

DYNAMICS OF THE TEICHMÜLLER FLOW ON COMPACT INVARIANT SETS

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ABSTRACT. Let $\mathcal{Q}(S)$ be the moduli space of area one holomorphic quadratic differentials for an oriented surface S of genus $g \geq 0$ with $m \geq 0$ punctures and $3g - 3 + m \geq 2$. We show that for every compact subset K of $\mathcal{Q}(S)$ the asymptotic growth rate $\delta(K)$ of the number of periodic orbits of the Teichmüller flow Φ^t which are contained in K is not bigger than $h = 6g - 6 + 2m$, and $\sup_K \delta(K) = h$. Similarly, h is the supremum of the topological entropies of the restriction of Φ^t to compact invariant subsets of $\mathcal{Q}(S)$.

1. INTRODUCTION

Let S be an oriented surface of finite type, i.e. S is a closed surface of genus $g \geq 0$ from which $m \geq 0$ points, so-called *punctures*, have been deleted. We assume that $3g - 3 + m \geq 2$, i.e. that S is not a sphere with at most four punctures or a torus with at most one puncture. We then call the surface S *nonexceptional*. Since the Euler characteristic of S is negative, the *Teichmüller space* $\mathcal{T}(S)$ of S is the quotient of the space of all complete hyperbolic metrics on S of finite volume under the action of the group of diffeomorphisms of S which are isotopic to the identity. The fibre bundle $\mathcal{Q}^1(S)$ over $\mathcal{T}(S)$ of all *holomorphic quadratic differentials* of area one can naturally be viewed as the unit cotangent bundle of $\mathcal{T}(S)$ for the *Teichmüller metric*. The *Teichmüller geodesic flow* Φ^t on $\mathcal{Q}^1(S)$ commutes with the action of the *mapping class group* $\text{Mod}(S)$ of all isotopy classes of orientation preserving self-homeomorphisms of S . Thus this flow descends to a flow on the quotient $\mathcal{Q}(S) = \mathcal{Q}^1(S)/\text{Mod}(S)$, again denoted by Φ^t . Recall that $\mathcal{Q}(S)$ is a non-compact orbifold.

In his seminal paper [V86], Veech showed that the asymptotic growth rate of the number of periodic orbits of the Teichmüller flow Φ^t on $\mathcal{Q}(S)$ is at least $h = 6g - 6 + 2m$ (we use here a normalization for the Teichmüller flow which differs from the one used by Veech). Recently Eskin and Mirzakhani announced a sharp counting result: They show that as $r \rightarrow \infty$, the number of periodic orbits for Φ^t of period at most r is asymptotic to e^{hr}/hr . An earlier partial result for the Teichmüller flow on the space of abelian differentials is due to Bufetov [Bu06].

In this note we are interested in the growth of the number of periodic orbits of the Teichmüller flow which are entirely contained in a fixed compact set. Namely,

Date: January 27, 2007.

AMS subject classification: 37A35, 30F60

Research partially supported by DFG SFB 611.

for a number $r > 0$ and a compact subset K of $\mathcal{Q}(S)$ define $n_K(r)$ (or $n_K^\square(r)$) to be the cardinality of the set of all periodic orbits for Φ^t of period at most r which are entirely contained in K (or which intersect K). Clearly $n_K^\square(r) \geq n_K(r)$ for all r . We show.

Theorem 1. (1) $\lim_{r \rightarrow \infty} \frac{1}{r} \log n_K^\square(r) \leq h$ for every compact subset K of $\mathcal{Q}(S)$.
 (2) For every $\epsilon > 0$ there is a compact subset $K \subset \mathcal{Q}(S)$ such that

$$\liminf_{r \rightarrow \infty} \frac{1}{r} \log n_K(r) \geq h - \epsilon.$$

The first part of Theorem 1 also follows from the results of Eskin and Mirzakhani. However, the proof given here is very short and easy.

For every compact Φ^t -invariant subset K of $\mathcal{Q}(S)$ the *topological entropy* $h_{\text{top}}(K)$ of the restriction of the Teichmüller flow to K is defined. By the variational principle, this topological entropy equals the supremum of the metric entropies of all Φ^t -invariant Borel probability measures on K .

Bufetov and Gurevich [BG07] showed that if we consider the Teichmüller flow on the moduli space $\mathcal{A}(S)$ of area one *abelian* differentials, then the supremum of the metric entropies over all Φ^t -invariant probability measures on $\mathcal{A}(S)$ is just the metric entropy of an invariant probability measure in the Lebesgue measure class, moreover this Lebesgue measure is the unique measure of maximal entropy.

The moduli space $\mathcal{Q}(S)$ of quadratic differentials also admits a Φ^t -invariant probability measure in the Lebesgue measure class [M82, V86]. The metric entropy of this measure equals h . Thus our second result can be viewed as a weak analog of the result of Bufetov and Gurevich for the Teichmüller flow on $\mathcal{Q}(S)$.

Theorem 2. For every compact Φ^t -invariant subset K of $\mathcal{Q}(S)$ we have $h_{\text{top}}(K) \leq h$ and

$$h = \sup\{h_{\text{top}}(K) \mid K \subset \mathcal{Q}(S) \text{ compact}\}.$$

The main tool we use for the proof of Theorem 1 and Theorem 2 is the *curve graph* of S and the relation between its geometry and the geometry of Teichmüller space. In Section 2 we introduce this curve graph $\mathcal{C}(S)$ of S . We define a map $\Upsilon_{\mathcal{T}} : \mathcal{T}(S) \rightarrow \mathcal{C}(S)$ which is coarsely equivariant with respect to the action of the mapping class group. We characterize quasi-geodesics in $\mathcal{T}(S)$ which are mapped to quasi-geodesics in $\mathcal{C}(S)$ and use this to obtain information on the geometry of the *thick* part of Teichmüller space. In Section 3 we investigate the Teichmüller flow Φ^t on a suitably chosen finite branched cover $\hat{\mathcal{Q}}(S)$ of $\mathcal{Q}(S)$. We show that the restriction of Φ^t to any compact invariant subset of $\hat{\mathcal{Q}}(S)$ is expansive. In Section 4 we use the results from Section 2 and Section 3 to establish a version of the Anosov closing lemma for the restriction of the Teichmüller flow to compact invariant subsets of $\hat{\mathcal{Q}}(S)$. The proof of the two theorems above is contained in Section 5.

2. QUASI-GEODESICS IN TEICHMÜLLER SPACE WHICH PROJECT TO QUASI-GEODESICS IN THE COMPLEX OF CURVES

Let S be an oriented surface of genus $g \geq 0$ with $m \geq 0$ punctures and $3g-3+m \geq 2$. The *curve graph* $\mathcal{C}(S)$ of S is the graph whose vertices are the free homotopy classes of *essential* simple closed curves on S , i.e. simple closed curves which are neither contractible nor freely homotopic into a puncture. Two such curves are joined by an edge if and only if they can be realized disjointly. Since $3g-3+m \geq 2$ by assumption, $\mathcal{C}(S)$ is connected (see [MM99] for this result of Harer). However, the curve graph is locally infinite. Namely, for every simple closed curve α on S the surface $S - \alpha$ which we obtain by cutting S open along α contains at least one connected component which either is of Euler characteristic at most -2 or is a once-holed torus, and such a component contains infinitely many pairwise distinct free homotopy classes of simple closed curves which viewed as curves in S are disjoint from α . In the sequel we often do not distinguish between a simple closed curve on S and its free homotopy class. Also, if we write $\alpha \in \mathcal{C}(S)$ then we always mean that α is an essential simple closed curve, i.e. α is a vertex in the curve graph $\mathcal{C}(S)$.

Providing each edge in $\mathcal{C}(S)$ with the standard euclidean metric of diameter 1 equips the curve graph with the structure of a geodesic metric space. Since $\mathcal{C}(S)$ is not locally finite, this metric space $(\mathcal{C}(S), d)$ is not locally compact. Masur and Minsky [MM99] showed that nevertheless its geometry can be understood quite explicitly. Namely, $\mathcal{C}(S)$ is hyperbolic of infinite diameter (see also [Bw06, H07a] for alternative shorter proofs). The mapping class group naturally acts on $\mathcal{C}(S)$ as a group of simplicial isometries.

The goal of this section is to relate the geometry of the thick part of Teichmüller space to the geometry of the curve graph in a quantitative way. To achieve this goal, we first define a map from the Teichmüller space into the curve graph. Namely, recall that for every marked hyperbolic metric $h \in \mathcal{T}(S)$, every essential free homotopy class α on S can be represented by a closed geodesic which is unique up to parametrization. This geodesic is simple if the free homotopy class admits a simple representative. The *h -length* $\ell_h(\alpha)$ of the class α is defined to be the length of its geodesic representative. Equivalently, $\ell_h(\alpha)$ equals the minimum of the h -lengths of all closed curves representing the class α .

A *pants decomposition* for S is a collection of $3g-3+m$ pairwise disjoint simple closed essential curves on S which decompose S into $2g-2+m$ *pairs of pants*. Here by a pair of pants we mean a planar surface which is homeomorphic to a three-holed sphere. By a classical result of Bers (see [B92]), there is a number $\chi_0 > 0$ only depending on the topological type of S such that for every complete hyperbolic metric h on S of finite volume there is a pants decomposition for S consisting of simple closed curves of h -length at most χ_0 . Define a map

$$(1) \quad \Upsilon_{\mathcal{T}} : \mathcal{T}(S) \rightarrow \mathcal{C}(S)$$

by associating to a complete hyperbolic metric h on S of finite volume an essential simple closed curve $\Upsilon_{\mathcal{T}}(h) \in \mathcal{C}(S)$ whose h -length is at most χ_0 . Denote again by d the Teichmüller distance on $\mathcal{T}(S)$. We have (compare [H06]).

Lemma 2.1. *There is a number $L > 1$ such that*

- (1) $d(\Upsilon_{\mathcal{T}}(g), \Upsilon_{\mathcal{T}}(h)) \leq Ld(g, h) + L$ for all $g, h \in \mathcal{T}(S)$.
- (2) $d(\Upsilon_{\mathcal{T}}(\varphi g), \varphi \Upsilon_{\mathcal{T}}(g)) \leq L$ for all $g \in \mathcal{T}(S), \varphi \in \text{Mod}(S)$.

Proof. By a result of Wolpert (see [IT99] for a detailed discussion), for every essential simple closed curve α on S and all $g, h \in \mathcal{T}(S)$ the Teichmüller distance between g and h is at least $|\log \ell_g(\alpha) - \log \ell_h(\alpha)|$. This implies that there is a constant $a > 1$ such that for all $g, h \in \mathcal{T}(S)$ with $d(g, h) \leq 1$ and every $\alpha \in \mathcal{C}(S)$ with $\ell_g(\alpha) \leq \chi_0$ we have $\ell_h(\alpha) \leq a\chi_0$.

By the collar lemma for hyperbolic surfaces (see [B92]), for any metric $h \in \mathcal{T}(S)$ the number of intersection points between any two simple closed geodesics of length at most $a\chi_0$ is bounded from above by a universal constant. On the other hand, the distance between two curves $\alpha, \beta \in \mathcal{C}(S)$ is bounded from above by the minimal number of intersection points between any representatives of α, β plus one [MM99, Bw06]. Thus the diameter in $\mathcal{C}(S)$ of the set of all simple closed curves of h -length at most $a\chi_0$ is bounded from above by a universal constant $L > 0$ not depending on h . Together with the consideration in the beginning of this proof, this shows that $d(\Upsilon_{\mathcal{T}}(g), \Upsilon_{\mathcal{T}}(h)) \leq L$ for all $g, h \in \mathcal{T}(S)$ with $d(g, h) \leq 1$. Since the Teichmüller metric is a geodesic metric, i.e. any two point in $\mathcal{T}(S)$ can be connected by a minimal geodesic, the first part of the lemma follows by induction on the distance in $\mathcal{T}(S)$.

The second part of the lemma is derived in the same way. Namely, by the properties of the action of $\text{Mod}(S)$ on $\mathcal{T}(S)$ and by the definition of the map $\Upsilon_{\mathcal{T}}$, for every $g \in \mathcal{T}(S)$ and every $\varphi \in \text{Mod}(S)$ the φg -lengths of the curves $\Upsilon_{\mathcal{T}}(\varphi g)$ and $\varphi \Upsilon_{\mathcal{T}}(g)$ are at most χ_0 . Thus as above, the distance between these curves in $\mathcal{C}(S)$ does not exceed L . \square

Let $J \subset \mathbb{R}$ be a closed connected subset, i.e. either J is a closed interval or a closed ray or the whole line. For some $p > 1$, a map $\gamma : J \rightarrow \mathcal{C}(S)$ is called a *p-quasi-geodesic* if for all $s, t \in J$ we have

$$(2) \quad d(\gamma(s), \gamma(t))/p - p \leq |s - t| \leq pd(\gamma(s), \gamma(t)) + p.$$

A map $\gamma : J \rightarrow \mathcal{C}(S)$ is called an *unparametrized p-quasi-geodesic* if there is a closed connected set $I \subset \mathbb{R}$ and a homeomorphism $\zeta : I \rightarrow J$ such that $\gamma \circ \zeta : I \rightarrow \mathcal{C}(S)$ is a p -quasi-geodesic. By a result of Masur and Minsky (Theorem 2.6 and Theorem 2.3 of [MM99]; the precise quantitative version which we will use is Theorem 4.1 of [H07a]), there is a number $p > 1$ such that the image under $\Upsilon_{\mathcal{T}}$ of every Teichmüller geodesic (i.e. every geodesic in $\mathcal{T}(S)$ with respect to the Teichmüller metric) is an unparametrized p -quasi-geodesic. However, in general it is not a quasi-geodesic with its proper parametrization.

For $\epsilon > 0$ let $\mathcal{T}(S)_{\epsilon}$ be the closed subset of $\mathcal{T}(S)$ of all hyperbolic metrics h for which the length of the shortest closed h -geodesic is at least ϵ . Informally we think of $\mathcal{T}(S)_{\epsilon}$ as the ϵ -thick part of Teichmüller space. The mapping class group preserves the set $\mathcal{T}(S)_{\epsilon}$ and acts on it cocompactly. Moreover, every $\text{Mod}(S)$ -invariant subset of $\mathcal{T}(S)$ on which $\text{Mod}(S)$ acts cocompactly is contained in $\mathcal{T}(S)_{\epsilon}$ for some $\epsilon > 0$.

Recall that the *Hausdorff distance* between two subsets A, B of a metric space X is the infimum of all numbers $r > 0$ such that A is contained in the r -neighborhood of B and B is contained in the r -neighborhood of A . It will be convenient to allow that the sets A, B are not necessarily closed and that the metric space X is unbounded. In the sequel we use the following terminology.

Definition 2.2. For $\epsilon > 0$, a *quasi-convex curve* in $\mathcal{T}(S)_\epsilon$ is a subset of $\mathcal{T}(S)$ whose Hausdorff distance to the image of a geodesic arc $\zeta : J \rightarrow \mathcal{T}(S)_\epsilon$ is at most $1/\epsilon$.

Note that a quasi-convex curve as defined above is not required to be a curve, i.e. the image of an interval in \mathbb{R} under a continuous map.

The main goal of this section is to show the following result of independent interest.

Theorem 2.3. (1) *For every $\nu > 1$ there is a constant $\epsilon = \epsilon(\nu) > 0$ with the following property. Let $J \subset \mathbb{R}$ be a closed connected set of diameter at least $1/\epsilon$ and let $\gamma : J \rightarrow \mathcal{T}(S)$ be a ν -quasi-geodesic. If $\Upsilon_{\mathcal{T}} \circ \gamma$ is a ν -quasi-geodesic in $\mathcal{C}(S)$ then $\gamma(J)$ is a quasi-convex curve in $\mathcal{T}(S)_\epsilon$.*
 (2) *For every $\epsilon > 0$ there is a constant $\nu(\epsilon) > 1$ with the following property. Let $\gamma : J \rightarrow \mathcal{T}(S)$ be a $1/\epsilon$ -quasi-geodesic in $\mathcal{T}(S)$ whose image $\gamma(J)$ is a quasi-convex curve in $\mathcal{T}(S)_\epsilon$; then $\Upsilon_{\mathcal{T}} \circ \gamma$ is a $\nu(\epsilon)$ -quasi-geodesic in $\mathcal{C}(S)$.*

For Teichmüller *geodesics*, the second part of the above theorem is due to Masur and Minsky [MM00].

We begin with establishing the (easier) second part of Theorem 2.3. For this we need the following simple no-retraction lemma for quasi-geodesics in the hyperbolic geodesic metric space $\mathcal{C}(S)$.

Lemma 2.4. *For $p > 1$ there is a constant $c = c(p) > 0$ with the following property. Let $\gamma : J \rightarrow \mathcal{C}(S)$ be any unparametrized p -quasi-geodesic; if $t_1 < t_2 < t_3 \in J$ then $d(\gamma(t_1), \gamma(t_3)) \geq d(\gamma(t_1), \gamma(t_2)) + d(\gamma(t_2), \gamma(t_3)) - c$.*

Proof. Let $p > 1$; by the definition of an unparametrized p -quasi-geodesic, it is enough to show the existence of a number $c > 0$ such that for every (parametrized) p -quasi-geodesic $\gamma : [0, n] \rightarrow \mathcal{C}(S)$ and all $0 < t < n$ we have $d(\gamma(0), \gamma(n)) \geq d(\gamma(0), \gamma(t)) + d(\gamma(t), \gamma(n)) - c$.

Since $\mathcal{C}(S)$ is a hyperbolic geodesic metric space, there is a constant $R > 0$ only depending on p with the following property. Let $n > 0$, let $\gamma : [0, n] \rightarrow \mathcal{C}(S)$ be any p -quasi-geodesic and let $\zeta : [0, m] \rightarrow \mathcal{C}(S)$ be a geodesic connecting $\gamma(0)$ to $\gamma(n)$. Then the Hausdorff distance between $\gamma[0, n]$ and $\zeta[0, m]$ is at most $R/2$. In particular, for every $t \in (0, n)$ there is a point $s \in (0, m)$ such that $d(\gamma(t), \zeta(s)) \leq R$. Thus we have $d(\gamma(0), \gamma(t)) + d(\gamma(t), \gamma(n)) \leq d(\zeta(0), \zeta(s)) + d(\zeta(s), \zeta(m)) + 2R = d(\gamma(0), \gamma(n)) + 2R$ which shows the lemma. \square

A *geodesic lamination* for a complete hyperbolic structure on S of finite volume is a *compact* subset of S which is foliated into simple geodesics. A geodesic lamination λ on S is called *minimal* if each of its half-leaves is dense in λ . Thus a simple closed

geodesic is a minimal geodesic lamination. A minimal geodesic lamination with more than one leaf has uncountably many leaves and is called *minimal arational*. A geodesic lamination λ is said to *fill up* S if every simple closed geodesic on S intersects λ transversely. This is equivalent to stating that the complementary components of λ are all topological discs or once punctured topological discs.

