LINES OF MINIMA IN OUTER SPACE

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Abstract. We define lines of minima in the thick part of Outer space for the free group F_n with $n \geq 3$ generators. We show that these lines of minima are contracting for the Lipschitz metric. Every fully irreducible outer automorphism of F_n defines such a line of minima. Now let Γ be a subgroup of the outer automorphism group of F_n which is not virtually abelian. We obtain that if Γ contains at least one fully irreducible element then for every $p \in (1, \infty)$ the second bounded cohomology group $H_b^2(\Gamma, \ell^p(\Gamma))$ is infinite dimensional.

1. Introduction

There are many resemblances between the *extended mapping class group* of a closed oriented surface S, i.e the outer automorphism group of the fundamental group of S, and the outer automorphism group $\mathrm{Out}(F_n)$ of a free group F_n with $n \geq 2$ generators. Most notably, $\mathrm{Out}(F_2)$ is just the extended mapping class group $GL(2, \mathbb{Z})$ of a torus. However, while in recent years the attempt to understand the mapping class group via the geometry of spaces on which it acts lead to a considerable gain of knowledge of the mapping class group, so far this approach has not been carried out successfully for $\mathrm{Out}(F_n)$.

The group $\mathrm{Out}(F_n)$ acts properly discontinuously on *Outer space* $\mathrm{CV}(F_n)$. This space consists of projective classes of marked metric graphs with fundamental group F_n and can be viewed as an equivalent of Teichmüller space for a closed surface S of higher genus. Teichmüller space admits several natural and quite well understood metrics which are invariant under the action of the extended mapping class group. The best known such metrics are the *Teichmüller metric* and the *Weil-Petersson metric*. In contrast, up to date there is no good geometric theory of Outer space. Only very recently Francaviglia and Martino [FM11] studied in a systematic way a natural $\mathrm{Out}(F_n)$-invariant metric on Outer space. This metric is the symmetrization of a non-symmetric geodesic metric, the so-called *Lipschitz metric*. The symmetric metric is not geodesic, and its analogue for Teichmüller space, the *Thurston metric*, also turned out to be harder to understand than the Teichmüller metric and the Weil-Petersson metric.

The viewpoint we take in this work is motivated by a slightly different approach to the geometry of Teichmüller space. Namely, *lines of minima* in Teichmüller space for a closed surface S of higher genus were defined and investigated by Kerckhoff [Ke92]. As for geodesics for the Teichmüller metric, such a line of minima is

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determined by two projective measured geodesic laminations which jointly fill up S. A line of minima uniformly fellow-travels its corresponding Teichmüller geodesic provided that this Teichmüller geodesic entirely remains in the *thick* part of Teichmüller space. There is also a very good control in the thin part of Teichmüller space though the uniform fellow traveller property is violated [CRS08].

More precisely, for a number $\epsilon > 0$, the ϵ-thick part $\mathcal{T}(S)_\epsilon$ of Teichmüller space for a closed oriented surface S of higher genus is the set of all hyperbolic metrics on S whose systole (i.e. the length of a shortest closed geodesic) is at least ϵ. Teichmüller geodesics γ entirely contained in $\mathcal{T}(S)_\epsilon$, and hence their corresponding lines of minima, have a *uniform contraction property*: If B is a closed metric ball in Teichmüller space for the Teichmüller metric or the Weil-Petersson metric which is disjoint from γ then the diameter of a shortest distance projection of B into γ is bounded from above by a constant only depending on ϵ [Mi96, BFu09].

Our main goal is to define such lines of minima for Outer space. We show that these lines of minima are uniform coarse geodesics for the symmetrized Lipschitz metric, and these coarse geodesics have the uniform contraction property (Corollary 5.3). We also observe that for every fully irreducible element $\varphi \in \mathrm{Out}(F_n)$ there is such a line of minima which is φ-invariant. This recovers a recent result of Algom-Kfir [AK08] who showed that axes for fully irreducible elements as defined by Handel and Mosher [HM06] have the contraction property.

As an application, we use the tools developpped in [H11] to show

Theorem. *Let $\Gamma < \mathrm{Out}(F_n)$ be a subgroup which is not virtually abelian and which contains at least one fully irreducible element. Then for every $p > 1$ the second bounded cohomology group $H_b^2(\Gamma, \ell^p(\Gamma))$ is infinite dimensional.*

Earlier Bestvina and Feighn [BF10] showed that a subgroup Γ as in the theorem has nontrivial second bounded cohomology with real coefficients. This is also immediate from our approach.

All constructions in this paper are equally valid for the action of the mapping class group on Teichmüller space. This leads for example to a new proof of the main result of [Mi96] avoiding completely the explicit use of Teichmüller theory. However, in this case our more abstract approach does not have any obvious advantage over the original arguments. The analogue of our main theorem for mapping class groups was derived with a different method in [H08].

2. Measured laminations and trees

In this section we introduce currents, trees and measured laminations for the free group F_n of rank $n \geq 3$. We single out an $\mathrm{Out}(F_n)$-invariant subset of the space of measured laminations which is used for the construction of lines of minima in the later sections. We continue to use the notations from the introduction.

The Cayley graph of F_n with respect to a fixed standard symmetric generating set is a regular simplicial tree which can be compactified by adding the *Gromov boundary* ∂F_n. This boundary is a compact totally disconnected topological space on which F_n acts as a group of homeomorphisms. It does not depend on the generating set up to F_n-equivariant homeomorphism. Every element $w \neq e \in F_n$ acts on ∂F_n with *north-south dynamics*. This means that w fixes precisely two points $a_+, a_- \in \partial F_n$, and for every neighborhood U_+ of a_+, U_- of a_- there is some $k > 0$ such that $w^k(\partial F_n - U_-) \subset U_+$ and $w^{-k}(\partial F_n - U_+) \subset U_-$.

