(1+\sqrt{5})r} + O\big(\|f\|_{0,\lambda} e^{-\lambda r}\big); \\ \hat{h}_{33}(r) & = b_1 e^{(1-\sqrt{5})r} + b_2 e^{(1+\sqrt{5})r} + O\big(\|f\|_{0,\lambda} e^{-\lambda r}\big); \\ e^r \hat{h}_{i3}(r) & = c_1^{(i)} e^{-r} \quad + c_2^{(i)} e^{3r} \quad + O\big(\|f\|_{0,\lambda} e^{-\lambda r}\big); \end{cases} \tag{9.22}$$

for some constants $a_1, a_2, b_1, b_2, c_1^{(i)}, c_2^{(i)} \in \mathbb{R}$. Note that $h \in L^2(C_{\mathrm{small}}) \subseteq L^1(C_{\mathrm{small}})$ since C_{small} has finite volume, and $\mathrm{vol}(T(r)) = O(e^{-2r})$, where $T(r) \subseteq C_{\mathrm{small}}$ is the torus all whose points have distance r to $\partial C_{\mathrm{small}}$. Hence $e^{-2r}|\hat{h}|(r) \in L^1(\mathbb{R}_{\geq 0})$. In particular, $e^{-2r}\mathrm{tr}(\hat{h})(r), e^{-2r}\hat{h}_{33}(r), e^{-2r}\big(e^r \hat{h}_{i3}(r)\big) \in L^1(\mathbb{R}_{\geq 0})$ $(i = 1, 2)$, and thus $a_2 = b_2 = c_2^{(i)} = 0$.

We know that $\max_{\partial C_{\mathrm{small}}} |h| = O(\|f\|_{0,\lambda})$ due to Proposition 7.5, and we have $|\hat{h}|(0) = O\big(\max_{\partial C_{\mathrm{small}}} |h|\big)$ by *ii)* of Lemma 9.12. Hence evaluating at $r = 0$ yields

$$a_1, b_1, c_1^{(i)} = O\big(\|f\|_{0,\lambda}\big).$$

As $\lambda < 1 < \sqrt{5} - 1$ this implies

$$|\mathrm{tr}(\hat{h})(r)|, |\hat{h}_{33}(r)|, |e^r \hat{h}_{i3}(r)| = O\big(\|f\|_{0,\lambda} e^{-\lambda r}\big). \tag{9.23}$$

By the last equation in (9.14) there is a quadratic polynomial Q_3 with roots 0 and 2 so that

$$\begin{aligned} Q_3\left(\frac{d}{dr}\right)(e^{2r}\hat{h}_{ij}) =& 2\delta_{ij}(\mathrm{tr}(\hat{h}) - \hat{h}_{33}) - 2e^{2r}(\widehat{f_c})_{ij} \\ =& O\big(\|f\|_{0,\lambda}e^{-\lambda r} + \varepsilon_0\psi(r)e^{-\lambda r}\big), \end{aligned}$$

where we used the growth rate of $|\hat{f}_c|$ in (9.17), and the one for $\mathrm{tr}(\hat{h}), \hat{h}_{33}$ in (9.23). Again invoking Lemma 9.14 and Lemma 9.15, and using Lemma 9.16 to estimate $\|\psi\|_{L^1(\mathbb{R}_{\geq 0})}$, we conclude

$$e^{2r}\hat{h}_{ij} = d_1^{(i,j)} + d_2^{(i,j)}e^{2r} + O\big(\|f\|_{0,\lambda}e^{-\lambda r}\big) \tag{9.24}$$

for some constants $d_1^{(i,j)}, d_2^{(i,j)} \in \mathbb{R}$. As before, $e^{-2r}|\hat{h}|(r) \in L^1(\mathbb{R}_{\geq 0})$ implies $d_2^{(i,j)} = 0$. Exactly as before, evaluating at $r = 0$ we obtain

$$d_1^{(i,j)} = O\big(\|f\|_{0,\lambda}\big). \tag{9.25}$$

Define an Einstein variation v in C_{small} by $v_{ij}(r) = d_1^{(i,j)}e^{-2r}$. Note that $\mathrm{tr}(\hat{h}) = \hat{h}_{33} + \mathrm{tr}(v)$, and that $\mathrm{tr}(v)$ is constant. Hence $\hat{h}_{33}(r), \mathrm{tr}(\hat{h})(r) \xrightarrow{r\to\infty} 0$ implies that $\mathrm{tr}(v) = 0$. Therefore, v is indeed a trivial Einstein variation. From (9.25) we know

$$|v| = O(\|f\|_{0,\lambda}).$$

Moreover, (9.23), (9.24), the fact that $d_2^{(i,j)} = 0$, and the definition of v imply

$$\|\hat{h} - v\|_{C^0_\lambda} = O\big(\|f\|_{0,\lambda}\big). \tag{9.26}$$

In particular, $|v| = O(\|f\|_{0,\lambda})$ and (9.26) yield

$$\sup_{C_{\mathrm{small}}} |\hat{h}| = O(\|f\|_{0,\lambda}). \tag{9.27}$$

We will need this fact in the proof of the surjectivity of $\mathcal{L}$. Until this point we did *not* need $\|h\|_{C^0} < \infty$, but only $h \in C^2\big(\mathrm{Sym}^2(T^*M)\big) \cap H^1_0(M)$.

From now on we use the assumption $\|h\|_{C^0} < \infty$, which implies $\|h\|_{C^1} < \infty$ due to Schauder estimates. Since $\lambda < 1$, we deduce from $v)$ of Lemma 9.12 that $\|h - \hat{h}\|_{C^0_\lambda} < \infty$ and hence $\|h-v\|_{C^0_\lambda} < \infty$. This proves the existence of a trivial Einstein variation as stated in Proposition 9.10. Uniqueness of such a trivial Einstein variation is clear because trivial Einstein variations have constant C^0-norm (with respect to g_{cusp}).

If we assume $\|h\|_{C^0}, \|f\|_{C^{0,\alpha}} \leq 1$, then these last considerations can be made more quantitative. Indeed, under this assumption we have $\|h\|_{C^1} = O(1)$ by Schauder estimates. Recall that $\mathrm{diam}(\partial C_{\mathrm{small}})$ is bounded by a universal constant due to the definition of the small part. Thus $|h - \hat{h}|(x) = O\big(e^{-r(x)}\big)$ by $v)$ of Lemma 9.12. Together with (9.26) this implies

$$e^{\lambda r(x)}|h - v|(x) = O(\|f\|_{0,\lambda} + e^{-(1-\lambda)r(x)}).$$

This finishes the proof of (9.12). As we already showed $|v| = O(\|f\|_{0,\lambda})$, this also yields (9.11). This completes the proof. □

The proof Proposition 9.11 is very similar to that of Proposition 9.10, so we only highlight the differences.

Let T be a Margulis tube of M with core geodesic γ. Denote by R the radius of T_{small}, that is, the distance from γ to $\partial T_{\text{small}}$. The model metric g_{cusp} given by Proposition 8.1 is only defined on $T_{\text{small}} \smallsetminus N_1(\gamma)$, and it satisfies $|g - g_{cusp}|_{C^2}(x) = O\big(\varepsilon_0 e^{-\eta r(x)} + e^{2(r(x)-R)}\big)$, where $r(x) = d(x, \partial T_{\text{small}})$. Define $\mathcal{L}_{cusp}$, f_c, and ψ exactly as in the case of a cusp (the only difference being that now ψ is only defined on $[0, R-1]$). As $\|f\|_{0,\lambda}, \|h\|_{C^0} \le 1$ by assumption, it holds $\|h\|_{C^2} = O(1)$ due to Schauder estimates. Thus we can estimate $|\widehat{f_c} - \hat{f}|(x) = O\big(\varepsilon_0 \psi(r) e^{-\lambda r} + e^{2(r-R)}\big)$. So $|\widehat{f_c}|$ satisfies the growth estimate

$$|\widehat{f_c}|(r) = O\big(\|f\|_{0,\lambda} W_\lambda(r) + \varepsilon_0 \psi(r) e^{-\lambda r} + e^{2(r-R)}\big), \tag{9.28}$$

where W_λ is the inverse weight entering the definition of $\|\cdot\|_{C^0_\lambda}$. In T_{small} it is given by $W_\lambda(r) = e^{-\lambda r} + e^{\lambda(r-R)}$. In contrast to the definition of the cusp, r now only takes values in a bounded interval. For this reason we can *not* get rid of the exponentially growing fundamental solutions as easily as in the case of a cusp. To remedy this, we evoke the following basic ODE lemma.

Lemma 9.17. *Let $Q \in \mathbb{R}[X]$ be a quadratic polynomial with distinct real roots $\lambda_2 \le 0 < \lambda_1$, $R \ge 2$, $\psi \in L^1([0, R-1])$ so that $\|\psi\|_{L^1([0,R-1])}$ is bounded by a universal constant, and let $\mathcal{W} : [0, R-1] \to \mathbb{R}$ be a function of the form*

$$\mathcal{W}(r) = \sum_{k=1}^{m} \beta_k e^{\mu_k r}$$

for some $\beta_k \in \mathbb{R}_{\ge 0}$, $\mu_k \notin \{\lambda_1, \lambda_2\}$, so that $\mathcal{W}(r)$ is bounded by a universal constant for $r \in [R-2, R-1]$. Let $y : [0, R-1] \to \mathbb{R}$ be a C^2 function with $|y| \le 1$ and

$$Q\left(\frac{d}{dr}\right)(y) = O\big(\mathcal{W}(r) + \psi(r) e^{-ar}\big)$$

for some $a \ge 0$. Then there is a universal constant C so that

$$|y|(r) \le C\Big(e^{\lambda_2 r}\big(|y|(0) + \mathcal{W}(0) + \|\psi\|_{L^1([0,R-1])}\big) + e^{\lambda_1(r-R)} + \mathcal{W}(r) + \|\psi\|_{L^1} e^{-ar}\Big) \tag{9.29}$$

for all $r \in [0, R-1]$. In particular,

$$|y|(r) \le C\Big(|y|(0) + \mathcal{W}(0) + \|\psi\|_{L^1([0,R-1])} + e^{\lambda_1(r-R)} + \mathcal{W}(r)\Big). \tag{9.30}$$

As in Lemma 9.14 and Lemma 9.15 the universal constant C is allowed to depend on λ_1, λ_2, and a, but *not* on R.

Proof. As $\mu_k \notin \{\lambda_1, \lambda_2\}$, and $\psi \in L^1([0, R-1])$, Lemma 9.14 and 9.15 imply

$$y(r) = A_1 e^{\lambda_1 r} + A_2 e^{\lambda_2 r} + O\big(\mathcal{W}(r) + \|\psi\|_{L^1} e^{-ar}\big) \tag{9.31}$$

for some constants $A_1, A_2 \in \mathbb{R}$. For ease of notation we abbreviate $R' := R - 1$. Since $|y|, \mathcal{W}, \|\psi\|_{L^1} e^{-ar} = O(1)$ on $[R'-1, R']$ (this uses $a \ge 0$), also $A_1 e^{\lambda_1 r} + A_2 e^{\lambda_2 r} = O(1)$ on

$[R'-1,R']$. Using this at $r=R'$ and $r=R'-1$ gives the linear system of equations

$$\begin{pmatrix} 1 & 1 \\ e^{-\lambda_1} & e^{-\lambda_2} \end{pmatrix} \begin{pmatrix} A_1 e^{\lambda_1 R'} \\ A_2 e^{\lambda_2 R'} \end{pmatrix} = \begin{pmatrix} O(1) \\ O(1) \end{pmatrix}.$$

The operator norm $\|L^{-1}\|_{\mathrm{op}}$ of the inverse of the linear operator $L = \begin{pmatrix} 1 & 1 \\ e^{-\lambda_1} & e^{-\lambda_2} \end{pmatrix}$ only depends on λ_1, λ_2. Hence $A_1 e^{\lambda_1 R'}$ and $A_2 e^{\lambda_2 R'}$ are bounded by a universal constant. We know $y(0) = A_1 + A_2 + O\big(\mathcal{W}(0) + \|\psi\|_{L^1}\big)$ due to (9.31). Invoking (9.31) once more we obtain

$$\begin{aligned} y(r) =& A_1 e^{\lambda_1 r} + A_2 e^{\lambda_2 r} \\ &+ O\big(\mathcal{W}(r) + \|\psi\|_{L^1} e^{-ar}\big) \\ =& e^{\lambda_1 (r-R')}\big(A_1 e^{\lambda_1 R'}\big) + e^{\lambda_2 r}\Big(y(0) - e^{-\lambda_1 R'}\big(A_1 e^{\lambda_1 R'}\big) + O\big(\mathcal{W}(0) + \|\psi\|_{L^1}\big)\Big) \\ &+ O\big(\mathcal{W}(r) + \|\psi\|_{L^1} e^{-ar}\big) \\ =& e^{\lambda_1 (r-R')} O(1) + e^{\lambda_2 r}\Big(y(0) + e^{-\lambda_1 R'} O(1) + O\big(\mathcal{W}(0) + \|\psi\|_{L^1}\big)\Big) \\ &+ O\big(\mathcal{W}(r) + \|\psi\|_{L^1} e^{-ar}\big), \end{aligned}$$

that is, there is a universal constant C so that

$$|y|(r) \le C\Big(e^{\lambda_1 (r-R')} + e^{\lambda_2 r}\big(|y|(0) + e^{-\lambda_1 R'} + \mathcal{W}(0) + \|\psi\|_{L^1}\big) + \mathcal{W}(r) + \|\psi\|_{L^1} e^{-ar}\Big)$$

for all $0 \le r \le R'$. As $\lambda_2 < \lambda_1$, so that $e^{\lambda_2 r} e^{-\lambda_1 R'} \le e^{\lambda_1 (r-R')}$, and $R' = R-1$, this completes the proof of (9.29). □

To apply Lemma 9.17 we need to make sure that the growth rates of $|\widehat{f_c}|$ are different from the fundamental solutions of the system of ODEs in (9.14). For this reason, we weaken the growth control on $|\widehat{f_c}|$ in (9.28) to

$$|\widehat{f_c}|(r) = O\big(\|f\|_{0,\lambda} W_\lambda(r) + \varepsilon_0 \psi(r) e^{-\lambda r} + e^{\frac{3}{2}(r-R)}\big). \tag{9.32}$$

Analogous to the case of a cusp, it still holds $\|\psi\|_{L^1([0,R-1])} = O(\|f\|_{0,\lambda})$. Indeed, the proof of Lemma 9.16 goes through without modification. Now the proof of Proposition 9.11 follows by applying Lemma 9.17 with

$$\mathcal{W}(r) = \|f\|_{0,\lambda} W_\lambda(r) + e^{\frac{3}{2}(r-R)} = \|f\|_{0,\lambda}\big(e^{-\lambda r} + e^{\lambda(r-R)}\big) + e^{\frac{3}{2}(r-R)}$$

and ψ defined in (9.16) componentwise to the linear system of ODEs $\mathcal{L}_{cusp}\hat{h} = \hat{f}_c$ given in (9.14). (To be precise: Similar to the proof of Proposition 9.11, one first applies Lemma 9.17 to the equations for $\mathrm{tr}(\hat{h}), \hat{h}_{33}, e^r \hat{h}_{i3}$, and then for $e^{2r}\hat{h}_{ij}$ one applies Lemma 9.17 to the above $\mathcal{W}$ + the growth of $\mathrm{tr}(\hat{h}), \hat{h}_{33}$ obtained from (9.29).) This gives growth estimates for $\hat{h}$. By $v)$ of Lemma 9.12 we again know $|h - \hat{h}|(x) = O(e^{-r(x)})$. Remembering that $r_\gamma = R - r$ will yield the estimate (9.13), thus finishing the proof of Proposition 9.11.

9.4. **A priori estimates.** The goal of this section is to prove the a priori estimate of Proposition 9.1, that is, that there exists a universal constant C so that

$$\|h\|_{2,\lambda;*} \leq C\|\mathcal{L}h\|_{0,\lambda}$$

for all $h \in C^{2,\alpha}_{\lambda}\big(\mathrm{Sym}^2(T^*M)\big)$. The norm $\|\cdot\|_{2,\lambda;*}$ is a mixture of weighted Sobolev norms and the decomposition norm $\|\cdot\|_{C^{2,\alpha}_{\lambda};*}$. By Remark 4.5, the weighted Sobolev-estimate $\|h\|_{H^2(M;\omega_x)} \leq C\|f\|_{L^2(M;\omega_x)}$ still holds if M is non-compact but has finite volume. Therefore, it suffices to prove an a priori estimate $\|h\|_{C^{2,\alpha}_{\lambda}(M);*} \leq C\|\mathcal{L}h\|_{0,\lambda}$. We do this in two steps. First we establish global $C^{2,\alpha}$-estimates, and then prove the estimate for the $*$-norm.

For the proof of a global $C^{2,\alpha}$-estimate, we adapt the arguments in [Bam12, Lemma 6.1].

Proposition 9.18. *For all $\alpha \in (0,1)$, $\Lambda \geq 0$, $\lambda \in (0,1)$, $b > 1$, $\delta \in (0,2)$, $r_0 \geq 1$, and $\eta \geq 2+\lambda$ there exist constants ε_0, $\bar{\epsilon}_0$ and $C > 0$ with the following property. Let M be a Riemannian 3-manifold of finite volume that satisfies*

$$|\sec + 1| \leq \varepsilon_0, \quad \|\nabla \mathrm{Ric}\|_{C^0(M)} \leq \Lambda,$$

and

$$\max_{\pi \subseteq T_xM} |\sec(\pi)+1|,\, |\nabla R|(x),\, |\nabla^2 R|(x) \leq \varepsilon_0 e^{-\eta d(x,\partial M_{\mathrm{small}})} \quad \text{for all } x \in M_{\mathrm{small}}.$$

Then for all $h \in C^{2,\alpha}_{\lambda}\big(Sym^2(TM)\big)$ it holds

$$\|h\|_{C^{2,\alpha}(M)} \leq C\|\mathcal{L}h\|_{0,\lambda},$$

where $\|\cdot\|_{0,\lambda}$ is the norm defined in (9.8) with respect to any $\bar{\epsilon} \leq \bar{\epsilon}_0$.

Proof. **Step 1 (Reducing the problem):** Let $\varepsilon_0 > 0$ be small enough so that one can apply Proposition 9.10, Proposition 9.11, and Lemma 9.19 below, and choose $\bar{\epsilon}_0 > 0$ so that Remark 4.6 applies. Due to Schauder estimates (Proposition 2.5) it suffices to prove a C^0-estimate.

Arguing by contradiction, assume that such a C^0-estimate does not hold. Then there exist a sequence of counterexamples, given by a sequence of finite volume Riemannian 3-manifolds (M^i, g^i) satisfying the curvature assumptions stated in Proposition 9.18, and tensors $h^i \in C^{2,\alpha}_{\lambda}\big(\mathrm{Sym}^2(T^*M)\big)$ so that

$$\|h^i\|_{C^0(M^i)} = 1 \quad \text{and} \quad \|\mathcal{L}_{g^i}h^i\|_{0,\lambda} \xrightarrow{i\to\infty} 0,$$

where $\|\cdot\|_{0,\lambda}$ is the norm defined in (9.8) with respect to some $\bar{\epsilon}^i \leq \bar{\epsilon}_0$. Abbreviate $f^i := \mathcal{L}_{g^i}h^i$. Choose points $x^i \in M^i$ with $|h^i|(x^i) \geq \frac{1}{2}$.

If $x^i \in M^i_{\mathrm{thick}}$, then Remark 4.6 shows $\|h^i\|_{C^0} \leq C\|f^i\|_0 \to 0$, where $\|\cdot\|_0$ is the norm defined in (4.3) with respect to $\bar{\epsilon}^i \leq \bar{\epsilon}_0$. This is a contradiction.

Hence $x^i \in M^i_{\mathrm{thin}}$ for all i large enough. By Proposition 7.5, and the definition (9.8) of $\|\cdot\|_{0,\lambda}$, it holds $|h^i|(x) \leq C\|f^i\|_{0,\lambda}$ for all $x \in M^i_{\mathrm{thin}} \smallsetminus M^i_{\mathrm{small}}$. Thus $x^i \in M^i_{\mathrm{small}}$ for all i large enough. After passing to a subsequence, either x^i is contained in a cusp $C_{l^i} \subseteq M^i$

for all i, or x^i is contained in a Margulis tube $T_{k^i} \subseteq M^i$ for all i. We distinguish these two cases.

Step 2 (Cusps): First we consider the case that x^i is contained in a cusp C_{l^i}. Abbreviate $r^i := d(x^i, \partial(C_{l^i})_{\text{small}})$. By the growth estimate (9.11) in Proposition 9.10 we know that

$$|h^i|(x^i) \le C\big(\|f^i\|_{0,\lambda} + e^{-r^i}\big)$$

for a universal constant C. As $\|f^i\|_{0,\lambda} \to 0$, this shows that for i large enough it holds

$$\frac{1}{2} \le |h^i|(x^i) \le \frac{1}{4} + Ce^{-r^i},$$

and thus $r^i \le \log(4C)$, that is, r^i is bounded by the universal constant $\bar{R} = \log(4C)$. By Lemma 7.10 there is some $C(\bar{R})$ so that

$$|h^i|(x) \le C(\bar{R})\|f^i\|_{0,\lambda}$$

for all $x \in N_{\bar{R}}(C_{l^i} \smallsetminus (C_{l^i})_{\text{small}})$. In particular, $|h^i|(x^i) \le C(\bar{R})\|f^i\|_{0,\lambda}$. However, this contradicts $|h^i|(x^i) \ge \frac{1}{2}$ and $\|f^i\|_{0,\lambda} \to 0$.

Step 3.1 (Radius of $(T_{k^i})_{\text{small}}$): The goal of this step is to show

$$R_{k^i} \to \infty \quad \text{as } i \to \infty,$$

where R_{k^i} is the radius of the small part $(T_{k^i})_{\text{small}}$, that is, the distance of the core geodesic γ_{k^i} to $\partial(T_{k^i})_{\text{small}}$. Assume that this is wrong. Then, after passing to a subsequence, it holds $R_{k^i} \le \bar{R}$ for all i for some $\bar{R} > 0$. Choose a finite cover $\hat{T}_{k^i} \to T_{k^i}$ so that $\ell(\hat{\gamma}_{k^i}) \in [1, 10]$. Note that the $\hat{T}_{k^i}$ have uniform lower bounds on the injectivity radius, uniform bounds on the sectional curvature and the covariant derivative of the Ricci tensor. Moreover, $d(\hat{x}^i, \hat{\gamma}_{k^i}) \le R_{k^i} \le \bar{R}$ where $\hat{x}^i$ is a preimage of x^i in $\hat{T}_{k^i}$. Therefore, after passing to a subsequence, it holds

$$(\hat{T}_{k^i}, \hat{x}^i) \xrightarrow{\text{pointed } C^{2,\beta}} (T^\infty, x^\infty),$$

where the convergence is pointed $C^{2,\beta}$-convergence for some $\beta \in (0, \alpha)$. Here T^∞ is a negatively curved tube of possibly finite length (see the discussion before Lemma 9.19 for our definition of a negatively curved tube). After passing to a subsequence we may assume that the limit $R_\infty = \lim_i R_{k^i} \in [0, \bar{R}]$ exists. Since the distance of $(T_{k^i})_{\text{small}}$ to M_{thick} is at least $\mu/2$ by Lemma 7.4, the length of T^∞, that is $\sup_{x \in T^\infty} d(x, \gamma_\infty)$, is at least $R_\infty + \mu/2$.

Denote the lifts of h^i and f^i to $\hat{T}_{k^i}$ by $\hat{h}^i$ and $\hat{f}^i$. Since $\|\hat{h}^i\|_{C^{2,\alpha}(M^i)}$ is uniformly bounded (due to Schauder estimates), after passing to a subsequence we may assume that $\hat{h}^i \to h^\infty$ in the pointed $C^{2,\beta}$-sense. From $|h^i(x^i)| \ge \frac{1}{2}$ it follows $|h^\infty(x^\infty)| \ge \frac{1}{2}$. Similarly, $\hat{f}^i \to 0$ in the pointed $C^{0,\beta}$-sense. Due to the stability of elliptic PDEs, we get $\mathcal{L}^\infty h^\infty = 0$.

Proposition 7.5 shows that

$$\max_{T_{k^i} \smallsetminus (T_{k^i})_{\text{small}}} |h^i| \le C\|f^i\|_{0,\lambda}.$$

The same still holds for $\hat{h}^i$. As $R_{k^i} \to R_\infty$, and $\|f^i\|_{0,\lambda} \to 0$, this implies $h^\infty = 0$ outside of $N_{R_\infty}(\gamma^\infty)$. In particular, this shows that h^∞ has compact support because the length of T^∞ is at least $R_\infty + \mu/2$. Note that $|\sec(M^i) + 1| \le \varepsilon_0$ implies $|\sec(T^\infty) + 1| \le \varepsilon_0$. By our choice of ε_0 we can apply Lemma 9.19 (or rather the comment following it) and conclude $h^\infty = 0$. This contradicts $|h^\infty(x^\infty)| \ge \frac{1}{2}$. Therefore, $R_{k^i} \to \infty$.

Step 3.2 (Estimate in a tube): By the growth estimate (9.13), we know that for all $x \in (T_{k^i})_{\text{small}} \smallsetminus N_1(\gamma_{k^i})$ it holds

$$|h^i|(x) \le C\Big(\|f^i\|_{0,\lambda} + e^{-r_{\partial T^i}(x)} + e^{-\frac{3}{2}r_{\gamma_{k^i}}(x)}\Big), \tag{9.33}$$

where $r_{\partial T^i}(x) = d\big(x, \partial(T_{k^i})_{\text{small}}\big)$ and $r_{\gamma_{k^i}}(x) = d(x, \gamma_{k^i})$. Recall that $|h^i(x^i)| \ge \frac{1}{2}$, and $\|f^i\|_{0,\lambda} \to 0$. Thus there is a subsequence so that either $r_{\gamma_{k^i}}(x^i)$ or $r_{\partial T^i}(x^i)$ stays bounded. We show that both these cases lead to a contradiction, thus completing the proof. Note that these two cases exclude each other because of Step 3.1.

Case 1: $r_{\partial T^i}(x^i)$ is bounded

Let $\bar{R} \in \mathbb{R}$ be so that $r_{\partial T^i}(x^i) \le \bar{R}$ for all $i \in \mathbb{N}$. In Step 3.1 we showed that the radius R_{k^i} of $(T_{k^i})_{\text{small}}$ goes to infinity. In particular, $N_{\bar{R}}(M^i \smallsetminus M^i_{\text{small}})$ is disjoint from the 1-neighbourhood of the core geodesic γ_{k^i} for all i large enough. Thus by applying Lemma 7.10 to the Margulis tube T_{k^i}, we obtain that there is some constant $C(\bar{R})$ so that

$$|h^i|(x) \le C(\bar{R})\|f^i\|_{0,\lambda}$$

for all $x \in N_{\bar{R}}(T_{k^i} \smallsetminus (T_{k^i})_{\text{small}})$. In particular, $|h^i|(x^i) \le C(\bar{R})\|f^i\|_{0,\lambda} \to 0$. But this contradicts $|h^i|(x^i) \ge \frac{1}{2}$.

Case 2: $r_{\gamma_i}(x^i)$ is bounded

Analogous to Step 3.1 (that is, going to appropriate covers and taking a convergent subsequence) we can construct a tensor h^∞ on a tube T^∞ with the following properties: it solves $\mathcal{L}^\infty h^\infty = 0$, and there is a point $x^\infty \in T^\infty$ so that $|h^\infty(x^\infty)| \ge \frac{1}{2}$. Moreover, since $R_{k^i} \to \infty$ the tube T^∞ is complete and of infinite length. As in Step 3.1 it holds $|\sec(T^\infty) + 1| \le \varepsilon_0$. Moreover, (9.33), $\|f^i\|_{0,\lambda} \to 0$, and $r_{\partial T^i}(x^i) \to \infty$ imply

$$|h^\infty|(x) \le Ce^{-\frac{3}{2}r_{\gamma^\infty}(x)}$$

for all $x \in T^\infty \smallsetminus N_1(\gamma_\infty)$. So we can apply Lemma 9.19 and conclude $h^\infty = 0$. However, this contradicts $|h^\infty(x^\infty)| \ge \frac{1}{2}$. $\square$

In the proof of Proposition 9.18 we used the next lemma several times. It is the analog of Proposition 8.3 in [Bam12]. However, since we are in the presence of pinched negative curvature there is a much shorter proof than that in [Bam12]. In its formulation, a complete negatively curved solid 3-torus is the quotient $T = \Gamma\backslash\tilde{M}$ of a simply connected negatively curved 3-manifold $\tilde{M}$ by an infinite cyclic group Γ of loxodromic isometries. The core geodesic of such a solid torus is the projection of the axis $\tilde{\gamma}$ of the elements of Γ.

Lemma 9.19. *There exists $\varepsilon_0 > 0$ with the following property. Let T be a complete solid 3-torus with $|\sec(T) + 1| \le \varepsilon_0$, and let h be a symmetric C^2-tensor that solves $\mathcal{L}h = 0$.*

Assume that for some constant $C > 0$

$$|h|(x) \le Ce^{-\frac{3}{2}r_\gamma(x)}$$

for all $x \in T \smallsetminus N_1(\gamma)$*, where* γ *is the core geodesic. Then* h *vanishes identically.*

The statement also holds if T is of finite length (that is, $T = \Gamma\backslash N_R(\tilde\gamma)$ for some $R > 0$) if we assume that h has compact support. Indeed, the same proof will go through since the the argument mostly relies on integration by parts.

Proof. We first argue that $\mathrm{tr}(h) = 0$. Taking the trace in $\mathcal{L}h = 0$ yields

$$\Delta\big(\mathrm{tr}(h)\big) + 2\mathrm{tr}(h) = 0.$$

By assumption, $|\mathrm{tr}(h)|(x) \to 0$ as $r_\gamma(x) \to \infty$. So there is some $x_0 \in T$ with $|\mathrm{tr}(h)|(x_0) = \|\mathrm{tr}(h)\|_{C^0(T)}$. By possibly replacing h with $-h$ we may assume that $\mathrm{tr}(h)(x_0) \ge 0$. Recall that by our sign convention $-\Delta u = \mathrm{tr}(\nabla^2 u)$. As $\mathrm{tr}(h)$ assumes its maximum at x_0, we have $-\Delta\big(\mathrm{tr}(h)\big)(x_0) \le 0$. Hence

$$\|\mathrm{tr}(h)\|_{C^0(T)} = \mathrm{tr}(h)(x_0) = -\frac{1}{2}\Delta\big(\mathrm{tr}(h)\big)(x_0) \le 0$$

and consequently $\mathrm{tr}(h) = 0$.

By regularity of solutions of elliptic equations and Schauder theory, h is smooth and all covariant derivatives $\nabla^k h$ satisfy the same kind of estimate, that is, $|\nabla^k h|(x) \le C_k e^{-\frac{3}{2}r_\gamma(x)}$. Jacobi field comparison shows that $\mathrm{area}(\partial N_R(\gamma)) = O(e^{\frac{5}{2}R})$ if $\varepsilon_0 > 0$ is small enough. Hence $\nabla^k h \in L^2(T)$ for all $k \in \mathbb{N}$. Therefore, [Gaf54] shows that one can apply integration by parts to h and all its derivatives.

The proof of the Poincaré inequality (Proposition 3.1) only needed integration by parts and some tensor calculus. Hence the Poincaré inequality also holds in the present situation, and we may apply it to h since h has vanishing trace. Thus if $\varepsilon_0 > 0$ is small enough it holds

$$\|h\|_{L^2(T)} \le \frac{1}{3 - c\varepsilon_0}\|\nabla h\|_{L^2(T)},$$

where $c = c(3)$ is the constant from Proposition 3.1. Recall that $\mathcal{L}h = \frac{1}{2}\Delta h + \frac{1}{2}\mathrm{Ric}(h) + 2h$. Since $\mathrm{tr}(h) = 0$, it follows from Lemma 3.2 that $\frac{1}{2}\langle \mathrm{Ric}(h), h\rangle \ge -\big(3 + c'\varepsilon_0\big)|h|^2$ for a constant $c' > 0$. Therefore, applying $(\cdot, h)_{L^2(T)}$ to the equation $\mathcal{L}h = 0$ yields (remembering that we are allowed to integrate by parts)

$$0 = \big(\mathcal{L}h, h\big)_{L^2(T)} \ge \frac{1}{2}\|\nabla h\|_{L^2(T)} - (1 + c'\varepsilon_0)\|h\|_{L^2(T)} \ge \left(\frac{3 - c\varepsilon_0}{2} - (1 + c'\varepsilon_0)\right)\|h\|_{L^2(T)}.$$

This implies $h = 0$ if we choose $\varepsilon_0 > 0$ small enough so that $\big(\frac{3 - c\varepsilon_0}{2} - (1 + c'\varepsilon_0)\big) > 0$. ☐

Our proof of the a priori estimate in Proposition 9.1 is a variation of that for [Bam12, Proposition 5.1]. We explain the central ideas first before presenting the complete proof. It suffices to prove an a priori estimate for the $\|\cdot\|_{C^0_{\lambda;*}}$-norm. Similar to the proof of Proposition 9.18 we assume that such an estimate does not hold. So there is a sequence

(M^i, g^i) of Riemannian manifolds satisyfing the assumptions of Proposition 9.1, and $h^i \in C^0_\lambda(\mathrm{Sym}^2(T^*M))$ so that

$$\|h^i\|_{C^0_\lambda;*} = 1 \quad \text{and} \quad \|f^i\|_{0,\lambda} \to 0,$$

where $f^i := \mathcal{L}_{g^i} h^i$. Consider the canonical decompositions

$$h^i = \bar{h}^i + \sum_k \rho_k u^i_k + \sum_l \varrho_l v^i_l,$$

given by Lemma 9.8 resp. Proposition 9.10, so that (up to a multiplicative universal constant) it holds

$$\|h^i\|_{C^0_\lambda;*} = \|\bar{h}^i\|_{C^0_\lambda} + \max_k |u^i_k| + \max_l |v^i_l|.$$

In this outline we only consider tubes. From Lemma 9.8 (or rather its proof) we know that $|u^i_k| = O(\|h^i\|_{C^0})$, and hence $|u^i_k| \to 0$ by Proposition 9.18. For simplicity assume that $\|\bar{h}^i\|_{C^0_\lambda} = 1$, and that there exists a point x^i in some Margulis tube T_{k^i} so that $|\bar{h}^i|_{C^0_\lambda}(x^i) = \frac{1}{W_\lambda(x^i)} |\bar{h}^i|_{C^0}(x^i) = 1$. Note that $x^i \in (T_{k^i})_{\mathrm{small}}$,

$$r_{\partial T}(x^i) \to \infty \quad \text{and} \quad r_{\gamma_{k^i}}(x^i) \to \infty,$$

where $r_\partial(x) = d(x, \partial(T_{k^i})_{\mathrm{small}})$ and $r_{\gamma_{k^i}}(x) = d(x, \gamma_{k^i})$ for the core geodesic γ_{k^i} of T_{k^i}. Indeed, otherwise the weight $\frac{1}{W_\lambda(x^i)}$ is bounded from above by some constant, and so Proposition 9.18 would yield a contradiction.

