DYNAMICS OF THE TEICHMÜLLER FLOW ON COMPACT INVARIANT SETS

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ABSTRACT. Let $\mathcal{Q}(S)$ be the moduli space of area one holomorphic quadratic differentials for an oriented surface S of genus $g \geq 0$ with $m \geq 0$ punctures and $3g - 3 + m \geq 2$. For a compact subset K of $\mathcal{Q}(S)$ let $\delta(K)$ be the asymptotic growth rate of the number of periodic orbits for the Teichmüller flow Φ^t which are contained in K. We relate $\delta(K)$ to the topological entropy of the restriction of Φ^t to K. Moreover, we show that $\sup_K \delta(K) = 6g - 6 + 2m$.

1. INTRODUCTION

An oriented surface S is called of finite type if S is a closed surface of genus $g \geq 0$ from which $m \geq 0$ points, so-called *punctures*, have been deleted. We assume that $3g - 3 + m \geq 2$, i.e. that S is not a sphere with at most four punctures or a torus with at most one puncture. We then call the surface S *nonexceptional*. Since the Euler characteristic of S is negative, the *Teichmüller space* $\mathcal{T}(S)$ of S is the quotient of the space of all complete hyperbolic metrics on S of finite volume under the action of the group of diffeomorphisms of S which are isotopic to the identity.

The fibre bundle $\mathcal{Q}^1(S)$ over $\mathcal{T}(S)$ of all *holomorphic quadratic differentials* of area one can naturally be viewed as the unit cotangent bundle of $\mathcal{T}(S)$ for the *Teichmüller metric*. The *Teichmüller geodesic flow* Φ^t on $\mathcal{Q}^1(S)$ commutes with the action of the *mapping class group* $\mathrm{Mod}(S)$ of all isotopy classes of orientation preserving self-homeomorphisms of S. Thus this flow descends to a flow on the quotient $\mathcal{Q}^1(S)/\mathrm{Mod}(S)$, again denoted by Φ^t. This quotient is a non-compact orbifold.

In his seminal paper [V86], Veech showed that the asymptotic growth rate of the number of periodic orbits of the Teichmüller flow Φ^t on $\mathcal{Q}^1(S)/\mathrm{Mod}(S)$ is at least $h = 6g - 6 + 2m$ (we use here a normalization for the Teichmüller flow which differs from the one used by Veech). Recently Eskin and Mirzakhani [EM08] obtained a sharp counting result: They show that as $r \to \infty$, the number of periodic orbits for Φ^t of period at most r is asymptotic to e^{hr}/hr. An earlier partial result for the Teichmüller flow on the space of abelian differentials is due to Bufetov [Bu09].

In this note we are interested in the dynamics of the restriction of the Teichmüller flow to a compact invariant set. For the formulation of our first result, a continuous

flow Φ^t on a compact metric space (X, d) is called *expansive* if there is a constant $\delta > 0$ with the following property. Let $x \in X$ and let $s : \mathbb{R} \to \mathbb{R}$ be any continuous function with $s(0) = 0$ and $d(\Phi^t(x), \Phi^{s(t)}(x)) < \delta$ for all t. If $y \in X$ is such that $d(\Phi^t(x), \Phi^{s(t)}(y)) < \delta$ for all t then $y = \Phi^\tau(x)$ for some $\tau \in \mathbb{R}$ [HK95]. Note that this definition of expansiveness does not depend on the choice of the metric d defining the topology on X so it makes sense to talk about an expansive flow on a compact metrizable space.

Let $\Gamma < \mathrm{Mod}(S)$ be a torsion free normal subgroup of $\mathrm{Mod}(S)$ of finite index. For example, the subgroup of all elements which act trivially on $H_1(S, \mathbb{Z}/3\mathbb{Z})$ has this property. Define $\hat{\mathcal{Q}}(S) = \mathcal{Q}^1(S)/\Gamma$. We show

Theorem 1. *The restriction of the Teichmüller flow to every compact invariant subset K of $\hat{\mathcal{Q}}(S)$ is expansive.*

Periodic orbits of expansive flows on compact spaces are separated, and their asymptotic growth rate can be related to the *topological entropy* of the flow.

For a compact Φ^t-invariant subset K of $\hat{\mathcal{Q}}(S)$ let $h_{\mathrm{top}}(K)$ be the topological entropy of the restriction of the Teichmüller flow to K. For any subset U of $\hat{\mathcal{Q}}(S)$ (or of $\mathcal{Q}^1(S)/\mathrm{Mod}(S)$) and for a number $r > 0$ define $n_U(r)$ (or $n_U^\cap(r)$) to be the cardinality of the set of all periodic orbits for Φ^t of period at most r which are entirely contained in U (or which intersect U). Clearly $n_U^\cap(r) \geq n_U(r)$ for all r. We show

Theorem 2. *Let $K \subset \hat{\mathcal{Q}}(S)$ be a compact Φ^t-invariant topologically transitive set. Then for every open neighborhood U of K we have*

$$\limsup_{r \to \infty} \frac{1}{r} \log n_K(r) \leq h_{\mathrm{top}}(K) \leq \liminf_{r \to \infty} \frac{1}{r} \log n_U(r).$$

It is not hard to see that Theorem 1 and Theorem 2 are equally valid for compact invariant sets on $\mathcal{Q}^1(S)/\mathrm{Mod}(S)$. However we did not find an argument which avoids using some differential geometric properties for the action of the mapping class group on Teichmüller space which is not in the spirit of this paper, so we omit a proof.

By the variational principle, the topological entropy of a flow Φ^t on a compact space K equals the supremum of the metric entropies of all Φ^t-invariant Borel probability measures on K. Bufetov and Gurevich [BG07] showed that the supremum of the topological entropies of the restriction of the Teichmüller flow to the moduli space of *abelian* differentials is just the metric entropy of the invariant probability measure in the Lebesgue measure class, moreover this Lebesgue measure is the unique measure of maximal entropy.

The following counting result shows that the entropy h of the Φ^t-invariant Lebesgue measure on $\mathcal{Q}^1(S)/\mathrm{Mod}(S)$ equals the supremum of the topological entropies of the restrictions of the Teichmüller flow to compact invariant sets.

Theorem 3.   (1) $\lim_{r \to \infty} \frac{1}{r} \log n_K^\cap(r) \leq h$ *for every compact subset K of* $\mathcal{Q}^1(S)/\mathrm{Mod}(S)$.

(2) *For every $\epsilon > 0$ there is a compact subset $K \subset \mathcal{Q}^1(S)/\text{Mod}(S)$ (or $K \subset \hat{\mathcal{Q}}(S)$) such that*

$$\liminf_{r \to \infty} \frac{1}{r} \log n_K(r) \geq h - \epsilon.$$

The first part of Theorem 3 is immediate from the results of Eskin and Mirzakhani, however the proof given here is very short and easy. The lower bound for the growth of periodic orbits which remain in a fixed compact set is the technically most involved part of this work.

The main tool we use for the proofs of the above resuls is the *curve graph* $\mathcal{C}(S)$ of S and the relation between its geometry and the geometry of Teichmüller space. In Section 2 we introduce the curve graph, and we summarize some results from [H10] in the form used in the later sections. In Section 3 we investigate the Teichmüller flow Φ^t on $\hat{\mathcal{Q}}(S)$ and we show Theorem 1. In Section 4 we use the results from Section 3 to establish a version of the Anosov closing lemma for the restriction of the Teichmüller flow to compact invariant subsets of $\hat{\mathcal{Q}}(S)$ and show Theorem 2. The proof of Theorem 3 is contained in Section 5.

2. The curve graph and its boundary

Let S be an oriented surface of genus $g \geq 0$ with $m \geq 0$ punctures and $3g-3+m \geq 2$. The *curve graph* $\mathcal{C}(S)$ of S is the graph whose vertices are the free homotopy classes of *essential* simple closed curves on S, i.e. simple closed curves which are neither contractible nor freely homotopic into a puncture. Two such curves are joined by an edge if and only if they can be realized disjointly. Since $3g-3+m \geq 2$ by assumption, $\mathcal{C}(S)$ is connected (see [MM99]). However, the curve graph is locally infinite. In the sequel we often do not distinguish between a simple closed curve on S and its free homotopy class. Also, if we write $\alpha \in \mathcal{C}(S)$ then we always mean that α is an essential simple closed curve, i.e. α is a vertex in the curve graph $\mathcal{C}(S)$.

Providing each edge in $\mathcal{C}(S)$ with the standard euclidean metric of diameter 1 equips the curve graph with the structure of a geodesic metric space. Since $\mathcal{C}(S)$ is not locally finite, this metric space $(\mathcal{C}(S), d)$ is not locally compact. Masur and Minsky [MM99] showed that nevertheless its geometry can be understood quite explicitly. Namely, $\mathcal{C}(S)$ is hyperbolic of infinite diameter. The mapping class group $\text{Mod}(S)$ naturally acts on $\mathcal{C}(S)$ as a group of simplicial isometries. A mapping class $\varphi \in \text{Mod}(S)$ is *pseudo-Anosov* if the cyclic subgroup of $\text{Mod}(S)$ generated by φ acts on the curve graph $\mathcal{C}(S)$ with unbounded orbits.

A *geodesic lamination* for a complete hyperbolic structure on S of finite volume is a *compact* subset of S which is foliated into simple geodesics. A geodesic lamination λ on S is called *minimal* if each of its half-leaves is dense in λ. Thus a simple closed geodesic is a minimal geodesic lamination. A minimal geodesic lamination with more than one leaf has uncountably many leaves and is called *minimal arational*. A geodesic lamination λ is said to *fill up* S if every simple closed geodesic on S intersects λ transversely. This is equivalent to stating that the complementary components of λ are all topological discs or once punctured topological discs.

A *measured geodesic lamination* is a geodesic lamination λ together with a translation invariant transverse measure. Such a measure assigns a positive weight to each compact arc in S which intersects λ nontrivially and transversely and whose endpoints are contained in complementary regions of λ. The geodesic lamination λ is called the *support* of the measured geodesic lamination; it consists of a disjoint union of minimal components. Vice versa, every minimal geodesic lamination is the support of a measured geodesic lamination.

The space $\mathcal{ML}$ of measured geodesic laminations on S can be equipped with the weak*-topology. Its projectivization $\mathcal{PML}$ is called the space of *projective measured geodesic laminations* and is homeomorphic to the sphere $S^{6g-7+2m}$. There is a continuous symmetric pairing $\iota : \mathcal{ML} \times \mathcal{ML} \to (0, \infty)$, the so-called *intersection form*, which satisfies $\iota(a\xi, b\eta) = ab\iota(\xi, \eta)$ for all $a, b \geq 0$ and all $\xi, \eta \in \mathcal{ML}$. By the Hubbard Masur theorem (see [Hu06]), for every $x \in \mathcal{T}(S)$ the space $\mathcal{PML}$ of projective measured geodesic laminations can naturally be identified with the projectivized cotangent space of $\mathcal{T}(S)$ at x. Moreover, a quadratic differential $q \in \mathcal{Q}^1(S)$ can be viewed as a pair $(\lambda, \nu) \in \mathcal{ML} \times \mathcal{ML}$ with $i(\lambda, \nu) = 1$ and the additional property that $i(\lambda, \zeta) + i(\nu, \zeta) > 0$ for all $\zeta \in \mathcal{ML}$.

Since $\mathcal{C}(S)$ is a hyperbolic geodesic metric space, it admits a *Gromov boundary* $\partial\mathcal{C}(S)$ which is a (non-compact) metrizable topological space equipped with an action of $\mathrm{Mod}(S)$ by homeomorphisms (see [BH99] for the definition of the Gromov boundary of a hyperbolic geodesic metric space and for references). Following Klarreich [Kl99] (see also [H06]), this boundary can naturally be identified with the space of all (unmeasured) minimal geodesic laminations which fill up S, equipped with the topology which is induced from the weak*-topology on $\mathcal{PML}$ via the measure forgetting map.

Now let $\mathcal{FML} \subset \mathcal{PML}$ be the $\mathrm{Mod}(S)$-invariant Borel subset of all projective measured geodesic laminations whose support is minimal and fills up S. The discussion in the previous paragraph shows that there is a continuous $\mathrm{Mod}(S)$-invariant surjection

$$(1) \hspace{3cm} F : \mathcal{FML} \to \partial\mathcal{C}(S)$$

which associates to a projective measured geodesic lamination in $\mathcal{FML}$ its support.

Since the curve graph is a hyperbolic geodesic metric space, for every $c \in \mathcal{C}(S)$ there is a *visual metric* δ_c of uniformly bounded diameter on the Gromov boundary $\partial\mathcal{C}(S)$ of $\mathcal{C}(S)$ (we refer to Chapter III.H of [BH99] for details of this construction and for references). These distances are related to the intrinsic geometry of $\mathcal{C}(S)$ as follows.

