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Teichmüller Theory

Ursula Hamenstädt

Introduction

The goal of these notes is to give an introduction to geometric aspects of Teichmüller theory for a closed surface S of genus $g \geq 2$. There are very good recent books devoted to this and related subjects, for example [**B92, D11, FM11, FlM07, HM98, Hu06, IT89, Mi95, M07**], and we refer to these books throughout. There are moreover many survey articles on various aspects of the theory, notably the articles in the recent "Handbook of Teichmüller theory", Vol.I,II, edited by A. Papadopoulos. We do not aim at duplicating what can be found in these books and survey articles beyond what is needed to make these notes fairly self-contained. Rather, we give a subjective and selective summary of some of the recent developments which introduce new tools and explore connections to some areas of mathematics which had not been looked at closely before.

Throughout, we consider a closed oriented surface S of genus $g \geq 2$. A *marked complex structure* on S is a pair consisting of a Riemann surface X and an orientation preserving diffeomorphism $\varphi : S \to X$. Two such marked complex structures (X, φ) and (X', φ') are *equivalent* if there is a biholomorphic map $F : X \to X'$ such that $F \circ \varphi$ is isotopic to φ'. The *Teichmüller space* $\mathcal{T}(S)$ of S is the space of all equivalence classes of marked complex structures on S.

Teichmüller space is a complex manifold which is biholomorphic to a bounded domain in $\mathbb{C}^{3g-3}$ (see [**IT89**]). As any complex manifold, it can be equipped with the *Kobayashi pseudo-distance* $d_{\mathcal{T}}$ which is invariant under the group of biholomorphic automorphisms.

It turns out that $\mathcal{T}(S)$ is *complete Kobayashi hyperbolic* which means that the Kobayashi pseudo-distance is a complete distance. Then this distance is the intrinsic path metric of a Finsler metric. The Finsler metric can be described explicitly, and it has some nice properties. For example, any two points in $\mathcal{T}(S)$ can be connected by a unique geodesic. The metric $d_{\mathcal{T}}$ is more commonly called the *Teichmüller metric*.

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The *mapping class group* $\mathrm{Mod}(S)$ of all isotopy classes of orientation preserving diffeomorphisms of S acts on $\mathcal{T}(S)$ properly discontinuously by precomposition of marking. This action preserves the complex structure on $\mathcal{T}(S)$. In fact, if $g > 2$ then $\mathrm{Mod}(S)$ equals the group of biholomorphic automorphisms of $\mathcal{T}(S)$ (see [**IT89**]- in the case $g = 2$ one has to divide $\mathrm{Mod}(S)$ by the hyperelliptic involution which acts trivially on Teichmüller space). In particular, $\mathrm{Mod}(S)$ acts on $(\mathcal{T}(S), d_{\mathcal{T}})$ as a group of isometries. As a consequence, the Teichmüller metric descends to a distance on the *moduli space* $\mathcal{M}_g = \mathcal{T}(S)/\mathrm{Mod}(S)$ of Riemann surfaces. This moduli space is a complex orbifold. If $g > 2$ then its singular locus is the projection of the set of all Riemann surfaces which admit a non-trivial biholomorphic automorphism.

In Section 2 we construct the Teichmüller space as a real analytic manifold using the fact that by the uniformization theorem, a complex structure on S can be identified with an isotopy class of hyperbolic metrics. Section 3 contains an account of the basic properties of the Teichmüller metric with an emphasis on quasi-conformal analysis.

In Section 4 we have a closer look at Teichmüller space as a complex manifold. We adopt the differential-geometric viewpoint and state the basic algebraic-geometric facts about the moduli space and the Torelli map without proof. We introduce complex geodesics and a second natural $\mathrm{Mod}(S)$-invariant metric on $\mathcal{T}(S)$, the so-called *Weil-Petersson metric*.

The Weil-Petersson metric is not complete. The completion of $\mathcal{T}(S)$ with respect to this metric can be described explicitly, and this is explained in Section 5. The completion of moduli space can be identified with the *Deligne-Mumford compactification* of $\mathcal{M}_g$. We introduce the curve complex of S as a combinatorial tool and discuss some results which relate geometric properties of the curve complex to geometric properties of Teichmüller space with the Teichmüller metric.

In the final section of this note we look at the $SL(2, \mathbb{R})$-action on the moduli space of quadratic differentials and relate some of its properties to geometric properties of Teichmüller space and the mapping class group. As an application, we observe that the projection of a complex geodesic in $\mathcal{T}(S)$ intersects a fixed compact subset of $\mathcal{M}_g$ not depending on the geodesic, and it is unbounded.

There is nothing new contained in these notes. Everything presented is looked at with the eyes of a differential geometer. This leads to omissions of results of fundamental importance, of viewpoints and references in a beautiful and overwhelmingly rich theory. Most of the main contributors to this theory will not be given proper credit. My apologies to all of them.

LECTURE 1

Hyperbolic Surfaces

In this section we introduce the Teichmüller space of a closed oriented surface S of genus $g \geq 2$ as the space of all marked hyperbolic structures on S. We discuss natural coordinates arising from hyperbolic geometry which equip Teichmüller space with a smooth (in fact real analytic) structure. Throughout, we use standard facts about the geometry of the hyperbolic plane, and we refer to the excellent treatment in [**B92**] for details.

The starting point is the observation that every closed surface of genus $g \geq 2$ admits a hyperbolic metric, i.e. a smooth Riemannian metric of constant Gauss curvature -1.

Definition 1.1. Let S be a closed oriented surface of genus $g \geq 2$. A *marked hyperbolic surface* is a pair (X, φ) where X is a closed oriented hyperbolic surface of genus $g \geq 2$ and $\varphi : S \to X$ is an orientation preserving diffeomorphism. Two such marked hyperbolic surfaces $(X, \varphi), (X', \varphi')$ are *equivalent* if there exists an isometry $g : X \to X'$ such that φ' and $g \circ \varphi$ are isotopic. The space of equivalence classes is called the *Teichmüller space* $\mathcal{T}(S)$ of S.

In the sequel we often drop the diffeomorphism φ which defines the marking from our notation if no confusion is possible.

The *mapping class group*

$$\mathrm{Mod}(S) = \mathrm{Diff}(S)^+ / \mathrm{Diff}_0^+(S)$$

of isotopy classes of orientation preserving diffeomorphisms of S acts on Teichmüller space by precomposition of marking: If ζ is a diffeomorphism of S then $\zeta(X, \varphi)$ is the point in $\mathcal{T}(S)$ which is given by the same hyperbolic surface X, but where the diffeomorphism φ has been replaced by $\varphi \circ \zeta^{-1}$. If η is isotopic to ζ then the marked hyperbolic structures $\eta(X, \varphi)$ and $\zeta(X, \varphi)$ are equivalent and hence this definition indeed defines an action of the mapping class group on $\mathcal{T}(S)$.

A closed curve α on a compact surface F, possibly with non-trivial boundary ∂F, is *essential* if α is not contractible and *non-peripheral*, i.e. not freely homotopic into the boundary. In the sequel we always assume that closed curves are essential.

Now let X be a marked hyperbolic surface. Let γ be an essential closed curve on S. Then the free homotopy class of γ can be represented by a unique closed geodesic on X. Here the identification of free homotopy classes on S with free homotopy classes on X is via the marking. The following observation is contained in Theorem 1.6.6 and Theorem 1.6.7 of [**B92**]. For its formulation, call a closed curve α on S *simple* if it is the image of an injective mapping $S^1 \to S$.

Proposition 1.2. *Suppose that α is a simple closed curve on S.*

(1) *The closed geodesic on X freely homotopic to α is simple.*
(2) *If β is another simple closed curve on S which is disjoint from α and not freely homotopic to α then the closed geodesics on X representing the free homotopy classes of α, β are disjoint.*

By Proposition 1.2, if α is a simple closed curve on S then we can cut X open along the closed geodesic $\hat{\alpha}$ freely homotopic to α. The result is a hyperbolic surface with two geodesic boundary circles.

There are now two possibilities. The first case is that α is *non-separating*, i.e. $S - \alpha$ is connected. Then the genus of $S - \alpha$ equals $g - 1$. If α is *separating* then $S - \alpha$ is disconnected. Its two connected components S_1, S_2 are surfaces of genus $g_1 \geq 0, g_2 \geq 0$, respectively, with connected boundary. The Euler characteristic of S equals the sum of the Euler characteristics of the components. Since the Euler characteristic of each component is negative, the Euler characteristic of each component is strictly bigger than the Euler characteristic of S. This reasoning also applies to essential simple closed curves on surfaces with boundary.

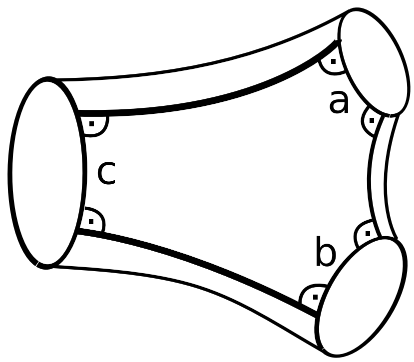
By Proposition 1.2, if the simple closed curve β can be realized disjointly from α and is not freely homotopic to α then the geodesic $\hat{\beta}$ on X freely homotopic to β is disjoint from the geodesic $\hat{\alpha}$. Thus we can successively decompose X into $2g - 2$ hyperbolic *pairs of pants*, i.e. bordered hyperbolic surfaces with geodesic boundary which are homeomorphic to a sphere with 3 holes. Namely, the Euler characteristic of a sphere with 3 holes equals -1, moreover a sphere with 3 holes does not contain any non-peripheral simple closed curve.

Definition 1.3. A *pants decomposition* $\mathcal{P}$ of S consists of $3g - 3$ disjoint simple closed curves which decompose S into $2g - 2$ *pairs of pants*.

By the above discussion, for every pants decomposition $\mathcal{P}$ of S and every marked hyperbolic surface (X, φ), the hyperbolic geodesics representing the pants curves of $\mathcal{P}$ decompose X into $2g - 2$ *hyperbolic pairs of pants*.

Given a hyperbolic pair of pants P, for each pair of distinct boundary geodesics γ_1, γ_2 there is a unique embedded geodesic arc connecting γ_1 to γ_2 which meets γ_1, γ_2 orthogonally at its endpoints. We call such an arc a *seam*. Every boundary geodesic contains precisely two endpoints of seams which decompose the boundary circle into two arcs of equal length. Cutting P open along the three seams results in two isometric right angled convex hyperbolic hexagons (this is Proposition 3.1.5 of [**B92**]).

Now for an arbitrarily prescribed triple (a, b, c) of positive numbers there is up to isometry a unique right angled convex hyperbolic hexagon with three pairwise non-consecutive sides of length a, b, c (Theorem 2.4.1 of [**B92**]). In particular, glueing two such hexagons along the remaining sides yields a hyperbolic pair of

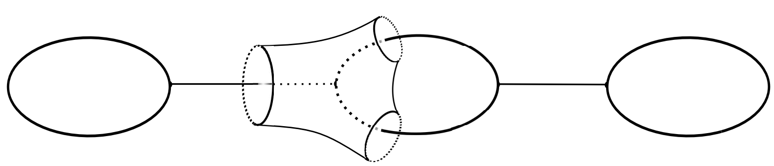


pants with geodesic boundary circles of length $2a, 2b, 2c$. Summarizing, we obtain Theorem 3.1.7 of [**B92**].

Proposition 1.4. *For any triple (a, b, c) of positive numbers, there is up to isometry a unique hyperbolic pair of pants with boundary circles of length a, b, c.*

To reconstruct the hyperbolic surface X from the pairs of pants $X - \mathcal{P}$ we have to remember how the surface was glued from the pairs of pants P_i which are the components of $X - \mathcal{P}$. Boundary circles of pairs of pants are glued in pairs. In particular, since each pair of pants has precisely three boundary circles, the glueing pattern can be represented by a trivalent graph. Each vertex of this graph represents one of the pairs of pants, and each edge represents one of the simple closed curves of the pants decomposition $\mathcal{P}$. The vertices representing the pairs of pants P_i, P_j are connected by an edge if and only if P_i, P_j have a common boundary circle. Vice versa, every connected trivalent graph with $2g - 2$ vertices determines a glueing pattern for pairs of pants to a closed surface of genus g (see Section 3.5 of [**B92**]).

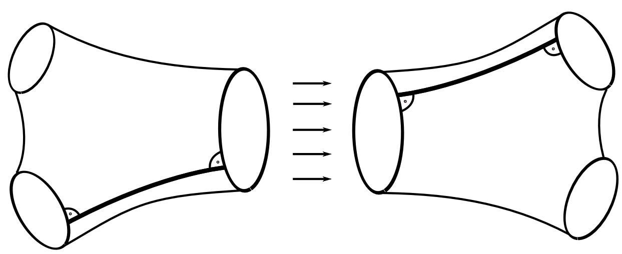
Definition 1.5. A *combinatorial type of pants decomposition* for a closed surface S of genus g is a trivalent graph with $2g - 2$ vertices.



A diffeomorphism of S maps disjoint simple closed curves on S to disjoint simple closed curves. Therefore the mapping class group acts on free homotopy classes of simple closed curves preserving disjointness. As a consequence, it acts on isotopy classes of pants decompositions of S (this statement uses some subtle properties of surface topology which are discussed in the appendix of [**B92**]). In the sequel we always consider simple closed curves and pants decompositions up to isotopy. If $\mathcal{P}$ is a pants decomposition and if $\varphi \in \mathrm{Mod}(S)$ then $\varphi\mathcal{P}$ is a pants decomposition of the same combinatorial type. Vice versa, any two pants decompositions of the same combinatorial type can be transformed into each other by an element of $\mathrm{Mod}(S)$. This follows easily from the fact that any two pairs of pants are diffeomorphic.

To reconstruct the marked hyperbolic metric on X we have to glue the hyperbolic pairs of pants with isometries of their boundary circles. This requires that the lengths of these boundary circles coincide. Now given two pairs of pants P_1, P_2 with a boundary curve γ_1, γ_2 of the same length, the seams of P_1, P_2 which end on γ_1, γ_2 determine two preferred ways of glueing P_1 to P_2 along γ_1, γ_2. Namely, we require that the identification of γ_1 with γ_2 is by an orientation reversing isometry (where the orientation of γ_i is the boundary orientation of P_i) and that the two endpoints of the seams on γ_1 are glued to the endpoints of the seams on γ_2. This is possible because the seams decompose the boundary circles γ_1, γ_2 into two arcs of equal length. The surface Y_0 obtained by this identification is a hyperbolic sphere with four holes and geodesic boundary.

There are more possibilities for glueing P_1 to P_2 along γ_1, γ_2. Namely, we can start rotating P_2 along γ_1 with unit speed. Thus if $\gamma_1, \gamma_2 : \mathbb{R} \to P_1, P_2$ are unit speed parametrizations of the boundary geodesics of P_1, P_2 defining the boundary orientation and so that $\gamma_1(0)$ and $\gamma_2(0)$ is a point on a seam, then for each t we can glue P_2 to P_1 by identifying $\gamma_1(s)$ with $\gamma_2(t-s)$. Denote the resulting hyperbolic surface by Y_t. Note that if $r > 0$ is the length of γ_i then the surface Y_r obtained by rotating by the angle 2π is isometric to the surface Y_0, but any initial marking has been changed by a full *Dehn twist* about the image of γ_i. We refer to Chapter 3 of [**FM11**] for a detailed discussion of Dehn twists. The surface $Y_{r/2}$ obtained by rotating by the angle π corresponds to the second preferred way of glueing P_1 to P_2 along γ_1, γ_2 (i.e. endpoints of seams are glued to endpoints of seams).



The following result is basic for Teichmüller theory. For its formulation, note first that every closed curve α on S defines a function ℓ_α on $\mathcal{T}(S)$ by associating to a marked hyperbolic surface X the length $\ell_\alpha(X)$ of the geodesic in the free homotopy class of α.

Let $\mathcal{P}$ be a fixed pants decomposition of S. Then the $3g-3$ pants curves $\alpha_1, \dots, \alpha_{3g-3}$ of $\mathcal{P}$ define $3g-3$ length functions $\ell_i = \ell_{\alpha_i}$ on $\mathcal{T}(S)$ with values in $\mathbb{R}_+$. Moreover, for a fixed initial choice of glueing the pants in such a way that endpoints of seams are identified, there are $3g-3$ twist parameters $\tau_i \in \mathbb{R}$ for the glueing arising as in the previous paragraph by rotating the two pairs of pants adjacent to α_i with unit speed. Using a surface for which endpoints of seams are identified with endpoints of seams as a basepoint for the marking, this construction associates to a $(6g-6)$-tuple of lengths- and twist parameters a unique marked hyperbolic surface.

The following statement is an extension of the above discussion.

Theorem 1.6. *For each pants decomposition $\mathcal{P}$ of S, the map which associates to a tuple*

$$(\ell_1, \dots, \ell_{3g-3}, \tau_1, \dots, \tau_{3g-3}) \in \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$$

of lengths and twist parameters the surface $(X, \varphi) \in \mathcal{T}(S)$ defined by these data is a bijection onto $\mathcal{T}(S)$.

The map $\mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3} \to \mathcal{T}(S)$ is called a system of *Fenchel-Nielsen coordinates* based at $\mathcal{P}$. The above discussion shows that Fenchel Nielsen coordinates define a surjective map from $\mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$ onto $\mathcal{T}(S)$. To show that this map is also injective we have to verify that distinct Fenchel Nielsen coordinates define surfaces X, X' which are not isometric with an isometry preserving the marking.

Clearly marked isometric surfaces have the same lengths parameters, so we have to study the effect of changing a twist parameter. To this end define the *twist vector field* $\mathbf{t}_\alpha$ of the component α of the pants decomposition $\mathcal{P}$ to be the vector field which integrates to the twist flow Ψ_t about α in Fenchel Nielsen coordinates. The image $\Psi_t X$ of X in Fenchel Nielsen coordinates equals the linear coordinate change obtained by adding t to the twist parameter of α.

Let β be any simple closed geodesic on X. If β does not intersect α then β is also a geodesic for $\Psi_t X$. In particular, in this case the length of β (i.e. the length of its geodesic representative) does not change along a flow line of the twist flow Ψ_t. In the case that β intersects α we can calculate the derivative of the length of β along the twist flow about α. The calculation is taken from [**K83**].

Proposition 1.7.

$$\mathbf{t}_\alpha \ell_\beta = \sum_{p \in \alpha \cap \beta} \cos \theta_p$$

where for each point $p \in \alpha \cap \beta$ the angle θ_p is the angle between α and β measured counter-clockwise from β to α.

Proof. We show the proposition in the case that β has a single point of intersection with α. Let α_1 be a lift of α to the hyperbolic plane $\mathbf{H}^2$ and let $p \in \alpha_1$ be a lift of the intersection point between α and β. There is a lift $\tilde{\beta}$ of β which meets p. For a fixed orientation of $\tilde{\beta}$ let q be the first intersection point distinct from p between $\tilde{\beta}$ and the preimage of α. Denote by α_2 the lift of α through q. The length of the subarc β_0 of $\tilde{\beta}$ connecting p to q is the length of β. Orient α_2 in such a way that the orientation of the basis β_0', α_2' of $T_q\mathbf{H}^2$ is positive.

After twisting along α the amount of t, the projection of β_0 to the twisted surface $\Psi_t X$ is not closed any more. To close it up, glide the endpoint of β_0 on α_2 the signed distance t. Let β_t be the resulting geodesic arc and let ℓ_{β_t} be its length. Then $t \to \beta_t$ is a variation of geodesic arcs with one fixed endpoint p and the second endpoint gliding along α_2. The first variational formula for geodesic lengths (see p.5 of [**CE75**]) states that the derivative at $t = 0$ of the lengths of these arcs equals the cosine of the oriented angle between the tangent of β_0 at q and the tangent of α_2 at q. Or, we have

$$\frac{d}{dt} \ell_{\beta_t}|_{t=0} = \cos \theta_p.$$

The geodesic arc β_t projects to a geodesic loop on the surface $\Psi_t X$. This loop has a breakpoint at its intersection point with α. The angle defect at this breakpoint (i.e. the angle between its incoming and outgoing angle) vanishes at $t = 0$.

By hyperbolic trigonometry [**B92**], the length of a closed geodesic γ' freely homotopic to a geodesic loop γ with an angle defect ζ at its breakpoint depends smoothly on the length of γ and the angle ζ, and the difference between the length of γ and γ' has a global minimum if $\zeta = 0$. Thus since the angle defect of β_t depends smoothly on t, the derivative at $t = 0$ of the length of the closed geodesic on $\Psi_t X$ freely homotopic to β coincides with the derivative at $t = 0$ of the function which associates to t the length of β_t. This completes the proof of the proposition. □

Corollary 1.8. (1) *For every simple closed curve β, the length function of β is continuously differentiable along Fenchel Nielsen twists.*
(2) $\mathbf{t}_\alpha \ell_\beta = -\mathbf{t}_\beta \ell_\alpha$ *for all α, β.*

Proof. By Proposition 1.7, we have to show that for every simple closed curve β which intersects a pants curve α of $\mathcal{P}$, the intersection angle between α and the geodesic representative of β varies continuously with the twist parameter for α at each intersection point between α and β.