A *geodesic current* for F_n is a locally finite Borel measure on

$$\partial F_n \times \partial F_n - \Delta = \partial^2(F_n)$$

(where Δ denotes the diagonal in $\partial F_n \times \partial F_n$) which is invariant under the action of F_n and under the flip $\iota : \partial^2(F_n) \to \partial^2(F_n)$ exchanging the two factors. The space $\mathrm{Curr}(F_n)$ of all geodesic currents equipped with the weak*-topology is a locally compact topological space which can be projectivized to the compact space $\mathcal{P}\mathrm{Curr}(F_n)$ of *projective currents*. The *outer automorphism group* $\mathrm{Out}(F_n)$ of F_n naturally acts on $\mathrm{Curr}(F_n)$ and on $\mathcal{P}\mathrm{Curr}(F_n)$ as a group of homeomorphisms.

If $w \neq e \in F_n$ is any indivisible element, i.e. an element which is not of the form $w = v^k$ for some $k \geq 2$, then the set of all pairs of fixed points in ∂F_n of all elements of F_n which are conjugate to w is a discrete F_n-invariant flip invariant subset of $\partial F_n \times \partial F_n - \Delta$. Thus the sum of the Dirac measures supported at these pairs of fixed points defines a geodesic current which we call *induced* by the conjugacy class $[w]$ of w. If $w = v^k$ for some $k \geq 2$ and some indivisible element $v \in F_n$ then we define the geodesic current induced by the conjugacy class $[w]$ to be the k-fold multiple of the geodesic current induced by the conjugacy class $[v]$ of v. Define a *weighted induced current* to be a geodesic current which is obtained by multiplying a geodesic current induced by a conjugacy class in $[F_n]$ by a positive weight. The set of weighted induced currents is invariant under the action of $\mathrm{Out}(F_n)$.

An element $w \neq e \in F_n$ is *primitive* if it belongs to some basis, i.e. if there is a decomposition of F_n into a free product of the form $< w > *H$ where $< w >$ is the infinite cyclic subgroup of F_n generated by w and where H is a free subgroup of F_n. A conjugacy class in F_n is *primitive* if one (and hence each) of its elements is primitive. The set of primitive elements is invariant under the action of the full automorphism group $\mathrm{Aut}(F_n)$ of F_n and hence $\mathrm{Out}(F_n)$ naturally acts on the set of all primitive conjugacy classes.

Definition 2.1. The space $\mathcal{ML}(F_n)$ of *measured laminations* is the closure in $\mathrm{Curr}(F_n)$ of the set of all currents which are weighted induced currents of primitive conjugacy classes.

The projectivization $\mathcal{PML}(F_n)$ of $\mathcal{ML}(F_n)$, equipped with the weak*-topology, is compact and invariant under the action of $\mathrm{Out}(F_n)$. Theorem B of [KL07] shows that $\mathcal{PML}(F_n)$ is the unique smallest non-empty closed $\mathrm{Out}(F_n)$-invariant subset of $\mathcal{P}\mathrm{Curr}(F_n)$.

Martin (Theorem 17 of [Ma95]) characterizes projective measured laminations which are induced by a conjugacy class in F_n as follows.

Proposition 2.2. *A projective current induced by a conjugacy class* $[\alpha]$ *in* F_n *is contained in* $\mathcal{PML}(F_n)$ *if and only if each element of* $[\alpha]$ *is contained in a proper free factor of* F_n.

A closed non-empty F_n-invariant subset of $\partial^2(F_n)$ which is moreover invariant under the flip ι is called a *topological lamination*. The space $\mathcal{L}$ of all topological laminations can be equipped with the *Chabauty topology*. With respect to this topology, $\mathcal{L}$ is compact. The group $\mathrm{Out}(F_n)$ acts on $\mathcal{L}$ as a group of homeomorphisms. Every nontrivial element $w \neq e \in F_n$ defines a point $[[w]] \in \mathcal{L}$ which is just the set of all pairs of fixed points of all elements of F_n which are conjugate to w. In other words, $[[w]]$ is the support of the current induced by the conjugacy class of w. Topological laminations of this form are called *rational*. The support of a measured lamination is a topological lamination.

We call a (projective) geodesic current supported in a topological lamination L a *(projective) transverse measure* for L. If L is the topological lamination defined by the conjugacy class of a primitive element $w \in F_n$ then L admits a unique projective transverse measure. This measure is just the projective measured lamination induced by the conjugacy class of w. Note that if $L_i \to L$ in $\mathcal{L}$ and if ζ_i is a projective geodesic current supported in L_i then up to passing to a subsequence, the projective geodesic currents ζ_i converge in $\mathcal{P}\mathrm{Curr}(F_n)$ to a projective geodesic current supported in L (we refer to [CHL08b] for a more precise discussion).

Remark 2.3. The definition of a topological lamination does not correspond to the definition of a geodesic lamination for closed surfaces. The correct analogue of a lamination in the surface case is a closed F_n-invariant subset of $\partial^2(F_n)$ which is contained in the Chabauty closure of those closed F_n-invariant subsets of $\partial^2(F_n)$ which consist of pairs of fixed points of elements in some primitive conjugacy class.

Let $cv(F_n)$ be the space of all minimal free and discrete isometric actions of F_n on $\mathbb{R}$-trees. Two such actions of F_n on $\mathbb{R}$-trees T and T' are identified in $cv(F_n)$ if there exists an F_n-equivariant isometry between T and T'. The quotient of a tree $T \in cv(F_n)$ under the action of F_n is a finite metric graph T/F_n without vertices of valence one or two whose fundamental group is marked isomorphic to F_n. The space $cv(F_n)$ admits a natural locally compact metrizable $\mathrm{Out}(F_n)$-invariant topology.

The *boundary* $\partial cv(F_n)$ of $cv(F_n)$ consists of all minimal *very small* isometric actions of F_n on $\mathbb{R}$-trees which either are non-simplicial or which are not free. Here an action is very small if and only if every nontrivial arc stabilizer is maximal cyclic and if tripod stabilizers are trivial. Again, any two such actions define the same point in $\partial cv(F_n)$ if there exists an F_n-equivariant isometry between them. The union $\overline{cv(F_n)} = cv(F_n) \cup \partial cv(F_n)$ has a natural $\mathrm{Out}(F_n)$-invariant topology such that $cv(F_n) \subset \overline{cv(F_n)}$ is open and dense.