Abbreviate $r := r_{\partial T}$. For simplicity assume that $r(x^i) = \frac{R_{k^i}}{2}$, where R_{k^i} is the radius of $(T_{k^i})_{\mathrm{small}}$, so that the weight $\frac{1}{W_\lambda}$ is maximal at x^i. Let $s(x) = r(x) - \frac{R_{k^i}}{2}$ be the radial function centered at x^i. By construction, the rescaled tensors $\underline{h}^i := \frac{1}{W_\lambda(x^i)} \bar{h}^i$ satisfy a growth estimate

$$|\underline{h}^i|(x) \le C\left(e^{-\lambda s(x)} + e^{\lambda s(x)}\right) \tag{9.34}$$

for some constant C (independent of i). As in the proof Proposition 9.18, after passing to suitable covers and taking a convergent subsequence, we obtain a two sided infinite hyperbolic cusp $(T^2 \times \mathbb{R}, g_{cusp})$ and a tensor field $\underline{h}^\infty$ only depending on the radial coordinate s so that $|\underline{h}^\infty|(x^\infty) = 1$, $\mathcal{L}^\infty \underline{h}^\infty = 0$, and

$$|\underline{h}^\infty|(x) \le C\left(e^{-\lambda s(x)} + e^{\lambda s(x)}\right).$$

As $\underline{h}^\infty$ only depends on s, $\mathcal{L}^\infty \underline{h}^\infty = 0$ is the linear system of ODEs in (9.14). If $|\lambda|$ is smaller than the absolute value of the non-zero exponential growth rates of the fundamental solutions of (9.14), the growth condition (9.34) implies that $\underline{h}^\infty$ is a trivial Einstein variation (see Lemma 9.20).

On the other hand, by definition, $\bar{h}^i$ satisfies

$$|\bar{h}^i|(x^i) \le |\bar{h}^i - u|(x^i)$$

for any trivial Einstein variation u on $(T_{k^i})_{\mathrm{small}}$ (see the proof of Lemma 9.8). This implies

$$|\underline{h}^\infty|(x^\infty) \le |\underline{h}^\infty - u|(x^\infty)$$

for any trivial Einstein variation u on $(T^2 \times \mathbb{R}, g_{cusp})$. But as $\underline{h}^\infty$ is itself a trivial Einstein variation, choosing $u = \underline{h}^\infty$ implies $|\underline{h}^\infty|(x^\infty) = 0$. But this contradicts $|\underline{h}^\infty|(x^\infty) = 1$ (which we know from $|\underline{h}^i|(x^i) = 1$).

From this outline the following is apparant. Inside the small part of a tube, the key requirement on the weight $\frac{1}{W_\lambda}$ is that away from its maximum, the decay rate of the weight is strictly smaller than the absolute value of the non-zero exponential growth rates of the fundamental solutions of the linear system of ODEs in (9.14).

Now we present the complete proof.

Proof of the a priori estimate in Proposition 9.1. **Step 1 (Set-Up):** As mentioned in the beginning of this section, we know that the weighted integral estimates hold, and so we only have to show $\|h\|_{C^{2,\alpha}_\lambda;*} \le C\|\mathcal{L}h\|_{0,\lambda}$ for a universal constant C. Because of the Schauder estimates for the $*$-norm (Lemma 9.7), it suffices to prove

$$\|h\|_{C^0_\lambda(M);*} \le C\|\mathcal{L}h\|_{0,\lambda}. \tag{9.35}$$

Denote by $\varepsilon_0, \bar{\epsilon}_0$ the constants obtained in Proposition 9.18.

Arguing by contradiction, we assume that a constant C as in (9.35) does not exist. Then there exists a sequence of finite volume Riemannian 3-manifolds (M^i, g^i) satisfying the curvature assumptions in Proposition 9.1, and tensor fields h^i such that

$$\|h^i\|_{C^0_\lambda(M^i);*} = 1 \text{ for all } i \in \mathbb{N} \quad \text{and} \quad \|f^i\|_{0,\lambda} \to 0, \text{ as } i \to \infty,$$

where as before we abbreviate $f^i := \mathcal{L}_{g^i} h^i$.

Consider the canonical decomposition

$$h^i = \bar{h}^i + \sum_k \rho^i_k u^i_k + \sum_l \varrho^i_l v^i_l,$$

where u^i_k and v^i_l are the canonical choices of trivial Einstein variations on a Margulis tube and a cusp, respectively, given by Lemma 9.8 and Proposition 9.10. So it holds

$$\|h^i\|_{C^0_\lambda(M^i);*} \le \|\bar{h}^i\|_{C^0_\lambda(M)} + \max_k |u^i_k| + \max_l |v^i_l| \le C\big(\|h^i\|_{C^0_\lambda(M^i);*} + \|f^i\|_{0,\lambda}\big)$$

for a universal constant C. We know $|u^i_k| \le C\|h^i\|_{C^0(M)}$ from Lemma 9.8 (or rather its proof) and $\|h^i\|_{C^0} \le C\|f^i\|_{0,\lambda}$ by Proposition 9.18, so that $\max_k |u^i_k| \to 0$. Also $\max_l |v^i_l| \to 0$ as we have $|v^i_l| = O(\|f^i\|_{0,\lambda})$ by Proposition 9.10. Together with Proposition 9.18, this shows that for all i large enough it holds

$$\frac{3}{4} \le \|\bar{h}^i\|_{C^0_\lambda} \le C \quad \text{and} \quad \|\bar{h}^i\|_{C^0(M)} \to 0, \text{ as } i \to \infty.$$

Choose $x^i \in M^i$ with $|\bar{h}^i|_{C^0_\lambda}(x^i) = \frac{1}{W_\lambda(x^i)}|\bar{h}^i|(x^i) \ge \frac{1}{2}$. Since the inverse weight function W_λ is constant to 1 outside M_{small}, it holds $|\bar{h}^i|_{C^0_\lambda}(x) = |\bar{h}^i|_{C^0}(x)$ for $x \notin M^i_{\text{small}}$ and hence $x^i \in M^i_{\text{small}}$. After passing to a subsequence, either x^i is contained in a cusp C_{l^i} for all i, or x^i is contained in a Margulis tube T_{k^i} for all i. We distinguish these two cases.

Step 2 (Cusps): We start by considering the case that x^i is contained in a cusp C_{l^i}. Abbreviate $r^i = d(x^i, \partial(C_{l^i})_{\text{small}})$. The growth estimate (9.12) of Proposition 9.10 shows that for some universal constant C, we have

$$\frac{1}{2} \leq |\bar{h}^i|_{C^0_\lambda}(x^i) \leq C\big(\|f^i\|_{0,\lambda} + e^{-(1-\lambda)r^i}\big),$$

and thus

$$\frac{1}{2} \leq \frac{1}{4} + Ce^{-(1-\lambda)r^i}$$

for large enough i. Hence $r^i \leq (1-\lambda)^{-1}\log(4C)$, that is, r^i is bounded by the universal constant $\bar{R} := (1-\lambda)^{-1}\log(4C)$. But then $\frac{1}{2} \leq |\bar{h}^i|_{C^0_\lambda}(x^i) \leq e^{\lambda\bar{R}}\|\bar{h}^i\|_{C^0} \to 0$, which is a contradiction.

Step 3 (Tubes): It remains to consider the case that x^i is contained in a Margulis tube T_{k^i}. Denote by γ_{k^i} the core geodesic of T_{k^i}. Set $\bar{f}^i := \mathcal{L}_{g^i}\bar{h}^i$. Since $\|\rho^i_k\|_{C^{2,\alpha}}$ is uniformly bounded, it follows from the proof of Lemma 9.7 that $\|\mathcal{L}_{g^i}(\sum_k \rho^i_k u^i_k)\|_{C^0_\lambda} \leq C\max_k|u^i_k| \to 0$, and the analogous statement holds for v^i_l. Thus

$$\|\bar{f}^i\|_{C^0_\lambda} \to 0, \text{ as } i \to \infty.$$

Recall that $\|\bar{h}^i\|_{C^0(M)} \to 0$, and $\frac{1}{W_\lambda(x^i)}|\bar{h}^i|_{C^0}(x^i) \geq \frac{1}{2}$ from the definition of x^i. Thus $W_\lambda(x^i) \to 0$. Hence, for i large enough, $x^i \in M^i_{\text{small}}$ and

$$r_{\partial T^i}(x^i), r_{\gamma_{k^i}}(x^i) \to \infty,$$

where $r_{\partial T^i}(x) = d\big(x, \partial(T_{k^i})_{\text{small}}\big)$ and $r_{\gamma_{k^i}}(x) = d(x,\gamma_{k^i})$. Abbreviate $r^i := r_{\partial T^i}(x^i)$.

Consider the torus $T^i(r^i) := \{x \in T_{k^i} \,|\, r_{\partial T^i}(x) = r^i\}$ in T_{k^i} containing x^i. Take a covering $\hat{T}^i \to (T_{k^i})_{\text{small}}$ such that

$$\operatorname{diam}\big(\hat{T}_{k^i}(r^i)\big) \leq 10 \quad \text{and} \quad \operatorname{inj}(\hat{x}^i) \geq 1,$$

where $\operatorname{diam}\big(\hat{T}^i(r^i)\big)$ is the diameter with respect to the intrinsic metric of $\hat{T}_{k^i}(r^i)$. Therefore, after passing to a subsequence,

$$\big(\hat{T}^i, \hat{x}^i\big) \xrightarrow{\text{pointed } C^{2,\beta}} \big(T^\infty, x^\infty\big),$$

where $\hat{x}^i$ is a preimage of x^i in $\hat{T}^i$ and where the convergence is pointed $C^{2,\beta}$-convergence for some $\beta \in (0,\alpha)$. From the curvature decay condition (9.1), and the fact that $r_{\partial T^i}(x^i), r_{\gamma_{k^i}}(x^i) \to \infty$, it follows that the limit manifold T^∞ is a hyperbolic cusp which is two-sided infinite, that is, $T^\infty = T^2 \times \mathbb{R}$ and

$$g^\infty = e^{-2r}g_{Flat} + dr^2$$

for some flat metric g_{Flat} on the torus T^2. Denote by $\hat{h}^i$ the pullback of $\bar{h}^i$ to $\hat{T}^i$, and analogously $\hat{f}^i$ shall denote the pullback of $\bar{f}^i$. Since going to covers does not change Hölder or C^k-norms, all the estimates on $\bar{h}^i$ and $\bar{f}^i$ still hold for $\hat{h}^i$ and $\hat{f}^i$.

Set $s^i(x) := r_{\partial T^i}(x) - r^i$. Then $s^i(x^i) = 0$ and $s^i \to s^\infty$, where s^∞ is the $\mathbb{R}$-coordinate in $T^\infty = T^2 \times \mathbb{R}$ with $s^\infty(x^\infty) = 0$.

Abbreviate $r(\cdot) := r_{\partial T^i}(\cdot)$, and write $R^i := R_{k^i}$ for the radius of $(T_{k^i})_{\text{small}}$, that is, the distance of the core geodesic to $\partial(T_{k^i})_{\text{small}}$. After passing to a subsequence, we may distinguish between the following three cases. We will show that each case leads to a contradiction, thus completing the proof.

Case 1: $R^i - 2r^i \to -\infty$

Note

$$e^{\lambda(R^i - r^i)}W_\lambda(x) = e^{\lambda(R^i-r^i)}\left(e^{-\lambda r(x)} + e^{\lambda(r(x)-R^i)}\right) = e^{\lambda(R^i-2r^i)}e^{-\lambda s^i(x)} + e^{\lambda s^i(x)}.$$

Define $\underline{h}^i$ to be the rescaled tensor $e^{\lambda(R^i-r^i)}\hat{h}^i$. As $\|\hat{h}^i\|_{C^0_\lambda} \le C$

$$|\underline{h}^i|(x) = e^{\lambda(R^i-r^i)}W_\lambda(x)\left|\frac{1}{W_\lambda(x)}|\hat{h}^i|(x)\right| \le C\big(e^{\lambda(R^i-2r^i)}e^{-\lambda s^i(x)} + e^{\lambda s^i(x)}\big).$$

Thus $\underline{h}^i$ is locally uniformly bounded near x^i. By Schauder-estimates, the same holds true for its derivatives. As a consequence, after passing to a subsequence we obtain that $\underline{h}^i \to \underline{h}^\infty$ in the pointed $C^{2,\beta'}$-sense for some $\beta' \in (0,\beta)$ where the symmetric $(0,2)$-tensor $\underline{h}^\infty$ on T^∞ satisfies

$$|\underline{h}^\infty|(x) \le Ce^{\lambda s^\infty(x)}.$$

Note $|\underline{h}^i|(\hat{x}^i) = \frac{1}{W_\lambda(\hat{x}^i)}|\hat{h}^i|(\hat{x}^i) \ge \frac{1}{2}$, and hence $|\underline{h}^\infty|(x^\infty) \ge \frac{1}{2}$. In particular, $\underline{h}^\infty$ is non-zero.

The same calculation as above shows that $\underline{f}^i = e^{\lambda(R^i-r^i)}\hat{f}^i$ satisfies

$$|\underline{f}^i|(x) \le \|f^i\|_{C^0_\lambda}\big(e^{\lambda(R^i-2r^i)}e^{-\lambda s^i(x)} + e^{\lambda s^i(x)}\big).$$

As $\|\hat{f}^i\|_{C^0_\lambda} \to 0$, this implies $\underline{f}^i \to 0$. By stability of elliptic equations the limit tensor $\underline{h}^\infty$ therefore solves $\mathcal{L}^\infty\underline{h}^\infty = 0$.

Note that the tensor $\underline{h}^\infty$ is obtained as a limit of tensors that are lifts of tensors on $(T_{k^i})_{\text{small}}$, and since the tubes $(T_{k^i})_{\text{small}}$ converge to a line in the pointed Gromov-Hausdorff topology, the limit tensor $\underline{h}^\infty$ only depends on s^∞ (we refer to Section 9.3 for the definition of what it means for a tensor to only depend on the $\mathbb{R}$-coordinate s). Lemma 9.20 below then shows that $\underline{h}^\infty = 0$, which contradicts $|\underline{h}^\infty|(x^\infty) \ge \frac{1}{2}$.

Case 2: $R^i - 2r^i \to \infty$

Compute

$$e^{\lambda r^i}W_\lambda(x) = e^{\lambda r^i}\left(e^{-\lambda r(x)} + e^{\lambda(r(x)-R^i)}\right) = e^{-\lambda s^i(x)} + e^{-\lambda(R^i-2r^i)}e^{\lambda s^i(x)}.$$

By the same arguments as in *Case 1*, after passing to a subsequence, the rescaled tensors $\underline{h}^i := e^{\lambda r^i}\hat{h}^i$ converge to some *non-zero* $\underline{h}^\infty$ which only depends on s^∞ and that satisfies

$$\mathcal{L}^\infty\underline{h}^\infty = 0 \quad\text{and}\quad |\underline{h}^\infty|(x) \le Ce^{-\lambda s^\infty(x)}$$

for some $C > 0$. Again by Lemma 9.20 below, this implies $\underline{h}^\infty = 0$, a contradiction.

Case 3: $R^i - 2r^i \to A \in \mathbb{R}$

The same calculation as in *Case 1* shows that the rescaled tensors $\underline{h}^i := e^{\lambda(R^i-r^i)}\hat{h}^i$ converge to some *non-zero* $\underline{h}^\infty$ that only depends on s^∞, satisfies the growth estimate

$$|\underline{h}^\infty|(x) \le C\big(e^Ae^{-\lambda s^\infty(x)} + e^{\lambda s^\infty(x)}\big),$$

and solves $\mathcal{L}^\infty\underline{h}^\infty = 0$. Lemma 9.20 shows that $\underline{h}^\infty$ is a trivial Einstein variation.

By the proof of Lemma 9.8, we know that $\bar{h}^i$ satisfies $|\bar{h}^i|(c^i_{k^i}) \leq |\bar{h}^i - u|(c^i_{k^i})$ for any trivial Einstein variation u on $(T_{k^i})_{\mathrm{small}}$. Here $c^i_{k^i} \in (T_{k^i})_{\mathrm{small}})$ is a chosen point with $r(c^i_{k^i}) = \frac{R^i}{2}$. Since C^0-norms and the space of trivial Einstein variations are invariant under passing to covers and scaling we also have

$$|\underline{h}^i|(\hat{c}^i_k) \leq |\underline{h}^i - u|(\hat{c}^i_k),$$

for every trivial Einstein variation u on $\hat{T}^i$ (notations are as above). By assumption, $r^i - \frac{R^i_k}{2}$ is bounded. Hence $d(\hat{x}^i, \hat{c}^i_{k^i})$ is also bounded. Therefore, there is a limit point $c^\infty \in T^\infty$. Note that any trivial Einstein variation on T^∞ is the limit of trivial Einstein variations on $\hat{T}^i$. Thus

$$|\underline{h}^\infty|(c^\infty) \leq |\underline{h}^\infty - u|(c^\infty)$$

for every trivial Einstein variation u on T^∞. Because $\underline{h}^\infty$ is itself a trivial Einstein variation, choosing $u = \underline{h}^\infty$ shows $\underline{h}^\infty(c^\infty) = 0$. As trivial Einstein variations have constant norm (with respect to a cusp metric), this implies $\underline{h}^\infty = 0$ everywhere. This is a contradiction. □

The following lemma was used at the end of the above proof to show that each of the three cases leads to a contradiction. The lemma plays the role of Proposition 7.1 in [Bam12]. Our proof is the same as that in [Bam12], but adapted to our context. Concerning terminology, we refer to Section 9.3 for the notion of a cusp metric, and what it means for a tensor to only depend on r.

Lemma 9.20. *Assume $T^2 \times \mathbb{R}$ is equipped with a hyperbolic cusp metric. Let h be a tensor that only depends on the $\mathbb{R}$-coordinate r, and that solves $\mathcal{L}h = 0$. If h satisfies*

$$|h|(r) \leq C\left(e^{-\lambda r} + e^{\lambda r}\right)$$

for some $C > 0$ and $\lambda \in (0,1)$, then h is a trivial Einstein variation. If moreover h satisfies

$$|h|(r) \leq Ce^{-\lambda r} \quad \text{for all } r \in \mathbb{R} \quad \text{or} \quad |h|(r) \leq Ce^{\lambda r} \quad \text{for all } r \in \mathbb{R},$$

then $h = 0$.

Proof. We first show $\mathrm{tr}(h) = 0$. From (9.15) we have $Q_1(\frac{\partial}{\partial r})\big(\mathrm{tr}(h)\big)(r) = 0$ for a quadratic polynomial Q_1 with roots $1 \pm \sqrt{5}$. Thus by applying Lemma 9.14 we get

$$\mathrm{tr}(h)(r) = A_+ e^{(1+\sqrt{5})r} + A_- e^{(1-\sqrt{5})r}$$

for some $A_+, A_- \in \mathbb{R}$. By assumption $|\mathrm{tr}(h)|(r) \leq C(e^{\lambda r} + e^{-\lambda r})$. Since $\lambda < 1 + \sqrt{5}$, taking $r \to \infty$ implies $A_+ = 0$. Similarly, since $\lambda < \sqrt{5} - 1$, taking $r \to -\infty$ shows $A_- = 0$. Thus $\mathrm{tr}(h) = 0$ everywhere.

Using (9.14), the same argument shows $h_{33} = 0$ and $h_{i3} = 0$ everywhere. For h_{i3} this uses the assumption $\lambda < 1$.

By (9.14) we have $Q_3(\frac{\partial}{\partial r})\big(e^{2r} h_{ij}\big) = 0$ for a quadratic polynomial Q_3 with roots 0 and 2. So invoking Lemma 9.14 yields

$$e^{2r} h_{ij} = A + Be^{2r}$$

for some $A, B \in \mathbb{R}$. As $|e^{2r} h_{ij}| \le |h|(r) \le C(e^{\lambda r} + e^{-\lambda r})$ and $\lambda < 2$, taking $r \to \infty$ implies $B = 0$. So $h_{ij} = Ae^{-2r}$ for some $A \in \mathbb{R}$.

With everything up to now we have shown that h is a trivial Einstein variation. Now assume that h either satisfies $|h|(r) \le C^{\lambda r}$ or $|h|(r) \le C^{-\lambda r}$. Note that trivial Einstein variations have constant norm (with respect to a cusp metric). Thus taking $r \to -\infty$ or $r \to \infty$ implies $|h| = 0$. □

9.5. **Surjectivity of $\mathcal{L}$.** In order to establish Proposition 9.1 we have to show that

$$\mathcal{L} : \left(C^{2,\alpha}_{\lambda}\big(\mathrm{Sym}^2(TM)\big), \|\cdot\|_{2,\lambda;*}\right) \longrightarrow \left(C^{0,\alpha}_{\lambda}\big(\mathrm{Sym}^2(TM)\big), \|\cdot\|_{0,\lambda}\right)$$

is an invertible operator, and that $\|\mathcal{L}^{-1}\|_{\mathrm{op}}$ is bounded by a universal constant. In Section 9.4 we proved that $\mathcal{L}$ satisfies an a priori estimate $\|h\|_{2,\lambda;*} \le C\|\mathcal{L}h\|_{0,\lambda}$. Therefore, to complete the proof of Proposition 9.1, we have to show that $\mathcal{L}$ is surjective. This will be done with the strategy used in the proof of Proposition 4.7, which had two main ingredients:

- (Weak solutions exist) For any f in the target space, the equation $\mathcal{L}(h) = f$ has a weak solution.
- (Regularity) If f is a smooth tensor in the target space and h is a solution of $\mathcal{L}h = f$, then h is contained in the source space.
- (Approximation) For any f in the target space, there is a sequence of smooth tensors $(f_i)_{i\in\mathbb{N}}$ converging to f;

In the setting of Section 4, the regularity part is immediate due to local (euclidean) regularity theory of elliptic PDEs. In our present non-compact setting this is no longer the case. Local regularity theory only shows that h is of a certain regularity in local coordiantes, but it does *not* necessarily mean that the globally defined norm $\|h\|_{2,\lambda;*}$ is finite. That this is indeed the case is shown in the following lemma. Here we assume that M satisfies the assumptions stated in Proposition 9.1. Also, we refer to Notation 2.2 for our convention of the O-notation.

Lemma 9.21. *Let $f \in C^{0,\alpha}_{\lambda}\big(\mathrm{Sym}^2(T^*M)\big)$ and $h \in C^2\big(\mathrm{Sym}^2(T^*M)\big) \cap H^1_0(M)$ be a solution of*

$$\mathcal{L}h = f.$$

Let C be a cusp of M, and $i_C \in (0,1)$ so that $i_C \le \mathrm{inj}(x)$ for all $x \in \partial C_{\mathrm{small}}$. Then it holds

$$\sup_{C_{\mathrm{small}}} |h| = O\Big(\frac{1}{i_C}\|f\|_{0,\lambda}\Big).$$

In particular, $\sup_M |h| < \infty$ and $h \in C^{2,\alpha}_{\lambda}\big(\mathrm{Sym}^2(T^*M)\big)$.

Because the boundary tori ∂C can have arbitrary large diameter, there can be no universal lower bound on i_C. The point of Lemma 9.21 is *not* to obtain a universal bound for $\|h\|_{C^0}$, but just that $h \in C^{2,\alpha}_{\lambda}\big(\mathrm{Sym}^2(T^*M)\big)$. Once this is known, one obtains universal estimates by the results of Section 9.4.

Proof. Fix a cusp C, and let g_{cusp} be the hyperbolic cusp metric on C_{small} given by Proposition 8.2. This satisfies $|g - g_{cusp}|_{C^2}(x) = O\big(\varepsilon_0 e^{-\eta r(x)}\big)$, where $r(x) = d(x, \partial C_{\text{small}})$, and η is the decay rate in the curvature decay condition (9.1). In particular, up to universal constants, the C^k-norms ($k \le 2$) with respect to g and g_{cusp} agree. Let $\hat{\cdot}$ be the averaging operation of the cusp metric g_{cusp} (see Lemma 9.12). First, we recall an estimate for $|\hat{h}|$ that was established in the proof of Proposition 9.10. In a second step we obtain estimates on $|h - \hat{h}|$. We do this by using De Giorgi–Nash–Moser estimates in the universal cover to reduce this problem to bounding L^2-norms in the universal cover $\tilde{M}$, which we obtain from weighted L^2-estimates in M. Throughout this proof, for $r \ge 0$ we denote by $T(r)$ the torus in C_{small} all of whose points have distance r to $\partial C_{\text{small}}$.

Step 1 (Estimating $|\hat{h}|$): In the proof of Proposition 9.10 we established (9.27), which states

$$\sup_{C_{\text{small}}} |\hat{h}| = O(\|f\|_{0,\lambda}). \tag{9.36}$$

Even though in the formulation of Proposition 9.10 we assume $\|h\|_{C^0} < \infty$, we pointed out that for (9.27) only the assumption $h \in C^2\big(\mathrm{Sym}^2(T^*M)\big) \cap H_0^1(M)$ is needed.

We also mention the following estimates that will be needed in Step 2. Denote by $\mathcal{L}_{cusp}$ the operator $\frac{1}{2}\Delta_L + 2\mathrm{id}$ with respect to the metric g_{cusp}. From $|g - g_{cusp}|_{C^2}(x) = O\big(\varepsilon_0 e^{-\eta r(x)}\big)$ it follows that

$$|\mathcal{L}_{cusp} h - f|_{C^0}(x) \le O\big(\varepsilon_0 e^{-\eta r(x)} |h|_{C^2}(x)\big). \tag{9.37}$$

Now *ii)* of Lemma 9.12 and (9.37) together yield

$$\big|\widehat{\mathcal{L}_{cusp} h} - \hat{f}\big|_{C^0}(x) = O\big(\varepsilon_0 \psi(r) e^{-(\eta-2)r}\big) \tag{9.38}$$

where $r = r(x)$, and $\psi \in L^1(\mathbb{R}_{\ge 0})$ is the function defined in (9.16).

Step 2 (Estimating $|h - \hat{h}|$): Since $\dim(M) = 3$, we can apply the estimates from Lemma 2.8 with $q = 4$. Hence the same argument that led to (4.9) shows that for all $x \in C_{\text{small}}$, and any lift $\tilde{x} \in \tilde{M}$ of x, it holds

$$|\tilde{h} - \tilde{\hat{h}}|_{C^0}(\tilde{x}) \le C\Big(\|\tilde{h} - \tilde{\hat{h}}\|_{L^2(B(\tilde{x},\rho))} + \|\mathcal{L}_{cusp}(\tilde{h} - \tilde{\hat{h}})\|_{L^2(B(\tilde{x},\rho))}\Big) \tag{9.39}$$

for some universal constant C. Here $\rho > 0$ is the universal radius appearing in the definition of the Hölder norms.

We want to bound these L^2-norms in $\tilde{M}$ by weighted L^2-norms in M. Towards this goal, note that for any $y \in M$ and lift $\tilde{y} \in \tilde{M}$ of y it holds $\#\big(\pi^{-1}(y) \cap B(\tilde{y}, 2\rho)\big) \le C \frac{1}{\mathrm{inj}(y)^2}$, where C is a universal constant, and $\pi : \tilde{M} \to M$ is the universal covering projection. Indeed, this follows by a simple area counting argument. Note that (up to universal constant) it is irrelevant whether the injectivity radius is taken with respect to g or g_{cusp}. Choose $i_C \in (0,1)$ so that $\mathrm{inj}(x) \ge i_C$ for all $x \in \partial C_{\text{small}}$. Since g_{cusp} is hyperbolic, the argument from the proof of Corollary 7.7 shows that $\mathrm{inj}(y) \ge e^{-d(y, \partial C_{\text{small}})} i_C$ for all $y \in C_{\text{small}}$. Therefore, there is a universal constant C so that for any function $u : M \to \mathbb{R}_{\ge 0}$,

$x \in C_{\mathrm{small}}$, and lift $\tilde{x} \in \tilde{M}$ of x it holds

$$\int_{B(\tilde{x},\rho)} \tilde{u}\, d\mathrm{vol}_{\tilde{g}_{cusp}} \leq C \frac{1}{i_C^2} \int_{B(x,\rho)} e^{2r(y)} u\, d\mathrm{vol}_{g_{cusp}}, \tag{9.40}$$

where $\tilde{u} = u \circ \pi$ and $r(y) = d(y, \partial C_{\mathrm{small}})$. See the claim in the proof of Proposition 7.5 for more details as to why this integral estimate follows from the preimage counting.

We now show how (9.40) can be used to estimate the L^2-norms in (9.39). We start with the first term. Inequality $(**)$ on p. 520 of [GKS07] shows that for any flat 2-torus T^2 of diameter 1, we have $\lambda_1(T^2) \geq e^{-2}$. Together with a scaling argument, this implies that if T^2 is a flat 2-torus of $\mathrm{diam}(T^2) \leq 1$, then $\lambda_1(T^2) \geq \frac{1}{\mathrm{diam}(T^2)^2} e^{-2}$. Therefore, for any function u it holds

$$\int_{T(r)} |u - \hat{u}|^2\, d\mathrm{vol}_{cusp} \leq e^2 \mathrm{diam}(T(r), g_{cusp})^2 \int_{T(r)} |\nabla u|^2\, d\mathrm{vol}_{cusp},$$

where $\hat{u}$ is the average of u over $T(r)$ (see the discussion before Lemma 9.12). Applying this componentwise and multiplying by e^{2r} implies

$$e^{2r} \int_{T(r)} |h - \hat{h}|^2_{C^0}\, d\mathrm{vol}_{cusp} \leq e^2 D^2 \int_{T(r)} |h|^2_{C^1}\, d\mathrm{vol}_{cusp},$$

where D is the universal constant appearing in the definition of the small part. Thus

$$\int_{C_{\mathrm{small}}} e^{2r} |h - \hat{h}|^2_{C^0}\, d\mathrm{vol}_{g_{cusp}} \leq e^2 D^2 \int_{C_{\mathrm{small}}} |h|^2_{C^1}\, d\mathrm{vol}_{cusp} = O(\|h\|^2_{H^2(C_{\mathrm{small}})}). \tag{9.41}$$

We know from the proof of Lemma 9.16 that $\|h\|_{H^1(C_{\mathrm{small}})} = O(\|f\|_{0,\lambda})$ (in fact, we even showed $\|h\|_{H^2(C_{\mathrm{small}})} = O(\|f\|_{0,\lambda})$). Thus (9.40) and (9.41) yield

$$\int_{B(\tilde{x},\rho)} |\tilde{h} - \tilde{\hat{h}}|^2\, d\mathrm{vol}_{\tilde{g}_{cusp}} = O\Big(\frac{1}{i_C^2} \|f\|^2_{0,\lambda}\Big). \tag{9.42}$$

This completes the bound of the first L^2-norm in (9.39).

Towards bounding the second L^2-norm in (9.39), we use the triangle inequality, (9.37), $ii)$ of Lemma 9.12, and (9.38) to estimate

$$\begin{aligned} |\mathcal{L}_{cusp}\tilde{h} - \mathcal{L}_{cusp}\tilde{\hat{h}}|(\tilde{x}) \leq& |\mathcal{L}_{cusp}\tilde{h} - \tilde{f}|(\tilde{x}) + |\tilde{f}|(\tilde{x}) + |\tilde{\hat{f}}|(\tilde{x}) + |\mathcal{L}_{cusp}\tilde{\hat{h}} - \tilde{\hat{f}}|(\tilde{x}) \\ =& O\Big(\varepsilon_0 e^{-\eta\tilde{r}} |\tilde{h}|_{C^2}(\tilde{x}) + \|f\|_{0,\lambda} e^{-\lambda\tilde{r}} + \varepsilon_0 \psi(r) e^{-(\eta-2)\tilde{r}}\Big), \end{aligned} \tag{9.43}$$

where $\tilde{r} = r \circ \pi$, and $\psi \in L^1(\mathbb{R}_{\geq 0})$ was defined in (9.16). Invoking (9.40), and using $\eta > 1$, we obtain

$$\begin{aligned} \int_{B(\tilde{x},\rho)} \big(e^{-\eta\tilde{r}} |\tilde{h}|_{C^2}\big)^2 d\mathrm{vol}_{\tilde{g}_{cusp}} \leq& C \frac{1}{i_C^2} \int_{B(x,\rho)} e^{-2(\eta-1)r} |h|^2_{C^2} d\mathrm{vol}_{g_{cusp}} \\ \leq& C \frac{1}{i_C^2} \int_{C_{\mathrm{small}}} |h|^2_{C^2} d\mathrm{vol}_{g_{cusp}} \\ =& O\Big(\frac{1}{i_C^2} \|f\|^2_{0,\lambda}\Big), \end{aligned} \tag{9.44}$$

where for the last inequality we used $\|h\|_{H^2(C_{\mathrm{small}})} = O(\|f\|_{0,\lambda})$ (this was established in the proof of Lemma 9.16). Also

$$\int_{B(\tilde{x},\rho)} e^{-2\lambda\tilde{r}}\, d\mathrm{vol}_{\tilde{g}_{cusp}} \le \mathrm{vol}(B(\tilde{x},\rho)) = O(1). \tag{9.45}$$

Recall that the function ψ is defined by $\psi(r) := \int_{T(r)} |h|_{C^2}$. Denote by $⨍_{T(r)}$ the average integral $\frac{1}{\mathrm{vol}_2(T(r))}\int_{T(r)}$. By the Jensen inequality $\left(⨍_{T(r)} |h|_{C^2}\right)^2 \le ⨍_{T(r)} |h|^2_{C^2}$, which implies $\left(e^r\psi(r)\right)^2 \le \int_{T(r)} |h|^2_{C^2}$. So $e^r\psi(r) \in L^2(\mathbb{R}_{\ge 0})$, and $\|e^r\psi\|_{L^2(\mathbb{R}_{\ge 0})} \le \|h\|_{H^2(C_{\mathrm{small}})} = O(\|f\|_{0,\lambda})$. Because ρ is universal, $\mathrm{area}(B(\tilde{x},\rho)\cap\{\tilde{y}\,|\,\tilde{r}(\tilde{y}) = s\})$ is bounded by a universal constant C for all s. Thus, as $\eta > 1$,

$$\begin{aligned}\int_{B(\tilde{x},\rho)} \left(e^{-(\eta-2)\tilde{r}}\psi\right)^2 \le & C\int_{r(x)-\rho}^{r(x)+\rho} e^{-2(\eta-1)r}(e^r\psi)^2\, dr \\ \le & C\int_{r(x)-\rho}^{r(x)+\rho} (e^r\psi)^2\, dr \\ = & O\left(\|f\|^2_{0,\lambda}\right).\end{aligned} \tag{9.46}$$

Combining (9.39), (9.42)-(9.46) yields

$$\sup_{C_{\mathrm{small}}} |h-\hat{h}| = O\Big(\frac{1}{i_C}\|f\|_{0,\lambda}\Big).$$

Together with (9.36) this implies $\sup_{C_{\mathrm{small}}} |h| = O\left(\frac{1}{i_C}\|f\|_{0,\lambda}\right)$.