A piecewise geodesic γ on a hyperbolic surface X with breakpoints on a simple closed geodesic α is smooth if and only if at each of the intersection points between γ and α, the incoming and the outgoing intersection angles between γ and α coincide. This is a closed condition which can be checked on the surface cut open along α.

As a consequence, as the twist parameter for α varies, the geodesic arcs on the surface cut open along α which close up to the geodesic representatives of the curve β vary continuously up to and including their endpoints.

To be more precise, consider for the moment a simple closed geodesic β which intersects the pants curve α in a single point. Let β_0 be the geodesic arc with both endpoints on α obtained by cutting β at its intersection point with α. Gliding the two endpoints of β_0 along α defines a smooth two-parameter family of geodesic arcs $\beta_{s,u}$ through $\beta_0 = \beta_{0,0}$. For all small s, t the arc $\beta_{s,s+t}$ defines a loop on the surface $\Psi_t X$. Requiring that the incoming and outgoing angles of $\beta_{s,u}$ at the endpoints coincide defines a smooth one-parameter subfamily A of the two-parameter family $\beta_{s,u}$. For each t, A intersects the arc $s \to \beta_{s,s+t}$ in a single point which depends continuously on t.

Together we conclude that the intersection angles of the geodesic representatives of the curve β on the surface $\Psi_t X$ vary continuously with t. The first part of the corollary now follows from Proposition 1.7.

For the second part, simply note that $\cos\theta_p = -\cos(\pi - \theta_p)$. □

The Dehn twists about the components of $\mathcal{P}$ define a free abelian subgroup Γ of $\mathrm{Mod}(S)$ of rank $3g - 3$. The group Γ acts on Fenchel-Nielsen coordinates for $\mathcal{P}$ by preserving the length parameters and acting on the twist parameters as a cocompact group of translations. Thus to show that Fenchel-Nielsen coordinates are injective, it suffices to find for every fixed tuple of length parameters and for a fixed fundamental domain D for the action of Γ on twist parameters a collection $\beta_1, \dots, \beta_{3g-3}$ of simple closed curves with the following properties.

(1) The curve β_i intersects α_i and no other curve from $\mathcal{P}$.
(2) The map which associates to $x \in D$ the tuple $(\ell_{\beta_1}(x), \dots, \ell_{\beta_{3g-3}}(x))$ is injective.

Proposition 1.7 suggests how this can be accomplished: For a given hyperbolic surface X, find simple closed geodesics on X with property 1) above which intersect the geodesic representatives of the pants curves with a very small angle.

To this end we note

Lemma 1.9. *Let α, β be simple closed geodesics on a hyperbolic surface X which intersect transversely. Let φ_α be the positive Dehn twist about α. Then as $k \to \infty$, the intersection angles between α and $\varphi_\alpha^k \beta$ measured counter-clockwise from $\varphi_\alpha^k \beta$ to α tend to zero.*

Proof. Assume for simplicity that α, β intersect in a single point p (the argument we give is valid in general). Let $\tilde{\alpha}$, $\tilde{\beta}$ be lifts of α, β to the hyperbolic plane which intersect in a preimage $\tilde{p}$ of p. Let $\xi, \nu \in \mathrm{PSL}(2, \mathbb{R})$ be the hyperbolic isometries which preserve $\tilde{\alpha}, \tilde{\beta}$ and whose conjugacy classes define α, β.

For each k the hyperbolic isometry $\xi^k \circ \nu$ defines the conjugacy class of the simple closed curve $\varphi_\alpha^k \beta$ on S. This isometry fixes a unique pair of points (a_k, b_k) in the ideal boundary $\partial \mathbf{H}^2$ of $\mathbf{H}^2$ which lie in the two different components of $\partial \mathbf{H}^2$ cut out by the endpoints (a, b) of $\tilde{\alpha}$. Assume without loss of generality that a is the attracting fixed point of ξ. By hyperbolic trigonometry, it now suffices to show that as $k \to \infty$ the attracting fixed point a_k of $\xi^k \circ \nu$ converges to a and its repelling fixed point b_k converges to $\nu^{-1} b$.

To see that this is the case it suffices to show the following. For every neighborhood U of a and V of $\nu^{-1}(b)$ there is some $k_0 > 0$ such that $\xi^k \circ \nu(U) \subset U$ and $(\xi^k \circ \nu)^{-1}(V) \subset V$ for all $k \geq k_0$. Now observe that the points $\nu^{-1} a, \nu^{-1} b$ are the fixed points of the hyperbolic isometry $\sigma_k = \nu^{-1} \circ \xi^k \circ \nu$. These fixed points are distinct from the fixed points a, b of ξ. Moreover, the translation length of the isometry σ_k tends to infinity as $k \to \infty$. Thus if U is a small neighborhood of a not containing $\nu^{-1} b$ and W is a neighborhood of $\nu^{-1} a$ not containing $\nu^{-1} b$ which is mapped by ν into U then for sufficiently large k, the set U is mapped by σ_k into W. Or, $\xi^k \circ \nu$ maps U into itself. The same argument is also valid for a neighborhood V of $\nu^{-1}(b)$ and the maps $(\xi^k \circ \nu)^{-1}$. This is what we wanted to show.

Note that the above argument can be made uniform: For any compact set $A \subset \mathrm{PSL}(2, \mathbb{R})$ of hyperbolic isometries whose fixed points are contained in the different components of $\partial \mathbf{H}^2 - \{a, b\}$, for any open neighborhood U of a not containing any fixed point of an element $\varphi \in A$ and every compact neighborhood $K \subset \partial \mathbf{H}^2 - \{a, b\}$ of the repelling fixed points of the elements in A, there is some $k_0 > 0$ so that for each $\varphi \in A$ and every $k \geq k_0$ the attracting fixed point of $\xi^k \circ \varphi$ is contained in U

and the repelling fixed point of $\xi^k \circ \varphi$ is contained in K. In particular, as $k \to \infty$ the angle of intersection with $\tilde{\alpha}$ of an axis of $\xi^k \circ \varphi$ is arbitrarily close to 0, uniformly in φ. □

Since intersection angles change continuously with twist parameters, the last paragraph in the proof of Lemma 1.9 shows that the estimate of intersection angles in Lemma 1.9 can be made uniform whenever the hyperbolic structure on S is modified by changing the twist parameter for α in Fenchel Nielsen coordinates by a bounded amount. From this observation, injectivity of the Fenchel-Nielsen coordinate map follows.

To be more precise, let $\mathcal{P}$ be a pants decomposition of S and let x, x' be two tuples of Fenchel Nielsen coordinates defining hyperbolic structures X, X' which are marked isometric. Then the length parameters of x, x' coincide. Let $\alpha \in \mathcal{P}$ and let τ, τ' be the twist parameters of x, x' for α. By Lemma 1.9, there is a simple closed curve β which is disjoint from $\mathcal{P} - \alpha$ and such that for each hyperbolic structure Y obtained from X by varying the twist parameter for α within the line segment connecting τ to τ', the geodesic representative of β on Y intersects α with an oriented angle smaller than $\pi/4$. Proposition 1.7 shows that the length of β is strictly increasing as the twist parameter is increasing. Since the length of β on X, X' coincides, the twist parameters τ, τ' coincide.

To summarize, Fenchel Nielsen coordinates for a pants decomposition $\mathcal{P}$ of S parametrize Teichmüller space. In particular, they define a topology on $\mathcal{T}(S)$. Our next goal is to check that this topology does not depend on the choice of the pants decomposition.

To this end let again $\mathcal{P}$ be a pants decomposition and let β be a simple closed curve which intersects one of the pants curves α but does not intersect any other pants curve. Let φ_α be the Dehn twist about α. Lemma 1.9 shows that the angle of intersection between α and the curve $\varphi_\alpha^k \beta$ tends to zero as $k \to \infty$.

By hyperbolic trigonometry [**B92**], two geodesics in the hyperbolic plane which intersect with a very small intersection angle remain uniformly close for a very large time. Or, the curve $\varphi_\alpha^k \beta$ remains in a small tubular neighborhood of α for a long time (i.e. it wraps around α many times) and therefore its length tends to infinity. This growth in length is uniform as we vary the Fenchel Nielsen coordinates within a fixed compact set. Since the length of $\varphi_\alpha^k \beta$ on the Riemann surface $X \in \mathcal{T}(S)$ equals the length of β on the surface $\varphi_\alpha^{-k}(X)$, the length of β is arbitrarily large on surfaces whose twist parameter for α in Fenchel Nielsen coordinates is large in absolute value provided that the length parameters are contained in a compact set (in fact, this is even the case if the length parameter of α is arbitrarily large or small).

For a marked hyperbolic surface X and $\epsilon > 0$ define a set $U(X, \epsilon) \subset \mathcal{T}(S)$ as follows. $X' \in U(X, \epsilon)$ if and only if $|\log \ell_c(X) - \log \ell_c(X')| < \epsilon$ for every simple closed curve c on S.

Lemma 1.10. *The sets $U(X, \epsilon)$ $(\epsilon > 0)$ define a neighborhood basis of X in Fenchel Nielsen coordinates.*

PROOF. (Sketch) Let $\mathcal{P}$ be a pants decomposition of S which defines Fenchel Nielsen coordinates. We first show that for every $X \in \mathcal{T}(S)$ and for every $\epsilon > 0$ there is a neighborhood of X in Fenchel Nielsen coordinates which is contained in $U(X, \epsilon)$.

Since a closed geodesic is the shortest curve in its free homotopy class, for this it suffices to show the following. Let $L > 1$ be arbitrary; then there is a neighborhood V of X in Fenchel Nielsen coordinates such that for every $Y \in V$, there is an L-bilipschitz diffeomorphism $F : X \to Y$ compatible with the markings. Namely, let γ be any closed geodesic on X of length $\ell_\gamma(X)$. Then $F(\gamma)$ is a closed curve on Y of length at most $L\ell_\gamma(X)$. Since F is compatible with the marking, $\ell_\gamma(Y)$ is the shortest length of a simple closed curve on Y freely homotopic to $F(\gamma)$ and hence $\ell_\gamma(Y) \le L\ell_\gamma(X)$. By symmetry, we also have $\ell_\gamma(X) \le L\ell_\gamma(Y)$.

Such a neighborhood can be obtained as follows. For a given number $L > 1$ and for every hyperbolic pair of pants P with boundary geodesics α_i of lengths a_i $(i = 1, 2, 3)$, find a number $\delta < \min\{a_i \mid i\}/2$ with the following property. Let $\tilde{a}_i \in (a_i - \delta, a_i + \delta)$ $(i = 1, 2, 3)$ and let $\tilde{P}$ be a hyperbolic pair of pants with boundary geodesics $\tilde{\alpha}_i$ of lengths $\tilde{a}_i$. Let moreover $s_i \in (-\delta, \delta)$; then there is an L-bilipschitz map $F : P \to \tilde{P}$ which maps a boundary geodesic of P parametrized by arc length to a boundary geodesic of $\tilde{P}$ parametrized proportional to arc length. Moreover, the endpoints of the seams on γ_i are mapped to points of oriented distance s_i to the endpoints of the seams on $\tilde{\gamma}_i$.

Cut X open along the geodesic representatives of the pants curves of $\mathcal{P}$. The maps described in the previous paragraph on the components of the cut open surface can be glued to an L-bilipschitz map respecting the markings of X onto any surface whose Fenchel Nielsen coordinates differ from the Fenchel Nielsen coordinates of X componentwise by at most δ.

As a consequence of this discussion, length functions are continuous in the topology defined by Fenchel Nielsen coordinates. Moreover, for every compact set K in Fenchel Nielsen coordinates, there is a number $L > 1$ and for any $X, Y \in K$ there is an L-bilipschitz map $X \to Y$ respecting the marking.

We are left with showing that any neighborhood V of X in Fenchel Nielsen coordinates contains a set of the form $U(X, \epsilon)$ for some $\epsilon > 0$. For this we argue by contradiction and we assume that there is a neighborhood V of X in Fenchel Nielsen coordinates and for each $i > 0$ there is some $X_i \in U(X, \frac{1}{i}) - V$. Then as $i \to \infty$, the length parameters in Fenchel Nielsen coordinates for the surfaces X_i converge to the lengths parameters for X. By the discussion preceding this lemma, the twist parameters of the surfaces X_i are bounded independent of i. As a consequence, after passing to a subsequence we may assume that the Fenchel Nielsen coordinates for X_i converge to the Fenchel Nielsen coordinates of some surface $Y \ne X$. By continuity of length functions in Fenchel Nielsen coordinates, we have $\ell_c(X) = \ell_c(Y)$ for every simple closed curve c. However, we observed in the proof of injectivity of Fenchel Nielsen coordinates that this implies that $Y = X$. This contradiction completes the proof of the lemma. $\square$

Since the sets $U(X,\epsilon)$ are defined independently of the choice of a pants decomposition, we conclude that transition maps for Fenchel Nielsen coordinates defined by distinct pants decompositions are homeomorphisms.

Now any $\varphi \in \mathrm{Mod}(S)$ maps Fenchel Nielsen coordinates for a pants decomposition $\mathcal{P}$ to Fenchel Nielsen coordinates for $\varphi(\mathcal{P})$ and therefore

Proposition 1.11. *$\mathcal{T}(S)$ has a $\mathrm{Mod}(S)$-invariant topology which is homeomorphic to $\mathbb{R}^{6g-6}$.*

The above discussion also indicates that Teichmüller space can be parametrized by suitably chosen length functions. We will not discuss this fact and rather state a useful easy property of length functions which can largely be generalized and analyzed quantitatively. In fact, for hyperbolic surfaces, the lengths of the closed geodesics are related to many other geometric invariants. We refer to [**B92**] for more information.

Lemma 1.12. *For every $X \in \mathcal{T}(S)$ and every $\ell > 0$ there are only finitely many closed geodesics on X of length at most ℓ.*

Proof. Let $X \in \mathcal{T}(S)$. Choose any finite collection $\mathcal{C} = \{\gamma_1, \dots, \gamma_k\}$ of simple closed geodesics on X whose union is a graph $G \subset X$ which decomposes X into simply connected regions. Then each free homotopy class on X can be represented by a closed edge-path on G. There are only finitely many homotopy classes of edge-paths of uniformly bounded combinatorial length, where the combinatorial length is the number of edges crossed through by the path.

Let γ be a closed geodesic on X which is distinct from a curve from $\mathcal{C}$. Then γ intersects each geodesic $\gamma_i \in \mathcal{C}$ transversely. It now suffices to show that the length of γ is bounded from below by a constant multiple of the number $\iota(\gamma, \mathcal{C})$ of these intersection points. Namely, since the components of $X - G$ are simply connected, there is an edge path on G which is homotopic to γ and whose combinatorial length is bounded from above by a constant multiple of the number $\iota(\gamma, \mathcal{C})$.

For this note that each curve from the collection $\mathcal{C}$ has a tubular neighborhood in X. Since the number of curves in $\mathcal{C}$ is finite, the width of each such neighborhood is bounded from below by a fixed number $\kappa > 0$. Then each essential intersection of γ with a component of $\mathcal{C}$ contributes at least 2κ to the length of γ. Now $\iota(\gamma, \mathcal{C}) \leq k \max_i \iota(\gamma, \gamma_i)$ and hence

$$\ell_\gamma(X) \geq 2\kappa\iota(\gamma, \mathcal{C})/k$$

which completes the proof of the lemma. □

For more details about the proof of the following proposition we refer to Section 6.3 of [**IT89**].

Proposition 1.13. *The action of $\mathrm{Mod}(S)$ on $\mathcal{T}(S)$ is properly discontinuous.*

PROOF. It suffices to show the following. For every $X \in \mathcal{T}(S)$ and every pants decomposition $\mathcal{P}$ for S there is a compact neighborhood K of X in Fenchel Nielsen coordinates for $\mathcal{P}$ so that $\varphi K \cap K \neq \emptyset$ only for finitely many $\varphi \in \mathrm{Mod}(S)$.

For this we claim that there is a compact neighborhood K of X in Fenchel Nielsen coordinates and there are finitely many pants decompositions $\mathcal{P}_1, \dots \mathcal{P}_k$ so that $\varphi\mathcal{P} \in \{\mathcal{P}_1, \dots \mathcal{P}_k\}$ for all $\varphi \in \mathrm{Mod}(S)$ with $\varphi(K) \cap K \neq \emptyset$.

Let $\ell = \max\{\ell_\alpha(X) \mid \alpha \in \mathcal{P}\}$. Let K be a compact neighborhood of X so that for every $Y \in K$ there is a marked 2-bilipschitz map $F : Y \to X$. If $\varphi \in \mathrm{Mod}(S)$ is such that $\varphi(Y) \in K$ for some $Y \in K$ then $\varphi\mathcal{P}$ is a pants decomposition whose pants curves have geodesic representatives on X of length at most 2ℓ (since φ just changes the marking). Thus the claim is an immediate consequence of Lemma 1.12.

Now observe that if φ_1, φ_2 map $\mathcal{P}$ to a fixed pants decomposition $\mathcal{P}_i$ then $\varphi_2^{-1} \circ \varphi_1$ preserves $\mathcal{P}$. Since every diffeomorphism of a pair of pants which preserves each of the boundary circles is isotopic to the identity (see[**B92**]), the stabilizer of $\mathcal{P}$ in $\mathrm{Mod}(S)$ equals the free abelian group of Dehn twists about the pants curves (we refer once more to [**FM11**] for a detailed account on Dehn twists). However, by construction of the Fenchel Nielsen coordinates, the action of the stabilizer of $\mathcal{P}$ in $\mathrm{Mod}(S)$ is properly discontinuous. From this the proposition follows. □

Example: Let P be a pair of pants all of whose boundary curves have the same length. Glue two boundary curves together so that endpoints of seams are glued to endpoints of seams, and glue two copies of the resulting one-holed torus identifying endpoints of seams with endpoints of seams. The resulting hyperbolic surface X of genus 2 has an isometric involution φ exchanging the two pairs of pants which form X so that X/φ is a torus. Thus $\varphi \in \mathrm{Mod}(S)$ is a non-trivial element which fixes X. Similar constructions can be carried out for surfaces of any genus $g \geq 2$ and show that the action of $\mathrm{Mod}(S)$ on $\mathcal{T}(S)$ is not free. For more details and bounds on the order of the isometry group of a hyperbolic surface we refer to [**B92**].

Fenchel Nielsen coordinates also define a real analytic structure on $\mathcal{T}(S)$, but this is more difficult to see. In the remainder of this section, we discuss the special case of the Teichmüller space $\mathcal{T}(S_0)$ of a four-holed sphere S_0. This is the space of marked hyperbolic structures on S_0 so that the boundary consists of four closed geodesics. Any simple closed curve α on S_0 defines a pants decomposition, and hence there are Fenchel Nielsen coordinates consisting of five length functions and one twist parameter about α.

Proposition 1.14. *Let S_0 be a four-holed sphere. Then Teichmüller space for S_0 has a structure of a real analytic manifold so that all length functions of simple closed geodesics are real analytic.*

PROOF. Let Y be a hyperbolic four-holed sphere. The universal covering of Y is a convex subset of the hyperbolic plane $\mathbf{H}^2$ with geodesic boundary. The fundamental group $\pi_1(Y)$ of Y is a free subgroup of $\mathrm{PSL}(2, \mathbb{R})$ which is generated by three hyperbolic isometries $\varphi_1, \varphi_2, \varphi_3 \in \mathrm{PSL}(2,\mathbb{R})$. The fixed points of these elements on the boundary $\partial \mathbf{H}^2$ of the hyperbolic plane are pairwise distinct.

The group $\mathrm{PSL}(2,\mathbb{R})$ acts triply transitive on the ideal boundary $\partial \mathbf{H}^2$ of the hyperbolic plane. If γ is an invariant geodesic for a hyperbolic isometry g and if $\varphi \in \mathrm{PSL}(2,\mathbb{R})$ then $\varphi\gamma$ is invariant for $\varphi \circ g \circ \varphi^{-1}$. On the other hand, two conjugate discrete torsion free subgroups of $\mathrm{PSL}(2,\mathbb{R})$ define the same marked hyperbolic surface. Therefore we may assume that in the upper half-plane model for $\mathbf{H}^2$, the fixed points of φ_1 on $\partial \mathbf{H}^2$ are the points $0, \infty$ and that 1 is a fixed point for φ_2. Since the group $\pi_1(Y)$ is generated by $\varphi_1, \varphi_2, \varphi_3$ and each element in $\mathrm{PSL}(2,\mathbb{R})$ is determined by three real parameters (=matrix entries), there are 6 real parameters which determine the conjugacy class of $\pi_1(Y) < \mathrm{PSL}(2,\mathbb{R})$.