Outer space $\mathrm{CV}(F_n)$ is the projectivization of $cv(F_n)$. If we define two trees T, T' to be equivalent if there is an F_n-equivariant homothety between T, T' then $\mathrm{CV}(F_n)$ is just the space of equivalence classes of points in $cv(F_n)$. The topology on $cv(F_n)$ descends to a natural locally compact topology on $\mathrm{CV}(F_n)$. The group $\mathrm{Out}(F_n)$ acts properly discontinuously on $\mathrm{CV}(F_n)$. Write $\partial\mathrm{CV}(F_n)$ to denote the projectivization of $\partial cv(F_n)$. Then $\overline{\mathrm{CV}(F_n)} = \mathrm{CV}(F_n) \cup \partial\mathrm{CV}(F_n)$ is a compact $\mathrm{Out}(F_n)$-space.

Notational convention: In the sequel we are going to use the following notations.

(1) An element of F_n is denoted by a small letter, and $[w]$ is the conjugacy class of $w \in F_n$.
(2) A point in $\overline{cv(F_n)} = cv(F_n) \cup \partial cv(F_n)$ is denoted by a capital letter, and $[T] \in \overline{\mathrm{CV}(F_n)}$ is the projectivization of $T \in \overline{cv(F_n)}$.
(3) A current is denoted by a Greek letter, and $[\nu]$ is the projectivization of $\nu \in \mathrm{Curr}(F_n)$.

For every $T \in \overline{cv(F_n)}$ and every $w \in F_n$, the *translation length* $\|w\|_T$ for the action of w on T is defined to be the *dilation*

$$\|w\|_T = \inf\{d(x, wx) \mid x \in T\}.$$

If $w' \in F_n$ is conjugate to w then $\|w'\|_T = \|w\|_T$ and hence the translation length $\|[w]\|_T$ of the conjugacy class $[w]$ of w is defined.

To every $T \in \partial cv(F_n)$ we can associate a topological lamination $L(T)$ of zero-length geodesics [CHL08a] as follows. For every $\epsilon > 0$ define $\Omega_\epsilon(T)$ to be the set of all elements $w \in F_n$ with translation length $\|w\|_T < \epsilon$. Denote by $L_\epsilon(T)$ the smallest F_n-invariant closed subset of $\partial^2(F_n)$ which contains all pairs of fixed points of each element in $\Omega_\epsilon(T)$. Then

$$L(T) = \cap_{\epsilon>0} L_\epsilon(T)$$

is a nonempty closed F_n-invariant flip invariant subset of $\partial^2 F_n$ which will be called the *zero lamination* of T. Note that two $\mathbb{R}$-trees $T, T' \in \partial cv(F_n)$ with the same projectivization have the same zero lamination. Thus the zero lamination is defined for points in $\partial\mathrm{CV}(F_n)$.

The following result is due to Kapovich and Lustig [KL09a, KL10b].

Proposition 2.4. (1) *There is a unique continuous* $\mathrm{Out}(F_n)$*-invariant* length pairing

$$\langle , \rangle : \overline{cv(F_n)} \times \mathrm{Curr}(F_n) \to [0, \infty)$$

which satisfies $\langle T, \eta\rangle = \|[w]\|_T$ *for every current* η *induced by an indivisible conjugacy class* $[w]$ *in* F_n *and for every* $T \in \overline{cv(F_n)}$.
(2) *If* $T \in \partial cv(F_n)$ *then* $\langle T, \nu\rangle = 0$ *if and only if* ν *is supported in the zero lamination of* T.

Remark 2.5. Kapovich and Lustig call the length pairing as defined above an intersection form.

Define $\mathcal{UML}' \subset \mathcal{PML}(F_n)$ to be the set of all projective measured laminations $[\nu]$ with the property that $\langle [T|, [\nu]\rangle = 0$ for precisely one projective tree $[T] \in \partial\mathrm{CV}(F_n)$. Note that this makes sense without referring to specific representatives of the projective classes. The projective tree $[T]$ is called *dual* to $[\nu]$. We denote by $\mathcal{UT}' \subset \partial\mathrm{CV}(F_n)$ the set of all projective trees which are dual to points in $\mathcal{UML}'$.

By invariance of the length pairing, the sets $\mathcal{UML}'$ and $\mathcal{UT}'$ are invariant under the action of $\mathrm{Out}(F_n)$. The assignment ω' which associates to $[\nu] \in \mathcal{UML}'$ the projective tree $\omega'([\nu]) \in \mathcal{UT}'$ which is dual to $[\nu]$ is $\mathrm{Out}(F_n)$-equivariant. However, this map is not injective as the following example shows. This example was provided by the referee of an earlier version of this paper.

Example 2.6. Let S be a compact surface of genus $g \geq 4$ with connected boundary ∂S. The fundamental group of S is the group F_{2g}. Let α be a simple closed separating curve on S which decomposes S into a surface S_0 of genus 2 and a surface S_1 of genus $g-2$ with boundary $\partial S \cup \alpha$. Let μ be a uniquely ergodic measured geodesic lamination on the surface S which is supported in S_1 and is maximal with this property. This means that after replacing the two boundary circles of S_1 by punctures, the support of μ decomposes S_1 into trigons and two once punctured monogons. Let T be the tree which is dual to μ. Since μ is uniquely ergodic, the tree T supports a unique transverse measure up to scaling.

Now let ν, ν' be two distinct currents whose support is the full group $\pi_1(S_0)$. Then the currents $\mu+\nu, \mu+\nu'$ are both dual to T and to no other tree. Namely, any tree dual to $\mu+\nu$ contains in its zero lamination the support of μ as well as the support of ν. Since the support of ν equals $\pi_1(S_0)$ and μ is uniquely ergodic, such a tree is topologically equivalent to T. But the tree T supports a unique transverse measure up to scaling and hence a tree dual to $\mu+\nu$ is projectively equivalent to T. This reasoning also applies to $\mu+\nu'$ and shows that a tree which is dual to $\mu+\nu, \mu+\nu'$ is projectively equivalent to T.