A *measured geodesic lamination* is a geodesic lamination λ together with a translation invariant transverse measure. Such a measure assigns a positive weight to each compact arc in S which intersects λ nontrivially and transversely and whose endpoints are contained in complementary regions of λ . The geodesic lamination λ is called the *support* of the measured geodesic lamination; it consists of a disjoint union of minimal components. Vice versa, every minimal geodesic lamination is the support of a measured geodesic lamination. The space \mathcal{ML} of measured geodesic laminations on S can be equipped with the weak*-topology. Its projectivization \mathcal{PML} is called the space of *projective measured geodesic laminations* and is homeomorphic to the sphere $S^{6g-7+2m}$. There is a continuous symmetric pairing $\iota : \mathcal{ML} \times \mathcal{ML} \rightarrow (0, \infty)$, the so-called *intersection form*, which satisfies $\iota(a\xi, b\eta) = ab\iota(\xi, \eta)$ for all $a, b \geq 0$ and all $\xi, \eta \in \mathcal{ML}$. By the Hubbard Masur theorem (see [Hu06]), for every $h \in \mathcal{T}(S)$ the space \mathcal{PML} of projective measured geodesic laminations can naturally be identified with the projectivized cotangent space of $\mathcal{T}(S)$ at h .

Since $\mathcal{C}(S)$ is a hyperbolic geodesic metric space, it admits a *Gromov boundary* $\partial\mathcal{C}(S)$ which is a (non-compact) metrizable topological space equipped with an action of $\text{Mod}(S)$ by homeomorphisms (see [BH99] for the definition of the Gromov boundary of a hyperbolic geodesic metric space and for references). Following Klarreich [Kl99] (see also [H06]), this boundary can naturally be identified with the space of all (unmeasured) minimal geodesic laminations which fill up S , equipped with the topology which is induced from the weak*-topology on \mathcal{PML} via the measure forgetting map.

The cotangent vector at $\gamma(0)$ of a Teichmüller geodesic $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)$ (which we always assume to be parametrized by arc length without further mentioning) is a holomorphic quadratic differential $q \in \mathcal{Q}^1(S)$ of area one for the Riemann surface $\gamma(0)$. The quadratic differential q corresponds to a pair (q_v, q_h) of measured geodesic laminations which satisfy $i(q_h, q_v) = 1$ and *jointly fill up* S . This means that $i(c, q_h) + i(c, q_v) > 0$ for every simple closed curve c on S . The measured geodesic lamination q_v is called the *vertical* measured geodesic lamination of q , and q_h is called *horizontal*. For every $t \in \mathbb{R}$ the unit cotangent vector of γ at $\gamma(t)$ is the quadratic differential $\Phi^t q$ defined by the pair $(e^t q_v, e^{-t} q_h)$. If the support of the vertical measured geodesic lamination q_v of q is minimal and fills up S then the unparametrized p -quasi-geodesic $t \rightarrow \Upsilon_{\mathcal{T}}(\gamma(t))$ in $\mathcal{C}(S)$ is of infinite diameter and converges to the support of q_v , viewed as a point in $\partial\mathcal{C}(S)$ [Kl99, H06].

The following lemma is implicitly contained in the work of Masur and Minsky [MM00]. Since we did not find a precise quantitative statement in the literature, we give a proof.

Lemma 2.5. *For every $\epsilon > 0$ there is a number $\nu_0 = \nu_0(\epsilon) > 0$ with the following property. Let $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$ be a Teichmüller geodesic; then the curve $\Upsilon_{\mathcal{T}} \circ \gamma : J \rightarrow \mathcal{C}(S)$ is a ν_0 -quasi-geodesic.*

Proof. Let $p > 1$ be such that the image under $\Upsilon_{\mathcal{T}}$ of every Teichmüller geodesic is an unparametrized p -quasi-geodesic in $\mathcal{C}(S)$; such a number exists by the results of Masur and Minsky [MM99, H07a]. Let $c = c(p) > 0$ be as in Lemma 2.4.

We claim that for every $\epsilon > 0$ there is a constant $k_0 = k_0(\epsilon) > 0$ with the following property. Let $k \geq k_0$ and let $\gamma : [0, k] \rightarrow \mathcal{T}(S)_\epsilon$ be a geodesic arc of length at least k_0 ; then $d(\Upsilon_{\mathcal{T}}(\gamma(0)), \Upsilon_{\mathcal{T}}(\gamma(k))) \geq 2c$.

To see that this is the case, we argue by contradiction and we assume otherwise. By Lemma 2.4 there is then a number $\epsilon > 0$ and for every $k > 0$ there is a geodesic arc $\gamma_k : [0, k] \rightarrow \mathcal{T}(S)_\epsilon$ such that $d(\Upsilon_{\mathcal{T}}\gamma_k(0), \Upsilon_{\mathcal{T}}\gamma_k(t)) \leq 3c$ for every $t \in [0, k]$.

Let $L > 0$ be as in Lemma 2.1. The action of $\text{Mod}(S)$ on $\mathcal{T}(S)_\epsilon$ is isometric and cocompact. Thus by the second part of Lemma 2.1, via replacing the constant $3c$ by $3c + 2L$ we may assume that the initial points $\gamma_k(0)$ ($k \geq 1$) of the geodesic arcs γ_k are contained in a fixed compact subset of $\mathcal{T}(S)_\epsilon$. Then the unit cotangent vectors $q_k \in \mathcal{Q}^1(S)$ of the geodesics γ_k at $\gamma_k(0)$ are contained in a compact subset K of $\mathcal{Q}^1(S)$. Consequently, by passing to a subsequence we may assume that the quadratic differentials q_k converge as $k \rightarrow \infty$ to a quadratic differential $q \in K$. Then the geodesic arcs γ_k converge locally uniformly as $k \rightarrow \infty$ to the geodesic ray $\gamma : [0, \infty) \rightarrow \mathcal{T}(S)$ with unit cotangent vector q at $\gamma(0)$. Since $\mathcal{T}(S)_\epsilon \subset \mathcal{T}(S)$ is closed and $\gamma_k[0, k] \subset \mathcal{T}(S)_\epsilon$ for all k , we have $\gamma[0, \infty) \subset \mathcal{T}(S)_\epsilon$. Moreover, the first part of Lemma 2.1 shows that $d(\Upsilon_{\mathcal{T}}\gamma(s), \Upsilon_{\mathcal{T}}\gamma(0)) \leq 3c + 4L$ for all $s \geq 0$.

Let $q_v \in \mathcal{ML}$ be the vertical measured geodesic lamination of the quadratic differential q which defines the geodesic γ . Since $\gamma[0, \infty) \subset \mathcal{T}(S)_\epsilon$, the geodesic γ projects into a compact subset of moduli space $\mathcal{T}(S)/\text{Mod}(S)$. Thus by a result of Masur [M82], the support of the measured geodesic lamination q_v fills up S . Then the curve $\Upsilon_{\mathcal{T}} \circ \gamma$ is an unparametrized quasi-geodesic in $\mathcal{C}(S)$ of *infinite* diameter (see [Kl99, H06]) which is a contradiction and shows for every $\epsilon > 0$ the existence of a constant $k_0 = k_0(\epsilon) > 0$ as claimed.

Let $\epsilon > 0$, let $k_0 = k_0(\epsilon)$, let $n > 0$ and let $\gamma : [0, k_0 n] \rightarrow \mathcal{T}(S)_\epsilon$ be any Teichmüller geodesic. The image under $\Upsilon_{\mathcal{T}}$ of every Teichmüller geodesic in $\mathcal{T}(S)$ is an unparametrized p -quasi-geodesic. Thus by the choice of c , for all $s \leq t \leq u$ we have

$$(3) \quad d(\Upsilon_{\mathcal{T}}\gamma(u), \Upsilon_{\mathcal{T}}\gamma(s)) \geq d(\Upsilon_{\mathcal{T}}\gamma(u), \Upsilon_{\mathcal{T}}\gamma(t)) + d(\Upsilon_{\mathcal{T}}\gamma(t), \Upsilon_{\mathcal{T}}\gamma(s)) - c.$$

On the other hand, for every integer $\ell < n$ we have $d(\Upsilon_{\mathcal{T}}\gamma(\ell k_0), \Upsilon_{\mathcal{T}}\gamma((\ell + 1)k_0)) \geq 2c$ by the choice of k_0 , and therefore by Lemma 2.4,

$$(4) \quad d(\Upsilon_{\mathcal{T}}\gamma((\ell + 1)k_0), \Upsilon_{\mathcal{T}}\gamma(s)) \geq d(\Upsilon_{\mathcal{T}}\gamma(\ell k_0), \Upsilon_{\mathcal{T}}\gamma(s)) + c \text{ for all } s \leq \ell k_0.$$

Inductively we deduce that $d(\Upsilon_{\mathcal{T}}\gamma(\ell k_0), \Upsilon_{\mathcal{T}}\gamma(m k_0)) \geq c|\ell - m|$ for all integers $\ell, m \leq n$. By the first part of Lemma 2.1, the map $\Upsilon_{\mathcal{T}} : \mathcal{T}(S) \rightarrow \mathcal{C}(S)$ is coarsely Lipschitz. It follows that $\Upsilon_{\mathcal{T}} \circ \gamma$ is a ν_0 -quasi-geodesic for a constant $\nu_0 > 0$ only depending on

ϵ (more precisely, we have $c|s-t|/k_0 - k_0L - L \leq d(\Upsilon_{\mathcal{T}}\gamma(s), \Upsilon_{\mathcal{T}}\gamma(t)) \leq L|s-t| + L$ for all $s, t \in [0, k_0n]$). This shows the lemma. \square

The following corollary shows the second part of Theorem 2.3.

Corollary 2.6. *For every $\epsilon > 0$ there is a number $\nu = \nu(\epsilon) > 1$ with the following property. Let $\gamma : J \rightarrow \mathcal{T}(S)$ be a $1/\epsilon$ -quasi-geodesic such that $\gamma(J)$ is a quasi-convex curve in $\mathcal{T}(S)_\epsilon$; then $\Upsilon_{\mathcal{T}} \circ \gamma : J \rightarrow \mathcal{C}(S)$ is a ν -quasi-geodesic.*

Proof. Let $\epsilon > 0$ and let $\gamma : J \rightarrow \mathcal{T}(S)$ be a $1/\epsilon$ -quasi-geodesic such that $\gamma(J)$ is a quasi-convex curve in $\mathcal{T}(S)_\epsilon$. Then there is a Teichmüller geodesic $\zeta : I \rightarrow \mathcal{T}(S)_\epsilon$ and a map $\rho : J \rightarrow I$ such that $d(\gamma(t), \zeta(\rho(t))) \leq 1/\epsilon$ for all t . Since by assumption the map γ is a $1/\epsilon$ -quasi-geodesic in $\mathcal{T}(S)$ and since ζ realizes the distance between any of its points, the map ρ is necessarily a b -quasi-isometry for a constant $b > 1$ only depending on ϵ .

By Lemma 2.5, the curve $\Upsilon_{\mathcal{T}} \circ \zeta : I \rightarrow \mathcal{C}(S)$ is a ν_0 -quasi-geodesic for a constant $\nu_0 > 0$ only depending on ϵ . Then the map $t \in J \rightarrow \zeta(\rho(t)) \in \mathcal{C}(S)$ is the composition of a b -quasi-isometry with a ν_0 -quasi-isometric embedding and hence it is a \tilde{b} -quasi-geodesic for a constant $\tilde{b} > 1$ only depending on ϵ . On the other hand, by the first part of Lemma 2.1 the map $\Upsilon_{\mathcal{T}}$ is coarsely Lipschitz and therefore the distances $d(\Upsilon_{\mathcal{T}}\gamma(t), \Upsilon_{\mathcal{T}}(\zeta \circ \rho(t)))$ are bounded from above by a constant only depending on ϵ . This shows that $\Upsilon_{\mathcal{T}} \circ \gamma$ is a ν -quasi-geodesic for a constant $\nu > 0$ only depending on ϵ . \square

To show the first part of Theorem 2.3 we begin again with a simple observation.

Lemma 2.7. *For every $\nu > 1$ there is a number $\epsilon_0 = \epsilon_0(\nu) > 0$ with the following properties. Let $\gamma : [0, n] \rightarrow \mathcal{T}(S)$ be a ν -quasi-geodesic whose image $\Upsilon_{\mathcal{T}} \circ \gamma$ in $\mathcal{C}(S)$ is a ν -quasi-geodesic. If $n \geq 1/\epsilon_0$ then $\gamma[0, n] \subset \mathcal{T}(S)_{\epsilon_0}$.*

Proof. Let $n > 0, \nu > 1$ and let $\gamma : [0, n] \rightarrow \mathcal{T}(S)$ be a ν -quasi-geodesic such that $\Upsilon_{\mathcal{T}} \circ \gamma$ is a ν -quasi-geodesic in $\mathcal{C}(S)$. Then we have

$$(5) \quad d(\Upsilon_{\mathcal{T}}(\gamma(t)), \Upsilon_{\mathcal{T}}(\gamma(s))) \geq |s-t|/\nu - \nu \text{ for all } s, t \in [0, n].$$

Let $R > 0$ be an upper bound for the diameter in $\mathcal{C}(S)$ of the collection of all simple closed curves on S whose length with respect to some metric $h \in \mathcal{T}(S)$ is at most χ_0 where $\chi_0 > 0$ is as before a fixed Bers constant for S . Let $[a, b] \subset [0, n]$ be an interval for which there is a simple closed curve $\alpha \in \mathcal{C}(S)$ so that $\ell_{\gamma(t)}(\alpha) \leq \chi_0$ for all $t \in [a, b]$ (here as before, $\ell_{\gamma(t)}(\alpha)$ is the $\gamma(t)$ -length of α). Then we have $d(\Upsilon_{\mathcal{T}}(\gamma(a)), \alpha) \leq R, d(\Upsilon_{\mathcal{T}}(\gamma(b)), \alpha) \leq R$ and therefore

$$(6) \quad |b-a| \leq 2\nu R + \nu^2.$$

Now by a result of Wolpert (see [IT99]), for all $\alpha \in \mathcal{C}(S)$ and all $h, h' \in \mathcal{T}(S)$ the distance between h and h' is at least $|\log \ell_h(\alpha) - \log \ell_{h'}(\alpha)|$. Thus if there is a point $t \in [0, n]$ and some $\alpha \in \mathcal{C}(S)$ with $\log(\ell_{\gamma(t)}(\alpha)) \leq \log(\chi_0) - 2\nu R - 2\nu^2$ then the $\gamma(s)$ -length of α is does not exceed χ_0 for every $s \in [0, n]$ with $|s-t| \leq 2\nu R + 2\nu^2$.

Consequently inequality (6) shows that $\Upsilon_{\mathcal{T}} \circ \gamma$ is *not* a ν -quasi-geodesic provided that $n \geq 4\nu R + 4\nu^2$. \square

The following lemma is the first part of Theorem 2.3. Its proof uses some arguments from Section 3.9 of [Mo03].

Lemma 2.8. *For every $\nu > 1$ there is a constant $\epsilon = \epsilon(\nu) > 0$ with the following property. Let $J \subset \mathbb{R}$ be a closed connected subset of diameter at least $1/\epsilon$ and let $\gamma : J \rightarrow \mathcal{T}(S)$ be a ν -quasi-geodesic such that $\Upsilon_{\mathcal{T}} \circ \gamma : J \rightarrow \mathcal{C}(S)$ is a ν -quasi-geodesic in $\mathcal{C}(S)$; then $\gamma(J)$ is a quasi-convex curve in $\mathcal{T}(S)_{\epsilon}$.*

Proof. For $\nu > 1$ define a ν -Lipschitz curve in $\mathcal{T}(S)$ to be a ν -Lipschitz map $\gamma : J \rightarrow \mathcal{T}(S)$ with respect to the standard metric on \mathbb{R} and the Teichmüller metric on $\mathcal{T}(S)$. Since $\mathcal{T}(S)$ is a smooth manifold and the Teichmüller metric is geodesic, every ν -quasi-geodesic $\gamma : J \rightarrow \mathcal{T}(S)$ can be replaced by a piecewise geodesic $\zeta : J \rightarrow \mathcal{T}(S)$ which is a 2ν -Lipschitz curve and which satisfies $d(\gamma(t), \zeta(t)) \leq 2\nu$ for all $t \in J$. Thus by the first part of Lemma 2.1 and via a change of notation, it is enough to show the statement of the lemma for ν -Lipschitz curves $\gamma : J \rightarrow \mathcal{T}(S)$ which are ν -quasi-geodesics and such that $\Upsilon_{\mathcal{T}} \circ \gamma$ is a ν -quasi-geodesic in $\mathcal{C}(S)$. Let $\epsilon_0 = \epsilon_0(\nu)$ is as in Lemma 2.7. In the sequel we also assume that the diameter $|J|$ of the set J is bigger than $\max\{1/\epsilon_0, \nu(\nu + 3)\}$; then $\gamma(J) \subset \mathcal{T}(S)_{\epsilon_0}$, and the distance in $\mathcal{C}(S)$ between the endpoints of $\Upsilon_{\mathcal{T}} \circ \gamma$ is at least 3.

The curve graph $\mathcal{C}(S)$ is hyperbolic, and $\Upsilon_{\mathcal{T}} \circ \gamma : J \rightarrow \mathcal{C}(S)$ is a ν -quasi-geodesic by assumption. Thus if J is not bounded from above (or below) then the points $\Upsilon_{\mathcal{T}}(\gamma(t))$ converge as $t \rightarrow \infty$ (or as $t \rightarrow -\infty$) to a point in the Gromov boundary $\partial\mathcal{C}(S)$ of $\mathcal{C}(S)$. We call this point the *right endpoint* (or the *left endpoint*) of $\Upsilon_{\mathcal{T}} \circ \gamma$.