In order to verify that we obtain 6 free parameters (i.e. there are no additional constraints) it suffices to show the following. Let $\tilde{\varphi}_i \in \mathrm{PSL}(2,\mathbb{R})$ be three elements which are close to φ_i with respect to the topology of $\mathrm{PSL}(2,\mathbb{R})$ as a quotient of $\mathrm{SL}(2,\mathbb{R})$. Then the elements $\tilde{\varphi}_i$ are hyperbolic and generate a free group which is the fundamental group of a hyperbolic four holed sphere with geodesic boundary.

An element in $\mathrm{PSL}(2,\mathbb{R})$ is hyperbolic if and only if it has a preimage in $\mathrm{SL}(2,\mathbb{R})$ whose trace is bigger than two. This clearly is an open condition.

Since Y is a four-holed sphere, there are six pairwise disjoint closed intervals $I_1, \dots, I_6 \subset \partial \mathbf{H}^2$ so that φ_i maps the exterior of I_i homeomorphically onto the interior of I_{i+3}. The intervals I_i, I_{i+3} contain the fixed points of φ_i. (This numbering of the intervals is not the numbering obtained by their counter-clockwise order on $\partial \mathbf{H}^2$). A nearby triple of hyperbolic elements $\tilde{\varphi}_i$ determines six new closed intervals $\tilde{I}_i$. The intervals $\tilde{I}_i$ can be chosen to depend continuously on the elements $\tilde{\varphi}_i$. In particular, for $\tilde{\varphi}_i$ sufficiently close to φ_i these intervals are pairwise disjoint.

A *ping pong argument* shows that any subgroup Γ of $\mathrm{PSL}(2,\mathbb{R})$ generated by three elements with this property is free and consists of hyperbolic elements. For this let $u = u_1 \cdots u_s$ be a nontrivial reduced word in the generators $\tilde{\varphi}_i$. Let $x \in \partial \mathbf{H}^2$ be a point in the complement of all intervals $\tilde{I}_j$. If $u_s = \tilde{\varphi}_j$ then $u_s x \in \tilde{I}_{j+3}$ and inductively, if $u_1 = \tilde{\varphi}_p$ then $u(x) \in \tilde{I}_{p+3}$ (indices are taken modulo six and $\tilde{\varphi}_{j+3} = \tilde{\varphi}_j^{-1}$). In particular, $u(x) \neq x$ and hence the group generated by $\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3$ is free (compare [**M88**]).

The same argument also shows that the group Γ generated by these elements is discrete. Namely, otherwise for the point $x \in \partial \mathbf{H}^2$ chosen as in the previous paragraph there is a sequence $v_j \subset \Gamma$ so that $v_j \neq e$ and that $v_j x \to x$. However, the discussion in the previous argument shows that for a compact neighborhood U of x in $\partial \mathbf{H}^2 - \cup_i \tilde{I}_i$ we have $vx \notin U$ for $v \neq e \in \Gamma$.

Now the length of a closed geodesic on the hyperbolic surface Y can be calculated from the trace of a lift to $\mathrm{SL}(2,\mathbb{R})$ of the corresponding element of $\mathrm{PSL}(2,\mathbb{R})$. Namely, if this trace equals $\lambda + \lambda^{-1}$ for some $\lambda > 1$ then the length is $2\log\lambda$ (we refer to [**B92**] for this fact). As a consequence, length functions are real analytic with respect to the analytic structure on $\mathcal{T}(S_0)$ defined by the matrix components. $\square$

Remark: Proposition 1.14 and its proof are equally valid for the Teichmüller space of a sphere with an arbitrary number $\ell \geq 4$ of geodesic boundary components. It can be extended to define coordinates for the Teichmüller space of closed surfaces as well. These so-called *Fricke coordinates* are important for many aspects of Teichmüller theory. For our purpose, however, they play no further role. We refer the reader to [**IT89**] for details and more.

Corollary 1.15. *Fenchel Nielsen coordinates are real analytic coordinates for the Teichmüller space of S_0. In particular, length functions are real analytic in Fenchel Nielsen coordinates.*

PROOF. Let α be a simple closed curve in S_0 and let $(\ell_1, \dots, \ell_4, \ell_5, \tau)$ be coordinates for $\mathcal{T}(S_0)$ defined by the 4 lengths ℓ_i of the boundary circles of S_0, the length ℓ_5 of α and the twist parameter τ. We claim that the map which associates to $X \in \mathcal{T}(S_0)$ its Fenchel Nielsen coordinates is real analytic.

By Proposition 1.14, for this it suffices to show that the twist parameter is an analytic function on $\mathcal{T}(S_0)$. This can be seen as follows.

Let $\tilde{\alpha}$ be a lift of α to $\mathbf{H}^2$. We may assume that in the upper half-plane model its endpoints are $0, \infty$. There are lifts β_1, β_2 of seams of the two components P_1, P_2 of $S_0 - \alpha$ with one endpoint on $\tilde{\alpha}$ and the second endpoint on a lift $\tilde{\beta}_i$ of a boundary curve of P_i distinct from α $(i = 1, 2)$. The geodesics $\tilde{\beta}_1, \tilde{\beta}_2$ are axes of hyperbolic isometries $\varphi_1, \varphi_2 \in \pi_1(S_0)$, and the segments β_1, β_2 realize the distance between the geodesic lines $\tilde{\alpha}$ and $\tilde{\beta}_1, \tilde{\beta}_2$.

As the isometries φ_1, φ_2 vary in $\mathrm{PSL}(2, \mathbb{R})$ in an analytic family, their axes vary in an analytic family as well. As a consequence, the shortest distance projections of these axes to the fixed geodesic $\tilde{\alpha}$ depend in a real analytic fashion on the elements. However, the signed distance between these projections is just the twist parameter τ of the Fenchel Nielsen coordinates in the analytic family. As a consequence, Fenchel Nielsen coordinates are real analytic.

To show that Fenchel Nielsen coordinates are real analytic coordinates for $\mathcal{T}(S_0)$ it now suffices to show that the map which associates to $X \in \mathcal{T}(S_0)$ its Fenchel Nielsen coordinates is of maximal rank differentiable at every point. This can easily be seen explicitly. We omit the proof and refer to [**A80, IT89**]. $\square$

We formulate without proof the corresponding result for the Teichmüller space of a closed surface S. We refer to [**A80**] and to [**IT89**] for a more comprehensive discussion and a proof.

Theorem 1.16. *Fenchel Nielsen coordinates define a* $\mathrm{Mod}(S)$*-invariant real analytic structure on $\mathcal{T}(S)$.*

Remark: Everything in this section is equally valid for marked hyperbolic metrics of finite area on a surface with punctures.

LECTURE 2

Quasiconformal Maps

In this section we discuss some analytic aspects of Teichmüller theory. We introduce quasiconformal maps and abelian and quadratic differentials, and we discuss Teichmüller's existence and uniqueness theorem.

Recall the upper half-plane model for the hyperbolic plane $\mathbf{H}^2$. An oriented hyperbolic surface can be defined as a surface S together with a covering of S by orientation preserving charts $\varphi : U \subset S \to \varphi(U) \subset \mathbf{H}^2$ so that chart transitions are orientation preserving isometries. Since an orientation preserving isometry of $\mathbf{H}^2$ is in particular a biholomorphic map, these charts define a complex structure on S. Thus we have

Proposition 2.1. *An oriented hyperbolic surface is a Riemann surface.*

By the uniformization theorem, the converse is also true. Namely, the *conformal class* of a complex structure on S is the space of all Riemannian metrics g on S with the property that the fibrewise multiplication with i in the tangent bundle of S preserves g. Since $i^2 = -1$, this means that multiplication by i in a fibre is just rotation by the angle $\pi/2$.

Proposition 2.2. *The conformal class of a complex structure on S contains a unique hyperbolic metric.*

Namely, the universal covering of a closed Riemann surface X of genus $g \geq 2$ is the unit disc $D \subset \mathbb{C}$ which is a model for the hyperbolic plane $\mathbf{H}^2$ (see [**FK80**]). Then X is the quotient of D by a discrete group of biholomorphic automorphisms of D. But the group of all biholomorphic automorphisms of $D = \mathbf{H}^2$ is just the group $\mathrm{PSL}(2, \mathbb{R})$ of hyperbolic isometries. In particular, there is a hyperbolic metric on X so that the orientation preserving charts determined by this metric are holomorphic for the complex structure. Or, the complex structure coincides with the structure constructed from the hyperbolic metric in the first paragraph of this section.

In Section 2 we saw that length functions of simple closed curves can be used to define a $\mathrm{Mod}(S)$-invariant topology on $\mathcal{T}(S)$. This topology is described by measuring the deviation from the existence of a marked isometry between two marked hyperbolic surfaces. (We remark here that Thurston initiated a systematic study of Teichmüller theory via (bi)-Lipschitz maps with optimal Lipschitz constants, and there is a substantial current activity in this line of investigation. We refer to [**PT07**] for a recent survey and references on this subject).

We can also define the topology on $\mathcal{T}(S)$ by measuring the deviation from the existence of a marked biholomorphic map between two marked Riemann surfaces. Recall that a map F between two Riemann surfaces is holomorphic if in complex charts we have $\bar\partial F = 0$.

Definition 2.3. Let X, X' be Riemann surfaces. For a number $K \geq 1$, a map $F : X \to X'$ is *K-quasiconformal* if F is a homeomorphism which is continuously differentiable, with differential of maximal rank, outside a finite set Σ of points and if

$$|\bar\partial F| \leq k|\partial F|$$

outside Σ where $0 \leq k < 1$ and $K = \frac{1+k}{1-k}$.

The geometric significance is as follows. Let g, g' be any Riemannian metric in the conformal class defined by the complex structure on X, X'. Then with respect to these metrics, the differential of F at $x \in X-\Sigma$ maps a round circle in the tangent space T_xX of X at x to an ellipse in the tangent space $T_{F(x)}X'$ of X' whose axes have length ratio bounded by K. Note that this is independent of the choice of the metrics g, g' in the conformal class defined by the complex structures. (In Lemma 4.8 of [**IT89**], this easy calculation is carried out in detail). As a consequence, the Jacobian of F at x is bounded from below by K^{-1} times the square of the operator norm of the differential of F at x.

Example: Since S is compact by assumption, every orientation preserving diffeomorphism $F : X \to X'$ is quasiconformal, i.e. there is a number $K > 0$ so that F is K-quasiconformal.

It turns out that particularly nice quasiconformal maps can be constructed using objects directly defined by the complex structure of the Riemann surface X. We next introduce these structures and these maps and establish some of their most important properties.

Definition 2.4. An *abelian differential* on a Riemann surface X is a nontrivial holomorphic one-form ω on X. Such a one-form is a holomorphic section of the *holomorphic cotangent bundle* $T'X$ of X.

In a holomorphic local coordinate z, a holomorphic one-form ω can be represented as $f(z)dz$ with a holomorphic function f. Now the exterior differential d for complex valued one-forms on $\mathbb{C}$ can be decomposed as $d = \partial + \bar\partial$ with $\partial\partial = \bar\partial\bar\partial = 0 = \partial\bar\partial + \bar\partial\partial$ (see p.77/78 of [**D11**] for a calculation in local coordinates) and hence $d\omega = df \wedge dz = 0$ since $\bar\partial f = 0$. As a consequence, ω defines a complex valued de Rham cohomology class on S.

Thus there is a natural homomorphism from the vector space $\Omega(X)$ of holomorphic one-forms on X into the first cohomology group $H^1(X, \mathbb{C})$ of X with complex coefficients which associates to $\omega \in \Omega(X)$ its de Rham cohomology class. The complex vector space $H^1(X, \mathbb{C})$ is isomorphic to $\mathbb{C}^{2g}$.

Proposition 2.5. *The homomorphism $\Omega(X) \to H^1(X, \mathbb{C})$ is an embedding onto a complex subspace of dimension g.*

PROOF. (Incomplete sketch) The complex Laplacian for smooth functions $f : X \to \mathbb{C}$ is defined by

$$\Delta = 2i\bar{\partial}\partial.$$

A function f is *harmonic* if $\Delta f = 0$. Similarly, a one-form α on X is *harmonic* if locally α is the differential of a harmonic function. Since $d = \partial + \bar{\partial}$ in a complex coordinate, a harmonic one-form is closed. In fact, a one-form α on X is harmonic if and only if it is both closed and coclosed (see p.27 of [**FK80**]).

Now a non-zero harmonic one-form α is not exact. Namely, otherwise $\alpha = df$ for a function f. Then f is harmonic and non-constant. By Stokes theorem, we have $\int d(f\bar{\partial}f) = 0 = \int \partial(f\bar{\partial}f)$, moreover $\bar{\partial}\partial = -\partial\bar{\partial}$ and therefore

$$\int f\Delta f = 0 = 2i\int \partial(f\bar{\partial}f) - f\partial\bar{\partial}f = 2i\int \partial f \wedge \bar{\partial}f = i\int df \wedge \overline{df}$$

which is possible only if df vanishes identically: In a local complex coordinate z, a complex valued one-form α can be written in the form $\alpha = adz + bd\bar{z}$ and hence $i\alpha \wedge \bar{\alpha} = i(a\bar{a} + b\bar{b})dz \wedge d\bar{z}$ which is a positive multiple of the euclidean area form at any point where α does not vanish (we refer to [**D11**] for more details).

By the Cauchy Riemann equations, a real valued function f on a domain in $\mathbb{C}$ is harmonic if and only if it is the real part of a holomorphic function. Thus a real one-form α on X is harmonic if and only if it is the real part of a holomorphic one-form. Since by the Hodge de Rham theorem every de Rham cohomology class of degree one can be represented by a unique harmonic one-form (see Theorem 6 of [**D11**]), this implies that the homomorphism $\Omega(X) \to H^1(X, \mathbb{C})$ is injective. Moreover, the $\mathbb{R}$-linear map which associates to a holomorphic one-form ω the de Rham cohomology class of its real part maps $\Omega(X)$ onto $H^1(X, \mathbb{R})$ and hence the complex dimension of the image $\Omega(X)$ in $H^1(X, \mathbb{C})$ equals g. $\square$

We also need related but more general objects which are defined as follows.

Definition 2.6. A *quadratic differential* on X is a holomorphic section of $T'X \otimes T'X$.

In local coordinates, q can be represented in the form $q = f(z)dz^2$ where f is a nontrivial holomorphic function. In particular, if $q \neq 0$ then q has only finitely many zeros.

Example: The square of every abelian differential is a quadratic differential.

The Riemann Roch formula (Theorem 7 of [**D11**]) implies

Theorem 2.7. *The complex vector space $\mathcal{Q}(X)$ of quadratic differentials on X is of dimension $3g - 3$.*

Quadratic differentials can be described as follows. Let $x \in X$ be a point so that $q \in \mathcal{Q}(X)$ does not vanish at x. Then with respect to a holomorphic coordinate u near x, the differential q can be represented in the form $f du^2$ with a non-vanishing holomorphic function f. Use a branch of the square root to write $q = g^2 du^2$ for a non-vanishing holomorphic function g (perhaps defined on a smaller neighborhood of x). Then locally near x, gdu is the differential of a holomorphic function z defined by

$$z(y) = \int_x^y g(u)du$$

so that

$$q = dz^2.$$

The coordinate z is unique up to translation and perhaps multiplication with -1, i.e. up to an euclidean isometry. In particular, every quadratic differential q defines an euclidean metric outside its zero set Σ. We refer to [**S84**] for a detailed account on this construction.

The line fields $\{v \in T(X - \Sigma) \mid q(v) \geq 0\}$ and $\{v \in T(X - \Sigma) \mid q(v) \leq 0\}$ define local foliations of $X - \Sigma$ called the *horizontal* and *vertical* foliation, respectively. In the distinguished coordinate z for q, these foliations are precisely the foliations into lines parallel to the real and imaginary axis. Rectangles for these transverse foliations are rectangles for the euclidean metric defined by q.

At a zero x we can write $q = g^p dz^2$ for some $p > 0$ and some holomorphic function g which has a simple zero at x. Taking appropriate roots shows that the metric has a standard $p+2$-pronged singularity at x (we refer to the standard reference [**S84**] for quadratic differentials for details). A neighborhood of such a singular point is a neighborhood of 0 in the following space. Take $p+2$ copies H_i ($i = 1, \dots, p+2$) of the upper half-plane and glue these half-planes along their boundaries in cyclic order so that the ray $\{\mathrm{Im} = 0, \mathrm{Re} \leq 0\}$ in the boundary of the half-plane H_i is glued to the corresponding ray in the boundary of the half-plane H_{i-1} with an orientation reversing isometry, and the ray $\{\mathrm{Im} = 0, \mathrm{Re} \geq 0\}$ in the boundary of H_i is glued to the corresponding ray in the boundary of H_{i+1}. In particular, the area of the singular euclidean metric defined by q is finite.

As a consequence, we can define a norm $\| \, \|$ on the space $\mathcal{Q}(X)$ by

$$\|q\| = \mathrm{area}(q) = \int_X |q|.$$

For a finite subset Σ of S, a foliation of $S - \Sigma$ is *orientable* if its tangent bundle admits a smooth nowhere vanishing section.

Lemma 2.8. *A quadratic differential is the square of a holomorphic one-form if and only if its horizontal and vertical foliations are orientable.*

Proof. If $q = \omega^2$ then $\omega > 0$ defines an orientation of the horizontal foliation, and $i\omega > 0$ defines an orientation of the vertical foliation.

On the other hand, if q has orientable foliations then away from the zeros of q we can take a square root of q defining this orientation, and these square roots are consistent under change of charts. $\square$

If the underlying complex structure of the quadratic differential is allowed to be arbitrary then it is very easy to construct quadratic differentials explicitly. For this purpose we define

Definition 2.9. Two simple closed curves α, β *bind* the surface S if α, β are in minimal position (i.e. they intersect in the minimal number of points in their free homotopy classes) and decompose S into simply connected regions.

The following construction is due to Thurston. We refer to Section 9 of [**V89**] for a slightly more general account.

Proposition 2.10. *A pair of simple closed curves on S which bind S defines a quadratic differential.*

Proof. We construct explicitly a quadratic differential from two simple closed curves α, β which bind S as follows.

Note first that α, β decompose S into polygons with an even number of sides. The sides alternate between subarcs of α and subarcs of β. Place an euclidean square of side length one over each intersection point between α and β. This can be done in such a way that the squares only meet along their sides or at their vertices, and that an intersection of sides corresponds to a component of $\alpha - \beta$ or a component of $\beta - \alpha$. This means that each side contains precisely one point of $\alpha - \beta$ or of $\beta - \alpha$, and each component of $\alpha - \beta$ or of $\beta - \alpha$ intersects the sides in precisely one point. Each component of $S - \{\alpha \cup \beta\}$ contains exactly one vertex. If this component is a $2p$-gon for some $p \geq 2$ then there are $2p$ squares coming together at that vertex.

The euclidean structures on the squares define a singular euclidean metric on S with one p-pronged singularity for each complementary polygon with $2p \geq 6$ sides. Away from the singularities, this structure is just given by the standard euclidean charts. It can be extended across the singular points since each such point is a standard p-pronged singularity.

Up to maps of the form $z \to \pm z + c$ for some $c \in \mathbb{C}$, each square has a preferred isometric embedding into $\mathbb{C}$ which maps the sides parallel to α to horizontal straight segments parallel to the real axis, and which maps the sides parallel to β to vertical straight segments parallel to the imaginary axis. Then transitions for these rectangles on their overlaps are of the form $z \to \pm z + c$ ($c \in \mathbb{C}$), and these transitions preserve the quadratic differential dz^2. As a consequence, these differentials define a quadratic differential q on the complement of the singular points which naturally extends to the singular points. $\square$

We can use quadratic differentials to construct quasiconformal maps as follows. Let q be a holomorphic quadratic differential with zero set Σ on a marked Riemann surface X. Then on $S - \Sigma$, the differential q defines an euclidean metric g and horizontal and vertical foliations. For $t > 0$ let g_t be the singular euclidean metric on $S - \Sigma$ obtained by stretching the horizontal direction by the factor $e^{t/2}$ and contracting the vertical direction by the factor $e^{-t/2}$. The metric extends to a singular euclidean metric on S with singular points at the zeros of q. This metric then defines a marked complex structure X_t on S and a holomorphic quadratic differential q_t for X_t. The marking of X_t is the composition of the marking of X with the stretch map. The stretch map $X \to X_t$ is quasi-conformal with constant e^t. It is called a *Teichmüller map* with initial differential q and stretch factor $e^{t/2}$. The area of q_t equals the area of q.