To complete the example we have to show that the currents $\mu+\nu, \mu+\nu'$ are contained in $\mathcal{ML}(F_n)$. For this note that by Proposition 2.2 and the fact that $\mathcal{ML}(F_n)$ is a *closed* subset of $\mathrm{Curr}(F_n)$, a current of the form $\zeta+\nu$ is contained in $\mathcal{ML}(F_n)$ for any weighted dual current ζ of a primitive conjugacy class in $\pi_1(S_1) < \pi_1(S)$. Since μ is a measured geodesic lamination on S_1, it can be approximated in the space of currents by weighted duals of primitive conjugacy classes in S_1. This completes the example.

Let

$$\begin{aligned}\mathcal{UML} = &\{[\nu] \in \mathcal{UML}' \mid \\ &\langle \omega'[\nu], [\zeta]\rangle = 0 \text{ for } [\zeta] \in \mathcal{PML}(F_n) \text{ only if } [\zeta] = [\nu]\}.\end{aligned}$$

By equivariance, the set $\mathcal{UML}$ is invariant under the action of $\mathrm{Out}(F_n)$. The restriction

$$\omega = \omega' | \mathcal{UML}$$

of the map ω' is a bijection of $\mathcal{UML}$ onto an $\mathrm{Out}(F_n)$-invariant subset $\mathcal{UT}$ of $\mathcal{UT}'$.

Remark 2.7.
(1) The set $\mathcal{UML}$ can be viewed as the analogue of the set of all projective classes of measured geodesic laminations for a surface of higher genus whose support fills up S and is uniquely ergodic (compare Example 2.6).
(2) The sets $\mathcal{UML}$ and $\mathcal{UT}$ are characterized by having unique duals. In other words, the definition is symmetric, and we could begin with defining a set of projective trees whose zero lamination supports a unique projective measured lamination etc. The discussion is completely formal and will be omitted.

An element $\alpha \in \mathrm{Out}(F_n)$ is called *fully irreducible* (or iwip for short) if there is no $k > 0$ such that α^k preserves a free factor of F_n.

The following result describes the action of an iwip element on the boundary $\partial\mathrm{CV}(F_n)$ of Outer space and on the space of projective measured laminations. Its first part is due to Levitt and Lustig [LL03], its second part is Theorem 36 of [Ma95] which is attributed to Bestvina.

Proposition 2.8. (1) *An iwip automorphism of F_n acts with north-south dynamics on the boundary $\partial\mathrm{CV}(F_n)$ of outer space.*
(2) *An iwip automorphism of F_n acts on $\mathcal{PML}(F_n)$ with north-south dynamics.*

Our first goal is to strengthen this duality between the action of an iwip element of $\mathrm{Out}(F_n)$ on projective trees and projective measured laminations by showing that the set $\mathcal{UT} \subset \partial\mathrm{CV}(F_n)$ contains all fixed points of iwip elements of $\mathrm{Out}(F_n)$. For the proof of this and for later use, we call an iwip automorphism $\alpha \in \mathrm{Out}(F_n)$ *non-geometric* if α does not admit any periodic conjugacy class in F_n. This is equivalent to stating that no power of α can be realized as a homeomorphism of a compact surface with fundamental group F_n (Theorem 4.1 of [BH92]).

Lemma 2.9. *Any fixed point of an iwip-automorphism on the boundary $\partial\mathrm{CV}(F_n)$ of Outer space is contained in $\mathcal{UT}$.*

Proof. Consider first a non-geometric iwip-automorphism α. Let $[T] \in \partial\mathrm{CV}(F_n)$ be the repelling fixed point for the action of α on the boundary of Outer space. By Proposition 5.6 of [CHL08b], the zero lamination $L([T])$ of $[T]$ is *uniquely ergodic*, i.e. it supports a unique projective transverse measure $[\nu]$. This projective transverse measure is a projective measured lamination [Ma95]. In particular, by the second part of Proposition 2.4 and the definitions, if $[\nu] \in \mathcal{UML}'$ then we also have $[\nu] \in \mathcal{UML}$ and $[T] \in \mathcal{UT}$.

Now let $\alpha \in \mathrm{Out}(F_n)$ be a geometric iwip element. By Theorem 4.1 of [BH92], there is a compact connected surface S with connected boundary and fundamental group F_n such that α can be represented by a pseudo-Anosov homeomorphism A of S. The repelling projective measured geodesic lamination $[\nu]$ for A determines up to scaling an action of $F_n = \pi_1(S)$ on an $\mathbb{R}$-tree T. The projectivization $[T]$ of this F_n-tree is just the repelling fixed point for the action of α on $\partial\mathrm{CV}(F_n)$. Moreover, $[\nu]$ is supported in the zero lamination of $[T]$.

The boundary of S defines a conjugacy class $[w]$ in F_n which is invariant under α. Any geodesic current supported in the zero lamination $L([T])$ of $[T]$ can be written in the form $a\nu + b\zeta$ where ν is a representative of the class $[\nu]$, where ζ is the current induced by the conjugacy class $[w]$ and where $a \geq 0, b \geq 0$. We claim that if $b > 0$ then the current $a\nu + b\zeta$ is *not* a measured lamination. Namely, otherwise for every $k > 0$ the current $\alpha^k(a\nu + b\zeta) = a\lambda^{-k}\nu + b\zeta$ is a measured lamination as well where $\lambda > 1$ is the expansion rate of α. Since the space of measured laminations is a closed subset of $\mathrm{Curr}(F_n)$ with respect to the weak*-topology, this implies that $\zeta \in \mathcal{ML}(F_n)$. However, the invariant conjugacy class $[w]$ for α is not contained in a proper free factor of F_n and hence this violates Proposition 2.2. As a consequence, if $[\nu] \in \mathcal{UML}'$ then also $[\nu] \in \mathcal{UML}$ and $[T] \in \mathcal{UT}$.

We are now left with showing that the projective measured lamination $[\nu] \in \mathcal{PML}(F_n)$ defined as above by the repelling fixed point $[T] \in \partial\mathrm{CV}(F_n)$ of an arbitrary iwip element $\alpha \in \mathrm{Out}(F_n)$ is contained in $\mathcal{UML}'$. For this let $\nu \in \mathcal{ML}(F_n)$ be a measured lamination which represents the projective class $[\nu]$. Assume to the contrary that there is a tree $T' \in \partial cv(F_n)$ with $[T'] \in \partial\mathrm{CV}(F_n) - \{[T]\}$ and $\langle T', \nu\rangle = 0$. Since $[T'] \neq [T]$ by assumption and since by the first part of Proposition 2.8 α acts with north-south dynamics on $\partial\mathrm{CV}(F_n)$, we have $\alpha^k[T'] \to [Q]$ $(k \to \infty)$ where $[Q]$ is the attracting fixed point for the action of α on $\partial\mathrm{CV}(F_n)$.