For a point $c \in \mathcal{C}(S)$, the *Gromov product* at c associates to points $x, y \in \mathcal{C}(S)$ the value

$$(x|y)_c = \frac{1}{2}(d(x, c) + d(y, c) - d(x, y)).$$

The Gromov product can be extended to a Gromov product $(\,|\,)_c$ for pairs of distinct points in $\partial\mathcal{C}(S)$ by defining

$$(2) \hspace{3cm} (\xi|\zeta)_c = \sup \liminf_{i,j \to \infty} (x_i|y_j)_c$$

where the supremum is taken over all sequences (x_i) and (y_j) in $\mathcal{C}(S)$ such that $\xi = \lim x_i$ and $\zeta = \lim y_j$. There are numbers $\beta > 0, \nu \in (0,1)$ such that

(3) $\nu e^{-\beta(\xi|\zeta)_c} \leq \delta_c(\xi, \zeta) \leq e^{-\beta(\xi|\zeta)_c}$ for all $\xi, \zeta \in \partial\mathcal{C}(S)$ and

$$\delta_c \leq e^{\beta d(c,a)} \delta_a \text{ for all } c, a \in \mathcal{C}(S).$$

The distances δ_c are equivariant with respect to the action of $\mathrm{Mod}(S)$ on $\mathcal{C}(S)$ and on $\partial\mathcal{C}(S)$. For $c \in \mathcal{C}(S)$, $\xi \in \partial\mathcal{C}(S)$ and $r > 0$ denote by $D_c(\xi, r) \subset \partial\mathcal{C}(S)$ the ball of radius r about ξ with respect to the distance function δ_c.

We will need a more precise quantitative relation between the distance functions δ_c $(c \in \mathcal{C}(S))$. Even though this property is well known, we did not find an explicit reference in the literature and we include a sketch of a proof.

For a formulation, for a number $m > 1$, an *m-quasi-geodesic* in a metric space (X, d) is a map $\gamma : J \to X$ such that

(4) $|s - t|/m - m \leq d(\gamma(s), \gamma(t)) \leq m|s - t| + m$ for all $s, t \in J$

where $J \subset \mathbb{R}$ is a closed connected set. Since $\mathcal{C}(S)$ is hyperbolic, every quasi-geodesic ray $\gamma : [0, \infty) \to \mathcal{C}(S)$ converges as $t \to \infty$ in $\mathcal{C}(S) \cup \partial\mathcal{C}(S)$ to an endpoint $\gamma(\infty) \in \partial\mathcal{C}(S)$.

Lemma 2.1. *For every $m > 1$ there are constants $a(m) > 1, b(m) > 0, \alpha_0(m) > 0$ with the following property. Let $\gamma : [0, \infty) \to \mathcal{C}(S)$ be an m-quasi-geodesic ray with endpoint $\gamma(\infty) \in \partial\mathcal{C}(S)$. Then for all $t \geq 0$ we have*

$$\delta_{\gamma(0)} \leq a(m) e^{-b(m)t} \delta_{\gamma(t)} \text{ on } D_{\gamma(t)}(\gamma(\infty), \alpha_0(m)).$$

Proof. Since $\mathcal{C}(S)$ is a hyperbolic geodesic metric space, there is a constant $p > 1$ depending on the hyperbolicity constant such that every point $c \in \mathcal{C}(S)$ can be connected to every point $\xi \in \partial\mathcal{C}(S)$ by a p-quasi-geodesic (for the particular case of the curve graph see [Kl99, H06, H10]). Similarly, any two points $\xi \neq \zeta \in \partial\mathcal{C}(S)$ can be joined by a p-quasi-geodesic.

Let $m \geq p$. By hyperbolicity, there is a number $r(m) > 0$ and for every m-quasi-geodesic triangle T in $\mathcal{C}(S)$ with vertices $c \in \mathcal{C}(S), \xi \neq \zeta \in \partial\mathcal{C}(S)$ there is a point $u \in \mathcal{C}(S)$ whose distance to each of the sides of T is at most $r(m)$. The (non-unique) point u is called a *center* of T. We claim that there is a number $\chi(m) > 0$ only depending on m and the hyperbolicity constant such that

(5) $\delta_c(\xi, \zeta) \in [\chi(m) e^{-\beta d(c,u)}, e^{-\beta d(c,u)}/\chi(m)].$

Namely, let $\gamma_1, \gamma_2 : [0, \infty) \to \mathcal{C}(S)$ be m-quasi-geodesic rays which connect c to ξ, ζ. There is a universal constant $b > 0$ only depending on the hyperbolicity constant for $\mathcal{C}(S)$ such that

(6) $(\gamma_1(\infty)|\gamma_2(\infty))_c - b \leq \liminf_{s,t\to\infty} (\gamma_1(s)|\gamma_2(t))_c \leq (\gamma_1(\infty)|\gamma_2(\infty))_c$

(Remark 3.17.5 in Chapter III.H of [BH99]).

By hyperbolicity, for sufficiently large s, t we have

$$|d(c, \gamma_1(s)) + d(c, \gamma_2(t)) - d(\gamma_1(s), \gamma_2(t)) - 2d(u, c)| \leq a$$

where $a > 0$ is a constant only depending on the hyperbolicity constant for $\mathcal{C}(S)$ and on m. Together with (2,3,6) this shows the estimate (5).

Now let $\gamma : [0, \infty) \to \mathcal{C}(S)$ be any m-quasi-geodesic ray with endpoint $\gamma(\infty) = \xi \in \partial\mathcal{C}(S)$, let $\zeta \neq \xi \in \partial\mathcal{C}(S)$ and let T be an m-quasi-geodesic triangle with side γ and vertices $\gamma(0), \xi, \zeta$. If $u \in \mathcal{C}(S)$ is a center for T and if $\sigma \geq 0$ is such that $d(u, \gamma(\sigma)) \leq r(m)$, then for every $s \in [0, \sigma]$ the distance between u and a center for any m-quasi-geodesic triangle with vertices $\gamma(s), \xi, \zeta$ is bounded from above by a constant only depending on m and the hyperbolicity constant for $\mathcal{C}(S)$. In particular, by the above discussion and the properties of an m-quasi-geodesic, there are constants $a(m) > 0, b(m) > 0$ such that

$$(7) \qquad \delta_{\gamma(0)}(\xi, \zeta) \leq a(m)e^{-b(m)s}\delta_{\gamma(s)}(\xi, \zeta) \text{ for every } s \in [0, \sigma].$$

From this the lemma follows. $\square$

By a result of Bers, there is a constant $\chi_0 = \chi_0(S) > 0$ such that for every complete hyperbolic metric x on S of finite volume there is a *pants decomposition* for S consisting of simple closed geodesics of x-length at most χ_0. Define a map

$$(8) \qquad\qquad\qquad \Upsilon_{\mathcal{T}} : \mathcal{T}(S) \to \mathcal{C}(S)$$

by associating to a marked hyperbolic metric $x \in \mathcal{T}(S)$ a simple closed curve $\Upsilon_{\mathcal{T}}(x)$ whose x-length $\ell_x(\Upsilon_{\mathcal{T}}(x))$ does not exceed χ_0. There are choices involved in the definition of $\Upsilon_{\mathcal{T}}(x)$, but for any two such choices and any x the distance between the images of x is uniformly bounded.

Denote by $d_{\mathcal{T}}$ the distance on $\mathcal{T}(S)$ defined by the Teichmüller metric. Lemma 2.2 of [H10] shows that there is a number $L > 1$ such that

$$(9) \qquad\qquad d(\Upsilon_{\mathcal{T}}(x), \Upsilon_{\mathcal{T}}(y)) \leq Ld_{\mathcal{T}}(x, y) + L \text{ for all } x, y \in \mathcal{T}(S).$$

Choose a smooth function $\sigma : [0, \infty) \to [0, 1]$ with $\sigma[0, \chi_0] \equiv 1$ and $\sigma[2\chi_0, \infty) \equiv 0$. For every $x \in \mathcal{T}(S)$ we obtain a finite Borel measure μ_x on $\mathcal{C}(S)$ by defining

$$(10) \qquad\qquad\qquad \mu_x = \sum_{\beta} \sigma(\ell_x(\beta))\delta_{\beta}$$

where δ_{β} denotes the Dirac mass at β. The total mass of μ_x is bounded from above and below by a universal positive constant, and the diameter of the support of μ_x in $\mathcal{C}(S)$ is uniformly bounded as well. The measures μ_x are equivariant with respect to the action of the mapping class group on $\mathcal{T}(S)$ and $\mathcal{C}(S)$, and they depend continuously on $x \in \mathcal{T}(S)$ in the weak*-topology. This means that for every bounded function $f : \mathcal{C}(S) \to \mathbb{R}$ the function $h \to \int f d\mu_x$ is continuous.

For $x \in \mathcal{T}(S)$ define a distance δ_x on $\partial\mathcal{C}(S)$ by

$$(11) \qquad\qquad\qquad \delta_x(\xi, \zeta) = \int \delta_c(\xi, \zeta)d\mu_x(c).$$

Clearly the metrics δ_x are equivariant with respect to the action of $\text{Mod}(S)$ on $\mathcal{T}(S)$ and $\partial\mathcal{C}(S)$. Moreover, there is a constant $\kappa > 0$ such that

$$(12) \qquad\qquad \delta_x \leq e^{\kappa d_{\mathcal{T}}(x,y)}\delta_y \text{ and } \kappa^{-1}\delta_x \leq \delta_{\Upsilon_{\mathcal{T}}(x)} \leq \kappa\delta_x$$

for all $x, y \in \mathcal{T}(S)$ (see p.230 and p.231 of [H09]).

The main theorem of [H10] relates the geometry of Teichmüller space to the geometry of the curve graph via the map $\Upsilon_{\mathcal{T}}$. For its formulation, denote for $\epsilon > 0$ by $\mathcal{T}(S)_\epsilon$ the set of all hyperbolic metrics whose *systole* (i.e. the shortest length of a closed geodesic) is at least ϵ. For sufficiently small ϵ the set $\mathcal{T}(S)_\epsilon$ is connected, and the mapping class group acts properly and cocompactly on $\mathcal{T}(S)_\epsilon$.

Theorem 2.2.   (1) *For every $L > 1$ there is a number $\epsilon = \epsilon(L) > 0$ with the following property. Let $J \subset \mathbb{R}$ be a closed connected set of length at least $1/\epsilon$ and let $\gamma : J \to \mathcal{T}(S)$ be an L-quasi-geodesic. If $\Upsilon_{\mathcal{T}} \circ \gamma$ is an L-quasi-geodesic in $\mathcal{C}(S)$ then there is a Teichmüller geodesic $\xi : J' \to \mathcal{T}(S)_\epsilon$ such that the Hausdorff distance between $\gamma(J)$ and $\xi(J')$ is at most $1/\epsilon$.*

(2) *For every $\epsilon > 0$ there is a number $L(\epsilon) > 1$ with the following property. Let $J \subset \mathbb{R}$ be a closed connected set and let $\gamma : J \to \mathcal{T}(S)$ be a $1/\epsilon$-quasi-geodesic. If there is a Teichmüller geodesic arc $\xi : J' \to \mathcal{T}(S)_\epsilon$ such that the Hausdorff distance between $\gamma(J)$ and $\xi(J')$ is at most $1/\epsilon$ then $\Upsilon_{\mathcal{T}} \circ \gamma$ is an $L(\epsilon)$-quasi-geodesic in $\mathcal{C}(S)$.*

Let $\mathcal{Q}^1(S)$ be the bundle of area one quadratic differentials over Teichmüller space $\mathcal{T}(S)$ for S. The mapping class group $\mathrm{Mod}(S)$ acts properly discontinuously on $\mathcal{Q}^1(S)$ as a group of bundle automorphisms. This action commutes with the action of the Teichmüller geodesic flow Φ^t.

An area one quadratic differential $q \in \mathcal{Q}^1(S)$ is determined by a pair (q_v, q_h) of measured geodesic laminations, the *vertical* and the *horizontal* measured geodesic lamination of q, respectively. For every $t \in \mathbb{R}$ the pair $(e^t q_v, e^{-t} q_h)$ corresponds to the quadratic differential $\Phi^t q$. The *strong stable manifold*

$$W^{ss}(q) \subset \mathcal{Q}^1(S)$$

of q is defined as the set of all quadratic differentials of area one whose vertical measured geodesic lamination coincides *precisely* with the vertical measured geodesic lamination of q. The *strong unstable* manifold

$$W^{su}(q) \subset \mathcal{Q}^1(S)$$

is the set of all quadratic differentials of area one whose horizontal measured geodesic lamination coincides precisely with the horizontal measured geodesic lamination of q. Define moreover the *stable manifold* $W^s(q)$ of q and the *unstable manifold* $W^u(q)$ of q by $W^s(q) = \cup_{t \in \mathbb{R}} \Phi^t W^{ss}(q)$ and $W^u(q) = \cup_{t \in \mathbb{R}} \Phi^t W^{su}(q)$. As q varies through $\mathcal{Q}^1(S)$, the manifolds $W^{ss}(q), W^{su}(q), W^s(q), W^u(q)$ define continuous foliations of $\mathcal{Q}^1(S)$ which are called the *strong stable*, the *strong unstable*, the *stable* and the *unstable foliation*. These foliations are invariant under the action of $\mathrm{Mod}(S)$ and under the action of the Teichmüller geodesic flow Φ^t.