As is customary in Riemannian geometry, for $t > 0$ and a quadratic differential $q \in \mathcal{Q}(X)$ of area one (= norm one) we associate to the quadratic differential tq of norm t the image $\Psi(tq)$ of the Teichmüller map with initial differential q and stretch factor $e^{t/2}$. This convention defines a map $\Psi : \mathcal{Q}(X) \to \mathcal{T}(S)$ with $\Psi(0) = X$. We call the image under Ψ of a ray in $\mathcal{Q}(X)$ a *Teichmüller geodesic* in $\mathcal{T}(S)$.

For two Riemann surfaces X, X' define

$$d_{\mathcal{T}}(X, X') = \frac{1}{2} \inf\{\log K \mid \text{there is a } K\text{-quasiconformal map } f : X \to X' \text{ respecting the marking}\}.$$

The next result implies that Teichmüller maps are optimal for this invariant. For its formulation, recall that a quadratic differential q determines an area element dA on S. Its total area is the norm $\|q\|$ of q. A *separatrix* for q is a maximal segment or ray (i.e. a geodesic for the singular euclidean metric) which begins at a singular point and does not contain any singular point in its interior. The proof of the following theorem is taken from [**FM11**].

Theorem 2.11. *Let $f : X \to X'$ be a quasiconformal map which is homotopic to a Teichmüller mapping with initial differential q and stretch factor L. Then*

$$\int_X |f_x| dA \geq L\|q\|$$

where f_x is the derivative of f in the horizontal direction and the norm is taken with respect to the singular euclidean metric of q.

PROOF. For $p \in X$ and $T > 0$ let

$$\delta(p, T) = \int_{-T}^{T} |f_x| dx$$

where the integration is over the horizontal arc α of length $2T$ centered at p. If such a horizontal arc passes through a singular point then $\delta(p, T)$ is undefined. Since there are only finitely many horizontal separatrices and hence the union of these separatrices has area zero, the value $\delta(p, T)$ is defined almost everywhere with respect to the area element dA of the quadratic differential q.

By Fubini's theorem,

$$\int_X \left(\int_{-T}^{T} |f_x| dx \right) dA = 2T \int_X |f_x| dA.$$

However, $\delta(p,T)$ is just the length of $f(\alpha)$ with respect to the singular euclidean metric defined by the terminal differential q' for the Teichmüller mapping. Thus

$$\int_X \left(\int_{-T}^{T} |f_x| dx \right) dA \geq \int_X (2LT - M) dA = (2LT - M)\|q\|$$

where $M > 0$ is a fixed constant which is obtained as follows.

The diameter of the singular euclidean metric defined by q' is finite. By assumption, there is a homotopy connecting f to the Teichmüller mapping with initial differential q and stretch factor L. For each horizontal arc α, such a homotopy determines a homotopy class of arcs connecting the endpoints of $f(\alpha)$ to the endpoints of the stretched arc. The q'-length of any q'-geodesic representative of such an arc is uniformly bounded, independent of α. If $M/2$ is an upper bound for these lengths then the estimate holds true for M.

As a consequence, we have

$$\int_X |f_x| dA \geq (L - \frac{M}{2T})\|q\|.$$

As $T \to \infty$ the theorem follows. □

Corollary 2.12. *A Teichmüller map between two marked Riemann surfaces X, X' is the unique $e^{2d_{\mathcal{T}}(X,X')}$-quasi-conformal map in its homotopy class mapping X to X'.*

Proof. Let q be an area one quadratic differential on a Riemann surface X and let X' be the image of the Teichmüller mapping with initial differential q and stretch factor L. If $f : X \to X'$ is K-quasi-conformal then with respect to the singular euclidean metrics on X, X', the Jacobian $\mathrm{Jac}(f)$ of f at any point which is both regular for the map f and the quadratic differential q is not smaller than K^{-1} times the square of the operator norm of its differential df.

Now the K-quasi-conformal map $f : X \to X'$ is a homeomorphism and therefore using the Cauchy Schwarz inequality, Theorem 2.11 and the fact that $|df| \geq |f_x|$, with the notations from Theorem 2.11 we have

$$1 = \int_X \mathrm{Jac}(f) dA \geq \int_X |df|^2 dA/K \geq \left(\int |f_x| dA \right)^2 /K \geq L^2/K.$$

This shows that the quasi-conformal constant K of f is at least L^2.

Equality only holds if equality holds true in the Cauchy-Schwarz inequality and if moreover $|f_x| = |df|$ almost everywhere. This implies that $|df| = |f_x| = L$ almost everywhere.

The same argument applies to the restriction of f to the vertical foliation. Thus since f is continuously differentiable on the complement of a finite set of points, if $K = L^2$ then the composition of f with the inverse of the Teichmüller map is an isometry for the singular euclidean metric defined by q which is continuously differentiable on the complement of a finite set of points. This isometry is moreover isotopic to the identity.

However, an isometry Φ of the singular euclidean metric defined by q which is isotopic to the identity is the identity.

Namely, since the metric defined by q is singular euclidean with singular points of cone angle bigger than 2π, it is a metric of non-positive curvature. An isotopy between an isometry Φ and the identity determines for each point $x \in S$ a homotopy class of an arc connecting x to $\Phi(x)$. The homotopy class can be represented by a unique geodesic $\alpha_x : [0,1] \to S$ depending continuously on x. By convexity, the map $\Phi_0 : x \to \alpha_x(\frac{1}{2})$ is distance non-increasing. If $\Phi \neq \mathrm{Id}$ then it is not isometric. Then Φ_0 is a map of degree one whose Jacobian is bounded from above by one and which is different from one on some nontrivial open set. This is impossible. □

Theorem 2.11 and its proof can also be used to classify annuli up to biholomorphic equivalence. Here an annulus is a compact Riemann surface which is biholomorphic to $A_a = [0,1] \times [0,a]/\sim$ where $(0,t) \sim (1,t)$ for all t. Define the *modulus* of A_a to be a. The reasoning in the proof of Theorem 2.11 shows that the modulus is an invariant of the complex structure of such annuli.

Corollary 2.13. *For $b \geq a$ the smallest quasiconformal dilatation of a quasiconformal map $A_a \to A_b$ equals b/a.*

If $A \subset \mathbb{C}$ is any annulus then uniformization implies that A is biholomorphic to A_a for some $a > 0$, and a is called the *modulus* of A. Corollary 2.13 immediately shows

Corollary 2.14. *The modulus classifies annuli up to biholomorphic equivalence.*

We can use Corollary 2.14 to relate lengths of geodesics for hyperbolic metrics directly to invariants of complex structures. As in Section 2, denote by ℓ_c the hyperbolic length function on $\mathcal{T}(S)$ of a simple closed curve c. Wolpert showed (see [**W10**] for references and for more details)

Proposition 2.15. *Let X, X' be hyperbolic surfaces and let $f : X \to X'$ be a K-quasiconformal map. Then for any simple closed curve c we have*

$$\frac{\ell_c(X)}{K} \leq \ell_c(X') \leq K\ell_c(X).$$

Proof. Let γ_1, γ_2 be the closed geodesics on X, X' in the free homotopy class of c. There is a covering A_1, A_2 of X, X' whose fundamental group is generated by γ_1, γ_2. If a_1, a_2 is the length of c on X, X' then we may assume that $A_i = \mathbf{H}^2/ < b_i >$ where $b_i(z) = e^{a_i}z$ (here as before, we use the upper half-plane $\{\mathrm{Im} > 0\}$ as a model for the hyperbolic plane).

There is a branch of the logarithm which maps the upper half-plane biholomorphically onto the infinite strip

$$\mathrm{Im} \in (0, \pi).$$

Under this identification, the group $< b_i >$ corresponds to the infinite cyclic group of translations generated by $z \to z + a_i$. As a consequence, the annulus A_i is biholomorphic to a standard flat cylinder of circumference a_i and height π. In other words, the modulus of A_i equals $m_i = \pi/a_i$.

A K-quasiconformal map $X \to X'$ lifts to a K-quasiconformal map of the cylinder A_1 onto the cylinder A_2. By Corollary 2.13, this just means that $m_2/K \leq m_1 \leq Km_2$ as claimed. $\square$

As an immediate consequence we obtain

Proposition 2.16. *$d_{\mathcal{T}}$ is a* $\mathrm{Mod}(S)$*-invariant metric on $\mathcal{T}(S)$ which defines the topology given by Fenchel Nielsen coordinates.*

Proof. Clearly $d_{\mathcal{T}}$ is a $\mathrm{Mod}(S)$-invariant pseudo-metric on $\mathcal{T}(S)$. Proposition 2.15 and Lemma 1.10 show that $d_{\mathcal{T}}$ is in fact a metric (i.e. we have $d_{\mathcal{T}}(X, X') > 0$ if $X \neq X'$) defining a topology which is finer than the topology induced by Fenchel Nielsen coordinates. This means that the identity $(\mathcal{T}(S), d_{\mathcal{T}}) \to \mathcal{T}(S)$ is continuous.

Thus for the proof of the proposition we are left with showing that the identity map $\mathcal{T}(X) \to (\mathcal{T}(X), d_{\mathcal{T}})$ is continuous as well. For this it suffices to show that in Fenchel Nielsen coordinates, if coordinate functions converge then the minimal dilatation of a quasiconformal map between the two surfaces defined by these coordinates converges to one. However, quasi-conformal maps with small quasiconformal dilatation between hyperbolic surfaces with nearby Fenchel Nielsen coordinates can be constructed explicitly using elementary hyperbolic geometry exactly as in the proof of Lemma 1.10. Namely, note that for some $L > 1$, an L-bilipschitz diffeomorphism F between two hyperbolic surfaces is L^2-quasi-conformal. $\square$

As before, let $\mathcal{Q}(X)$ be the vector space of quadratic differentials on the Riemann surface X. Recall the definition of the map $\Psi : \mathcal{Q}(X) \to \mathcal{T}(S)$. The following important result is much more difficult than the rather elementary facts discussed so far. We are not going to give a proof but rather indicate an elementary strategy to this end which we hope gives some geometric intuition why the result holds true. For this and later purpose, we define

Definition 2.17. A *saddle connection* of a quadratic differential q is a straight line segment for the singular euclidean metric defined by q which connects two singular points and does not have a singular point in its interior.

Thus a saddle connection is a compact separatrix.

In a sequel we mean by a triangulation of a surface S a decomposition of S into triangles whose vertices are not necessarily distinct. The following simple observation will be used several times.

Lemma 2.18. *Let q be a quadratic differential on S. Then there exists a triangulation of S consisting of saddle connections whose vertex set is the set of singular points of S.*

Proof. Throughout this proof, distances are taken with respect to the singular euclidean metric.

Let $B \subset S$ be any embedded graph whose vertices are singular points for q and whose edges are saddle connections. If there is a component C of $S - B$ which is not simply connected then choose an arc $\alpha \subset \overline{C}$ with endpoints at singular points which is not homotopic into B with fixed endpoints, which does not cross through an edge of B and which is shortest with this property. Then α is a piecewise geodesic for the singular euclidean metric. Since the length of α is minimal, it does not contain an edge from the graph B. Moreover, it does not meet the interior of an edge of B since otherwise α has a breakpoint at such an interior point with nontrivial angle defect and hence it can be shortened by a local homotopy. As a consequence, α is a saddle connection whose interior is disjoint from the interiors of the edges of B. In other words, $B \cup \alpha$ is an embedded graph whose vertices are singular points and whose edges are saddle connections.

By successively adding edges to B with the above procedure, one constructs an embedded graph B' in S whose edges are saddle connections and which decomposes S into simply connected regions. Any such region can be subdivided into a union of triangles by first connecting any singular point in the interior to a boundary vertex with a saddle connection of minimal length and then subdividing the resulting euclidean polygons. □

Theorem 2.19. *The map Ψ is continuous.*

PROOF. The standard proof evokes the measurable Riemann mapping theorem which is beyond the scope of these notes. We refer to Section 11 of [**FM11**] for a nice account. Another reference is [**IT89**].

There is a partial result which can be proven with elementary methods. Namely, let $q_i, q \in \mathcal{Q}(X)$ and assume that $q_i \to q$. Since q_i, q are holomorphic sections of the same line bundle over X, the zeros of q_i converge to the zeros of q. Assume that the zeros of q are *simple*, i.e. all singular points for the singular euclidean metric defined by q are 3-pronged singularities. Then for large i, the zeros of q_i are simple as well.

By Lemma 2.18 we can choose a triangulation T of X by saddle connections for q which contains all singular points as vertices.

Every triangle of the triangulation T is isometric to an euclidean triangle. For large enough i there is a corresponding triangulation T_i of the surface S whose edges consist of saddle connections for q_i. This triangulation is constructed by choosing a diffeomorphism F of S isotopic to the identity which maps the singular points of q to the singular points of q_i and replacing the image under F of each edge from T by the unique shortest arc for q_i in the same homotopy class with fixed endpoints. Since such an arc is a concatenation of saddle connections, for large enough i these arcs are saddle connections.

Let $\tilde{q}, \tilde{q}_i$ be the terminal quadratic differentials of the Teichmüller maps $\Psi(q)$, $\Psi(q_i)$. Since the stretch maps are affine in euclidean coordinates, the triangulations of S constructed in the previous paragraph define triangulations of S whose edges are saddle connections for $\tilde{q}, \tilde{q}_i$. As $i \to \infty$, the side lengths of the triangles for $\tilde{q}_i$ converge to the side lengths of the corresponding triangles for $\tilde{q}$.

Choose a number $r > 0$ so that the circular discs of radius $2r$ about the singular points of $\tilde{q}$ are pairwise disjoint. Note that such a disc is up to isometry determined by its radius and the order of the singular point. The intersection of each triangle with the complement of the discs of radius r about the singular points is the complement of discs of radius r about the vertices of the triangles.

Since the side lengths of the triangles for $\tilde{q}_i$ converge to the side lengths of the triangles for $\tilde{q}$ as $i \to \infty$, for large i one can construct explicitly a bilipschitz diffeomorphism between the truncated triangles whose bilipschitz constants tend to one as $i \to \infty$. This can be done in such a way that the restrictions of these maps to the truncated sides match up. Moreover, we can require that they match up with suitably constructed bilipschitz maps between the circles of radius r which are isometries near the singular point and whose bilipschitz constant tends to one as $i \to \infty$. Then these maps can be glued to a quasi-conformal map between the Riemann surfaces $\Psi(\tilde{q}_i)$, $\Psi(\tilde{q})$, with constant tending to one as $i \to \infty$ (compare the proof of Lemma 1.10).

A variation of this elementary argument can also be used in the general case, but apart from its intuitive appeal, it does not have any advantage over the elegant proof using the important measurable Riemann mapping theorem. □

Lemma 2.20. Ψ *is proper.*

Proof. Since Ψ is continuous, it suffices to show that the preimage under Ψ of a compact set in $\mathcal{T}(S)$ is bounded in $\mathcal{Q}(X)$.

To this end let $\kappa(Y) = d_{\mathcal{T}}(X, Y)$. By Proposition 2.16, κ is a continuous function on $\mathcal{T}(S)$. In particular, if $K \subset \mathcal{T}(S)$ is compact then κ assumes a maximum $M > 0$ on K. By Corollary 2.12, this implies that $\|q\| \leq M$ for every $q \in \mathcal{Q}(X)$ with $\Psi(q) \in K$. The lemma follows. □

Now Ψ is an injective continuous proper map between topological manifolds homeomorphic to $\mathbb{R}^{6g-6}$. By invariance of domain, the map Ψ is in fact a homeomorphism. As a consequence, we obtain

Corollary 2.21. *Any two points in $\mathcal{T}(S)$ can be connected by a unique Teichmüller geodesic.*

LECTURE 3

Complex Structures, Jacobians and the Weil Petersson Form

The goal of this section is a differential geometric look at the complex geometry of Teichmüller space and moduli space. We also evoke without proof some connections to the algebraic geometry of the moduli space of curves.

We begin with having a closer look at the vector space $\Omega(X)$ of holomorphic one-forms on a Riemann surface X. By integration along loops, any 1-form $0 \neq \omega \in \Omega(X)$ defines a homomorphism $\pi_1(S) \to \mathbb{C}$ called the *period map*.

Choose a basis $a_1, \dots, a_g, b_1, \dots, b_g$ for $H_1(S, \mathbb{Z})$ so that $a_i \cdot b_j = \delta_{ij}$ where $\cdot$ is the homology intersection form. Such a basis is called a *symplectic basis*. An example can be constructed as follows. Let Y be a $4g$-gon with a counter-clockwise numbering of the edges. For each $0 \leq i \leq g-1$ identify the edge $a_i = 4i+1$ with the edge $a_i^{-1} = 4i+3$ with an orientation reversing homeomorphism, and identify the edge $b_i = 4i+2$ with the edge $b_i^{-1} = 4i+4$ with an orientation reversing homeomorphism. The result is a closed surface S of genus g. The image in S of each oriented edge of Y is a simple closed curve which is identified with its edge label, and these curves intersect in a single point. The curves $a_1, \dots, a_g, b_1, \dots, b_g$ define a symplectic basis of $H_1(S, \mathbb{Z})$.

Now assume that the above polygon Y is obtained by cutting the Riemann surface X open along smooth simple closed loops intersecting in a single point. Let α_i, β_i be the periods of ω for the corresponding symplectic basis of $H_1(S, \mathbb{Z})$. On the polygon Y, ω is the differential df of a holomorphic function f. By Stokes' theorem,

$$\int_X \omega \wedge \bar{\omega} = \int_Y df \wedge \bar{\omega} = \int_{\partial Y} f \wedge \bar{\omega}.$$

Observe that if p, p' are two points which are identified under the glueing of the sides of Y and if p projects to an interior points of a_i and p' projects to an interior point of a_i^{-1} then

$$f(p') - f(p) = \int_{b_i} \omega = \beta_i.$$

Similarly, if u, u' are identified under the glueing and if u projects to an interior point of b_i and u' projects to an interior point of b_i^{-1} then $f(u') - f(u) = -\alpha_i$. We therefore have

$$\int_{a_i}(f(p)-f(p'))\wedge\bar\omega = -\beta_i\bar\alpha_i \text{ and}$$
$$\int_{b_i}(f(u)-f(u'))\wedge\bar\omega = \alpha_i\bar\beta_i.$$

Since the curve a_i occurs precisely twice along ∂Y, together this yields the formula

$$\int_S \omega\wedge\bar\omega = \sum \alpha_i\bar\beta_i - \beta_i\bar\alpha_i.$$

The formula implies that if the a-periods of ω vanish then so does ω. But there are exactly g a-periods and hence the linear map $\omega\to(\int_{a_1}\omega,\dots,\int_{a_g}\omega)$ is a linear isomorphism $\Omega(X)\to\mathbb{C}^g$ (recall from Proposition 2.5 that $\Omega(X)$ is isomorphic to $\mathbb{C}^g$; in fact, the above discussion coincides with the discussion in the proof of Proposition 2.5, only the viewpoint taken here is slightly different). Or, for a given symplectic basis (a_i,b_j) of $H_1(S,\mathbb{Z})$, there is a canonical basis of $\Omega(X)$ such that $\omega_i(a_j)=\delta_{ij}$. Then the *Riemann period matrix* of X is given by

$$Z_{i,j}=\omega_i(b_j).$$

Thus if $\omega\in\Omega(X)$ has a-periods α_i, the b-periods are $\beta_i=\sum_j Z_{ij}\alpha_j$, or, equivalently, $\beta=Z\alpha$. Together we get

$$|\omega|^2=\frac{i}{2}\int\omega\wedge\bar\omega=\frac{i}{2}(\alpha\bar Z\bar\alpha-\bar\alpha Z\alpha)=\mathrm{Im}(\bar\alpha Z\alpha).$$

As a consequence, the matrix Z is symmetric, and its imaginary part is positive definite. This is one half of what is called *Riemann binary relations* in the literature (see e.g p.262 of [**Mi95**] for more information).

For a given symplectic basis (a_i,b_j) of $H_1(S,\mathbb{Z})$ and every $X\in\mathcal{T}(S)$ let $B(X)$ be the basis of $\Omega(X)$ defined by $\omega_i(a_j)=\delta_{ij}$. This defines a trivialization of the space

$$\mathcal{H}=\cup_{X\in\mathcal{T}(S)}\Omega(X)$$

which equips $\mathcal{H}$ with the structure of a complex vector bundle on $\mathcal{T}(S)$. Any other symplectic basis of $H_1(S,\mathbb{Z})$ defines a new trivialization which differs from the initial one by a bundle isomorphism.

Indeed, that this is pointwise the case is immediate from the above discussion. That these pointwise isomorphisms depend smoothly on the basepoint requires an argument. One fairly easy way to see that this is the case is to observe that the Laplacian depends smoothly on the marked hyperbolic structure in Fenchel Nielsen coordinates, compare Proposition 2.5.