A simple closed curve $\alpha \in \mathcal{C}(S)$ supports a unique projective measured geodesic lamination which we denote by $[\alpha]$. Similarly, for a measured geodesic lamination $\lambda \in \mathcal{ML}$ we denote by $[\lambda]$ the projective class of λ . Following Mosher [Mo03], we say that the projective measured geodesic lamination $[\alpha]$ defined by a simple closed curve $\alpha \in \mathcal{C}(S)$ is *realized* at some $t \in J$ if the length of α with respect to the metric $\gamma(t) \in \mathcal{T}(S)$ is at most χ_0 where as before, χ_0 is a Bers constant for S . Note that the number of projective measured geodesic laminations which are realized at a given point $t \in J$ is uniformly bounded and that $[\Upsilon_{\mathcal{T}}(\gamma(t))]$ is realized at $\gamma(t)$. Similarly, we say that the projectivization $[\lambda]$ of a measured geodesic lamination λ is realized at an infinite “endpoint” of J if the support of λ equals the corresponding right or left endpoint of the quasi-geodesic $\Upsilon_{\mathcal{T}} \circ \gamma : J \rightarrow \mathcal{C}(S)$ in the Gromov boundary $\partial\mathcal{C}(S)$ of $\mathcal{C}(S)$, viewed as a minimal geodesic lamination. The set of projective measured geodesic laminations which are realized at an infinite endpoint of J is a nonempty closed subset of \mathcal{PML} [Kl99, H06]. We call a projective measured geodesic lamination which is realized at a (finite or infinite) endpoint of J an *endpoint lamination*.

The assignment $t \rightarrow \Upsilon_{\mathcal{T}}(\gamma(t))$ is a ν -quasi-geodesic in $\mathcal{C}(S)$ by assumption, and the diameter in $\mathcal{C}(S)$ of the set of all curves of length at most χ_0 with respect to some fixed hyperbolic metric $h \in \mathcal{T}(S)$ is bounded from above by a universal constant. Any two curves $\alpha, \beta \in \mathcal{C}(S)$ with $d(\alpha, \beta) \geq 3$ *jointly fill up* S , i.e. are

such that every simple closed essential curve $\zeta \in \mathcal{C}(S)$ intersects either α or β transversely (see [MM99, Bw06]). Hence by the assumption on the lower bound for the diameter of the parameter interval J , any two projective measured geodesic laminations $[\alpha], [\beta]$ which are realized at the two distinct endpoints of J jointly fill up S . This is also valid if J is unbounded since in this case an endpoint lamination $[\lambda]$ at an infinite endpoint of J jointly fills up S with every projective measured geodesic lamination whose support does not coincide with the support of $[\lambda]$.

Any pair of distinct points $[\lambda] \neq [\mu] \in \mathcal{PML}$ which jointly fill up the surface S define a unique Teichmüller geodesic whose cotangent line consists of area one quadratic differentials with vertical measured geodesic lamination contained in the class $[\lambda]$ and with horizontal measured geodesic lamination contained in the class $[\mu]$. Therefore, for every ν -quasi-geodesic $\zeta : J \rightarrow \mathcal{T}(S)$ with $|J| \geq 1/\epsilon_0$ such that $\Upsilon_{\mathcal{T}} \circ \zeta$ is a ν -quasi-geodesic in $\mathcal{C}(S)$, any pair of projective measured geodesic laminations $[\lambda], [\mu]$ realized at the two (possibly infinite) endpoints of J defines a unique Teichmüller geodesic $\eta([\lambda], [\mu])$.

Choose a smooth function $\sigma : [0, \infty) \rightarrow [0, 1]$ with $\sigma[0, \chi_0] \equiv 1$ and $\sigma[2\chi_0, \infty) \equiv 0$. For every $h \in \mathcal{T}(S)$ we obtain a finite Borel measure μ_h on $\mathcal{C}(S)$ by defining

$$(7) \quad \mu_h = \sum_{\beta} \sigma(\ell_h(\beta)) \delta_{\beta}$$

where δ_{β} denotes the Dirac mass at β . The total mass of μ_h is bounded from above and below by a universal positive constant, and the diameter of the support of μ_h in $\mathcal{C}(S)$ is uniformly bounded as well. The measures μ_h are equivariant with respect to the action of the mapping class group on $\mathcal{T}(S)$ and $\mathcal{C}(S)$, and they depend continuously on $h \in \mathcal{T}(S)$ in the weak*-topology. This means that for every bounded function $f : \mathcal{C}(S) \rightarrow \mathbb{R}$ the function $h \rightarrow \int f d\mu_h$ is continuous.

Define a symmetric non-negative function ρ on $\mathcal{T}(S) \times \mathcal{T}(S)$ by

$$(8) \quad \rho(h, h') = \int_{\mathcal{C}(S) \times \mathcal{C}(S)} d(\cdot, \cdot) d\mu_h \times d\mu_{h'} / \mu_h(\mathcal{C}(S)) \mu_{h'}(\mathcal{C}(S)).$$

Clearly the function ρ is continuous and invariant under the diagonal action of $\text{Mod}(S)$. Moreover, there is a universal constant $a > 0$ such that

$$(9) \quad \rho(h, h')/a - a \leq d(\Upsilon_{\mathcal{T}}(h), \Upsilon_{\mathcal{T}}(h')) \leq a\rho(h, h') + a \text{ for all } h, h' \in \mathcal{T}(S).$$

As a consequence, for every $\nu > 1$ there is a constant $p = p(\nu) > 1$ with the following property. If $\gamma : J \rightarrow \mathcal{T}(S)$ is such that $\Upsilon_{\mathcal{T}} \circ \gamma$ is a ν -quasi-geodesic, then γ is a p -quasi-geodesic with respect to the function ρ . By this we mean that

$$(10) \quad \rho(\gamma(s), \gamma(t))/p - p \leq |s - t| \leq p\rho(\gamma(s), \gamma(t)) + p \text{ for all } s, t \in J.$$

Vice versa, for every $p > 1$ there is a constant $\nu = \nu(p) > 1$ such that if $\gamma : J \rightarrow \mathcal{T}(S)$ is a Lipschitz curve which is a p -quasi-geodesic with respect to ρ , then $\Upsilon_{\mathcal{T}} \circ \gamma$ is a ν -quasi-geodesic in $\mathcal{C}(S)$.

Let $h \in \mathcal{T}(S)$ and let $\mu \in \mathcal{ML}$ be a measured geodesic lamination. The product of the transverse measure for μ together with the length element of h defines a measure on the support of μ whose total mass is called the h -length of μ ; we denote it by $\ell_h(\mu)$. The function $(h, \mu) \in \mathcal{T}(S) \times \mathcal{ML} \rightarrow (0, \infty)$ is continuous. Following

Mosher [Mo03], for $p > 1$ define Γ_p to be the set of all triples $(\gamma : J \rightarrow \mathcal{T}(S), \lambda_+, \lambda_-)$ with the following properties.

- (1) $0 \in J$ and the diameter of J is at least $\max\{1/\epsilon_0, \nu(p)(\nu(p) + 3)\}$ where $\epsilon_0 = \epsilon_0(\nu(p))$ is as in Lemma 2.7.
- (2) $\gamma : J \rightarrow \mathcal{T}(S)$ is a p -Lipschitz curve which is a p -quasi-geodesic with respect to the function ρ .
- (3) $\lambda_+, \lambda_- \in \mathcal{ML}$ are measured geodesic laminations of $\gamma(0)$ -length 1, and the projective measured geodesic lamination $[\lambda_+]$ is realized at the right endpoint of J , the projective measured geodesic lamination $[\lambda_-]$ is realized at the left endpoint of J .

We equip Γ_p with the product topology, using the weak*-topology on \mathcal{ML} for the second and the third component of the triple and the compact-open topology for the arc $\gamma : J \rightarrow \mathcal{T}(S)$. Note that this topology is metrizable.

We follow Mosher (Proposition 3.17 of [Mo03]) and show that the action of $\text{Mod}(S)$ on Γ_p is cocompact. Namely, recall from Lemma 2.7 that for every $(\gamma : J \rightarrow \mathcal{T}(S), \lambda_+, \lambda_-) \in \Gamma_p$ the image $\gamma(J)$ is contained in $\mathcal{T}(S)_{\epsilon_0}$ where $\epsilon_0 > 0$ is as in the first part of the definition of Γ_p . Since $\text{Mod}(S)$ acts isometrically and cocompactly on $\mathcal{T}(S)_{\epsilon_0}$ and since the function ρ is invariant under the diagonal action of $\text{Mod}(S)$, it is enough to show that the subset of Γ_p consisting of triples with the additional property that $\gamma(0)$ is contained in a fixed compact subset A of $\mathcal{T}(S)_{\epsilon_0}$ is compact. Now the topology on Γ_p is metrizable and hence this follows if every sequence in Γ_p contained in the subset $\{(\gamma : J \rightarrow \mathcal{T}(S), \lambda_+, \lambda_-) \in \Gamma_p \mid \gamma(0) \in A\}$ has a convergent subsequence.

By the Arzela-Ascoli theorem, the set of p -Lipschitz maps $\gamma : J \rightarrow \mathcal{T}(S)_{\epsilon_0}$ where $J \subset \mathbb{R}$ is a closed connected subset containing 0 and such that $\gamma(0) \in A$ is compact with respect to the compact open topology. Moreover, the function ρ on $\mathcal{T}(S) \times \mathcal{T}(S)$ is continuous and hence if $\gamma_i : J_i \rightarrow \mathcal{T}(S)$ converges locally uniformly to $\gamma : J \rightarrow \mathcal{T}(S)$ and if γ_i is a p -quasi-geodesic with respect to ρ for all i then the same is true for γ . Since the function on $\mathcal{T}(S) \times \mathcal{ML}$ which assigns to a metric $h \in \mathcal{T}(S)$ and a measured geodesic lamination $\mu \in \mathcal{ML}$ the h -length of μ is continuous and since for every fixed $h \in \mathcal{T}(S)$ the set of measured geodesic laminations of h -length 1 is compact and naturally homeomorphic to \mathcal{PML} , the action of $\text{Mod}(S)$ on Γ_p is indeed cocompact provided that the following holds: If $\gamma_i : J_i \rightarrow \mathcal{T}(S)_{\epsilon_0}$ ($i > 0$) is a sequence of p -Lipschitz curves which converge locally uniformly to $\gamma : J \rightarrow \mathcal{T}(S)_{\epsilon_0}$, if for each i the projective measured geodesic lamination $[\lambda_i]$ is realized at the right endpoint of J_i and if $[\lambda_i] \rightarrow [\lambda]$ in \mathcal{PML} ($i \rightarrow \infty$) then $[\lambda]$ is realized at the right endpoint of J .

To see that this is indeed the case, assume first that $J \cap [0, \infty) = [0, b]$ for some $b \in (0, \infty)$. Then for sufficiently large i we have $J_i \cap [0, \infty) = [0, b_i]$ with $b_i \in (0, \infty)$ and $b_i \rightarrow b$. Thus $\gamma_i(b_i) \rightarrow \gamma(b)$ ($i \rightarrow \infty$) and therefore for sufficiently large i there is only a *finite* number of curves $\alpha \in \mathcal{C}(S)$ whose length with respect to one of the metrics $\gamma_j(b_j), \gamma(b)$ ($j \geq i$) is at most χ_0 . By passing to a subsequence we may assume that there is a simple closed curve $\alpha \in \mathcal{C}(S)$ with $[\lambda_j] = [\alpha]$ for all large j . The $\gamma_j(b_j)$ -length of α is at most χ_0 for all sufficiently large j and hence the same is

true for the $\gamma(b)$ -length of α by continuity of the length function. As a consequence, the limit $[\lambda] = [\alpha]$ of the sequence $([\lambda_i])$ is realized at the right endpoint $\gamma(b)$ of γ .

In the case that $[0, \infty) \subset J$ we argue as before. Assume first that $b_i < \infty$ for all i and that $b_i \rightarrow \infty$. Then $[\lambda_i]$ is the projective class of a simple closed curve $\alpha_i \in \mathcal{C}(S)$. For each i , the curve α_i is contained in a ball about $\Upsilon_{\mathcal{T}}(\gamma_i(b_i))$ of radius $R > 0$ independent of i . Since the curves γ_i converge uniformly on compact sets to γ , as $i \rightarrow \infty$ longer and longer subsegments of the curve γ are uniformly fellow-traveled by the curves γ_i . Since the map $\Upsilon_{\mathcal{T}}$ is coarsely Lipschitz and since the maps $t \rightarrow \Upsilon_{\mathcal{T}}(\gamma_i(t))$ are uniform quasi-geodesics in $\mathcal{C}(S)$, this implies that as $i \rightarrow \infty$, longer and longer subsegments of the quasi-geodesic $\Upsilon_{\mathcal{T}} \circ \gamma$ in $\mathcal{C}(S)$ are uniformly fellow-traveled by the quasi-geodesics $\Upsilon_{\mathcal{T}} \circ \gamma_i$. By hyperbolicity and the definition of the topology on the union of a hyperbolic geodesic metric space with its Gromov boundary (see [BH99] for details), this implies that as $i \rightarrow \infty$, the simple closed curves $\alpha_i \in \mathcal{C}(S)$ converge in $\mathcal{C}(S) \cup \partial\mathcal{C}(S)$ to the endpoint $\mu \in \partial\mathcal{C}(S)$ of the quasi-geodesic ray $\Upsilon_{\mathcal{T}} \circ \gamma$. As a consequence, up to passing to a subsequence the projective measured geodesic laminations $[\alpha_i] \subset \mathcal{PM}\mathcal{L}$ converge to a projective measured geodesic lamination supported in μ (we refer to [K199, H06] for a detailed discussion of this fact). But $[\alpha_i] = [\lambda_i] \rightarrow [\lambda]$ in $\mathcal{PM}\mathcal{L}$ by assumption and hence the lamination $[\lambda]$ is realized at the right (infinite) endpoint of J .

The same argument can also be applied in the case that $b_i = \infty$ for infinitely many i and shows that the right endpoints $\beta_i \in \partial\mathcal{C}(S)$ of the quasi-geodesics $\Upsilon_{\mathcal{T}} \circ \gamma_i$ converge in $\partial\mathcal{C}(S)$ to the right endpoint of $\Upsilon_{\mathcal{T}} \circ \gamma$. As before, this implies that the limit $[\lambda]$ of the projective measured laminations $[\lambda_i]$ ($i \rightarrow \infty$) is realized at the right infinite endpoint of J (see [H06]). This shows that the sufficient condition stated above for cocompactness of the action of $\text{Mod}(S)$ on Γ_p is satisfied.

Now we follow Section 3.10 of [Mo03]. Each point $(\gamma : J \rightarrow \mathcal{T}(S), \lambda_+, \lambda_-) \in \Gamma_p$ determines the geodesic $\eta([\lambda_+], [\lambda_-])$ in $\mathcal{T}(S)$. The unit cotangent line of this geodesic is the set q_t of quadratic differentials with vertical and horizontal measured geodesic laminations $(e^t \lambda_+, e^{-t} \lambda_-) / i(\lambda^+, \lambda^-)$.

For $(\gamma : J \rightarrow \mathcal{T}(S), \lambda_+, \lambda_-)$ let $\sigma(\gamma, \lambda_+, \lambda_-) \in \mathcal{T}(S)$ be the point on the geodesic $\eta([\lambda_+], [\lambda_-])$ which is the projection of the quadratic differential defined by the pair $(\lambda_+, \lambda_-) / i(\lambda^+, \lambda^-)$. The map taking $(\gamma : J \rightarrow \mathcal{T}(S), \lambda_+, \lambda_-)$ to $(\gamma(0), \sigma(\gamma, \lambda_+, \lambda_-)) \in \mathcal{T}(S) \times \mathcal{T}(S)$ is continuous and equivariant with respect to the natural action of $\text{Mod}(S)$ on Γ_p and on $\mathcal{T}(S) \times \mathcal{T}(S)$. Since the action of $\text{Mod}(S)$ on Γ_p is cocompact, the same is true for the action of $\text{Mod}(S)$ on the image of this map (see [Mo03] for a similar reasoning). Thus the distance between $\gamma(0)$ and $\sigma(\gamma, \lambda_+, \lambda_-)$ is bounded from above by a universal constant $b > 0$.

Let again $(\gamma : J \rightarrow \mathcal{T}(S), \lambda_+, \lambda_-) \in \Gamma_p$. For each $s \in J$ define

$$a_-(s) = \frac{1}{\ell_{\gamma(s)}(\lambda_-)}, \quad a_+(s) = \frac{1}{\ell_{\gamma(s)}(\lambda_+)}$$

where as before, $\ell_{\gamma(s)}(\lambda_{\pm})$ is the $\gamma(s)$ -length of λ_{\pm} . These are continuous functions of $s \in J$. For $s \in J$ define the shift $\gamma^s(t) = \gamma(t + s)$; then the ordered triple $(\gamma^s, a_+(s)\lambda_+, a_-(s)\lambda_-)$ lies in the $\text{Mod}(S)$ -cocompact set Γ_p and hence the distance between $\gamma(s)$ and a suitably chosen point on the geodesic $\eta([\lambda_+], [\lambda_-])$ is at most

b. As a consequence, the arc γ is contained in the b -neighborhood of the geodesic $\eta([\lambda_+], [\lambda_-])$. Since the curve γ is p -Lipschitz, this implies that the Hausdorff distance between $\gamma(J)$ and a subarc of $\eta([\lambda_+], [\lambda_-])$ is uniformly bounded. In other words, $\gamma(J)$ is a quasi-convex curve in $\mathcal{T}(S)_{\epsilon_0}$ which shows the lemma. \square

3. COMPACT INVARIANT SETS ARE EXPANSIVE

Let again S be an oriented surface of genus $g \geq 0$ with $m \geq 0$ punctures and where $3g - 3 + m \geq 2$. Let $\mathcal{Q}^1(S)$ be the bundle of area one quadratic differentials over Teichmüller space $\mathcal{T}(S)$ for S . The mapping class group $\text{Mod}(S)$ acts properly discontinuously on $\mathcal{Q}^1(S)$ as a group of bundle automorphisms. This action commutes with the action of the Teichmüller geodesic flow Φ^t . However, the action of $\text{Mod}(S)$ on $\mathcal{Q}^1(S)$ is not free and the quotient space $\mathcal{Q}(S) = \mathcal{Q}^1(S)/\text{Mod}(S)$ is a non-compact orbifold rather than a manifold.

To overcome this (mainly technical) difficulty we choose a torsion free normal subgroup Γ of $\text{Mod}(S)$ of finite index. For example, the group of all elements which act trivially on $H_1(S, \mathbb{Z}/3\mathbb{Z})$ has this property. Define

$$(11) \quad \hat{\mathcal{Q}}(S) = \mathcal{Q}^1(S)/\Gamma$$

and let $\Pi : \mathcal{Q}^1(S) \rightarrow \hat{\mathcal{Q}}(S)$ be the canonical projection. Since the action of Γ on $\mathcal{Q}^1(S)$ is free, the map Π is a covering. The Teichmüller flow Φ^t acts on $\hat{\mathcal{Q}}(S)$.