For every $q \in \mathcal{Q}^1(S)$ the map which associates to a quadratic differential its vertical measured geodesic lamination restricts to a homeomorphism of the unstable manifold $W^u(q)$ containing q onto an open dense subset of $\mathcal{ML}$. The space $\mathcal{ML}$ admits a natural $\mathrm{Mod}(S)$-invariant measure in the Lebesgue measure class which lifts to a locally finite measure on $W^u(q)$ in the Lebesgue measure class. The induced family of conditional measures $\tilde{\lambda}_u$ on strong unstable manifolds transform under the Teichmüller flow by $d\tilde{\lambda}_{\Phi^t q} \circ \Phi^t = e^{ht} d\tilde{\lambda}_q$ where as before, $h = 6g - 6 + 2m$. The measures $\tilde{\lambda}_q$ are equivariant under the action of the mapping class

group. Moreover, they are conditional measures for a $\mathrm{Mod}(S)$-invariant locally finite measure $\tilde{\lambda}$ on $\mathcal{Q}^1(S)$ in the Lebesgue measure class. The measure $\tilde{\lambda}$ is the lift of a *finite* measure λ on $\mathcal{Q}^1(S)/\mathrm{Mod}(S)$ which is Φ^t-invariant and mixing. The measure λ gives full measure to the set of all quadratic differentials whose horizontal and vertical measured geodesic laminations are uniquely ergodic and fill up S (see [M82, V86] for details). Moreover, by the Poincaré recurrence theorem, λ-almost every $q \in \mathcal{Q}^1(S)/\mathrm{Mod}(S)$ is *recurrent*, i.e. it is contained in the ω-limit set of its own orbit under the Teichmüller flow.

Let

(13) $$\pi : \mathcal{Q}^1(S) \to \mathcal{PML}$$

be the map which associates to a quadratic differential its vertical projective measured geodesic lamination. For every $q \in \mathcal{Q}^1(S)$ the restriction of the projection π to $W^{su}(q)$ is a homeomorphism of $W^{su}(q)$ onto the open subset of $\mathcal{PML}$ of all projective measured geodesic laminations μ which together with $\pi(-q)$ *jointly fill up* S, i.e. are such that for every measured geodesic lamination $\eta \in \mathcal{ML}$ we have $i(\mu, \eta) + i(\pi(-q), \eta) \neq 0$ (note that this makes sense even though the intersection form i is defined on $\mathcal{ML}$ rather than on $\mathcal{PML}$). The measure class of the push-forward under π of the measure $\tilde{\lambda}_q$ on $W^{su}(q)$ does not depend on q and defines a $\mathrm{Mod}(S)$-invariant ergodic measure class on $\mathcal{PML}$.

3. Compact invariant sets are expansive

Let again $\mathcal{Q}^1(S)$ be the bundle of area one quadratic differentials over Teichmüller space $\mathcal{T}(S)$ of an oriented surface of genus $g \geq 0$ with $m \geq 0$ punctures and where $3g - 3 + m \geq 2$. The mapping class group $\mathrm{Mod}(S)$ acts on $\mathcal{Q}^1(S)$, but this action is not free and the quotient space $\mathcal{Q}^1(S)/\mathrm{Mod}(S)$ is a non-compact orbifold rather than a manifold.

To overcome this (mainly technical) difficulty we choose a torsion free normal subgroup Γ of $\mathrm{Mod}(S)$ of finite index. For example, the group of all elements which act trivially on $H_1(S, \mathbb{Z}/3\mathbb{Z})$ has this property. Define $\hat{\mathcal{Q}}(S) = \mathcal{Q}^1(S)/\Gamma$ and let

(14) $$\Pi : \mathcal{Q}^1(S) \to \hat{\mathcal{Q}}(S)$$

be the canonical projection. Since the action of Γ on $\mathcal{Q}^1(S)$ is free, the map Π is a covering. The Teichmüller flow Φ^t acts on $\hat{\mathcal{Q}}(S)$.

The goal of this section is to show

Theorem 3.1. *The restriction of the Teichmüller flow to every compact invariant subset K of $\hat{\mathcal{Q}}(S)$ is expansive.*

We begin with the construction of a convenient metric on $\mathcal{Q}^1(S)$ and $\hat{\mathcal{Q}}(S)$ inducing the usual topology. To this end, call a distance d on a space X a *length metric* if the distance between any two points is the infimum of the lengths of all paths connecting these points. In the sequel denote by

(15) $$P : \mathcal{Q}^1(S) \to \mathcal{T}(S)$$

the canonical projection.

Lemma 3.2. *There is a complete* $\mathrm{Mod}(S)$*-invariant length metric d on* $\mathcal{Q}^1(S)$ *with the following properties.*

(1) *The metric d induces the usual topology.*
(2) *The canonical projection* $P : (\mathcal{Q}^1(S), d) \to \mathcal{T}(S)$ *is distance non-decreasing where* $\mathcal{T}(S)$ *is equipped with the Teichmüller metric.*
(3) *Every orbit of the Teichmüller flow with its natural parametrization is a minimal geodesic for d parametrized by arc length.*

Proof. By the Hubbard Masur theorem (see [Hu06] for a presentation of this celebrated and by now classical result), the restriction of the canonical projection $P : \mathcal{Q}^1(S) \to \mathcal{T}(S)$ to every unstable (or stable) manifold in $\mathcal{Q}^1(S)$ is a homeomorphism onto $\mathcal{T}(S)$. Thus the Teichmüller metric on $\mathcal{T}(S)$ lifts to a length metric on the leaves of the stable and of the unstable foliation.

Call a path $\rho : [0, 1] \to \mathcal{Q}^1(S)$ *admissible* if there is a finite partition $0 = t_0 < \cdots < t_k = 1$ such that the restriction of ρ to each interval $[t_{i-1}, t_i]$ is entirely contained in a stable or in an unstable manifold. For each such admissible path ρ we can define its *length* to be the sum of the lengths with respect to the lifts of the Teichmüller metric of the subsegments of ρ entirely contained in a stable or an unstable manifold. For $q_0, q_1 \in \mathcal{Q}^1(S)$ define $d(q_0, q_1)$ to be the infimum of the lengths of all admissible paths connecting q_0 to q_1. Then d is a (a priori non-finite) distance function on $\mathcal{Q}^1(S)$ which satisfies the second requirement in the lemma. The third property holds true since the projection to $\mathcal{T}(S)$ of an orbit of the Teichmüller flow is a Teichmüller geodesic of the same length and hence realizes the distance between its endpoints. By the second property for d and the definition, the d-length of every path in $\mathcal{Q}^1(S)$ which is entirely contained in a stable or an unstable manifold coincides with the length of its projection to $\mathcal{T}(S)$. As a consequence, the metric d is a length metric.

We are left with showing that d induces the usual topology on $\mathcal{Q}^1(S)$. For this let $q \in \mathcal{Q}^1(S)$ and let $\epsilon > 0$. We have to show that the ϵ-ball about q for the distance d contains a neighborhood of q in $\mathcal{Q}^1(S)$. For this denote for $z \in \mathcal{Q}^1(S)$ and $r > 0$ by $B^i(q, r)$ the open r-ball about q in $W^i(q)$ with respect to the lift of the Teichmüller metric ($i = s, u$). For each $z \in B^s(q, \epsilon/2)$, the open ball $B^u(z, \epsilon/2)$ of radius $\epsilon/2$ about z in $W^u(z)$ is an open neighborhood of z in $W^u(z)$ whose closure is compact and depends continuously on z in the Hausdorff topology for compact subsets of $\mathcal{Q}^1(S)$ by the Hubbard Masur theorem. Then $U = \cup_{z \in B^s(q, \epsilon/2)} B^u(z, \epsilon/2)$ is an open neighborhood of q in $\mathcal{Q}^1(S)$. Moreover by construction, U is contained in the ϵ-ball about q with respect to the distance function d. This completes the proof of the lemma. $\qquad\square$

The distance d on $\mathcal{Q}^1(S)$ constructed in Lemma 3.2 induces a metric on $\hat{\mathcal{Q}}(S)$, again denote by d, via

$$(16) \qquad\qquad d(x, y) = \inf\{d(\tilde{x}, \tilde{y}) \mid \Pi(\tilde{x}) = \Pi(\tilde{y})\}$$

where as before, $\Pi : \mathcal{Q}^1(S) \to \hat{\mathcal{Q}}(S)$ is the canonical projection. In the sequel we always use these distance functions on $\mathcal{Q}^1(S)$ and $\hat{\mathcal{Q}}(S)$ without further mentioning.

With these preparations, we can prove Theorem 3.1.

Proof of Theorem 3.1. Let $K \subset \hat{\mathcal{Q}}(S)$ be a compact Φ^t-invariant subset and let $\tilde{K} \subset \mathcal{Q}^1(S)$ be the preimage of K under the canonical projection $\Pi : \mathcal{Q}^1(S) \to \hat{\mathcal{Q}}(S)$. Let as before $P : \mathcal{Q}^1(S) \to \mathcal{T}(S)$ be the canonical projection. By the second part of Theorem 2.2, there is a constant $p > 0$ only depending on K such that for every $q \in \tilde{K}$ the curve $t \to \Upsilon_{\mathcal{T}}(P\Phi^t q)$ is a p-quasi-geodesic in $\mathcal{C}(S)$, i.e. we have

$$(17) \quad |t - s|/p - p \leq d(\Upsilon_{\mathcal{T}}(P\Phi^t q), \Upsilon_{\mathcal{T}}(P\Phi^s q)) \leq p|t - s| + p \quad \text{for all} \quad s, t \in \mathbb{R}.$$

Since by inequality (9) the map $\Upsilon_{\mathcal{T}}$ is coarsely Lipschitz, this shows that lifts of orbits of $\Phi^t | K$ which are contained in the same unstable manifold diverge linearly in forward direction. We therefore just have to relate distances in the curve graph to distances in $\mathcal{Q}^1(S)$ for the metric d in a quantitative way. The remainder of the argument establishes such a control.

Let $\mathcal{F} : \mathcal{Q}^1(S) \to \mathcal{Q}^1(S)$ be the *flip* $q \to \mathcal{F}(q) = -q$. This flip is equivariant with respect to the action of the mapping class group and hence it descends to a continuous involution of $\hat{\mathcal{Q}}(S)$ which we denote again by $\mathcal{F}$. Recall that the Gromov boundary $\partial\mathcal{C}(S)$ of $\mathcal{C}(S)$ can be identified with the set of all (unmeasured) minimal geodesic laminations on S which fill up S. Let again $\pi : \mathcal{Q}^1(S) \to \mathcal{PML}$ be the canonical projection. If $q \in \tilde{K} \cup \mathcal{F}(\tilde{K})$ then the support $F(\pi q)$ of the vertical measured geodesic lamination of q is *uniquely ergodic*, which means that $F(\pi q)$ admits a unique transverse measure up to scale, and $F(\pi q)$ is minimal and fills up S [M82]. Moreover, the p-quasi-geodesic $t \to \Upsilon_{\mathcal{T}}(P\Phi^t q)$ converges in $\mathcal{C}(S) \cup \partial\mathcal{C}(S)$ to $F(\pi q)$ (the latter statement follows from the explicit identification of $\partial\mathcal{C}(S)$ with the set of minimal geodesic laminations which fill up S established in [Kl99, H06], see also [H10]).

Write $A = \pi(\tilde{K} \cup \mathcal{F}(\tilde{K}))$. Then A is a Γ-invariant Borel subset of $\mathcal{PML}$. By the consideration in the previous paragraph, the restriction to A of the map $F : \mathcal{FML} \to \partial\mathcal{C}(S)$ introduced in (1) of Section 2 is a Γ-equivariant continuous injection

$$F_A : A \to \partial\mathcal{C}(S)$$

which associates to a projective measured geodesic lamination contained in A its support. In other words, we can identify the set A with a subset of $\partial\mathcal{C}(S)$.

Recall from equation (11) the definition of the distances δ_x $(x \in \mathcal{T}(S))$ on the Gromov boundary $\partial\mathcal{C}(S)$ of $\mathcal{C}(S)$. Since the map $F_A : A \to \partial\mathcal{C}(S)$ is injective, for every $q \in \mathcal{Q}^1(S)$ the function $(\xi, \zeta) \in A \times A \to \delta_{Pq}(F_A \xi, F_A \zeta)$ is a distance on A. For simplicity of notation, we denote this distance again by δ_{Pq}. The topology on A defined by this distance is just the subspace topology of A as a subset of $\mathcal{PML}$ (this is the result of [Kl99]).