A mapping class defines a symplectic automorphism of $H_1(S,\mathbb{Z})$ and therefore the group $\mathrm{Mod}(S)$ acts on this bundle by bundle automorphisms. In fact, the above trivializations define on $\mathcal{H}$ a $\mathrm{Mod}(S)$-invariant real analytic structure as well. The bundle $\mathcal{H}\to\mathcal{T}(S)$ is called the *Hodge bundle*.

Since the imaginary part of the period matrix Z is positive definite the map

$$H_1(S,\mathbb{Z})\to\Omega(X)^*$$

defined by the periods is non-degenerate (here $\Omega(X)^*$ is the dual of $\Omega(X)$). In particular, its image is a lattice in $\Omega(X)^*$, and $\Omega(X)^*/H_1(S,\mathbb{Z})$ is a complex torus.

To summarize, the period matrix determines a marked complex torus where the marking is defined by the choice of a symplectic basis of $H_1(S,\mathbb{Z})$. The (real) symplectic structure on $H_1(S,\mathbb{Z})$ defined by the intersection form $\cdot$ is called the *principal polarization*. Recall that the intersection form is given by $a\wedge b=(a\cdot b)[S]$ where $[S]$ denotes the fundamental cycle of S.

For each $X\in\mathcal{T}(S)$ there is an anti-symmetric real bilinear form $\langle,\rangle$ on $\Omega(X)$ defined by

$$\langle\omega,\zeta\rangle=\mathrm{Im}\frac{i}{2}\int\omega\wedge\bar{\zeta}.$$

This form clearly is invariant under the complex multiplication J (here J is just standard multiplication with the imaginary i), i.e. we have $\langle J\omega,J\zeta\rangle=\langle\omega,\zeta\rangle$ for all ω,ζ. Moreover, $(u,v)\to\langle Ju,v\rangle$ is an inner product defining an hermitian metric with imaginary part $\langle,\rangle$. A complex multiplication J with these two properties is called *compatible* with the symplectic structure $\langle,\rangle$.

The Hermitian metric determined by the inner product $\langle J\cdot,\cdot\rangle$ defines a complex linear isomorphism $\Omega^*(X)\to\Omega(X)$ which maps the principal polarization on $H_1(S,\mathbb{Z})\subset\Omega(X)^*$ to the real bilinear form $\langle,\rangle$ on $\Omega(X)$. Namely, separating into real and imaginary part and identifying $\Omega(X)^*$ with a subspace of $H_1(S,\mathbb{C})$, the isomorphism $\Omega^*(X)\to\Omega(X)$ is just the isomorphism given by Poincaré duality. In particular, the complex torus defined by the period matrix can be viewed as a quotient of $\Omega(X)$.

As the Hermitian metric on $\Omega(X)$ is determined by the complex structure on $\Omega(X)$ and the principal polarization, the unitary group $U(g)$ for this metric does not depend on any choices made. Moreover, it preserves the complex torus defined by the period matrix.

The symplectic group $\mathrm{Sp}(2g,\mathbb{R})$ acts simply transitively on the space of symplectic bases of $\mathbb{R}^{2g}$. It also acts transitively on the space of complex structures compatible with the symplectic structure. The standard identification of $\mathbb{R}^{2g}$ with $\mathbb{C}^g$ determines a particular symplectic basis, namely the standard basis e_i,f_j $(i,j=1,\dots,g)$ of $\mathbb{R}^{2g}$ where $e_1,\dots,e_g$ is a complex basis of $\mathbb{C}^g$ and where the symplectic structure $\langle,\rangle$ satisfies $\langle e_i,e_j\rangle=\langle f_i,f_j\rangle=0$ and $\langle e_i,f_j\rangle=\delta_{ij}$.

The stabilizer in $\mathrm{Sp}(2g,\mathbb{R})$ of the standard complex structure on $\mathbb{C}^g$ is just the unitary group $\mathrm{U}(g)$. Thus the *Siegel upper half-space* $\mathrm{Sp}(2g,\mathbb{R})/\mathrm{U}(g)$ is the space of complex tori which are quotients of $\mathbb{R}^{2g}$ equipped with a compatible complex structure by a lattice generated by a symplectic basis. With this interpretation, the map which associates to a marked Riemann surface X the complex torus defined by the period matrix for a given symplectic basis of $H_1(S,\mathbb{Z})$ can be viewed as a map from Teichmüller space $\mathcal{T}(S)$ into the Siegel upper half-space.

By Proposition 1.13 the mapping class group $\mathrm{Mod}(S)$ acts properly discontinuously on Teichmüller space. The quotient orbifold $\mathcal{M}_g=\mathcal{T}(S)/\mathrm{Mod}(S)$ is called the *moduli space*.

The automorphism group $\mathrm{Sp}(2g,\mathbb{Z})$ of the integral homology $H_1(S,\mathbb{Z})$ of S equipped with the intersection form changes the marking of the complex torus defined by the period matrix of $\Omega(X)$. The action of a mapping class on the first homology $H_1(S,\mathbb{Z})$ defines a surjective homomorphism $\mathrm{Mod}(S) \to \mathrm{Sp}(2g,\mathbb{Z})$ (we refer to [**FM11**] for surjectivity) and hence the map which associates to a marked Riemann surface the marked complex torus defined by the period matrix descends to a map $\mathcal{M}_g \to \mathrm{Sp}(2g,\mathbb{Z})\backslash\mathrm{Sp}(2g,\mathbb{R})/\mathrm{U}(g)$ which is called the *Torelli map*. The Torelli map does not depend on any choices made. It associates to an unmarked Riemann surface X its *Jacobian* $\mathrm{Jac}(X)$ which is the unmarked complex torus defined by a period matrix and equipped with its principal polarization.

The following is due to Torelli. A proof can be found in Section III of [**FK80**].

Theorem 3.1. *The Torelli map* $\mathcal{M}_g \to \mathrm{Sp}(2g,\mathbb{Z})\backslash\mathrm{Sp}(2g,\mathbb{R})/\mathrm{U}(g)$ *is injective.*

The Siegel upper half-space is a Hermitean symmetric space of non-compact type. In particular, it has an $\mathrm{Sp}(2g,\mathbb{R})$-invariant complex structure given by a system of charts with values in $\mathbb{C}^m$. Its complex dimension equals $\frac{g(g+1)}{2}$ and hence for $g \geq 4$, this dimension exceeds half of the real dimension $3g-3$ of the Teichmüller space. Moreover, for $g \geq 3$ the Torelli map is *not* an embedding in the sense of differentiable orbifolds. Namely, it is branched at the *hyperelliptic locus* which is the set of *hyperelliptic curves*, i.e. Riemann surfaces X with a biholomorphic involution τ so that $X/\tau = \mathbb{C}P^1$. In particular, its differential fails to be of maximal rank at the hyperelliptic locus. The well known *Schottky problem* asks for determining the image of the Torelli map (see [**HM98**] for an account of what was known about this problem 15 years ago).

The usual way to analyze this and related questions is via complex algebraic geometry. This requires the existence of a $\mathrm{Mod}(S)$-invariant complex structure on $\mathcal{T}(S)$. Such a structure indeed exists (see [**IT89, Hu06**]), but it is difficult to construct.

Instead we will adopt a differential geometric viewpoint and explore a (much weaker) structure.

Definition 3.2. An *almost complex structure* on $\mathcal{T}(S)$ is a continuous $(1,1)$-tensor field J on the tangent bundle of $\mathcal{T}(S)$ with $J^2 = -\mathrm{Id}$.

It follows from the discussion in Section 3 that for every $X \in \mathcal{T}(S)$ the complex vector space $\mathcal{Q}(X)$ can be identified with the cotangent space of $\mathcal{T}(S)$ at X (why this is the cotangent space rather than the tangent space is not apparent from our discussion. Instead we refer to [**Hu06**]). Moreover,

$$\mathcal{Q}(S) = \cup_X \mathcal{Q}(X)$$

is a smooth (in fact real analytic) vector bundle over $\mathcal{T}(S)$ on which $\mathrm{Mod}(S)$ acts as a group of bundle automorphisms (again this requires an argument which we do not give here. Again we refer to [**IT89, Hu06**]).

For $X \in \mathcal{T}(S)$, the complex structure on $\mathcal{Q}(X)$ is given by the usual multiplication of a quadratic differential with a complex number. These fibrewise complex structures define a complex structure on the bundle $\mathcal{Q}(S)$.

If $q \in \mathcal{Q}(X)$ then locally $q = f dz^2$ for a holomorphic function f. The complex conjugate of q then equals $\bar{q} = \bar{f} d\bar{z}^2$. In particular, if u is another quadratic differential then $q\bar{u}$ is locally of the form $\alpha dz^2 d\bar{z}^2$ for a complex valued function α. Now the hyperbolic metric h on X is a tensor field of the form $\beta dz d\bar{z}$ and hence we can define

$$g(q_1, q_2) = \int_S \frac{q_1 \bar{q}_2}{h}.$$

This construction defines an Hermitian inner product on the cotangent bundle $\mathcal{Q}(S)$ of $\mathcal{T}(S)$ which is called the *Weil-Petersson metric*. The Weil-Petersson metric is invariant under the action of the mapping class group. It determines a $\mathrm{Mod}(S)$-equivariant bundle isomorphism $T\mathcal{T}(S) \to \mathcal{Q}(S)$ which maps the complex structure on $\mathcal{Q}(S)$ to an almost complex structure J on $\mathcal{T}(S)$.

An almost complex structure J on a manifold M is called *integrable* if it can be defined by charts with values in $\mathbb{C}^m$ for some $m > 0$ in the sense that in these charts, the almost complex structure is just multiplication with i in the tangent bundle. The following fundamental result is quite involved and will not be proven in these notes. Proofs can be found in [**Hu06, IT89**].

Theorem 3.3.
1. *The almost complex structure J on $\mathcal{T}(S)$ is integrable.*
2. *The Torelli map is holomorphic.*

Since the almost complex structure J is $\mathrm{Mod}(S)$-invariant, it follows that $\mathrm{Mod}(S)$ acts on $\mathcal{T}(S)$ properly discontinuously as a group of biholomorphic automorphisms.

The Hermitian metric g on $\mathcal{T}(S)$ defines a two-form

$$\omega(q, z) = \mathrm{Im}\, g(q, z)$$

which is called the *Weil-Petersson form*. The most important properties of the Weil-Petersson metric and the form ω are summarized in the following theorem. We refer to [**IT89, W10, DW07**] for more details and for references.

Theorem 3.4.
1. *ω is closed, i.e. the Weil-Petersson metric is Kähler.*
2. *The Weil-Petersson metric has negative sectional curvature which is neither bounded from below nor bounded from above by a negative constant.*
3. *The Weil-Petersson metric is incomplete.*
4. *Any two points in $\mathcal{T}(S)$ can be connected by a unique Weil-Petersson geodesic.*
5. *Lengths functions are convex along Weil-Petersson geodesics.*
6. *The group of orientation preserving isometries for the Weil-Petersson metric is the mapping class group.*

The strength of this result is illustrated by the following short proof of the *Nielsen realization theorem*. This result was first established by Kerckhoff [**K83**]. The proof below is due to Wolpert (see [**W10**]).

Theorem 3.5. *Let $\Gamma < \mathrm{Mod}(S)$ be a finite subgroup. Then there is $X \in \mathcal{T}(S)$ such that $\Gamma(X) = X$.*

Proof. Choose a Γ-invariant finite set $\mathcal{C}$ of simple closed curves on S so that $\sum_{c\in\mathcal{C}}\ell_c$ is a proper function on $\mathcal{T}(S)$ where as before, $\ell_c(X)$ is the length of the simple closed curve c on the hyperbolic surface X. Such a set of curves can easily be found as follows. Let $\mathcal{P}$ be a pants decomposition of S. For each pants curve $\alpha\in\mathcal{P}$ choose a simple closed curve β_α which is disjoint from $\mathcal{P}-\alpha$ and which intersects α in one or two points depending on whether the component containing α of $S-(\mathcal{P}-\alpha)$ is a bordered torus or a four-holed sphere. Let $\mathcal{C}_0$ be the union of these curves. It follows from the discussion in Section 2 that $\sum_{c\in\mathcal{C}_0}\ell_c$ is a proper function on $\mathcal{T}(S)$. Let $\mathcal{C}=\Gamma\mathcal{C}_0$.

For sufficiently large $b>0$ the set $K=\{\sum_{c\in\mathcal{C}}\ell_c\leq b\}\subset\mathcal{T}(S)$ is compact and non-empty. Since length functions are convex along Weil-Petersson geodesics, this set is moreover convex for the Weil-Petersson metric (i.e. the geodesic segment connecting any two points in K is contained in K), and by Γ-invariance of $\mathcal{C}$, it is invariant under Γ.

The restriction to K of the Weil-Petersson metric is a smooth Riemannian metric of negative sectional curvature. Thus K has a unique center for this metric which can be found as follows. For each $p\in K$ let $\kappa(p)=\max\{d_{WP}(p,u)\mid u\in K\}$ where d_{WP} denotes the distance defined by the Weil-Petersson metric. The function $p\to\kappa(p)$ is continuous. The center of K then is the unique point $p\in K$ for which $\kappa(p)$ is minimal. Here uniqueness follows from convexity of K and the fact that the Weil-Petersson metric is negatively curved and therefore distance functions are strictly convex along Weil-Petersson geodesics. Since K is Γ-invariant and Γ preserves d_{WP}, this center is Γ-invariant. □

Recall from Corollary 1.8 the definition of the Fenchel-Nielsen twist vector field $\mathbf{t}_\alpha$ for a simple closed curve α on S. The following important result is due to Wolpert and can be found in [**W10**]. Its proof uses harmonic Beltrami differentials and exceeds the scope of these notes.

Theorem 3.6. *Let α be a simple closed curve. Then*

$$2\mathbf{t}_\alpha=J\mathrm{grad}\ell_\alpha$$

and $2\omega(\cdot,\mathbf{t}_\alpha)=d\ell_\alpha$.

Corollary 3.7. *ω is invariant under the Fenchel Nielsen twist flow.*

Proof. Since ω is closed, the Lie derivative $L_{\mathbf{t}_\alpha}\omega$ of ω in direction of $\mathbf{t}_\alpha$ equals

$$2L_{\mathbf{t}_\alpha}\omega=2d(\mathbf{t}_\alpha\lrcorner\omega)=-d(d\ell_\alpha)=0.$$

□

Wolpert uses this to show (Theorem 3.14 of [**W10**])

Theorem 3.8. *In Fenchel Nielsen coordinates,*

$$\omega=\frac{1}{2}\sum_i d\ell_i\wedge d\tau_i.$$

PROOF. Let $\mathcal{P}$ be a pants decomposition of S defining the Fenchel Nielsen coordinates. For $i \neq j$ we have

$$2\omega(\text{grad}\ell_j, \mathbf{t}_i) = d\ell_i(\text{grad}\ell_j) = \delta_{ij}.$$

Our goal is to express ω in the length-twist parameters at a given surface X. Since ω is invariant under the Fenchel Nielsen twist flow, we can modify the surface X along the flow lines of the (commuting) twist flows about the pants curves of $\mathcal{P}$ to a surface X' with vanishing twist coordinates without changing the expression for ω in length-twist parameters. Then for X', the seams of a pairs of pants from $S - \mathcal{P}$ are glued to the seams of the neighboring pants.

A pair of pants has an orientation reversing isometric reflection which preserves all seams pointwise and exchanges the two right angled hexagons which together form the pair of pants. By the choice of X', the involution of each individual pair of pants extends to an isometric reflection ρ of X'. The reflection reverses the orientation of the twist vector fields and preserves the length coordinates.

The reflection ρ reverses the complex structure on $\mathcal{T}(S)$ and preserves the real part of the Weil-Petersson metric and hence ω is anti-invariant under ρ. But X' is a fixed point for ρ, moreover its differential at X' preserves the tangents of the length parameter and reverses the tangents of the twist parameters. In particular, the ρ-anti-invariant subspace of the second exterior power of the tangent space of $\mathcal{T}(S)$ at X' is spanned by the two-forms $d\ell_i \wedge d\tau_j$. As a consequence, in the expression for ω the coefficients of the two-forms $d\tau_i \wedge d\tau_j$ and $d\ell_i \wedge d\ell_j$ vanish. The theorem now follows from the second paragraph of this proof. $\square$

As $\mathcal{T}(S)$ is a complex manifold, we can ask for *holomorphic discs* in $\mathcal{T}(S)$. Such a disc is a holomorphic map $D \to \mathcal{T}(S)$ where $D \subset \mathbb{C}$ is the standard unit disc or, equivalently, the hyperbolic plane $\mathbf{H}^2$. Thus we are looking for holomorphic maps $\mathbf{H}^2 \to \mathcal{T}(S)$. Particular such maps can easily be constructed.

Let $q \in \mathcal{Q}(S)$ be arbitrary. Outside its set Σ of zeros, the quadratic differential q defines a singular euclidean metric. This metric determines a family of charts $U_j \subset S - \Sigma \to \varphi_j U_j \subset \mathbb{C}$, unique up to translation and perhaps reflection (multiplication by -1). Chart transitions are translations, perhaps composed with multiplication by -1. The charts are holomorphic for the complex structure X defined by q. For every $A \in \text{SL}(2,\mathbb{R})$ we can postcompose these charts with A. The resulting family $A \circ \varphi_i$ of charts has chart transitions $(A \circ \varphi_i) \circ (A \circ \varphi_j)^{-1} = A(\varphi_i \circ \varphi_j^{-1})A^{-1}$ which are translations, perhaps composed with -1. As a consequence, these charts define a new complex structure and a new quadratic differential. If A is a diagonal matrix, then the new Riemann surface is just the image of X under a Teichmüller map defined by q. This construction defines an action of $\text{SL}(2,\mathbb{R})$ on $\mathcal{Q}(S)$ preserving the norm (=area) of the differential.

If $A \in \text{SO}(2)$ then the new charts for S differ from the old ones by a rotation by some angle θ. This amounts to replacing q by $e^{i\theta}q$. This multiplication does not change the complex structure on the underlying Riemann surface and therefore the map $\text{SL}(2,\mathbb{R}) \to \mathcal{Q}(S)$ projects to a map $\text{SL}(2,\mathbb{R})/\text{SO}(2) = \mathbf{H}^2 \to \mathcal{T}(S)$. We call such a map a *complex geodesic*. Note that the orbits of the action of the diagonal subgroup of $\mathrm{SL}(2,\mathbb{R})$ project to Teichmüller geodesics on $\mathcal{T}(S)$. In particular, a complex geodesic is totally geodesic for the Teichmüller metric. This means that any geodesic segment connecting two points in the disc is entirely contained the disc.

Lemma 3.9. *A complex geodesic is holomorphic for the almost complex structure J and totally geodesic for the Teichmüller metric.*

Proof. Let $P : \mathcal{Q}(S) \to \mathcal{T}(S)$ be the canonical projection. By construction, the differential at $z = Pq$ of a complex geodesic $\sigma : \mathbf{H}^2 \to \mathcal{T}(S)$ associates to $e^{i\theta} \in S^1 \subset \mathbb{C} = T_i\mathbf{H}^2$ the cotangent at z of the Teichmüller geodesic with initial velocity $e^{i\theta}q$. But this just means that this differential satisfies $J \circ d\sigma = d\sigma \circ i$, i.e. it is complex linear. □

The mapping class group acts on $\mathcal{T}(S)$ preserving the complex geodesics. Every real Teichmüller geodesic is contained in a unique complex geodesic.

The *Kobayashi pseudo-metric* of a complex manifold M is defined as follows. For any two points $x, y \in M$ let $\rho_1(x,y)$ be the infimum of the hyperbolic distances in $\mathbf{H}^2$ between two points x_0, y_0 so that there is a holomorphic map $f : \mathbf{H}^2 \to M$ with $f(x_0) = x, f(y_0) = y$. For any positive integer n we put

$$\rho_n(x,y) = \inf \sum_{i=0}^{n-1} \rho_1(x_i, x_{i+1})$$

where the infimum is over all n-chains with $x_0 = x, x_n = y$. Finally put $\rho(x,y) = \inf_n \rho_n(x,y)$. Clearly ρ is symmetric and satisfies the triangle inequality, so it is a pseudo-metric.

The significance of this construction is as follows. If $\varphi : M \to N$ is any holomorphic map then φ is distance non-increasing for the Kobayashi pseudo-metric. In particular, the Kobayashi pseudo-metric is invariant under all biholomorphic automorphisms of M.

The above discussion immediately implies one part of the following theorem due to Royden (see [**IT89**]).

Theorem 3.10. *The Kobayashi pseudo-metric on $\mathcal{T}(S)$ equals the Teichmüller metric.*

Proof. Any two points $x, y \in \mathcal{T}(S)$ are contained in a complex geodesic. This means that there is a holomorphic map $\mathbf{H}^2 \to \mathcal{T}(S)$ which is an isometric embedding for the hyperbolic metric on $\mathbf{H}^2$ and the Teichmüller metric on $\mathcal{T}(S)$. Thus by definition, the Kobayashi pseudo-metric is not bigger than the Teichmüller metric.