A continuous flow Φ^t on a compact metric space (X, d) is called *expansive* if there is a constant $\delta > 0$ with the following property. Let $x \in X$ and let $s : \mathbb{R} \rightarrow \mathbb{R}$ be any continuous function with $s(0) = 0$ and $d(\Phi^t(x), \Phi^{s(t)}(x)) < \delta$ for all t . If $y \in X$ is such that $d(\Phi^t(x), \Phi^{s(t)}(y)) < \delta$ for all t then $y = \Phi^\tau(x)$ for some $\tau \in \mathbb{R}$ [HK95]. Note that this definition of expansiveness does not depend on the choice of the metric d defining the topology on X . The goal of this section is to show.

Theorem 3.1. *The restriction of the Teichmüller flow to every compact invariant subset K of $\hat{\mathcal{Q}}(S)$ is expansive.*

The main tool for the proof of Theorem 3.1 is the curve graph $\mathcal{C}(S)$ of S and the results of Section 2. We continue to use the assumptions and notations from Section 2, and we begin with collecting some properties of the Gromov boundary of the curve graph which are needed in the proof.

Since the curve graph is a hyperbolic geodesic metric space, for every $c \in \mathcal{C}(S)$ there is a *visual metric* δ_c on the Gromov boundary $\partial\mathcal{C}(S)$ of $\mathcal{C}(S)$ of uniformly bounded diameter (we refer to Chapter III.H of [BH99] for details of this construction and for references). These distances are related to the intrinsic geometry of $\mathcal{C}(S)$ as follows.

For a point $c \in \mathcal{C}(S)$, the *Gromov product* at c associates to points $x, y \in \mathcal{C}(S)$ the value

$$(12) \quad (x|y)_c = \frac{1}{2}(d(x, c) + d(y, c) - d(x, y)).$$

The Gromov product can be extended to a Gromov product $(|)_c$ for pairs of distinct points in $\partial\mathcal{C}(S)$ by defining

$$(13) \quad (\xi|\zeta)_c = \sup \liminf_{i,j \rightarrow \infty} (x_i|y_j)_c$$

where the supremum is taken over all sequences (x_i) and (y_j) in $\mathcal{C}(S)$ such that $\xi = \lim x_i$ and $\zeta = \lim y_j$. There are numbers $\beta > 0, \nu \in (0, 1)$ such that

$$(14) \quad \nu e^{-\beta(\xi|\zeta)_c} \leq \delta_c(\xi, \zeta) \leq e^{-\beta(\xi|\zeta)_c} \text{ for all } \xi, \zeta \in \partial\mathcal{C}(S) \text{ and} \\ \delta_c \leq e^{\beta d(c,a)} \delta_a \text{ for all } c, a \in \mathcal{C}(S).$$

The distances δ_c are equivariant with respect to the action of $\text{Mod}(S)$ on $\mathcal{C}(S)$ and on $\partial\mathcal{C}(S)$. We will need a more precise quantitative relation between the distance functions δ_c ($c \in \mathcal{C}(S)$). Even though this property is well known, we did not find an explicit reference in the literature and we include a sketch of a proof. For a formulation, for $c \in \mathcal{C}(S)$, $\xi \in \partial\mathcal{C}(S)$ and $r > 0$ denote by $B_c(\xi, r) \subset \partial\mathcal{C}(S)$ the ball of radius r about ξ with respect to the distance function δ_c . Recall that every quasi-geodesic ray $\gamma : [0, \infty) \rightarrow \mathcal{C}(S)$ converges as $t \rightarrow \infty$ in $\mathcal{C}(S) \cup \partial\mathcal{C}(S)$ to an endpoint $\gamma(\infty) \in \partial\mathcal{C}(S)$.

Lemma 3.2. *For every $m > 1$ there are constants $a(m) > 1, b(m) > 0, \alpha_0(m) > 0$ with the following property. Let $\gamma : [0, \infty) \rightarrow \mathcal{C}(S)$ be an m -quasi-geodesic ray with endpoint $\gamma(\infty) \in \partial\mathcal{C}(S)$. Then for all $t \geq 0$ we have*

$$\delta_{\gamma(0)} \leq a(m) e^{-b(m)t} \delta_{\gamma(t)} \text{ on } B_{\gamma(t)}(\gamma(\infty), \alpha_0(m)).$$

Proof. Since $\mathcal{C}(S)$ is a hyperbolic geodesic metric space, there is a constant $p > 1$ depending on the hyperbolicity constant such that every point $c \in \mathcal{C}(S)$ can be connected to every point $\xi \in \partial\mathcal{C}(S)$ by a p -quasi-geodesic. In the particular case of the curve graph, such a quasi-geodesic can be obtained as a reparametrization of a curve of the form $t \rightarrow \Upsilon_{\mathcal{T}}(\rho(t))$ where ρ is a Teichmüller geodesic in $\mathcal{T}(S)$ (compare the discussion in Section 2 and see [Kl99, H06]). Similarly, any two points $\xi \neq \zeta \in \partial\mathcal{C}(S)$ can be joined by a p -quasi-geodesic.

There is a universal constant $b > 0$ only depending on the hyperbolicity constant for $\mathcal{C}(S)$ with the following property (Remark 3.17.5 in Chapter III.H of [BH99]). Let $\gamma_i : [0, \infty) \rightarrow \mathcal{C}(S)$ ($i = 1, 2$) be quasi-geodesic rays with $e = \gamma_1(0) = \gamma_2(0)$ converging to $\gamma_1(\infty) \neq \gamma_2(\infty) \in \partial\mathcal{C}(S)$; then we have

$$(15) \quad (\gamma_1(\infty)|\gamma_2(\infty))_e - b \leq \liminf_{t \rightarrow \infty} (\gamma_1(t)|\gamma_2(t))_e \leq (\gamma_1(\infty)|\gamma_2(\infty))_e.$$

Let $m \geq p$. By hyperbolicity, there is a number $r(m) > 0$ and for every m -quasi-geodesic triangle T in $\mathcal{C}(S)$ with vertices $c \in \mathcal{C}(S), \xi \neq \zeta \in \partial\mathcal{C}(S)$ there is a point $u \in \mathcal{C}(S)$ whose distance to each of the sides of T is at most $r(m)$. The (non-unique) point u is called a *center* of T . Via enlarging the constant $r(m)$ we may assume that for all $a, b \in \mathbb{R}$ the Hausdorff distance between any m -quasi-geodesic arc $\gamma : [a, b] \rightarrow \mathcal{C}(S)$ and any geodesic arc in $\mathcal{C}(S)$ connecting its endpoints $\gamma(a), \gamma(b)$ is at most $r(m)$. An application of inequality (15) to the sides γ_1, γ_2 of T connecting c to ξ, ζ together with the properties (14) of the distances δ_c then implies the existence of a constant $\chi(m) > 0$ with the following property. Let u be a center

of an m -quasi-geodesic triangle in $\mathcal{C}(S)$ with vertices $c \in \mathcal{C}(S), \xi \neq \zeta \in \partial\mathcal{C}(S)$; then we have

$$(16) \quad \delta_c(\xi, \zeta) \in [\chi(m)e^{-\beta d(c,u)}, e^{-\beta d(c,u)}/\chi(m)].$$

Now let $\gamma : [0, \infty) \rightarrow \mathcal{C}(S)$ be any m -quasi-geodesic ray with endpoint $\gamma(\infty) = \xi \in \partial\mathcal{C}(S)$, let $\zeta \neq \xi \in \partial\mathcal{C}(S)$ and let T be an m -quasi-geodesic triangle with side γ and vertices $\gamma(0), \xi, \zeta$. If $u \in \mathcal{C}(S)$ is a center for T and if $\sigma \geq 0$ is such that $d(u, \gamma(\sigma)) \leq r(m)$, then for every $s \in [0, \sigma]$ the distance between u and a center for any m -quasi-geodesic triangle with vertices $\gamma(s), \xi, \zeta$ is bounded from above by a constant only depending on m and the hyperbolicity constant for $\mathcal{C}(S)$. In particular, by the above discussion and the properties of an m -quasi-geodesic, there are constants $a(m) > 0, b(m) > 0$ such that

$$(17) \quad \delta_{\gamma(0)}(\xi, \zeta) \leq a(m)e^{-b(m)t} \delta_{\gamma(s)}(\xi, \zeta) \text{ for every } s \in [0, \sigma].$$

From this the lemma easily follows. \square

For $h \in \mathcal{T}(S)$ let μ_h be the finite Borel measure on $\mathcal{C}(S)$ defined in equation (7) of Section 2. Recall that the total mass of μ_h is bounded from above and below by a positive constant not depending on h , and the diameter of the support of μ_h in $\mathcal{C}(S)$ is uniformly bounded as well.

For $h \in \mathcal{T}(S)$ define a distance δ_h on $\partial\mathcal{C}(S)$ by

$$(18) \quad \delta_h(\xi, \zeta) = \int \delta_c(\xi, \zeta) d\mu_h(c).$$

Clearly the metrics δ_h are equivariant with respect to the action of $\text{Mod}(S)$ on $\mathcal{T}(S)$ and $\partial\mathcal{C}(S)$. Moreover, there is a constant $\kappa > 0$ such that

$$(19) \quad \delta_h \leq e^{\kappa d(h,z)} \delta_z$$

for all $h, z \in \mathcal{T}(S)$ (here as before, d denotes the Teichmüller metric). Namely, the function $\sigma : [0, \infty) \rightarrow [0, 1]$ used in Section 2 to define the measures μ_h is smooth, with uniformly bounded differential. By a result of Wolpert (see [IT99]), for every simple closed curve $c \in \mathcal{C}(S)$, the function $h \rightarrow \log \ell_h(c)$ on $\mathcal{T}(S)$ is smooth, with uniformly bounded differential with respect to the norm induced by the Teichmüller metric. Since σ is supported in $[0, 2\chi_0]$, this implies that for each $c \in \mathcal{C}(S)$ the function $h \rightarrow \sigma(\ell_h(c))$ on $\mathcal{T}(S)$ is smooth, with uniformly bounded differential. As a consequence, for all $\xi \neq \eta \in \partial\mathcal{C}(S)$ the function $h \rightarrow \delta_h(\xi, \eta)$ is smooth, and the differential of its logarithm is uniformly bounded with respect to the Teichmüller norm, independent of ξ, η . From this and the definitions, the estimate (19) above is immediate. Via enlarging the constant κ we may also assume that

$$(20) \quad \kappa^{-1} \delta_h \leq \delta_{\mathcal{T}(h)} \leq \kappa \delta_h$$

for every $h \in \mathcal{T}(S)$.

As in Section 2, for an area one quadratic differential $q \in \mathcal{Q}^1(S)$ we denote by q_v, q_h the vertical and the horizontal measured geodesic lamination of q , respectively. For every $t \in \mathbb{R}$ the pair $(e^t q_v, e^{-t} q_h)$ corresponds to the quadratic differential $\Phi^t q$. The *strong stable manifold* $W^{ss}(q) \subset \mathcal{Q}^1(S)$ of q is defined

as the set of all quadratic differentials of area one whose vertical measured geodesic lamination coincides *precisely* with the vertical measured geodesic lamination of q . The *strong unstable* manifold $W^{su}(q)$ is the set of all quadratic differentials of area one whose horizontal measured geodesic lamination coincides precisely with the horizontal measured geodesic lamination of q . Define moreover the *stable manifold* $W^s(q)$ of q and the *unstable manifold* $W^u(q)$ of q by $W^s(q) = \cup_{t \in \mathbb{R}} \Phi^t W^{ss}(q)$ and $W^u(q) = \cup_{t \in \mathbb{R}} \Phi^t W^{su}(q)$. As q varies through $\mathcal{Q}^1(S)$, the manifolds $W^{ss}(q)$, $W^{su}(q)$, $W^s(q)$, $W^u(q)$ define continuous foliations of $\mathcal{Q}^1(S)$ which are called the *strong stable*, the *strong unstable*, the *stable* and the *unstable foliation*. These foliations are invariant under the action of $\text{Mod}(S)$ and under the action of the Teichmüller geodesic flow Φ^t . Hence they descend to continuous Φ^t -invariant foliations on the quotient $\hat{\mathcal{Q}}(S) = \mathcal{Q}^1(S)/\Gamma$ and to singular Φ^t -invariant foliations on the quotient $\mathcal{Q}(S) = \mathcal{Q}^1(S)/\text{Mod}(S)$.

We use these foliations to define a distance function on $\hat{\mathcal{Q}}(S)$ which is particularly well suited for the proof of Theorem 3.1. Recall that a distance d on a space X is called a *length metric* if the distance between any two points is the infimum of the lengths of all paths connecting these points. In the sequel denote by

$$(21) \quad P : \mathcal{Q}^1(S) \rightarrow \mathcal{T}(S)$$

the canonical projection.

Lemma 3.3. *There is a complete $\text{Mod}(S)$ -invariant length metric d on $\mathcal{Q}^1(S)$ with the following properties.*

- (1) *The metric d induces the usual topology.*
- (2) *The canonical projection $P : (\mathcal{Q}^1(S), d) \rightarrow \mathcal{T}(S)$ is distance non-decreasing where $\mathcal{T}(S)$ is equipped with the Teichmüller metric.*
- (3) *Every orbit of the Teichmüller flow with its natural parametrization is a minimal geodesic for d parametrized by arc length.*

Proof. By the Hubbard Masur theorem (see [Hu06] for a presentation of this celebrated and by now classical result), the restriction of the canonical projection $P : \mathcal{Q}^1(S) \rightarrow \mathcal{T}(S)$ to every unstable (or stable) manifold in $\mathcal{Q}^1(S)$ is a homeomorphism onto $\mathcal{T}(S)$. Thus the Teichmüller metric on $\mathcal{T}(S)$ lifts to a length metric on the leaves of the stable and of the unstable foliation. Call a path $\rho : [0, 1] \rightarrow \mathcal{Q}^1(S)$ *admissible* if there is a finite partition $0 = t_0 < \dots < t_k = 1$ such that the restriction of ρ to each interval $[t_{i-1}, t_i]$ is entirely contained in a stable or in an unstable manifold. For each such admissible path ρ we can define its *length* to be the sum of the lengths with respect to the lifts of the Teichmüller metric of the subsegments of ρ entirely contained in a stable or an unstable manifold. For $q_0, q_1 \in \mathcal{Q}^1(S)$ define $d(q_0, q_1)$ to be the infimum of the lengths of all admissible paths connecting q_0 to q_1 . Then d is a (a priori non-finite) distance function on $\mathcal{Q}^1(S)$ which satisfies the second and the third requirement in the lemma. By the second property for d and the definition, the d -length of every path in $\mathcal{Q}^1(S)$ which is entirely contained in a stable or an unstable manifold coincides with the length of its projection to $\mathcal{T}(S)$. As a consequence, the metric d is a length metric.

We are left with showing that d induces the usual topology on $\mathcal{Q}^1(S)$. For this let $q \in \mathcal{Q}^1(S)$ and let $\epsilon > 0$. We have to show that the ϵ -ball about q for the

distance d contains a neighborhood of q in $\mathcal{Q}^1(S)$. For this denote for $z \in \mathcal{Q}^1(S)$ and $r > 0$ by $B^i(q, r)$ the open r -ball about q in $W^i(q)$ with respect to the lift of the Teichmüller metric ($i = s, u$). For each $z \in B^s(q, \epsilon/2)$, the open ball $B^u(z, \epsilon/2)$ of radius $\epsilon/2$ about z in $W^u(z)$ is an open neighborhood of z in $W^u(z)$ whose closure is compact and depends continuously on z in the Hausdorff topology for compact subsets of $\mathcal{Q}^1(S)$ by the Hubbard Masur theorem. Then $U = \cup_{z \in B^s(q, \epsilon/2)} B^u(z, \epsilon/2)$ is an open neighborhood of q in $\mathcal{Q}^1(S)$. Moreover by construction, U is contained in the ϵ -ball about q with respect to the distance function d . This completes the proof of the lemma. \square

The distance d on $\mathcal{Q}^1(S)$ constructed in Lemma 3.3 induces a metric on $\hat{\mathcal{Q}}(S)$, again denote by d , via

$$(22) \quad d(x, y) = \inf\{d(\tilde{x}, \tilde{y}) \mid \Pi(\tilde{x}) = \Pi(\tilde{y})\}$$

where $\Pi : \mathcal{Q}^1(S) \rightarrow \hat{\mathcal{Q}}(S)$ is the canonical projection. In the sequel we always use the above distance function on $\mathcal{Q}^1(S)$ and $\hat{\mathcal{Q}}(S)$ without further mentioning.

Let

$$(23) \quad \pi : \mathcal{Q}^1(S) \rightarrow \mathcal{PML}$$

be the map which associates to a quadratic differential its vertical projective measured geodesic lamination. For every $q \in \mathcal{Q}^1(S)$ the restriction of the projection π to $W^{su}(q)$ is a homeomorphism of $W^{su}(q)$ onto the open subset of \mathcal{PML} of all projective measured geodesic laminations μ which together with $\pi(-q)$ *jointly fill up* S , i.e. are such that for every measured geodesic lamination $\eta \in \mathcal{ML}$ we have $i(\mu, \eta) + i(\pi(-q), \eta) \neq 0$ (note that this makes sense even though the intersection form i is defined on \mathcal{ML} rather than on \mathcal{PML}). With these preparations, we can prove Theorem 3.1.

Proof of Theorem 3.1. Let $K \subset \hat{\mathcal{Q}}(S)$ be a compact Φ^t -invariant subset and let $\tilde{K} \subset \mathcal{Q}^1(S)$ be the preimage of K under the canonical projection $\Pi : \mathcal{Q}^1(S) \rightarrow \hat{\mathcal{Q}}(S)$. Let as before $P : \mathcal{Q}^1(S) \rightarrow \mathcal{T}(S)$ be the canonical projection. By the second part of Theorem 2.3, there is a constant $p > 0$ only depending on K such that for every $q \in \tilde{K}$ the curve $t \rightarrow \Upsilon_{\mathcal{T}}(P\Phi^t q)$ is a p -quasi-geodesic in $\mathcal{C}(S)$, i.e. we have

$$(24) \quad |t - s|/p - p \leq d(\Upsilon_{\mathcal{T}}(P\Phi^t q), \Upsilon_{\mathcal{T}}(P\Phi^s q)) \leq p|t - s| + p \quad \text{for all } s, t \in \mathbb{R}.$$

Let $\mathcal{F} : \mathcal{Q}^1(S) \rightarrow \mathcal{Q}^1(S)$ be the *flip* $q \rightarrow \mathcal{F}(q) = -q$. This flip is equivariant with respect to the action of the mapping class group and hence it descends to a continuous involution of $\hat{\mathcal{Q}}(S)$ which we denote again by \mathcal{F} . Recall that the Gromov boundary $\partial\mathcal{C}(S)$ of $\mathcal{C}(S)$ can be identified with the set of all (unmeasured) minimal geodesic laminations on S which fill up S . Let again $\pi : \mathcal{Q}^1(S) \rightarrow \mathcal{PML}$ be the canonical projection. If $q \in \tilde{K} \cup \mathcal{F}(\tilde{K})$ then the support $F(\pi q)$ of the vertical measured geodesic lamination of q is *uniquely ergodic*, which means that $F(\pi q)$ admits a unique transverse measure up to scale, and $F(\pi q)$ is minimal and fills up S [M82]. Moreover, the p -quasi-geodesic $t \rightarrow \Upsilon_{\mathcal{T}}(P\Phi^t q)$ converges in $\mathcal{C}(S) \cup \partial\mathcal{C}(S)$ to $F(\pi q)$ (the latter statement follows from the explicit identification of $\partial\mathcal{C}(S)$ with the set of minimal geodesic laminations which fill up S established in [K199, H06]).