For $q \in \tilde{K} \cup \mathcal{F}(\tilde{K})$ denote by $D_q(\pi(q), r) \subset A$ the ball of radius r in A about $\pi(q)$ with respect to this distance. By continuity of the projection π and the map F_A and by the relation (12) between the distances δ_x and δ_y for $x, y \in \mathcal{T}(S)$, the ball $D_q(\pi(q), r)$ depends continuously on q in the following sense. For every

$q \in \tilde{K} \cup \mathcal{F}(\tilde{K})$, every $r > 0$ and every $\epsilon \in (0, r)$ there is a neighborhood U of q in $\tilde{K} \cup \mathcal{F}(\tilde{K})$ such that for every $u \in U$ we have

$$(18) \qquad D_u(\pi(u), r - \epsilon) \subset D_q(\pi(q), r) \subset D_u(\pi(u), r + \epsilon).$$

By inequalities (12) and (17) and by Lemma 2.1, there are numbers $\alpha > 0, a > 1, b > 0$ such that for every $q \in \tilde{K}$ and for all $t > 0$ we have

$$(19) \qquad \delta_{P\Phi^{-t}q} \le ae^{-bt}\delta_{Pq} \text{ on } D_q(\pi(q), 2\alpha) \text{ and}$$

$$\delta_{P\Phi^{t}q} \le ae^{-bt}\delta_{Pq} \text{ on } D_{\mathcal{F}(q)}(\pi(\mathcal{F}q), 2\alpha).$$

For $q \in \mathcal{Q}^1(S)$ and $\beta > 0$ denote by $B(q, \beta)$ the ball of radius β about q in $\mathcal{Q}^1(S)$ with respect to the length metric d defined in Lemma 3.2. Since the projection π is continuous, for every $q \in \tilde{K} \cup \mathcal{F}(\tilde{K})$ there is a number $\epsilon(q) > 0$ such that $\pi(B(q, \epsilon(q)) \cap (\tilde{K} \cup \mathcal{F}\tilde{K})) \subset D_q(\pi(q), \alpha)$ where $\alpha > 0$ is as in the inequalities (19). By the continuity properties (18) of the balls $D_u(\pi(u), r)$ for $u \in \tilde{K} \cup \mathcal{F}(\tilde{K})$ and $r > 0$, by invariance under the action of the group $\Gamma < \mathrm{Mod}(S)$ and cocompactness we can find a universal number $\beta_0 > 0$ such that

$$(20) \qquad \pi(B(q, \beta_0) \cap (\tilde{K} \cup \mathcal{F}\tilde{K})) \subset D_q(\pi(q), \alpha) \text{ for all } q \in \tilde{K} \cup \mathcal{F}(\tilde{K}).$$

Denote again by d the distance on $\hat{\mathcal{Q}}(S)$ induced in equation (16) from the distance on $\mathcal{Q}^1(S)$. Since $K \cup \mathcal{F}(K)$ is compact and $\Pi : \mathcal{Q}^1(S) \to \hat{\mathcal{Q}}(S)$ is a covering, there is a number $\beta < \beta_0$ such that for every $q \in K \cup \mathcal{F}(K)$ and every lift $\tilde{q}$ of q to $\mathcal{Q}^1(S)$ the ball $B(q, \beta)$ in $\hat{\mathcal{Q}}(S)$ of radius β about q is the homeomorphic image under Π of the ball $B(\tilde{q}, \beta)$. Now the orbits of Φ^t are geodesics for the distance d. Hence if $x \in K$, if $s : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $s(0) = 0$ and $d(\Phi^t q, \Phi^{s(t)}q) < \beta/2$ for all t then by the choice of β we have $|t - s(t)| < \beta/2$ for all t.

This implies that for this function s, if $u \in K$ is such that $d(\Phi^t q, \Phi^{s(t)}u) < \beta/2$ for all t then $d(\Phi^t u, \Phi^{s(t)}u) < \beta/2$ (since $|t - s(t)| < \beta/2$ and orbits of the Teichmüller flow are geodesics) and hence $d(\Phi^t u, \Phi^t q) < \beta$ for all t by the triangle inequality. In particular, for a lift $\tilde{q} \in \mathcal{Q}^1(S)$ of q and a lift $\tilde{u}$ of u with $d(\tilde{q}, \tilde{u}) < \beta$ we have $d(\Phi^t \tilde{q}, \Phi^t \tilde{u}) < \beta$ for all $t \in \mathbb{R}$.

Let $W^s_{\mathrm{loc}}(q)$ be the connected component containing q of the intersection of $B(q, \beta)$ with the stable manifold $W^s(q)$ of q. We claim that $u \in W^s_{\mathrm{loc}}(q)$. For this assume otherwise. Let again $\tilde{q}$ be a preimage of q in $\mathcal{Q}^1(S)$ and let $\tilde{u} \in \mathcal{Q}^1(S)$ be the preimage of u with $d(\tilde{q}, \tilde{u}) < \beta$. If $\pi(\tilde{q}) \ne \pi(\tilde{u})$ then $\delta_{P\tilde{q}}(\pi(\tilde{q}), \pi(\tilde{u})) > 0$ and by continuity, our choice of α and the estimates (19), there is a number $t > 0$ such that $\delta_{P\Phi^t\tilde{q}}(\pi(\tilde{q}), \pi(\tilde{u})) = \alpha$. On the other hand, for every $s \in [0, t]$ the distance in $\mathcal{Q}^1(S)$ between $\Phi^s\tilde{q}, \Phi^s\tilde{u}$ is smaller than β which is a contradiction to the choice of $\beta < \beta_0$ and the relation (20).

In the same way we conclude that u is contained in the intersection of $B(q, \beta)$ with the local unstable manifold of q. Now the intersection of the local stable manifold $W^s_{\mathrm{loc}}(q)$ with the local unstable manifold $W^u_{\mathrm{loc}}(q)$ is contained in the orbit of q under the Teichmüller flow Φ^t which completes the proof of Theorem 3.1. $\square$

For a compact Φ^t-invariant subset K of $\hat{\mathcal{Q}}(S)$ let $h_{\mathrm{top}}(K)$ be the topological entropy of the restriction of Φ^t to K. For $r > 0$ let moreover $n_K(r)$ be the number of periodic orbits of Φ^t of period at most r which are contained in K. Since by Theorem 3.1 the restriction of the flow Φ^t to K is expansive, by Proposition 3.2.14 of [HK95] we have

Corollary 3.3. *Let $K \subset \hat{\mathcal{Q}}(S)$ be a compact Φ^t-invariant set; then*

$$\limsup_{r \to \infty} \frac{1}{r} \log n_K(r) \leq h_{\mathrm{top}}(K).$$

4. An Anosov closing lemma

The goal of this section is to establish a version of the Anosov closing lemma for the restriction of the Teichmüller flow to a compact invariant set $K \subset \hat{\mathcal{Q}}(S)$. The classical Anosov closing lemma roughly states that for a hyperbolic flow on a closed Riemannian manifold, a closed curve consisting of sufficiently long orbit segments which are connected at the endpoints by sufficiently short arcs is closely fellow-traveled by a periodic orbit.

We continue to use the assumptions and notations from Section 2 and Section 3. In particular, we always use the distances on $\mathcal{Q}^1(S)$ and $\hat{\mathcal{Q}}(S)$ defined in Lemma 3.2 and in equation (16). For a precise formulation of our version of an Anosov closing lemma for the Teichmüller flow, using the assumptions and notations from Section 2 and Section 3 we define.

Definition 4.1. For $n > 0, \epsilon > 0$, an (n,ϵ)-*pseudo-orbit* for the Teichmüller flow Φ^t on $\hat{\mathcal{Q}}(S)$ consists of a sequence of points $q_0, q_1, \ldots, q_k \in \hat{\mathcal{Q}}(S)$ and a sequence of numbers $t_0, \ldots, t_{k-1} \in [n, \infty)$ with the following properties.

(1) For every $j \leq k$ the 2ϵ-neighborhood of q_j is contained in a contractible subset of $\hat{\mathcal{Q}}(S)$.
(2) For every $j < k$ we have $d(\Phi^{t_j} q_j, q_{j+1}) \leq \epsilon$.

The pseudo-orbit is *contained* in a compact set K if for all i and all $t \in [0, t_i]$ we have $\Phi^t q_i \in K$. The pseudo-orbit is called *closed* if $q_0 = q_k$.

An (n, ϵ)-pseudo-orbit $q_0, \ldots, q_k$ determines an essentially unique arc connecting q_0 to q_k which we call a *characteristic arc*. Namely, by assumption, for each $j < k$ the 2ϵ-neighborhood of q_{j+1} is contained in a contractible subset of $\hat{\mathcal{Q}}(S)$. Hence the homotopy class with fixed endpoints in $\hat{\mathcal{Q}}(S)$ of an arc of length smaller than 2ϵ connecting $\Phi^{t_j} q_j$ to q_{j+1} is unique. We define a characteristic arc of the (n, ϵ)-pseudo-orbit to be an arc connecting q_0 to q_k which is obtained by successively joining the endpoint of the orbit segment $\{\Phi^t q_j \mid 0 \leq t \leq t_j\}$ to q_{j+1} with an arc of length smaller than 2ϵ which is parametrized on the unit interval $(j = 0, \ldots, k-1)$. The points q_i $(1 \leq i \leq k)$ are called the *breakpoints* of the characteristic arc of the pseudo-orbit. The *characteristic homotopy class* of the pseudo-orbit is the homotopy class with fixed endpoints of a characteristic arc connecting q_0 to q_k. Note that this is independent of the choice of a characteristic arc. If the pseudo-orbit is closed then it determines a closed characteristic curve and hence a free homotopy

class of closed curves in $\hat{\mathcal{Q}}(S)$ which we call the *characteristic free homotopy class* of the closed pseudo-orbit.

By abuse of notation, denote again by $P : \hat{\mathcal{Q}}(S) \to \mathcal{T}(S)/\Gamma$ the canonical projection.

Definition 4.2. An (n, ϵ)-pseudo-orbit $q_0, \ldots, q_k$ as in Definition 4.1 is *δ-shadowed* by an orbit segment $\zeta = \{\Phi^t q \mid t \in [0, \tau]\}$ for some $q \in \hat{\mathcal{Q}}(S)$ and some $\tau > 0$ if the following holds.

(1) There is a number $\alpha \leq \delta$ such that the α-neighborhoods of Pq_0, Pq_k in $\mathcal{T}(S)/\Gamma$ with respect to the projection of the Teichmüller metric are contractible and contain $Pq, P\Phi^\tau q$.
(2) There is a lift $\tilde{\zeta}$ to $\mathcal{Q}^1(S)$ of the orbit segment ζ and a lift $\tilde{\gamma}$ to $\mathcal{Q}^1(S)$ of a characteristic arc γ for the pseudo-orbit with the following properties. The distance between the endpoints of $P\tilde{\gamma}, P\tilde{\zeta}$ is at most α, and the Hausdorff distance between $\tilde{\gamma}$ and $\tilde{\zeta}$ is at most δ.

A closed pseudo-orbit in $\hat{\mathcal{Q}}(S)$ is *δ-shadowed* by a periodic orbit if in addition to the above requirements the orbit $\{\Phi^t q \mid t \in [0, \tau]\}$ is closed.

Note that every point in the orbifold $\mathcal{Q}1(S)/\text{Mod}(S)$ admits a contractible neighborhood, so we can use the above definition is the same way for the Teichmüller flow on the orbifold $\mathcal{Q}^1(S)/\text{Mod}(S)$.

Using Definition 4.1 and Definition 4.2 we can now formulate a version of the Anosov closing lemma for the Teichmüller flow which is the main result of this section.

Theorem 4.3. *For every compact Φ^t-invariant set $K \subset \hat{\mathcal{Q}}(S)$ there are numbers $\epsilon_1 = \epsilon_1(K) > 0$, $n = n(K) > 0$, $b = b(K) > 0$ such that every (n, ϵ_1)-pseudo-orbit contained in K is b-shadowed by an orbit. Moreover, for every $\delta > 0$ there is a number $\epsilon_2 = \epsilon_2(K, \delta) < \epsilon_1$ such that a closed (n, ϵ_2)-pseudo-orbit contained in K is δ-shadowed by a periodic orbit which is contained in the characteristic free homotopy class of the closed pseudo-orbit.*

For the proof of Theorem 4.3 we need the following technical preparation which is also used in Section 5. To this end, recall that a point $q \in \mathcal{Q}^1(S)/\text{Mod}(S)$ is recurrent if q is contained in the ω-limit set of its own orbit under the Teichmüller flow. For some $m > 1$, an *unparametrized m-quasi-geodesic* in $\mathcal{C}(S)$ is an arc $\gamma : J \to \mathcal{C}(S)$ with the property that there is an orientation preserving homeomorphism $\varphi : I \to J$ such that $\gamma \circ \varphi : I \to \mathcal{C}(S)$ is an m-quasi-geodesic. We show

Lemma 4.4.　(1) *For every compact Φ^t-invariant set $K \subset \hat{\mathcal{Q}}(S)$ there are numbers $\epsilon_0 = \epsilon_0(K) > 0, n_0 = n_0(K) > 0, \ell_0 = \ell_0(K) > 1$ depending on K with the following property. Let $q_0, \ldots, q_k \in \hat{\mathcal{Q}}(S)$ be an (n_0, ϵ_0)-pseudo-orbit contained in K and let $\tilde{\gamma}$ be a lift to $\mathcal{Q}^1(S)$ of a characteristic arc connecting q_0 to q_k. Then the arc $t \to \Upsilon_\mathcal{T}(P\tilde{\gamma}(t))$ is an ℓ_0-quasi-geodesic in $\mathcal{C}(S)$.*

(2) *There is a number $m > 1$ and for every recurrent point $q \in \mathcal{Q}^1(S)/\mathrm{Mod}(S)$ there are numbers $\epsilon_0(q) > 0, n_0(q) > 0$ with the following properties. Let $q_0, \ldots, q_k$ be an $(n_0(q), \epsilon_0(q))$-pseudo-orbit with $d(q_i, q) \leq \epsilon_0(q)$ for all i and let $\tilde{\gamma}$ be a lift to $\mathcal{Q}^1(S)$ of a characteristic arc γ connecting q_0 to q_k. Then the arc $t \to \Upsilon_{\mathcal{T}}(P\tilde{\gamma}(t))$ is an unparametrized m-quasi-geodesic in $\mathcal{C}(S)$.*

Proof. Recall from (3) and (11) of Section 2 the definition of the distance functions δ_c $(c \in \mathcal{C}(S))$ and δ_x $(x \in \mathcal{T}(S))$ on the Gromov boundary $\partial \mathcal{C}(S)$ of $\mathcal{C}(S)$. By hyperbolicity, every quasi-geodesic ray $\zeta : [0, \infty) \to \mathcal{C}(S)$ converges as $t \to \infty$ to a point $R(\zeta) \in \partial \mathcal{C}(S)$. Moreover, for every $p > 1$ there are numbers $m(p) > 0, \alpha(p) > 0, \beta(p) > 1$ with the following property.