Choose a continuous section $\Sigma : \partial\mathrm{CV}(F_n) \to \partial cv(F_n)$; such a section was for example constructed by Skora and White [S89, W91] (or see the definition below). Then $\Sigma(\alpha^k[T']) \to \Sigma[Q]$, moreover we have $\langle \Sigma[Q], \nu\rangle > 0$. Hence by continuity of the length pairing on $\partial cv(F_n) \times \mathrm{Curr}(F_n)$ [KL09a] and by naturality under scaling, we conclude that $\langle \Sigma(\alpha^k[T']), \nu\rangle > 0$ and hence $\langle \alpha^k(\Sigma[T']), \nu\rangle > 0$ for sufficiently large k. Then $\langle \Sigma[T'], \alpha^{-k}\nu\rangle = \langle \alpha^k(\Sigma[T']), \nu\rangle > 0$ by invariance of the length pairing under the action of $\mathrm{Out}(F_n)$ which contradicts the fact that $\alpha\nu = \rho\nu$ for some $\rho > 0$ and $\langle T', \nu\rangle = 0$. □

Remark 2.10. (1) The proof of Lemma 2.9 implies the second part of Proposition 2.8. However, we used Proposition 5.6 of [CHL08b] which in turn uses a weak version of Martin's result. We also used the work of Levitt and Lustig [LL03] which appeared after Martin's thesis.

(2) In general, there may be points $[T] \neq [T'] \in \partial\mathrm{CV}(F_n)$ with the same zero lamination (see [CHL07] for a detailed account on this issue). Lemma 2.9 implies that for the fixed point $[T]$ of an iwip element, there is no projective tree $[T] \neq [T'] \in \partial\mathrm{CV}(F_n)$ whose zero lamination coincides with the zero lamination of $[T]$.

We equip $\mathcal{UML}$ with the topology as a subspace of $\mathcal{PML}(F_n)$, and we equip $\mathcal{UT}$ with the topology as a subspace of $\partial\mathrm{CV}(F_n)$.

Denote by

$$cv_0(F_n) \subset cv(F_n)$$

the subspace of F_n-trees $T \in cv(F_n)$ whose quotient graphs T/F_n have volume one.

For the remainder of this section, choose an arbitrary tree

$$T_0 \in cv_0(F_n) \tag{1}$$

and define

$$\Lambda(T_0) = \{\nu \in \mathcal{ML}(F_n) \mid \langle T_0, \nu\rangle = 1\}. \tag{2}$$

We call a lamination $\nu \in \Lambda(T_0)$ *normalized* for T_0, or simply normalized if the choice of T_0 is clear from the context. The next lemma is immediate from the continuity of the length pairing. Recall that the spaces $\mathcal{ML}(F_n)$ and $\mathcal{PML}(F_n)$ are equipped with the weak*-topology.

Lemma 2.11. *$\Lambda(T_0)$ is a continuous section of the fibration*

$$\mathcal{ML}(F_n) \to \mathcal{PML}(F_n).$$

For each fixed $T_0 \in cv_0(F_n)$, the function $a : cv_0(F_n) \times \Lambda(T_0) \to (0, \infty)$ defined by $a(T, \zeta)\zeta \in \Lambda(T)$ is continuous.

For $T_0 \in cv_0(F_n)$ there is a dual section

$$\Sigma(T_0) = \{T \in cv(F_n) \cup \partial cv(F_n) \mid \max\{\langle T, \nu\rangle \mid \nu \in \Lambda(T_0)\} = 1\} \tag{3}$$

of the fibration $cv(F_n) \cup \partial cv(F_n) \to \mathrm{CV}(F_n) \cup \partial \mathrm{CV}(F_n)$.

Remark 2.12. A point $T \in \Sigma(T_0) \cap cv(F_n)$ can be written in the form $T = bT'$ where $T' \in cv_0(F_n)$ and where $b > 0$ has a geometric meaning. Namely, if d_L denotes the one-sided Lipschitz metric on $cv_0(F_n)$ then $b = e^{-d_L(T_0,T')}$. We refer to Corollary 4.2 for details.

For later reference we formulate the analogue of Lemma 2.11 for the section $\Sigma(T_0)$, but the proof is left to the reader.

Lemma 2.13. $\Sigma(T_0)$ *is a continuous section of the fibration*

$$cv(F_n) \cup \partial cv(F_n) \to \mathrm{CV}(F_n) \cup \partial \mathrm{CV}(F_n).$$

For each fixed $T_0 \in cv_0(F_n)$ *the function* $b : cv_0(F_n) \times \Sigma(T_0) \to (0, \infty)$ *defined by* $b(S, T)T \in \Sigma(S)$ *is continuous.*

The following comments summarize some properties of the sections $\Lambda(T_0)$ and $\Sigma(T_0)$ which will be frequently used in the sequel.

Remark 2.14. Let $T_0 \in cv_0(F_n)$ be arbitrary.

(1) The spaces $\Lambda(T_0)$ and $\Sigma(T_0)$ are compact.
(2) $0 \le \langle T, \nu\rangle \le 1$ for all $T \in \Sigma(T_0)$ and all $\nu \in \Lambda(T_0)$. Moreover, $\langle T, \nu\rangle = 0$ only if $T \in \partial cv(F_n)$.

We use these sections to show

Lemma 2.15. *The map* $\omega : \mathcal{UML} \to \mathcal{UT}$ *is an* $\mathrm{Out}(F_n)$*-equivariant homeomorphism.*

Proof. The map $\omega : \mathcal{UML} \to \mathcal{UT}$ is clearly an $\mathrm{Out}(F_n)$-equivariant bijection and hence we just have to show that ω is continuous and open.