Write $A = \pi(\tilde{K} \cup \mathcal{F}(\tilde{K}))$. Then A is Γ -invariant Borel subset of \mathcal{PML} . By the consideration in the previous paragraph, there is a natural Γ -equivariant continuous injection $F : A \rightarrow \partial\mathcal{C}(S)$ which associates to a projective measured geodesic lamination contained in A its support. In other words, we can identify the set A with subset of $\mathcal{C}(S)$.

Recall from equation (18) the definition of the distances δ_h ($h \in \mathcal{T}(S)$) on the Gromov boundary $\partial\mathcal{C}(S)$ of $\mathcal{C}(S)$. Since the map $F : A \rightarrow \partial\mathcal{C}(S)$ is injective, for every $q \in \mathcal{Q}^1(S)$ the function $(x, y) \in A \times A \rightarrow \delta_{Pq}(Fx, Fy)$ is a distance on A . The topology on A defined by this distance is just the subspace topology of A as a subset of \mathcal{PML} . For $q \in \tilde{K} \cup \mathcal{F}(\tilde{K})$ denote by $B_q(\pi(q), r) \subset A$ the ball of radius r in A about $\pi(q)$ with respect to this distance. By continuity of the projection π and the map F and by the relation (19) between the distances δ_h and δ_z for $h, z \in \mathcal{T}(S)$, the ball $B_q(\pi(q), r)$ depends continuously on q in the following sense. For every $q \in \tilde{K} \cup \mathcal{F}(\tilde{K})$, every $r > 0$ and every $\epsilon < r$ there is a neighborhood U of q in $\tilde{K} \cup \mathcal{F}(\tilde{K})$ such that for every $z \in U$ we have

$$(25) \quad B_z(\pi(z), r - \epsilon) \subset B_q(\pi(q), r) \subset B_z(\pi(z), r + \epsilon).$$

By inequalities (20) and (24) and by Lemma 3.2, there are numbers $\alpha_0 > 0$, $a > 1$, $b > 0$ such that for every $q \in \tilde{K}$ and for all $t > 0$ we have

$$(26) \quad \begin{aligned} \delta_{P\Phi^{-t}q} &\leq ae^{-bt}\delta_{Pq} \text{ on } B_q(\pi(q), 2\alpha_0) \text{ and} \\ \delta_{P\Phi^tq} &\leq ae^{-bt}\delta_{Pq} \text{ on } B_{\mathcal{F}(q)}(\pi(\mathcal{F}q), 2\alpha_0). \end{aligned}$$

For $q \in \mathcal{Q}^1(S)$ and $\beta > 0$ denote by $B(q, \beta)$ the ball of radius β about q in $\mathcal{Q}^1(S)$ with respect to the length metric d defined in Lemma 3.3. Since the projection π is continuous, for every $q \in \tilde{K} \cup \mathcal{F}(\tilde{K})$ there is a number $\epsilon(q) > 0$ such that $\pi(B(q, \epsilon(q)) \cap (\tilde{K} \cup \mathcal{F}(\tilde{K}))) \subset B_q(\pi(q), \alpha_0)$ where $\alpha_0 > 0$ is as in the inequalities (26). By the continuity properties (25) of the balls $B_z(\pi(z), r)$ for $z \in \tilde{K} \cup \mathcal{F}(\tilde{K})$ and $r > 0$, by invariance under the action of the group $\Gamma < \text{Mod}(S)$ and cocompactness we can find a universal number $\beta_0 > 0$ such that

$$(27) \quad \pi(B(q, \beta_0) \cap (\tilde{K} \cup \mathcal{F}(\tilde{K}))) \subset B_q(\pi(q), \alpha_0) \text{ for all } q \in \tilde{K} \cup \mathcal{F}(\tilde{K}).$$

Denote again by d the distance on $\hat{\mathcal{Q}}(S)$ induced in equation (22) from the distance on $\mathcal{Q}^1(S)$. Since $K \cup \mathcal{F}(K)$ is compact and $\Pi : \mathcal{Q}^1(S) \rightarrow \hat{\mathcal{Q}}(S)$ is a covering, there is a number $\beta < \beta_0$ such that for every $q \in K \cup \mathcal{F}(K)$ and every lift \tilde{q} of q to $\mathcal{Q}^1(S)$ the ball $B(q, \beta)$ in $\hat{\mathcal{Q}}(S)$ of radius β about q is the homeomorphic image under Π of the ball $B(\tilde{q}, \beta)$. Now the orbits of Φ^t are geodesics for the distance d . Hence if $x \in K$, if $s : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function such that $s(0) = 0$ and $d(\Phi^t x, \Phi^{s(t)} x) < \beta/2$ for all t then by the choice of β we have $|t - s(t)| < \beta/2$ for all t .

This implies that if $y \in K$ is such that $d(\Phi^t x, \Phi^{s(t)} y) < \beta/2$ for all t then $d(\Phi^t y, \Phi^{s(t)} y) < \beta/2$ and hence $d(\Phi^t y, \Phi^t x) < \beta$ for all t by the triangle inequality. In particular, for a lift $\tilde{x} \in \mathcal{Q}^1(S)$ of x and a lift \tilde{y} of y with $d(\tilde{x}, \tilde{y}) < \beta$ we have $d(\Phi^t \tilde{x}, \Phi^t \tilde{y}) < \beta$ for all $t \in \mathbb{R}$.

Let $W_{\text{loc}}^s(x)$ be the connected component containing x of the intersection of $B(x, \beta)$ with the stable manifold $W^s(x)$ of x . We claim that $y \in W_{\text{loc}}^s(x)$. For this assume otherwise. Let again \tilde{x} be a preimage of x in $\mathcal{Q}^1(S)$ and let $\tilde{y} \in \mathcal{Q}^1(S)$ be the preimage of y with $d(\tilde{x}, \tilde{y}) < \beta$. If $\pi(\tilde{x}) \neq \pi(\tilde{y})$ then by continuity, our choice of α_0 and the estimates (26), there is a number $t > 0$ such that $\delta_{P\Phi^t\tilde{x}}(\pi(\tilde{x}), \pi(\tilde{y})) = \alpha_0$. On the other hand, for every $s \in [0, t]$ the distance in $\mathcal{Q}^1(S)$ between $\Phi^s\tilde{x}, \Phi^s\tilde{y}$ is smaller than β which is a contradiction to the choice of β . In the same way we conclude that y is contained in the intersection of $B(x, \beta)$ with the local unstable manifold of x . This completes the proof of Theorem 3.1. \square

For a compact Φ^t -invariant subset K of $\hat{\mathcal{Q}}(S)$ let $h_{\text{top}}(K)$ be the topological entropy of the restriction of Φ^t to K . For $r > 0$ let moreover $n_K(r)$ be the number of periodic orbits of Φ^t which are contained in K . Since by Theorem 3.1 the restriction of the flow Φ^t to K is expansive, by Proposition 3.2.14 of [HK95] we have.

Corollary 3.4. *Let $K \subset \hat{\mathcal{Q}}(S)$ be a compact Φ^t -invariant set; then*

$$\limsup_{r \rightarrow \infty} \frac{1}{r} \log n_K(r) \leq h_{\text{top}}(K).$$

4. AN ANOSOV CLOSING LEMMA

The goal of this section is to establish a version of the Anosov closing lemma for the restriction of the Teichmüller flow to a compact invariant set $K \subset \hat{\mathcal{Q}}(S)$. The classical Anosov closing lemma roughly states that for a hyperbolic flow on a closed Riemannian manifold, a closed curve consisting of sufficiently long orbit segments which are connected at the endpoints by sufficiently short arcs is closely fellow-traveled by a periodic orbit. For a precise formulation of our version of an Anosov closing lemma for the Teichmüller flow, using the assumptions and notations from Section 2 and Section 3 we define.

Definition 4.1. For $n > 0, \epsilon > 0$, an (n, ϵ) -pseudo-orbit for the Teichmüller flow Φ^t on $\hat{\mathcal{Q}}(S)$ consists of a sequence of points $x_0, x_1, \dots, x_k \in \hat{\mathcal{Q}}(S)$ and a sequence of numbers $t_0, \dots, t_{k-1} \in [n, \infty)$ with the following properties.

- (1) For every $j \leq k$ the 2ϵ -neighborhood of x_j is contained in a contractible subset of $\hat{\mathcal{Q}}(S)$.
- (2) For every $j < k$ we have $d(\Phi^{t_j}x_j, x_{j+1}) \leq \epsilon$.

The pseudo-orbit is *contained* in a compact set K if for all i and all $t \in [0, t_i]$ we have $\Phi^t x_i \in K$. The pseudo-orbit is called *closed* if $x_0 = x_k$.

An (n, ϵ) -pseudo-orbit x_0, \dots, x_k determines an essentially unique arc connecting x_0 to x_k which we call a *characteristic arc*. Namely, by assumption, for each $j < k$ the 2ϵ -neighborhood of x_{j+1} is contained in a contractible subset of $\hat{\mathcal{Q}}(S)$ and hence the homotopy class with fixed endpoints in $\hat{\mathcal{Q}}(S)$ of an arc of length smaller than 2ϵ connecting $\Phi^{t_j}x_j$ to x_{j+1} is unique. We define a characteristic arc of the (n, ϵ) -pseudo-orbit to be an arc connecting x_0 to x_k which is obtained by successively joining the endpoints of the orbit segments $\{\Phi^t x_j \mid 0 \leq t \leq t_j\}$ with a family of

arcs of length smaller than 2ϵ connecting $\Phi^{t_j}x_j$ to x_{j+1} and which are parametrized on the unit interval. The points x_i ($1 \leq i \leq k$) are called the *breakpoints* of the characteristic arc of the pseudo-orbit. The *characteristic homotopy class* of the pseudo-orbit is the homotopy class with fixed endpoints of a characteristic arc connecting x_0 to x_k . Note that this is independent of the choice of a characteristic arc. If the pseudo-orbit is closed then it determines a closed characteristic curve and hence a free homotopy class of closed curves in $\hat{\mathcal{Q}}(S)$ which we call the *characteristic free homotopy class* of the closed pseudo-orbit.

By abuse of notation, denote again by $P : \hat{\mathcal{Q}}(S) \rightarrow \mathcal{T}(S)/\Gamma$ the canonical projection.

Definition 4.2. An (n, ϵ) -pseudo-orbit x_0, \dots, x_k as in Definition 4.1 is δ -*shadowed* by an orbit segment $\zeta = \{\Phi^t q \mid t \in [0, \tau]\}$ for some $q \in \hat{\mathcal{Q}}(S)$ and some $\tau > 0$ if the following holds.

- (1) There is a number $\alpha \leq \delta$ such that the α -neighborhoods of Px_0, Px_k in $\mathcal{T}(S)/\Gamma$ with respect to the projection of the Teichmüller metric are contractible and contain $q, \Phi^\tau q$.
- (2) There is a lift $\tilde{\zeta}$ to $\mathcal{Q}^1(S)$ of the orbit segment ζ and a lift $\tilde{\gamma}$ to $\mathcal{Q}^1(S)$ of a characteristic arc γ for the pseudo-orbit with the following properties. The distance between the endpoints of $P\tilde{\gamma}, P\tilde{\zeta}$ is at most α , and the Hausdorff distance between $\tilde{\gamma}$ and $\tilde{\zeta}$ is at most δ .

A closed pseudo-orbit in $\hat{\mathcal{Q}}(S)$ is δ -shadowed by a periodic orbit if in addition to the above requirements the orbit $\{\Phi^t q \mid t \in [0, \tau]\}$ is closed.

Note that every point in the orbifold $\mathcal{Q}(S)$ admits a contractible neighborhood, so we can use the above definition in the same way for the Teichmüller flow on the orbifold $\mathcal{Q}(S)$.

Using Definition 4.1 and Definition 4.2 we can now formulate a version of the Anosov closing lemma for the Teichmüller flow which is the main result of this section.

Theorem 4.3. *For every compact Φ^t -invariant set $K \subset \hat{\mathcal{Q}}(S)$ there are numbers $\epsilon_1 = \epsilon_1(K) > 0$, $n = n(K) > 0$, $b = b(K) > 0$ such that every (n, ϵ_1) -pseudo-orbit contained in K is b -shadowed by an orbit. Moreover, for every $\delta > 0$ there is a number $\epsilon_2 = \epsilon_2(K, \delta) < \epsilon_1$ such that a closed (n, ϵ_2) -pseudo-orbit contained in K is δ -shadowed by a periodic orbit which is contained in the characteristic free homotopy class of the closed pseudo-orbit.*

For the proof of Theorem 4.3 we need the following technical preparation which is also used in Section 5. Call a point $q \in \mathcal{Q}(S)$ is *recurrent* if q is contained in the ω -limit set of its own orbit under the Teichmüller flow. We show.

Lemma 4.4. (1) *For every compact Φ^t -invariant set $K \subset \hat{\mathcal{Q}}(S)$ there are numbers $\epsilon_0 = \epsilon_0(K) > 0$, $n_0 = n_0(K) > 0$, $\ell_0 = \ell_0(K) > 1$ depending on K with the following property. Let $x_0, \dots, x_k \in \mathcal{Q}(S)$ be an (n_0, ϵ_0) -pseudo-orbit contained in K and let $\tilde{\gamma}$ be a lift to $\mathcal{Q}^1(S)$ of a characteristic arc*

connecting x_0 to x_k . Then the arc $t \rightarrow \Upsilon_{\mathcal{T}}(P\tilde{\gamma}(t))$ is an ℓ_0 -quasi-geodesic in $\mathcal{C}(S)$.

- (2) There is a number $m > 1$ and for every recurrent point $q \in \mathcal{Q}(S)$ there are numbers $\epsilon_0(q) > 0, n_0(q) > 0$ with the following properties. Let x_0, \dots, x_k be an $(n_0(q), \epsilon_0(q))$ -pseudo-orbit with $d(x_i, q) \leq \epsilon_0(q)$ for all i and let $\tilde{\gamma}$ be a lift to $\mathcal{Q}^1(S)$ of a characteristic arc γ connecting x_0 to x_k . Then the arc $t \rightarrow \Upsilon_{\mathcal{T}}(P\tilde{\gamma}(t))$ is an unparametrized m -quasi-geodesic in $\mathcal{C}(S)$. Moreover, if $s < t$ are such that $\tilde{\gamma}(s), \tilde{\gamma}(t)$ are lifts of two distinct breakpoints of γ then $d(\Upsilon_{\mathcal{T}}(\tilde{\gamma}(s)), \Upsilon_{\mathcal{T}}(\tilde{\gamma}(t))) \geq 2c(m)$ where $c(m) > 0$ is as in Lemma 2.4.

Proof. Recall from (14) and (18) of Section 3 the definition of the distance functions δ_c ($c \in \mathcal{C}(S)$) and δ_x ($x \in \mathcal{T}(S)$) on the Gromov boundary $\partial\mathcal{C}(S)$ of $\mathcal{C}(S)$. By hyperbolicity, every quasi-geodesic ray $\zeta : [0, \infty) \rightarrow \mathcal{C}(S)$ converges as $t \rightarrow \infty$ to a point $F(\zeta) \in \partial\mathcal{C}(S)$. Moreover, for every $\ell > 1$ there are numbers $m(\ell) > 0, \alpha(\ell) > 0, p(\ell) > 1$ with the following property. Let $k > 0$ and let $\zeta_0, \dots, \zeta_{k-1} : \mathbb{R} \rightarrow \mathcal{C}(S)$ be infinite ℓ -quasi-geodesics. Let $L > 1$ be as in Lemma 2.1. Suppose that for every $j \leq k-2$ there is a number $T_j > m(\ell)$ such that $d(\zeta_j(T_j), \zeta_{j+1}(0)) \leq 2L$. For each $j \leq k-1$ let $\rho_j : [0, 1] \rightarrow \mathcal{C}(S)$ be any map whose image is contained in the $2L$ -neighborhood of $\zeta_j(T_j)$ and let $\tilde{\zeta}_j$ be the composition of $\zeta_j[0, T_j]$ with ρ_j parametrized in the natural way on $[0, T_j + 1]$. If $\delta_{\zeta_{j+1}(0)}(F(\zeta_j), F(\zeta_{j+1})) < \alpha(\ell)$ for all j then the curve $\zeta : [0, \sum_i T_i + k] \rightarrow \mathcal{C}(S)$ defined by

$$(28) \quad \zeta(t) = \tilde{\zeta}_j(t - \sum_{i=0}^{j-1} T_i - j) \text{ for } t \in [\sum_{i=0}^{j-1} T_i + j, \sum_{i=0}^j T_i + j + 1]$$

is a $p(\ell)$ -quasi-geodesic.

Let $K \subset \hat{\mathcal{Q}}(S)$ be a compact Φ^t -invariant set and let $\tilde{K} \subset \mathcal{Q}^1(S)$ be the preimage of K under the natural projection. By Theorem 2.3 there is a number $\ell > 1$ such that for every $q \in \tilde{K}$ the assignment $t \rightarrow \Upsilon_{\mathcal{T}}(P\Phi^t q)$ ($t \in \mathbb{R}$) is an ℓ -quasi-geodesic.