For the last assertions, note that M has only finitely many cusps because M has finite volume. The proven estimate and the compactness of $M \smallsetminus \bigcup_C C_{\mathrm{small}}$ immediately imply that $\sup_M |h| < \infty$. To conclude $h \in C^{2,\alpha}_\lambda\left(\mathrm{Sym}^2(T^*M)\right)$ we have to show that $\|h\|_{C^{2,\alpha}_\lambda;*} < \infty$. As $\|\mathcal{L}h\|_{C^{0,\alpha}_\lambda} < \infty$ by assumption, the Schauder estimates for the $*$-norm (Lemma 9.7) reduce the problem to showing $\|h\|_{C^0_\lambda;*} < \infty$. Due to the compactness of $M \smallsetminus \bigcup_C C_{\mathrm{small}}$, it suffices to show $\|h\|_{C^0_\lambda;*} < \infty$ in each rank 2 cusp of M. But this follows from Proposition 9.10 since $\|h\|_{C^0} < \infty$. □

Recall from the introduction of this section that establishing surjectivity of $\mathcal{L}$ requires an approximation and a regularity result. Lemma 9.21 is the regularity statement. The approximation result is given by the next lemma.

Lemma 9.22. *Let $f \in C^{0,\alpha}_\lambda\left(\mathrm{Sym}^2(T^*M)\right)$ and $\beta \in (0,\alpha)$. Then there is a sequence $(f_\varepsilon)_{\varepsilon>0} \subseteq C^\infty\left(\mathrm{Sym}^2(T^*M)\right)$ so that $\lim_{\varepsilon\to 0}\|f - f_\varepsilon\|_{C^{0,\beta}_\lambda(M)} = 0$.*

Proof. It is well known that for $u \in C^{0,\alpha}(\mathbb{R}^n)$ there is a sequence $(u_\varepsilon)_{\varepsilon>0} \subseteq C^\infty(\mathbb{R}^n)$ so that $\|u - u_\varepsilon\|_{C^{0,\beta}} \to 0$ as $\varepsilon \to 0$. Moreover, if u has compact support inside some open set $\Omega \subseteq \mathbb{R}^n$, the u_ε can be chosen to have compact support in Ω too.

Denote by r the distance function to $M \smallsetminus M_{\mathrm{small}}$. For $k \ge 0$ define $U_k := r^{-1}\big((k-1,k+1)\big)$. Choose a partition of unity $(\eta_k)_{k\ge 0}$ subordinate to the cover $\{U_k\}_{k\ge 0}$. By applying the above approximation result locally, we see that for each k there is $f^{(k)}_\varepsilon$ so that

$\operatorname{supp}(f_\varepsilon^{(k)}) \subseteq U_k$ and $\|(\eta_k f) - f_\varepsilon^{(k)}\|_{C^{0,\beta}} \le \frac{\varepsilon}{2} e^{-\lambda(k+1)}$. Then $f_\varepsilon := \sum_{k=0}^\infty f_\varepsilon^{(k)}$ has the desired property. Indeed, let $x \in M$ be arbitrary and and choose $k_0 \in \mathbb{N}$ so that $k_0 \le r(x) < k_0 + 1$. Then U_{k_0} and U_{k_0+1} are the only sets of the cover $\{U_k\}_{k \ge 0}$ which may contain x. Thus

$$\begin{aligned} |f - f_\varepsilon|_{C^{0,\beta}}(x) &\le |(\eta_{k_0} f) - f_\varepsilon^{(k_0)}|_{C^{0,\beta}}(x) + |(\eta_{k_0+1} f) - f_\varepsilon^{(k_0+1)}|_{C^{0,\beta}}(x) \\ &\le \frac{\varepsilon}{2} e^{-\lambda(k_0+1)} + \frac{\varepsilon}{2} e^{-\lambda(k_0+2)} \\ &\le \varepsilon e^{-\lambda r(x)} \\ &\le \varepsilon W_\lambda(x), \end{aligned}$$

and hence $\|f - f_\varepsilon\|_{C_\lambda^{0,\beta}(M)} = \sup_{x \in M} \frac{1}{W_\lambda(x)} |f - f_\varepsilon|_{C^{0,\beta}}(x) \le \varepsilon$. □

Now we are ready to present the proof of Proposition 9.1.

Proof of Proposition 9.1. As the a priori estimate was established in Section 9.4, it remains to show that $\mathcal{L}$ is surjective. The same argument as in Proposition 4.7 shows that for any $f \in L^2(\operatorname{Sym}^2(T^*M))$, there is a weak solution $h \in H_0^1(\operatorname{Sym}^2(T^*M))$ of $\mathcal{L}f = h$. Fix $f \in C_\lambda^{0,\alpha}(\operatorname{Sym}^2(T^*M))$. Invoking Lemma 9.22 we obtain a sequence $(f_i)_{i \in \mathbb{N}} \subseteq C^\infty(\operatorname{Sym}^2(T^*M))$ so that $\|f - f_i\|_{C_\lambda^{0,\frac{\alpha}{2}}} \to 0$ as $i \to \infty$. Choose weak solutions $h_i \in H_0^1(\operatorname{Sym}^2(T^*M))$ of $\mathcal{L}h_i = f_i$. Then $h_i \in C^\infty(\operatorname{Sym}^2(T^*M))$ and $\mathcal{L}h_i = f_i$ holds in the classical sense. Moreover, Lemma 9.21 implies $h_i \in C_\lambda^{0,\frac{\alpha}{2}}(\operatorname{Sym}^2(T^*M))$. Note that the norms $\|\cdot\|_{C_\lambda^{0,\frac{\alpha}{2}}}$ and $\|\cdot\|_{0,\lambda}$ are equivalent on $C_\lambda^{0,\frac{\alpha}{2}}(\operatorname{Sym}^2(T^*M))$ (but with a non-universal constant). Therefore, the a priori estimate from Proposition 9.18 gives

$$\|h_i - h_j\|_{C^2} \le C \|f_i - f_j\|_{C_\lambda^{0,\frac{\alpha}{2}}} \to 0 \quad \text{as } i, j \to \infty$$

for some (non-universal) constant C. So $(h_i)_{i \in \mathbb{N}} \subseteq C^2(\operatorname{Sym}^2(T^*M))$ is a Cauchy sequence. Let $h \in C^2(\operatorname{Sym}^2(T^*M))$ be the limit tensor field. As M has finite volume, C^2-convergence implies H^1-convergence. Thus $h \in H_0^1(\operatorname{Sym}^2(T^*M))$ and $\mathcal{L}h = f \in C_\lambda^{0,\alpha}(\operatorname{Sym}^2(T^*M))$. Invoking Lemma 9.21 once more yields $h \in C_\lambda^{2,\alpha}(\operatorname{Sym}^2(T^*M))$. Therefore, $\mathcal{L}$ is a surjective mapping from $C_\lambda^{2,\alpha}(\operatorname{Sym}^2(T^*M))$ to $C_\lambda^{0,\alpha}(\operatorname{Sym}^2(T^*M))$. □

10. Proof of the pinching theorem without lower injectvity radius bound

We can now finally state and prove the full version of Theorem 2.

Theorem 10.1. *For all $\alpha \in (0,1)$, $\Lambda \ge 0$, $\lambda \in (0,1)$, $\delta \in (0,2)$, $r_0 \ge 1$, $b > 1$, and $\eta \ge 2 + \lambda$ there exist constants $\varepsilon_0 = \varepsilon(\alpha, \Lambda, \lambda, \delta, r_0, b, \eta) > 0$ and $C = C(\alpha, \Lambda, \lambda, \delta, r_0, b, \eta) > 0$ with the following property. Let M be a 3-manifold that admits a complete Riemannian metric $\bar{g}$ satisfying the following conditions for some $\varepsilon \le \varepsilon_0$:*

i) $\operatorname{vol}(M, \bar{g}) < \infty$*;*
ii) $-1 - \varepsilon \le \sec_{(M,\bar{g})} \le -1 + \varepsilon$*;*

iii) It holds

$$\max_{\pi \subseteq T_x M} |\sec(\pi) + 1|, \, |\nabla R|(x), \, |\nabla^2 R|(x) \leq \varepsilon e^{-\eta d(x, \partial M_{\mathrm{small}})}$$

for all $x \in M_{\mathrm{small}}$*;*

iv) $\|\nabla \mathrm{Ric}(\bar{g})\|_{C^0(M,\bar{g})} \leq \Lambda$*;*

v) It holds

$$\int_M e^{-(2-\delta) r_x(y)} |\mathrm{Ric}(\bar{g}) + 2\bar{g}|^2_{\bar{g}} \, d\mathrm{vol}_{\bar{g}}(y) \leq \varepsilon^2$$

for all $x \in M$ *with*

$$\int_{B(x,2r_0) \smallsetminus B(x,r_0)} e^{-(2-\delta) r_x(y)} \, d\mathrm{vol}_{\bar{g}}(y) > \varepsilon_0,$$

where $r_x(y) = d_{\bar{g}}(x,y)$*;*

vi) It holds

$$e^{b d(x, M_{\mathrm{thick}})} \int_M e^{-(2-\delta) r_x(y)} |\mathrm{Ric}(\bar{g}) + 2\bar{g}|^2_{\bar{g}} \, d\mathrm{vol}_{\bar{g}}(y) \leq \varepsilon^2$$

for all $x \in M_{\mathrm{thin}} \smallsetminus M_{\mathrm{small}}$*.*

Then there exists a hyperbolic metric g_{hyp} *on* M *so that*

$$\|g_{\mathrm{hyp}} - \bar{g}\|_{2,\lambda;*} \leq C \varepsilon^{1-\alpha},$$

where $\|\cdot\|_{2,\lambda;*}$ *is the norm defined in (9.7) with respect to the metric* $\bar{g}$ *and the constants* $\alpha, \lambda, b, \varepsilon_0, \delta, r_0$*.*

Moreover, if for some $\beta \leq 1 - \frac{1}{2}\delta$ *and* $U \subseteq M$ *it holds*

$$\int_M e^{-(2-\delta) r_x(y)} |\mathrm{Ric}(\bar{g}) + 2\bar{g}|^2_{\bar{g}} \, d\mathrm{vol}_{\bar{g}}(y) \leq \varepsilon^{2(1-\alpha)} e^{-2\beta \, \mathrm{dist}_{\bar{g}}(x, U \cup \partial M_{\mathrm{rhick}})} \quad \text{for all } x \in M_{\mathrm{thick}},$$

then

$$|g_{\mathrm{hyp}} - \bar{g}|_{C^{2,\alpha}}(x) \leq C \varepsilon^{1-\alpha} e^{-\beta \mathrm{dist}_{\bar{g}}(x, U \cup \partial M_{\mathrm{thick}})} \quad \text{for all } x \in M_{\mathrm{thick}}.$$

In particular, if $\bar{g}$ *is already hyperbolic outside a region* $U \subseteq M$*, and if*

$$\int_U |\mathrm{Ric}(\bar{g}) + 2\bar{g}|^2_{\bar{g}} \, d\mathrm{vol}_{\bar{g}} \leq \varepsilon^2,$$

then it holds

$$|g_{\mathrm{hyp}} - \bar{g}|_{C^{2,\alpha}}(x) \leq C \varepsilon^{1-\alpha} e^{-(1-\frac{1}{2}\delta) \mathrm{dist}_{\bar{g}}(x, U \cup \partial M_{\mathrm{thick}})} \quad \text{for all } x \in M_{\mathrm{thick}}.$$

The following slight generalisation follows from Remark 9.2.

Remark 10.2. Theorem 10.1 holds for all $\lambda \in (0,1)$ and $\eta > 1$.

Proof. Since the proof is basically identical to that of Theorem 5.1 we only sketch it and highlight differences. For $R > 0$ we denote by $\bar{B}(0,R) \subseteq C^{2,\alpha}_{\lambda}\big(\mathrm{Sym}^2(T^*M)\big)$ the closed ball of radius R around the 0-section with respect to the norm $\|\cdot\|_{2,\lambda;*}$. Analogous to the proof of Theorem 5.1 we consider

$$\Psi : \bar{B}(0,R) \to C^{2,\alpha}_{\lambda}\big(\mathrm{Sym}^2(T^*M)\big), h \mapsto h - \mathcal{L}^{-1}\Phi(\bar{g} + h),$$

where Φ is the Einstein operator defined in (2.1), and $\mathcal{L} = (D\Phi)_{\bar{g}}$. We want to show that for $R > 0$ small enough, Ψ is $\frac{1}{2}$-Lipschitz. Since $\|\mathcal{L}^{-1}\|_{\mathrm{op}}$ is bounded by a universal constant (Proposition 9.1), it suffices to show that for any $h \in B(0,R)$ and any $h' \in C^{2,\alpha}_\lambda(\mathrm{Sym}^2(T^*M))$ it holds

$$\|(D\Phi)_{\bar{g}+h}(h') - (D\Phi)_{\bar{g}}(h')\|_{0,\lambda} \le c(R)\|h'\|_{0,\lambda;*}$$

for some universal $c(R)$ with $c(R) \to 0$ as $R \to 0$. Indeed, this can be shown by the exact same argument as in [Bam12, pages 898 and 899].

Fix $R > 0$ such that Ψ is $\frac{1}{2}$-Lipschitz. Then it follows from the definition (9.7) of $\|\cdot\|_{2,\lambda;*}$, the conditions $ii) - v)$ and Proposition 9.1 that $\|\Psi(0)\|_{2,\lambda;*} \le C\varepsilon^{1-\alpha}$, which is at most $\frac{R}{2}$ if $\varepsilon_0 > 0$ is small enough. Remembering that in dimension three Einstein metrics have constant sectional curvature, the existence of a hyperbolic metric g_{hyp} now follows exactly as in the proof of Theorem 5.1.

To show the improved estimate, note that (5.1) still holds for points in the thick part of M, that is, for all $h \in C^{2,\alpha}_\lambda(\mathrm{Sym}^2(T^*M))$ and all $x \in M_{\mathrm{thick}}$ it holds

$$|h|_{C^{2,\alpha}}(x) \le C_0\Big(|\mathcal{L}h|_{C^{0,\alpha}}(x) + \|\mathcal{L}h\|_{L^2(M;\omega_x)}\Big)$$

for a universal constant C_0. Also choose C_0 large enough so that $\|\Psi(0)\|_{2,\lambda;*} \le \frac{1}{2}C_0\varepsilon^{1-\alpha}$. Abbreviate $R_\varepsilon = C_0\varepsilon^{1-\alpha}$, so that Ψ restricts to a $\frac{1}{2}$-Lipschitz endomorphism of $\bar{B}(0,R_\varepsilon)$. For an appropriatly chosen $C_1 \gg C_0$ define

$$\mathcal{U} := \left\{h \in \bar{B}(0,R_\varepsilon)\,|\, h \text{ satisfies (10.1) for all } x \in M_{\mathrm{thick}} \text{ and (10.2) for all } x \in M\right\},$$

where the inequalities (10.1) and (10.2) appearing in the definition of $\mathcal{U}$ are

$$|h|_{C^{2,\alpha}}(x) \le C_1\varepsilon^{1-\alpha}e^{-\beta\,\mathrm{dist}_{\bar{g}}(x,U\cup\partial M_{\mathrm{thick}})} \tag{10.1}$$

and

$$\|h\|_{H^2(M;\omega_x)} \le C_1\varepsilon^{1-\alpha}e^{-\beta\,\mathrm{dist}_{\bar{g}}(x,U\cup\partial M_{\mathrm{thick}})} \tag{10.2}$$

The rest of the proof of Theorem 5.1 carries over with only one small additional observation. For $h \in \mathcal{U}$ the estimate (10.1) even holds for all $y \in N_\rho(M_{\mathrm{thick}})$. Here $\rho > 0$ is the radius appearing in the definition of the Hölder norms. Indeed, if $y \in N_\rho(M_{\mathrm{thick}}) \setminus M_{\mathrm{thick}}$, then $d(y,\partial M_{\mathrm{thick}}) \le \rho$, and thus

$$|h|_{C^{2,\alpha}}(y) \le \|h\|_{2,\lambda;*} \le C_0\varepsilon^{1-\alpha} \le \left(C_0e^\rho\right)\varepsilon^{1-\alpha}e^{-(1-\frac{1}{2}\delta)d(y,U\cup\partial M_{\mathrm{thick}})}$$

So $h \in \mathcal{U}$ satisfies (10.1) for all $y \in N_\rho(M_{\mathrm{thick}})$ if we choose $C_1 > C_0e^\rho$. We really need the estimate (10.1) in an enlarged region because in order to check that $\mathcal{U}$ is Ψ-invariant, it is necessary to control $\max_{y\in B(x,\rho)}|h|^2_{C^{2,\alpha}}(y)$ for $x \in M_{\mathrm{thick}}$ (see (5.7)). □

Analogous to Remark 5.4, Remark 9.3 implies the following.

Remark 10.3. Theorem 10.1 also holds when M is non-orientable.

11. Drilling and filling

In the first part of this section we consider a hyperbolic 3-manifold M of finite volume and a collection $T_1,\dots,T_k$ of Margulis tubes in M. Each T_i is a solid 3-torus whose boundary ∂T_i is a flat 2-torus which is locally isometrically embedded in M. The *meridian* of the tube T_i is an essential simple closed curve α_i on ∂T_i which is homotopically trivial in M. It is unique up to free homotopy. Up to homotopy equivalence and hence isometry (by the Mostow rigidity theorem), the manifold M is uniquely determined by the homotopy type of $M - \cup_i T_i$ and the choice of these meridians. The core curve of the tube T_i is a primitive closed geodesic $\beta_i \subset T_i$.

Drilling of the geodesics β_i, that is, removal of the geodesic β_i for each i, defines a new manifold $\hat{M}$. Brock and Bromberg (Theorem 6.1 of [BB02]) showed that if the sum of the lengths of the geodesics β_i is sufficiently small, then the manifold $\hat{M}$ admits a complete hyperbolic metric of finite volume for which each Margulis tube T_i about one of the geodesics β_i has been replaced by a rank two cusp C_i. Furthermore, the hyperbolic metric on $\hat{M} - \cup_i C_i$ is L-bilipschitz to the hyperbolic metric on $M - \cup_i T_i$ for a number L which tends to one as the sum of the lengths of the geodesics β_i tends to zero.

The work [BB02] does not give an effective upper bound for the total length of the geodesics β_i for which the drilling result holds true, nor is the dependence of the bilipschitz constant L on this total length explicit. Such effective bounds were recently obtained by Futer, Purcell and Schleimer [FPS21].

The first main goal of this section is to establish a version of the drilling result of Brock and Bromberg [BB02] as an application of our main theorem. As in [BB02], our result is not effective, but it allows the drilling of an arbitrary number of geodesics, with only a universal upper length bound for each of them, provided that these geodesics are sufficiently sparsely distributed in M.

For an application of our methods, it is more convenient to control a Margulis tube via the length of its meridian on the boundary torus and not via the length of the core geodesic. Thus we begin with comparing the information on meridional length with the information on the length of the core geodesic.

Let us consider for the moment an arbitrary Margulis tube T with core geodesic β of length $\ell > 0$ and boundary ∂T in some hyperbolic 3-manifold M. If $R > 0$ is the radius of the tube, that is, the distance of the core geodesic β to the boundary torus ∂T, then the meridian of T is a simple closed geodesic on the flat torus ∂T of length $2\pi \sinh R$. In particular, since the injectivity radius of ∂T roughly equals the Margulis constant for hyperbolic 3-manifolds, the radius R is bounded from below by a universal positive constant. Cutting ∂T open along a meridian yields a flat cylinder with boundary length $2\pi \sinh R$ and height $\ell \cosh R$. The area of ∂T equals $2\pi\ell \sinh R \cosh R$.

In general, the relation between the length ℓ of the core geodesic of a Margulis tube and the radius R of the tube is delicate. The following effective bound is a special case of Theorem 1.1 of [FPS19b].

Theorem 11.1 (Futer, Purcell and Schleimer). *Let $\epsilon \leq 0.3$ be a Margulis constant for hyperbolic 3-manifolds. Let M be a hyperbolic 3-manifold and let $N \subset M$ be a Margulis* *tube whose core geodesic has length $\ell < 8\epsilon^2$. Then the radius of the tube is at least* $\operatorname{arcosh}\frac{\epsilon}{\sqrt{8\ell}}$.

Theorem 11.1 makes the idea effective that a very short core geodesic is contained in a Margulis tube of very large radius. As a consequence, finding non-effective upper length bounds on closed geodesics which we can drill from a hyperbolic manifold, with effective control on the geometry of the resulting manifold with cusps, is equivalent to finding non-effective lower bounds on tube radii about closed geodesics which allow for geometrically controlled drilling.

We show

Theorem 11.2 (The drilling theorem)**.** *For any $\varepsilon > 0$, $\kappa \in (0,1)$ and $m > 0$ there exists a number $R = R(\varepsilon,\kappa,m) > 0$ with the following property. Let M be a finite volume hyperbolic 3-manifold, and let $T_1,\dots,T_k$ be a family of Margulis tubes in M. Let $R_i > 0$ be the radius of the tube T_i, and let β_i be its core geodesic. If for each $r > 0$ and each $x \in M$ we have $\#\{i \mid \operatorname{dist}(x,T_i) \le r\} \le me^{\kappa r}$ and if $R_i \ge R$ for all i, then the manifold $\hat{M}$ obtained from M by drilling each of the geodesics β_i admits a complete hyperbolic metric of finite volume, and the restriction of this hyperbolic metric to the complement of the cusps C_i obtained from the drilling is ε-close in the C^2-topology to the metric on $M - \cup_{i\le k} T_i$.*

Remark 11.3. Our drilling theorem is weaker than Theorem 6.1 of [BB02] as we require that the manifold M is of finite volume rather than just geometrically finite. Furthermore, in contrast to the work [FPS21], our estimates are not effective. But it is also stronger than the results obtained in [BB02, FPS21] as it allows for drilling of an arbitrary number of closed geodesics, contained in Margulis tubes of tube radius larger than a fixed constant, provided that the tubes are sufficiently sparsely distributed in the manifold M, and it gives better geometric control on the drilled manifold.

In fact, the improved estimate in Theorem 10.1 and the estimate (11.4), which will be obtained during the proof of Theorem 11.2, immediately imply the following. For any $\delta > 0$ there exists $R = R(\delta,\varepsilon,\kappa,m) > 0$ so that on the thick part of the drilled manifold, the hyperbolic metric is $\varepsilon e^{-(1-\frac{1}{2}\kappa-\delta)\operatorname{dist}(\cdot,M_{\text{thin}})}$-close to the original metric. In particular, by choosing $\delta \le \frac{1}{2}(1-\kappa)$ one can always arrange that on the thick part of the drilled manifold, the hyperbolic metric is $\varepsilon e^{-\frac{1}{2}\operatorname{dist}(\cdot,M_{\text{thin}})}$-close to the original metric.

Proof of Theorem 11.2. We split the proof into several steps.

Step 1 (Construction of an approximating metric): Let us consider the boundary ∂T of a Margulis tube T in a hyperbolic 3-manifold of finite volume. This is a flat torus, and the meridian of the tube T defines a foliation of ∂T by closed geodesics. In polar coordinates (r,θ,y) about the core geodesic β of the Margulis tube, the hyperbolic metric can be written as

$$g = dr^2 + (\sinh r)^2 d\theta^2 + (\cosh r)^2 dy^2$$

where dy is a one-form on T which vanishes on the immersed totally geodesics hyperbolic planes which intersect the core geodesic of the tube orthogonally and where $r > 0$ is the radial distance from the core curve of the tube T. The bilinear form $(\sinh r)^2 d\theta^2 +$

$(\cosh r)^2 dy^2$ defines the flat metric on the level tori $\{r = \text{const}\}$ of the distance function from the core geodesic.

Choose a smooth function $\sigma : \mathbb{R} \to [0,1]$ such that $\sigma \equiv 1$ in a neighbourhood of $(-\infty, -1]$ and $\sigma \equiv 0$ in a neighbourhood of $[0,\infty)$. Let $\hat{R}$ be the radius of the Margulis tube T. In the coordinates (r,θ,y), define a new metric $\hat{g}$ on a tubular neighborhood $C \subset T$ about ∂T of radius 1 by

$$\begin{aligned}\hat{g} =& dr^2 + \left(\left(1-\sigma(r-\hat{R})\right)\sinh(r) + \sigma(r-\hat{R})\frac{1}{2}e^r\right)^2 d\theta^2 \\ &+ \left(\left(1-\sigma(r-\hat{R})\right)\cosh(r) + \sigma(r-\hat{R})\frac{1}{2}e^r\right)^2 dy^2.\end{aligned}$$

Since the first and second derivatives of σ are uniformly bounded and the derivatives of the function $e^r/2 - \sinh(r) = e^{-r}/2$ equal $\pm e^{-r}/2$ and similarly for $e^r/2 - \cosh(r)$, the metric $\hat{g}$ is $a_1 e^{-2\hat{R}}$-close to g in the C^2-topology on C where $a_1 > 0$ is a universal constant. Furthermore, as the metric $\hat{g}$ coincides with the hyperbolic metric g of M in the complement of the Margulis tube T, it can be viewed as a metric on $M - \hat{T}$ which is $a_1 e^{-2\hat{R}}$-close to g in the C^2-topology, where $\hat{T} = T - C$. The sectional curvature of $\hat{g}$ equals -1 outside of the collar C, and it is contained in the interval $[-1 - a_2 e^{-2\hat{R}}, -1 + a_2 e^{-2\hat{R}}]$ for a universal constant $a_2 > 0$ not depending on T or M. The same argument shows that $\|\nabla \mathrm{Ric}(\hat{g})\|_{C^0} \le \Lambda$ for a universal constant $\Lambda > 0$.

Near the boundary component of C which is contained in the interior of the tube T, the metric $\hat{g}$ is of the form $dr^2 + e^{2r} g_0$ where g_0 is a fixed flat metric on the distance tori to ∂T for the metric g. Such a warped product metric is the local model for a hyperbolic metric on a rank two cusp. Thus we can glue a hyperbolic rank two cusp to the interior boundary component of the collar C in such a way that the resulting manifold $\hat{M}$ is obtained from M by drilling the closed geodesic β, and $\hat{M}$ is equipped with a complete Riemannian metric $\bar{g}$ whose restriction to $\hat{M} - \hat{T}$ coincides with $\hat{g}$ and is hyperbolic in the complement of the collar C. In particular, $\hat{g}$ coincides with g on $M - T$ (using the natural embedding of $M - T$ into $\hat{M}$).

The same construction can be done for all the Margulis tubes $T_1, \ldots, T_k$ simultaneously. That is, the manifold $\hat{M}$ which is obtained from M by drilling the closed geodesics $\beta_1, \ldots, \beta_k$ admits a complete finite volume Riemannian metric $\bar{g}$ such that

- $\bar{g} = g$ on $M \setminus \bigcup_{i=1}^k T_i$;
- $|\bar{g} - g|_{C^2} \le a e^{-2R}$ in $M \setminus \bigcup_{i=1}^k \hat{T}_i$;
- $|\sec(\bar{g}) + 1| \le a e^{-2R_i}$ inside the cusp that is obtained by drilling β_i;
- $\bar{g}$ is hyperbolic outside the union of the collars $\bigcup_{i=1}^k C_i$;
- $\|\nabla \mathrm{Ric}(\bar{g})\|_{C^0(M,\bar{g})} \le \Lambda$.

Here a and Λ are both universal constants.

Step 2 (Reducing the problem): For sufficiently large R_i, all the conditions in Theorem 2 besides the integral estimate are clearly satisfied. Thus it remains to prove the integral estimate. Since $\mathrm{area}(\partial T_i) = 2\pi \ell(\beta_i) \cosh(R_i) \sinh(R_i)$ where R_i is the radius of the tube T_i, we have

$$\mathrm{vol}_{\bar{g}}(C_i) \le ce^{R_i}$$

for a universal constant c, where C_i is the collar around ∂T_i. Namely, if R is large, then the same holds true for $\sinh(R)$. Thus a shortest closed geodesic on T_i, whose length equals twice the Margulis constant μ and hence roughly equals one, is different from a meridian and has to cross through the cylinder of height $\ell(\beta_i)\cosh(R_i)$ obtained by cutting T_i open along a meridian. As a consequence, its length is at least $\ell(\beta_i)\cosh(R_i)$, that is, we have $\ell(\beta_i)\cosh(R_i) \le 2\mu$. Together with the curvature estimate, this implies

$$\int_{C_i} |\mathrm{Ric}(\bar{g}) + 2\bar{g}|^2_{\bar{g}}\, d\mathrm{vol}_{\bar{g}} \le C\mathrm{vol}(C_i)\left(e^{-2R_i}\right)^2 \le Ce^{-3R_i} \le Ce^{-3R}, \tag{11.1}$$

where in the last inequality we used the assumption that the radius R_i of T_i is at least R.

Write $\delta_i(x) := \mathrm{dist}(x, \partial T_i)$ and $r_x(y) = d(x,y)$. As C_i has width 1, for all $x \in M$, all $i \le k$ and all $y \in C_i$ it holds $\delta_i(x) \le d(x,y) + 1 = r_x(y) + 1$, whence $\lceil \delta_i(x) \rceil \le r_x(y) + 2$. Therefore, we deduce from (11.1)

$$\int_{\hat{M}} e^{-(2-\delta)r_x(y)} |\mathrm{Ric}(\bar{g}) + 2\bar{g}|^2_{\bar{g}}\, d\mathrm{vol}_{\bar{g}}(y) \le C\left(\sum_{i=1}^{k} e^{-(2-\delta)\lceil \delta_i(x) \rceil}\right) e^{-3R}. \tag{11.2}$$

Here $\delta \in (0,2)$ is a small constant that will be determined later.

Step 3 (Estimating the sum): By assumption, we have $\#\{i \,|\, \delta_i(x) \le r\} \le me^{\kappa r}$. Observe that by choosing $\kappa' > \kappa$, if $\bar{R} = \bar{R}(\kappa, \kappa')$ is big enough, then by the second bullet point at the end of Step 1, after replacing κ by κ', this estimate also holds true in $\hat{M}$. Choose once and for all such a number $\kappa' \in (\kappa, 1)$. From now on all distances are taken in $\hat{M}$ and with respect to $\bar{g}$.

To estimate the sum in (11.2) we need to analyze the functions $\lceil \delta_i(x) \rceil$. Observe that $\lceil \delta_i(x) \rceil$ also satisfies the growth condition. Indeed, since for all $u \in \mathbb{R}$ and $r \in \mathbb{N}$ it holds $u \le r$ if and only if $\lceil u \rceil \le r$, we have

$$\#\{i \,|\, \lceil \delta_i(x) \rceil \le r\} = \#\{i \,|\, \delta_i(x) \le r\} \le me^{\kappa' r}$$

for all $r \in \mathbb{N}$. Fix $x \in M$. Let $0 < r_0 < \ldots < r_l$ be an enumeration of $\{\lceil \delta_i(x) \rceil \,|\, i = 1, \ldots, k\}$. So

$$\sum_{i=1}^{k} e^{-(2-\delta)\lceil \delta_i(x) \rceil} = \sum_{j=0}^{l} \#\{i \,|\, \lceil \delta_i(x) \rceil = r_j\} e^{-(2-\delta)r_j} \le m \sum_{j=0}^{l} e^{-(2-\delta-\kappa')r_j}. \tag{11.3}$$

As $(r_j)_{j=0,\ldots,l}$ is an increasing sequence of natural numbers and since $r \mapsto e^{-(2-\delta-\kappa')r}$ is decreasing (assuming $\delta + \kappa' < 2$), we get

$$\sum_{j=0}^{l} e^{-(2-\delta-\kappa')r_j} \le \int_{r_0-1}^{\infty} e^{-(2-\delta-\kappa')}\, dr = \frac{1}{2-\delta-\kappa'} e^{-(2-\delta-\kappa')(r_0-1)}. \tag{11.4}$$

Finally note that $d(x, \hat{M}_{\text{thick}}) \le d(x, \partial T_i)$ since $\partial T_i \subseteq \hat{M}_{\text{thick}}$, and hence $d(x, \hat{M}_{\text{thick}}) \le r_0$. Therefore, if we choose $\delta = \delta(\kappa) \in (0,2)$ and $b = b(\kappa) > 1$ small enough so that $\delta + \kappa' + b < 2$, then

$$e^{bd(x,\hat{M}_{\text{thick}})} \int_{\hat{M}} e^{-(2-\delta)r_x(y)} |\mathrm{Ric}(\bar{g}) + 2\bar{g}|^2_{\bar{g}}\, d\mathrm{vol}_{\bar{g}}(y) \le Ce^{-3R}$$

by combining (11.2), (11.3) and (11.4). For R large enough, this will be at most ε^2. Therefore, we can evoke Theorem 2 to complete the proof. □

Controlled Dehn filling can be done with precisely the same argument. The following is a version of a theorem of Hodgson and Kerckhoff [HK08]. In its formulation, we start with a finite volume hyperbolic 3-manifold M and a collection of cusps $C_1, \ldots, C_k$ whose boundaries ∂C_i are 2-tori. Our goal is to fill these cusps by replacing them by solid tori, and show that if the lengths on ∂C_i of the meridians of the solid tori are sufficiently large, then the filled in manifold admits a hyperbolic metric which is close to the metric of M on the complement of the cusps. Our result is weaker than the result in [HK08] as the filling constants we find are not effective. On the other hand, similarly to Theorem 11.2 and unlike the results of [HK08], the filling constants do not depend on the number of cusps to be filled but only on sparsity of these cusps in M. Furthermore, the lower length bound on the meridional geodesic γ we use for the filling is the actual length of γ on the flat torus T and not its *normalized length*, defined to be the length of γ on the flat torus T' which is obtained from T by rescaling the metric so that the volume of T' equals one.

Theorem 11.4 (The filling theorem). *For any $\varepsilon > 0$, $\kappa \in (0,1)$ and $m > 0$ there exists a number $L = L(\varepsilon, \kappa, m) > 0$ with the following property. Let M be a finite volume hyperbolic 3-manifold, $C_1, \ldots, C_k \subseteq M$ be a finite collection of torus cusps, and assume that for each $r > 0$ and each $x \in M$ we have $\#\{i \mid \operatorname{dist}(x, C_i) \leq r\} \leq me^{\kappa r}$. For each $i \leq k$ let α_i be a flat simple closed geodesic in ∂C_i of length $L_i \geq L$. Then the manifold obtained from M by filling the cusps C_i, with meridian α_i, is hyperbolic, and the restriction of its metric to the complement of the Margulis tubes obtained from the filling is ε-close to the metric on $M - \cup_i C_i$.*

Proof. The proof is analogous to the proof of Theorem 11.2. Namely, let $\alpha_i \subset \partial C_i$ be a closed geodesic. Then α_i defines a foliation of ∂C_i by closed geodesics, and there is a dual orthogonal foliation by geodesics (which are not necessarily closed). Let us assume that the length of α_i equals $e^{R_i}/2$ for a number $R_i > 0$. Reversing the argument in the proof of Theorem 11.2, define a new metric on the tubular neighborhood $N(\partial C_i, 1)$ of radius one about ∂C_i in the cusp C_i as follows. Write the hyperbolic metric in horospherical coordinates in the form $dt^2 + e^{2t}dx^2 + e^{2t}dy^2$ where the euclidean coordinates (x, y) on the flat torus are such that the horizontal lines are the geodesics parallel to α_i and the vertical lines define the orthogonal foliation.