Let $T_0 \in cv_0(F_n)$ be a simplicial F_n-tree with quotient of volume one and let $\Lambda = \Lambda(T_0)$ and $\Sigma = \Sigma(T_0)$ as defined in equalities (2),(3). By Lemma 2.11, the set $\mathcal{UML}$ is naturally homeomorphic to a subset Λ_0 of Λ, and $\mathcal{UT}$ is homeomorphic to a subset Σ_0 of Σ. The bijection ω then induces a bijection $\omega_0 : \Lambda_0 \to \Sigma_0$.

Both $\mathcal{PML}(F_n)$ and $\mathrm{CV}(F_n) \cup \partial\mathrm{CV}(F_n)$ are compact and metrizable topological spaces and hence the same holds true for Λ, Σ. Thus to show continuity of ω, it suffices to show that if $(\nu_i) \subset \Lambda_0$ is any sequence converging to some $\nu \in \Lambda_0$ then $\omega_0(\nu_i) \to \omega_0(\nu)$. Via passing to a subsequence, we may assume that $\omega_0(\nu_i) \to T$ for some $T \in \Sigma$. Since $\langle \omega_0(\nu_i), \nu_i\rangle = 0$ for all i, by continuity of the length pairing we have $\langle T, \nu\rangle = 0$ and hence $T = \omega_0(\nu)$.

To show that ω_0 is open, it suffices to show that $\omega_0^{-1} : \Sigma_0 \to \Lambda_0$ is continuous. However, this follows from continuity of the length pairing as in the above argument. ∎

Remark 2.16. It can be shown that $\mathcal{UML}$ and $\mathcal{UT}$ are Borel subsets of $\mathcal{PML}(F_n)$ and $\partial\mathrm{CV}(F_n)$. Since this fact is not needed in the sequel, we omit a proof.

Let $M \subset \partial\mathrm{CV}(F_n)$ be the closure of the set $\mathcal{UT}$. Since $\mathcal{UT}$ is invariant under $\mathrm{Out}(F_n)$, the same holds true for M. As an easy consequence of Lemma 2.15 we obtain:

Corollary 2.17. *The action of* $\mathrm{Out}(F_n)$ *on* M *is minimal.*

Proof. Lemma 2.9 shows that the set $\mathcal{F} \subset \partial\mathrm{CV}(F_n)$ of fixed points of iwip-elements of $\mathrm{Out}(F_n)$ is contained in $\mathcal{UT}$. This set is invariant under the action of $\mathrm{Out}(F_n)$ and hence its image under the homeomorphism ω^{-1} from Lemma 2.15 is invariant as well. Since the action of $\mathrm{Out}(F_n)$ on $\mathcal{PML}(F_n)$ is minimal, Lemma 2.15 then implies that $\mathcal{F}$ is dense in $\mathcal{UT}$ and hence in M.

By the first part of Proposition 2.8, every iwip element acts with north-south dynamics on $\partial\mathrm{CV}(F_n)$. Moreover, there is no global fixed point for the action of $\mathrm{Out}(F_n)$ on M. Thus the closure of every $\mathrm{Out}(F_n)$-orbit on M contains $\mathcal{F}$. Since $\mathcal{F} \subset M$ is dense, this shows the lemma. ☐

Remark 2.18. The proof of Corollary 2.17 also implies that a closed $\mathrm{Out}(F_n)$-invariant minimal subset of $\partial\mathrm{CV}(F_n)$ is unique. Again, the argument is an immediate consequence of the work of Levitt and Lustig [LL03]. Earlier Guirardel [G00] described a minimal invariant set for the action of $\mathrm{Out}(F_n)$ explicitly and established uniqueness with a different argument.

For $\epsilon > 0$ define the *ϵ-thick part* of $cv_0(F_n)$ to be the set

$$\mathrm{Thick}_\epsilon(F_n) \subset cv_0(F_n) \tag{4}$$

of all simplicial trees $T \in cv_0(F_n)$ with the additional property that the smallest translation length on T of any element $w \neq e$ of F_n is at least ϵ. For sufficiently small $\epsilon > 0$ the set $\mathrm{Thick}_\epsilon(F_n)$ is a closed connected $\mathrm{Out}(F_n)$-invariant subset of $cv_0(F_n)$ on which $\mathrm{Out}(F_n)$ acts cocompactly.

The following simple observation will be used several times in the sequel. For its formulation, recall from (3) the definition of the set $\Sigma(T_0)$ for some tree $T_0 \in \mathrm{Thick}_\epsilon(F_n)$.

Lemma 2.19. *Let* $(T_i) \subset \mathrm{Thick}_\epsilon(F_n)$ *be a sequence such that for every compact set* $K \subset \mathrm{Thick}_\epsilon(F_n)$*, we have* $T_i \in K$ *only for finitely many* i*. For each* i *let* $a_i > 0$ *be such that* $a_i T_i \in \Sigma(T_0)$*. Then* $a_i \to 0$ $(i \to \infty)$*.*

Proof. We follow Kapovich and Lustig [KL09a].

Let $(T_i) \subset \mathrm{Thick}_\epsilon(F_n)$ be a sequence such that for every compact set K, we have $T_i \in K$ only for finitely many i. For each i let $a_i > 0$ be such that $a_i T_i \in \Sigma(T_0)$. Since $\Sigma(T_0)$ is compact, after passing to a subsequence we may assume that $a_i T_i \to T$ for some $T \in \Sigma(T_0)$. Since (T_i) exits every compact set we have $T \in \partial cv(F_n)$.