The other inequality is much harder. We refer to [**IT89**] for a proof. □

A complex geodesic (or *Teichmüller disc*) is a special case of the following

Definition 3.11. A *holomorphic family of Riemann surfaces* is a holomorphic surjection $P : U \to T$ of complex manifolds such that each fibre is a Riemann surface and that P admits horizontally holomorphic trivializations.

Example: For any Riemann surface S and every complex manifold M, $S \times M$ is a holomorphic family of Riemann surfaces.

Note that if $P \to T$ is a holomorphic family of Riemann surfaces, then the holomorphic cotangent bundle of the fibre is a holomorphic line bundle over P. Moreover, the Hodge bundle induces a holomorphic vector bundle over P etc. We refer to the book [**HM98**] for a careful discussion of this and related results.

Theorem 3.12. *The functor which associates to a complex manifold T the set of isomorphism classes of holomorphic families of Riemann surfaces $P : Y \to T$ with an equivalence class of markings is equivalent to the functor which associates to T the set of holomorphic maps $T \to \mathcal{T}(S)$.*

Remark: The above theorem is false without the requirement that the marking is being remembered. The difficulty is the existence of surfaces with non-trivial automorphisms. This results in the non-existence of a fine moduli space of Riemann surfaces which causes substantial difficulties for an algebraic geometric approach. We refer once more to [**HM98**] for a comprehensive discussion of this problem and the various ways to overcome it.

LECTURE 4

The Curve Graph and the Augmented Teichmüller Space

In Section 4 we saw that there is a natural complex structure on Teichmüller space which descends to a complex structure on moduli space. It turns out that moduli space admits a natural compactification, the so-called Deligne-Mumford compactification. The goal of this section is to give a differential geometric description of this compactification and relate its structure to the geometry of Teichmüller space.

The idea is as follows. Consider a set of Fenchel Nielsen coordinates for a pants decomposition $\mathcal{P}$ of S. For a given pants curve α, we can shrink the length of α to zero and obtain a surface where the curve α has been degenerated to a pair of cusps. The next observation is an immediate consequence of the classical collar lemma in hyperbolic geometry (see [**B92**]). It is used to understand this degeneration.

Lemma 4.1. *For every $\epsilon > 0$ there is a number $M(\epsilon) > 0$ with the following property. Let α be a simple closed curve on the hyperbolic surface X of length at most ϵ. Then α is the core curve of an embedded annulus of width and modulus at least $M(\epsilon)$. Moreover, $M(\epsilon) \to \infty$ as $\epsilon \to 0$.*

PROOF. Let γ be a simple closed geodesic of length ϵ in a hyperbolic surface X. Let A be the cover of X with fundamental group $< \gamma >$. In the proof of Proposition 2.15 we saw that A is an annulus of modulus π/ϵ.

Let $\tilde{\gamma}$ be the lift of γ to A. Define $w(\epsilon)$ by $\sinh w(\epsilon) \sinh(\epsilon/2) = 1$. An explicit calculation shows that the cylinder about $\tilde{\gamma}$ of width $w(\epsilon)$ embeds into X (we refer to [**B92**] for details). Its modulus can be calculated as in the proof of Proposition 2.15 using a branch of the logarithm. It goes to infinity as $\epsilon \to 0$. □

As a consequence, the lengths of two simple closed curves on X can both be small only if the curves can be realized disjointly. This suggests to look more closely at the combinatorial structure of the set of all simple closed curves on S.

Definition 4.2. The *curve complex* $\mathcal{C}(S)$ of S is the simplicial complex whose vertices are isotopy classes of simple closed curves on S and where a k-tuple of such curves spans a simplex if and only if they can be realized disjointly.

Example: 1) Let α be a simple closed non-separating curve. Then $S - \alpha$ is a surface of genus $g - 1$ with two boundary components. In particular, there are infinitely many distinct free homotopy classes of simple closed curves in $S - \alpha$. As a consequence, $\mathcal{C}(S)$ is locally infinite.

2) Any k-tuple of mutually disjoint simple closed curves can be completed to a pants decomposition of S. Thus any k-simplex in $\mathcal{C}(S)$ is a face of a simplex of maximal dimension $3g - 4$.

The mapping class group $\mathrm{Mod}(S)$ acts on free homotopy classes of simple closed curves preserving disjointness and hence it acts on $\mathcal{C}(S)$ as a group of simplicial automorphisms. The quotient $\mathcal{C}(S)/\mathrm{Mod}(S)$ is a finite simplicial complex. Namely, for any two non-separating simple closed curves α, β there is a mapping class which maps α to β. If α, β are separating then there is a mapping class which maps α to β if and only if the disconnected surfaces $S - \alpha, S - \beta$ are homeomorphic. Since these surfaces are determined up to homeomorphism by the minimum of the genus of one of their components, the number of $\mathrm{Mod}(S)$-orbits of separating curves equals $[g/2]$. As a consequence, $\mathcal{C}(S)/\mathrm{Mod}(S)$ has only $[g/2] + 1$ vertices. Finiteness of the number of simplices of dimension bigger than zero is seen in the same way.

The following important result is due to Ivanov (see [**I02**] for more and for references) with some cases due to Korkmaz and Luo.

Theorem 4.3. *For $g \geq 3$ the automorphism group of the curve complex is the extended mapping class group of S.*

Here the extended mapping class group is the group of isotopy classes of all diffeomorphisms of S including orientation reversing diffeomorphisms.

Consider again a pants decomposition $\mathcal{P}$ and its associated system of Fenchel Nielsen coordinates $(\ell_1, \dots, \ell_{3g-3}, \tau_1, \dots, \tau_{3g-3})$. Let α_i be the pants curve corresponding to the length parameter ℓ_i. Fixing all coordinates but the length function ℓ_1 and letting ℓ_1 go to zero defines a smooth curve of hyperbolic surfaces which degenerate to a surface with two cusps:

The tuple $(\ell_2, \dots, \ell_{3g-3}, \tau_2, \dots, \tau_{3g-3})$ defines as before Fenchel Nielsen coordinates for a surface with two cusps (which may be disconnected if the pants curve α_1 is separating). The twist parameter τ_1 is not defined any more.

Replacing simple closed curves by punctures can simultaneously be done with every tuple of disjoint simple closed curves, i.e. with all vertices of a simplex σ in $\mathcal{C}(S)$. We denote by $\mathcal{S}(\sigma)$ the corresponding Teichmüller space of punctured Riemann surfaces. This Teichmüller space is defined exactly in the same way as the Teichmüller space of a closed surface. In particular, Fenchel Nielsen coordinates define a real analytic structure which is invariant under the mapping class group. Note that if σ is a maximal simplex whose vertices define a pants decomposition then $\mathcal{S}(\sigma)$ consists of a unique point.

Definition 4.4. *Augmented Teichmüller space* is $\overline{\mathcal{T}}(S) = \mathcal{T}(S) \cup_{\sigma \in \mathcal{C}(S)} \mathcal{S}(\sigma)$.

We can define a topology on $\overline{\mathcal{T}}(S)$ as follows. Let $\sigma \in \mathcal{C}(S)$ be a simplex of dimension $k-1$. Let $\mathcal{P}$ be a pants decomposition of S with pants curves α_i $(1 \leq i \leq 3g-3)$ with the property that σ is spanned by the curves $\alpha_1, \dots, \alpha_k$. Let $(\ell_1, \dots, \ell_{3g-3}, \tau_1, \dots, \tau_{3g-3})$ be Fenchel Nielsen coordinates for the pants decomposition $\mathcal{P}$. Then every $X \in \mathcal{S}(\sigma)$ determines a tuple of Fenchel Nielsen coordinates

$$(\ell_{k+1}, \dots, \ell_{3g-3}, \tau_{k+1}, \dots, \tau_{3g-3}).$$

Define a sequence $(X_i) \subset \overline{\mathcal{T}}(S)$ to converge to X if the following holds. Let $(\ell_1^i, \dots, \ell_{3g-3}^i, \tau_1^i, \dots, \tau_{3g-3}^i)$ be the Fenchel Nielsen coordinates of X_i with respect to the pants decomposition $\mathcal{P}$. We require that $\ell_j^i \to 0$ $(1 \leq j \leq k)$ and $\ell_j^i \to \ell_j, \tau_j^i \to \tau_j$ $(j \geq k+1)$. Since Fenchel Nielsen coordinates parametrize Teichmüller space for all hyperbolic surfaces, this definition does not depend on the choice of the pants decomposition $\mathcal{P}$ extending σ. It defines a topology on $\overline{\mathcal{T}}(S)$ which is invariant under the action of the mapping class group $\mathrm{Mod}(S)$.

Proposition 4.5. $\overline{\mathcal{T}}(S)$ *is a stratified non-locally compact* $\mathrm{Mod}(S)$*-space.*

Here the stratification is the decomposition of $\overline{\mathcal{T}}(S)$ into subsets $\mathcal{S}_j$ called *strata*. Each stratum $\mathcal{S}_j$ is a (topological) manifold of dimension n_j. The closure of a stratum is the union of the stratum with a collection of strata of smaller dimension. A stratum in $\overline{\mathcal{T}}(S)$ is defined by a simplex $\mathcal{S}(\sigma) \subset \mathcal{C}(S)$ (which is allowed to be empty). Its closure is the union of $\mathcal{S}(\sigma)$ with the sets $\mathcal{S}(\sigma')$ where σ' runs through all simplices in $\mathcal{C}(S)$ which contain σ as a face. $\overline{\mathcal{T}}(S)$ is not locally compact since a neighborhood basis of a point $X \in \mathcal{S}(\sigma)$ consists of sets with the property that for a pants decomposition $\mathcal{P}$ extending σ, the Fenchel Nielsen length parameters for the curves in σ are small but the twist parameters are arbitrary.

In spite of the fact that $\overline{\mathcal{T}}(S)$ is not locally compact we have

Theorem 4.6. $\overline{\mathcal{T}}(S)/\mathrm{Mod}(S)$ *is compact. It is called the* Deligne Mumford compactification *of moduli space.*

For the proof of compactness one uses an observation of independent interest.

Lemma 4.7. *There is a number* $\chi = \chi(g) > 0$ *so that every closed hyperbolic surface of genus g admits a pants decomposition with pants curves of length at most* χ.

Proof. (Sketch, a detailed proof can be found in Chapter 5 of [**B92**] which also gives explicit bounds.)

Let X be a hyperbolic surface. By Lemma 4.1, any very short simple closed geodesic γ on X is contained in an embedded annulus A of very large width. Moreover, if γ' is any very short simple closed curve which is freely homotopic to γ then γ' is contained in A. In other words, in the complement C of the standard annuli about the very short simple closed geodesics, the injectivity radius of X is bounded from below by a universal constant $12r_0 > 0$. We may assume that these annuli are convex with smooth boundary of constant curvature and uniformly bounded length. Then C is a possibly disconnected surface with smooth boundary which is diffeomorphic to the surface obtained from X by cutting X open along the very short simple closed geodesics. In particular, C has at most $2g-2$ components.

By the Gauss Bonnet theorem, the area of a hyperbolic surface X of genus $g \geq 2$ equals $2\pi(2g-2)$. The area of a hyperbolic disc of radius $r > 0$ is $2\pi \sinh r$ and hence the cardinality of a maximal collection of disjoint embedded discs of radius r_0 in X with center in C is at most $(2g-2)/\sinh r_0$.

Let $x_1, \dots, x_k$ be the set of centers of these discs. The open discs D_i of radius $3r_0$ and center x_i cover C. Two such discs intersect only if the distance between their centers is at most $6r_0$. Since the injectivity radius at a center point is at least $12r_0$, if two such discs intersect then their union is contained in a contractible disc. In particular, their centers can be connected by a unique minimal geodesic of length at most $6r_0$.

The union of all geodesic arcs connecting centers of intersecting discs from the collection $\{D_i\}$ is a graph $G \subset X$ of uniformly bounded length. The inclusion $G \to X$ maps the fundamental group of G onto the fundamental group of C. Namely, every closed curve ρ in C travels successively through a chain of discs among the discs D_i so that any two adjacent discs intersect. Let $D_{i_1}, \dots, D_{i_k}$ be this chain of discs. For each $j \leq k$ choose some t_j so that $t_j < t_{j+1}$, that $\rho(t_j) \in D_{i_j}$ and such that $\rho[t_j, t_{j+1}] \subset D_{i_j} \cup D_{i_{j+1}}$. Connect $\rho(t_j)$ by a path α_j entirely contained in D_{i_j} to the center x_{i_j} of D_{i_j}. Then $\alpha_{j+1} \circ \rho[t_j, t_{j+1}] \circ \alpha_j^{-1}$ is a path connecting x_{i_j} to $x_{i_{j+1}}$ which is entirely contained in $D_{i_j} \cup D_{i_{j+1}}$. Therefore this path is homotopic with fixed endpoints to an edge of G. Now ρ is homotopic to the concatenation of these paths and hence ρ is homotopic to an edge path in G.

Since the number of edges of G is uniformly bounded, the fundamental group of C can be generated by simple edge loops in G whose lengths are bounded from above by a universal constant, and these loops pairwise intersect in uniformly few points. Then there also is a pants decomposition consisting of curves of uniformly bounded length. Such a pants decomposition contains every very short simple closed curve on X. The lemma follows. □

A pants decomposition of a hyperbolic surface X with the properties stated in Lemma 4.7 is called a *Bers decomposition* of X. The number χ is called a *Bers constant*.

Let as before $\mathcal{M}_g = \mathcal{T}(S)/\mathrm{Mod}(S)$ be the moduli space of S. Bers decompositions are used in the following

Proposition 4.8. *Let $(x_i) \subset \mathcal{M}_g$ be a sequence which exits every compact set. Then the shortest length of a simple closed curve on x_i tends to zero as $i \to \infty$.*

PROOF. We show that a sequence of surfaces $(x_i) \subset \mathcal{M}_g$ so that the shortest length ρ_i of a simple closed curve on x_i is bounded away from zero is relative compact in moduli space.

For this let $\tilde{X}_i \in \mathcal{T}(S)$ be a preimage of x_i and let $\mathcal{P}_i$ be a Bers decomposition for $\tilde{X}_i$. Since there are only finitely many combinatorial types of pants decompositions for S and any two pants decompositions of the same combinatorial type can be mapped to each other with an element of $\mathrm{Mod}(S)$, up to passing to a subsequence and changing the lift $\tilde{X}_i$ we may assume that these pants decompositions all coincide. Let $\mathcal{P}$ be this fixed pants decomposition. Let ℓ_j $(j = 1, \dots, 3g-3)$ be the length functions of the pants curves of $\mathcal{P}$. The lengths $\ell_j(\tilde{X}_i)$ are bounded from above and below by a positive constant independent of i.

Dehn twists about the pants curves act with compact quotient on the twist parameters. In other words, up to modifying the preimages of the points x_i by suitably chosen Dehn multitwists about the pants curves of $\mathcal{P}$, these preimages are contained in a compact subset of $\mathcal{T}(S)$. Therefore the sequence $(x_i) \subset \mathcal{M}_g$ is relative compact. □

To complete the proof that $\overline{\mathcal{T}}(S)/\mathrm{Mod}(S)$ is compact, observe that if the sequence $(x_i) \subset \mathcal{M}_g$ exits every compact set then by Proposition 4.8, up to passing to a subsequence we may assume that for each i there is at least one simple closed curve on x_i whose length is at most $1/i$.

Assume as in the proof of Proposition 4.8 that there are lifts $\tilde{X}_i \in \mathcal{T}(S)$ of x_i which admit a fixed pants decomposition $\mathcal{P}$ as a Bers decomposition. Let $\epsilon_0 > 0$ be sufficiently small that for any hyperbolic surface X, any two distinct simple closed geodesics on X of length at most ϵ_0 are disjoint and moreover every simple closed geodesic of length at most ϵ_0 is contained in a Bers decomposition of X. Such a number exists since a very short simple closed geodesic is the core curve of an annulus of very large width. For sufficiently small $\epsilon < \epsilon_0$ and sufficiently large i let σ_i be the simplex in the curve complex corresponding to the curves of $\tilde{X}_i$-length at most ϵ. Then σ_i is a face of the maximal simplex defined by $\mathcal{P}$ and hence by passing to a subsequence, we may assume that σ_i does not depend on i.

Let σ be this fixed simplex. The above argument implies that up to passing to another subsequence, $(\tilde{X}_i)$ converges to a point in the stratum $\mathcal{S}(\sigma)$ and hence (x_i) converges in $\overline{\mathcal{T}}(S)/\mathrm{Mod}(S)$. This completes the proof that $\overline{\mathcal{T}}(S)/\mathrm{Mod}(S)$ is compact.

We also have (see [**W10**] for references)

Theorem 4.9. *Augmented Teichmüller space is the completion of $\mathcal{T}(S)$ with respect to the Weil-Petersson metric. Strata are convex for the completed Weil-Petersson metric.*

The *curve graph* $\mathcal{CG}(S)$ is the one-skeleton of the curve complex. It is a locally infinite graph. Providing each edge with a metric of length one defines the structure of a metric space on $\mathcal{CG}(S)$. The mapping class group acts on this metric graph as a group of simplicial isometries.

The curve graph can be used to obtain information on the geometry of Teichmüller space and the mapping class group. To this end we look more closely at the geometry of $\mathcal{CG}(S)$.

Let $\iota(c,d)$ be the geometric intersection number between two simple closed curves c,d. By definition, $\iota(c,d)$ is the minimal number of intersection points between any two curves c',d' which are freely homotopic to c,d. We call curves which realize this intersection number *in minimal position.*

Lemma 4.10. *The distance in $\mathcal{CG}(S)$ between two simple closed curves c,d does not exceed $\iota(c,d)+1$.*

Proof. Let c,d be simple closed curves in minimal position. We may assume that c and d intersect. If c and d intersect in a single point then the boundary of a tubular neighborhood of $c\cup d$ in S is a simple closed curve which is disjoint from both c and d. Thus the distance between c and d is at most two.

If $\iota(c,d)\geq 2$ then let β be a component of $d-c$. If β leaves and returns to the same side of c then the concatenation of β with a subarc of c connecting the endpoints of β is an essential simple closed curve disjoint from c whose intersection number with d is strictly smaller than the intersection number between c and d.

If β leaves and returns to different sides of c and if $\beta\neq d$ then a similar argument produces a simple closed curve c' whose intersection number with d does not exceed $\iota(c,d)-2$ and which intersects c in at most one point. We saw above that the distance between c and c' does not exceed 2. The claim now follows by induction on $\iota(c,d)$. $\square$

Choose a map

$$\Upsilon:\mathcal{T}(S)\to\mathcal{CG}(S)$$

which associates to a hyperbolic metric a simple closed curve of length at most χ where as before, χ is a Bers constant for S.

Define a map Ψ between metric spaces to be *coarsely Lipschitz* if there is some $L>0$ so that $d(\Psi(x),\Psi(y))\leq Ld(x,y)+L$. Proposition 2.15 and Lemma 4.10 are used to show

Proposition 4.11. *The map Υ is coarsely Lipschitz.*

Proof. Let γ be a simple closed curve on X of length at most χ. By Lemma 4.1, there is a universal number $c>0$ such that the length of every simple closed curve β on X is at least $c\iota(\gamma,\beta)$.

By Proposition 2.15, if $d_{\mathcal{T}}(X,X')\leq 1$ then $\ell_\beta(X)\leq e^4\chi$ for every curve β with $\ell_\beta(X')\leq\chi$. Then the intersection number between γ,β does not exceed $e^4\chi/c$ and therefore by Lemma 4.10, the distance between γ and β in $\mathcal{CG}(S)$ is at most $e^4\chi/c+1$. $\square$

In fact, much more is true. The following fundamental result is due to Masur and Minsky [**MM99**]. For its formulation, we say that a map $\gamma:\mathbb{R}\to\mathcal{CG}(S)$ is an *L-quasi-geodesic* if

$$|s-t|/L-L\leq d(\gamma(s),\gamma(t))\leq L|s-t|+L.$$

The map is an *unparametrized L-quasi-geodesic* if there is a homeomorphism $\rho : \mathbb{R} \to \mathbb{R}$ so that $\gamma \circ \rho$ is an L-quasigeodesic.

A geodesic metric space X is *hyperbolic* if there exists a constant $\delta > 0$ with the following property. Let Δ be a geodesic triangle in X with sides a, b, c. Then the side c is contained in the δ-neighborhood of $a \cup b$.