Let $\sigma : \mathbb{R} \to [0,1]$ be a smooth function which vanishes on $[0, \infty)$ and equals 1 on $(-\infty, -1]$. Define a metric $\hat{g}_i$ on C_i by

$$\hat{g}_i = g - (\sigma(t)e^{-R_i - t}/2)dx^2 + (\sigma(t)e^{-R_i - t}/2)dy^2.$$

Then the metric $\hat{g}_i$ on the distance tori of distance bigger than one can be written in orthogonal coordinates as $(\sinh(R_i - t))^2 dx^2 + (\cosh(R_i - t))^2 dy^2$. In particular, if $\ell_i > 0$ is such that the height of the cylinder obtained by cutting the boundary component of $N(\partial C_i, 1)$ distinct from ∂C_i open along α_i equals $\ell_i \cosh(R_i - 1)$, then we can glue a hyperbolic tube to this boundary with core curve of radius ℓ_i and meridian α_i.

The resulting manifold $\hat{M}$ is obtained from M by Dehn filling of α_i. Furthermore, it is equipped with a Riemannian metric of curvature in the interval $[-1-\epsilon,-1+\epsilon]$ which is of constant curvature -1 outside of the sets $N(\partial C_i,1)$, and the integral of the traceless Ricci curvature fulfills the assumptions in Theorem 11.2.

Thus the arguments in the proof of Theorem 11.2 apply without change and show the theorem. □

12. Effective hyperbolization I

A *handlebody of genus* $g\geq 1$ is a compact 3-manifold H which is diffeomorphic to the connected sum of g solid tori and whose boundary is a closed oriented surface $\partial H=\Sigma$ of genus g. It is characterized up to marked homotopy equivalence by its *disk set*, which can be thought of as the collection of all essential simple closed curves on ∂H which bound embedded disks in H. Equivalently, the disk set is the set of all essential simple closed curves in ∂H which are homotopic to zero in H.

If we glue two handlebodies H_1,H_2 along their boundaries with an orientation reversing diffeomorphism $f:\partial H_1\to\partial H_2$, then the resulting 3-manifold is closed and oriented. Up to homotopy and hence diffeomorphism, it only depends on the isotopy class of f, in fact, only on the double coset of this isotopy class in the mapping class group of ∂H which allows for precomposition of f with an element of the *handlebody group* of H_1 and postcomposition of f with an element of the handlebody group of H_2.

The *curve graph* $\mathcal{CG}(\Sigma)$ of $\Sigma=\partial H$ is the graph whose vertices are isotopy classes of simple closed curves on Σ and where two such curves can be connected by an edge of length one if they can be realized disjointly. The curve graph of Σ is known to be a hyperbolic geodesic metric graph. The disk set of H determines a full subgraph $\mathcal{D}$ of $\mathcal{CG}(\Sigma)$ whose vertex set is the disk set of H. This subgraph is uniformly *quasi-convex* [MM04] in $\mathcal{CG}(\Sigma)$. This means that there exists a number $k>0$ only depending on the genus of the surface Σ such that for any two disks $a,b\in\mathcal{D}$, any geodesic in $\mathcal{CG}(\Sigma)$ connecting a to b is contained in the k-neighborhood of $\mathcal{D}$.

The *Hempel distance* of the Heegaard splitting f is defined to be the distance in $\mathcal{CG}(\Sigma)$ between the disk set $\mathcal{D}_2$ of H_2 and the image $\mathcal{D}_1$ under the gluing map f of the disk set of H_1. If the Hempel distance is at least three, then the manifold M_f is known to be aspherical and atoroidal [Hem01] and hence by the geometrization theorem, it admits a hyperbolic metric. The goal of this section is to give an effective construction of such a metric not depending on any earlier hyperbolization result provided that the gluing map f fulfills some combinatorial requirement which for example is satisfied for random 3-manifolds. We refer to [HV22] for a detailed account on the geometry of random 3-manifolds.

To introduce the combinatorial condition, note that since $\mathcal{D}_i$ is a quasi-convex subset of the curve graph and the curve graph is hyperbolic [MM99], if the distance between $\mathcal{D}_1$ and $\mathcal{D}_2$ is larger than a constant $b>0$ only depending on the hyperbolicity constant of $\mathcal{CG}(\Sigma)$ (which does not depend on Σ) and of the quasi-convexity constant for the embedding $\mathcal{D}_i\to\mathcal{CG}(\Sigma)$ (which only depends on Σ), then there is a coarsely well defined shortest geodesic ζ in $\mathcal{CG}(\Sigma)$ connecting $\mathcal{D}_1$ to $\mathcal{D}_2$. This means that ζ is a geodesic in

$\mathcal{CG}(\Sigma)$ which connects a point $c_1 \in \mathcal{D}_1$ to a point $c_2 \in \mathcal{D}_2$, and if ν is any geodesic in $\mathcal{CG}(\Sigma)$ connecting a point in $\mathcal{D}_1$ to a point in $\mathcal{D}_2$, then ζ is entirely contained in the r-neighborhood of ν where $r > 0$ is a constant only depending on Σ. For the remainder of this section, we always assume that $d_{\mathcal{CG}}(\mathcal{D}_1, \mathcal{D}_2) \geq b$.

For a proper essential connected subsurface S of Σ different from a pair of pants and an annulus, define the *arc and curve graph* $\mathcal{CG}(S)$ of S to be the graph whose vertices are essential simple closed curves in S or essential arcs with endpoints in the boundary ∂S of S. Two such arcs or curves are connected by an edge of length 1 if they can be realized disjointly. If S is an annulus then this construction has to be modified. As we do not need more precise information here, we omit a more detailed discussion which can be found in [MM00]. For a simple closed curve $c \in \mathcal{CG}(\Sigma)$ with an essential intersection with S, the *subsurface projection* of c into $\mathcal{CG}(S)$ is the union of all intersection components of c with S (properly interpreted if S is an annulus).

By the above discussion and Theorem 3.1 of [MM00], there exists a number $p > 0$ with the following property. Let as before ζ be a shortest geodesic in $\mathcal{CG}(\Sigma)$ connecting $\mathcal{D}_1$ to $\mathcal{D}_2$. Let us assume that there exists a proper essential connected subsurface $S \subset \Sigma$ whose boundary ∂S consists of a collection of simple closed curves whose distance to each of the endpoints of ζ is at least p. Let us also assume that the diameter of the subsurface projection of the endpoints of ζ into S equals $k \geq 2p$. Then for any pair $(a_1, a_2) \in \mathcal{D}_1 \times \mathcal{D}_2$, the diameter of the subsurface projection of a_1, a_2 into S is at least $k - p \geq p$. Furthermore, any geodesic in $\mathcal{CG}(\Sigma)$ connecting a_1 to a_2 passes through a simple closed curve which is disjoint from S.

The following is the main result of this section. For its formulation, we define an ε-*model metric* on a closed ashperical atoroidal 3-manifold M to be a metric which fulfills the assumptions in Theorem 10.1 for the control constant ε.

Theorem 12.1. *For every $k \geq 2p, \varepsilon > 0$ there exists a number $b = b(\Sigma, k, \varepsilon) > 0$ with the following property. Let $\mathcal{D}_1, \mathcal{D}_2$ be the disk sets of the manifold M_f. Assume that a minimal geodesic ζ in $\mathcal{CG}(\Sigma)$ connecting $\mathcal{D}_1$ to $\mathcal{D}_2$ contains a subsegment $\hat{\zeta}$ of length at least b whose endpoints do not have any subsurface projection of diameter at least k into any subsurface of Σ. Then M_f admits an explicit ε-model metric which is ε-close in the C^2-topology to a hyperbolic metric.*

We begin the proof of Theorem 12.1 by recalling some results from [HV22].

The geometry of the curve graph of the surface Σ is coarsely tied to the geometry of the *Teichmüller space* $\mathcal{T}(\Sigma)$ of Σ. Namely, there is a (coarsely well-defined) $\mathrm{Mod}(\Sigma)$-equivariant Lipschitz map $\Upsilon : \mathcal{T}(\Sigma) \to \mathcal{CG}(\Sigma)$, called the *systole map*, that associates to every marked hyperbolic structure $X \in \mathcal{T}(\Sigma)$ a shortest geodesic $\Upsilon(X)$ on it. It follows from Masur-Minsky [MM99] (see Lemma 3.3 of [MM00] for a precise account, and note that small extremal length of a closed curve on a Riemann surface is equivalent to small hyperbolic length) that there exists a constant $L > 1$ only depending on Σ such that for every Teichmüller geodesic $\gamma : I \to \mathcal{T}(\Sigma)$ (here I is a connected subset of $\mathbb{R}$), the composition $\Upsilon \circ \gamma : I \to \mathcal{CG}(\Sigma)$ is an *unparameterized L-quasi-geodesic*. This means that there exists a homeomorphism $\rho : J \to I$ such that the composition $\Upsilon \circ \gamma \circ \rho$ is an L-quasi-geodesic in $\mathcal{CG}(\Sigma)$. Moreover, if we restrict our attention to the *δ-thick part*

$\mathcal{T}_\delta(\Sigma)$ of Teichmüller space of all hyperbolic metrics whose *systole*, that is, the length of a shortest closed geodesic, is at least δ, then the situation improves: In [Ham10] it is shown that for every $\delta > 0$ there exist $L_\delta > 1$ such that if γ is parameterized by arc length on an interval I of length at least L_δ and if $\gamma(I) \subset \mathcal{T}_\delta(\Sigma)$, then $\Upsilon \circ \gamma$ is a *parameterized* L_δ-quasi-geodesic.

Using the notations from Theorem 12.1, let us now assume that for some $k \geq 2p$, a shortest geodesic ζ connecting $\mathcal{D}_1$ to $\mathcal{D}_2$ contains a subsegment $\hat{\zeta}$ with the property that there does not exist *any* proper essential subsurface S of Σ for which the diameter of the subsurface projection of the endpoints of $\hat{\zeta}$ into S is larger than k. If the length of $\hat{\zeta}$ is sufficiently large, then we shall construct from this segment two convex cocompact hyperbolic metrics on a handlebody which contain large almost isometric regions whose injectivity radius is bounded from below by a universal constant. We then glue the handlebodies along these regions with a map which is close to an isometry for these metrics and construct a closed 3-manifold diffeomorphic to M_f with an ϵ-model metric which can be deformed to a hyperbolic metric using Theorem 2.

To implement this program, we follow [HV22] and introduce a notion of *relative bounded combinatorics* and *height*. Fix a sufficiently small threshold $\delta > 0$. Denote by $d_{\mathcal{T}}$ the distance on $\mathcal{T}(\Sigma)$ for the Teichmüller metric.

Definition 12.2 (Relative Bounded Combinatorics)**.** Consider $Y, X \in \mathcal{T}(\Sigma)$. We say that (Y, X) has *relative δ-bounded combinatorics* with respect to the handlebody H with disk set $\mathcal{D}$ if the Teichmüller geodesic $[Y, X]$ connecting Y to X is contained in $\mathcal{T}_\delta(\Sigma)$ and if

$$d_{\mathcal{CG}}(\mathcal{D}, \Upsilon(Y)) + d_{\mathcal{CG}}(\Upsilon(Y), \Upsilon(X)) \leq d_{\mathcal{CG}}(\mathcal{D}, \Upsilon(X)) + \frac{1}{\delta}.$$

The *height* of the pair (Y, X) is $d_{\mathcal{T}}(Y, X)$.

A *convex cocompact* metric on a handlebody H is a complete hyperbolic metric on the interior of H with the following property. The hyperbolic metric determines up to conjugacy an embedding of the fundamental group of H (which is the free group with g generators) into $PSL(2, \mathbb{C})$. The image group Γ acts on the boundary $\partial \mathbb{H}^3$ of hyperbolic 3-space, preserving a decomposition of $\partial \mathbb{H}^3$ into the *limit set* $\Lambda(\Gamma)$ and the *domain of discontinuity* $\Omega(\Gamma)$.

The quotient $\mathbb{H}^3 \cup \Omega(\Gamma)/\Gamma$ is compact and homeomorphic to the handlebody H. Moreover, as the action of Γ on $\Omega(\Gamma)$ preserves the conformal structure, the quotient $\Omega(\Gamma)/\Gamma$ is the surface Σ equipped with a conformal structure $X \in \mathcal{T}(\Sigma)$. Up to isometry, the convex cocompact metric on H is determined by X, and the corresponding hyperbolic handlebody will be denoted by $H(X)$. The *convex core* $\mathcal{CC}(H(X))$ of $H(X)$ is the quotient of the convex hull of $\Lambda(\Gamma)$ in $\mathbb{H}^3$ by the action of Γ, with boundary $\partial\mathcal{CC}(H(X))$. The convex core $\mathcal{CC}(H(X))$ is homeomorphic to the handlebody H.

A *product region* in a convex cocompact hyperbolic handlebody $H(X)$ with boundary surface Σ is a codimension 0 submanifold $U \subset H(X)$ contained in the convex core $\mathcal{CC}(H(X))$ of $H(X)$ which is homeomorphic to $\Sigma \times [0,1]$ with a homeomorphism whose restriction to each surface $\Sigma \times \{s\}$ is homotopic to the inclusion $\Sigma \to \partial\mathcal{CC}(H(X)) \subset H(X)$. If U is such a product region then we can define the *width* $\mathrm{width}(U) = \inf\{d(x, y) \mid x \in$

$\partial^+ U, y \in \partial U^-\}$ where $\partial^\pm U$ are the two boundary components of U. If the width of the product region is at least D and the diameter is at most $2D$ then we say that the product region has *size* D (see Section 5 of [HV22]).

A product region $U \subset H(X)$ can be used to decompose the handlebody $H(X)$ into two connected components. The first component is the closed subset of $H(X) - U$ containing $H(X) - \mathcal{CC}(H(X))$ (it is straightforward that there are no choices made in this construction). The second component is its complement, which is an open subset of H containing U. We define the *gluing block* H_U of $H(X)$ to be the component containing U.

Fix a number $\alpha \in (0,1)$. For a number $\xi > 0$, define a ξ-*almost isometry* between two Riemannian manifolds $(M_1, \rho_1), (M_2, \rho_2)$ to be a smooth map $\Phi : M_1 \to M_2$ such that $\|\rho_1 - \Phi^* \rho_2\|_{C^2} < \xi$. Our main technical tool is the following Theorem 5.12 of [HV22]. In its formulation, $d_{\mathcal{CG}}$ denotes as before the distance in the curve graph of $\Sigma = \partial H_i$.

Theorem 12.3 (The gluing theorem). *For $\delta > 0$ there exists $\iota = \iota(\delta) > 0, D = D(\delta) > 0$, and for $\xi > 0$ there exists $h_{\text{gluing}}(\delta, \xi) > 0$ such that the following holds true. Let H_1, H_2 be two handlebodies of genus g and let $f : \partial H_1 \to \partial H_2$ be a gluing map. Let $[Y, X] \subset \mathcal{T}_\delta(\Sigma)$ be a geodesic segment satisfying the following relative bounded combinatorics and large heights properties:*

- *$d_{\mathcal{T}}(Y, X) \in [h, 2h]$ for some $h > h_{\text{gluing}}(\delta, \xi)$.*
- *If $\mathcal{D}_1$ denotes the disk set of the handlebody H_1, then the pair (Y, X) satisfies*

$$d_{\mathcal{CG}}(\Upsilon(X), \Upsilon(Y)) + d_{\mathcal{CG}}(\Upsilon(Y), \mathcal{D}_1) \leq d_{\mathcal{CG}}(\Upsilon(X), \mathcal{D}_1) + \frac{1}{\delta}.$$

  The same holds true for the pair $(f^{-1}X, f^{-1}Y)$ and the disk set $\mathcal{D}_2$ of H_2.

Consider $N_1 = H(Y), N_2 = H(f^{-1}X)$. Then there exist:

- *Product regions $U_j \subset \mathcal{CC}(N_j)$ of size D for $j = 1, 2$. We denote by $N_j^0 \subset \mathcal{CC}(N_j)$ the gluing blocks they define.*
- *An orientation reversing ξ-almost isometric diffeomorphism $\Phi : U_1 \to U_2$ for $j = 1, 2$ in the homotopy class of f.*

In particular, we can form the 3-manifold

$$X_f = N_1^0 \cup_{\Phi : U_1 \to U_2} N_2^0$$

obtained from the disjoint union of N_1^0, N_2^0 by identifying a point $x \in N_1^0$ with its image under Φ in N_2^0. This manifold is diffeomorphic to $M_f = H_1 \cup_f H_2$. Denote by Ω the image in X_f of $U_1 \cup U_2$. The manifold X_f comes equipped with a Riemannian metric ρ with the following properties:

i) The sectional curvature of ρ is contained in the interval $(-1 - \xi, -1 + \xi)$, and it is constant -1 on $X_f - \Omega$.
ii) The diameter of Ω is at most $2D$, and the injectivity radius on Ω is at least ι.
iii) The two components of $X_f - \Omega$ are isometric to the complement in $\mathcal{CC}(N_j)$ of collar neighborhoods about the boundary of $\mathcal{CC}(N_j)$ of uniformly bounded diameter (depending on h and hence on ξ, δ).

We call the metric constructed in Theorem 12.3 from the convex cocompact handlebodies $H(Y), H(f^{-1}X)$ and the gluing map f a *ξ-model metric with δ-bounded combinatorics*. Theorem 12.3 then can be restated as saying that if M_f fulfills the assumption stated in Theorem 12.3, then it admits a ξ-model metric with δ-bounded combinatorics. Note that the lower injectivity radius bound on Ω is not explicitly stated in Theorem 5.12 of [HV22], but is discussed in Section 5.3 of [HV22]. The metric has constant curvature -1 outside of the open subset Ω, called the *gluing region* in the sequel. Furthermore, since it is constructed by gluing two almost isometric hyperbolic metrics with a gluing function all of whose derivatives are uniformly bounded (and in fact small depending on the geometric data which enter in the construction), the covariant derivative of Ric is pointwise uniformly bounded by a constant only depending on δ.

By construction, the gluing region contains an open subset diffeomorphic to $\Sigma \times (0,1)$ whose diameter is bounded from above by a constant $2D > 0$ only depending on δ, whose injectivity radius is bounded from below by a constant ι only depending on δ and is such that the distance between the two boundary surfaces of this set is at least D. This implies that its volume is contained in the interval $[v^{-1}, v]$ for a number $v > 0$ only depending on the genus g of the handlebody and on δ.

Theorem 2 from the introduction can be used to promote a model metric to a hyperbolic metric which is close to the model metric in the C^2-topology. We summarize this as follows.

Proposition 12.4. *For all $\varepsilon > 0, k > 0$ there exist numbers $b = b(\varepsilon, k) > 0$, $\xi = \xi(\varepsilon, k) > 0$, $\delta = \delta(\varepsilon, k) > 0$ and $v = v(\varepsilon, k) > 0$ with the following properties. Let $f : \partial H_1 \to \partial H_2$ be a gluing map und use this to define the disk sets $\mathcal{D}_1, \mathcal{D}_2$. Assume that a shortest geodesic ζ connecting $\mathcal{D}_1$ to $\mathcal{D}_2$ contains a subsegment $\hat{\zeta}$ of length at least b such that for any proper essential subsurface S of Σ, the diameter of the subsurface projection of the endpoints of $\hat{\zeta}$ into S is at most k. Then the manifold M_f admits a ξ-model metric with δ-bounded combinatorics which is ε-close in the C^2-topology to a hyperbolic metric. If $\tilde{\zeta}$ is another subsegment of ζ of length at least b which has the same properties as $\hat{\zeta}$ and is disjoint from $\hat{\zeta}$, then these two segments determine a submanifold of M_f diffeomorphic to $\Sigma \times [0,1]$ whose volume for the hyperbolic metric on M_f is at least $v(d_{\mathcal{CG}}(\hat{\zeta}, \tilde{\zeta}))$.*

Proof. The distance formula Theorem 6.12 of [MM00] and its variation for the Teichmüller metric together with the main result of [Ham10] and Lemma 6.7 and Lemma 6.8 of [HV22] shows that for every $k > 0$ there are numbers $m_0 = m_0(k)$, $\sigma = \sigma(k) > 0$ and $L = L(k) > 1$ with the following property.

Let $m \geq 3m_0$ and let $\eta : [0, m] \to \mathcal{CG}(\Sigma)$ be a geodesic with the property that there exists no proper essential subsurface S of Σ such that the diameter of the projection of the endpoints $\eta(0), \eta(m)$ of η into the arc and curve graph of S is larger than k. Let $X, Y \in \mathcal{T}(\Sigma)$ be such that the Y-length of the curve $\eta(0)$ is not larger than a Bers constant for Σ, and that the same holds true for the X-length of the curve $\eta(m)$. Let $[Y, X]$ be the Teichmüller geodesic connecting Y to X. Then there exists a subsegment $[Y_0, X_0] \subset [Y, X]$ entirely contained in $\mathcal{T}_\sigma(\Sigma)$ with the property that $\Upsilon|[Y_0, X_0]$ is an L-quasi-geodesic connecting a point in the L-neighborhood of $\eta(m_0)$ to a point in the

L-neighborhood of $\eta(m-m_0)$. In particular, we have

$$d_{\mathcal{CG}}(\eta(m_0),\Upsilon(Y_0)) + d_{\mathcal{CG}}(\Upsilon(Y_0),\Upsilon(X_0)) \leq d_{\mathcal{CG}}(\eta(m_0),\Upsilon(X_0)) + L$$

and similarly for $\eta(m-m_0)$ and $\Upsilon(X_0)$.

It follows from the construction in the previous paragraph that up to a uniform adjustment of constants, if η is a subsegment of a minimal geodesic connecting $\mathcal{D}_1$ to $\mathcal{D}_2$, then the pair (Y_0, X_0) has relative σ-bounded combinatorics with respect to the handlebody H_1 with disk set $\mathcal{D}_1$, and (X_0, Y_0) has relative σ-bounded combinatorics with respect to the handlebody H_2 with disk set $\mathcal{D}_2$ in the sense of Definition 12.2. The height $d_{\mathcal{T}}(Y_0, X_0)$ is bounded from below by $(m-2m_0-2L)/c-c$ for a universal constant $c>0$ by the fact that the image of $[Y_0, X_0]$ under Υ is an L-quasi-geodesic connecting two points in $\mathcal{CG}(\Sigma)$ of distance at least $m-2m_0-2L$ and the fact that Υ is coarsely c-Lipschitz. As a consequence, for any $\xi>0$, if $h = h_{\text{gluing}}(\sigma,\xi)>0$ is as in Theorem 12.3 and if $m > ch + 2m_0 + 2L + c^2$, then this height is at least h.

Recall that the diameter D and hence the volume of the gluing region Ω in the statement of Theorem 12.3 for $\delta = \sigma$ and ξ is bounded from above by a constant which only depends on σ but not on ξ. Since the sectional curvature of the model metric is contained in the interval $[-1-C\xi, -1+C\xi]$, we know that for a given number $\varepsilon>0$ and the fixed number σ which only depends on k, there exists a number $\xi_0 = \xi_0(\epsilon,\sigma)>0$ such that if $\xi<\xi_0$, then the ξ-model metric with relative σ-bounded combinatorics on M_f fulfills the assumptions in Theorem 2 for this number ε. An application of Theorem 2 then shows that there is a hyperbolic metric on M_f in the ε-neighborhood of the model metric in the C^2-topology.

We are left with the volume estimate. To this end note that the construction of the Teichmüller segment $[Y_0, X_0]$ which gave rise to the gluing region Ω only used a sufficiently long subsegment $\hat{\zeta}$ of a minimal geodesic in $\mathcal{CG}(\Sigma)$ connecting $\mathcal{D}_1$ to $\mathcal{D}_2$. Let us now assume that $\tilde{\zeta}$ is a second such subsegment which is disjoint from $\hat{\zeta}$, and let us assume that it is contained in the subsegment of ζ connecting $\mathcal{D}_1$ to $\hat{\zeta}$. Let $[W,V]$ be a Teichmüller geodesic segment constructed from $\tilde{\zeta}$ as in the first paragraph of this proof, and let $[W_0,V_0]\subset[W,V]$ be the subsegment in $\mathcal{T}_\sigma(\Sigma)$ found with the argument in the second paragraph of this proof. By Proposition 4.1 of [HV22], the convex cocompact handlebody $H(Y)$ which entered the above construction contains a submanifold N which is ξ-almost isometric to a submanifold N_0 of the quasi-fuchsian manifold $Q(V_0,Y_0)$ defined by the marked hyperbolic surfaces V_0, Y_0, and this submanifold contains the complement of a collar of uniformly bounded height about the boundary in its convex core $\mathcal{CC}(Q(V_0,Y_0))$. By Theorem 12.3, the submanifold N of $H(Y)$ is isometrically embedded in M_f, equipped with the model metric constructed as above from the segment $\hat{\zeta}$.

The model manifold theorem [Min10] or earlier work of Brock [Bro03] shows that there exists a constant $\rho>0$ such that the volume of the manifold N_0 and hence the volume of N is bounded from below by $\rho d_{WP}(V_0,Y_0)$, where d_{WP} is the distance in $\mathcal{T}(\Sigma)$ induced by the Weil Petersson metric. Namely, by [Bro03], the volume of the quasi-fuchsian manifold $Q(V_0,Y_0)$ is bounded from below by $\rho' d_{WP}(V_0,Y_0)$ for a constant $\rho' = \rho'(\Sigma)$, and $\mathrm{vol}(Q(V_0,Y_0)-N_0)$ is uniformly bounded. As by bounded combinatorics and the explicit construction, the Weil Petersson distance $d_{WP}(V_0, W_0)$ is large, the volume estimate for N follows from an adjustment of constants.

Now there exist a number $C > 0$ such that $d_{WP}(V_0, Y_0) \geq C d_{\mathcal{CG}}(\Upsilon(V_0), \Upsilon(Y_0))$ [Bro03] and hence the volume of the submanifold N of M_f with respect to the model metric is bounded from below by $\rho C d_{\mathcal{CG}}(\Upsilon(V_0), \Upsilon(Y_0)) \geq \rho C d_{\mathcal{CG}}(\hat{\zeta}, \tilde{\zeta})$. Thus Theorem 2 implies that the same holds true for M_f, equipped with the hyperbolic metric (recall the convention of adjusting constants). This completes the proof of the proposition. □

Remark 12.5. The volume estimate in Proposition 12.4 is far from being sharp. Namely, in the proof, we used the fact that under suitable assumptions on the gluing map f, the hyperbolic manifold M_f contains an embedded subset which is almost isometric to the complement of a collar of uniformly bounded height about the boundary in $\mathcal{CC}(Q(V_0, Y_0))$ where $Q(V_0, Y_0)$ is a quasi-fuchsian manifold whose conformal boundaries V_0, Y_0 are contained in the thick part of Teichmüller space. By a result of Brock [Bro03], the volume of $Q(V_0, Y_0)$ is proportional to the Weil-Petersson distance $d_{WP}(V_0, Y_0)$ between V_0, Y_0 in $\mathcal{T}(\Sigma)$, and the ratio $d_{\mathcal{CG}}(\Upsilon(V_0), \Upsilon(Y_0))/d_{WP}(V_0, Y_0)$ can be arbitrarily small.

Since the Weil-Petersson distance between $V_0, Y_0 \in \mathcal{T}(\Sigma)$ is proportional to the distance in the pants graph between shortest pants decompositions for V_0, Y_0 [Bro03], this leads us to conjecture that the volume of a hyperbolic 3-manifold M_f with Heegaard surface Σ of minimal genus is proportional to the minimal distance in the pants graph between two pants decompositions $P_1 \subset \mathcal{D}_1$ and $P_2 \subset \mathcal{D}_2$, with constants only depending on Σ.

13. A priori geometric bounds for closed hyperbolic manifolds

The goal of this section is to obtain some geometric control on a closed hyperbolic 3-manifold M_f constructed by gluing two handlebodies H_1, H_2 with boundary $\partial H_1 = \partial H_2 = \Sigma$ with a gluing map f which does not fulfill the combinatorial condition in Theorem 12.1. This leads to the proof of Theorem 6 from the introduction. We always assume that the Hempel distance of the Heegaard splitting is at least 3. This rules out the existence of trivial handles in the Heegaard surface.

As before, denote by $\mathcal{D}_1, \mathcal{D}_2$ the disk sets of M_f, viewed as subsets of the curve graph $\mathcal{CG}(\Sigma)$ of Σ. Call a proper essential subsurface Y of Σ *strongly incompressible* in M_f if the distance in $\mathcal{CG}(\Sigma)$ between ∂Y and $\mathcal{D}_1 \cup \mathcal{D}_2$ is at least three. This implies that the boundary ∂Y of Y consists of simple closed curves in Σ which are not homotopic to zero in M_f. More concretely, we have

Lemma 13.1. *Let $Y \subset \Sigma$ be a strongly incompressible subsurface.*

i) For any boundary component γ of Y, the inclusion $\Sigma \smallsetminus \gamma \to M_f \smallsetminus \gamma$ is π_1-injective.
ii) If $\alpha \subset Y$ is an embedded essential arc with endpoints on ∂Y, then α is not homotopic in M_f into ∂Y keeping the endpoints in ∂Y.
iii) If α, β are two disjoint non-homotopic essential arcs in Y, then α, β are not homotopic in M_f keeping the endpoints in ∂Y.

Proof. The first statement of the lemma follows from Dehn's lemma [Hem76], applied to the complement of a small open tubular neighborhood N of γ in M_f. Namely, $\Sigma - N$ is a properly embedded bordered surface in $M_f - N$, and hence if there is an essential closed curve in $\Sigma - \gamma$ which is contractible in $M_f - \gamma$, then there is an essential simple closed curve in $\Sigma - \gamma$ which bounds a disk in M_f. But this contradicts the fact that any diskbounding simple closed curve in Σ has essential intersections with γ.

Now let $\alpha \subset Y$ be an embedded essential arc with endpoints on the same component ζ of ∂Y. Then each component of the boundary of a small neighborhood of $\alpha \cup \zeta$ in Y is an essential simple closed curve in Y. This curve is not homotopic to zero in M_f by the discussion in the previous paragraph. But this means that α is not homotopic in M_f into ∂Y keeping the endpoints in ∂Y.

Similarly, if α_1, α_2 are two disjoint non-homotopic arcs in Y with endpoints on the same (not necessarily distinct) boundary components ζ_1, ζ_2 of Y, then their union with suitable chosen subarcs of $\zeta_1 \cup \zeta_2$ defines an essential simple closed curve in Y to which the above discussion applies. Thus such arcs can not be homotopic in M_f keeping the endpoints in ∂Y. □

Denote by $d_{\mathcal{CG}}$ the distance in the curve graph of Σ. Lemma 13.1 does not state that for a subsurface $Y \subset \Sigma$ with $d_{\mathcal{CG}}(\partial Y, \mathcal{D}_1 \cup \mathcal{D}_2) \geq 3$, the inclusion $Y \to M_f$ is π_1-injective. However we have the following weaker statement.

For its formulation, for a proper essential subsurface Y of Σ denote by d_Y the distance in the arc and curve graph of Y, and diam_Y denotes the diameter of subsets of this graph. Furthermore, if α_1, α_2 are simple closed curves in Σ which have an essential intersection with Y, then we write $d_Y(\alpha_1, \alpha_2)$ to denote the distance in the arc and curve graph of Y between the *subsurface projections* of α_1, α_2, that is, the components of $\alpha_i \cap Y$.

Lemma 13.2. *There exists a number $p = p(\Sigma) > 4$ with the following property. Let $Y \subset \Sigma$ be a strongly incompressible subsurface whose boundary ∂Y, as a geodesic multicurve in M_f, fulfills $d_{\mathcal{CG}}(\partial Y, \mathcal{D}_1 \cup \mathcal{D}_2) \geq p$. If $h : \Sigma \times [0,1] \to M_f$ is any homotopy of the inclusion which preserves ∂Y and if $h_1 : \Sigma \to \Sigma$ is a homotopy equivalence, then h_1 induces the identity on $\pi_1(\Sigma)$.*

Proof. By [MM00], there exists a number $p = p(\Sigma) > 4$ with the following property. Let α, β be simple closed curves on Σ and let $Y \subset \Sigma$ be a subsurface which has an essential intersection with α, β and such that $d_Y(\alpha, \beta) \geq p - 1$; then any geodesic in $\mathcal{CG}(\Sigma)$ connecting α to β has to pass through a curve disjoint from Y.

Since by [MM04], the disk sets $\mathcal{D}_1, \mathcal{D}_2$ are uniformly quasi-convex subsets of $\mathcal{CG}(\Sigma)$, this implies that up to increasing p, the following holds true. Let $Y \subset \Sigma$ be any proper essential subsurface such that $d_{\mathcal{CG}}(\partial Y, \mathcal{D}_i) \geq p$ $(i = 1, 2)$; then $\mathrm{diam}_Y(\mathcal{D}_i) \leq p$.

Let $Y \subset \Sigma$ be such a subsurface and let $h : \Sigma \times [0,1] \to M$ be a homotopy of the inclusion $h_0 : \Sigma \to M$ which fixes ∂Y. Assume that h_1 is a homotopy equivalence of Σ onto $h_0(\Sigma)$. Then h_1 defines a mapping class $\varphi \in \mathrm{Mod}(W)$, the mapping class group of the component W. We claim that φ induces the identity on $\pi_1(W)$.

Namely, as W is a surface with non-empty boundary, the group $\mathrm{Mod}(W)$ does not have elements of finite order. Thus if φ is not trivial, then either φ is a pseudo-Anosov mapping class of W, or φ preserves a non-trivial multicurve $\beta \subset W$. Furthermore, there exists a subsurface Z of W which is preserved by φ, and if Z is not an annulus, then the restriction of φ to Z is a pseudo-Anosov mapping class, and if Z is an annulus, then the restriction of φ to Z is a Dehn twist. The mapping class φ induces the identity on $\pi_1(W)$ if and only if it is a composition of Dehn twists about the boundary curves of W or is trivial.

To see that this is the case, note that by composition, for each $k \geq 1$, the mapping class φ^k can also be represented by a homotopy of Σ which preserves ∂Y. But $d_{\mathcal{CG}}(\partial Y, \mathcal{D}_1 \cup \mathcal{D}_2) \geq p \geq 4$ and hence since ∂W is disjoint fromm ∂Y, we have $d_{\mathcal{CG}}(\partial W, \mathcal{D}_1 \cup \mathcal{D}_2) \geq p-1$. In particular, any diskbounding simple closed curve in Σ has an essential intersection with Z. Furthermore, by the choice of p, we have $\mathrm{diam}_Y(\mathcal{D}_i) \leq p$ $(i = 1,2)$.