Let $k > 0$ and let $\zeta_0, \ldots, \zeta_{k-1} : \mathbb{R} \to \mathcal{C}(S)$ be infinite p-quasi-geodesics. Let $L > 1$ be as in inequality (9). Suppose that for every $j \leq k-2$ there is a number $T_j > m(p)$ such that $d(\zeta_j(T_j), \zeta_{j+1}(0)) \leq 2L$. For each $j \leq k-1$ let $\rho_j : [0, 1] \to \mathcal{C}(S)$ be any map whose image is contained in the $2L$-neighborhood of $\zeta_j(T_j)$ and let $\tilde{\zeta}_j$ be the composition of $\zeta_j[0, T_j]$ with ρ_j parametrized in the natural way on $[0, T_j + 1]$. If $\delta_{\zeta_{j+1}(0)}(R(\zeta_j), R(\zeta_{j+1})) < \alpha(p)$ for all j then the curve $\zeta : [0, \sum_i T_i + k] \to \mathcal{C}(S)$ defined by

$$(21) \qquad \zeta(t) = \tilde{\zeta}_j(t - \sum_{i=0}^{j-1} T_i - j) \text{ for } t \in [\sum_{i=0}^{j-1} T_i + j, \sum_{i=0}^{j} T_i + j + 1]$$

is a $\beta(p)$-quasi-geodesic.

Let $K \subset \hat{\mathcal{Q}}(S)$ be a compact Φ^t-invariant set and let $\tilde{K} \subset \mathcal{Q}^1(S)$ be the preimage of K under the natural projection. By the second part of Theorem 2.2 there is a number $p > 1$ such that for every $q \in \tilde{K}$ the assignment $t \to \Upsilon_{\mathcal{T}}(P\Phi^t q)$ $(t \in \mathbb{R})$ is a p-quasi-geodesic.

For $q \in \tilde{K}$ we have $F(\pi(q)) = R(t \to \Upsilon_{\mathcal{T}}(P\Phi^t q)) \in \partial \mathcal{C}(S)$. Let $\kappa > 0$ be as in (12) of Section 2. By continuity, for every $q \in \tilde{K}$ there is a number $\epsilon(q) > 0$ such that for every point $\tilde{q} \in \tilde{K}$ which is contained in the $2\epsilon(q)$-neighborhood of q the δ_{Pq}-distance between $F(\pi(q))$ and $F(\pi(\tilde{q}))$ is smaller than $\alpha(p)/\kappa$ where $\alpha(p) > 0$ is as in the first paragraph of this proof. By continuity, invariance under the action of the mapping class group on $\mathcal{Q}^1(S)$ and $\partial \mathcal{C}(S)$ and cocompactness of the action of Γ on $\tilde{K}$, there is a number $\epsilon_0 \in (0, 1/2)$ which has this property for all $q \in \tilde{K}$ (compare the proof of Theorem 3.1 for a similar statement).

Let $m(p) > 0$ be as in the first paragraph of this proof. Let $\tilde{\gamma}$ be the lift to $\mathcal{Q}^1(S)$ of a characteristic arc of an $(m(p), \epsilon_0)$-pseudo-orbit $q_0, q_1, \ldots, q_k$ contained in $K \subset \hat{\mathcal{Q}}(S)$. Then $\tilde{\gamma}$ is a composition of curves $\tilde{\gamma}_i$ $(i = 1, \ldots, k)$ where the curve $\tilde{\gamma}_i$ is a lift to $\mathcal{Q}^1(S)$ of an orbit segment for the Teichmüller flow of length at least $m(p)$ beginning at x_{i-1} and an arc of length at most $2\epsilon_0 < 1$ parametrized on $[0, 1]$, with endpoint x_i. By the choice of $m(p)$, of $\epsilon_0 > 0$ and by the construction of the curves $\tilde{\gamma}_i$, the curve $\Upsilon_{\mathcal{T}}(P\tilde{\gamma})$ is of the form described in the second paragraph of this proof and hence it is a $\beta(p)$-quasi-geodesic in $\mathcal{C}(S)$. This shows the first part of the lemma.

The second part of the lemma follows in the same way. By [MM99] there is a number $\ell > 0$ such that for every $\tilde{q} \in \mathcal{Q}^1(S)$ the map $t \to \Upsilon_{\mathcal{T}}(P\Phi^t\tilde{q})$ is an unparametrized ℓ-quasi-geodesic (this number $\ell > 0$ does not coincide with the number $p > 1$ above, see also [H10]). Let $m(\ell) > 0, \alpha(\ell) > 0, p(\ell) > 1$ be as in the first paragraph of this proof. Let $q \in \mathcal{Q}^1(S)/\mathrm{Mod}(S)$ be a recurrent point and let $\tilde{q} \in \mathcal{Q}^1(S)$ be a lift of q. Then the vertical measured geodesic lamination of $\tilde{q}$ is uniquely ergodic and fills up S [M82], and the unparametrized ℓ-quasi-geodesic $t \to \Upsilon_{\mathcal{T}}(P\Phi^t\tilde{q})$ is of infinite diameter. By continuity and by Lemma 2.4 of [H10] there is a neighborhood V of $\tilde{q}$ in $\mathcal{Q}^1(S)$ and a number $T(q) > 0$ such that

$$(22) \qquad d(\Upsilon_{\mathcal{T}}(P\Phi^t u), \Upsilon_{\mathcal{T}}(Pu)) \geq \ell m(\ell) + \ell \text{ for all } u \in V \text{ and all } t \geq T(q).$$

In particular, if $u \in V$ and if $\rho_u : [0, a) \to [0, \infty)$ $(a \in (0, \infty])$ is a homeomorphism with the property that the map $t \to \Upsilon_{\mathcal{T}}(P\Phi^{\rho_u(t)}u)$ is a *parametrized* ℓ-quasi-geodesic in $\mathcal{C}(S)$ then $\rho_u(m(\ell)) \leq T(q)$.

The subset $\mathcal{U}$ of $\mathcal{Q}^1(S)$ of all points with uniquely ergodic vertical measured geodesic lamination which fills up S is dense [M82]. Each $u \in \mathcal{U}$ defines a point $F\pi(u) \in \partial\mathcal{C}(S)$ which is just the endpoint of the infinite unparametrized ℓ-quasi-geodesic $t \to \Upsilon_{\mathcal{T}}(P\Phi^t u)$. The map $u \in \mathcal{U} \to F\pi(u) \in \partial\mathcal{C}(S)$ is continuous. Thus we can find a number $\epsilon_0(q) \in (0, 1/2)$ which is small enough that the $2\epsilon_0(q)$-neighborhood W of $\tilde{q}$ in $\mathcal{Q}^1(S)$ is contained in V and that for every point $u \in W \cap \mathcal{U}$ the $\delta_{P\tilde{q}}$-distance between $F(\pi(\tilde{q}))$ and $F(\pi(u))$ is smaller than $\alpha(\ell)/\kappa$ where as before, $\kappa > 0$ is as in (12) in Section 2. The second part of the lemma holds true for the numbers $\epsilon_0(q) > 0, T(q) > 0$.   □

Proof of Theorem 4.3. Let $K \subset \hat{\mathcal{Q}}(S)$ be any compact Φ^t-invariant set and let $\tilde{K}$ be the preimage of K in $\mathcal{Q}^1(S)$ under the natural projection. Let $\epsilon_0 = \epsilon_0(K) \in (0, 1/2), n_0 = n_0(K) > 0$ be as in the first part of Lemma 4.4. Let $q_0, \ldots, q_k$ be an (n_0, ϵ_0)-pseudo-orbit for Φ^t which is contained in K and let $t_0, \ldots, t_{k-1} \in [n_0, \infty)$ be as in the definition of a pseudo-orbit such that $d(\Phi^{t_i}q_i, q_{i+1}) \leq \epsilon_0$ for $i < k$. Let γ be a characteristic arc of this pseudo-orbit which is parametrized on $[0, \sum_{i=0}^{k-1} t_i + k]$ in such a way that for each j the restriction of γ to $[\sum_{i<j} t_i + j, \sum_{i<j+1} t_i + j]$ is a reparametrization of the orbit segment $\{\Phi^t q_j \mid t \in [0, t_j]\}$ by a translation. The points $q_1, \ldots, q_{k-1}$ are the breakpoints of the characteristic arc.

Let $\tilde{\gamma}$ be a lift of γ to $\mathcal{Q}^1(S)$. By Lemma 4.4, the assignment $t \to \Upsilon_{\mathcal{T}}(P\tilde{\gamma}(t))$ is an ℓ_0-quasi-geodesic in $\mathcal{C}(S)$ for a number $\ell_0 > 1$ only depending on K. The map $t \to P\tilde{\gamma}(t) \in (\mathcal{T}(S), d_T)$ is one-Lipschitz. Since by inequality (9) there is a number $L > 0$ such that $d_T(Pq, Pz) \geq d(\Upsilon_{\mathcal{T}}(Pq), \Upsilon_{\mathcal{T}}(Pz))/L - L$ for all $q, z \in \mathcal{Q}^1(S)$, we conclude that the curve $t \to P\tilde{\gamma}(t)$ is a uniform quasi-geodesic in $\mathcal{T}(S)$. By the first part of Theorem 2.2, this implies that there is a Teichmüller geodesic whose Hausdorff distance to $P\tilde{\gamma}$ is bounded from above by a universal constant. As a consequence, the Hausdorff distance between $\tilde{\gamma}$ and the tangent line of this geodesic is bounded from above by a universal constant $b > 0$.

A mapping class $g \in \mathrm{Mod}(S)$ is pseudo-Anosov if the cyclic subgroup of $\mathrm{Mod}(S)$ generated by g acts on the curve graph $\mathcal{C}(S)$ with unbounded orbits. In this case the conjugacy class of g can be represented by a closed orbit for the Teichmüller flow Φ^t on $\mathcal{Q}^1(S)/\mathrm{Mod}(S)$, and it can be represented by a closed orbit for the Teichmüller

flow on $\hat{\mathcal{Q}}(S)$ if the conjugacy class of g is contained in the normal subgroup Γ of $\mathrm{Mod}(S)$. Assume now that the (n_0, ϵ_0)-pseudo-orbit $q_0, \ldots, q_k$ contained in K is closed. Let $\tilde{\gamma}$ be a lift to $\mathcal{Q}^1(S)$ of a closed characteristic arc γ for the pseudo-orbit. By the first part of Lemma 4.4, the curve $t \to \Upsilon_{\mathcal{T}}(P\tilde{\gamma}(t))$ is an infinite ℓ_0-quasi-geodesic in $\mathcal{C}(S)$ which is invariant under an element $g \in \Gamma < \mathrm{Mod}(S)$ of the mapping class group. The mapping class g acts on this quasi-geodesic as a translation and hence it is pseudo-Anosov. As a consequence, there is a unique g-invariant Teichmüller geodesic in $\mathcal{T}(S)$ whose cotangent line in $\mathcal{Q}^1(S)$ projects to a periodic orbit of Φ^t in $\hat{\mathcal{Q}}(S)$ which defines the free homotopy class of γ. In other words, there is a closed orbit for Φ^t in $\hat{\mathcal{Q}}(S)$ which is freely homotopic to γ.

By the first part of Theorem 2.2, applied to the biinfinite quasi-geodesic $P\tilde{\gamma}$ in $\mathcal{T}(S)$, this orbit is contained in a compact subset $C_0 \supset K$ of $\hat{\mathcal{Q}}(S)$ not depending on the pseudo-orbit. Moreover, it b-shadows the pseudo-orbit for a number $b > 0$ only depending on K, n_0, ϵ_0. This shows the first part of Theorem 4.3.

Let $C \subset C_0$ be the Φ^t-invariant subset of C_0 of all points whose Φ^t-orbit is entirely contained in C_0. The periodic orbit defined by the conjugacy class of the pseudo-Anosov element g is contained in C. Let $\tilde{C}$ be the preimage of C in $\mathcal{Q}^1(S)$. Then every lift to $\mathcal{Q}^1(S)$ of a periodic orbit in $\hat{\mathcal{Q}}(S)$ determined as above by a closed (n_0, ϵ_0)-pseudo-orbit contained in K is contained in $\tilde{C}$.

Let again $\pi : \mathcal{Q}^1(S) \to \mathcal{PML}$ be the canonical projection. Write $A = \pi(\tilde{C} \cup \mathcal{F}(\tilde{C}))$ where $\mathcal{F} : \mathcal{Q}^1(S) \to \mathcal{Q}^1(S)$ is the flip $q \to \mathcal{F}(q) = -q$. As in Section 3, let $F_A = F|A : A \to \partial\mathcal{C}(S)$ be the measure forgetting injection. For $q \in \tilde{C} \cup \mathcal{F}(\tilde{C})$ let δ_{Pq} be the distance on $\partial\mathcal{C}(S)$ defined in equation (11) and denote by $D_q(\pi(q), r)$ the ball of radius r about $\pi(q)$ in A with respect to the distance $(x, y) \in A \times A \to \delta_{Pq}(F_A x, F_A y) \in [0, \infty)$ which we denote again by δ_{Pq} (compare the proof of Theorem 3.1).