Theorem 4.12. *The curve graph is hyperbolic. The image under Υ of a Teichmüller geodesic is a uniform unparametrized quasi-geodesic.*

This result turned out to be very important not only for an understanding of the geometry of the mapping class group, but also for an understanding of the Teichmüller metric and the behavior of Teichmüller geodesics. To give a glimpse of what has been recently achieved in this direction, denote for $\epsilon > 0$ by $\mathcal{T}(S)_\epsilon$ the *ϵ-thick part* of Teichmüller space. This set consists of all hyperbolic metrics on S for which the shortest length of a closed geodesic is at least ϵ. Clearly $\mathcal{T}(S)_\epsilon$ is invariant under $\mathrm{Mod}(S)$, moreover by Proposition 4.8, the action of $\mathrm{Mod}(S)$ on $\mathcal{T}(S)_\epsilon$ is cocompact.

A closed unbounded subset B of $\mathcal{T}(S)$ is called *coarsely convex* if there is a number $r > 0$ such that for any two points $X, Y \in B$ the Teichmüller geodesic connecting X to Y is contained in the r-neighborhood of B (note that this definition is meaningless if B is bounded). With this terminology we have [**H10**]

Proposition 4.13. *The restriction of the map Υ to a coarsely convex subset B of $\mathcal{T}(S)_\epsilon$ is a quasi-isometry onto a coarsely convex subset of $\mathcal{CG}(S)$.*

As a consequence, coarsely convex subsets of $\mathcal{T}(S)_\epsilon$ are hyperbolic for the Teichmüller metric.

There is a particularly well known consequence of this statement. Namely, a *pseudo-Anosov mapping class* is an element $g \in \mathrm{Mod}(S)$ which preserves a Teichmüller geodesic γ and acts on it as a non-trivial translation. The geodesic γ is called the *axis* of the pseudo-Anosov element. Since g acts cocompactly on its axis, for some $\epsilon > 0$ the geodesic is entirely contained in $\mathcal{T}(S)_\epsilon$. Now a Teichmüller geodesic is coarsely convex and hence this shows the "only if" part of the following statement.

Corollary 4.14. *A mapping class is pseudo-Anosov if and only if it acts on $\mathcal{CG}(S)$ with unbounded orbits.*

The "if" part of the corollary is a consequence of the fact that for a mapping class $g \in \mathrm{Mod}(S)$ which is not pseudo-Anosov there is some $k \geq 1$ so that g^k fixes a simple closed curve (see [**FM11**]). In particular, every g-orbit on $\mathcal{CG}(S)$ is bounded.

In the thin part of Teichmüller space, the Teichmüller metric is very far from being hyperbolic. Indeed, Minsky's product region theorem [**M96**] shows that "coarse positive curvature" prevails in the thin part of Teichmüller space.

For the formulation of his result, let again $\mathcal{P}$ be a pants decomposition of S and let $\sigma \subset \mathcal{P}$ be a simplex in $\mathcal{C}(S)$ with $u \geq 1$ components $\gamma_1, \ldots, \gamma_u$. For some sufficiently small $\delta > 0$ let $V \subset \mathcal{T}(S)$ be the region where the length of each of the components γ_i is at most δ. Let S_0 be the (possibly disconnected) surface obtained by replacing each of the components γ_i of γ by a pair of punctures. The $(6g-6-2u)$-tuple of Fenchel Nielsen coordinates at pants curves in $\mathcal{P}-\sigma$ defines Fenchel Nielsen coordinates for S_0. The natural coordinate forgetful map determines a projection $\Pi_0 : V \to \mathcal{T}(S_0)$.

For each component γ_i of γ let $\mathbf{H}_i$ be a copy of the upper half-plane equipped with the hyperbolic metric. Define a map $\Pi_i : V \to \mathbf{H}_i$ by $\Pi_i(x) = (\tau_i, 1/\ell_i)$ where (τ_i, ℓ_i) are the twist and length parameters for γ_i. In Fenchel Nielsen coordinates, the positive Dehn twist about the curve γ_i preserves the length and twist parameter of the pants curves $\zeta \in \mathcal{P}-\gamma_i$ and projects to the transformation $(\tau_i, \ell_i) \to (\tau_i+\ell_i, \ell_i)$. For two points $x, y \in V$ define

$$d_P(x,y) = \max\{d_{\mathcal{T}(S_0)}(\Pi_0(x), \Pi_0(y)), d_{\mathbf{H}_i}(\Pi_i(x), \Pi_i(y))\}$$

where $d_{\mathcal{T}(S_0)}$ denotes the Teichmüller distance on $\mathcal{T}(S_0)$ and where $d_{\mathbf{H}_i}$ is the hyperbolic metric on $\mathbf{H}_i$. The following is Theorem 6.1 of [**M96**].

Theorem 4.15. *There is a constant $a > 0$ only depending on δ such that for all $x, y \in V$,*

$$|d_{\mathcal{T}(S)}(x,y) - d_P(x,y)| \le a.$$

More recently, this result has been used by Rafi [**R07b**] to establish a coarse distance formula for the Teichmüller metric.

The product region theorem is silent about the precise behavior of Teichmüller geodesics connecting any two given points in $\mathcal{T}(S)$- the latter are unique while geodesics for an L^∞-metric are not. However, coarsely such geodesics can be understood explicitly [**R10**]. An example of possible behavior is illustrated in the following [**H12**]

Theorem 4.16. *For every $\epsilon > 0$ there is a pseudo-Anosov element $g \in \mathrm{Mod}(S)$ whose axis is entirely contained in $\mathcal{T}(S) - \mathcal{T}(S)_\epsilon$.*

In fact, in imperfect analogy to Weil-Petersson geodesics (we refer once more to [**W10**] for references), Teichmüller geodesics in the thin part or Teichmüller space approach Teichmüller geodesics in boundary strata [**H11, R10**].

LECTURE 5

Geometry and Dynamics of Moduli Space

In this final section we relate some geometric properties of moduli space to dynamics of the Teichmüller flow.

A basic question concerning the geometry of moduli space is as follows: To what extend does $\mathcal{M}_g$ equipped with the Teichmüller metric resemble a locally symmetric space of finite volume?

The Torelli map embeds the moduli space $\mathcal{M}_g$ into the finite volume locally symmetric space $\mathrm{Sp}(2g,\mathbb{Z})\backslash\mathrm{Sp}(2g,\mathbb{R})/\mathrm{U}(g)$. However, the dimension of this locally symmetric space is $\frac{g(g+1)}{2}$ and hence for g large, the image of $\mathcal{M}_g$ is of high codimension. Moreover, the kernel of the induced map on orbifold fundamental groups is the *Torelli group*, a group which is large and complicated (in fact, for $g = 2$ the Torelli group is an infinitely generated free group, see [**I02**] for this result of Mess).

The following result is due to Kaimanovich-Masur and Masur-Farb. Another proof was given later by Bestivina and Fujiwara. We refer to [**I02**] for a history and references.

Theorem 5.1. *Let* Γ *be an irreducible lattice in a semisimple Lie group of non-compact type and rank at least two. Then every homomorphism* $\Gamma \to \mathrm{Mod}(S)$ *has finite image.*

An important geometric property of a locally symmetric space Y of non-compact type and finite volume is the following. Let $r \geq 1$ be the rank of Y and let $T \subset Y$ be an embedded flat torus of rank r. Then there is a compact set $K \subset Y$ such that T can not be homotoped outside K [**PS09**]. If $r = 1$ then this means that a closed geodesic can not be homotoped outside K.

The mapping class group $\mathrm{Mod}(S)$ is finitely presented and hence it can be equipped with a word metric defined by a finite generating set. Such a word metric is unique up to bilipschitz equivalence. The *geometric rank* of the mapping class group is defined to be the maximal dimension of a quasi-isometrically embedded euclidean space. Since any abelian subgroup of $\mathrm{Mod}(S)$ is quasi-isometrically embedded [**FLM91**], the geometric rank is not smaller than the number $3g-3$ of components of a pants decomposition of S. Namely, the group of Dehn twists about the components of a pants decomposition is free abelian of rank $3g-3$. In fact, the geometric rank of $\mathrm{Mod}(S)$ equals $3g-3$ [**BM08**]. There should be a similar statement for Teichmüller space with the Teichmüller metric as indicated by the product region theorem, but we are not aware of such a statement in the literature.

In contrast to locally symmetric spaces, using the Deligne Mumford compactification of moduli space it is easy to show

Proposition 5.2. *For every compact set $K \subset \mathcal{M}_g$, an embedded torus in $\mathcal{M}_g$ whose fundamental group is the abelian group of Dehn twists about the components of a pants decomposition can be homotoped into $\mathcal{M}_g - K$.*

Proof. Let $\mathcal{P}$ be a pants decomposition of S. For any $\epsilon > 0$, a torus as in the proposition can be represented by the projection to $\mathcal{M}_g$ of the set of all points in $\mathcal{T}(S)$ whose Fenchel Nielsen coordinates with respect to $\mathcal{P}$ have lengths parameters equal to ϵ. □

It turns out, however, that *complex* geodesics in $\mathcal{M}_g$ (i.e. projections of complex geodesics in $\mathcal{T}(S)$) are much more rigid. The remainder of this section is devoted to the proof of

Theorem 5.3.
(1) *There is a compact set $K \subset \mathcal{M}_g$ so that every complex geodesic intersects K.*
(2) *A complex geodesic is unbounded in moduli space.*

The (easier) second part of the theorem is due to Masur [**M86**]. The strategy is to use the geometry of the singular euclidean metrics on S of the quadratic differentials defining the complex geodesic. For this we have to understand a bit more explicitly the topology of $\mathcal{Q}(S)$. We begin with organizing quadratic differentials as follows.

Definition 5.4. A *stratum* of quadratic or abelian differentials consists of differentials with the same number and orders of zeros.

Each stratum is a complex orbifold. *Relative periods* of *abelian* differentials can be used to define coordinates on strata. By this we mean the following.

Let ω be any abelian differential. Choose a triangulation T of S whose set of vertices is the set of singular points of ω and whose edges are saddle connections for ω. The existence of such a triangulation was shown in Lemma 2.18. The tangent vector of each edge from the triangulation is a vector in $\mathbb{C}$. These vectors determine the singular euclidean metric defined by ω as well as the horizontal and vertical line fields for ω and hence they determine ω.

A nearby abelian differential ζ in the stratum has nearby singular points of the same orders and hence the triangulation T of S is isotopic to a triangulation T' whose vertices are singular points for ζ and whose edges are saddle connections. Thus we obtain a new set of vectors from ζ by integrating the differential over the edges of T'. As a consequence, integration over the edges of T defines an embedding of a neighborhood of ω in its stratum into $\mathbb{C}^r$ for some large $r > 0$, and this embedding defines a topology on strata which is independent of any choices made. Indeed, changing the triangulation amounts to an affine change of coordinates. As another consequence, strata of abelian differentials are complex manifolds. Note that this structure of a complex manifold is compatible with the structure on the

Hodge bundle $\mathcal{H}$ defined by period coordinates as discussed in Section 4. In particular, the closure of a stratum of abelian differentials is a union of strata in $\mathcal{H}$ of smaller dimension obtained by merging two or more zeros of the differentials. (Using the not very difficult fact that every Riemann surface admits an abelian differential with simple zeros (see p.98 of [**FK80**]), it seems possible to use the natural complex structure on the maximal stratum of abelian differentials to give a direct and elementary proof that Teichmüller space admits a Mod(S)-invariant integrable complex structure).

Strata of quadratic differentials can be treated in the same way. Namely, given a quadratic differential q which is not the square of a holomorphic one-form, there is a two-sheeted cover $\hat{S}$ of S ramified at some of the zeros of q such that q lifts to an abelian differential $\hat{q}$ on $\hat{S}$. Namely, for a simple closed loop α on S not passing through a singular point, define the *holonomy* by parallel transport of a non-zero vector v along α with respect to the singular euclidean metric defined by q. Then the image of v either equals v or $-v$, and this association of sign defines a homology class in $H_1(S - \Sigma, \mathbb{Z}/2\mathbb{Z})$ where Σ is the set of singular points of q. The double cover of $S - \Sigma$ associated to this class determines a branched cover of S with the desired properties.

Any nearby quadratic differential in the stratum of q has the same orientation cover and hence a neighborhood of q in its stratum injects into a neighborhood of $\hat{q}$ in its stratum. The closure of a stratum in $\mathcal{Q}(S)$ is a union of strata of smaller dimension obtained by merging two or more zeros of the differentials.

Strata are invariant under scaling. Their intersections with the space $\mathcal{Q}^1(S)$ of area one quadratic differentials will be called strata in $\mathcal{Q}^1(S)$. Such strata in $\mathcal{Q}^1(S)$ are invariant under the natural $\mathrm{SL}(2,\mathbb{R})$-action on $\mathcal{Q}^1(S)$ and under the action of the mapping class group. Therefore strata in $\mathcal{Q}^1(S)$ project to strata in the moduli space $\mathcal{Q}^1(S)/\mathrm{Mod}(S) = \mathcal{V}$ of area one quadratic differentials.

Compact subsets of strata in $\mathcal{V}$ are easy to understand. The next definition is used to this end.

Definition 5.5. The *extremal length* of a simple closed curve α on a Riemann surface X is defined to be

$$E(\alpha) = \sup_{\rho} \frac{\ell_\rho(\alpha)^2}{\mathrm{area}_\rho(S)}$$

where the supremum is taken over all Borel metrics ρ in the conformal class of X and where $\ell_\rho(\alpha)$ is a shortest ρ-length of a curve in the free homotopy class of α.

If S contains a cylinder of modulus $a > 0$ then the extremal length of the core curve of the cylinder is at most $1/a$ (we refer to [**IT89**] for references).

Proposition 5.6. *A closed subset K of a stratum in $\mathcal{V}$ is compact if and only if there is a number $\epsilon > 0$ so that no $q \in K$ has a saddle connection of length smaller than ϵ.*

Proof. Let C be a stratum in $\mathcal{V}$ and let $K \subset C$ be compact. Assume that there is a sequence $(q_i) \subset K$ so that for each i there is a saddle connection γ_i for q_i of length less than $1/i$. By passing to a subsequence we may assume that $q_i \to q \in K$. But then two zeros of q_i have collided to a single zero of q and hence $q \notin C$, a contradiction.

For the reverse direction, let K be a closed subset of a stratum $C \subset \mathcal{V}$ so that there is no saddle connection of length smaller than ϵ for any point in K. We have to show that K is compact. For this note that for $q \in K$ there is no essential simple closed curve of q-length at most ϵ. Namely, by the Arzela Ascoli theorem, any closed curve on S is freely homotopic to a closed geodesic for the singular euclidean metric defined by q whose length is minimal in its free homotopy class. Such a closed geodesic either is a concatenation of saddle connections and hence its length is bigger than ϵ by assumption, or it does not pass through a singular point. Since the metric q is euclidean, in the latter case the geodesic is the core curve of a flat cylinder foliated by closed geodesics of the same length. A boundary circle of a maximal such cylinder contains a singular point and hence it is a path composed of saddle connections (we refer to [**S84**] for more details). Once again, the length of the curve is at least ϵ.

Now the q-length of any simple closed curve on S is at least ϵ, and the area of q equals one. This implies that the extremal length of any simple closed curve on the Riemann surface X defined by q is bounded from below by ϵ^2. As a consequence, there is no embedded cylinder in X of modulus bigger than $1/\epsilon^2$. On the other hand, a short hyperbolic geodesic on a surface S is the core curve of an embedded cylinder whose modulus tends to infinity as the length of the curve tends to zero. This implies that K projects to a compact subset of $\mathcal{M}_g$ and hence it is relative compact in $\mathcal{V}$.

If there is an accumulation point of K in $\mathcal{V}$ which is not contained in C then the shortest length of a saddle connection for points in K is not bounded from below by a positive constant. The proposition follows. $\square$

The *Teichmüller flow* Φ^t on $\mathcal{Q}^1(S)$ is the flow defined by the action of the diagonal subgroup of $\mathrm{SL}(2,\mathbb{R})$. Thus

$$\Phi^t q = \begin{pmatrix} e^{t/2} & \\ & e^{-t/2} \end{pmatrix} q.$$

This flow commutes with the action of the mapping class group and hence it descends to a flow on $\mathcal{V}$ which is called again the Teichmüller flow.

Corollary 5.7. *Let q be a quadratic differential which admits a vertical saddle connection. Then the image of q under the Teichmüller flow does not have an accumulation point in the interior of the stratum containing q.*

Proof. Let q be a quadratic differential with a vertical saddle connection of length s. Then for every $t > 0$, the differential $\Phi^t q$ has a vertical saddle connection of length $e^{-t/2}s$. Now use Proposition 5.6. $\square$

If q is any quadratic differential and if v is any saddle connection for q then there is some $\theta \in [0, 2\pi)$ so that v is vertical for $e^{i\theta}q$. Thus Corollary 5.7 shows that an $\mathrm{SL}(2,\mathbb{R})$-orbit is not relative compact in any stratum. However, there are quadratic differentials with arbitrarily short saddle connections which project to a fixed compact subset of moduli space and hence this does not imply that a complex geodesic intersects the complement of every compact subset of moduli space. That this nevertheless holds true is due to Masur [**M86**].

Proposition 5.8. *Let $H \subset \mathcal{M}_g$ be a complex geodesic and let $K \subset \mathcal{M}_g$ be any compact set. Then $H \cap (\mathcal{M}_g - K) \neq \emptyset$.*

PROOF. Let $q \in \mathcal{V}$ be an area one quadratic differential. Let $E \subset \mathcal{V}$ be the set of accumulation points of the $\mathrm{SL}(2,\mathbb{R})$-orbit of q, i.e. the set of all points z so that there is an unbounded sequence $(g_i) \subset \mathrm{SL}(2,\mathbb{R})$ such that $g_i q \to z$. Then E is a closed $\mathrm{SL}(2,\mathbb{R})$-invariant subset of $\mathcal{V}$. We may assume that $E \neq \emptyset$.

For $u \in \mathcal{V}$ with zeros of order k_i define $\mathcal{O}(u) = \sum_i (k_i - 1)$. Let $E_{\max} = \{u \in E \mid \mathcal{O}(u) \geq \mathcal{O}(z) \text{ for all } z \in E\}$. Since E is closed, the set $E_{\max}$ is closed as well. Namely, if q is a quadratic differential with s zeros of order k_i, then any nearby differential has at least s zeros. By the Gauss Bonnet theorem, the sum of the orders of the zeros is constant which implies that $\mathcal{O}(z) < \mathcal{O}(q)$ if z has more than s zeros. It now suffices to show that $E_{\max}$ is not compact.

For this we argue by contradiction and we assume that $E_{\max}$ is compact. For a quadratic differential u let $d(u)$ be the shortest length of a saddle connection for the singular euclidean metric defined by u. Let $d_0 = \inf_{u \in E_{\max}} d(u)$.

If $d_0 = 0$ then choose a sequence $u_i \subset E_{\max}$ so that $d(u_i) \to 0$. Since we assume that $E_{\max}$ is compact, by passing to a subsequence we may assume that $u_i \to u \in E_{\max}$. But then two zeros of u_i connected by a short saddle connection collide to a single zero in u. As a consequence, we have $\mathcal{O}(u) > \mathcal{O}(u_i)$ (see the discussion above) which contradicts the definition of $E_{\max}$.

If $d_0 > 0$ then by the same reasoning, there is some $u \in E_{\max}$ with $d(u) = d_0$. Let γ be a saddle connection of length d_0 on u. There is some $\theta_0 \in [0, 2\pi]$ so that γ is vertical for $e^{i\theta_0}u$. Since the Teichmüller flow contracts vertical distances, there is some $s > 0$ so that $\Phi^s e^{i\theta_0}u$ has a saddle connection of length $d_0/2$.

The set E is $\mathrm{SL}(2,\mathbb{R})$-invariant and therefore $\Phi^s e^{i\theta_0}u \in E$. However, the $\mathrm{SL}(2,\mathbb{R})$-action preserves the strata of $\mathcal{V}$ and hence we have $\Phi^s e^{i\theta_0}u \in E_{\max}$. This is a contradiction to the definition of d_0. The proposition follows. □

A degeneration of quadratic differentials which is particularly easy to understand can be described as follows. A *metric cylinder* for a singular euclidean metric q is an embedded cylinder in S which is foliated by closed geodesics for q. If these geodesics are vertical then their lengths decrease and the height of the cylinder increases under the Teichmüller flow Φ^t. Lemma 4.1 then shows that the hyperbolic length of the core curve of the cylinder tends to zero as $t \to \infty$. In particular, the underlying family of Riemann surfaces leaves every compact subset of moduli space.

Masur [**M86**] showed the following result which is much stronger than Proposition 5.8 (see also the survey [**MT02**]).

Theorem 5.9. *For every quadratic differential q, the set of directions θ so that $e^{i\theta}q$ has a vertical metric cylinder is dense in the unit circle.*

The *horocyle flow* h_t ($t \in \mathbb{R}$) is the flow on $\mathcal{V}$ defined by the action of the unipotent subgroup

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \tag{1}$$

of $\mathrm{SL}(2,\mathbb{R})$.