Since φ is induced by a homotopy of Σ in M, it has to preserve the diskbounding curves in Σ as this set is determined by the topology of M. Now if φ preserves the non-peripheral subsurface $Z \subset W$ and acts on Z as a pseudo-Anosov mapping class (here we include the case that Z is an annulus and φ is a Dehn twist) then

$$\mathrm{diam}_Z(\mathcal{D}_1 \cup \mathcal{D}_2, \varphi^k(\mathcal{D}_1 \cup \mathcal{D}_2)) \to \infty \quad (k \to \infty).$$

As $\phi^k(\mathcal{D}_1 \cup \mathcal{D}_2) = \mathcal{D}_1 \cup \mathcal{D}_2$, and as $\mathrm{diam}_Z(\mathcal{D}_1 \cup \mathcal{D}_2) < \infty$, this is a contradiction which shows the lemma. ☐

Using an idea of Minsky [Min00], we establish an a priori upper bound for the total length of the boundary ∂Y of Y for the hyperbolic metric on M_f in terms of the diameter of the subsurface projection of $\mathcal{D}_1, \mathcal{D}_2$ into Y.

More precisely, for a simple closed multi-curve γ on Σ, denote by $\ell_f(\gamma)$ the sum of the minimal lengths of representatives of the free homotopy classes of the components of γ in the hyperbolic manifold M_f. By convention, we have $\ell_f(c) = 0$ for any curve on Σ which is homotopically trivial in M_f. In the statement of the following result and later on, $p \geq 4$ is the constant from Lemma 13.2.

Theorem 13.3 (A priori length bounds). *There exists a number $p = p(\Sigma) \geq 3$, and for every $\epsilon > 0$ there exists a number $k = k(\Sigma, \epsilon) > 0$ with the following property. Let M_f be a hyperbolic 3-manifold with Heegaard surface Σ, Hempel distance at least 4 and disk sets $\mathcal{D}_1, \mathcal{D}_2$. If $Y \subset \Sigma$ is a proper essential subsurface of Σ, with $d_{\mathcal{CG}}(\partial Y, \mathcal{D}_1 \cup \mathcal{D}_2) \geq p$ and $\mathrm{diam}_Y(\mathcal{D}_1 \cup \mathcal{D}_2) \geq k$, then $\ell_f(\partial Y) \leq \epsilon$.*

Theorem 13.3 can be thought of as a version of Theorem B of [Min00] in a different setting. The fact that Heegaard surfaces in M_f are compressible requires however a substantial modification of the proof.

Following [Min00], the main tool for the proof of Theorem 13.3 are *pleated surfaces*. The pleated surfaces we are interested in are maps $g : \Sigma \to M_f$ in the homotopy class of the inclusion $\Sigma \to M_f$ together with a hyperbolic metric σ on Σ satisfying the following two conditions.

- g is *path-isometric* with respect to σ.
- There exists a σ-geodesic lamination λ on Σ whose leaves are mapped to geodesics by g. In the complement of λ, g is totally geodesic.

The geodesic lamination λ is called the *pleating lamination* of g. We refer to [Min00] for more details on pleated surfaces as used in our context. We call the hyperbolic metric σ on Σ which has the above properties the metric *induced* by the map g. Note that this makes sense since σ is indeed the pull-back of the hyperbolic metric on M_f by g (properly interpreted on the pleating locus).

We fix now once and for all a constant κ_0 which is smaller than a Margulis constant for hyperbolic surfaces and with the following properties (see p.139 of [Min00] for details).

(P1) For any hyperbolic surface S with geodesic boundary ∂S, any two essential properly embedded arcs τ, τ' in S with endpoints on ∂S whose lengths are at most κ_0 are either homotopic keeping endpoints in ∂S, or they are disjoint.

(P2) If α is a simple closed geodesic on a hyperbolic surface S and if α contains a point $x \in S$ of injectivity radius smaller than κ_0, then x is contained in a Margulis tube A of S, and either α equals the core curve of A, or the subarc of $A \cap \alpha$ containing x crosses through A, that is, it connects the two distinct boundary components of A.

The second property follows from the fact that a closed geodesic in a hyperbolic surface which enters sufficiently deeply into a Margulis tube but is not entirely contained in the tube either crosses through the tube, or it has self-intersections.

A *bridge arc* for an essential proper non-annular subsurface $Y \subset \Sigma$ is an embedded arc $\alpha \subset Y$ with both endpoints on ∂Y which is not homotopic in M_f into ∂Y keeping the endpoints in ∂Y. For a hyperbolic metric σ on Σ, define a *minimal proper arc* to be a bridge arc τ for Y which is minimal in σ-length among all such arcs. The following is a version of Lemma 4.1 of [Min00].

Lemma 13.4 (Lemma 4.1 of [Min00]). *There exists a number $D_1 = D_1(\Sigma) > 0$ with the following property. Let $Y \subset \Sigma$ be a proper essential non-annular subsurface which is strongly incompressible for the hyperbolic 3-manifold M_f. Then for every $\gamma \in \mathcal{D}_1 \cup \mathcal{D}_2$ there exists a pleated surface g_γ in the homotopy class of the inclusion $\Sigma \to M_f$ mapping ∂Y geodesically, with induced metric $\sigma(g_\gamma)$, such that for any minimal proper arc τ in $(Y, \sigma(g_\gamma))$ we have*

$$d_Y(\gamma, \tau) \le D_1.$$

Proof. The proof of Lemma 4.1 of [Min00] carries over with no essential modification. Namely, let $\gamma \subset \Sigma$ be a simple closed curve which defines an element of $\mathcal{D}_1 \cup \mathcal{D}_2$. By assumption on Y, the curve γ has an essential intersection with ∂Y.

Modify γ by spinning it about ∂Y. That is, let $\mathcal{T}_{\partial Y}$ be the mapping class that performs one positive Dehn twist about each component of ∂Y. The sequence of curves $\mathcal{T}^n_{\partial Y}(\gamma)$ converge, as $n \to \infty$, to a finite-leaved lamination λ whose non-compact leaves spiral about ∂Y and whose closed leaves are precisely ∂Y. Since the distance in $\mathcal{CG}(\Sigma)$ between γ and ∂Y is at least three, the complement $\Sigma - (\partial Y \cup \gamma)$ is a union of simply connected components and hence the complementary regions of λ are simply connected as well. Add finitely many leaves to λ so that the resulting lamination λ' is maximal. To simplify notations, we identify λ with λ'.

The lamination λ is the pleating lamination of a pleated surface g_λ in M_f mapping λ geodesically, with induced metric σ_λ (compare [Min00]). Namely, let η be a component of ∂Y. Then η is not homotopic to zero in M_f and hence it can be represented by a unique closed geodesic $\hat{\eta}$. Let $\hat{M}_f$ be the covering of M_f whose fundamental group is infinitely cyclic and generated by the loop $\hat{\eta}$. Then $\hat{M}_f$ is a solid torus with core curve the geodesic $\hat{\eta}$, and η lifts to a closed curve in $\hat{M}_f$. Mapping a point p on $\eta \subset \hat{M}_f$ to its shortest distance projection to $\hat{\eta}$ and connecting p to its image by a geodesic arc determines a canonical homotopy of η to $\hat{\eta}$ which projects to a homotopy in M_f. Using this homotopy, any essential arc in Y with endpoints on η can be extended to an arc with endpoints on $\hat{\eta}$.

Thus the intersection arcs of the simple closed curve γ with ∂Y, assumed without loss of generality to be essential, define a collection of arcs in M_f with boundary on the collection $\hat{\partial} Y$ of geodesics in M_f homotopic to ∂Y. By the second part of Lemma 13.1, such an extended arc is not homotopic into $\hat{\partial} Y$ keeping the endpoints in $\hat{\partial} Y$, and by the third part of Lemma 13.1, two such extended arcs are homotopic in M_f keeping the endpoints in $\hat{\partial} Y$ only if the corresponding arcs in Y are homotopic keeping the endpoints in ∂Y. Namely, two distinct such arcs are disjoint up to homotopy. Using the above homotopy which deforms η to $\hat{\eta}$, this yields that each such arc can be represented by a unique nontrivial geodesic arc in M_f with endpoints in $\hat{\partial} Y$ which meets $\hat{\partial} Y$ orthogonally at the endpoints

Now spinning γ about the boundary components of ∂Y corresponds to turning the endpoints of the geodesic arcs with boundary on $\hat{\partial} Y$ about the components of $\hat{\partial} Y$. Taking a limit as the number of turns goes to infinity results in replacing the arcs by infinite geodesics which spiral about the components of $\partial \hat{Y}$. These geodesics define the geometric realization of the lamination λ in M_f. After adding finitely many leaves, the lamination λ decomposes Σ into finitely many ideal triangles. These triangles bound totally geodesic immersed ideal triangles in M_f whose union defines the pleated surface g_λ.

Denote by R_λ the complement in (Σ, σ_λ) of the κ_0-Margulis tubes whose cores are components of λ (and hence of ∂Y), where κ_0 is the constant chosen above with properties (P1) and (P2). Realize the diskbounding curve $\gamma \subset \Sigma$ by its geodesic representative for σ_λ. Denoting by $\ell_{\sigma_\lambda}(\alpha)$ the length of a geodesic arc α for the hyperbolic metric σ_λ, Theorem 3.5 and formula (4.3) of [Min00] show that

$$\ell_{\sigma_\lambda}(\gamma \cap R_\lambda) \leq 2C\iota(\gamma, \partial Y)$$

for a universal constant $C = C(\Sigma, \kappa_0) > 0$ where $\iota(\gamma, \partial Y)$ is the geometric intersection number. Note that Theorem 3.5 of [Min00] holds true as stated for homotopically trivial curves in M_f, that is, for curves of vanishing length.

As on p.139 of [Min00], it now follows that there exists at least one component arc of $\gamma \cap Y \cap R_\lambda$ of length at most $4C$. Given a minimal proper arc τ for (Y, σ_λ), Lemma 2.1 of [Min00] then bounds $d_Y(\gamma, \tau)$ from above by a universal constant. This is what we wanted to show. □

Following once more [Min00], we next turn to the proof of an analogue of Lemma 13.4 for essential incompressible annuli. For its formulation, define a *bridge arc* for a simple closed curve α in Σ to be an embedded arc in Σ with both endpoints in α which meets α only at its endpoints and is not homotopic into α keeping the endpoints in α. Let σ be a hyperbolic metric on Σ and let α be a simple closed geodesic for this metric. Define a *mininmal curve crossing* α to be a simple closed curve constructed in the following way.

Pick one side of α in Σ and let τ be a minimal length primitive bridge arc for α, that is, a bridge arc whose interior is disjoint from α, that is incident to α on this side. Let τ' be a minimal length primitive bridge arc for α that is incident to α on the other side. If one of these arcs meets α on both sides then put $\tau = \tau'$. Choose β to be a minimal length shortest simple closed curve that can be represented as a concatenation of τ, τ' (if they are different) and arcs on α. Thus β crosses through α once or twice.

Lemma 13.5 (Lemma 4.3 of [Min00]). *Given $\epsilon > 0$ there exists $D_2 = D_2(\Sigma, \epsilon) > 0$ such that for a closed hyperbolic 3-manifold M_f the following holds. Let $Y \subset \Sigma$ be a proper essential strongly incompressible annulus with core curve α so that $\ell_f(\alpha) \geq \epsilon$. Then for every $\gamma \in \mathcal{D}_1 \cup \mathcal{D}_2$, there exists a pleated surface g_γ, with induced metric $\sigma(g_\gamma)$, mapping α geodesically and the property that a minimal curve β for $\sigma(g_\gamma)$ crossing α satisfies*

$$d_Y(\gamma, \beta) \leq D_2.$$

Proof. The proof of Lemma 4.3 of [Min00] is valid without any change. Construct a lamination λ from α and $\gamma \in \mathcal{D}_1 \cup \mathcal{D}_2$ by spinning γ about α. As in the proof of Lemma 13.4, by the assumption on the distance in $\mathcal{CG}(\Sigma)$ between α and γ, all complementary components of λ are simply connected. Adding finitely many leaves to λ yields a maximal lamination and hence a pleated surface g_λ, with metric σ_λ. As in Lemma 13.4, it follows from Theorem 3.5 of [Min00] that

$$\ell_{\sigma_\lambda}(\gamma) \leq 2C\iota(\alpha, \gamma)$$

for a universal constant $C = C(\tau, \epsilon) > 0$.

The remainder of the argument in the proof of Lemma 4.3 of [Min00] only uses the geometry of σ_λ on Σ and does not use any information on the hyperbolic 3-manifold containing the pleated surface g_λ. It is thus valid without any adjustment. □

In [Min00], the proof of a version of Theorem 13.3 is completed by proving a universal upper length bound for minimal bridge arcs for pleated surfaces constructed from proper essential strongly incompressible subsurfaces $Y \subset \Sigma$ with big boundary length (this is the core of the proof of Lemma 4.2 and Lemma 4.4 of [Min00]). This step requires a substantial modification for Heegaard surfaces. We formulate what we need in the following proposition. For the remainder of this section, M_f always denotes a hyperbolic 3-manifold with Heegaard surface Σ, Hempel distance at least 4 and disk sets $\mathcal{D}_1, \mathcal{D}_2$. Recall that for any proper incompressible subsurface $Y \subset \Sigma$, any simple closed curve c on Σ with $d_{\mathcal{CG}}(c, \partial Y) \geq 3$ gives rise to a pleated surface containing ∂Y in its pleating lamination. The number $p > 4$ is as in Lemma 13.2.

Proposition 13.6. *For any $\epsilon > 0$ there exists a number $D_3 = D_3(\Sigma, \epsilon) > 0$ with the following property. Let $Y \subset \Sigma$ be a proper essential subsurface with $d_{\mathcal{CG}}(\partial Y, \mathcal{D}_1 \cup \mathcal{D}_2) \geq p$, and let g_1, g_2 be a pair of pleated surfaces in the homotopy class of the inclusion $\Sigma \to M_f$ mapping ∂Y geodesically which are constructed from diskbounding simple closed curves $\gamma_1 \in \mathcal{D}_1, \gamma_2 \in \mathcal{D}_2$. Let $\sigma(g_1), \sigma(g_2)$ be the induced hyperbolic metrics on Σ and let τ_1, τ_2 be minimal proper arcs for $\sigma(g_1), \sigma(g_2)$ if Y is not an annulus, or minimal curves crossing α for $\sigma(g_1), \sigma(g_2)$ if Y is an annulus. If $\ell_f(\partial Y) \geq \epsilon$ then*

$$d_Y(\tau_1, \tau_2) \leq D_3.$$

We are now ready to deduce Theorem 13.3 from Lemma 13.4, Lemma 13.5 and Proposition 13.6.

Proof of Theorem 13.3. Let $Y \subset \Sigma$ be a proper essential subsurface with $d_{\mathcal{CG}}(\partial Y, \mathcal{D}_1 \cup \mathcal{D}_2) \geq p$. Let $\epsilon > 0$ and assume that $\ell_f(\partial Y) \geq \epsilon$. Let $\gamma_1 \in \mathcal{D}_1, \gamma_2 \in \mathcal{D}_2$ be two diskbounding simple closed curves in Σ.

If Y is non-annular then apply Lemma 13.4 to obtain two pleated surfaces mapping ∂Y geodesically, and minimal proper arcs τ_1, τ_2 in Y with respect to the two induced metrics on Σ, with $d_Y(\gamma_i, \tau_i) \leq D_1$ $(i = 1, 2)$. Proposition 13.6 then implies that $d_Y(\gamma_1, \gamma_2) \leq 2D_1 + D_3(\Sigma, \epsilon)$.

If Y is an annulus, then apply Lemma 13.5 to obtain two pleated surfaces mapping the core curve α of Y to a geodesic, and minimal curves β_1, β_2 crossing α with respect to the two induced metrics on Σ, with $d_Y(\beta_i, \gamma_i) \leq D_2 = D_2(\Sigma, \epsilon)$ for $i = 1, 2$. An application of Proposition 13.6 shows as before that $d_Y(\gamma_1, \gamma_2) \leq 2D_2 + D_3(\Sigma, \epsilon)$. This completes the proof of Theorem 13.3. ☐

We are left with proving Proposition 13.6 which is the main technical result of this section.

To facilitate notations, call a system $X \subset \Sigma$ of nontrivial homotopically distinct disjoint simple closed curves in Σ *stongly incompressible in M_f* if $d_{\mathcal{CG}}(X, \mathcal{D}_1 \cup \mathcal{D}_2) \geq 3$. This notion is compatible with the notion of a strongly incompressible subsurface of Σ. Note that X is strongly incompressible if and only if the same holds true for each of its components.

If $X \subset \Sigma$ is a strongly incompressible curve system, then we denote by $\mathcal{P}(X)$ the collection of all pleated surfaces in M_f in the homotopy class of the inclusion $\Sigma \to M_f$, with pleating lamination a complete (that is, maximal and approximable in the Hausdorff topology by simple closed geodesics) finite geodesic lamination whose minimal components are precisely the components of X.

An important fact is that for any strongly incompressible curve system $X \subset \Sigma$, any two pleated surfaces $g, h \in \mathcal{P}(X)$ can be deformed into each other with a homotopy consisting of surfaces with controlled geometry. To make this precise, we define $\mathcal{L}(X)$ to be the collection of all maps $g : \Sigma \to M_f$ in the homotopy class of the inclusion with the following additional property. There exists a hyperbolic metric $\sigma(g)$ on Σ such that for this metric, the map g is one-Lipschitz and maps each component of X isometrically onto its geodesic representative in M_f. Note that the metric $\sigma(g)$ on Σ is part of the data which define a point in $\mathcal{L}(X)$ although it may not be unique. Note also that we have $\mathcal{P}(X) \subset \mathcal{L}(X)$. If g is a pleated surface then $\sigma(g)$ is assumed to be the hyperbolic metric on Σ defined by g.

A *path* in $\mathcal{L}(X)$ is a continuous map $h : \Sigma \times [a, b] \to M_f$ for some interval $[a, b] \subset \mathbb{R}$ such that for each $s \in [a, b]$, there is a marked hyperbolic metric $\sigma(s)$ on Σ depending continuously on s and such that the map $h_s : x \in \Sigma \to h_s(x) = h(x, s) \in M$ is a point in $\mathcal{L}(X)$ for the metric $\sigma(s)$. If a point of the path, say the point h_a, is a pleated surface, then we require that $\sigma(a) = \sigma(h_a)$.

Following Section 3 of [Min00], define two pleated surfaces $f, g : \Sigma \to M_f$ to be *homotopic relative to a common pleating lamination μ* if μ is a sublamination of the pleating lamination of f, g and if f and g are homotopic by a family of maps which fixes μ pointwise.

The following is a slight strengthening of a well know construction which goes back to Thurston (see p.140 of [Min00] and [Can93] for more on earlier accounts). Recall from Lemma 13.1 that for an incompressible curve system $X \subset \Sigma$, an essential arc α in Σ with endpoints on ∂X lifts to an arc in the universal covering $\mathbb{H}^3$ of M_f which connects two distinct lifts of the components of X containing the endpoints of α. The second part of the lemma extends Lemma 3.3 of [Min00].

Lemma 13.7. *Let $X \subset \Sigma$ be a strongly incompressible curve system. Then any $g, h \in \mathcal{P}(X)$ can be connected by a path in $\mathcal{L}(X)$. Furthermore, g, h are homotopic relative to any common pleating lamination $\mu \subset X$.*

Proof. Let us first consider two pleated surfaces $g_0, g_1 \in \mathcal{P}(X)$ which are related by a *diagonal move*. By this we mean the following. The pleating lamination λ of g_0 is an extension of X. It decomposes Σ into ideal triangles whose sides spiral about X. Isolated leaves of λ do not belong to X. Removal of such an isolated leaf α results in a geodesic lamination λ' whose complementary components are ideal triangles and one ideal quadrangle Q. The leaf α connects two opposite vertices of Q and subdivides Q into two ideal triangles. We assume that the pleating lamination for g_1 is obtained from λ by replacing α by the diagonal β of Q connecting the other two opposite vertices.

Our goal is to construct a path in $\mathcal{L}(X)$ connecting g_0 to g_1. To this end let $\tilde{g}_0 : \mathbb{H}^2 \to \mathbb{H}^3$ be a lift of g_0 to the universal coverings $\mathbb{H}^2$ of Σ and $\mathbb{H}^3$ of M_f. Let $\tilde{Q} \subset \mathbb{H}^2$ be a lift of the ideal quadrangle Q. The image $\tilde{g}_0(\tilde{Q})$ of $\tilde{Q}$ under the map $\tilde{g}_0$ is the union of two ideal triangles which are glued along a common side. Let $(a_1, a_2, a_3, a_4) \subset \partial\mathbb{H}^3$ be the ordered collection of points in the ideal boundary $\partial\mathbb{H}^3$ of $\mathbb{H}^3$ which are the images of the ordered vertices of $\tilde{Q}$. This ordered quadruple of points spans an ideal tetrahedron $T \subset \mathbb{H}^3$. Note that by the third part of Lemma 13.1, this tetrahedron is non-degenerate. The map $\tilde{g}_0$ maps $\tilde{Q}$ onto the union $\tilde{Q}_0$ of two adjacent sides of T. The image of $\tilde{Q}$ under a suitably chosen lift $\tilde{g}_1$ of g_1 equals the union $\tilde{Q}_1$ of the remaining two adjacent sides of T. Four of the six edges of T are the sides of $\tilde{g}_0(\tilde{Q})$, and the remaining two edges are the images under $\tilde{g}_0, \tilde{g}_1$ of the lifts $\tilde{\alpha}, \tilde{\beta}$ of the diagonals α, β of Q to $\tilde{Q}$. The restriction of $\tilde{g}_0, \tilde{g}_1$ to $\tilde{Q}$ is a path isometry onto $\tilde{Q}_0, \tilde{Q}_1$, respectively.

The piecewise totally geodesic quadrangle $\tilde{Q}_0$ is equipped with an intrinsic hyperbolic metric. Let $\tilde{\beta}_0 \subset \tilde{Q}_0$ be the intrinsic geodesic which connects the 2 ideal vertices of $\tilde{Q}_0$ which are different from the endpoints of $\tilde{g}_0(\tilde{\alpha})$. Then $\tilde{\beta}_0$ is a piecewise geodesic line in $\mathbb{H}^3$ which intersects the geodesic $\tilde{g}_0(\tilde{\alpha})$ in a single point x_0. The point x_0 is the finite vertex of a partition of $\tilde{Q}_0$ into 4 totally geodesic triangles with one vertex at x_0 and two ideal vertices. The total cone angle, that is, the sum of the angles at x_0 of these triangles, equals 2π. Construct in the same way a point $x_1 \in \tilde{Q}_1$ as the intersection point between the two intrinsic geodesics connecting the two pairs of opposite ideal vertices of $\tilde{Q}_1$. As before, x_1 is the finite vertex of a partition of $\tilde{Q}_1$ into 4 totally geodesic triangles with total cone angle 2π at x_1.

Connect x_0 to x_1 by a geodesic arc $\gamma : [0,1] \to T \subset \mathbb{H}^3$ parameterized proportional to arc length on $[0,1]$. For each $t \in [0,1]$ consider the union $\tilde{Q}_t$ of the 4 totally geodesic triangles $A_i(t)$ $(i = 1,\dots,4)$ with one vertex at $\gamma(t)$ which have the same ideal vertices as the triangles which subdivide $\tilde{Q}_0$. Note that this notation is consistent with the above definition of $\tilde{Q}_0, \tilde{Q}_1$. Each of the 4 boundary geodesics of $\tilde{Q}_0$ is contained in precisely one of the triangles from the collection $\tilde{Q}_t$. If we choose the labels of the triangles $A_i(t)$ in such a way that for each i, the triangles $A_i(t)$ contain the same boundary geodesic of $\tilde{Q}_0$ for all t, then these triangles depend continuously on t. Since T is the convex hull of its ideal vertices and $\gamma \subset T$, the total cone angle at $\gamma(t)$ of the union of these triangles is at least 2π, and it is 2π at the endpoints $x_0 = \gamma(0)$, $x_1 = \gamma(1)$ of γ.

For $t \in [0,1]$ let $q(t) \geq 0$ be such that the total cone angle of $\tilde{Q}_t$ at $\gamma(t)$ equals $2\pi(1 + q(t))$. Denote by $\nu_i(t)$ the angle of the triangle $A_i(t) \subset \tilde{Q}_t$ at $\gamma(t)$. Let $\hat{\nu}_i(t) = \nu_i(t)/(1 + q(t)) \leq \nu_i(t)$ $(i = 1,2,3,4)$; we have $\sum_i \hat{\nu}_i(t) = 2\pi$ for all t. Let $B_i(t)$ be the hyperbolic triangle with two ideal vertices and one vertex of angle $\hat{\nu}_i(t)$. Note that there exists a natural isometric embedding of $A_i(t)$ into $B_i(t)$ so that the image contains the biinfinite side of $B_i(t)$. This embedding is unique if we require that the finite vertex of $A_i(t)$ is contained in the minimal geodesic $\xi_i(t)$ of $B_i(t)$ which connects the finite vertex of $B_i(t)$ to the opposite side. Denote the image of $A_i(t)$ under this embedding again by $A_i(t)$.

By the choice of the angles $\hat{\nu}_i(t)$, the triangles $B_i(t)$ can be glued along their sides which are adjacent to the finite vertex cyclically in the order prescribed by the order of the triangles $A_i(t)$ in the polygon $\tilde{Q}_t$ to a hyperbolic ideal quadrangle $B(t)$ with a distinguished vertex $q(t)$. The ideal quadrangle $B(t)$ contains the union $A(t)$ of the triangles $A_i(t)$. This construction does not depend on choices and hence depends continuously on t. Moreover, $B(t) - A(t)$ is a region which is star shaped with respect to the point $q(t)$. This region consists of the interior of an embedded relatively compact quadrangle $C(t)$, with an ideal triangle attached to each of its sides.

By invariance under the action of $\pi_1(\Sigma)$, the hyperbolic quadrangle $B(t)$ determines a hyperbolic metric $\sigma(t)$ on Σ depending continuously on t, and $\sigma(0) = \sigma(g_0), \sigma(1) = \sigma(g_1)$. Thus we are left with constructing a continuous map $h : \Sigma \times [0,1] \to M_f$ such that for each t, the restriction of h to $\Sigma \times \{t\}$ is a one-Lipschitz map $(\Sigma, \sigma(t)) \to M_f$ mapping λ' geodesically.

There is a natural 1-Lipschitz map $B(t) \to \tilde{Q}_t$ which maps each of the triangles $A_i(t)$ isometrically, collapses the complementary quadrangle $C(t)$ to a point and collapses the ideal triangles attached to the sides of $C(t)$ to one of its infinite length sides by collapsing a geodesic arc contained in one of these triangles with endpoints on the two distinct infinite sides of the triangle to a point if its endpoints are identified in $\tilde{Q}_t$. This construction defines a one-Lipschitz map $(\Sigma, \sigma(t)) \to M_f$ depending continuously on t and mapping X isometrically. As for $t = 0$ and $t = 1$ the collapsing map equals the identity, we obtain a path in $\mathcal{L}(X)$ connecting g_0 to g_1 provided that g_0 and g_1 are related by a diagonal move.

Note that this construction does not use any information on the pleating locus of g_0, g_1 beyond the information that these pleating loci differ by a diagonal move. Moreover, it yields a path $g_s \in \mathcal{L}(X)$ whose restriction to the intersection of the pleating loci of g_0, g_1 is the identity. In particular, the pleated surfaces g_0, g_1 are homotopic relative to the pleating lamination X.

To complete the proof of the lemma we are left with showing that any two pleated surfaces $g_0, g_1 \in \mathcal{P}(X)$ can be connected by a finite chain of pleated surfaces in $\mathcal{P}(X)$ so that any two consecutive pleated surfaces in the chain are related by a diagonal move. That this is possible is an immediate consequence of a result of Hatcher [Hat91] (compare [Can93] and [Min00] for more details about this fact). By concatenation, this shows that any two pleated surfaces in $\mathcal{P}(X)$ can be connected by a path in $\mathcal{L}(X)$, and g_0, g_1 are homotopic relative to the pleating lamination X. □

Remark 13.8. The proof of Lemma 13.4 together with Lemma 13.7 and Lemma 13.2 yield additional information on M_f. Namely, let as before $Y \subset \Sigma$ be a strongly incompressible subsurface and let $\alpha \subset \Sigma - \partial Y$ be any system of pairwise disjoint non-homotopic arcs with endpoints on ∂Y which decompose Y into the maximal possible number of simply connected regions. Then α determines a pleated surface g in M_f in the homotopy class of the inclusion $\Sigma \to M_f$ whose pleating lamination contains ∂Y as the union of its minimal components. This pleated surface only depends on ∂Y and the homotopy classes of the components of α as arcs in M_f with boundary on ∂Y. If α_1, α_2 are two such arc systems, and if α_1 contains an arc ζ_1 which is homotopic in M_f relative to ∂Y to an arc ζ_2 from α_2, then ζ_1 and ζ_2 determine the same isolated leaf of the pleating lamination of the pleated surface in M_f constructed from α_1, α_2. Since by the proof of Lemma 13.4 the pleated surfaces constructed in this way are naturally homotopic to the inclusion $\Sigma \to M_f$, Lemma 13.2 yields that the arcs ζ_1, ζ_2 are in fact homotopic in Σ.

The strategy is now to obtain geometric information on minimal proper arcs or minimal curves for a pleated surface $g \in \mathcal{P}(\partial Y)$ from information on the geodesic representative $\partial Y \subset M_f$. In the following elementary observation, $\ell_f(c)$ denotes as before the length in M_f of a geodesic representative of a multicurve c in Σ.

Lemma 13.9. *For every $\epsilon > 0$ there exists a number $L = L(\epsilon) > 0$ with the following property. Let $Y \subset \Sigma$ be a proper essential strongly incompressible non-annular subsurface with $\ell_f(\partial Y) \geq \epsilon$; then for any $g \in \mathcal{L}(\partial Y)$, the $\sigma(g)$-length of a minimal proper arc for Y is at most L. Moreover, for any $\kappa_1 > 0$ there exists a number $R_1 = R_1(\kappa_1) > 0$ with the following property. If $\beta \subset \partial Y$ is a component with $\ell_f(\beta) \geq R_1$ then there exists a bridge arc for Y with one endpoint on β and of $\sigma(g)$-length at most κ_1.*

Proof. Let $m \in [1, 3g-3]$ be the number of components of ∂Y. By the collar theorem for hyperbolic metrics on Σ, if β is a component of ∂Y of length at least ϵ/m, then the supremum ρ of the $\sigma(g)$-heights of a half-collar about β is bounded from above by a number $L/2$ only depending on ϵ/m. By the choice of ρ, the boundary of a half-collar of radius ρ about β can not be retracted into β. Hence there exists a radial geodesic segment of length ρ emanating from the side of β determined by the half-collar whose endpoint either is the endpoint of another such segment or is contained in β. In both cases, we find an essential arc τ with endpoints on β whose length does not exceed 2ρ.

Using the half-collar at the side of β contained in Y, we can assume that a neighborhood of at least one endpoint of the essential arc τ is contained in Y. Since the second endpoint of τ is contained in β, we conclude that there is a (possibly proper) subarc of τ which is a bridge arc for Y, and the length of this arc is at most $2\rho \leq L$ as claimed.

Using again the collar lemma for hyperbolic metrics on Σ (or a standard area estimate), for a given number $\kappa_1 > 0$, if $R_1 > 0$ is sufficiently large then the height of a half-collar about a simple closed geodesic of length at least R_1 for any hyperbolic metric on Σ is smaller than $\kappa_1/2$. By the above discussion, this implies that if ∂Y contains a component β of length at least R_1, then there exists a bridge arc for Y of length at most κ_1 with one endpoint on β. This completes the proof of the lemma. □

We use Lemma 13.7 and Lemma 13.9 to establish the following version of Lemma 4.2 of [Min00]. In its formulation, $\ell_f(\partial Y)$ denotes as before the length of ∂Y with respect to the hyperbolic metric on M_f.

Lemma 13.10. *For all $\epsilon > 0, R > 0$ there exists a number $k_0 = k_0(\epsilon, R) > 0$ with the following property. Let $Y \subset \Sigma$ be a proper essential strongly incompressible subsurface and assume that $\ell_f(\partial Y) \leq R$. If* $\mathrm{diam}_Y(\mathcal{D}_1 \cup \mathcal{D}_2) \geq k_0$*, then*

$$\ell_f(\partial Y) < \epsilon.$$

Proof. Let $R > 0, \epsilon < R$ be fixed and assume that $\ell_f(\partial Y) \in [\epsilon, R]$. If Y is not an annulus, then by Lemma 13.9, there exists a number $L = L(\epsilon) > 0$ such that for every $g \in \mathcal{L}(\partial Y)$, the $\sigma(g)$-length of a minimal proper arc for Y is at most L.

We claim that there also is a uniform upper length bound $L' = L'(\epsilon, R)$ for a minimal curve for Y if Y is an annulus. Namely, let Y be an annulus with core curve c. If σ is a hyperbolic metric on Σ such that the σ-length of c is contained in the interval $[\epsilon, R]$, then the σ-height of a half-collar about c is bounded from above by a constant $\rho > 0$ only depending on ϵ. Thus there exists a proper arc τ for σ of length at most 2ρ with both endpoints on c. If τ leaves and returns at the two different sides of c, then the endpoints of τ can be connected by a subarc of c of length at most $R/2$ to yield a simple closed curve crossing through c of length at most $2\rho + R/2$. If all proper arcs τ for c of length at most 2ρ leave and return to the same side of c, then we can find such a proper arc τ leaving and returning to a fixed side of c, and a second arc τ' leaving and returning to the other side of c. Furthermore, we may assume that τ and τ' are disjoint. The endpoints of τ and τ' can be connected by disjoint subarcs of c to yield a simple closed curve crossing through c of length at most $4\rho + R$.

We first show the lemma in the case that Y is not an annulus. Following the proof of Lemma 4.2 of [Min00], let $g_0, g_1 \in \mathcal{P}(\partial Y)$ be two pleated surfaces and assume that g_0 is constructed from Y and a maximal system A of arcs with endpoints in ∂Y which contain the intersection arcs with Y of a disk from the disk set $\mathcal{D}_2$, and that g_1 is constructed from Y and a maximal system B of arcs which contain the intersection arcs with Y of a disk from the disk set $\mathcal{D}_1$. By Lemma 13.7, these pleated surfaces can be connected in $\mathcal{L}(\partial Y)$ by a path g_t ($t \in [0,1]$). Let $\sigma(g_t)$ be the corresponding path in the Teichmüller space $\mathcal{T}(\Sigma)$ of Σ connecting $\sigma(g_0)$ to $\sigma(g_1)$. Given any bridge arc τ for Y, let $E_\tau \subset [0,1]$ denote the set of t-values for which τ is homotopic rel ∂Y to a minimal proper arc with respect to $\sigma(g_t)$. Continuity of the metrics $\sigma(g_t)$ in t implies that E_τ is closed, and the family $\{E_\tau\}$ covers $[0,1]$. The path g_t determines a coarsely well defined map Ψ from the interval $[0,1]$ into the arc and curve graph of Y. This map associates to $t \in [0,1]$ the set of all τ with $t \in E_\tau$.