For $q \in \tilde{K}$ write $f(q) = F(t \rightarrow \Upsilon_{\mathcal{T}}(P\Phi^t q)) \in \partial\mathcal{C}(S)$. Recall [Kl99, H06] that $f(q)$ is just the support of the vertical measured geodesic lamination of q . The map $f : \tilde{K} \rightarrow \partial\mathcal{C}(S)$ is continuous [Kl99, H06]. Let $\kappa > 0$ be as in (20) of Section 3. By continuity, for every $q \in \tilde{K}$ there is a number $\epsilon(q) > 0$ such that for every point $\tilde{q} \in \tilde{K}$ which is contained in the $\epsilon(q)$ -neighborhood of q the δ_{Pq} -distance between $f(q)$ and $f(\tilde{q})$ is smaller than $\alpha(\ell)/\kappa$ where $\alpha(\ell) > 0$ is as in the first paragraph of this proof. By continuity, invariance under the action of the mapping class group on $\mathcal{Q}^1(S)$ and $\partial\mathcal{C}(S)$ and cocompactness of the action of Γ on \tilde{K} , there is a number $\epsilon_0 \in (0, 1/2)$ which has this property for all $q \in \tilde{K}$.

Let $m(\ell) > 0$ be as in the first paragraph of this proof. If $\tilde{\gamma}$ is the lift to $\mathcal{Q}^1(S)$ of a characteristic arc of an $(m(\ell), \epsilon_0)$ -pseudo-orbit contained in $K \subset \hat{\mathcal{Q}}(S)$ then $\tilde{\gamma}$ is a composition of curves γ_i where the curve γ_i consists of an orbit segment $\xi_i \subset \tilde{K}$ for the Teichmüller flow of length at least $m(\ell)$ and an arc of length at most $2\epsilon_0 < 1$ parametrized on $[0, 1]$ and issuing from the endpoint of ξ_i . By the choice of $m(\ell)$, of $\epsilon_0 > 0$ and by the construction of the curves γ_i , the curve $\Upsilon_{\mathcal{T}}(P\tilde{\gamma})$ is of the form described in the beginning of this proof and hence it is a $p(\ell)$ -quasi-geodesic in $\mathcal{C}(S)$. This shows the first part of the lemma.

The second part of the lemma follows in the same way. Namely, recall from [MM99] that there is a number $\ell > 0$ such that for every $\tilde{q} \in \mathcal{Q}^1(S)$ the curve $t \rightarrow \Upsilon_{\mathcal{T}}(P\Phi^t\tilde{q})$ is an unparametrized ℓ -quasi-geodesic (this number $\ell > 0$ does not coincide with the constant in the second paragraph of this proof). Let $m(\ell) > 0, \alpha(\ell) > 0, p(\ell) > 1$ be as in the first paragraph of this proof. Let $q \in \mathcal{Q}(S)$ be a recurrent point and let $\tilde{q} \in \mathcal{Q}^1(S)$ be a lift of q . Then the vertical measured geodesic lamination of \tilde{q} is uniquely ergodic and fills up S [M82], and the unparametrized ℓ -quasi-geodesic $t \rightarrow \Upsilon_{\mathcal{T}}(P\Phi^t\tilde{q})$ is infinite. By continuity, there is a number $T(q) > 0$ and there is an open neighborhood V of \tilde{q} in $\mathcal{Q}^1(S)$ such that

$$(29) \quad d(\Upsilon_{\mathcal{T}}(P\Phi^{T(q)}z), \Upsilon_{\mathcal{T}}(Pz)) \geq \ell m(\ell) + \ell + c(p(\ell)) \text{ for all } z \in V$$

where $c(p(\ell)) > 0$ is as in Lemma 2.4. We then have $d(\Upsilon_{\mathcal{T}}(P\Phi^t z), \Upsilon_{\mathcal{T}}(Pz)) \geq \ell m(\ell) + \ell$ for all $z \in V$ and all $t \geq T(q)$. In particular, if $z \in V$ and if $\rho_z : [0, a] \rightarrow [0, \infty)$ ($a \in (0, \infty]$) is a homeomorphism with the property that the map $t \rightarrow \Upsilon_{\mathcal{T}}(P\Phi^{\rho_z(t)}z)$ is a *parametrized* ℓ -quasi-geodesic in $\mathcal{C}(S)$ then $\rho_z(m(\ell)) \leq T(q)$.

The subset \mathcal{U} of $\mathcal{Q}^1(S)$ of all points with uniquely ergodic vertical measured geodesic lamination is dense [M82]. Each $z \in \mathcal{U}$ defines a point $F\pi(z) \in \partial\mathcal{C}(S)$ which is just the endpoint of the infinite unparametrized ℓ -quasi-geodesic $t \rightarrow \Upsilon_{\mathcal{T}}(P\Phi^t z)$. The map $z \in \mathcal{U} \rightarrow F\pi(z) \in \partial\mathcal{C}(S)$ is continuous. Thus we can find a number $\epsilon_0(q) \in (0, 1/2)$ which is small enough that the $2\epsilon_0(q)$ -neighborhood W of \tilde{q} is contained in V and that for every point $z \in W \cap \mathcal{U}$ the $\delta_{P\tilde{q}}$ -distance between $f(\tilde{q})$ and $f(\tilde{z})$ is smaller than $\alpha(\ell)/\kappa$ where as before, $\kappa > 0$ is as in (20) in Section 3. The second part of the lemma holds true for the numbers $\epsilon_0(q) > 0, T(q) > 0$. \square

Proof of Theorem 4.3. Let $K \subset \hat{\mathcal{Q}}(S)$ be any compact Φ^t -invariant set and let \tilde{K} be the preimage of K in $\mathcal{Q}^1(S)$ under the natural projection. Let $\epsilon_0 = \epsilon_0(K) \in (0, 1/2), n_0 = n_0(K) > 0$ be as in the first part of Lemma 4.4. Let x_0, \dots, x_k be an (n_0, ϵ_0) -pseudo-orbit for Φ^t which is contained in K and let $t_0, \dots, t_{k-1} \in [n_0, \infty)$ be as in the definition of a pseudo-orbit such that $d(\Phi^{t_i}x_i, x_{i+1}) \leq \epsilon_0$ for $i < k$. Let γ be a characteristic arc of this pseudo-orbit which is parametrized on $[0, \sum_{i=0}^{k-1} t_i + k]$ in such a way that for each j the restriction of γ to $[\sum_{i<j} t_i + j, \sum_{i<j+1} t_i + j]$ is a reparametrization of the orbit segment $\{\Phi^t x_j \mid t \in [0, t_j]\}$ by a translation. The points x_1, \dots, x_{k-1} are the breakpoints of the characteristic arc.

Let $\tilde{\gamma}$ be a lift of γ to $\mathcal{Q}^1(S)$. By Lemma 4.4, the assignment $t \rightarrow \Upsilon_{\mathcal{T}}(P\tilde{\gamma}(t))$ is an ℓ_0 -quasi-geodesic in $\mathcal{C}(S)$ for a number $\ell_0 > 1$ only depending on K . The map $t \rightarrow P\tilde{\gamma}(t) \in \mathcal{T}(S)$ is one-Lipschitz. Since by Lemma 2.1 there is a number $L > 0$ such that $d(Pq, Pz) \geq d(\Upsilon_{\mathcal{T}}(Pq), \Upsilon_{\mathcal{T}}(Pz))/L - L$ for all $q, z \in \mathcal{Q}^1(S)$, we conclude that the curve $t \rightarrow P\tilde{\gamma}(t)$ is a uniform quasi-geodesic in $\mathcal{T}(S)$. By the first part of Theorem 2.3, this implies that the Hausdorff distance between $P\tilde{\gamma}$ and the Teichmüller geodesic with the same endpoints is bounded from above by a universal constant. This Teichmüller geodesic is contained in $\mathcal{T}_{\sigma}(S)$ for a constant $\sigma > 0$ only depending on K . As a consequence, the Hausdorff distance between $\tilde{\gamma}$ and the tangent line of this geodesic is bounded from above by a universal constant $b > 0$.

Recall that a mapping class $g \in \text{Mod}(S)$ is *pseudo-Anosov* if the cyclic subgroup of $\text{Mod}(S)$ generated by g acts on the curve graph $\mathcal{C}(S)$ with unbounded orbits.

In this case the conjugacy class of g can be represented by a closed orbit for the Teichmüller flow Φ^t on $\mathcal{Q}(S)$, and it can be represented by a closed orbit for the Teichmüller flow on $\hat{\mathcal{Q}}(S)$ if the conjugacy class of g is contained in the normal subgroup Γ of $\text{Mod}(S)$. Assume now that the (n_0, ϵ_0) -pseudo-orbit x_0, \dots, x_k contained in K is closed and let $\tilde{\gamma}$ be a lift to $\mathcal{Q}^1(S)$ of a closed characteristic arc γ , parametrized on $[0, \sum_i t_i + k]$. By Lemma 4.4, the curve $t \rightarrow \Upsilon_{\mathcal{T}}(P\tilde{\gamma}(t))$ is an infinite ℓ_0 -quasi-geodesic in $\mathcal{C}(S)$ which is invariant under an element $g \in \Gamma < \text{Mod}(S)$ of the mapping class group. The mapping class g acts on this quasi-geodesic as a translation and hence it is pseudo-Anosov. As a consequence, there is a unique g -invariant Teichmüller geodesic in $\mathcal{T}(S)$ whose cotangent line in $\mathcal{Q}^1(S)$ projects to a periodic orbit of Φ^t in $\hat{\mathcal{Q}}(S)$ which defines the free homotopy class of γ . In other words, there is a closed orbit for Φ^t in $\hat{\mathcal{Q}}(S)$ which is freely homotopic to γ .

By the first part of Theorem 2.3, applied to the biinfinite quasi-geodesic $P\tilde{\gamma}$ in $\mathcal{T}(S)$, this orbit is contained in a compact subset $C_0 \supset K$ of $\hat{\mathcal{Q}}(S)$ not depending on the pseudo-orbit. Let $C \subset C_0$ be the Φ^t -invariant subset of C_0 of all points whose Φ^t -orbit is entirely contained in C_0 . The periodic orbit defined by the conjugacy class of g is contained in C . Let \tilde{C} be the preimage of C in $\mathcal{Q}^1(S)$. Then every lift to $\mathcal{Q}^1(S)$ of a periodic orbit in $\hat{\mathcal{Q}}(S)$ determined as above by a closed (n_0, ϵ_0) -pseudo-orbit contained in K is contained in \tilde{C} .

Let again $\pi : \mathcal{Q}^1(S) \rightarrow \mathcal{PM}\mathcal{L}$ be the canonical projection and write $A = \pi(\tilde{C} \cup \mathcal{F}(\tilde{C}))$ where $\mathcal{F} : \mathcal{Q}^1(S) \rightarrow \mathcal{Q}^1(S)$ is the flip $q \rightarrow \mathcal{F}(q) = -q$. As in Section 3, let $F : A \rightarrow \partial\mathcal{C}(S)$ be the measure forgetting injection. For $q \in \pi^{-1}(A)$ let δ_{Pq} be the distance on $\partial\mathcal{C}(S)$ defined in equation (18) of Section 3 and denote by $B_q(\pi(q), r)$ the ball of radius r about $\pi(q)$ in A with respect to the distance $(x, y) \in A \times A \rightarrow \delta_{Pq}(Fx, Fy) \in [0, \infty)$.

By the second part of Theorem 2.3, applied to the projection into $\mathcal{T}(S)$ of the preimage \tilde{C} of the compact Φ^t -invariant set $C \subset \hat{\mathcal{Q}}(S)$, by Lemma 3.2 and by inequality (20) of Section 3, there are numbers $\alpha_0 < 1/2, a > 1, b > 0$ such that for every $q \in \tilde{C}$ and for all $t > 0$ we have

$$(30) \quad \delta_{P\Phi^{-t}q} \leq ae^{-bt}\delta_{Pq} \text{ on } B_q(\pi(q), 4\alpha_0).$$

Moreover, for every $\alpha < \alpha_0$ there is a number $\beta = \beta(\alpha) < 1$ such that for every $q \in \tilde{C}$ we have $A \cap \pi B(q, \beta) \subset B_q(\pi(q), \alpha)$ where $B(q, \beta)$ is the ball of radius β about q in $\mathcal{Q}^1(S)$ (compare the proof of Theorem 3.1).

Let $n = \max\{n_0, \log(4a)/b\}$, let $\alpha < \alpha_0$ and let $\beta = \min\{\epsilon_0, \beta(\alpha), \kappa^{-1} \log 2\}$ where $\kappa > 0$ is as in inequality (19) of Section 3. We claim that for a lift $\tilde{\gamma} : [0, T] \rightarrow \mathcal{Q}^1(S)$ of a characteristic arc of any (n, β) -pseudo-orbit contained in K we have $\delta_{P\tilde{\gamma}(0)}(\pi\tilde{\gamma}(0), \pi\tilde{\gamma}(T)) \leq \alpha$. To see this we proceed by induction on the number of breakpoints of the pseudo-orbit. The case that there is a no breakpoint is trivial, so assume that the claim is known whenever the number of breakpoints of the pseudo-orbit is at most $k-1 \geq 0$. Let $\tilde{\gamma}$ be a lift to $\mathcal{Q}^1(S)$ of a characteristic arc of an (n, β) -pseudo-orbit contained in K with k breakpoints. Let $t_0 \geq n+1$ be such that $\gamma(t_0)$ is the first breakpoint of γ . By assumption and the choice of

the parametrization of a characteristic arc we have $d(\tilde{\gamma}(t_0), \tilde{\gamma}(t_0 - 1)) \leq \beta$. Since $\tilde{\gamma}(t_0) \in \tilde{C}, \tilde{\gamma}(t_0 - 1) \in \tilde{C}$, by the choice of β we have

$$(31) \quad \delta_{P\tilde{\gamma}(t_0)}(\pi\tilde{\gamma}(t_0), \pi\tilde{\gamma}(t_0 - 1)) \leq \alpha,$$

moreover the distances $\delta_{P\tilde{\gamma}(t_0)}, \delta_{P\tilde{\gamma}(t_0-1)}$ are 2-bilipschitz equivalent (recall that the projection $P : \mathcal{Q}^1(S) \rightarrow \mathcal{T}(S)$ is distance non-increasing).

Now $\pi\tilde{\gamma}(T) \in B_{\tilde{\gamma}(t_0)}(\pi\tilde{\gamma}(t_0), \alpha)$ by the induction hypothesis and therefore

$$(32) \quad \delta_{P\tilde{\gamma}(t_0-1)}(\pi\tilde{\gamma}(t_0 - 1), \pi\tilde{\gamma}(T)) \leq 4\alpha.$$

On the other hand, since $\alpha \leq \alpha_0$, since $n \geq \log(4a)/b$ and since $\pi\tilde{\gamma}(t_0 - 1) = \pi\tilde{\gamma}(0)$ we infer from the estimate (30) that

$$(33) \quad \delta_{P\tilde{\gamma}(t_0-1)}(\pi\tilde{\gamma}(t_0 - 1), \pi\tilde{\gamma}(T)) \geq 4\delta_{P\tilde{\gamma}(0)}(\pi\tilde{\gamma}(0), \tilde{\gamma}(T)).$$

Together this implies the claim.

Note moreover that the argument in the previous paragraph together with the estimate (30) also shows that

$$(34) \quad \delta_{\tilde{\gamma}(t)}(\pi\tilde{\gamma}(t), \pi\tilde{\gamma}(T)) \leq 4a\alpha$$

for all $t \in [0, T]$ with the additional property that $\tilde{\gamma}(t) \in \tilde{C}$ (note that this holds true for every t such that $\tilde{\gamma}(t)$ projects to an orbit segment defining the pseudo-orbit).

Consider again a lift $\tilde{\gamma}$ to $\tilde{\mathcal{Q}}^1(S)$ of a closed characteristic curve γ for an (n, β) -pseudo-orbit contained in K . The curve $\Upsilon_{\mathcal{T}}(P\tilde{\gamma})$ is a uniform quasi-geodesic in $\mathcal{C}(S)$ which is invariant under a pseudo-Anosov element $g \in \Gamma < \text{Mod}(S)$ on $\partial\mathcal{C}(S)$ whose conjugacy class defines the free homotopy class of γ . The oriented cotangent line of the axis of g is contained in \tilde{C} . If $z \in \tilde{C}$ is a point in this cotangent line then $\pi(z) \in A$ is a fixed point for the action of g on \mathcal{PML} . An inductive application to longer and longer subsegments of $\tilde{\gamma}$ of the argument which lead to the estimate (34) shows that for every $t \in \mathbb{R}$ such that $\tilde{\gamma}(t) \in \tilde{C}$ the fixed point $\pi(z) \in A$ of g is contained in the ball $B_{\tilde{\gamma}(t)}(\pi(\tilde{\gamma}(t)), 4a\alpha)$. The same argument also shows that the fixed point $\pi(-z)$ for the action of g is contained in $B_{-\tilde{\gamma}(t)}(\pi(-\tilde{\gamma}(t)), 4a\alpha)$. The periodic orbit on $\hat{\mathcal{Q}}(S)$ defined by g is contained in the compact Φ^t -invariant subset $C \supset K$ of $\hat{\mathcal{Q}}(S)$ determined above.

By the considerations in Theorem 3.1 and its proof, applied to the compact Φ^t -invariant subset C of $\hat{\mathcal{Q}}(S)$, this means that for every $\delta > 0$ there is a constant $\beta > 0$ only depending on K with the following property. Let x_0, \dots, x_k be a closed (n, β) -pseudo-orbit contained in K . Then there is a closed orbit for Φ^t contained in C whose Hausdorff distance to a closed characteristic curve defined by the pseudo-orbit is at most δ . From this Theorem 4.3 follows. \square

The Anosov closing lemma implies the existence of many periodic orbits near any non-wandering point of a compact Φ^t -invariant subset K of $\hat{\mathcal{Q}}(S)$. However, as for compact invariant hyperbolic sets in the usual sense of smooth dynamical systems (see [HK95]), these periodic orbits are in general not contained in K . The next lemma is an adaptation of Corollary 6.4.19 of [HK95] and shows that the periodic orbits can be chosen to be contained in an arbitrarily small neighborhood of K .

Lemma 4.5. *Let K be a compact Φ^t -invariant subset of $\hat{\mathcal{Q}}(S)$ and let U be an open neighborhood of K . Then every non-wandering point $q \in K$ is an accumulation point of periodic points of Φ^t whose orbits are entirely contained in U .*

Proof. We adapt the short argument in the proof of Corollary 6.4.19 of [HK95]. Namely, let K be a compact Φ^t -invariant subset of $\hat{\mathcal{Q}}(S)$ and let U be an open neighborhood of K .