The following is a non-quantitative version of a result of Minsky and Weiss [**MW02**]. Together with Proposition 5.8 it implies Theorem 5.3.

Theorem 5.10. *For every stratum $C \subset \mathcal{V}$ there is a compact subset K of C which is intersected by the orbit of the horocycle flow through every quadratic differential $q \in C$ without horizontal saddle connection.*

Example: Let q be a quadratic differential which defines a flat metric consisting of a single cylinder which is foliated by vertical simple closed geodesics. This cylinder is bounded by vertical saddle connections (see [**S84**] for examples). Assume moreover that there is a closed horizontal curve. Then there is a closed orbit of the horocycle flow through q. Choosing such quadratic differentials carefully results in closed orbits for the horocycle flow which are disjoint from any given compact subset K of the stratum containing q. Thus the statement of Theorem 5.10 does not hold for all quadratic differentials in C. This behavior is familiar for other unipotent flows. An example of such a flow is the horocycle flow on the unit tangent bundle of a non-compact hyperbolic surface of finite volume (which is defined to be the action of same unipotent subgroup of $\mathrm{PSL}(2,\mathbb{R})$). For this flow every orbit is closed or equidistributed for the Haar measure (see [**MW02**] for an overview and references), and closed orbits exist arbitrarily far out in the cusps.

Slightly stronger versions of Theorem 5.10 can be found in the appendix of [**LM08**] and of [**H09**]. Beyond this, not much is known about the horocycle flow on strata (see however [**SW04**]).

The remainder of these notes is devoted to the proof of Theorem 5.10. We follow [**MW02**]. The proof uses an idea from the theory of unipotent flows: make effective use of slow divergence of orbits to find a point on the orbit which does not have a short saddle connection.

We begin with having a closer look at an orbit of the horocycle flow through an area one quadratic differential q without horizontal saddle connection. Let δ be a saddle connection for q. Then δ is a straight line segment for the singular euclidean metric defined by q. Its tangent can be decomposed into its horizontal part $x = x(\delta, q)$ and its vertical part $y = y(\delta, q)$. Since q does not have a horizontal saddle connection, we have $y \neq 0$ and hence $h_t(x, y) = (x + ty, y) \neq (x, y)$ for $t \neq 0$

(note that the image under h_t of a saddle connection for q is a saddle connection for $h_t q$). Write

$$\ell(\delta, q) = \max\{|x(\delta, q)|, |y(\delta, q)|\}$$

and $\ell_{q,\delta}(t) = \ell(\delta, h_t q)$. The next easy lemma describes these functions explicitly and gives a quantitative version of the idea of slow divergence of orbits of the horocycle flow.

Lemma 5.11. (1) *There are $t_0 \in \mathbb{R}, c > 0$ so that*

$$\ell_{q,\delta}(t) = \max\{c, c|t - t_0|\}.$$

(2) *Let $f, \tilde{f}$ be two functions of the form $t \to \max\{\mathrm{c}, \mathrm{c}|\mathrm{t} - \mathrm{t_0}|\}$ for some $c > 0$. Suppose that for some $b > 0$ and $s \in \mathbb{R}$ we have $f(s) < b, \tilde{f}(s) < b$. Then possibly after exchanging f and $\tilde{f}$, $f(t) < 3b$ whenever $\tilde{f}(t) < b$.*

Proof. Put $c = |y| > 0$ and $t_0 = -x/y$. The first part of the lemma follows.

For the second part, let c, t_0 and $\tilde{c}, \tilde{t}_0$ be the constants for $f, \tilde{f}$. Assume that $0 < c \leq \tilde{c}$.

By the hypothesis on s, t we have $\tilde{c}|s - \tilde{t}_0| < b, \tilde{c}|t - \tilde{t}_0| < b$ and hence

$$\tilde{c}|t - s| \leq \tilde{c}|t - \tilde{t}_0| + \tilde{c}|\tilde{t}_0 - s| < 2b.$$

Now $f(s) < b$ and therefore $c < b$. If t_0 is between s and t then

$$c|t - t_0| \leq \tilde{c}|t - t_0| \leq \tilde{c}|s - t| < 2b.$$

If t_0 is not between s and t then

$$c|t - t_0| = c|s - t_0| + c|t - s| < f(s) + 2b < 3b.$$

In either case we get $f(t) < 3b$. $\square$

Denote by $|A|$ the Lebesgue measure of a subset of the real line. From the first part of Lemma 5.11 one obtains the following. Let δ be any saddle connection, let $\theta > 0$, $I \subset \mathbb{R}$ be an interval and let

$$I_{\delta,\theta} = \{s \in I \mid \ell_{q,\delta}(s) < \theta\}$$

and

$$\|\ell_{q,\delta}\|_I = \sup_{s \in I} \ell_{q,\delta}(s).$$

Then

$$\frac{|I_{\delta,\theta}|}{|I|} \leq 2\left(\frac{\theta}{\|\ell_{q,\delta}\|_I}\right).$$

Namely, using the above notation, if $\theta \leq c$ then $I_{\delta,\theta} = \emptyset$. If $\theta > c$ then either $I \cap \{t \geq t_0\} \subset I_{\delta,\theta}$ or for $b = \sup\{s \in I\}$ we have $\frac{|I_{\delta,\theta} \cap \{t \geq t_0\}|}{|I \cap \{t \geq t_0\}|} = \frac{\theta}{c|b - t_0|} \leq \frac{\theta}{\|\ell_{q,\delta}\|_I}$.

From this we deduce

Lemma 5.12. *Assume that there is some $M > 0$, there is an interval I, a number $\rho > 0$ and a collection Δ of saddle connections for q with the following properties.*

(1) *For any $t \in I$, $\ell_{q,\delta}(t) < \rho$ for at most M saddle connections $\delta \in \Delta$.*

(2) $\|\ell_{q,\delta}\|_I \geq \rho$ *for every* $\delta \in \Delta$.

Then

$$|\{s \in I \mid \ell_{q,\delta}(s) < \rho/4M^2 \text{ for some } \delta \in \Delta\}| < |I|/2M.$$

Proof. Let

$$J = \int_I \sharp\{\delta \in \Delta \mid \ell_{q,\delta}(s) < \rho\} ds.$$

The integrand of J is bounded from above by M and therefore $J \leq M|I|$. Let $\beta \leq \rho/4M^2$. By the discussion after Lemma 5.11, for every $\delta \in \Delta$ we have $|I_{\delta,\beta}| \leq 2|I_{\delta,\rho}|\frac{\beta}{\rho}$. Since the set of all saddle connections is countable, the sum

$$\sum_{\delta \in \Delta} |I_{\delta,\rho}|$$

is well defined and equals J. Together we obtain

$$J \geq [2\frac{\beta}{\rho}]^{-1} \sum_{\delta \in \Delta} |I_{\delta,\beta}|.$$

Since $\beta \leq \rho/4M^2$ we conclude that $\sum_{\delta\in\Delta} |I_{\delta,\beta}| \leq |I|/2M$. □

While for every hyperbolic metric on S there are at most $3g-3$ short simple closed curves, there is no uniform bound on the number of short saddle connections for a quadratic differential. Examples can be constructed as follows.

Let T_1, T_2 be two flat tori of area one. For a small number $\delta > 0$, cut T_2 open along an embedded line segment of length δ. For each i let T_2^i be the flat torus obtained from T_2 by scaling the flat metric by the constant $1/i$. Cut T_1 open along a line segment of length δ/i and glue the tori T_1 and T_2^i along the two slits of length δ/i to obtain a surface T^i of genus 2. The euclidean metrics on the tori T_1 and T_2^i define a singular euclidean metric on T^i which is the metric of an abelian differential ω^i. The images in T^i of the endpoints of the slits in T_1, T_2^i are singular points for ω^i, and the slits are saddle connections. As $i \to \infty$, the length of any saddle connection of ω^i which is entirely contained in T_2^i tends zero (with the obvious interpretation). Now observe that there are countably many such saddle connections.

To overcome this difficulty one looks at "isolating" saddle connections which play the role of the slits in the above example. One verifies that the number of isolating saddle connections is uniformly bounded and that moreover in the absence of isolating saddle connections, there are no short saddle connections at all.

To define such isolating saddle connection, we observe

Lemma 5.13. *There is a number* $M > 0$ *so that the cardinality of any set of saddle connections with mutually disjoint interior is at most* M.

PROOF. Assume that q has k singular points. Let $\alpha_1, \dots, \alpha_s$ be saddle connections with disjoint interior. As in the proof of Lemma 2.18, by adding a finite additional collection of saddle connections we may assume that the arcs α_i decompose S into triangles in such a way that all singular points of q occur as vertices. We do not require that the vertices of these triangles are distinct.

The number of edges of this triangulation of S equals s. Any edge is contained in the boundary of 2 triangles, and every triangle has 3 sides. Euler's formula shows that $\frac{2}{3}s - s + k = 2 - 2g$ and hence $s = 3(2g - 2 + k)$. □

Let $\mathcal{L}_q$ be the set of all saddle connections for q. For $r \leq M$ define

$$\mathcal{E}_r = \{E \subset \mathcal{L}_q \mid E \text{ consists of } r \text{ segments with disjoint interior}\}.$$

For $E \in \mathcal{E}_r$ define $S(E)$ to be the closure of the simply connected components of $S - \cup_{\delta \in E}\delta$. Then $S(E)$ is a possibly empty subsurface of S whose boundary $\partial S(E)$ is contained in E. Let $W(E)$ be the union of $\partial S(E)$ with those saddle connections in E whose interiors are disjoint from $S(E)$. Then $W(E)$ is a closed subset of S which is empty only if $S(E) = S$.

The next lemma indicates how to find "isolating" saddle connections: As saddle connections on the boundary of regions of S which can be decomposed by short saddle connections into simply connected components. Note that the constant θ in the assumptions of the lemma is a gap for lengths of saddle connections. The main remaining task for the completion of the proof of Theorem 5.10 will then be to show the existence of such a gap number to which the lemma can be applied.

Lemma 5.14. *Suppose that $E \in \mathcal{E}_r$ consists of segments of length $\leq \theta$. Suppose furthermore that there is no saddle connection of length $\leq 4\theta$ whose interior is disjoint from E. Then any saddle connection δ' which intersects the interior of a saddle connection $\delta \in W(E)$ has length at least 2θ.*

PROOF. Let δ be a saddle connection in $W(E)$ and assume that the saddle connection δ' intersects δ in an interior point p. Let $\omega \subset \delta'$ be a subsegment of δ' in $S - (S(E) \cup E)$ with one endpoint p and such that the second endpoint of ω either is an interior point of a saddle connection in E or is a singular point. Then ω is contained in a component C of $S - E$ which is not simply connected.

The component C is an oriented surface with singular euclidean metric with cone points of cone angles bigger than 2π and finite area. Its metric completion $\overline{C}$ is a compact surface with piecewise geodesic boundary and finitely many singular points. Each boundary arc with the boundary orientation is an oriented saddle connection in S. The closure of C in S can be obtained from $\overline{C}$ by identifying some of the boundary arcs with an orientation reversing isometry. The boundary of $\overline{C}$ contains the saddle connection $\delta \in W(E)$.

Our goal is to show that the length of ω is at least 2θ. For this we argue by contradiction and we assume that the length of ω is smaller than 2θ.

Assume furthermore for the moment that the second endpoint of ω is a singular point. Let ρ_1, ρ_2 be the two subarcs of the saddle connection $\delta \subset \overline{C}$ connecting p

to the two endpoints x_1, x_2 of δ. The concatenation of ω with ρ_1, ρ_2 is a broken geodesic in $\overline{C}$ with endpoints at singular points and length smaller than 3θ.

The shortest arc ρ_i' in $\overline{C}$ which is homotopic with fixed endpoints to the concatenation of ω with ρ_i is composed of saddle connections all of whose lengths are smaller than 3θ. The saddle connections either are contained in E or their interiors are disjoint from E. By the assumption in the lemma, these saddle connections are all contained in E.

Now ω is a smooth geodesic arc in $\overline{C}$ and hence it is the unique arc of minimal length in its homotopy class with fixed endpoints. The concatenation $\rho_i^{-1} \circ \rho_i'$ is homotopic to ω with fixed endpoints. In particular, ρ_i', ρ_i, ω bound an embedded disc Ω_i in $\overline{C}$. Since ρ_i' is minimal, the angle at each singular point of $\partial\Omega_i$ which is distinct from x_i and the endpoints of ω is at least π. Since the cone angle at each interior singular point of Ω_i is bigger than 2π, by the Gauss Bonnet theorem the angle at x_i is smaller than π.

As a consequence, the concatenation $\delta \circ \rho_1'$ is not length minimizing. By construction, ρ_2' is a shortest arc in $\overline{C}$ with the same endpoints which is homotopic to $\rho_1' \circ \delta$. A homotopy between these two arcs with fixed endpoints covers a disc which necessarily contains ω. By the definition of $S(E)$, this region is contained in $S(E)$ which is a contradiction to the choice of ω.

If both endpoints of ω are interior points of saddle connections in E then we argue in exactly the same way. In this case we construct from ω and the subarcs of the saddle connections containing the endpoints of ω a simply connected quadrangle $Q \subset S(E)$. Two opposite sides of Q are the saddle connections $\delta, \delta' \in W(E)$ containing the endpoints of ω. The other two sides are geodesic arcs in $S - (S(E) \cup E)$ homotopic to the concatenation of ω with the two subsegments of δ, δ' which are to the left (or right) of ω for some choice of orientation. $\square$

Lemma 5.14 shows that under the assumption of the existence of a "gap" the surface S can be decomposed into subsurfaces bounded by paths which are composed of short saddle connections, and these subsurfaces either are small in size or the short saddle connections they contain is a forest, i.e. a finite union of trees.

Our next goal is to find points on the orbit $h_t q$ which satisfy the assumptions in Lemma 5.14 for a suitably chosen gap constant. Let $M > 0$ be as in Lemma 5.13 and for $r \in \{1, \dots, M\}$ define

$$\alpha_r(t) = \min_{E \in \mathcal{E}_r} \max\{\ell_{q,\delta}(t) \mid \delta \in E\}.$$

Note that $\alpha_1(t)$ is just the shortest length of a saddle connection for $h_t q$.

Lemma 5.15. *There is a number $\theta_0 > 0$ with the following property. Let $E \in \mathcal{E}_r$ be such that $S(E) = S$; then for each t, E contains at least one saddle connection of $h_t q$-length at least θ_0. In particular, we have $\alpha_M(t) \geq \theta_0$ for all t.*

PROOF. Let E be a collection of saddle connections with pairwise disjoint interiors which decompose S into simply connected regions. Each of these regions is a singular euclidean polygon. The number of these polygons is uniformly bounded. Since the area of the singular euclidean metric defined by $h_t q$ equals one, there is at least one side of these polygons whose length is bounded from below by a constant $\theta_0 > 0$ only depending on S.

This shows the first part of the lemma, and the second part follows from the fact that by Lemma 5.13, any collection of M saddle connections with pairwise disjoint interior decomposes S into simply connected components. □

In the following lemma, the number $T > 0$ depends in an essential way on the initial quadratic differential q. Note that this corresponds to the fact that the amount of time the h_t-orbit through q spends before entering the fixed compact set whose existence is stated in Theorem 5.10 can be arbitrarily large.

Lemma 5.16. *There is a number $T > 0$ such that $\|\ell_{q,\delta}\|_{[0,T]} \geq \theta_0$ for every saddle connection δ.*

PROOF. Since q is an arbitrary but fixed quadratic differential without horizontal saddle connection there are only finitely many saddle connections for q whose length is smaller than θ_0. By Lemma 5.11, for each of these saddle connections δ there is a number $\tau(\delta) > 0$ so that $\ell_{q,\delta}(\tau(\delta)) \geq \theta_0$. The maximum T of these numbers $\tau(\delta)$ satisfies the requirement in the lemma. □

Write $I = [0, T]$. For $k \leq M - 1$ define

$$L_k = \theta_0/(48M^2)^{M-k}$$

and for each t let

$$r(t) = \max\{k \mid \alpha_k(t) < L_k\}.$$

If $E \in \mathcal{E}_{r(t)}$ is any collection of $r(t)$ disjoint saddle connections for $h_t q$ of length at most L_k then any saddle connection whose interior is disjoint from E has length at least $48M^2 L_k$. The sets

$$V_k = \{t \in I \mid r(t) = k\}\,(k = 1, \dots, M-1)$$

are disjoint, and their union equals the set V of all $t \in I$ so that $h_t q$ has a saddle connection of length smaller than L_1. In particular, there is some $k \leq M - 1$ such that

$$|V_k| \geq \frac{|V|}{M-1}.$$

For $\delta \in \mathcal{L}_q$, let $H(\delta)$ be the set of all $t \in I$ so that $\ell_{q,t}(t) < L_k$, and whenever $\delta \cap \delta' \neq \emptyset$ for some $\delta' \in \mathcal{L}_q$ then

$$\ell_{q,\delta'}(t) \geq 24M^2 L_k.$$

Thus if $t \in H(\delta)$ then δ satisfies the conclusion of Lemma 5.14 for $h_t q$ and hence it is "isolating". Define

$$\mathcal{F}_0 = \{\delta \in \mathcal{L}_q \mid V_k \cap H(\delta) \neq \emptyset\}.$$

In other words, $\mathcal{F}_0$ is the set of all saddle connections which are "isolating" for some time in V_k. The following statement is the main remaining step towards Theorem 5.10.

Lemma 5.17. *For all* $t \in I$,

$$\sharp\{\delta \in \mathcal{F}_0 \mid \ell_{q,\delta}(t) \leq 4M^2 L_k\} \leq M.$$

PROOF. All we have to show is the following. If δ, δ' are any two saddle connections with $H(\delta) \cap V_k \neq \emptyset, H(\delta') \cap V_k \neq \emptyset$ and if $\ell_{q,\delta}(t) \leq 4M^2 L_k, \ell_{q,\delta'}(t) \leq 4M^2 L_k$ then δ, δ' are disjoint.

Now choose $s \in V_k \cap H(\delta)$ so that $\ell_{q,\delta}(s) < L_k < 4M^2 L_k$; such an s exists by the assumption on δ. After perhaps exchanging δ and δ', the second part of Lemma 5.11 shows that $\ell_{q,\delta'}(s) < 24M^2 L_k$ (the factor 2 in this estimate comes from the fact that the function f measures the sup of the vertical and horizontal component rather than the euclidean metric) and hence the claim follows from the definition of the set $\mathcal{F}_0$. □

Now

$$V_k \subset \cup_{\delta \in \mathcal{L}_q} H(\delta).$$

Namely, if $t \in V_k$ then there is a collection $E \subset \mathcal{E}_k$ of k saddle connections of length at most L_k with disjoint interior so that any additional saddle connection whose interior is disjoint from E has length at least $48M^2 L_k$. By Lemma 5.15 and the choice of L_k, the union $S(E)$ of the closures of the simply connected components of $S - E$ is not all of S. Then $W(E) \neq \emptyset$ and by Lemma 5.14, we have $t \in H(\delta)$ for every $\delta \in W(E)$.

Apply Lemma 5.12 to the interval $I = [0, T]$ and to $\rho = 4M^2 L_k < \theta_0$. It follows that $|V_k| \leq |I|/2M$. Since $|V_k| \geq |V|/M - 1$ this shows that $I - V$ is non-empty. This completes the proof of Theorem 5.10.

Particularly nice complex geodesics are complex geodesics whose stabilizers in $\mathrm{Mod}(S)$ act with cofinite volume on the geodesic. We refer to [**HS06**] for an introduction to this fascinating subject.

The proof of Theorem 5.10 relies on a careful analysis of the geometry of a singular euclidean metric defined by a quadratic differential and its change under the horocycle flow.

Rafi [**R07a**] obtained a fairly precise understanding of such singular euclidean metrics. To explain his result, define for a number $\epsilon > 0$ an *ϵ-thick piece* of a hyperbolic surface X to be a component of $X - \cup\alpha$ where α runs through the simple closed geodesics of length at most ϵ.

Provided that $\epsilon > 0$ is sufficiently small, if $q \in \mathcal{Q}(X)$ then for every ϵ-thick piece Y there is an associated (perhaps degenerate) subset Y_q of S. This set is bounded by q-geodesics freely homotopic to the boundary circles of Y. Moreover, Y_q is disjoint from the interior of any flat cylinder whose core curve is freely homotopic to a boundary circle of Y. Rafi showed [**R07a**]

Theorem 5.18. *For every ϵ-thick piece Y of X there is a number $\lambda(Y) > 0$ so that the q-length of every essential closed curve $\alpha \subset Y$ equals $\lambda(Y)\ell_\alpha(X)$ up to a multiplicative constant which is bounded independent of X.*

In other words, maximal subsets of X which can be decomposed into simply connected regions by short saddle connections correspond to ϵ-thick pieces Y of X for which the scaling constant $\lambda(Y)$ is small.

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