Following p.141 of [Min00], we observe that if $E_\tau \cap E_{\tau'} \neq \emptyset$ then up to homotopy, τ intersects τ' in at most one point and hence the distance in the arc and curve graph of Y between τ, τ' is at most 2. Thus the coarsely well defined map Ψ has the following property. If $\tau \in \Psi[0,1]$, and if $\Psi[0,1]$ consists of more than one point, then any $\tau' \in \Psi[0,1] - \{\tau\}$ fulfills $d_Y(\tau,\tau') \leq 2$. Together with Lemma 13.4, this shows that $\Psi[0,1]$ contains a sequence of arcs $\tau_0, \tau_1, \cdots, \tau_n$ so that $d_Y(\tau_0, A) \leq D_1$, $d_Y(\tau_n, B) \leq D_1$ and $1 \leq d_Y(\tau_i, \tau_{i+1}) \leq 2$ for all i. As a consequence, it holds $n \geq d_Y(\mathcal{D}_1, \mathcal{D}_2)/2 - 2D_1$. We also may assume that the arcs τ_i are pairwise non-homotopic as arcs in Y with endpoints in ∂Y.

By Remark 13.8, if two such arcs τ_i, τ_j are homotpic in M_f keeping the endpoints in ∂Y, then τ_i, τ_j are homotopic in Σ keeping the endpoints in ∂Y. Together this implies the following. Among the bridge arcs τ_i of Y, there are at least $d_Y(\mathcal{D}_1, \mathcal{D}_2)/2 - 2D_1 = q$ arcs which are pairwise non-homotopic in M_f keeping the endpoints in ∂Y.

Since the maps $g_t \in \mathcal{L}(\partial Y)$ are one-Lipschitz, the union of ∂Y with the homotopy classes of minimal proper arcs for the metrics $\sigma(g_t)$, viewed as arcs in M_f with boundary in ∂Y via the 1-Lipschitz maps $g_t : \Sigma \to M_f$, can be represented in M_f by a 1-complex V with at most m components where $m \leq 3|\chi(\Sigma)|/2$ is the number of components of ∂Y. The diameter in M_f of each component of V is at most $R + mL$. Each such minimal proper arc τ together with one or two segments of $\partial Y - \tau$ gives rise to a loop in this one-complex V of length at most $2R+2L$. Up to homotopy rel ∂Y, these based loops are images by the inclusion $\Sigma \hookrightarrow M_f$ of *simple* closed curves contained in Y. Such a simple closed curve is a component of the boundary of a small neighborhood of the union of τ with the components of ∂Y containing the endpoints of τ.

Given a component V_0 of V, choose a basepoint x for V_0 in a component of ∂Y contained in V_0. Connecting each of the loops in V_0 constructed in the previous paragraph to x determines a collection of based loops in M_f which up to homotopy are images of based simple loops contained in Y. The length of each such loop is at most $3R+(2m+2)L$. By Lemma 13.1, no two distinct of these loops are homotopic in M_f.

As bridge arcs τ, τ' for Y which are homotopically distinct in M_f give rise to homotopy classes which do not have a common power and hence which do not commute, a standard application of the Margulis lemma gives an upper bound $M = M(3R + (2m+2)L)$ for the number of such elements of $\pi_1(M_f)$ which can translate any point a distance $3R + (2m+2)L$ or less (see p.141 of [Min00] for more details). As a consequence, the number q of homotopy classes of arcs obtained from the above construction is at most M. Together we conclude that

$$d_Y(\mathcal{D}_1, \mathcal{D}_2) \leq 2q + 4D_1 \leq 2M + 4D_1.$$

This complete the proof of the lemma in the case that $Y \subset \Sigma$ is not an annulus.

If Y is an annulus then the above argument carries over in the same way, where the arc and curve graph is now the arc graph of the annular cover of Σ whose fundamental group equals the fundamental group of Y. We refer to Lemma 4.4 of [Min00] for more details of this argument which is valid in our context with as only addition the above counting estimates for homotopy classes of arcs in M_f with endpoints in the core curve of Y. An application of Lemma 13.5 then completes the proof of the lemma. □

Let $\kappa_0 > 0$ be a constant which has properties (P1) and (P2) from the beginning of this section. By possibly decreasing κ_0, we may assume that it is a Margulis constant for hyperbolic surfaces and hyperbolic 3-manifolds. In the sequel we always assume that the thin part of a hyperbolic 3-manifold is determined by such a constant κ_0.

We next investigate strongly incompressible surfaces $Y \subset \Sigma$ with the property that the geodesic representatives of their boundaries ∂Y enter deeply into a Margulis tube of M_f away from the core curve of the tube. For a number $\nu < \kappa_0$ and a Margulis tube T for M_f, we call the set $T^{<\nu}$ of all points in T of injectivity radius smaller than ν the *ν-thin part of T*.

Lemma 13.11. *There exists a number $\nu_0 < \kappa_0$ with the following property. Let $T \subset M_f$ be a Margulis tube and let $Y \subset \Sigma$ be a strongly incompressible subsurface whose boundary ∂Y, as a geodesic multicurve in M_f, intersects $T^{<\nu_0}$ in the complement of the core geodesic of T. Then for every map $g \in \mathcal{L}(\partial Y)$, and any choice $\sigma(g)$ of a corresponding hyperbolic metric, there exists a Margulis tube for $\sigma(g)$ whose core curve $\alpha \subset \Sigma$ intersects Y in a bridge arc τ of length smaller than κ_0 if Y is not an annulus, or which is a simple closed curve crossing through ∂Y if Y is an annulus. Moreover, one of the following not mutually exclusive possibilities is satisfied.*

i) Up to homotopy, $g(\alpha)$ bounds a disk $D \subset T \subset M_f$.
ii) $g(\alpha)$ is homotopic to a nontrivial multiple of the core curve of T. Furthermore, there exists a diskbounding simple closed curve β on Σ which is disjoint from α.
iii) Up to homotopy, any component of the intersection of $g(\Sigma)$ with the 1-neighborhood of $T^{<\nu_0}$ is an annulus, and this annulus is the image under g of a Margulis tube for $\sigma(g)$. The core curve of each such tube is mapped by g to a curve homotopic to a nontrivial multiple of the core curve of T.

Proof. For the fixed choice of a Margulis constant $\kappa_0 > 0$ for hyperbolic surfaces, there exists a number $\ell > 0$ only depending on Σ and κ_0 such that for any hyperbolic metric on Σ, the diameter of any component of the κ_0-thick part of Σ is at most ℓ. Let $\nu_0 > 0$ be sufficiently small that the 2ℓ-neighborhood of the ν_0-thin part $T^{<\nu_0}$ of a Margulis tube T for M_f is entirely contained in T.

Let $Y \subset \Sigma$ be a strongly incompressible surface with boundary ∂Y and let $g \in \mathcal{L}(\partial Y)$. There exists a decomposition of Σ into thick and thin components for the metric $\sigma(g)$. The thin components are Margulis tubes about closed $\sigma(g)$-geodesics of length smaller than $2\kappa_0$. By the choice of ℓ and ν_0, for any Margulis tube $T \subset M_f$, any component of the thick part of Σ for the metric $\sigma(g)$ whose image under g intersects $T^{<\nu_0}$ is mapped by g into T.

Assume that the geodesic multicurve $\partial Y \subset M_f$ intersects the ν_0-thin part $T^{<\nu_0}$ of a Margulis tube $T \subset M_f$ in the complement of the core curve of the tube. Since ∂Y is a union of closed geodesics in M_f and the only closed geodesic in M_f which is entirely contained in the tube T is the core curve of the tube, ∂Y intersects the boundary ∂T of T, which is a torus smoothly embedded in M_f.

Let $W_0 \subset \Sigma$ be the union of all components of the $\sigma(g)$-thick part of Σ whose images under g do *not* intersect $T^{<\nu_0}$, and let $W \subset \Sigma$ be the union of W_0 with all Margulis tubes for $\sigma(g)$ whose images under g do not intersect $T^{<\nu_0}$. Then the closure Z of $\Sigma - W$ is a closed nonempty essential subsurface of Σ. The surface Z is a union of components of the thick part of Σ and some Margulis tubes. Each component of the thick part of Z is mapped by g into the tube T. Since the fundamental group of T is cyclic, and the map g induces a surjection $g_* : \pi_1(\Sigma) \to \pi_1(M_f)$, this implies that the subsurface Z of Σ is proper. Note that by assumption, the surface Z is intersected by ∂Y.

Now $g(W)$ is disjoint from the core curve of T, and the complement of the core curve of T deformation retracts onto the boundary ∂T of T. Thus up to modifying g with a homotopy and replacing T by the complement in T of a suitably chosen collar about ∂T, we may assume that $g(W) \cap T = \emptyset$.

There are now two possibilities. In the first case, Z contains a component Z_0 of the $\sigma(g)$-thick part of Σ. Then we have $g(Z_0) \subset T$.

Let $x_0 \in \partial Z_0$ and consider the map $g_*^{Z_0} : \pi_1(Z_0, x_0) \to \pi_1(M_f, g(x_0))$. Since $Z_0 \subset \Sigma$ is a properly embeded connected surface different from an annulus, its fundamental group $\pi_1(Z_0)$ is a free group with at least two generators. As $\pi_1(T)$ is infinite cyclic, the kernel of $g_*^{Z_0}$ is nontrival. Now $Z_0 \subset \Sigma$ is a proper subsurface of the embedded Heegaard surface Σ. Therefore the loop theorem Theorem 4.2 of [AR04] (see also Theorem 4.2 of [Hem76]) shows that there exists a simple closed curve $c \subset Z_0$ such that $g(c)$ is homotopically trivial in M_f and hence in T.

To be more precise, since $g(Z_0) \subset T$, if $\alpha \subset Z_0$ is any closed curve such that $g(\alpha)$ is contractible in M_f, then $g(\alpha)$ is contractible in T. Thus there exists a homotopy of $g(\alpha)$ to the trivial curve which does not intersect $g(W) \subset M_f - T$, and, consequently, there exists a homotopy of $\alpha \subset Z_0 \subset \Sigma \subset M_f$ to the trivial curve which does not intersect $W \subset M_f$.

Cutting M_f open along $W \subset \Sigma$ yields a manifold N whose boundary ∂N consists of two copies of W, glued along the boundary. Up to homotopy, each component of $\Sigma - W = Z$ is a properly embedded surface in N. In particular, this holds true for the component $\hat{Z}_0$ of $\Sigma - W$ containing Z_0 (a priori, Z_0 may be a proper subsurface of $\hat{Z}_0$). Note that $\hat{Z}_0$ is an oriented, two-sided properly embedded subsurface of N which is different from a disk and a 2-sphere.

Since a loop in $Z_0 \subset \hat{Z}_0$ which is contractible in M_f is contractible in $M_f - W$, it is contractible in N. Therefore the homomorphism $\pi_1(\hat{Z}_0) \to \pi_1(N)$ induced by the inclusion $\hat{Z}_0 \to N$ is not injective. The loop theorem Theorem 4.2 of [AR04] then shows that there is a simple closed curve $c \subset \hat{Z}_0$ which bounds an embedded disk in N and hence in M_f.

If c is either peripheral in $\hat{Z}_0$ or the core curve of a Margulis tube for $\sigma(g)$ contained in $\hat{Z}_0$, then c is a core curve of a Margulis tube for $\sigma(g)$ which is diskbounding in M_f. Identify c with its geodesic representative for $\sigma(g)$. As the subsurface $Y \subset \Sigma$ is strongly incompressible by assumption, its boundary ∂Y has to cross through the diskbounding simple closed curve c. If Y is not an annulus and if $\xi \subset \partial Y$ is an embedded arc crossing through c, then a subarc of c connecting $\xi \cap c$ with the point in $c \cap \partial Y$ which is closest along c and leaves ξ at the side of ξ contained in Y is a bridge arc for Y of $\sigma(g)$-length at most κ_0. If Y is an annulus, c is a simple closed curve of $\sigma(g)$-length less than κ_0 which crosses through ∂Y. This shows that the first possibility stated in the lemma is fulfilled. Note that in the case that $\hat{Z}_0$ is a 3-holed sphere, the only simple closed curves in $\hat{Z}_0$ are the boundary curves and hence the simple closed curve c in $\hat{Z}_0$ is automatically peripheral.

On the other hand, if no diskbounding simple closed curve $c \subset \hat{Z}_0$ is the core curve of a Margulis tube for $\sigma(g)$, then the second case in statement of the lemma holds true. Namely, in this case a peripheral curve $d \subset \hat{Z}_0$ is the core curve of a Margulis tube and disjoint from c. Furthermore, the curve d is mapped by g into T and hence it is homotopic to a nontrivial multiple of the core curve of T. Since $d_{\mathcal{CG}}(\partial Y, \mathcal{D}_1 \cup \mathcal{D}_2) \geq 3$ by assumption, the multicurve ∂Y has to cross through d. We then find a bridge arc for Y of $\sigma(g)$-length smaller than κ_0 which is a subarc of d, or, if Y is an annulus, choose d as a simple curve crossing through ∂Y of length smaller than κ_0. Thus the second possibility in the statement of the lemma is fulfilled. This completes the analysis of the case when the image under g of the $\sigma(g)$-thick part of Σ intersects $T^{<\nu_0}$.

If the image under g of the $\sigma(g)$-thick part of Σ does not intersect $T^{<\nu_0}$, then each intersection point of $g(\Sigma)$ with $T^{<\nu_0}$ is contained in the image of a Margulis tube for $\sigma(g)$. Since ∂Y intersects $T^{<\nu_0}$ in the complement of the core curve of T, there is simple closed curve $\alpha \subset \Sigma$ which is freely homotopic to the core curve of one of these Margulis tubes, of $\sigma(g)$-length at most κ_0, which intersects ∂Y and which is mapped by g into T. As before, if $g(\alpha)$ is contractible in M_f then we are in the situation described in the first case of the lemma. Otherwise $g(\alpha)$ is homotopic in T to a multiple of the core curve of T, and we conclude that the third case described in the lemma is fulfilled. This completes the proof of the lemma. ☐

In the following lemma, the constant $p > 4$ is as in Lemma 13.2.

Lemma 13.12. *There exist numbers $k_1 = k_1(\Sigma) > 0$, and $\nu_1 < \nu_0$ with the following properties. Assume that the Hempel distance $d_{\mathcal{CG}}(\mathcal{D}_1, \mathcal{D}_2)$ for M_f is at least 4. Let $Y \subset \Sigma$ be a strongly incompressible subsurface whose boundary ∂Y, as a geodesic multicurve in M_f, intersects the ν_1-thin part $T^{<\nu_1}$ of a Margulis tube $T \subset M_f$ in the complement of the core geodesic of T and fulfills $d_{\mathcal{CG}}(\partial Y, \mathcal{D}_1 \cup \mathcal{D}_2) \geq p$. Then we have*

$$\mathrm{diam}_Y(\mathcal{D}_1 \cup \mathcal{D}_2) \leq k_1.$$

Proof. The number $p > 4$ was chosen so that the following holds true [MM00]. Let α, β be simple closed curves on Σ and let $Y \subset \Sigma$ be a subsurface which has an essential intersection with α, β and 0 such that $d_Y(\alpha, \beta) \geq p$; then any geodesic in $\mathcal{CG}(\Sigma)$ connecting α to β has to pass through a curve disjoint from Y. Furthermore, if $Y \subset \Sigma$ is any proper essential subsurface such that $d_{\mathcal{CG}}(\partial Y, \mathcal{D}_i) \geq p$ $(i = 1, 2)$; then $\mathrm{diam}_Y(\mathcal{D}_i) \leq p$.

Let $\ell > 0$ be an upper bound for the diameter of a component of the thick part of a hyperbolic metric on Σ for a Margulis constant $\kappa_0 > 0$ as in (P1),(P2). For $\nu_0 > 0$ as in Lemma 13.11, let $\nu_1 < \nu_0$ be sufficiently small that the neighborhood of radius ℓ of the

ν_1-thin part $M_f^{<\nu_1}$ of M_f is contained in the ν_0-thin part $M_f^{<\nu_0}$ of M_f. Let us assume that $Y \subset \Sigma$ is a proper essential subsurface with $d_{\mathcal{CG}}(\partial Y, \mathcal{D}_i) \geq p$ for $i = 1, 2$ and such that ∂Y intersects the ν_1-thin part $T^{<\nu_1}$ of a Margulis tube T in the complement of the core curve of T.

Consider first the case that Y is not an annulus. Let $\gamma_i \in \mathcal{D}_i$ for $i = 1, 2$ and $g, h \in \mathcal{P}(\partial Y)$ be pleated surfaces for γ_1, γ_2 as in Lemma 13.4 . By Lemma 13.7, we can connect g, h by a path h_s ($s \in [0,1], h_0 = g, h_1 = h$) in $\mathcal{L}(\partial Y)$. Let $\sigma(h_s)$ be a corresponding path in Teichmüller space connecting $\sigma(g)$ to $\sigma(h)$. We showed in Lemma 13.11 that for each s, there exists a bridge arc τ_s for Y of $\sigma(h_s)$-length smaller than κ_0 which is contained in a simple closed curve α_s on Σ of $\sigma(h_s)$-length at most κ_0, and α_s is homotopic to the core curve of a Margulis tube for $\sigma(h_s)$ and crossed through by ∂Y. Furthermore, up to homotopy, we may assume that $h_s(\alpha_s) \subset T^{<\nu_0}$. By the choice of κ_0, the bridge arc τ_s is disjoint from a minimal proper arc for Y and the metric $\sigma(h_s)$ (see p. 139 of [Min00] for more details).

For each s there exists a connected open neighborhood V_s of s in $[0,1]$ so that the curve α_s has the properties stated in the previous paragraph for each $t \in V_s$. By compactness, the interval $[0,1]$ can be covered by finitely many of the sets V_s. Thus we may assume that there is a partition $0 = s_0 < \cdots < s_n = 1$ such that for each $i < n$, the interval $[s_i, s_{i+1}]$ is contained in $V_i = V_{s_i}$. Then for each $s \in [s_i, s_{i+1}]$ the curve α_{s_i} is of length smaller than κ_0 for the metric $\sigma(h_s)$. In particular, by the choice of κ_0, the curves α_{s_i} and $\alpha_{s_{i+1}}$ are disjoint.

Assume that the number n of partition points of $[0,1]$ is minimal with the above property. This then implies that for all i, the curve α_i is not homotopic to α_{i+1}. If $n = 1$, or, equivalently, if $\alpha_{s_i} = \alpha_{s_j}$ for all i, j, then there exists a bridge arc τ for Y which up to homotopy is of length at most κ_0 for each of the metrics h_s. Since this bridge arc is of distance at most 1 in the arc and curve graph of Y to a $\sigma(h_s)$-minimal proper arc for Y, it then follows from the choice of g, h and Lemma 13.4 that the diameter of the subsurface projection of $\gamma_1 \cup \gamma_2$ into Y is at most $2D_1 + 2$.

If $n \geq 2$ then by minimality, we can not find a simple closed curve in Σ which is the core curve of a Margulis tube for $\sigma(h_s)$ with the properties stated above for all $s \in [s_i, s_{i+2}]$ and all i. In particular, we have $\alpha_{s_i} \neq \alpha_{s_{i+1}}$ for all i. Furthermore, there is at least one $s \in [s_{i+1}, s_{i+2}]$ such that for the metric $\sigma(h_s)$, the curve α_{s_i} is not the core curve of a Margulis tube with the properties stated above. Now α_{s_i} is crossed through by ∂Y and is mapped by h_{s_i} into $T^{<\nu_0}$, furthermore we may assume that the restrictions to ∂Y of the maps h_s coincide. As a consequence, there is some $s \in [s_{i+1}, s_{i+2}]$ and a component of the (closure of the) thick part of $\sigma(h_s)$ whose image under the map h_s intersects $T^{\leq\nu_0}$. Let $s \geq s_{i+1}$ be the smallest number with this property. By continuity, the curve α_{s_i} is the core curve of a Margulis tube for $\sigma(h_s)$.

By the choice of ν_0, by Lemma 13.11 and the definition of the set V_i, there exists a diskbounding simple closed curve c_i on Σ which is disjoint from the core curve of any Margulis tube for $\sigma(h_s)$ and hence which is disjoint from α_{s_i}. Note that we may have $c_i = \alpha_{s_i}$. This curve then belongs to one of the disk sets $\mathcal{D}_1, \mathcal{D}_2$. Moreover, property (1) or (2) in Lemma 13.11 holds true for $\sigma(h_s)$.

Using this argument inductively, we conclude that either $n = 1$ and $\mathrm{diam}_Y(\mathcal{D}_1 \cup \mathcal{D}_2) \le 2D_1 + 2$ by the beginning of the proof, or for each $i \ge 1$, the curve α_{s_i} and hence the bridge arc τ_{s_i} for Y is disjoint from a diskbounding simple closed curve c_i on Σ.

Since $\alpha_{s_i}, \alpha_{s_{i+1}}$ are disjoint for all i, we have $d_{\mathcal{CG}}(c_i, c_{i+1}) \le 3$. But $d_{\mathcal{CG}}(\mathcal{D}_1, \mathcal{D}_2) \ge 4$ by assumption, and therefore if $c_i \in \mathcal{D}_j$ for $j = 1$ or $j = 2$ then the same holds true for c_{i+1}. By induction on i, we deduce that up to renaming, we have $c_i \in \mathcal{D}_1$ for all i.

As a consequence, by the choice of p, we have $\mathrm{diam}_Y(\cup_i c_i) \le p$. Lemma 13.4 then shows that $\mathrm{diam}_Y(\gamma_1 \cup \gamma_2) \le p + 2(D_1 + 2)$. Since $\gamma_i \in \mathcal{D}_i$ for $i = 1, 2$ were arbitrarily chosen and $\mathrm{diam}_Y(\mathcal{D}_j) \le p$, this yields that $\mathrm{diam}_Y(\mathcal{D}_1 \cup \mathcal{D}_2) \le p + 2(D_1 + 2)$ which is what we wanted to show.

The argument in the case that Y is an annulus is identical to the above discussion, with the only difference that in each step, the bridge arc τ for Y is replaced by the simple closed curve α crossing through ∂Y. Additional details will be left to the reader. ☐

From now on we always assume that the Hempel distance of the manifold M_f is at least 4, and we let $p > 0$ be the number from Lemma 13.2. We use Lemma 13.12 to control the diameters of the subsurface projections of $\mathcal{D}_1 \cup \mathcal{D}_2$ into subsurfaces Y whose boundaries have large diameter in M_f.

Lemma 13.13. *There exist numbers $R_2 = R_2(\Sigma) > 0, k_2 = k_2(\Sigma) > 0$ with the following property. Let $Y \subset \Sigma$ be a strongly incompressible subsurface with $d_{\mathcal{CG}}(\partial Y, \mathcal{D}_1 \cup \mathcal{D}_2) \ge p$. If ∂Y contains a component β whose diameter in M_f is at least R_2, then*

$$\mathrm{diam}_Y(\mathcal{D}_1 \cup \mathcal{D}_2) \le k_2.$$

Proof. Let $\nu_1 < \kappa_0$ be as in Lemma 13.12. Choose $R_2 > 0$ sufficiently large that the following holds true. Consider a hyperbolic metric σ on Σ, and let $x, y \in \Sigma$ be two points of distance at least R_2; then any path in Σ connecting x to y crosses through a Margulis tube whose core curve has length smaller than ν_1.

Assume that ∂Y has a component β whose diameter in M_f is at least R_2. By Lemma 13.12, it suffices to consider the case that on β, the injectivity radius of M_f is bounded from below by $\nu_1/2$.

By the choice of R_2, for every $g \in \mathcal{L}(\partial Y)$ and corresponding hyperbolic metric $\sigma(g)$, there exists a simple closed curve α on Σ of $\sigma(g)$-length smaller than ν_1 which is crossed through by β. Since the injectivity radius of M_f on β is at least $\nu_1/2$, the curve $g(\alpha)$ bounds a disk in M_f. In other words, α is contained in one of the disk sets for M_f, say the disk set $\mathcal{D}_1$.

As in the proof of Lemma 13.12, we find that if Y is not an annulus, then a subarc of α is a bridge arc for Y. In other words, there exists a bridge arc for Y of $\sigma(g)$-length smaller than $\nu_1 < \kappa_0$ which is a subarc of the diskbounding simple closed curve α. If Y is an annulus then we choose α as a simple closed curve of length smaller than κ_0 which crosses through ∂Y.

We argue now as in the proof of Lemma 13.12. Let $\gamma_i \in \mathcal{D}_i$ $(i = 1, 2)$ and let $g, h \in \mathcal{P}(\partial Y)$ be pleated surfaces for γ_1, γ_2 as in Lemma 13.4 or Lemma 13.5. Connect g, h by a path h_s $(0 \le s \le 1)$ in $\mathcal{L}(\partial Y)$. For each s choose a diskbounding simple closed curve α_s for h_s of $\sigma(h_s)$-length smaller than ν_1 which is crossed through by β. The curve α_s

contains a bridge arc for Y of $\sigma(h_s)$-length smaller than κ_0. By continuity, there exists an open neighborhood V_s of s in $[0,1]$ so that for every $t \in V_s$, the $\sigma(h_t)$-length of α_s is smaller than κ_0.

Cover the interval $[0,1]$ by finitely many of the sets V_s. This covering can be used to find a partition $0 = s_0 < \cdots < s_n = 1$ such that for all i, we have $[s_i, s_{i+1}] \subset V_{s_i}$. Now for each i, if α_{s_i} is different from $\alpha_{s_{i+1}}$, then the curves $\alpha_{s_i}, \alpha_{s_{i+1}}$ are core curves of Margulis tubes for the same metric $h_{s_{i+1}}$ and hence they are disjoint. This implies that if $\alpha_{s_i} \in \mathcal{D}_j$ ($j \in \{1,2\}$) then the same holds true for $\alpha_{s_{i+1}}$. As a consequence, if $\alpha_0 \in \mathcal{D}_j$ then so is α_1.

By assumption on Y, we have $\mathrm{diam}_Y(\mathcal{D}_j) \le p$ (see the proof of Lemma 13.12). Using once more Lemma 13.4 and Lemma 13.5, this implies as in the proof of Lemma 13.12 that $d_Y(\gamma_1, \gamma_2) \le p + 2(D_1 + 1)$ if Y is not an annulus, and $d_Y(\gamma_1, \gamma_2) \le p + 2(D_2 + 1)$ otherwise. This is what we wanted to show. ☐

The proof of Lemma 13.13 uses the assumption that the diameter of a component β of ∂Y in M_f is large to conclude that for any $g \in \mathcal{L}(\partial Y)$, a component of ∂Y crosses through a Margulis tube for $\sigma(g)$ whose core curve is diskbounding. It is not used elsewhere in the proof. Thus the statement of the lemma can be extended in the following way. For a multicurve $\partial Y \subset \Sigma$, define a *simplicial path* $g_s \subset \mathcal{L}(\partial Y)$ between two pleated surfaces $(\Sigma, g_0), (\Sigma, g_1)$ to be a path which consists of pleated surfaces connected by a diagonal exchange path as in Lemma 13.7. We assume for convenience that such a path is parameterized on the interval $[0,1]$, but there are no other requirements for the parameterization. Let as before $\kappa_0 > 0$ be a constant with properties (P1),(P2). We say that such a simplicial path $g_s \subset \mathcal{L}(\partial Y)$ is *thick-thin incompatible* if for every s there exists a simple closed curve in $(\Sigma, \sigma(g_s))$ of length smaller than $\kappa_0/10$ whose image under g_s is contractible in M. The constants $R_2 > 0, p > 0$ in the following lemma are as in Lemma 13.13.

Lemma 13.14. *Let $Y \subset \Sigma$ be an essential subsurface with $d_{\mathcal{CG}}(\partial Y, \mathcal{D}_1 \cup \mathcal{D}_2) \ge p$ and let $s \in [0,1] \to g_s \in \mathcal{L}(\partial Y)$ be a thick-thin incompatible path. Then for $\ell = 1$ or $\ell = 2$ and each s, there exists a bridge arc τ_s for Y of $\sigma(g_s)$-length at most κ_0 which is the subsurface projection of a diskbounding curve in $\mathcal{D}_\ell$, or there is a diskbounding curve in $\mathcal{D}_\ell$ of $\sigma(g_s)$-length at most κ_0 crossing through ∂Y if Y is an annulus. In particular, the distance in the arc and curve graph of Y between τ_0, τ_1 is at most p.*

Proof. By uniform quasi-convexity of the disk set $\mathcal{D}_\ell$ in $\mathcal{CG}(\Sigma)$ ($\ell = 1,2$) and the lower bound p on the distance $d_{\mathcal{CG}}(\partial Y, \mathcal{D}_1 \cup \mathcal{D}_2)$, the diameter of the subsurface projection of $\mathcal{D}_i$ into Y is bounded from above by p (see the proof of Lemma 13.13).

Now if Y is not an annulus, if $g \in \mathcal{L}(\partial Y)$ and if there exists a simple closed curve c in Σ of $\sigma(g)$-length at most κ_0 whose image under the map g is contractible in M, then ∂Y crosses through c and hence there exists a bridge arc for Y of $\sigma(g)$-length at most κ_0 which is a subarc of a curve in $\mathcal{D}_\ell$ for $\ell = 1$ or $\ell = 2$. The path g_s then coarsely determines in this way a sequence of elements of $\mathcal{D}_1 \cup \mathcal{D}_2$ containing such short bridge arcs where we may assume that two consecutive of these elements are disjoint. Thus this path is entirely contained in $\mathcal{D}_\ell$ for $\ell = 1$ or $\ell = 2$.

As a consequence, there are bridge arcs for $\sigma(g_0), \sigma(g_1)$ of length at most κ_0 which are projections of curves in $\mathcal{D}_\ell$. Then their distance in the arc and curve graph of Y is at most p. $\square$

Using what we established so far, we are left with analyzing subsurfaces Y whose boundary ∂Y have a component β with $\ell_f(\beta) \geq R_2$ and diameter in M which is smaller than R_2 together with maps $g \in \mathcal{L}(\partial Y)$ whose thin part is mapped to the thin part of M. Compressibility of the Heegaard surface $\Sigma \subset M$ causes considerable difficulties, and the remainder of this section is devoted to overcoming this problem. The strategy was laid out by Thurston [Thu86a] and is based on an argument by contradiction.

To set up the proof, consider a sequence (M_n, x_n) $(n \geq 1)$ of pointed hyperbolic 3-manifolds so that the injectivity radius $\mathrm{inj}(x_n)$ of M_n at x_n is bounded from below by a positive constant not depending on n. We say that the sequence *converges* to a pointed hyperbolic 3-manifold (M, x) *in the pointed geometric topology* if for every $R > 0, \xi > 0$ there is a number $n(R, \xi) > 0$, and for every $n \geq n(R, \xi)$ there exists a smooth embedding, the *approximation map* $k_n : U_n \subset M \to M_n$, such that k_n is defined on the ball $B_M(x, R) \subset U_n$ of radius R centered at $x \in M$, it sends $k_n(x) = x_n$, and the restriction of k_n to $B_M(x, R)$ satisfies $\|\rho_M - k_n^* \rho_{M_n}\|_{C^2(B_M(x,R))} \leq \xi$ where ρ_M, ρ_{M_n} are the metric tensors on M, M_n. We then say that the restriction of k_n to $B(x, R)$ is *ξ-almost isometric.*

The following is well known (see e.g. Chapter E of [BP92] for details).

Proposition 13.15. *If (M_n, x_n) is a sequence of pointed hyperbolic 3-manifolds such that $\mathrm{inj}(x_n) \geq \chi > 0$ for all n, then up to passing to a subsequence, the sequence (M_n, x_n) converges in the pointed geometric topology to a pointed hyperbolic 3-manifold (M, x).*

We can also consider convergence of pleated surfaces in the pointed geometric topology. The following lemma establishes a first control on such pleated surfaces.

Lemma 13.16. *Let M_n be a sequence of closed hyperbolic 3-manifolds with Heegaard surface Σ of Hempel distance at least 4. Assume that there exists a number $\chi > 0$ and for each n a pleated surface $g_n : (\Sigma, \sigma(g_n)) \to M_n$ homotopic to the inclusion with the following properties.*

(1) There exists a point $x_n \in \Sigma$ with $\mathrm{inj}(x_n) \geq \chi$ and $\mathrm{inj}(g_n(x_n)) \geq \chi$.
(2) The thin part of $(\Sigma, \sigma(g_n))$ consist of annuli whose core curves are of distance at least 3 to $\mathcal{D}_1 \cup \mathcal{D}_2$.

Then up to passing to a subsequence, the pointed manifolds $(M_n, g_n(x_n))$ converge in the pointed geometric topology to a pointed hyperbolic 3-manifold $(M, \hat{x})$, and the pointed pleated surfaces $g_n : (\Sigma, x_n) \to (M_n, g_n(x_n))$ converge to a pleated surface $g : W \to M$ where W is a finite volume hyperbolic surface homeomorphic to an essential subsurface of Σ of negative Euler characteristic, and g maps cusps of W to cusps of M. Furthermore, if $W \neq \Sigma$ then g is π_1-injecctive.

Proof. Fix as before a Margulis constant $\kappa_0 < \chi$ for hyperbolic surfaces. Since there are only finitely many topological types of subsurfaces of Σ, after passing to a subsequence we may assume that there exists a connected subsurface $W_n \subset \Sigma$ containing x_n with geodesic boundary such that the following holds true.

- The homeomorphism type of the surfaces W_n does not depend on n.
- If $\hat{W}_n$ denotes the component of the κ_0-thick part of $(\Sigma, \sigma(g_n))$ containing x_n, then the inclusion $\hat{W}_n \to W_n$ is a homotopy equivalence.
- The $\sigma(g_n)$-length of each boundary component of W_n tends to 0 as $n \to \infty$.

Using the first property above, we may identify each W_n with a fixed subsurface W of Σ. The surface W has negative Euler characteristic and may coincide with Σ.

By assumption, the image under g_n of no boundary component α of $g_n(W)$ is contractible in M_n. Since g_n is 1-Lipschitz, this implies that for each n, the curve $g_n(\alpha)$ is homotopic to a closed geodesic $\hat{\alpha}_n$ in M. As $n \to \infty$, the lengths of the curves $g_n(\alpha)$ and hence of $\hat{\alpha}_n$ tend to zero.

Since the $\sigma(g_n)$-diameter of any component of the κ_0-thick part of W_n is bounded from above by a constant only depending on Σ, it follows from Proposition 13.15 and the Arzela Ascoli theorem that up to passing to another subsequence, we may assume that the pointed manifolds $(M_n, g_n(x_n))$ converge in the pointed geometric topology to a pointed hyperbolic 3-manifold $(M, \hat{x})$, and the pleated surfaces $g_n|W_n : (W_n, x_n) \to M_n$ converge in the pointed geometric topology to a pleated surface $g : (W, x) \to (M, \hat{x})$ where W is obtained from W_n by replacing each boundary component by a puncture. The surface W is equipped with a complete finite volume hyperbolic metric. Furthermore, since the length of $g_n(\alpha)$ tends to 0 as $n \to \infty$, the map g maps cusps of W to cusps of M.