By the second part of Theorem 2.2, applied to the projection into $\mathcal{T}(S)$ of the preimage $\tilde{C}$ of the compact Φ^t-invariant set $C \subset \hat{\mathcal{Q}}(S)$, by Lemma 2.1 and by inequality (12) of Section 2, there are numbers $\alpha_0 < 1/2, a > 1, b > 0$ such that for every $q \in \tilde{C}$ and for all $t > 0$ we have

(23) $$\delta_{P\Phi^{-t}q} \le ae^{-bt}\delta_{Pq} \text{ on } D_q(\pi(q), 4\alpha_0).$$

Moreover, for every $\alpha < \alpha_0$ there is a number $\beta = \beta(\alpha) < 1$ such that for every $q \in \tilde{C}$ we have $A \cap \pi B(q, \beta) \subset D_q(\pi(q), \alpha)$ where $B(q, \beta)$ is the ball of radius β about q in $\mathcal{Q}^1(S)$ (compare the proof of Theorem 3.1).

Let $n = \max\{n_0, \log(4a)/b\}$, let $\alpha < \alpha_0$ and let $\sigma = \min\{\epsilon_0, \beta(\alpha), \kappa^{-1}\log 2\}$ where $\kappa > 0$ is as in inequality (12). We claim that for a lift $\tilde{\gamma} : [0, T] \to \mathcal{Q}^1(S)$ of a characteristic arc γ of any (n, σ)-pseudo-orbit contained in K we have

$$\delta_{P\tilde{\gamma}(0)}(\pi\tilde{\gamma}(0), \pi\tilde{\gamma}(T)) \le \alpha.$$

To see this we proceed by induction on the number of breakpoints of the pseudo-orbit. The case that there is a no breakpoint is trivial, so assume that the claim is known whenever the number of breakpoints of the pseudo-orbit is at most $k-1 \ge 0$. Let $\tilde{\gamma}$ be a lift to $\mathcal{Q}^1(S)$ of a characteristic arc γ of an (n, σ)-pseudo-orbit contained

in K with k breakpoints. Let $t_0 \geq n + 1$ be such that $\gamma(t_0)$ is the first breakpoint of γ. By assumption and the choice of the parametrization of a characteristic arc we have $d(\tilde{\gamma}(t_0), \tilde{\gamma}(t_0 - 1)) \leq \sigma$. Since $\tilde{\gamma}(t_0) \subset \tilde{K} \subset \tilde{C}, \tilde{\gamma}(t_0 - 1) \in \tilde{C}$, by the choice of σ we have

$$(24) \qquad \delta_{P\tilde{\gamma}(t_0)}(\pi\tilde{\gamma}(t_0), \pi\tilde{\gamma}(t_0 - 1)) \leq \alpha,$$

moreover the distances $\delta_{P\tilde{\gamma}(t_0)}, \delta_{P\tilde{\gamma}(t_0-1)}$ are 2-bilipschitz equivalent (recall that the projection $P : \mathcal{Q}^1(S) \to \mathcal{T}(S)$ is distance non-increasing).

Now $\pi\tilde{\gamma}(T) \in D_{\tilde{\gamma}(t_0)}(\pi\tilde{\gamma}(t_0), \alpha)$ by the induction hypothesis and therefore

$$(25) \qquad \delta_{P\tilde{\gamma}(t_0-1)}(\pi\tilde{\gamma}(t_0 - 1), \pi\tilde{\gamma}(T)) \leq 4\alpha.$$

On the other hand, since $\alpha \leq \alpha_0$, since $n \geq \log(4a)/b$ and since $\pi\tilde{\gamma}(t_0 - 1) = \pi\tilde{\gamma}(0)$ we infer from the estimate (23) that

$$(26) \qquad \delta_{P\tilde{\gamma}(t_0-1)}(\pi\tilde{\gamma}(t_0 - 1), \pi\tilde{\gamma}(T)) \geq 4\delta_{P\tilde{\gamma}(0)}(\pi\tilde{\gamma}(0), \tilde{\gamma}(T)).$$

Together this implies the claim.

Note that the argument in the previous paragraph together with the estimate (23) also shows that

$$(27) \qquad \delta_{\tilde{\gamma}(t)}(\pi\tilde{\gamma}(t), \pi\tilde{\gamma}(T)) \leq 4a\alpha$$

for *all* $t \in [0, T]$ with the additional property that $\tilde{\gamma}(t) \in \tilde{C}$ (namely this holds true for every t such that $\tilde{\gamma}(t)$ projects to an orbit segment defining the pseudo-orbit).

Let again $\tilde{\gamma}$ be a biinfinite lift to $\tilde{\mathcal{Q}}^1(S)$ of a closed characteristic curve γ for an (n, σ)-pseudo-orbit contained in K. The curve $\Upsilon_{\mathcal{T}}(P\tilde{\gamma})$ is a uniform quasi-geodesic in $\mathcal{C}(S)$ which is invariant under a pseudo-Anosov element $g \in \Gamma < \mathrm{Mod}(S)$ on $\partial\mathcal{C}(S)$. The conjugacy class of γ defines the free homotopy class of γ. The oriented cotangent line of the axis of g is contained in $\tilde{C}$. If $z \in \tilde{C}$ is a point in this cotangent line then $\pi(z) \in A$ is a fixed point for the action of g on $\mathcal{PML}$. An inductive application to longer and longer subsegments of $\tilde{\gamma}$ of the argument which lead to the estimate (27) shows that for every $t \in \mathbb{R}$ such that $\tilde{\gamma}(t) \subset \tilde{K} \subset \tilde{C}$ the fixed point $\pi(z) \in A$ of g is contained in the ball $D_{\tilde{\gamma}(t)}(\pi(\tilde{\gamma}(t)), 4a\alpha)$. The same argument also shows that the fixed point $\pi(-z)$ for the action of g is contained in $D_{-\tilde{\gamma}(t)}(\pi(-\tilde{\gamma}(t)), 4a\alpha)$. The periodic orbit on $\hat{\mathcal{Q}}(S)$ defined by g is contained in the compact Φ^t-invariant subset $C \supset K$ of $\hat{\mathcal{Q}}(S)$ determined above.

By the considerations in Theorem 3.1 and its proof, applied to the compact Φ^t-invariant subset C of $\hat{\mathcal{Q}}(S)$, this means that for every $\delta > 0$ there is a constant $\beta > 0$ only depending on K with the following property. Let $q_0, \ldots, q_k$ be a closed (n, β)-pseudo-orbit contained in K. Then there is a closed orbit for Φ^t contained in C whose Hausdorff distance to a closed characteristic curve defined by the pseudo-orbit is at most δ. From this Theorem 4.3 follows. $\square$

The Anosov closing lemma implies the existence of many periodic orbits near any non-wandering point of a compact Φ^t-invariant subset K of $\hat{\mathcal{Q}}(S)$. However, as for compact invariant hyperbolic sets in the usual sense of smooth dynamical systems (see [HK95]), these periodic orbits are in general not contained in K. The next corollary is an immediate adaptation of Corollary 6.4.19 of [HK95] and

shows that the periodic orbits can be chosen to be contained in an arbitrarily small neighborhood of K.

Corollary 4.5. *Let K be a compact Φ^t-invariant subset of $\hat{\mathcal{Q}}(S)$ and let U be an open neighborhood of K. Then every non-wandering point $q \in K$ is an accumulation point of periodic points of Φ^t whose orbits are entirely contained in U.*

In the case of a topologically transitive compact invariant set $K \subset \hat{\mathcal{Q}}(S)$ we can say more.

Lemma 4.6. *Let K be a compact Φ^t-invariant topologically transitive subset of $\hat{\mathcal{Q}}(S)$. Then for every $\sigma > 0$ there is a periodic orbit for Φ^t whose Hausdorff-distance to K (as subsets of $\hat{\mathcal{Q}}(S)$) is at most σ.*

Proof. Let $K \subset \hat{\mathcal{Q}}(S)$ be a compact Φ^t-invariant topologically transitive set and let $\sigma > 0$. Let $n = n(K) > 0$, $\epsilon_2 = \epsilon_2(K, \sigma/2) < \sigma/2$ be as in Theorem 4.3. Since K is topologically transitive by assumption, there is some $q \in K$ and there is some $T > n$ such that $d(q, \Phi^T q) < \epsilon_2$ and that moreover the Hausdorff distance between the set K and its subset $B = \{\Phi^t q \mid 0 \leq t \leq T\}$ is at most $\sigma/2$.

By Theorem 4.3, applied to the closed (n, ϵ_2)-pseudo-orbit defined by the orbit segment $\{\Phi^t q \mid 0 \leq t \leq T\}$, there is a periodic orbit for Φ^t whose Hausdorff distance to B is at most $\sigma/2$. This means that the Hausdorff distance between this orbit and the set K is at most σ and shows the lemma. $\square$

For a compact Φ^t-invariant subset $K \subset \hat{\mathcal{Q}}(S)$ denote by $h_{\text{top}}(K)$ the topological entropy of the restriction of Φ^t to K. For an arbitrary subset $U \subset \hat{\mathcal{Q}}(S)$ and a number $r > 0$ let $n_U(r)$ be the number of all periodic orbits of Φ^t of period at most r which are contained in U. The following corollary is another fairly immediate consequence of Theorem 4.3. Together with Corollary 3.3 it shows Theorem 2 from the introduction.

Corollary 4.7. *Let $K \subset \hat{\mathcal{Q}}(S)$ be a compact Φ^t-invariant topologically transitive set. Then for every open neighborhood U of K we have*

$$h_{\text{top}}(K) \leq \lim \inf_{r \to \infty} \frac{1}{r} \log n_U(r).$$

Proof. Let $K \subset \hat{\mathcal{Q}}(S)$ be a topologically transitive compact Φ^t-invariant set and let U be an open neighborhood of K. Then there is a number $\beta > 0$ such that U contains the β-neighborhood of K.

Let $\delta < \beta$ be sufficiently small that the δ-neighborhood of every point in K is contained in a contractible subset of $\hat{\mathcal{Q}}(S)$. Let $n = n(K) > 0, \epsilon_2 = \epsilon_2(K, \delta/8) < 1$ be as in Theorem 4.3. Since the Teichmüller flow on K is topologically transitive by assumption, by compactness of $K \times K$ there is a number $N > n$ with the following property. Let $q, q' \in K$; then there is some $u \in K$ and some $T \in [n, N]$ with $d(u, q') < \epsilon_2$ and $d(\Phi^T u, q) < \epsilon_2$.

A subset E of K is called (m, δ)-*separated* for some $m \geq 0$ if for any two points $q \neq u \in E$ we have

$$(28) \qquad\qquad d(\Phi^t q, \Phi^t u) \geq \delta \text{ for some } t \in [0, m].$$

Let $m > n$ and let $E_m \subset K$ be any (m, δ)-separated set. Let $q \in E_m$. By the choice of $N > n$ there is some $u \in K$ and some $T \in [n, N]$ such that $d(u, \Phi^m q) < \epsilon_2$ and $d(\Phi^T u, q) < \epsilon_2$. By Theorem 4.3, the closed (n, ϵ_2)-pseudo-orbit q, u, q is $\delta/8$-shadowed by a periodic orbit which defines the characteristic free homotopy class of the pseudo-orbit. Since periodic orbits for Φ^t in $\hat{\mathcal{Q}}(S)$ minimize the length in their free homotopy class, the length of the periodic orbit does not exceed $m + N + 2\epsilon_2$. Moreover, by the choice of δ this periodic orbit is contained in U. There is a point $\zeta(q)$ on the orbit with $d(q, \zeta(q)) \leq \delta/8$. In other words, there is a map ζ which associates to every point $q \in E_m$ a point $\zeta(q) \in U$ whose orbit under Φ^t is entirely contained in U and is periodic of period at most $m + N + 2\epsilon_2$.

Since the points in the set E_m are (m, δ)-separated by assumption and the orbits of Φ^t are geodesics parametrized by arc length, the orbit segments $c(q) = \cup_{t \in (-\delta/8, \delta/8)} \Phi^t \zeta(q)$ ($q \in E_m$) are pairwise disjoint. Thus for a fixed periodic orbit γ for Φ^t of length at most $m + N + 2\epsilon_2$ there are at most $4(m + N + 2)/\delta$ distinct points $q \in E_m$ with $\zeta(q) \in \gamma$. As a consequence, there are at least $\delta \operatorname{card}(E_m)/4(m + N + 2)$ distinct periodic orbits of period at most $m + N + 2$ in U. This shows that the asymptotic growth as $m \to \infty$ of the maximal cardinality of an (m, δ)-separated subset of K does not exceed the asymptotic growth of the numbers $n_U(r)$ as $r \to \infty$. The corollary is now an immediate consequence from the definition of the topological entropy of a continuous flow on a compact space (recall also from Theorem 3.1 that the Teichmüller flow on K is expansive and hence for all sufficiently small $\delta > 0$ its topological entropy is just the asymptotic growth rate of maximal (m, δ)-separated sets as $m \to \infty$). $\qquad\square$

5. Lower bounds for the number of periodic orbits

In this section we complete the proof of Theorem 3 from the introduction. For this we continue to use the assumptions and notations from Sections 2 and 3. In particular, we always denote by d_T the Teichmüller metric on Teichmüller space $\mathcal{T}(S)$ for S.