If $a_i \not\to 0$ $(i \to \infty)$ then after passing to a subsequence, we may assume that $a_i \geq a > 0$ for all i. Since $T \in \partial cv(F_n)$ there are nontrivial elements in F_n acting on

T with arbitrarily small translation length. This means that there exists a sequence $(g_j) \subset F_n - \{e\}$ with

$$\lim_{j\to\infty} \|g_j\|_T = 0. \tag{5}$$

However, $T_i \in \mathrm{Thick}_\epsilon(F_n)$ and hence $\|g_j\|_{T_i} \geq \epsilon$ for all i, j. Hence for all i, j we have

$$a_i\|g_j\|_{T_i} \geq a\epsilon.$$

On the other hand, for all j we have $a_i\|g_j\|_{T_i} \to \|g_j\|_T$ $(i \to \infty)$ which contradicts (5) above. Thus indeed $\lim_{i\to\infty} a_i = 0$. This shows the lemma. □

Let $[\mathrm{Thick}_\epsilon(F_n)] \subset \mathrm{CV}(F_n)$ be the projectivization of $\mathrm{Thick}_\epsilon(F_n)$ and denote by $\overline{[\mathrm{Thick}_\epsilon(F_n)]}$ the closure of $[\mathrm{Thick}_\epsilon(F_n)]$ in $\mathrm{CV}(F_n) \cup \partial CV(F_n)$. Then

$$\partial[\mathrm{Thick}_\epsilon(F_n)] = \overline{[\mathrm{Thick}_\epsilon(F_n)]} - [\mathrm{Thick}_\epsilon(F_n)]$$

is a closed $\mathrm{Out}(F_n)$-invariant subset of $\partial\mathrm{CV}(F_n)$.

For a simplicial tree $T \in \mathrm{Thick}_\epsilon(F_n)$ call a primitive conjugacy class $[w]$ in F_n *basic* for T if $[w]$ can be represented by a loop in T/F_n of length at most two. For example, if $[w]$ can be represented by a loop which travels through each edge of T/F_n at most twice then $[w]$ is basic for T. The following duality statement is motivated by Teichmüller theory.

Lemma 2.20. *Let $C \subset \overline{[\mathrm{Thick}_\epsilon(F_n)]}$ be a closed set. Denote by $V \subset \mathcal{PML}(F_n)$ the set of all projective measured laminations which are induced by a basic primitive conjugacy class for a tree $S \in \mathrm{Thick}_\epsilon(F_n)$ with $[S] \in C$ and let $\overline{V}$ be the closure of V in $\mathcal{PML}(F_n)$. Then*

$$\overline{V} - V \subset \{[\mu] \mid \exists\, [T] \in C \cap \partial[\mathrm{Thick}_\epsilon(F_n)], \langle [T], [\mu]\rangle = 0\}.$$

Proof. Let $C \subset \overline{[\mathrm{Thick}_\epsilon(F_n)]}$ be a closed set. Let $V \subset \mathcal{PML}(F_n)$ be the set of all projective measured laminations which are induced by a basic primitive conjugacy class for some tree $T \in \mathrm{Thick}_\epsilon(F_n)$ with $[T] \in C$. Let $([\alpha_i]) \subset V$ be a sequence which converges to some $[\alpha] \in \overline{V}$. For every $i \geq 0$ let $T_i \in \mathrm{Thick}_\epsilon(F_n)$ be such that $[T_i] \in C$ and that $[\alpha_i]$ is induced by a primitive conjugacy class $[w_i]$ in F_n which is basic for T_i. After passing to a subsequence we may assume that $[T_i] \to [T_\infty] \in C$.

Consider first the case that $[T_\infty] \in [\mathrm{Thick}_\epsilon(F_n)] \subset \mathrm{CV}(F_n)$. Let $T_\infty \in cv_0(F_n)$ be the representative tree with quotient of volume one; then $T_\infty \in \mathrm{Thick}_\epsilon$ and $T_i \to T_\infty$. In particular, for sufficiently large i there is a 3/2-Lipschitz marked homotopy equivalence $T_i/F_n \to T_\infty/F_n$. Thus for sufficiently large i the class $[w_i]$ can be represented by a loop in T_∞/F_n of length at most 3. However, the number of conjugacy classes which can be represented by a loop in T_∞/F_n of length at most 3 is finite. This means that there is a conjugacy class $[w]$ in F_n so that $[w_i] = [w]$ for infinitely many i. Then $[\alpha] = [\alpha_i]$ for at least one i and hence $[\alpha] \in V$.

In the case that $[T_\infty] \in \partial\mathrm{CV}(F_n)$ let $T_\infty \in \Sigma(T_0) \cap \partial cv(F_n)$ be a representative of $[T_\infty]$. For $i \geq 0$ let $a_i > 0$ be such that

$$a_iT_i \in \Sigma(T_0).$$

Then $a_iT_i \to T_\infty$ $(i \to \infty)$, and since $T_i \in \text{Thick}_\epsilon(F_n)$ for all i, Lemma 2.19 implies that $a_i \to 0$ $(i \to \infty)$.

Let $\alpha_i \in \Lambda(T_0)$ be the representative of $[\alpha_i]$. Then $\alpha_i = b_i\alpha_i'$ where α_i' is induced by a primitive conjugacy class which is basic for T_i and $b_i > 0$. Now $T_0 \in \text{Thick}_\epsilon(F_n)$ and hence $\langle T_0, \alpha_i'\rangle \geq \epsilon$ for all i. This shows that $b_i \leq 1/\epsilon$ for all i.

By compactness, we may assume that $\alpha_i \to \alpha \in \Lambda(T_0)$ where α is a representative of the class $[\alpha]$. Since $\langle T_i, \alpha_i'\rangle \leq 2$ by assumption, we have $\langle T_i, \alpha_i\rangle \leq 2/\epsilon$ for all i. But $\langle a_iT_i, \alpha_i\rangle \to \langle T_\infty, \alpha\rangle$ and $a_i \to 0$ $(i \to \infty)$ and therefore $\langle T_\infty, \alpha\rangle = 0$ by continuity. This shows that $[\alpha]$ is supported in the zero lamination of T_∞ and completes the proof of the lemma. ☐

Corollary 2.21. *Let $C \subset \overline{[\text{Thick}_\epsilon(F_n)]}$ be a closed set and let $[T] \in \mathcal{UT} - C$. Then $\omega^{-1}([T]) \in \mathcal{UML}$ is not contained in the closure of the set of all projective measured laminations which are induced by a basic primitive conjugacy class for some tree $S \in \text{Thick}_\epsilon(F_n)$ with $[S] \in C$.*

Proof. If $C \subset \overline{[\text{Thick}_\epsilon(F_n)]}$ is a closed set and if $[T] \in \mathcal{UT} - C$ then $\omega^{-1}([T])$ is not supported in the zero lamination of any tree $[S] \in C$. Together with Lemma 2.20, this shows the corollary. ☐

3. Contracting pairs

In this section we use the length pairing to single out sets of pairs of distinct points in $\mathcal{PML}$ whose corresponding lengths functions determine lines in Outer space with strong contraction properties. Examples of such pairs include all pairs of fixed points of iwip elements in $\text{Out}(F_n)$. To define these pairs we establish first some properties of sums of length functions on Outer space.