We are left with showing that if $W \neq \Sigma$, then g is π_1-injective. Thus assume that W has at least one cups. We know that g maps cusps in W to cusps in M and hence a closed curve $\alpha \subset W$ whose image under g is contractible in M is essential in W. Assume to the contrary that $\alpha \subset W$ is such a curve.

Denote by $D \subset \mathbb{C}$ the closed unit disk. There exists a continuous map $\psi : D \to M$ with $\psi(\partial D) = g(\alpha)$. By compactness of D and hence of $\psi(D) \subset M$, for large enough n the set $\psi(D) \subset M$ is contained in the domain U_n of the almost isometric map k_n which determine the geometric convergence $(M_n, g_n(x_n)) \to (M, \hat{x})$, and hence $k_n(g(\alpha)) \subset M_n$ is contractible as well.

But α is contained in W, and a simple closed curve ξ going around a cups of W is a geometric limit of boundary curves of W_n which are of distance at least 3 from $\mathcal{D}_1 \cup \mathcal{D}_1$. Convergence of the maps $g_n : W_n \subset \Sigma \to M_n$ to the map $g : W \to M$ yields that for large enough n, the image $k_n g(\xi)$ of $g(\xi)$ under k_n is homotopic to the image under the map $g_n|W_n : W_n \to M_n$ of a boundary component of W_n, and $k_n g(\alpha)$ is contractible in $M_n - g_n(\partial W_n)$.

On the other hand, since $d_{\mathcal{CG}}(\partial W_n, \mathcal{D}_1 \cup \mathcal{D}_2) \geq 3$ by assumption, Lemma 13.1 shows that the surface $g_n(W_n)$ is incompressible in $M - g_n(\partial W_n)$. This is a contradiction which shows that indeed, g is π_1-injective and completes the proof of the lemma. □

We use Lemma 13.16 to establish the following relative version of Thurston's uniform injectivity result. Let $\mathbf{P}(M)$ be the projectivized tangent bundle of the hyperbolic 3-manifold M. If $g : (\Sigma, \sigma(g)) \to M$ is a pleated surface, with pleating lamination λ, then there is a map $p : \lambda \to \mathbf{P}(M)$ which associates to $x \in \lambda$ the tangent line of $g(\lambda)$ at $g(x)$. The number $R_2 > 0$ in the statement of the proposition is the number from Lemma 13.13.

Proposition 13.17. *For $b > 0$ there exists a number $R_3 = R_3(b) > R_2$, and for all $\epsilon > 0$ there exists a number $\delta = \delta(b, \epsilon) > 0$ with the following property. Let M be a closed hyperbolic 3-manifold with Heegaard surface Σ and Hempel distance at least 4. Let $X \subset \Sigma$ by a multicurve with $d_{\mathcal{CG}}(X, \mathcal{D}_1 \cup \mathcal{D}_2) \geq p$. Assume that the diameter in M of the geodesic representative of each component of X is at most R_2 and that X has a component β whose length in M at least R_3. Let $g \in \mathcal{P}(X)$ be a pleated surface with all core curves of Margulis tubes incompressible in M and assume that there exists a simple closed curve α on Σ disjoint from X of $\sigma(g)$-length at most b. Then*

$$d_{\sigma(g)}(x, y) \leq \epsilon \text{ for all } x, y \in \beta \text{ with } d_{\mathbf{P}(M)}(p(x), p(y)) \leq \delta$$

where $d_{\sigma(g)}$ denotes the distance function of the hyperbolic metric $\sigma(g)$ on $\Sigma \supset \beta$.

Proof. Following the strategy of [Thu86a], assume to the contrary that the proposition does not hold true. Then there are numbers $b > 0, \epsilon > 0$ for which no $R_3(b) > 0, \delta(b, \epsilon) > 0$ as in the statement of the proposition exists. This means that there exists a sequence of counterexamples, consisting of a sequence of hyperbolic 3-manifolds M_n with Heegaard surface Σ, multicurves $X_n \subset \Sigma$ with $d_{\mathcal{CG}}(X_n, \mathcal{D}_1 \cup \mathcal{D}_2) \geq p$, pleated surfaces $g_n : \Sigma \to M_n$ whose pleating lamination contains $X_n \supset \beta_n$ and the following properties.

- The length of β_n is at least n.
- The diameter in M_n of each component of X_n is at most R_2.
- The core curve of each Margulis tube of $\sigma(g_n)$ is not homotopic to zero.
- There exists a simple closed curve $\alpha_n \subset \Sigma$ disjoint from X_n of $\sigma(g_n)$-length at most b.
- There are points $x_n, y_n \in \beta_n$ with $d_{\sigma(g_n)}(x_n, y_n) \geq \epsilon$ and $d_{\mathbf{P}(M)}(p(x_n), p(y_n)) \leq 1/n$.

Since the diameter in M_n of the curve β_n is at most R_2 and the length of β_n tends to infinity with n, for sufficiently large n the curve β_n is not the core curve of a Margulis tube and β_n is contained in the χ-thick part of M_n for a constant $\chi > 0$ only depending on R_2. Since core curves of Margulis tubes in Σ for $\sigma(g_n)$ are not null-homotopic in M_n, we can apply Lemma 13.16. It yields that by passing to a subsequence, we may assume that the pointed pleated surfaces $g_n : (\Sigma, x_n) \to (M_n, g_n(x_n))$ converge in the geometric topology to a pointed pleated surface $g : (W, x) \to (M, g(x))$ where W is a finite volume hyperbolic surface homeomorphic to the interior of an essential subsurface of Σ and g maps cusps to cusps. The pleating lamination of g contains the geometric limit β of the simple closed curves β_n.

By Lemma 13.16, if $W \neq \Sigma$ then the map $g : W \to M$ is π_1-injective. If $W = \Sigma$ then there exists a simple closed curve α on Σ of $\sigma(g)$-length at most b which is disjoint from a geometric limit β of the curves β_n. Furthermore, the argument in the proof of Lemma 13.16 yields that the map $g : \Sigma - \alpha \to M - g(\alpha)$ is π_1-injective. No modification of the argument is required. Let $Z = W$ if $W \neq \Sigma$ or $Z = \Sigma - \alpha$ otherwise where we identify α with its geodesic representative for the metric $\sigma(g)$.

Assume for the moment that $W = \Sigma$. By the collar lemma for hyperbolic surfaces, the distance in $(Z, \sigma(g))$ between α and the geodesic lamination β is bounded from below by a positive constant $\rho = \rho(b) < \chi/2$ only depending on b. Modify g with a homotopy which equals the identity on the complement of the $\rho/2$-neighborhood of α to a map $\hat{g}$ which maps α to the complement of the $\rho/2$-neighborhood of $g(\beta)$. This can be done in such a way that the map $\hat{g}$ is L-Lipschitz for some $L > 0$ and that it maps W into $M^{>\chi/2}$.

Resume the general case that includes $W \neq \Sigma$. If $W \neq \Sigma$ then put $L = 1$ and $\alpha = g(\alpha) = \varnothing$. Consider the diagonal lamination $\beta \times \beta \subset Z \times Z$. If $x_1, x_2 \in \beta$ are two points which have the same image under p, then the leaves ℓ_1, ℓ_2 of β through x_1, x_2 are mapped to the same geodesics in $\hat{g}(Z) \subset M$. Since the lamination β and hence $\beta \times \beta$ is compact, the leaf $\ell_1 \times \ell_2$ of $\beta \times \beta$ enters a small neighborhood $U \subset Z \times Z$ infinitely often. We may assume that the diameter of U for the product metric on $Z \times Z$ is smaller than $\rho/2L$.

From each return of $\ell_1 \times \ell_2$ to U one can construct closed loops in Z based at x_1, x_2 by connecting the endpoints of the subarcs of ℓ_1, ℓ_2 determined by these return times by an arc of length at most $\rho/2L$. The two resulting closed curves are not homotopic in Z. As the map $\hat{g}$ is L-Lipschitz, their images under $\hat{g}$ are obtained from each other by concatenation with a loop of length smaller than $\rho/2 < \chi/4$, based at a point in $g(\beta)$. But the injectivity radius of M on $g(\beta)$ is a least $\chi/2$. This implies that the images under $\hat{g}$ of these loops are homotopic with a homotopy entirely contained in the $\rho/2$-neighborhood of $g(\beta)$. Now the $\rho/2$-neighborhood of $g(\beta)$ is contained in $M - \hat{g}(\alpha)$. Since the map $\hat{g} : Z \setminus \alpha \to M \setminus \hat{g}(\alpha)$ is π_1-injective, we deduce as on p.232 of [Thu86a] that the leaves ℓ_1, ℓ_2 of β are identical.

That this leads to a contradiction to the assumption $d_{\sigma(g_n)}(x_n, y_n) \geq \epsilon$ for all n follows from the arguments on p.232-233 of [Thu86a] which work directly with compact subsurfaces filled by limit laminations and uses nowhere that the underlying surface is closed or of finite volume. This completes the proof of the proposition. □

Remark 13.18. It follows from the proof of Proposition 13.17 that under the assumption in the proposition, there exists a constant $\rho = \rho(b) > 0$ such that the restriction of a map $g \in \mathcal{L}(X)$ to the ρ-neighborhood of β is incompressible within the ρ-neighborhood of $g(\beta) \subset M$. The point here is that ρ only depends on b. Furthermore, the conclusion of the proposition also holds true for any element $g \in \mathcal{L}(X)$ which is contained in some simplicial path in $\mathcal{L}(X)$. Namely, the argument only used that the maps considered are one-Lipschitz and map the boundary ∂Y of the subsurface Y isometrically.

Recall that if $d_{\mathcal{CG}}(\partial Y, \mathcal{D}_1 \cup \mathcal{D}_2) \geq p$ then a simple closed curve on Σ which is disjoint from ∂Y is not homotopic to zero in M_f and hence it has a geodesic representative in M_f.

Corollary 13.19. *For every $b > 0$ there exists a number $R_4 = R_4(b) > R_3(b)$ with the following property. Let $\partial Y \subset \Sigma$ be a subsurface with $d_{\mathcal{CG}}(\partial Y, \mathcal{D}_1 \cup \mathcal{D}_2) \geq p$ and assume that there exists a component β of ∂Y whose geodesic representative in M is contained in a subset of M of diameter at most R_2 and has length at least R_4. Let $g_s \subset \mathcal{L}(\partial Y)$ be a simplicial path and assume that for each s, the core curve of any Margulis tube for $\sigma(g_s)$ is incompressible in M. Assume furthermore that for each s there exists a simple* *closed curve on* $(\Sigma, \sigma(g_s))$ *disjoint from* ∂Y *of* $\sigma(g_s)$*-length at most* b*. If* Y *is not an annulus then there exists a bridge arc* τ *for* Y *whose length is smaller than* κ_0 *for each of the metrics* $\sigma(g_s)$*. If* Y *is an annulus then there exists a simple closed curve crossing through* Y *which intersects a minimal curve crossing through* Y *for each of the metrics* $\sigma(g_s)$ *in at most 2 points.*

Proof. As in the proof of Proposition 13.17, there exists a universal constant $\chi < \kappa_0$ only depending on R_2 with the following property. Let $g \in \mathcal{L}(\partial Y)$ be such that the core curves of Margulis tubes for $\sigma(g)$ are incompressible in M. Then for any point $x \in \beta$, the injectivity radius of $\sigma(g)$ at x is at least χ.

Let $\delta = \delta(b, \chi/4) > 0$ be as in Proposition 13.17. By standard hyperbolic geometry (see [Min00] for details), there exists a number $R_4 = R_4(b, \chi/4) > 0$ with the property that for any closed geodesic ζ in a hyperbolic 3-manifold M whose diameter in M is at most R_2 and whose length is at least R_4, there are three points $z_1, z_2, z_3 \in \zeta$ with $d_{\mathbf{P}(M)}((pT\zeta)(z_i), (pT\zeta)(z_j)) < \delta$ and such that the distance along ζ between z_i, z_j is larger than 2χ $(i = 1, 2, 3)$. Here $pT\zeta$ denotes the projectivized tangent line of ζ.

Let $g_s \subset \mathcal{L}(\partial Y)$ be a simplicial path as in the statement of the corollary for this number R_4. For each s the restriction of the map $g_s : (\Sigma, \sigma(g_s)) \to M$ to the $\sigma(g_s)$-geodesic $\beta \subset \partial Y$ is an isometry onto a geodesic $\hat{\beta} \subset M$. If the diameter of $\hat{\beta}$ in M is at most R_2 and its length is at least R_4 then by the previous paragraph, there are points $z_1, z_2, z_3 \in \hat{\beta}$ with the following property. Let $x_i \in \beta$ be the preimage of z_i under g_s; then $d_{\mathbf{P}(M)}(px_i, px_j) < \delta$ and the distance along β between x_i, x_j is larger than 2χ. It then follows from Proposition 13.17 and Remark 13.18 that $d_{\sigma(g_s)}(x_i, x_j) < \chi/4$ for all s $(i, j = 1, 2, 3)$.

Since the injectivity radius of $\sigma(g_s)$ at x_i, x_j is at least χ for all s, the points x_i, x_j can be connected in $(\Sigma, \sigma(g_s))$ by a unique minimal geodesic arc α_s of length at most $\chi/4$. Since the metrics $\sigma(g_s)$ depend continuously on s, the arc depends continuously on s and hence its homotopy class with fixed endpoints is independent on s. See also Lemma 13.7 and Remark 13.8 for a similar statement.

Consider first the case that Y is not an annulus. If α_s is contained in Y, then α_s is a bridge arc for Y with the required properties. If α_s has a proper subsegment contained in Y with one endpoint an endpoint of α_s, then the same argument applies to this subsegment. Now the points x_2, x_3 are contained in the $\chi/4$-neighborhood of x_1, and the simple closed geodesic β crosses through these points. Since β does not have self-intersections, locally the subarcs of β through the points x_i decompose a suitably chosen disk neighborhood of x_1 into two strips with boundary in β which are separated by a subarc of β, say the subarc through x_j, and two half-disks. Then the subarc of β through x_j divides a small disk about x_j into two half-disks, at least one of which is contained in Y. Thus for either $\ell = j + 1$ or $\ell = j - 1$ (indices are taken modulo 3), the initial segment of the minimal geodesic connecting x_j to x_ℓ is contained in Y. Its first intersection with ∂Y defines a bridge arc for Y of length smaller than $\chi/2 < \kappa_0/2$, and up to homotopy keeping the endpoints in ∂Y, the length of this arc is smaller than $\kappa_0/2$ for all s. This shows the corollary in the case that Y is not an annulus.

Following once more [Min00], the argument in the case that Y is an annulus is analogous. Namely, the above argument shows that for each of the two sides of the geodesic representative β of the core curve of Y we can find a homotopy class of an arc τ with endpoints in β which leaves β from the chosen side and whose length is smaller than $\kappa_0/2$ for each of the metrics g_s. If the arc τ leaves and returns to different sides of β then its concatenation with a subarc of β determines a simple closed curve crossing through Y with the properties we are looking for. Otherwise there are two such arcs leaving and returning to distinct sides of Y, and the union of these arcs with subarcs of ∂Y define a simple closed curve with the desired properties. This follows once more from the fact that for a given hyperbolic metric, any two bridge arcs for a subsurface Y of length at most κ_0 are disjoint up to homotopy keeping the endpoints on ∂Y. $\square$

We are left with analyzing pleated surfaces (Σ, g) whose pleating locus contains the boundary ∂Y of a subsurface $Y \subset \Sigma$ with the following properties.

(1) $d_{\mathcal{CG}}(\partial Y, \mathcal{D}_1 \cup \mathcal{D}_2) \geq p$.
(2) The diameter of $(\Sigma, \sigma(g))$ is uniformly bounded.
(3) The length in $(\Sigma, \sigma(g))$ of any simple closed curve disjoint from ∂Y is large.

A geodesic lamination β on a surface Σ is said to *fill* Σ if all complementary components of β are simply connected. We have

Lemma 13.20. *Let (Σ, σ_n) be a sequence of hyperbolic surfaces of uniformly bounded diameter. Let $\beta_n \subset (\Sigma, \sigma_n)$ be a simple closed multicurve whose length tends to infinity with n and assume that the same holds true for the length of any simple closed curve disjoint from β_n. Then up to passing to a subsequence and the action of the mapping class group, the triple $(\Sigma, \sigma_n, \beta_n)$ converges in the geometric topology to a triple (Σ, σ, β) where σ is a hyperbolic metric on Σ of uniformly bounded diameter and β is a geodesic lamination which fills Σ.*

Proof. Since by assumption the diameters of the hyperbolic metrics σ_n are bounded from above by a universal constant and since the mapping class group acts cocompactly on the thick part of Teichmüller space, up to the action of the mapping class group we may assume that the hyperbolic metrics σ_n converge in Teichmüller space to a hyperbolic metric σ.

The space of geodesic laminations on (Σ, σ) equipped with the Hausdorff topology on compact subsets of Σ is compact. Thus up to passing to a subsequence, the geodesic laminations β_n converge as $n \to \infty$ in the Hausdorff topology to a geodesic lamination β. We have to show that β fills Σ.

Namely, otherwise there is a simple closed curve $\alpha \subset \Sigma$ disjoint from β. As $\beta_n \to \beta$ in the Hausdorff topology, either the curve α is disjoint from β_n for all sufficiently large n, or α is a component of β. Now the length of α for the metric σ is close to the length of α for σ_n and hence by the assumption that the σ_n-length of any closed curve disjoint from β_n tends to infinity with n, the curve α can not be disjoint from β_n for large n.

We conclude that any simple closed curve α which is disjoint from β is a component of β. Moreover, as β is a limit in the Hausdorff topology of simple closed curves with an essential intersection with α, if $A \subset \Sigma$ is any annular neighborhood of α then β intersects both components of $A - \alpha$. But this just means that β fills Σ and completes the proof of the lemma. $\square$

Lemma 13.21. *For $D > 0, b > D$ there exists a number $R_5 = R_5(D,b) > 0$ with the following property. Let $Y \subset \Sigma$ be an essential subsurface with $d_{\mathcal{CG}}(\partial Y, \mathcal{D}_1 \cup \mathcal{D}_2) \geq p$ and let $(\Sigma, g_s) \subset \mathcal{L}(\partial Y)$ be a simplicial path with the property that the diameter of $\sigma(g_s)$ is at most D for all s. Assume that the $\sigma(g_0)$-length of every simple closed curve on Σ disjoint from ∂Y is at least R_5. Then the following holds true.*

(1) The $\sigma(g_1)$-length of every simple closed curve disjoint from ∂Y is at least b.
(2) If Y is not an annulus then there exists a bridge arc for Y of length at most κ_0 for each of the metrics $\sigma(g_s)$. If Y is an annulus then there exists a simple closed curve crossing through Y which intersects a minimal curve crossing through $\sigma(g_s)$ in at most 2 points for all s.

Proof. Again we argue by contradiction and we assume that there are $D > 0, b > 0$ such that a number $R_5 = R_5(D,b) > 0$ with property (1) in the lemma does not exist. We then obtain a sequence of counter examples, consisting of the following data.

(a) A hyperbolic 3-manifold M_n with Heegaard surface Σ and an essential subsurface $Y_n \subset \Sigma$ with $d_{\mathcal{CG}}(\partial Y_n, \mathcal{D}_1 \cup \mathcal{D}_2) \geq p$ whose boundary has diameter at most D in M_n.
(b) A simplicial path $(\Sigma, g_n^s) \subset \mathcal{L}(\partial Y_n)$ ($s \in [0,1]$) such that the $\sigma(g_n^s)$-diameter of Σ is bounded from above by D for all n, s and that the $\sigma(g_n^0)$-length of every simple closed curve on Σ disjoint from ∂Y_n tends to infinity with n.
(c) A simple closed curve α_n disjoint from ∂Y_n whose $\sigma(g_n^1)$-length is at most b.

Choosing a point $z_n \in M_n$ on the geodesic representative of ∂Y_n, by passing to a subsequence we may assume that the pointed hyperbolic manifolds (M_n, z_n) converge in the pointed geometric topology to a pointed hyperbolic 3-manifold (M, z). Note to this end as before that the injectivity radius of M_n at z_n is bounded from below by a universal constant.

By assumption, the diameters of the hyperbolic metrics $\sigma(g_n^s)$ are uniformly bounded. Thus for each n the images $g_n^s(\Sigma) \subset M_n$ are contained in a compact subset of M_n of diameter bounded from above by a constant independent of n. Namely, the maps g_n^s are one-Lipschitz and their images $g_n^s(\Sigma)$ contain the geodesic representatives of ∂Y_n.

Since for all n the maps g_n^s are homotopic within a fixed compact subset of M_n, and by Lemma 13.7 and Remark 13.8 they all define the same marked homotopy class of maps $\Sigma \to M_n$, up to passing to a subsequence and the action of the mapping class group, the almost isometric maps $k_n : U_n \subset M \to M_n$ whose existence follows from the assumption on geometric convergence $M_n \to M$ define a fixed marked homotopy class of maps $h : \Sigma \to M$. If (Σ, g) is any geometric limit of one of the maps (Σ, g_n^s) as $n \to \infty$, then this limit is contained in this fixed marked homotopy class.

For large enough n, the geodesic multicurve ∂Y_n of uniformly bounded diameter in M_n defines via the almost isometric map k_n a geodesic multicurve $\partial \hat{Y}_n$ in M which is the image of ∂Y_n under the preferred homotopy class of maps $\Sigma \to M$. The length in M of these multicurves tends to infinity as $n \to \infty$. Using the preferred homotopy class of maps $\Sigma \to M$, for large enough n each pleated surface (Σ, g_n^s) in the simplicial path $g_n^s \subset \mathcal{L}(\partial Y_n)$ determines via the map k_n a pleated surface $(\Sigma, \hat{g}_n^s)$ in M. In particular, this holds true for $s = 0,1$. Furthermore, by the explicit construction of simplicial paths in $\mathcal{L}(\partial Y_n)$, we also obtain a corresponding simplicial path $\hat{g}_n^s \subset \mathcal{L}(\partial \hat{Y}_n)$. In particular, the maps $\hat{g}_n^0, \hat{g}_n^1 : \Sigma \to M$ are homotopic relative to the common pleating lamination $\partial \hat{Y}_n$.

Using once more the almost isometric maps k_n, the length of any simple closed curve on $(\Sigma, \sigma(\hat{g}_n^0))$ disjoint from $\partial \hat{Y}_n$ tends to infinity with n. Thus by Lemma 13.20, up to passing to another subsequence we may assume that the multicurves $\partial \hat{Y}_n$ converge as $n \to \infty$ in the geometric topology to a geodesic lamination $\hat{\mu}^0$ on Σ which fills Σ.

On the other hand, by assumption, for each n there exists a simple closed curve on $(\Sigma, \sigma(g_n^1))$ of length at most b which is disjoint from ∂Y_n. Via the almost isometric maps k_n, for large enough n the same holds true (for a perhaps slightly larger constant) for $(\Sigma, \sigma(\hat{g}_n^1))$. By passing once more to a subsequence, this property passes on to a limiting pleated surface $(\Sigma, \hat{g}^1)$. In other words, there exists a simple closed curve c on Σ which is disjoint from a limit $\hat{\mu}^1$ of the geodesic laminations $\partial \hat{Y}_n$ on Σ. Since $\hat{\mu}^1$ is a limit of $\partial \hat{Y}_n$ in the Hausdorff topology, we deduce that c is disjoint from $\partial \hat{Y}_n$ for all large enough n.

But a limiting lamination $\hat{\mu}^0$ for $\sigma(\hat{g}_n^0)$ fills Σ. Since by Remark 13.8 the pleated surfaces $\hat{g}_n^0$ and $\hat{g}_n^1$ induce the same maps $\pi_1(\Sigma) \to \pi_1(M)$ as marked homomorphisms, this implies that $\partial \hat{Y}_n$ has an essential intersection with c for all large enough n. Recall that this is a topological property. This is a contradiction and yields that there exists a number $R_5 = R_5(D, b) > 0$ for which property (1) stated in the proposition holds true.

We are left with showing that up to perhaps enlarging R_5, property (2) is satisfied as well. Again we argue by contradiction and we assume that the statement does not hold true. Then there is a sequence of counter examples with properties (a) and (b) from the beginning of this proof and that moreover no such bridge arcs or simple closed curves exist. By what we established so far, we may assume that as $n \to \infty$ and up to the action of the mapping class group, the hyperbolic metrics $\sigma(g_n^0)$ and $\sigma(g_n^1)$ on Σ converge in Teichmüller space to metrics σ^0, σ^1, and the geodesic laminations $\beta_n \subset (\Sigma, \sigma(g_n^i))$ $(i = 0,1)$ converge to laminations $\hat{\mu}^0, \hat{\mu}^1$ which fill Σ. The pleated surfaces g_n^0, g_n^1 converge in the pointed geometric topology to pleated surfaces $g^0, g^1 : \Sigma \to M$ in the same homotopy class which map μ^0, μ^1 leafwise isometrically to the same geodesic lamination $\mu \subset M$. This lamination is a geometric limit of the geodesic representatives of the preimages $\hat{\beta}_n$ of the geodesics β_n under the almost isometric metric maps k_n which define the geometric convergence.

It follows from the above discussion that the laminations μ^0, μ^1 coincide as marked geodesic lamination on Σ. Denote this lamination by μ for convenience. Let $\hat{g}^0, \hat{g}^1$ be two limiting pleated surfaces which are limits of the sequence g_n^0, g_n^1, respectively. Let $Q \subset \Sigma$ be a complementary component of the filling geodesic lamination μ which is contained in the pleating lamination for $\hat{g}^0, \hat{g}^1$ and let ℓ_1, ℓ_2 be two oriented boundary leaves of Q which are backward asymptotic. Then for a given $\epsilon > 0$, there are points $x \in \ell_1, y \in \ell_2$ of distance at most $\epsilon/2$ for both $\hat{g}^0, \hat{g}^1$. For large enough n such that $\hat{g}^i(\Sigma)$ is contained in the domain of the map k_n, there are points on ∂Y_n close to the images of x, y under k_n of distance at most ϵ for both g_n^0, g_n^1 and such that there exists a bridge arc with endpoints on ∂Y_n which is of length at most ϵ for both g_n^0, g_n^1. If this bridge arc is contained in Y_n then this is a contradiction. This is in particular the case if ∂Y_n is a non-separating simple closed curve.

We are left with showing that the latter property can always be assumed. Namely, if Y_n is not an annulus then $Y_n \subset \Sigma$ is a subsurface of negative Euler characteristic. The pleating lamination λ_n for (Σ, g_n^0) decomposes Y_n into ideal triangles. A limiting ideal triangle is contained in the limit of the subsurfaces Y_n and hence the above construction applied to a complementary polygon which is contained in the limit of the surfaces Y_n yields the desired property. This completes the proof of the lemma in the case that Y is not an annulus.

The argument for the case that Y is an annulus is completely analogous and will be omitted. □

The next proposition combines what we established so far to a subsurface projection bound for subsurfaces with large length geodesic realization. In its formulation, the constants $R_2 > 0, k_2 > 0, p > 0$ are as in Lemma 13.13.

Proposition 13.22. *There exists a number $R_6 = R_6(\Sigma) > R_2$ with the following property. Let $Y \subset \Sigma$ be a proper essential subsurface. Assume that $d_{\mathcal{CG}}(\partial Y, \mathcal{D}_1 \cup \mathcal{D}_2) \geq p$, that the diameter in M_f of each component of ∂Y is at most R_2 and that ∂Y contains a component β of length at least R_6; then*

$$\mathrm{diam}_Y(\mathcal{D}_1 \cup \mathcal{D}_2) \leq k_2.$$

Proof. We only show the proposition for non-annular subsurfaces, the claim for annuli follows from exactly the same argument.

Thus let Y be a proper essential subsurface of Σ with $d_{\mathcal{CG}}(\partial Y, \mathcal{D}_1 \cup \mathcal{D}_2) \geq p$. Assume that the diameter in M_f of each component of ∂Y is at most R_2 where $R_2 > 0$ is as in Lemma 13.13.

Choose a number $b_0 > 0$ which is larger than 10 times the Bers constant for Σ. For this number b_0 let $b_1 = R_5(R_2, b)$ be as in Lemma 13.21. Define $b_2 = R_5(R_2, b_1)$. Note that by Lemma 13.21, if (Σ, g) is a pleated surface in the homotopy class of the inclusion of a Heegaard surface whose pleating lamination contains ∂Y for an essential subsurface Y of Σ, if the diameter of $\sigma(g)$ is at most R_2, the length of ∂Y is at least b_2 and if there exists a simple closed curve disjoint from ∂Y of length at most b_1, then all pleated surfaces in a path consisting of surfaces of diameter at most R_2 contain a simple closed curve disjoint from ∂Y of length at most b_2. Let $R_6 = R_4(b_2)$ be as in Proposition 13.17.

Assume that the length of some component of ∂Y is at least R_6. Choose diskbounding simple closed curves $c_i \in \mathcal{D}_{i+1}$ $(i = 0, 1)$ and use these curves to construct pleated surfaces $g_0, g_1 : \Sigma \to M$ with pleating lamimation defined by spinning c_i about ∂Y. Connect g_0 to g_1 by a simplicial path $g_s \subset \mathcal{L}(\partial Y)$. It follows from the assumption on Y that there exists a partition $0 = t_0 < \cdots < t_n = 1$ of $[0, 1]$ such that the path $g_i = g|[t_{i-1}, t_i]$ has one of the following properties.

(1) If i is even then for each $s \in [t_{i-1}, t_i]$ there exists a Margulis tube for $\sigma(g_s)$ with core curve of length at most $\kappa_0/10$, and this core curve is diskbounding.

(2) If i is odd then either for each $s \in [t_{i-1}, t_i]$ the diameter of $\sigma(g_s)$ is at most R_2 and for all s there exists a simple closed curve disjoint from ∂Y whose g_s-length is at most b_2, or for all s the shortest $\sigma(g_s)$-length of a simple closed curve disjoint from Y is at least b_1.

Now by Corollary 13.19 and Lemma 13.21, for each odd i there exists a bridge arc for Y of $\sigma(g_s)$-length smaller than κ_0 for each $s \in [t_{i-1}, t_i]$. Furthermore, for each even i there exists a bridge arc of length at most κ_0 contained in a diskbounding curve, and each of these bridge arcs is a subarc of a curve from $\mathcal{D}_j$ for $j = 1$ or $j = 2$. But as the distance in $\mathcal{CG}(\Sigma)$ between $\mathcal{D}_1$ and $\mathcal{D}_2$ is at least 4 by assumption, for consecutive even i the disk sets which give rise to the short bridge arcs coincide. As a consequence, either there is a short bridge arc persisting along the path, or all the short bridge arcs are uniformly close in the arc and curve graph of Y to a bridge arc contained in a diskbounding curve from a fixed disk set, say the set $\mathcal{D}_1$. This shows the proposition. □

Proof of Theorem 13.3. Let $R = R_6 > R_2$ where R_2, R_6 are as in Lemma 13.13 and Proposition 13.22. For this number R and the given number $\epsilon > 0$ let $k = k(R, \epsilon) > 0$ be as in Lemma 13.10. Assume that the diameter of the subsurface projection of $\mathcal{D}_1 \cup \mathcal{D}_2$ into Y is at least k. By Lemma 13.13 and Proposition 13.22, we know that the length of ∂Y is at most R. But then an application of Lemma 13.10 shows that this length is in fact smaller than ϵ. This is what we wanted to show. □

14. Effective hyperbolization II

The goal of this section is to complete the proof of effective hyperbolization for closed 3-manifolds M_f with large Hempel distance. Theorem 12.1 takes care of the case that a minimal geodesic in the curve graph of the Heegaard surface Σ connecting the two disk sets $\mathcal{D}_1, \mathcal{D}_2$ for M_f has a sufficiently long subsegment with bounded combinatorics (no large subsurface projections). Thus we are left with considering manifolds for which there are proper strongly incompressible subsurfaces $Y \subset \Sigma$ so that the diameter of the subsurface projection of $\mathcal{D}_1 \cup \mathcal{D}_2$ into Y is larger than some a priori fixed constant.

Our strategy is to choose two of these subsurfaces of Σ, say the surfaces Y_1, Y_2, whose boundaries are sufficiently far away in the curve graph of Σ from both $\mathcal{D}_1 \cup \mathcal{D}_2$ and from each other, and to choose a boundary curve α_1 of Y_1. Drilling the curve α_1 from M_f results in a non-compact manifold M_1 with one end C_1 homeomorphic to $T^2 \times (0, \infty)$ where T^2 denotes a 2-torus. Proposition 3.1 of [FSV19] shows that the manifold M_1 is irreducible, atoroidal and Haken and hence it admits a complete finite volume hyperbolic metric by Thurston's geometrization theorem for Haken manifolds [Thu86a], [Thu86b]. The end C_1 of M_1 is a cusp for this hyperbolic metric. The torus $T^2 = \partial C_1$ inherits a flat metric from the hyperbolic metric on M_1.

By the Dehn filling theorem Theorem 11.4, removal of C_1 and gluing a solid torus to the boundary ∂C_1 of $M_1 - C_1$ in such a way that the meridian for the gluing is sufficiently long for the flat metric on ∂C_1 yields a closed hyperbolic manifold N_1. We show that for a suitable choice of the meridian for the gluing, the Heegaard surface Σ for M_f also is a Heegaard surface for N_1, and we can control distances in the curve graph of Σ and sizes of subsurface projections for the disk sets of both M_f and N_1. In particular, we observe that the Heegaard distance of N_1 coincides with the Heegaard distance of M_f, and the diameter of the subsurface projection of the disk sets $\mathcal{D}_1 \cup \mathcal{D}_2$ of M_f into the subsurface Y_2 of Σ essentially coincides with the diameter of the subsurface projections of the disk sets of N_1.

By Theorem 13.3, the length of the boundary ∂Y_2 of Y_2 for the hyperbolic metric on N_1 is bounded from above by a constant only depending on the size of the subsurface projection of $\mathcal{D}_1 \cup \mathcal{D}_2$ into Y_2, but not on the filling meridian defining N_1. Thus if the diameter of this subsurface projection is large, then this length is smaller than any a priori chosen constant.

Do the above construction with the manifold M_f and a boundary curve α_2 of the subsurface Y_2 of Σ. Drilling α_2 from M_f yields an irreducible atoroidal Haken manifold M_2 with one end C_2 which admits a complete finite volume hyperbolic metric. Filling the cusp using a suitably chosen long meridian on the boundary of C_2 results in a hyperbolic manifold N_2. Using again Theorem 13.3, the length of the boundary ∂Y_1 of Y_1 in N_2 is smaller than any a priori chosen constant provided that the diameter of the subsurface projection of $\mathcal{D}_1 \cup \mathcal{D}_2$ into Y_1 is sufficiently large, independent of the choice of the filling of the cusp of M_2.

But this means the following. Let $\hat{M}_f$ be the manifold obtained from M_f by drilling both curves α_1, α_2. This manifold admits a finite volume hyperbolic metric with two rank two cusps $\hat{C}_1, \hat{C}_2$. There exists a uniform lower bound for the lengths of the curves on the boundaries $\partial\hat{C}_1, \partial\hat{C}_2$ of $\hat{C}_1, \hat{C}_2$ corresponding to the meridians for the filling of $\hat{C}_1, \hat{C}_2$ which gives rise to M_f, and this lower bound only depends on the diameters of the subsurface projections of the disk sets of M_f into Y_1, Y_2. respectively. As a consequence, we can use Theorem 11.4 to fill both cusps and construct a hyperbolic metric on M_f.