By compactness, there is a number $\beta > 0$ such that the set U contains the β -neighborhood of K . Let $q \in K$ be a non-wandering point for Φ^t . Let $\delta < \beta$ and let $n = n(K) > 0$, $\epsilon_2 = \epsilon_2(K, \delta) < \delta$ be as in Theorem 4.3. Assume without loss of generality that the $3\epsilon_2$ -neighborhood of q in $\hat{\mathcal{Q}}(S)$ is contained in a contractible subset of $\hat{\mathcal{Q}}(S)$. Let $V \subset K$ be the intersection with K of the $\epsilon_2/2$ -neighborhood of q . Since q is non-wandering by assumption, there is some $N > n$ such that $\Phi^N(V) \cap V \neq \emptyset$. Let $x_0 \in V$ be such that $x_1 = \Phi^N x_0 \in V$. Then $d(x_0, x_1) < \epsilon_2$ and hence the orbit segment $\{\Phi^t x_0 \mid 0 \leq t \leq N\}$ defines a closed (n, ϵ_2) -pseudo-orbit contained in K . By Theorem 4.3 and by the choice of the constants ϵ_2, n , this pseudo-orbit is δ -shadowed by a periodic orbit for Φ^t . This periodic orbit is then entirely contained in the $\delta < \beta$ -neighborhood of K and hence it is contained in U by the choice of β . Moreover, the distance between the point q and the periodic orbit is bounded from above by 2δ . This implies the lemma. \square

In the case of a topologically transitive compact invariant set $K \subset \hat{\mathcal{Q}}(S)$ we can say more.

Lemma 4.6. *Let K be a compact Φ^t -invariant topologically transitive subset of $\hat{\mathcal{Q}}(S)$. Then for every $\sigma > 0$ there is a periodic orbit for Φ^t whose Hausdorff distance to K (as subsets of $\hat{\mathcal{Q}}(S)$) is at most σ .*

Proof. Let $K \subset \hat{\mathcal{Q}}(S)$ be a compact topologically transitive set and let $\sigma > 0$. Let $n = n(K) > 0$, $\epsilon_2 = \epsilon_2(K, \sigma/2) < \sigma/2$ be as in Theorem 4.3. Since K is topologically transitive by assumption, there is some $q \in K$ and there is some $T > n$ such that $d(q, \Phi^T q) < \epsilon_2$ and that moreover the Hausdorff distance between the set K and its subset $B = \{\Phi^t q \mid 0 \leq t \leq T\}$ is at most $\sigma/2$.

By Theorem 4.3, applied to the closed (n, ϵ_2) -pseudo-orbit defined by the orbit segment $\{\Phi^t q \mid 0 \leq t \leq T\}$, there is a periodic orbit for Φ^t whose Hausdorff distance to B is at most $\sigma/2$. This means that the Hausdorff distance between this orbit and the set K is at most σ and shows the lemma. \square

For a compact Φ^t -invariant subset $K \subset \hat{\mathcal{Q}}(S)$ denote by $h_{\text{top}}(K)$ the topological entropy of the restriction of Φ^t to K . For an arbitrary subset $U \subset \hat{\mathcal{Q}}(S)$ and a number $r > 0$ let $n_U(r)$ be the number of all periodic orbits of Φ^t of period at most r which are contained in U . The following corollary is another fairly immediate consequence of Theorem 4.3.

Corollary 4.7. *Let $K \subset \hat{\mathcal{Q}}(S)$ be a compact Φ^t -invariant topologically transitive set. Then for every open neighborhood U of K we have*

$$h_{\text{top}}(K) \leq \liminf_{r \rightarrow \infty} \frac{1}{r} \log n_U(r).$$

Proof. Let $K \subset \hat{\mathcal{Q}}(S)$ be a topologically transitive compact Φ^t -invariant set and let U be an open neighborhood of K . Then there is a number $\beta > 0$ such that U contains the β -neighborhood of K .

Let $\delta < \beta$ be sufficiently small that the δ -neighborhood of every point in K is contained in a contractible subset of $\hat{\mathcal{Q}}(S)$. Let $n = n(K) > 0$, $\epsilon_2 = \epsilon_2(K, \delta/8) < 1$ be as in Theorem 4.3. Since the Teichmüller flow on K is topologically transitive by assumption, by compactness of $K \times K$ there is a number $N > n$ with the following property. Let $x, x' \in K$; then there is some $y \in K$ and some $T \in [n, N]$ with $d(y, x') < \epsilon_2$ and $d(\Phi^T y, x) < \epsilon_2$.

Recall that a subset E of K is called (m, δ) -separated for some $m \geq 0$ if for any two points $x \neq y \in E$ we have

$$(35) \quad d(\Phi^t x, \Phi^t y) \geq \delta \text{ for some } t \in [0, m].$$

Let $m > n$ and let $E_m \subset K$ be any (m, δ) -separated set. Let $x \in E_m$. By the choice of $N > n$ there is some $y \in K$ and some $T \in [n, N]$ such that $d(y, \Phi^m x) < \epsilon_2$ and $d(\Phi^T y, x) < \epsilon_2$. By Theorem 4.3, the closed (n, ϵ_2) -pseudo-orbit x, y, x is $\delta/8$ -shadowed by a periodic orbit which defines the characteristic free homotopy class of the pseudo-orbit. Since periodic orbits for Φ^t in $\hat{\mathcal{Q}}(S)$ minimize the length in their free homotopy class, the length of the periodic orbit does not exceed $m + N + 2\epsilon_2$. Moreover, by the choice of δ this periodic orbit is contained in U . There is a point $\zeta(x)$ on the orbit with $d(x, \zeta(x)) \leq \delta/8$. In other words, there is a map ζ which associates to every point $x \in E_m$ a point $\zeta(x) \in U$ whose orbit under Φ^t is entirely contained in U and periodic of period at most $m + N + 2\epsilon_2$.

Since the points in the set E_m are (m, δ) -separated by assumption and the orbits of Φ^t are geodesics parametrized by arc length, the orbit segments $c(x) = \cup_{t \in (-\delta/8, \delta/8)} \Phi^t \zeta(x)$ ($x \in E_m$) are pairwise disjoint. Thus for a fixed periodic orbit γ for Φ^t of length at most $m + N + 2\epsilon_2$ there are at most $4(m + N + 2)/\delta$ distinct points $x \in E_m$ with $\zeta(x) \in \gamma$. As a consequence, there are at least $\delta \text{card}(E_m)/4(m + N + 2)$ distinct periodic orbits of period at most $m + N + 2$ in U . This shows that the asymptotic growth as $m \rightarrow \infty$ of the maximal cardinality of an (m, δ) -separated subset of K does not exceed the asymptotic growth of the numbers $n_U(r)$ as $r \rightarrow \infty$. The corollary is now an immediate consequence from the definition of the topological entropy of a continuous flow on a compact space (recall also from Theorem 3.1 that the Teichmüller flow on K is expansive and hence for all sufficiently small $\delta > 0$ its topological entropy is just the asymptotic growth rate of maximal (n, δ) -separated sets as $m \rightarrow \infty$). \square

5. PROOF OF THE THEOREMS

In this section we show Theorem 1 and Theorem 2 from the introduction. For this we continue to use the assumptions and notations from Sections 2 and 3. In particular, we always denote by d the Teichmüller metric on Teichmüller space $\mathcal{T}(S)$ for S .

The *Poincaré series with exponent* $\alpha > 0$ at a point $x \in \mathcal{T}(S)$ is defined to be the series

$$(36) \quad \sum_{g \in \text{Mod}(S)} e^{-\alpha d(x, gx)}.$$

The *critical exponent* of $\text{Mod}(S)$ is the infimum of all numbers $\alpha > 0$ such that the Poincaré series with exponent α converges. Note that this critical exponent does not depend on the choice of x . Athreya, Bufetov, Eskin and Mirzakhani [ABEM06] showed that the critical exponent of the Poincaré series equals $h = 6g - 6 + 2m$ and that the Poincaré series diverges at the critical exponent.

For $r > 0$ and for a compact set $K \subset \mathcal{Q}(S)$ let $n_K^\square(r)$ be the number of all periodic orbits for the Teichmüller flow of period at most r which *intersect* K . The next lemma is the first part of Theorem 1.

Lemma 5.1. *For every compact subset K of $\mathcal{Q}(S)$ we have*

$$\limsup_{r \rightarrow \infty} \frac{1}{r} \log n_K^\square(r) \leq 6g - 6 + 2m.$$

Proof. Let \hat{K} be any compact subset of the *moduli space* $\mathcal{M}(S) = \mathcal{T}(S)/\text{Mod}(S)$ which is the closure of an open set. Let $K_1 \subset \mathcal{T}(S)$ be a relative compact fundamental domain for the action of $\text{Mod}(S)$ on the preimage \tilde{K} of \hat{K} . Let D be the diameter of K_1 and let $x \in K_1$ be any point. Let $g \in \text{Mod}(S)$ be a pseudo-Anosov element whose *axis* (i.e. the unique g -invariant Teichmüller geodesic on which g acts as a translation) projects to a closed geodesic γ in moduli space which intersects \hat{K} . Then there is a point $\tilde{x} \in K_1$ which lies on the axis of a conjugate of g which we denote again by g for simplicity. By the properties of an axis, the length $\ell(\gamma)$ of the closed geodesic γ equals $d(\tilde{x}, g\tilde{x})$. On the other hand, we have

$$(37) \quad d(x, gx) \leq d(\tilde{x}, g\tilde{x}) + 2d(x, \tilde{x}) \leq \ell(\gamma) + 2D$$

by the definition of D and the choice of \tilde{x} . Therefore, if we denote by $K \subset \mathcal{Q}(S)$ the preimage of $\hat{K} \subset \mathcal{M}(S)$ under the natural projection and if we define $N(r)$ for $r > 0$ to be the number of all $g \in \text{Mod}(S)$ with $d(x, gx) \leq r$, then we have

$$(38) \quad \limsup_{r \rightarrow \infty} \frac{1}{r} \log n_K^\square(r) \leq \limsup_{r \rightarrow \infty} \frac{1}{r} \log N(r).$$

Since the critical exponent of the Poincaré series equals $6g - 6 + 2m$, for every $\epsilon > 0$ the Poincaré series converges at the exponent $\alpha = 6g - 6 + 2m + \epsilon$. Let $c(\alpha) > 0$ be its value. Then for every $r > 0$, the cardinality of the set $\{g \in \text{Mod}(S) \mid d(x, gx) \leq r\}$ does not exceed $c(\alpha)e^{\alpha r}$ (note the term in the Poincaré series corresponding to such an element of $\text{Mod}(S)$ is *not smaller* than $e^{-\alpha r}$). This

shows that $\limsup \frac{1}{r} \log N(r) \leq 6g - 6 + 2m + \epsilon$. Since $\epsilon > 0$ and the compact set $\hat{K} \subset \mathcal{M}(S)$ were arbitrarily chosen, the lemma follows. \square

As an immediate consequence we obtain the first part of Theorem 2 from the introduction.

Corollary 5.2. *Let $K \subset \mathcal{Q}(S)$ be a compact Φ^t -invariant topologically transitive set. Then $h_{\text{top}}(K) \leq 6g - 6 + 2m$.*

Proof. Let $K \subset \mathcal{Q}(S)$ be a compact Φ^t -invariant topologically transitive set, let $q \in K$ be a point whose orbit under Φ^t is dense in K and let $\hat{q} \in \hat{\mathcal{Q}}(S)$ be a preimage of q under the natural projection $\Theta : \hat{\mathcal{Q}}(S) \rightarrow \mathcal{Q}(S)$. Let \hat{K} be the closure of the orbit of \hat{q} ; then \hat{K} is a compact Φ^t -invariant topologically transitive set which is mapped by Θ onto K . By equivariance of the Teichmüller flow under the projection Θ and by Corollary 4.7, for every open relative compact neighborhood U of \hat{K} we have

$$(39) \quad h_{\text{top}}(K) \leq h_{\text{top}}(\hat{K}) \leq \liminf_{r \rightarrow \infty} \frac{1}{r} \log n_U(r).$$

Now the projection Θ maps periodic orbits for Φ^t in U of period at most r to periodic orbits for Φ^t of period at most r which are contained in the relative compact set $\Theta(U) \subset \mathcal{Q}(S)$. If the periodic orbits $\gamma_1 \neq \gamma_2$ in U are mapped to the same periodic orbit in $\Theta(U)$ then there is some element g from the factor group $G = \text{Mod}(S)/\Gamma$ which maps γ_1 to γ_2 . Since G is finite, the number of distinct periodic orbits in U which are mapped to the single orbit in $\Theta(U)$ is uniformly bounded. Therefore by Lemma 5.1 we have

$$(40) \quad \liminf_{r \rightarrow \infty} \frac{1}{r} \log n_U(r) \leq \liminf_{r \rightarrow \infty} \frac{1}{r} \log n_{\Theta(U)}(r) \leq 6g - 6 + 2m.$$

This shows the corollary. \square

Now we are ready for the proof of the second part of Theorem 1 from the introduction. Together with Corollary 3.4, it also implies Theorem 2.

Proposition 5.3. *For every $\epsilon > 0$ there is a compact Φ^t -invariant subset K of $\mathcal{Q}(S)$ with*

$$\liminf_{r \rightarrow \infty} \frac{1}{r} \log n_K(r) \geq 6g - 6 + 2m - \epsilon.$$

Proof. Let $\mathcal{FML} \subset \mathcal{PML}$ be the $\text{Mod}(S)$ -invariant Borel subset of all projective measured geodesic laminations whose support is minimal and fills up S . Recall from [K199, H06] that there is a continuous $\text{Mod}(S)$ -equivariant surjection

$$(41) \quad F : \mathcal{FML} \rightarrow \partial\mathcal{C}(S)$$

which associates to a projective measured geodesic lamination in \mathcal{FML} its support. Define

$$(42) \quad \mathcal{A} = \pi^{-1}\mathcal{FML} \subset \mathcal{Q}^1(S).$$

Let λ be the Φ^t -invariant probability measure on $\mathcal{Q}(S)$ in the Lebesgue measure class constructed in [M82, V86]. This measure is ergodic and mixing under the Teichmüller flow, with full support. In particular, the Φ^t -orbit of every density point $q \in \mathcal{Q}(S)$ for λ returns to every neighborhood of q for arbitrarily large times. The measure λ lifts to a $\text{Mod}(S)$ -invariant Φ^t -invariant Radon measure $\tilde{\lambda}$ on $\mathcal{Q}^1(S)$ of full support which gives full measure to the $\text{Mod}(S)$ -invariant Borel set \mathcal{A} [M82].

The Lebesgue measure $\tilde{\lambda}$ on $\mathcal{Q}^1(S)$ is absolutely continuous with respect to the strong unstable foliation. More precisely, for every $q \in \mathcal{Q}^1(S)$ there is a natural conditional measure $\tilde{\lambda}_q$ for $\tilde{\lambda}$ on the strong unstable manifold $W^{su}(q)$, and these conditional measures transform under the Teichmüller flow via $d\tilde{\lambda}_{\Phi^t q} \circ \Phi^t = e^{ht} d\tilde{\lambda}_q$ where $h = 6g - 6 + 2m$ as before. The image under the projection π of the measure $\tilde{\lambda}_q$ on $W^{su}(q)$ is a locally finite Borel measure λ_q on the open dense subset of $\mathcal{PM}\mathcal{L}$ of all projective measured geodesic laminations which together with $\pi(-q)$ jointly fill up S . The measures λ_q are all absolutely continuous, and they depend continuously on $q \in \mathcal{Q}^1(S)$ in the weak topology. Moreover, for each q the measure λ_q gives full measure to the set $\mathcal{FM}\mathcal{L}$ and hence it can be mapped via the surjection F to a measure on $\partial\mathcal{C}(S)$ which we denote again by λ_q .

Recall from (18) and (19) in Section 3 the definition of the distances δ_h ($h \in \mathcal{T}(S)$) on $\partial\mathcal{C}(S)$ and their properties. For $q \in \mathcal{A}$ and $\chi > 0$ define $D(q, \chi) \subset \partial\mathcal{C}(S)$ to be the closed δ_{Pq} -ball of radius χ about $F\pi(q) \in \partial\mathcal{C}(S)$.

Let $q_0 \in \mathcal{Q}(S)$ be a density point for the Lebesgue measure λ and let q_1 be a lift of q_0 to $\mathcal{Q}^1(S)$. Assume without loss of generality that Pq_1 is not fixed by any element of $\text{Mod}(S)$. This is possible since the set of points in $\mathcal{T}(S)$ which are stabilized by a non-trivial element of $\text{Mod}(S)$ is closed and nowhere dense and since the Lebesgue measure is of full support.

Let $m > 1$ be as in the second part of Lemma 4.4. We may assume that the image under the map $\Upsilon_{\mathcal{T}}$ of every Teichmüller geodesic in $\mathcal{T}(S)$ is an unparametrized m -quasi-geodesic in $\mathcal{C}(S)$. Let $\kappa > 0$ be as in inequality (19). By Lemma 3.2 and the inequality (20), there is a number $\alpha > 0$ depending on m and there is a neighborhood V of q_1 in $\mathcal{Q}^1(S)$ of diameter at most $\log 2/\kappa$ and a number $T_0 > 0$ such that for every $t \geq T_0$ and every $z \in V \cap \mathcal{A}$ we have $\delta_{P\Phi^t z} \geq 16\delta_{Pz}$ on $D(\Phi^t z, \alpha)$. Since q_0 is recurrent and hence the vertical measured geodesic lamination of q_1 is uniquely ergodic and fills up S , by Lemma 3.2 of [H07a] there is a constant $\chi \leq \alpha/4$ such that $F\pi(V \cap \mathcal{A} \cap W^{su}(q_1)) \supset D(q_1, \chi)$.

Let $\epsilon_0 = \epsilon_0(q_0) > 0$ be as in the second part of Lemma 4.4. We may assume that the ϵ_0 -neighborhood of q_0 is contained in a contractible subset of $\mathcal{Q}(S)$. By continuity, there is a compact neighborhood $K \subset V$ of q_1 with the following properties.

- (1) The diameter of K does not exceed $\max\{\epsilon_0, (\log 2)/\kappa\}$.
- (2) $F \circ \pi(K \cap \mathcal{A}) \subset D(q_1, \chi/4)$.

By the second requirement for K , if $z, u \in K \cap \mathcal{A}$ then $\delta_{Pq_1}(F\pi(u), F\pi(z)) \leq \chi/2$. The first property of K together with the relation (20) for the distances δ_h ($h \in \mathcal{T}(S)$) then implies that $\delta_{Pu}(F\pi(z), F\pi(u)) \leq \chi$ and $D(z, \chi) \subset D(u, 4\chi)$.