We begin with establishing the first part of Theorem 3 which is immediate from the work of Eskin and Mirzakhani [EM08]. Since the proof is short and easy, we include it for completeness.

The *Poincaré series with exponent* $\alpha > 0$ at a point $x \in \mathcal{T}(S)$ is defined to be the series

$$(29) \qquad\qquad \sum_{g \in \operatorname{Mod}(S)} e^{-\alpha d(x, gx)}.$$

The *critical exponent* of $\operatorname{Mod}(S)$ is the infimum of all numbers $\alpha > 0$ such that the Poincaré series with exponent α converges. Note that this critical exponent does not depend on the choice of x. Athreya, Bufetov, Eskin and Mirzakhani [ABEM06]

showed that the critical exponent of the Poincaré series equals $h = 6g - 6 + 2m$ and that the Poincaré series diverges at the critical exponent.

For $r > 0$ and for a compact set $K \subset \mathcal{Q}^1(S)/\mathrm{Mod}(S)$ let $n_K^{\cap}(r)$ be the number of all periodic orbits for the Teichmüller flow of period at most r which *intersect* K. The next lemma is the first part of Theorem 1.

Lemma 5.1. *For every compact subset K of $\mathcal{Q}^1(S)/\mathrm{Mod}(S)$ we have*

$$\limsup_{r \to \infty} \frac{1}{r} \log n_K^{\cap}(r) \leq 6g - 6 + 2m.$$

Proof. Let $\hat{K}$ be any compact subset of the *moduli space* $\mathcal{M}(S) = \mathcal{T}(S)/\mathrm{Mod}(S)$ which is the closure of an open set. Let $K_1 \subset \mathcal{T}(S)$ be a relative compact fundamental domain for the action of $\mathrm{Mod}(S)$ on the preimage $\tilde{K}$ of $\hat{K}$ in $\mathcal{T}(S)$. Let D be the diameter of K_1 and let $x \in K_1$ be any point. Let $g \in \mathrm{Mod}(S)$ be a pseudo-Anosov element whose *axis* (i.e. the unique g-invariant Teichmüller geodesic on which g acts as a translation) projects to a closed geodesic γ in moduli space which intersects $\hat{K}$. Then there is a point $\tilde{x} \in K_1$ which lies on the axis of a conjugate of g which we denote again by g for simplicity. By the properties of an axis, the length $\ell(\gamma)$ of the closed geodesic γ equals $d_T(\tilde{x}, g\tilde{x})$. On the other hand, we have

$$(30) \qquad d_T(x, gx) \leq d_T(\tilde{x}, g\tilde{x}) + 2d_T(x, \tilde{x}) \leq \ell(\gamma) + 2D$$

by the definition of D, the choice of $\tilde{x}$ and invariance of the Teichmüller metric under the action of $\mathrm{Mod}(S)$. Therefore, if we denote by $K \subset \mathcal{Q}^1(S)/\mathrm{Mod}(S)$ the preimage of $\hat{K} \subset \mathcal{M}(S)$ under the natural projection and if we define $N(r)$ for $r > 0$ to be the number of all $g \in \mathrm{Mod}(S)$ with $d(x, gx) \leq r$, then we have

$$(31) \qquad \limsup_{r \to \infty} \frac{1}{r} \log n_K^{\cap}(r) \leq \limsup_{r \to \infty} \frac{1}{r} \log N(r).$$

Since the critical exponent of the Poincaré series equals $6g - 6 + 2m$, for every $\epsilon > 0$ the Poincaré series converges at the exponent $\alpha = 6g - 6 + 2m + \epsilon$. Let $c(\alpha) > 0$ be its value. Then for every $r > 0$, the cardinality of the set $\{g \in \mathrm{Mod}(S) \mid d(x, gx) \leq r\}$ does not exceed $c(\alpha)e^{\alpha r}$ (note that the term in the Poincaré series corresponding to such an element of $\mathrm{Mod}(S)$ is *not smaller* than $e^{-\alpha r}$). This shows that $\limsup \frac{1}{r} \log N(r) \leq 6g - 6 + 2m + \epsilon$. Since $\epsilon > 0$ and the compact set $\hat{K} \subset \mathcal{M}(S)$ were arbitrarily chosen, the lemma follows. $\square$

As an immediate consequence we obtain

Corollary 5.2. *Let $K \subset \mathcal{Q}^1(S)/\mathrm{Mod}(S)$ be a compact Φ^t-invariant topologically transitive set. Then $h_{\mathrm{top}}(K) \leq 6g - 6 + 2m$.*

Proof. Let $K \subset \mathcal{Q}^1(S)/\mathrm{Mod(S)}$ be a compact Φ^t-invariant topologically transitive set, let $q \in K$ be a point whose orbit under Φ^t is dense in K and let $\hat{q} \in \hat{\mathcal{Q}}(S)$ be a preimage of q under the natural projection $\Theta : \hat{\mathcal{Q}}(S) \to \mathcal{Q}^1(S)/\mathrm{Mod}(S)$. Let $\hat{K}$ be the closure of the orbit of $\hat{q}$; then $\hat{K}$ is a compact Φ^t-invariant topologically transitive set. By equivariance of the Teichmüller flow under the projection Θ, this

set is mapped by Θ onto K. Moreover, by Corollary 4.7, for every open relative compact neighborhood U of $\hat{K}$ we have

$$(32) \qquad h_{\text{top}}(K) \leq h_{\text{top}}(\hat{K}) \leq \liminf_{r \to \infty} \frac{1}{r} \log n_U(r).$$

Now the projection Θ maps periodic orbits for Φ^t in U of period at most r to periodic orbits for Φ^t of period at most r which are contained in the relative compact set $\Theta(U) \subset \mathcal{Q}^1(S)/\text{Mod}(S)$. If the periodic orbits $\gamma_1 \neq \gamma_2$ in U are mapped to the same periodic orbit in $\Theta(U)$ then there is some element g from the factor group $G = \text{Mod}(S)/\Gamma$ which maps γ_1 to γ_2. Since G is finite, the number of distinct periodic orbits in U which are mapped to the single orbit in $\Theta(U)$ is uniformly bounded. Therefore by Lemma 5.1 we have

$$(33) \qquad \liminf_{r \to \infty} \frac{1}{r} \log n_U(r) \leq \liminf_{r \to \infty} \frac{1}{r} \log n_{\Theta(U)}(r) \leq 6g - 6 + 2m.$$

This shows the corollary. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Now we are ready for the proof of the second part of Theorem 2 from the introduction.

Proposition 5.3. *For every $\epsilon > 0$ there is a compact Φ^t-invariant subset K of $\mathcal{Q}^1(S)/\text{Mod}(S)$ with*

$$\liminf_{r \to \infty} \frac{1}{r} \log n_K(r) \geq 6g - 6 + 2m - \epsilon.$$

Proof. As in Section 2, let $\mathcal{FML} \subset \mathcal{PML}$ be the $\text{Mod}(S)$-invariant Borel subset of all projective measured geodesic laminations whose support is minimal and fills up S and let $F : \mathcal{FML} \to \partial\mathcal{C}(S)$ be the continuous $\text{Mod}(S)$-equivariant surjection which associates to a projective measured geodesic lamination in $\mathcal{FML}$ its support. Let $\pi : \mathcal{Q}^1(S) \to \mathcal{PML}$ be the natural projection as defined in (13) and define

$$(34) \qquad\qquad\qquad \mathcal{A} = \pi^{-1}\mathcal{FML} \subset \mathcal{Q}^1(S).$$

Let λ be the Φ^t-invariant probability measure on $\mathcal{Q}^1(S)/\text{Mod}(S)$ in the Lebesgue measure class constructed in [M82, V86]. This measure is ergodic and mixing under the Teichmüller flow, with full support. In particular, the Φ^t-orbit of λ-almost every point $q \in \mathcal{Q}^1(S)/\text{Mod}(S)$ returns to every neighborhood of q for arbitrarily large times. The measure λ lifts to a $\text{Mod}(S)$-invariant Φ^t-invariant Radon measure $\tilde{\lambda}$ on $\mathcal{Q}^1(S)$ of full support which gives full measure to the $\text{Mod}(S)$-invariant Borel set $\mathcal{A}$ [M82].

The Lebesgue measure $\tilde{\lambda}$ on $\mathcal{Q}^1(S)$ is absolutely continuous with respect to the strong unstable foliation. More precisely, for every $q \in \mathcal{Q}^1(S)$ there is a natural conditional measure $\tilde{\lambda}_q$ for $\tilde{\lambda}$ on the strong unstable manifold $W^{su}(q)$, and these conditional measures transform under the Teichmüller flow via $d\tilde{\lambda}_{\Phi^t q} \circ \Phi^t = e^{ht} d\tilde{\lambda}_q$ where $h = 6g - 6 + 2m$ as before. The image under the projection π of the measure $\tilde{\lambda}_q$ on $W^{su}(q)$ is a locally finite Borel measure λ_q on the open dense subset of $\mathcal{PML}$ of all projective measured geodesic laminations which together with $\pi(-q)$ jointly fill up S. The measures λ_q are all absolutely continuous, and they depend continuously on $q \in \mathcal{Q}^1(S)$ in the weak*-topology. Moreover, for each q the measure λ_q gives

full measure to the set $\mathcal{FML}$ and hence it can be mapped via the surjection F to a measure on $\partial\mathcal{C}(S)$ which we denote again by λ_q.

Recall from (11) and (12) the definition of the distances δ_x ($x \in \mathcal{T}(S)$) on $\partial\mathcal{C}(S)$ and their properties. For $q \in \mathcal{A}$ and $\chi > 0$ define

$$D(q, \chi) \subset \partial\mathcal{C}(S)$$

to be the closed δ_{Pq}-ball of radius χ about $F\pi(q) \in \partial\mathcal{C}(S)$. Note that $D(q, \chi)$ contains the image under the map F of the ball $B_{\delta_{Pq}}(\pi(q), \chi)$ used in Section 4. Note moreover that $D(q, \chi)$ depends on q and not only on its center $F\pi(q)$ and the radius χ since for $t \neq 0$ the distances δ_{Pq} and $\delta_{P\Phi^t q}$ do not coinicide,

Let $q_0 \in \mathcal{Q}^1(S)/\mathrm{Mod}(S)$ be a typical point for the Lebesgue measure λ (so that the Birkhoff ergodic theorem holds true for q_0) and let q_1 be a lift of q_0 to $\mathcal{Q}^1(S)$. Assume without loss of generality that Pq_1 is not fixed by any element of $\mathrm{Mod}(S)$. This is possible since the set of points in $\mathcal{T}(S)$ which are stabilized by a non-trivial element of $\mathrm{Mod}(S)$ is closed and nowhere dense and since the Lebesgue measure is of full support.

Let $m > 1$ be as in the second part of Lemma 4.4. We may assume that the image under the map $\Upsilon_{\mathcal{T}}$ of every Teichmüller geodesic in $\mathcal{T}(S)$ is an unparametrized m-quasi-geodesic in $\mathcal{C}(S)$. Since q_0 is a typical point for the Lebesgue measure, the unparametrized quasi-geodesic $t \to \Upsilon_{\mathcal{T}}(P\Phi^t q_1)$ is of infinite diameter.

Let $\kappa > 0$ be as in inequality (12). By Lemma 2.1 and the inequality (12), there is a number $\alpha > 0$ depending on m and there is a neighborhood V of q_1 in $\mathcal{Q}^1(S)$ of diameter at most $\log 2/\kappa$ and a number $T_0 > 0$ such that

(35) $\delta_{P\Phi^t u} \geq 16\delta_{Pu}$ on $D(\Phi^t u, \alpha)$ for $t \geq T_0$, $u \in V \cap \mathcal{A}$.

Since q_0 is recurrent and hence the vertical measured geodesic lamination of q_1 is uniquely ergodic and fills up S, by the second part of Lemma 3.2 of [H09] there is a number $\chi \leq \alpha/4$ such that

(36) $F\pi(V \cap \mathcal{A} \cap W^{su}(q_1)) \supset D(q_1, \chi)$.

Let $\epsilon_0 = \epsilon_0(q_0) > 0$ be as in the second part of Lemma 4.4. We may assume that the ϵ_0-neighborhood of q_0 is contained in a contractible subset of $\mathcal{Q}^1(S)/\mathrm{Mod}(S)$. By continuity, there is a compact neighborhood $K \subset V$ of q_1 with the following properties.

(1) The diameter of K does not exceed $\max\{\epsilon_0, (\log 2)/\kappa\}$.
(2) $F \circ \pi(K \cap \mathcal{A}) \subset D(q_1, \chi/4)$.

By the second requirement for K, if $q, u \in K \cap \mathcal{A}$ then $\delta_{Pq_1}(F\pi(u), F\pi(q)) \leq \chi/2$. The first property of K together with the relation (12) for the distances δ_x ($x \in \mathcal{T}(S)$) then implies that $\delta_{Pu}(F\pi(q), F\pi(u)) \leq \chi$ and $D(q, \chi) \subset D(u, 4\chi)$.