To implement this strategy we have to assure that suitably chosen Dehn surgeries about a boundary curve of a strongly incompressible subsurface $Y \subset \Sigma$ yield manifolds N with the same Heegaard surface Σ as M_f, and we have to control the disk sets of the surgered manifold as well as their distances in the curve graph of Σ.

To set up this control we use Theorem 3.1 of [MM00]. For a proper essential subsurface $Y \subset \Sigma$, we denote as before by d_Y the distance in the arc and curve graph of Y.

Theorem 14.1 (Masur-Minsky). *There exist constants $m = m(\Sigma) > 0, p = p(\Sigma) < m$ with the following properties. Let $\alpha, \beta \in \mathcal{CG}(\Sigma)$ be two simple closed curves and let $Y \subset \Sigma$ be a proper essential subsurface. If $d_Y(\alpha, \beta) \geq m$, then any geodesic $\zeta : [0, n] \to \mathcal{CG}(\Sigma)$ connecting $\alpha = \zeta(0)$ to $\beta = \zeta(n)$ has to pass through a curve $\zeta(j)$ (for some $j \in [1, n-1]$) which is disjoint from Y. Furthermore, if $j \in [3, n-3]$ and if $a \in [0, j-3]$, $b \in [j+3, n]$ then*

$$|d_Y(\zeta(a), \zeta(b)) - d_Y(\alpha, \beta)| \leq p.$$

Here the last part of Theorem 14.1 follows from the fact that a geodesic in $\mathcal{CG}(\Sigma)$ can contain at most three simple closed curves disjoint from a subsurface Y, and their mutual distance is at most 2. Thus if $j \in [3, n-3]$ and if $a \in [0, j-3]$, $b \in [j+3, n]$ then up to adjusting constants, Theorem 3.1 of [MM00] shows that $d_Y(\zeta(0), \zeta(a)) \leq p/2$, $d_Y(\zeta(b, \zeta(n)) \leq p/2$ and hence the statement follows from the triangle inequality.

Using the constants m, p from Theorem 14.1, let us assume that $Y \subset \Sigma$ is a proper essential subsurface which is strongly incompressible for M_f and such that the diameter of the subsurface projection of $\mathcal{D}_1 \cup \mathcal{D}_2$ into Y is at least $2m$. Assume also that $d_{\mathcal{CG}}(\partial Y, \mathcal{D}_i) \geq 3$. Let $\zeta : [0,n] \to \mathcal{CG}(\Sigma)$ be a minimal geodesic in $\mathcal{CG}(\Sigma)$ connecting $\mathcal{D}_1$ to $\mathcal{D}_2$. Choose *markings* μ_1, μ_2 for Σ whose pants decompositions are composed of curves in $\mathcal{D}_1, \mathcal{D}_2$ and which contain $\zeta(0), \zeta(n)$. Here a marking consists of a pants decomposition of Σ together with a system of spanning curves, one for each component α of the pants decomposition P, which is disjoint from $P - \alpha$ and intersects α in the minimal number of points (one or two, depending on the topological type of the component of $\Sigma - (P - \alpha)$ containing α).

The marking graph is the graph whose vertices are markings and whose edges are given by so-called *elementary moves*, consisting of removal of one of the marking curves and replacing it by another curve while keeping all the remaining curves from the marking (see [MM00] for a detailed discussion). Choose a simplicial path μ_s ($s \in [0,u]$) in the marking graph of Σ so that each point of μ contains a point of ζ. We require moreover that the pants decomposition of the endpoints μ_0 of μ consists of curves in the disk set $\mathcal{D}_1$ and contains $\zeta(0)$, and that the pants decomposition of the endpoint μ_u of μ consists of curves in the disk set $\mathcal{D}_2$ and contains the endpoints of $\zeta(n)$ of ζ. Since ζ passes through a curve disjoint from Y, we may assume that there exists a point in μ which contains the boundary of Y as part of the pants decomposition. View the 3-manifold M_f as being glued from two handlebodies H_1, H_2 of genus g and the manifold $\Sigma \times [1,2]$, where $\Sigma \times \{1\}$ is equipped with the marking μ_0, and $\Sigma \times \{2\}$ is equipped with the marking μ_u. In this way the manifold M_f is completely determined by the pair of markings (μ_0, μ_u) of Σ.

Let T be a solid torus. Its boundary ∂T contains the meridian as a distinguished homotopy class of a simple closed curve c, characterized by being contractible in T. A *longitude* for T is a simple closed curve on ∂T which intersects c in a unique point and is isotopic to the core curve of T, that is, to a generator of the fundamental group of T.

Let α be a boundary curve of the strongly incompressible subsurface Y of Σ. Then α is not contractible in M_f and hence it is the core curve of a solid torus $T \subset M_f$. Choose as a longitude on the boundary of T the curve $\alpha \subset \Sigma$. This construction associates to the torus $T \subset M_f$ with core curve α a preferred meridian-longitude pair (c, α) on ∂T. Theorem 6.2 of [Com96] now states the following.

Theorem 14.2 (Comar)**.** *Let (c, α) be a preferred meridian-longitude pair in the boundary of a tube in M_f with core curve $\alpha \subset Y$. Let $\mathcal{T}_\alpha$ be the left Dehn twist about $\alpha \subset \Sigma$. Then the manifold defined by the pair of markings $(\mu_0, \mathcal{T}_\alpha^q \mu_u)$ is obtained from M_f by $(1,q)$-Dehn surgery along α for all q.*

To control the diameter of subsurface projections of the disk sets of the Dehn surgered manifolds we begin with some preliminary discussion about Dehn twists.

Lemma 14.3. *Let $k > 0$, let $A \subset \Sigma$ be an annulus and let $\mathcal{T}_A$ be the left Dehn twist about the core curve of A. Let $c, d \in \mathcal{CG}(\Sigma)$ be simple closed curves which have an essential intersection with A. Then there exists a number $q \in \mathbb{Z}$ such that the diameter of the subsurface projection of $c, \mathcal{T}_A^q d$ into A is contained in $[k, k+2]$. Up to perhaps replacing q by $q \pm 1$, the number q is unique if we require in addition that its absolute value is minimal with this property.*

The ambiguity in the choice of q in the last statement of the lemma reflects the fact that the subsurface projection into an annulus is only well defined up to the ambiguity of possibly adding a single positive or negative twist.

Proof. The subsurface projection into A of two simple closed curves c, d on Σ is defined as follows. Equip Σ with an auxiliary hyperbolic metric. Consider the covering V of Σ with fundamental group $\pi_1(A)$. This covering is an annulus, equipped with a complete hyperbolic metric. Both ends of V have infinite volume and hence the ideal boundary of V consists of two disjoint circles. Since c, d intersect A essentially, there are components $\tilde{c}, \tilde{d}$ of a lift of c, d to V which are arcs abutting on the two distinct components of the ideal boundary of V. Up to an additive constant of ± 1, the diameter of the subsurface projection of c, d into A then equals the number of essential intersections of $\tilde{c}, \tilde{d}$.

Let $\hat{d}$ be an essential arc in V with the same endpoints as $\tilde{d}$ which is disjoint from $\tilde{c}$ except perhaps at its endpoints. Up to homotopy with fixed endpoints, we have $\hat{d} = \mathcal{T}_A^n \tilde{d}$ for some $n \in \mathbb{Z}$; note that this makes sense since the Dehn twist $\mathcal{T}_A$ lifts to V. Write $q = n - k$ if $n \geq 0$, and write $q = n + k$ if $n < 0$. Then $\mathcal{T}_A^q(\tilde{d})$ has $k \pm 1$ intersections with $\tilde{c}$. Thus the diameter of the subsurface projections into A of the curves $c, \mathcal{T}_A^q d$ is contained in the interval $[k-1, k+1]$. Furthermore, up to perhaps replacing q by $q \pm 1$, the number q is the unique number of minimal absolute value with this property. This shows the lemma. ☐

Let $\zeta : [0, n] \to \mathcal{CG}(\Sigma)$ be a minimal geodesic connecting $\mathcal{D}_1$ to $\mathcal{D}_2$ where as before, $\mathcal{D}_1 \cup \mathcal{D}_2$ are the disk sets of M_f. Let $Y \subset \Sigma$ be a proper essential strongly incompressible subsurface such that $d_Y(\mathcal{D}_1, \mathcal{D}_2) \geq m + 2p$ where $m, p > 0$ are as in Theorem 14.1. Let $\alpha \subset \partial Y$ be a boundary curve and let $\mathcal{T}_\alpha$ be the left Dehn twist about α. By the choice of m, there exists $j > 0$ be such that $\zeta(j)$ is disjoint from Y. Let $\ell > 2m$ and let $q \in \mathbb{Z}$ be as in Lemma 14.3 of minimal absolute value such that the diameter of the subsurface projection of $\zeta(0), \mathcal{T}_\alpha^q \zeta(n)$ into the annulus $A \subset \Sigma$ with core curve α is contained in the interval $[\ell, \ell + 2]$.

Since the Dehn twist $\mathcal{T}_\alpha^q$ fixes the curve $\zeta(j)$, we can define a modified path $\zeta_\alpha : [0, n] \to \mathcal{CG}(\Sigma)$ by

$$\zeta_\alpha(u) = \begin{cases} \zeta(u) & \text{for } u \leq j \\ \mathcal{T}_\alpha^q(\zeta(u)) & \text{for } u \geq j \end{cases}.$$

Note that by Theorem 14.2, the curve ζ_α connects the disk sets $\hat{\mathcal{D}}_1, \hat{\mathcal{D}}_2$ of the manifold $\hat{M}$ obtained from M by $(1, q)$-Dehn surgery along α. We have

Lemma 14.4. *i) The path ζ_α is a geodesic in $\mathcal{CG}(\Sigma)$.*

ii) If $Y \subset \Sigma$ is not an annulus then $d_Y(\zeta_\alpha(0), \zeta_\alpha(n)) = d_Y(\zeta(0), \zeta(n))$.

iii) Let $Z \subset \Sigma$ be a proper incompressible subsurface such that $d_{\mathcal{CG}}(\partial Y, \partial Z) \geq 5$. Assume that $d_Z(\zeta(0), \zeta(n)) \geq m + 2p$ and that $\ell \in (0, n)$ is such that $\zeta(\ell)$ is disjoint from Z. Then $|\ell - j| \geq 3$, and

$$d_Z(\zeta(0), \mathcal{T}_\alpha^q \zeta(n)) \geq d_Z(\zeta(0), \zeta(n)) - 2p \text{ if } \ell < j,$$
$$d_{\mathcal{T}_\alpha^q Z}(\zeta(0), \mathcal{T}_\alpha^q \zeta(n)) \geq d_Z(\zeta(0), \zeta(n)) - 2p \text{ if } \ell > j.$$

Proof. The path ζ_α has the same length as ζ. We claim that it is a geodesic in $\mathcal{CG}(\Sigma)$.

To show the claim let $\beta : [0,b] \to \mathcal{CG}(\Sigma)$ be a geodesic connecting $\beta(0) = \zeta_\alpha(0)$ to $\beta(b) = \zeta_\alpha(n)$. Its length b is at most the length n of the path ζ_α. Now note that if Y is not an annulus, then $d_Y(\zeta(0), \mathcal{T}_\alpha^q \zeta(n))$ coincides with $d_Y(\zeta(0), \zeta(n))$, and if Y is an annulus then $d_Y(\zeta(0), \mathcal{T}_\alpha^q \zeta(n)) \geq 2m$ by construction. Thus any geodesic in $\mathcal{CG}(\Sigma)$ connecting $\zeta(0) = \zeta_\alpha(0)$ to $\zeta_\alpha(n) = \mathcal{T}_\alpha^q(\zeta(n))$ has to pass through a curve disjoint from Y.

Let $a \in [0,b]$ be such that $\beta(a)$ is disjoint from Y. Then $\beta(a)$ is left fixed by $\mathcal{T}_\alpha^{-q}$ and therefore we can define an edge path $\hat{\beta}$ of length b connecting $\zeta(0)$ to $\zeta(n)$ by

$$\hat{\beta}(u) = \begin{cases} \beta(u) & \text{for } u \leq a \\ \mathcal{T}_\alpha^{-q}\beta(u) & \text{for } u \geq a \end{cases}.$$

As ζ is a geodesic, the length b of $\hat{\beta}$ is not smaller than the length n of ζ. Thus we have $b = n$ and consequently ζ_α is a geodesic as claimed. This shows the first part of the lemma.

The second part of the lemma follows from the fact that the projection of a simple closed curve c with an essential intersection with a proper essential non-annular subsurface Y of Σ equals the union of the intersection arcs $c \cap Y$. Thus if c is replaced by $T_\alpha^q(c)$ for a boundary component α of Y, then these subsurface projections coincide.

To show the third part of the lemma, assume without loss of generality that $\ell < j$, the case $\ell > j$ follows from an application of $\mathcal{T}_\alpha^q$. Since the distance in $\mathcal{CG}(\Sigma)$ between ∂Z and ∂Y is at least 5, and a curve disjoint from $\partial Z, \partial Y$ has distance at most 1 to $\partial Z, \partial Y$, we have $|j - \ell| \geq 3$ and hence Theorem 14.1 shows that

$$d_Z(\zeta(0), \zeta(j)) \geq d_Z(\zeta(0), \zeta(n)) - p \geq m + p.$$

Now the restriction of the geodesic ζ_α to $[0,j]$ coincides with the restriction of the geodesic ζ, and hence the same estimate holds true for ζ_α as well. As ζ_α is a geodesic, and $\zeta_\alpha[0,j]$ passes through a curve disjoint from Z, the subsegment $\zeta_\alpha[j,n]$ does not pass through a curve disjoint from Z. Theorem 14.1 then shows that $d_Z(\zeta_\alpha(0), \zeta_\alpha(n)) \geq d_Z(\zeta_\alpha(0), \zeta_\alpha(j)) - p \geq m - p$. But $d_Z(\zeta_\alpha(0), \zeta_\alpha(j)) = d_Z(\zeta(0), \zeta(j))$ and consequently we have $d_Z(\zeta_\alpha(0), \zeta_\alpha(n)) \geq d_Z(\zeta(0), \zeta(n)) - 2p$ as claimed. □

We are now ready to complete the proof of Theorem 5 from the introduction.

Theorem 14.5. *For every $g \geq 2$ there exist numbers $R = R(g) > 0$ and $C = C(g) > 0$ with the following property. Let M_f be a closed 3-manifold with Heegaard surface Σ of genus g, gluing map f and disk sets $\mathcal{D}_1 \cup \mathcal{D}_2$, and assume that $d_{\mathcal{CG}}(\mathcal{D}_1, \mathcal{D}_2) \geq R$. Then M_f admits a hyperbolic metric, and the volume of M_f for this metric is at least $C d_{\mathcal{CG}}(\mathcal{D}_1, \mathcal{D}_2)$.*

Proof. Fix a Margulis constant μ for hyperbolic 3-manifolds and let $\xi \in (0, 1/4)$ be a sufficiently small constant. Let $L = L(\xi, 1/100, 2) > 0$ be as in Theorem 11.4. By Theorem 11.1, there exists a constant $\varepsilon > 0$ such that the following holds true. If M is any hyperbolic 3-manifold and if $N \subset M$ is a hyperbolic solid torus whose core geodesic has length less than ε, then the length of the meridian of the tube on its boundary is at least $2L$. For this number ε let $k = k(\Sigma, \varepsilon) > 0$ be as in Theorem 13.3. For the number k and the above number $\xi > 0$ let $b = b(\Sigma, k, \xi) > 0$ be as in Theorem 12.1. Let $p \geq 3$ be as in Theorem 13.3.

Consider a closed 3-manifold M_f, constructed from a gluing map $f : \partial H_1 \to \partial H_2$. Assume that the Hempel distance of M_f, that is, the distance in $\mathcal{CG}(\Sigma)$ between the disk sets $\mathcal{D}_1, \mathcal{D}_2$, is larger than $2b + 2p + 3$. There are two possibilities.

In the first case, a minimal geodesic ζ in $\mathcal{CG}(\Sigma)$ connecting $\mathcal{D}_1$ to $\mathcal{D}_2$ contains a subsegment of length at least b whose endpoints do not have any subsurface projections of diameter at least k. Then M_f fulfilles the assumptions in Proposition 12.4 and the existence of a hyperbolic metric on M_f is an immediate consequence of Proposition 12.4.

In the second case, no such subsegment exists. Then there exist at least two distinct proper essential incompressible subsurfaces Y_1, Y_2 of Σ whose boundaries have distance at least 5 in $\mathcal{CG}(\Sigma)$, distance at least p from $\mathcal{D}_1 \cup \mathcal{D}_2$ and such that the diameter of the subsurface projection of $\mathcal{D}_1 \cup \mathcal{D}_2$ into Y_1, Y_2 is at least k. Namely, in this case there exists such a subsurface Y_1, and there exist one or two points on the minimal geodesic ζ in $\mathcal{CG}(\Sigma)$ connecting $\mathcal{D}_1$ to $\mathcal{D}_2$ which are disjoint from Y_1, and these points are of distance one in $\mathcal{CG}(\Sigma)$. Such a point $\zeta(m)$ decomposes the geodesic into two subsegments, one of which has length at least $b + p + 1$. We then use this subsegment to find a second subsurface Y_2 of Σ with these properties and whose boundary if of distance at least 5 to the boundary of Y_1 in the curve graph of Σ.

Let α_1, α_2 be a boundary component of Y_1, Y_2. Its distance in $\mathcal{CG}(\Sigma)$ from $\mathcal{D}_1 \cup \mathcal{D}_2$ is at least $p \geq 3$ and hence by Proposition 3.1 of [FSV19], the 3-manifold M_i obtained by drilling α_i is irreducible atoroidal and Haken, with a single end C_i which is homeomorphic to $T \times [0, \infty)$ where $T = \partial C_i$ is a 2-torus. The simple closed curve $\alpha_i \subset \partial Y_i$ determines a distinguished free homotopy class β_i on the boundary ∂C_i of C_i, chosen so that the 3-manifold obtained by removing C_i and gluing a solid torus to the boundary ∂C_i of $M_i - C_i$ with meridian β_i is just the manifold M_f. We call the curve β_i the *meridian* of M_f in the sequel. By Theorem 14.2, the manifold M_i is obtained from M_f by $(1, \infty)$-Dehn surgery along a preferred meridian-longitude pair for the boundary of a tube about α_i in M_f.

By Thurston's hyperbolization theorem for irreducible atoroidal Haken manifolds (see [Thu86a], [Thu86b]), M_i admits a complete finite volume hyperbolic metric for which the end C_i is a rank two cusp. Replace M_i by a Dehn filling $\hat{M}_i$ which is obtained from M_f by $(1, q)$-Dehn surgery along a preferred meridian-longitude pair for the boundary of a tube about α_i in M_f. Theorem 11.4 shows that for sufficiently large q, the manifold $\hat{M}_i$ admits a hyperbolic metric which is close to the metric of M_i away from the cusp C_i.

Since $\hat{M}_i$ is obtained from M_f by $(1, q)$-surgery along along a preferred meridian-longitude pair for M_f, Theorem 14.2 shows that the manifold $\hat{M}_i$ admits a Heegaard decomposition with the same Heegaard surface Σ as M_f. Let $\hat{\mathcal{D}}_1, \hat{\mathcal{D}}_2$ be the disk sets of $\hat{M}_i$ for this Heegaard decomposition. By Lemma 14.4, the Heegaard distance of $\hat{M}_i$ coincides with the Heegaard distance of M_f, and the diameter of the subsurface projection of $\hat{\mathcal{D}}_1 \cup \hat{\mathcal{D}}_2$ into Y_{i+1} equals the diameter of the subsurface projection of $\mathcal{D}_1 \cup \mathcal{D}_2$ up to a uniformly bounded additive error (indices are taken modulo 2).

As a consequence, the proper incompressible subsurface Y_{i+1} fulfills the assumption of Theorem 13.3 for the hyperbolic manifold $\hat{M}_i$ with Heegaard surface Σ. An application of Theorem 13.3 shows that the length of the curve α_{i+1} in $\hat{M}_i$ is less than ε. In particular, if we consider the meridian of the Margulis tube in $\hat{M}_i$ with core curve α_{i+1}, viewed as a curve on the boundary of the Margulis tube defined by α_{i+1} in $\hat{M}_i$, then the length of this meridian is at least $2L$, independent of the filling slope for the Dehn filling of M_i which gives rise to $\hat{M}_i$.

Exchanging the roles of M_1 and M_2 in this argument shows the following. For sufficiently large q, the manifold N obtained from M_f by $(1,q)$ Dehn surgery at both α_1, α_2 is hyperbolic, and the lengths of the simple closed curves on the boundaries of the surgered Margulis tubes which correspond to the meridians in M_f (that is, which are obtained from N by $(1,-q)$-surgery) are at least $2L$. Thus Theorem 11.4 shows that we can modify N by Dehn surgery with slope $(1,-q)$ at both α_1, α_2. The resulting manifold is diffeomorphic to M_f, and it carries a hyperbolic metric whose restriction to the κ_0-thick part of M_f is ξ-close in the C^2-topology to the restriction of the hyperbolic metric of N, where $\kappa_0 > 0$ is the constant with properties (P1),(P2) used in Section 13. This completes the proof that M_f admits a hyperbolic metric.

We are left with controlling the volume of this metric. Using the constant $k = k(\varepsilon) > 0$ from Theorem 13.3, and for this number k the integer $b = b(\Sigma, k, \varepsilon)$ from Theorem 12.1, we find the following. Denote by n the Hempel distance $d_{\mathcal{CG}}(\mathcal{D}_1, \mathcal{D}_2)$ of M_f. Let $\zeta : [0,n] \to \mathcal{CG}(\Sigma)$ be a shortest geodesic in $\mathcal{CG}(\Sigma)$ connecting $\mathcal{D}_1$ to $\mathcal{D}_2$. Let $Y_1, \ldots, Y_s$ be the subsurfaces of Σ with $d_{\mathcal{CG}}(\partial Y_i, \mathcal{D}_1 \cup \mathcal{D}_2) \geq p$ such that the diameter of the subsurface projection of $\mathcal{D}_1 \cup \mathcal{D}_2$ into Y_i is at least k. The geodesic ζ passes through simple closed curves disjoint from Y_i.

Subdivide ζ into segments of length b. Let ℓ_0, ℓ_1, respectively, the smallest and largest integer such that for the segment $[\ell_j b, (\ell_j + 1)b]$, there exists no $u \in [\ell_j b, (\ell_j + 1)b)$ so that $\zeta(u)$ is disjoint from one of the surfaces Y_i $(j = 0, 1)$. There are now two possibilities. In the first case, we have $\ell_1 - \ell_0 \geq n/2b$. By Proposition 12.4 and its proof, we conclude that in this case, the volume of M_f is at least $v(d_{\mathcal{CG}}(\zeta(b(\ell_0+1)), \zeta(b(\ell_1-1))) \geq vb(\ell_1 - \ell_0 - 2) \geq vn/2 - 2vb$ which gives the required bound up to adjusting constants.

On the other hand, if $\ell_1 - \ell_0 \leq n/2b$ then each of the segments $\zeta|_{[b\ell, b(\ell+1))}$ for $\ell < \ell_0$ or $\ell > \ell_1$ contains at least one curve which is disjoint from a subsurface with large subsurface projection. There are at least $\ell_0 + \lfloor \frac{n}{b} \rfloor - \ell_1 \geq \lfloor n/2b \rfloor$ such segments. For each of these segments $[kb, (k+1)b)$, there exists at least one subsurface Y_k such that $d_{Y_k}(\zeta(0), \zeta(n))$ is large, and such that $\zeta(u)$ is disjoint from Y_k for some $u \in [kb, (k+1)b)$. Now if $\zeta(u), \zeta(s)$ are both disjoint from Y_k, then $|u - s| \leq 2$ and hence if $Y_{k_1} = Y_{k_2}$ then $|k_1 - k_2| \leq 2$. As a consequence, the number s of such distinct subsurfaces is at least $\lfloor n/2b \rfloor / 2$.

By Theorem 13.3, for each $i \leq s$ the total length of the geodesic representatives of the boundary curves ∂Y_i of the surface Y_i is not bigger than ε and therefore a boundary component of Y_i is the core curve of a Margulis tube in M_f. These Margulis tubes are pairwise disjoint, and their volumes are bounded from below by a fixed number $w > 0$ as this is already true for the one-neighborhoods of their boundary tori. In other words, each of the tubes contributes at least the fixed amount w to the volume of M_f, independent of any choices. Adding up shows that the volume of M_f and is at least Cn where $C > 0$ is a constant only depending on b and hence only depending on Σ. □

Remark 14.6. Our construction for manifolds with Heegaard splitting Σ and large subsurface projection of the disk sets $\mathcal{D}_1 \cup \mathcal{D}_2$ into a proper essential non-annular subsurface Y of Σ gives less information than the article [FSV19]. Namely, in contrast to these results, we do not obtain any information on the shape of boundary tori of Margulis tubes arising from such large subsurface projections which are reminiscent of the model manifold theorem for quasi-fuchsian groups in [Min10].

References

[AR04] I. R. Aitchison and J. H. Rubinstein. Localising Dehn's lemma and the loop theorem in 3-manifolds. *Mathematical Proceedings of the Cambridge Philosophical Society*, 137:281 – 292, 2004.

[And90] M. T. Anderson. Convergence and rigidity of manifolds under Ricci curvature bounds. *Inventiones mathematicae*, 102(2):429–446, 1990.

[And06] M. T. Anderson. Dehn Filling and Einstein Metrics in Higher Dimensions. *Journal of Differential Geometry*, 73(2):219 – 261, 2006.

[AFLMR07] D. Azagra, J. Ferrera, F. López-Mesas, and Y. Rangel. Smooth approximation of Lipschitz functions on Riemannian manifolds. *Journal of Mathematical Analysis and Applications*, 326(2):1370–1378, 2007.

[BGS85] W. Ballmann, M. Gromov, and V. Schroeder. *Manifolds of Nonpositive Curvature*. Progress in Mathematics. Birkhäuser Boston, 1985.

[Bam12] R. H. Bamler. Construction of Einstein metrics by generalized Dehn filling. *Journal of the European Mathematical Society*, 14:887–909, 2012.

[BP92] R. Benedetti and C. Petronio. *Lectures on Hyperbolic Geometry*. Universitext (Berlin. Print). Springer Berlin Heidelberg, 1992.

[Ber66] M. Berger. Sur les variétés d'Einstein compactes. In *Comptes Rendus de la IIIe Réunion du Groupement des Mathématiciens d'Expression Latine (Namur, 1965)*, pages 35–55. Librairie Universitaire, Louvain, 1966.

[Bes08] A. L. Besse. *Einstein Manifolds*. Classics in Mathematics. Springer, 2008. Reprint of the 1987 edition.

[Biq00] O. Biquard. *Métriques d'Einstein asymptotiquement symétriques*. Number 265 in Astérisque. Société mathématique de France, 2000.

[Bro03] J. F. Brock. The Weil-Petersson Metric and Volumes of 3-Dimensional Hyperbolic Convex Cores. *Journal of the American Mathematical Society*, 16(3):495–535, 2003.

[BB02] J. F. Brock and K. W. Bromberg. On the density of geometrically finite Kleinian groups. *Acta Mathematica*, 192:33–93, 2002.

[BCM12] J. F. Brock, R. D. Canary, and Y. N. Minsky. The classification of Kleinian surface groups, II: The Ending Lamination Conjecture. *Annals of Mathematics*, 176(3):1–149, 2012.

[BMNS16] J. F. Brock, Y. N. Minsky, H. Namazi, and J. Souto. Bounded combinatorics and uniform models for hyperbolic 3-manifolds. *Journal of Topology*, 9(2):451–501, 2016.

[Can93] R. D. Canary. Algebraic Convergence of Schottky Groups. *Transactions of the American Mathematical Society*, 337(1):235–258, 1993.

[CE08] J. Cheeger and D. G. Ebin. *Comparison Theorems in Riemannian Geometry*. AMS Chelsea Publishing, 2008.

[CGT82] J. Cheeger, M. Gromov, and M. Taylor. Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds. *Journal of Differential Geometry*, 17(1):15 – 53, 1982.

[Com96] T. Comar. Hyperbolic Dehn surgery and convergence of Kleinian groups. *University of Michigan*, 1996. Dissertation.

[dC92] M. P. do Carmo. *Riemannian Geometry*. Mathematics (Boston, Mass.). Birkhäuser, 1992.

[dC16] M. P. do Carmo. *Differential Geometry of Curves and Surfaces: Revised and Updated Second Edition*. Dover Books on Mathematics. Dover Publications, 2016.

[Esc87] J. H. Eschenburg. Comparison theorems and hypersurfaces. *manuscripta mathematica*, 59:295–323, 1987.

[Eva10] L.C. Evans. *Partial Differential Equations*. Graduate studies in mathematics. American Mathematical Society, 2010.

[FSV19] P. Feller, A. Sisto, and G. Viaggi. Uniform models and short curves for random 3-manifolds. *arXiv preprint arXiv:1910.09486*, 2019.

[FP20] J. Fine and B. Premoselli. Examples of compact Einstein four-manifolds with negative curvature. *Journal of the American Mathematical Society*, 33(4):991–1038, 2020.

[FPS19a] D. Futer, J. Purcell, and S. Schleimer. Effective bilipschitz bounds on drilling and filling. *arXiv preprint arXiv:1907.13502*, 2019.

[FPS19b] D. Futer, J. Purcell, and S. Schleimer. Effective distance between nested Margulis tubes. *Transactions of the American Mathematical Society*, 372(6):4211–4237, 2019.

[FPS21] D. Futer, J. Purcell, and S. Schleimer. Effective drilling and filling of tame hyperbolic 3-manifolds. *arXiv preprint arXiv:2104.09983*, 2021.

[Gaf54] M. P. Gaffney. A Special Stokes's Theorem for Complete Riemannian Manifolds. *Annals of Mathematics*, 60(1):140–145, 1954.

[GHL04] S. Gallot, D. Hulin, and J. Lafontaine. *Riemannian Geometry*. Universitext. Springer Berlin Heidelberg, 3. edition, 2004.

[GT01] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Classics in Mathematics. Springer-Verlag, 2001. Reprint of the 1998 edition.

[Gro78] M. Gromov. Manifolds of negative curvature. *Journal of Differential Geometry*, 13(2):223 – 230, 1978.

[GKS07] M. Gromov, M. Katz, and S. Semmes. *Metric Structures for Riemannian and Non-Riemannian Spaces*. Modern Birkhäuser Classics. Birkhäuser Boston, 2007.

[Ham10] U. Hamenstädt. Stability of quasi-geodesics in Teichmüller space. *Geometriae Dedicata*, 146:101–116, 2010.

[HV22] U. Hamenstädt and G. Viaggi. Small eigenvalues of random 3-manifolds. *Transactions of the American Mathematical Society*, 375(6):3795–3840, 2022.

[Har82] P. Hartman. *Ordinary Differential Equations*. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics, 2. edition, 1982.

[Hat91] A. Hatcher. On triangulations of surfaces. *Topology and its Applications*, 40(2):189–194, 1991.

[HIH77] E. Heintze and H. C. Im Hof. Geometry of horospheres. *Journal of Differential Geometry*, 12(4):481 – 491, 1977.

[Hem76] J. Hempel. *3-manifolds*. AMS Chelsea Publishing Series. Princeton University Press, 1976.

[Hem01] J. Hempel. 3-Manifolds as viewed from the curve complex. *Topology*, 40(3):631–657, 2001.

[HK08] C. Hodgson and S. Kerckhoff. The shape of hyperbolic Dehn surgery space. *Geometry & Topology*, 12(2):1033–1090, 2008.

[HQS12] X. Hu, J. Qing, and Y. Shi. Regularity and rigidity of asymptotically hyperbolic manifolds. *Advances in Mathematics*, 230(4):2332–2363, 2012.

[JMM10] J. Johnson, Y. Minsky, and Y. Moriah, Heegaard splittings with large subsurface distance, *Algebr. Geom. Topol.*, 10(4):2251–2275, 2010.

[JK82] J. Jost and H. Karcher. Geometrische Methoden zur Gewinnung von A-Priori-Schranken für harmonische Abbildungen. *Manuscripta Mathematica*, 40:27–77, 1982.

[KY09] D. Knopf and A. Young. Asymptotic Stability of the Cross Curvature Flow at a Hyperbolic Metric. *Proceedings of the American Mathematical Society*, 137(2):699–709, 2009.

[Koi78] N. Koiso. Nondeformability of Einstein metrics. *Osaka Journal of Mathematics*, 15(2):419 – 433, 1978.

[MM99] H. A. Masur and Y. N. Minsky. Geometry of the complex of curves I: Hyperbolicity. *Inventiones mathematicae*, 138:103–149, 1999.

[MM00] H. A. Masur and Y. N. Minsky. Geometry of the complex of curves II: Hierarchical structure. *Geometric & Functional Analysis GAFA*, 10:902–974, 2000.

[MM04] H. A. Masur and Y. N. Minsky. Quasiconvexity in the curve complex. In *In the tradition of Ahlfors and Bers, III*, volume 355 of *Contemp. Math.*, pages 309–320. Amer. Math. Soc., Providence, RI, 2004.

[MO90] M. Min-Oo. Almost Einstein manifolds of negative Ricci curvature. *Journal of Differential Geometry*, 32(2):457 – 472, 1990.

[Min00] Y. N. Minsky. Kleinian groups and the complex of curves. *Geometry & Topology*, 4:117–148, 2000.

[Min10] Y. N. Minsky. The classification of Kleinian surface groups, I: models and bounds. *Annals of Mathematics*, 171(1):1–107, 2010.

[Pet16] P. Petersen. *Riemannian Geometry*. Springer, 3. edition, 2016.

[PW97] P. Petersen and G. Wei. Relative Volume Comparison with Integral Curvature Bounds. *Geometric And Functional Analysis*, 7(6):1031–1045, 1997.

[Shc83] S. A. Shcherbakov. The degree of smoothness of horospheres, radial fields, and horosphericalcoordinates on a Hadamard manifold. *Dokl. Akad. Nauk SSSR*, 271(5):1078–1082, 1983.

[Thu86a] W. P. Thurston. Hyperbolic Structures on 3-manifolds, I: Deformation of acylindrical manifolds. *Annals of Mathematics*, 124:203 – 246, 1986.

[Thu86b] W. P. Thurston. Hyperbolic Structures on 3-manifolds, III: Deformation of 3-manifolds with incompressible boundary. *arXiv:math/9801058*, 1986.

[Tia] G. Tian. A Pinching Theorem on Manifolds with Negative Curvature. unpublished manuscript.

[Top06] P. Topping. *Lectures on the Ricci Flow*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2006.

[Via21] G. Viaggi. Volumes of random 3-manifolds. *Journal of Topology*, 14(2):504–537, 2021.

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