Following [F69], a *Borel covering relation* for a Borel subset C of a topological space X is a family \mathcal{V} of pairs (x, V) where $V \subset X$ is a Borel set, where $x \in V$ and such that

$$(43) \quad C \subset \bigcup \{V \mid (z, V) \in \mathcal{V} \text{ for some } z \in C\}.$$

For $\chi > 0$ and the neighborhood $K \subset \mathcal{Q}^1(S)$ of q_1 as above define

$$(44) \quad \mathcal{V}_{q_0, \chi, K} = \{(F\pi(q), gD(q_1, \chi)) \mid \\ q \in W^{su}(q_1) \cap \mathcal{A}, g \in \text{Mod}(S), gK \cap \bigcup_{t>0} \Phi^t q \neq \emptyset\}.$$

By Proposition 3.5 of [H07b], via possibly decreasing the size of χ and K we may assume that the covering relation $\mathcal{V}_{q_0, \chi, K}$ is a *Vitali relation* for the measure λ_{q_1} on $\partial\mathcal{C}(S)$.

Since the measures λ_z and the distances δ_{Pz} on $\partial\mathcal{C}(S)$ depend continuously on $z \in \mathcal{Q}^1(S)$, there is a number $a \leq \lambda_{q_1} D(q_1, \chi/4)$ such that $\lambda_z D(u, \chi) \in [a, a^{-1}]$ for all $z \in K, u \in K \cap \mathcal{A}$. By the transition properties for the measures λ_z and invariance under the action of the mapping class group, if $g \in \text{Mod}(S)$, if $z \in W^{su}(q_1)$ and if $t > 0$ are such that $\Phi^t z \in gK$ for some $t > 0$ then $\lambda_{q_1}(gD(q_1, \chi)) \in [ae^{-ht}, e^{-ht}/a]$.

Let $n_0 = n_0(q_0) > 0$ be as in the second part of Lemma 4.4. Let $\epsilon > 0$ and let $T(\epsilon) > \max\{T_0, n_0\} + 2$ be sufficiently large that $\int_{T(\epsilon)}^{\infty} e^{-\epsilon s} ds \leq e^{-2h} a^2$. Since the covering relation $\mathcal{V}_{q_0, \chi, K}$ is a Vitali relation for λ_{q_1} , there is a covering of λ_{q_1} -almost all of $D(q_1, \chi/4)$ by pairwise disjoint sets from the relation of the form $V(g, t) = (F\pi(z), gD(q_1, \chi))$ where $z \in W^{su}(q_1) \cap \mathcal{A}, F\pi(z) \in D(q_1, \chi/4), g \in \text{Mod}(S)$ and where $t \geq T(\epsilon)$ is such that $\Phi^t z \in gK$. By the choice of χ , we have $z \in V$ and therefore from the assumption $T(\epsilon) \geq T_0$ we deduce that $D(\Phi^t z, 4\chi) \subset D(z, \chi/4) \subset D(q_1, \chi)$. On the other hand, we have $\Phi^t z \in gK$ and hence

$$(45) \quad gD(q_1, \chi) \subset D(\Phi^t z, 4\chi) \subset D(q_1, \chi).$$

The total λ_{q_1} -mass of the balls from the covering is at least $a \leq \lambda_{q_1} D(q_1, \chi/4)$. Therefore there is a number $T > T(\epsilon) + 2$ such that the total volume of those balls $V(g, t)$ from the covering which correspond to a parameter $t \in [T-2, T-1]$ is at least $e^{-\epsilon T} e^{2h}/a$. Now the λ_{q_1} -volume of each such ball is at most $e^{-h(T-2)}/a$ and hence the number of these balls is at least $e^{(h-\epsilon)T}$.

Let $\{g_1, \dots, g_k\} \subset \text{Mod}(S)$ be the subset of $\text{Mod}(S)$ defining these balls. For any i, j the set $g_j g_i D(q_1, \chi)$ is contained in $g_j D(q_1, \chi)$. In particular, for every fixed $j \leq k$ the sets $g_j g_i D(q_1, \chi)$ ($i = 1, \dots, k$) are pairwise disjoint. By induction, we conclude that for any two distinct words $w_1 = g_{i_1} \cdots g_{i_\ell}$ and $w_2 = g_{j_1} \cdots g_{j_m}$ in the letters g_1, \dots, g_k , viewed as elements of $\text{Mod}(S)$, the images of $D(q_1, \chi)$ under w_1, w_2 are either disjoint or properly contained in each other. This shows that the elements g_1, \dots, g_k generate a free semi-subgroup Λ of $\text{Mod}(S)$.

Since $T(\epsilon) \geq n_0$, each word w of length $\ell \geq 1$ in the letters g_1, \dots, g_k defines a closed (n_0, ϵ_0) -pseudo-orbit x_0, \dots, x_ℓ in $\mathcal{Q}(S)$ with $d(x_i, q_0) < \epsilon_0$. This pseudo-orbit consists of the successive projections to $\mathcal{Q}(S)$ of flow lines $\{\Phi^t z \mid t \in [0, \tau]\}$ where $z \in V \cap W^{su}(q_1) \cap \mathcal{A}$ and $\tau \in [T-2, T-1]$ are such that $F\pi(z) \in g_j D(q_1, \chi/4)$

for some $j \leq k$ and $\Phi^\tau z \in g_j K$. Thus by the second part of Lemma 4.4, if $\tilde{\gamma}$ is a lift to $\mathcal{Q}^1(S)$ of a characteristic arc of such a pseudo-orbit then $\Upsilon_{\mathcal{T}}(\tilde{\gamma})$ is a biinfinite unparametrized m -quasi-geodesic in $\mathcal{C}(S)$ which is invariant under the element of $\Lambda \subset \text{Mod}(S)$ defined by w . In particular, this element is pseudo-Anosov, and its conjugacy class defines the characteristic free homotopy class of the closed pseudo-orbit. The length of the periodic orbit of Φ^t determined by w does not exceed the length of a characteristic closed curve for the pseudo-orbit and hence its is not bigger than $T\ell$. Moreover, since by the choice of n_0 for any $s < t$ with the property that $\tilde{\gamma}(s), \tilde{\gamma}(t)$ project to distinct breakpoints of γ the distance between $\Upsilon_{\mathcal{T}}(\tilde{\gamma}(s)), \Upsilon_{\mathcal{T}}(\tilde{\gamma}(t))$ is at least $2c(m)$, it follows from Lemma 2.4 that the unparametrized m -quasi-geodesic $\Upsilon_{\mathcal{T}}(\tilde{\gamma})$ is in fact a *parametrized p -quasi-geodesic* for some $p > m$. Using once more the first part of Theorem 2.3, this implies that the axis of the element of $\Lambda \subset \text{Mod}(S)$ defined by w passes through a fixed compact neighborhood B of P_{q_1} in $\mathcal{T}(S)$, and the projection of its unit tangent line to $\mathcal{Q}(S)$ is a periodic orbit for Φ^t which is contained in a compact subset C_0 of $\mathcal{Q}(S)$ not depending on w . If we denote by C the closed subset of C_0 of all points $z \in C_0$ whose orbit under Φ^t is entirely contained in C_0 then each of these orbits is contained in C .

The above argument does not immediately imply that the asymptotic growth rate of the number of periodic orbits in C is at least $h - \epsilon$. Namely, periodic orbits of the Teichmüller flow on $\mathcal{Q}(S)$ are in one-to-one correspondence to *conjugacy classes* of pseudo-Anosov elements in $\text{Mod}(S)$. Thus if we want to count periodic orbits for Φ^t in $\mathcal{Q}(S)$ using the semi-subgroup Λ of $\text{Mod}(S)$ constructed above, then we have to identify those elements of Λ which are conjugate in $\text{Mod}(S)$.

For this recall that the axis of each element of the semi-subgroup Λ of $\text{Mod}(S)$ passes through the fixed compact neighborhood B of P_{q_1} . Thus if γ, ζ is the axis of $u, w \in \Lambda$ and if u, w are conjugate in $\text{Mod}(S)$ then there is some $b \in \text{Mod}(S)$ with $w = b^{-1}ub$ and the following additional property. Let $\gamma[0, \tau]$ be a fundamental domain for the action of u on γ and such that $\gamma(0) \in B$. Such a fundamental domain always exists, perhaps after a reparametrization of γ . Then there is some $t \in [0, \tau]$ such that $b^{-1}\gamma(t) \in B$.

As a consequence, the number of all elements $w \in \Lambda$ which are conjugate to a fixed element $u \in \Lambda$ is bounded from above by the number of elements $b \in \text{Mod}(S)$ with $bB \cap \gamma[0, \tau] \neq \emptyset$. In particular, if D is the diameter of B then this number does not exceed the cardinality of the set

$$(46) \quad \{b \in \text{Mod}(S) \mid d(bP_{q_1}, \gamma[0, \tau]) \leq D\}.$$

However, this cardinality is bounded from above by a universal multiple of τ . Therefore there is a constant $c > 0$ such that for all sufficiently large $r > 0$ the number of periodic orbits of Φ^t contained in C of length at most r is not smaller than $e^{(h-\epsilon)r}/cr$. This completes the proof of the proposition. \square

Remarks:

1. Since the measure entropy of the Φ^t -invariant Lebesgue measure on $\mathcal{Q}(S)$ equals $h = 6g - 6 + 2m$, Corollary 3.4, Proposition 5.3 and the variational principle imply that the entropy of this Lebesgue measure equals the supremum of the

entropies of all Φ^t -invariant Borel probability measures on $\mathcal{Q}(S)$ which give full mass to a compact set. For the action of the Teichmüller flow Φ^t on the moduli space $\mathcal{A}(S)$ of abelian differentials, Bufetov and Gurevich [BG07] recently obtained a much stronger result. Namely, they show that the Φ^t -invariant Lebesgue measure on $\mathcal{A}(S)$ is the unique measure of maximal entropy for $\Phi^t|_{\mathcal{A}(S)}$.

2. The abundance of orbits of the Teichmüller flow which entirely remain in some compact set (depending on the orbit) was earlier established by Kleinbock and Weiss [KW04]. They show that this set is of full Hausdorff dimension.

We conclude this note with some easy remarks on counting of arbitrary periodic orbits for the Teichmüller flow in $\mathcal{Q}(S)$. Namely, for a point $x \in \mathcal{M}(S)$ define $s(x)$ to be the length of the shortest closed geodesic on the hyperbolic surface x . For a periodic geodesic $\gamma : [0, 1] \rightarrow \mathcal{M}(S)$ write $s(\gamma) = \min\{s(\gamma(t)) \mid t \in [0, 1]\}$. We have.

Lemma 5.4. *There is a constant $\kappa > 0$ such that the length of every closed geodesic in $\mathcal{M}(S)$ is at least $(-\log s(\gamma) - \kappa)/(3g - 3 + m)$.*

Proof. Let $\chi > 0$ be sufficiently small that for every complete hyperbolic metric h on S of finite volume and every closed geodesic c for the metric h of length at most χ , the length of every closed h -geodesic which intersects c nontrivially is bigger than 2χ . Via an appropriate choice of the constant $\kappa > 0$ in the statement of the lemma, it is enough to show the lemma for closed geodesics γ in moduli space with $s(\gamma) \leq \chi$.

Thus let $\gamma : [0, 1] \rightarrow \mathcal{M}(S)$ be such a closed geodesic parametrized proportional to arc length. Via reparametrization, assume that $s(\gamma) = s(\gamma(0))$. Let $\tilde{\gamma} : \mathbb{R} \rightarrow \mathcal{T}(S)$ be a lift of γ to $\mathcal{T}(S)$ which is the axis of the pseudo-Anosov element $g \in \text{Mod}(S)$ and let c be an essential simple closed curve on $\tilde{\gamma}(0)$ of length $s(\gamma) \leq \chi$. We claim that the length of the geodesic arc $\tilde{\gamma}[0, 3g - 3 + m]$ is at least $\log \chi - \log s(\gamma)$. Namely, the function $\mathbb{R} \rightarrow (0, \infty)$ which associates to $t \in \mathbb{R}$ the length $\ell_{\tilde{\gamma}(t)}(c)$ of the geodesic representative of the simple closed curve c with respect to the hyperbolic metric $\tilde{\gamma}(t)$ is smooth, and the derivative of its logarithm does not exceed the length of the tangent vector of $\tilde{\gamma}$ (see [IT99]). Thus if the length of $\tilde{\gamma}[0, 3g - 3 + m]$ is smaller than $\log \chi - \log s(\gamma)$ then the length of the curve c with respect to the hyperbolic structure $\tilde{\gamma}(i)$ is smaller than χ for $i = 0, \dots, 3g - 3 + m$. By our choice of χ , for every i there are at most $3g - 3 + m$ distinct simple closed geodesics of length at most χ for the hyperbolic metric $\tilde{\gamma}(i)$. However, since for each i the length of $g^i(c)$ on the hyperbolic surface $\tilde{\gamma}(i)$ equals $s(\gamma)$, the length of $g^i c$ on $\tilde{\gamma}(3g - 3 + m)$ does not exceed χ . As a consequence, there is some $i \in \{1, \dots, 3g - 3 + m\}$ such that the curves c and $g^i c$ coincide. This contradicts the fact that g is pseudo-Anosov. The lemma follows. \square

As a corollary, we obtain a somewhat more precise version of a counting result of Veech [V86], with a new proof. For its formulation, for $r > 0$ let $n(r)$ be the number of periodic orbits of the Teichmüller flow of period at most r . Eskin and Mirzakhani [EM08] recently showed that $n(r)$ is asymptotic to e^{hr}/hr . Here we include an easy much weaker result.

Corollary 5.5. $\limsup_{r \rightarrow \infty} \frac{1}{r} \log n(r) \leq (6g - 6 + 2m)(6g - 5 + 2m).$

Proof. For $\alpha > 0$ let $\mathcal{M}(S)_\alpha$ be the set of all hyperbolic metrics on S up to isometry whose systole is at least α , i.e. $\mathcal{M}(S)_\alpha$ is the projection of the subset $\mathcal{T}(S)_\alpha$ defined in Section 3. By Lemma 5.4, there is a number $\epsilon > 0$ such that for every closed geodesic γ in moduli space of length at most r there is a point $z \in \mathcal{M}(S)_\epsilon$ whose distance to γ is not bigger than $(3g - 3 + m)r$. Namely, every $h \in \mathcal{T}(S)$ can be connected to $\mathcal{T}(S)_\epsilon$ with a geodesic of length at most $\log s(h) + C$ where $C > 0$ is a universal constant (see [Mi96]).

Let $R > 0$ be the diameter of $\mathcal{M}(S)_\epsilon$. For $x \in \mathcal{T}(S)_\epsilon$ there is a lift $\tilde{\gamma}$ of γ to $\mathcal{T}(S)$ which passes through a point y of distance at most $(3g - 3 + m)r + R$ to x . Let $g \in \text{Mod}(S)$ be the pseudo-Anosov map preserving $\tilde{\gamma}$ which defines γ . We then have $d(gx, x) \leq r + 2(3g - 3 + m)r + 2R = (6g - 5 + 2m)r + 2R$ and hence our corollary follows from the fact that the critical exponent of $\text{Mod}(S)$ equals $h = 6g - 6 + 2m$. \square

Acknowledgement: Part of this work was done while I visited the Bernoulli center in Lausanne. I thank Alain Valette for inviting me there, and I thank the center for its hospitality and the excellent working conditions it provided.

REFERENCES

- [ABEM06] J. Athreya, A. Bufetov, A. Eskin, M. Mirzakhani, *Lattice point asymptotics and volume growth on Teichmüller space*, arXiv:math.DS/0610715.
- [BH99] M. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*, Springer Grundlehren 319, Springer, Berlin 1999.
- [Bw06] B. Bowditch, *Intersection numbers and the hyperbolicity of the curve complex*, J. reine angew. Math. 598 (2006), 105–129.
- [Bu06] A. Bufetov, *Logarithmic asymptotics for the number of periodic orbits of the Teichmüller flow on Veech’s space of zippered rectangles*, arXiv:math.DS/0511035.
- [BG07] A. Bufetov, B. Gurevich, *Existence and uniqueness of the measure of maximal entropy for the Teichmüller flow on the moduli space of abelian differentials*, arXiv:math.DS/0703020.
- [B92] P. Buser, *Geometry and spectra of compact Riemann surfaces*, Birkhäuser, Boston 1992.
- [CEG87] R. Canary, D. Epstein, P. Green, *Notes on notes of Thurston*, in “Analytical and geometric aspects of hyperbolic space”, edited by D. Epstein, London Math. Soc. Lecture Notes 111, Cambridge University Press, Cambridge 1987.
- [EM08] A. Eskin, M. Mirzakhani, to appear.
- [F69] H. Federer, *Geometric measure theory*, Springer Grundlehren, Berlin Heidelberg 1969.
- [H06] U. Hamenstädt, *Train tracks and the Gromov boundary of the complex of curves*, in “Spaces of Kleinian groups” (Y. Minsky, M. Sakuma, C. Series, eds.), London Math. Soc. Lec. Notes 329 (2006), 187–207.
- [H07a] U. Hamenstädt, *The geometry of the curve complex and of Teichmüller space*, in ”Handbook of Teichmüller theory”, (A. Papadopoulos, ed.), European Math. Soc. 2007, 447–467.
- [H07b] U. Hamenstädt, *Invariant Radon measures on measured lamination space*, arXiv:math.DS/0703602.
- [HK95] B. Hasselblatt, A. Katok, *Introduction to the modern theory of dynamical systems*, Cambridge University Press, Cambridge 1995.
- [Hu06] J. Hubbard, *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 1. Teichmüller theory*. Matrix Editions, Ithaca, NY, 2006.

- [IT99] Y. Iwayoshi, M. Taniguchi, *An introduction to Teichmüller spaces*, Springer, Tokyo 1999.
- [K199] E. Klarreich, *The boundary at infinity of the curve complex and the relative Teichmüller space*, unpublished manuscript, 1999.
- [KW04] D. Kleinbock, B. Weiss, *Bounded geodesics in moduli space*, Int. Math. Res. Not. 30 (2004), 1551–1560.
- [M80] H. Masur, *Uniquely ergodic quadratic differentials*, Comm. Math. Helv. 55 (1980), 255–266.
- [M82] H. Masur, *Interval exchange transformations and measured foliations*, Ann. Math. 115 (1982), 169–201.
- [MM99] H. Masur, Y. Minsky, *Geometry of the complex of curves I: Hyperbolicity*, Invent. Math. 138 (1999), 103–149.
- [MM00] H. Masur, Y. Minsky, *Geometry of the complex of curves II: Hierarchical structure*, GAFA 10 (2000), 902–974.
- [Mi96] Y. Minsky, *Extremal length estimates and product regions in Teichmüller space*, Duke Math. J. 83 (1996), 249–286.
- [Mo03] L. Mosher, *Stable Teichmüller quasigeodesics and ending laminations*, Geom. Top. 7 (2003), 33–90.
- [V86] W. Veech, *The Teichmüller geodesic flow*, Ann. Math. 124 (1986), 441–530.

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