Following [F69], a *Borel covering relation* for a Borel subset C of a topological space X is a family $\mathcal{V}$ of pairs (x, V) where $V \subset X$ is a Borel set, where $x \in V$ and such that

$$(37) \qquad C \subset \bigcup \{V \mid (z, V) \in \mathcal{V} \text{ for some } z \in C\}.$$

For $\chi > 0$ and the neighborhood $K \subset \mathcal{Q}^1(S)$ of q_1 as above define

$$(38) \qquad \mathcal{V}_{q_0, \chi, K} = \{(F\pi(q), gD(q_1, \chi)) \mid$$
$$q \in W^{su}(q_1) \cap \mathcal{A}, g \in \mathrm{Mod}(S), gK \cap \cup_{t > 0} \Phi^t q \neq \emptyset\}.$$

By Proposition 3.5 of [H09], via possibly decreasing the size of χ and K we may assume that the covering relation $\mathcal{V}_{q_0, \chi, K}$ is a *Vitali relation* for the measure λ_{q_1} on $\partial \mathcal{C}(S)$. In our context, this means that for every $T > 0$ there is a covering of λ_{q_1}-almost all of $D(q_1, \chi/4)$ by pairwise disjoint sets from the relation of the form

$$V(g, t) = (F\pi(z), gD(q_1, \chi))$$

where $z \in W^{su}(q_1) \cap \mathcal{A}, F\pi(z) \in D(q_1, \chi/4)$, $g \in \mathrm{Mod}(S)$ and where $t \geq T$ is such that $\Phi^t z \in gK$ (we refer to Section 3 of [H09] for a detailed discussion).

Since the measures λ_q and the distances δ_{Pq} on $\partial \mathcal{C}(S)$ depend continuously on $u \in \mathcal{Q}^1(S)$, there is a number $a \leq \lambda_{q_1} D(q_1, \chi/4)$ such that $\lambda_q D(u, \chi) \in [a, a^{-1}]$ for all $q \in K, u \in K \cap \mathcal{A}$. By the transition properties for the measures λ_u and invariance under the action of the mapping class group, if $g \in \mathrm{Mod}(S)$, if $u \in W^{su}(q_1)$ and if $t > 0$ are such that $\Phi^t u \in gK$ for some $t > 0$ then $\lambda_{q_1}(gD(q_1, \chi)) \in [ae^{-ht}, e^{-ht}/a]$ (compare the discussion in [H09]).

Let $n_0 = n_0(q_0) > 0$ be as in the second part of Lemma 4.4. Let $\epsilon > 0$ and let $T(\epsilon) > \max\{T_0, n_0\} + 2$ be sufficiently large that

$$\int_{T(\epsilon)}^{\infty} e^{-\epsilon s} ds \leq e^{-2h} a^2.$$

Since $\mathcal{V}_{q_0, \chi, K}$ is a Vitali relation for λ_{q_1}, there is a covering of λ_{q_1}-almost all of $D(q_1, \chi/4)$ by pairwise disjoint sets from the relation of the form

$$V(g, t) = (F\pi(u), gD(q_1, \chi))$$

where $u \in W^{su}(q_1) \cap \mathcal{A}, F\pi(u) \in D(q_1, \chi/4)$, $g \in \mathrm{Mod}(S)$ and where $t \geq T(\epsilon)$ is such that $\Phi^t u \in gK$. By the inclusion (36), we have $u \in V$ and therefore from the assumption $T(\epsilon) \geq T_0$ and the estimate (35) we deduce that $D(\Phi^t u, 4\chi) \subset D(u, \chi/4) \subset D(q_1, \chi)$. On the other hand, we have $\Phi^t u \in gK$ and hence

$$(39) \qquad gD(q_1, \chi) \subset D(\Phi^t u, 4\chi) \subset D(q_1, \chi).$$

The total λ_{q_1}-mass of the balls from the covering is at least $\lambda_{q_1} D(q_1, \chi/4) \geq a$. Therefore there is a number $T > T(\epsilon) + 2$ such that the total volume of those balls $V(g, t)$ from the covering which correspond to a parameter $t \in [T - 2, T - 1]$ is at least $e^{-\epsilon T} e^{2h}/a$. Now the λ_{q_1}-volume of each such ball is at most $e^{-h(T-2)}/a$ and hence the number of these balls is at least $e^{(h-\epsilon)T}$.

Let $\{g_1, \ldots, g_k\} \subset \mathrm{Mod}(S)$ be the subset of $\mathrm{Mod}(S)$ defining these balls. By (39) above, for any i, j we have

$$g_j g_i D(q_1, \chi) \subset g_j D(q_1, \chi).$$

Thus the sets $g_i g_i D(q_1, \chi)$ $(i, j = 1, \ldots, k)$ are pairwise disjoint. Namely, the sets $g_j D(q_1, \chi)$ $(j = 1, \ldots, k)$ are pairwise disjoint, and for each j the sets $g_j g_i D(q_1, \chi) \subset g_j D(q_1, \chi)$ $(i = 1, \ldots, k)$ are pairwise disjoint as well. By induction, we conclude that for any two distinct words $w_1 = g_{i_1} \cdots g_{i_\ell}$ and $w_2 = g_{j_1} \cdots g_{j_m}$ in the letters $g_1, \ldots, g_k$, viewed as elements of $\mathrm{Mod}(S)$, the images of $D(q_1, \chi)$ under w_1, w_2 are either disjoint or properly contained in each other. This shows that the elements $g_1, \ldots, g_k$ generate a free semi-subgroup Λ of $\mathrm{Mod}(S)$.

Since $T(\epsilon) \geq n_0$, each word w of length $\ell \geq 1$ in the letters $g_1, \ldots, g_k$ defines a closed (n_0, ϵ_0)-pseudo-orbit $u_0, \ldots, u_\ell$ in $\mathcal{Q}^1(S)/\mathrm{Mod}(S)$ with $d(u_i, q_0) < \epsilon_0$. This pseudo-orbit consists of the successive projections to $\mathcal{Q}^1(S)/\mathrm{Mod}(S)$ of flow lines $\{\Phi^t u \mid t \in [0, \tau]\}$ where $u \in V \cap W^{su}(q_1) \cap \mathcal{A}$ and $\tau \in [T - 2, T - 1]$ are such that $F\pi(u) \in g_j D(q_1, \chi/4)$ for some $j \leq k$ and $\Phi^\tau u \in g_j K$. Thus by the second part of Lemma 4.4, if $\tilde{\gamma}$ is a lift to $\mathcal{Q}^1(S)$ of a characteristic arc of such a pseudo-orbit then $\Upsilon_{\mathcal{T}}(\tilde{\gamma})$ is a biinfinite unparametrized m-quasi-geodesic in $\mathcal{C}(S)$ which is invariant under the element of $\Lambda \subset \mathrm{Mod}(S)$ defined by w. In particular, this element is pseudo-Anosov, and its conjugacy class defines the characteristic free homotopy class of the closed pseudo-orbit.

The length of the periodic orbit of Φ^t determined by w does not exceed the length of a characteristic closed curve for the pseudo-orbit and hence its is not bigger than $T\ell$. Moreover, since by the choice of n_0 for any $s < t$ with the property that $\tilde{\gamma}(s), \tilde{\gamma}(t)$ project to distinct breakpoints of γ the distance between $\Upsilon_{\mathcal{T}}(\tilde{\gamma}(s)), \Upsilon_{\mathcal{T}}(\tilde{\gamma}(t))$ is at least $2c(m)$, it follows from Lemma 2.4 of [H10] that the unparametrized m-quasi-geodesic $\Upsilon_{\mathcal{T}}(\tilde{\gamma})$ is in fact a *parametrized p-quasi-geodesic* for some $p > m$. Using once more the first part of Theorem 2.2, this implies that the axis of the element of $\Lambda \subset \mathrm{Mod}(S)$ defined by w passes through a fixed compact neighborhood B of Pq_1 in $\mathcal{T}(S)$, and the projection of its unit tangent line to $\mathcal{Q}^1(S)/\mathrm{Mod}(S)$ is a periodic orbit for Φ^t which is contained in a compact subset C_0 of $\mathcal{Q}^1(S)/\mathrm{Mod}(S)$ not depending on w. If we denote by C the closed subset of C_0 of all points $z \in C_0$ whose orbit under Φ^t is entirely contained in C_0 then each of these orbits is contained in C.

The above argument does not immediately imply that the asymptotic growth rate of the number of periodic orbits in C is at least $h - \epsilon$. Namely, periodic orbits of the Teichmüller flow on $\mathcal{Q}^1(S)/\mathrm{Mod}(S)$ correspond to *conjugacy classes* of pseudo-Anosov elements in $\mathrm{Mod}(S)$. Thus if we want to count periodic orbits for Φ^t in $\mathcal{Q}^1(S)/\mathrm{Mod}(S)$ using the semi-subgroup Λ of $\mathrm{Mod}(S)$ constructed above, then we have to identify those elements of Λ which are conjugate in $\mathrm{Mod}(S)$.

For this recall that the axis of each element of the semi-subgroup Λ of $\mathrm{Mod}(S)$ passes through the fixed compact neighborhood B of Pq_1. Thus if γ, ζ is the axis of $v, w \in \Lambda$ and if v, w are conjugate in $\mathrm{Mod}(S)$ then there is some $b \in \mathrm{Mod}(S)$ with $w = b^{-1}vb$ and the following additional property. Let $\gamma[0, \tau]$ be a fundamental domain for the action of v on γ and such that $\gamma(0) \in B$. Such a fundamental

domain always exists, perhaps after a reparametrization of γ. Then there is some $t \in [0, \tau]$ such that $b^{-1}\gamma(t) \in B$.

As a consequence, the number of all elements $w \in \Lambda$ which are conjugate to a fixed element $v \in \Lambda$ is bounded from above by the number of elements $b \in \text{Mod}(S)$ with $bB \cap \gamma[0, \tau] \neq \emptyset$. In particular, if D is the diameter of B then this number does not exceed the cardinality of the set

$$(40) \qquad \{b \in \text{Mod}(S) \mid d_{\mathcal{T}}(bPq_1, \gamma[0, \tau]) \leq D\}.$$

However, this cardinality is bounded from above by a universal multiple of τ. Therefore there is a constant $c > 0$ such that for all sufficiently large $r > 0$ the number of periodic orbits of Φ^t contained in C of length at most r is not smaller than $e^{(h-\epsilon)r}/cr$. This completes the proof of the proposition. $\square$

Remarks:

1. Proposition 5.3 is equally valid, with identical proof, for the Teichmüller flow Φ^t on $\hat{\mathcal{Q}}(S)$. Together with Theorem 2 it implies that the metric entropy h of the unique Φ^t-invariant Lebesgue measure on $\hat{\mathcal{Q}}(S)$ in the Lebesgue measure class equals the supremum of the topological entropies of the restriction of Φ^t to compact invariant subsets of $\hat{\mathcal{Q}}(S)$. In [BG07], this fact was established for the Teichmüller flow on the moduli space of abelian differentials using symbolic dynamics.

2. The abundance of orbits of the Teichmüller flow which entirely remain in some compact set (depending on the orbit) was earlier established by Kleinbock and Weiss [KW04]. They show that this set is of full Hausdorff dimension.

References

[ABEM06] J. Athreya, A. Bufetov, A. Eskin, M. Mirzakhani, *Lattice point asymptotics and volume growth on Teichmüller space*, arXiv:math.DS/0610715.

[BH99] M. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*, Springer Grundlehren 319, Springer, Berlin 1999.

[Bu09] A. Bufetov, *Logarithmic asymptotics for the number of periodic orbits of the Teichmüller flow on Veech's space of zippered rectangles*, Mosc. Math. J. 9 (2009), no. 2, 245–261.

[BG07] A. Bufetov, B. Gurevich, *Existence and uniqueness of the measure of maximal entropy for the Teichmüller flow on the moduli space of abelian differentials*, arXiv:math.DS/0703020.

[EM08] A. Eskin, M. Mirzakhani, *Counting closed geodesics in moduli space*, arXiv:0811.2362.

[F69] H. Federer, *Geometric measure theory*, Springer Grundlehren, Berlin Heidelberg 1969.

[H06] U. Hamenstädt, *Train tracks and the Gromov boundary of the complex of curves*, in "Spaces of Kleinian groups" (Y. Minsky, M. Sakuma, C. Series, eds.), London Math. Soc. Lec. Notes 329 (2006), 187–207.

[H09] U. Hamenstädt, *Invariant Radon measures on measured lamination space*, Invent. Math. 176 (2009), 223–273.

[H10] U. Hamenstädt, *Stability of quasi-geodesics in Teichmüller space*, Geom. Dedicata (published online Nov. 2009).

[HK95]   B. Hasselblatt, A. Katok, *Introduction to the modern theory of dynamical systems*, Cambridge University Press, Cambridge 1995.

[Hu06]   J. Hubbard, *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 1. Teichmüller theory*. Matrix Editions, Ithaca, NY, 2006.

[Kl99]   E. Klarreich, *The boundary at infinity of the curve complex and the relative Teichmüller space*, unpublished manuscript, 1999.

[KW04]   D. Kleinbock, B. Weiss, *Bounded geodesics in moduli space*, Int. Math. Res. Not. 30 (2004), 1551–1560.

[M82]   H. Masur, *Interval exchange transformations and measured foliations*, Ann. Math. 115 (1982), 169-201.

[MM99]   H. Masur, Y. Minsky, *Geometry of the complex of curves I: Hyperbolicity*, Invent. Math. 138 (1999), 103-149.

[V86]   W. Veech, *The Teichmüller geodesic flow*, Ann. Math. 124 (1986), 441–530.

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