Fix some small $\epsilon > 0$. For the formulation of the following definition, recall from (4) in Section 2 the definition of the set $\text{Thick}_\epsilon(F_n)$.

Definition 3.1. A family $\mathcal{F}$ of nonnegative functions ρ on $cv_0(F_n)$ is called *uniformly proper* if for every $c > 0$ there is a compact subset $A(c)$ of $\text{Thick}_\epsilon(F_n)$ such that $\rho^{-1}[0, c] \cap \text{Thick}_\epsilon(F_n) \subset A(c)$ for every $\rho \in \mathcal{F}$.

Call a pair $(\mu, \nu) \in \mathcal{ML}(F_n)^2$ *positive* if the function $T \to \langle T, \nu + \mu\rangle$ is positive on $cv(F_n) \cup \partial cv(F_n)$. For $T \in cv_0(F_n)$ recall from Section 2 the definitions (2), (3) of the compact sets $\Lambda(T)$ and $\Sigma(T)$ before and after Lemma 2.11. These sets are the main tools throughout this section.

Lemma 3.2. *Let $K \subset \mathcal{ML}(F_n) \times \mathcal{ML}(F_n)$ be a compact set consisting of positive pairs. Then the family of functions $\{\langle \cdot, \mu + \mu'\rangle \mid (\mu, \mu') \in K\}$ on $cv_0(F_n)$ is uniformly proper.*

Proof. Let $K \subset \mathcal{ML}(F_n) \times \mathcal{ML}(F_n)$ be as in the lemma. Let $T_0 \in \text{Thick}_\epsilon(F_n)$ and let $\Sigma = \Sigma(T_0)$. By assumption, we have $\langle T, \nu + \nu' \rangle > 0$ for every $T \in \Sigma$ and every $(\nu, \nu') \in K$. By continuity of the length pairing and compactness of Σ and K there is then a number $\delta > 0$ such that $\langle T, \mu + \mu' \rangle \geq \delta$ for every $(\mu, \mu') \in K$ and every $T \in \Sigma$.

Let $c > 0$ and let $A(c) = \{T \in \text{Thick}_\epsilon(F_n) \mid \min\{\langle T, \mu + \mu' \rangle \mid (\mu, \mu') \in K\} \leq c\}$. Then $A(c)$ is a closed subset of $\text{Thick}_\epsilon(F_n)$. Our goal is to show that $A(c)$ is compact.

For this assume otherwise. Since $\text{Thick}_\epsilon(F_n)$ is locally compact, there is then a sequence $(T_i) \subset A(c)$ such that for every compact set $B \subset \text{Thick}_\epsilon(F_n)$, we have $T_i \in B$ only for finitely many i, and for each i there is some $(\mu_i, \mu_i') \in K$ with

$$\langle T_i, \mu_i + \mu_i' \rangle \leq c. \tag{6}$$

Let $a_i > 0$ be such that $a_i T_i \in \Sigma$. Since Σ is compact, after passing to a subsequence we may assume that

$$\lim_{i \to \infty} a_i T_i = T$$

in Σ. Moreover, since K is compact, after passing to another subsequence we may assume that $(\mu_i, \mu_i') \to (\mu, \mu') \in K$. Then $\langle T, \mu + \mu' \rangle \geq \delta$.

Lemma 2.19 shows that $\lim_{i \to \infty} a_i = 0$. On the other hand, since $\langle T, \mu + \mu' \rangle \geq \delta$, using once more continuity of the length function we infer from (6) that

$$0 = \lim_{i \to \infty} a_i \langle T_i, \mu_i + \mu_i' \rangle = \lim_{i \to \infty} \langle a_i T_i, \mu_i + \mu_i' \rangle = \langle T, \mu + \mu' \rangle \geq \delta.$$

This is a contradiction and shows that $\{\langle \cdot, \mu + \mu' \rangle \mid (\mu, \mu') \in K\}$ is indeed uniformly proper. □

A pair of projective measured laminations $([\mu], [\nu]) \in \mathcal{PML}(F_n) \times \mathcal{PML}(F_n) - \Delta$ is called *positive* if for any representatives μ, ν of $[\mu], [\nu]$ the function $\langle \cdot, \mu + \nu \rangle$ on $cv(F_n) \cup \partial cv(F_n)$ is positive. A tree $T \in cv(F_n) \cup \partial cv(F_n)$ is called *balanced* for a positive pair $(\mu, \nu) \in \mathcal{ML}(F_n)^2$ if $\langle T, \mu \rangle = \langle T, \nu \rangle$. Note that this only depends on the projective class of T. The set

$$\text{Bal}(\mu, \nu) \subset cv(F_n) \cup \partial cv(F_n)$$

of all balanced trees for (μ, ν) is a closed subset of $cv(F_n) \cup \partial cv(F_n)$ which is disjoint from the set of trees on which either μ or ν vanishes. Let moreover

$$\text{Min}_\epsilon(\mu + \nu) \subset \text{Thick}_\epsilon(F_n)$$

be the set of all points for which the restriction of the function $T \to \langle T, \mu + \nu \rangle$ to $\text{Thick}_\epsilon(F_n)$ assumes a minimum.

Remark 3.3. It follows from continuity of the function $T \to \langle T, \mu + \nu \rangle$, local compactness of $\text{Thick}_\epsilon(F_n)$ and Lemma 3.2 that the set $\text{Min}_\epsilon(\mu + \nu)$ is non-empty and compact.

From now on we fix a sufficiently small $\epsilon > 0$. The definition of a B-contracting pair below depends on this number ϵ. However, as it will become apparent in the proof of Proposition 3.10, a pair which is B-contracting for a given choice of ϵ is B'-contracting for another choice ϵ'. Thus even though the quantitative control of a B-contracting pair depends on the choice of ϵ, the large-scale properties do not.

The reason for fixing a number $\epsilon > 0$ for the construction is two-fold. First, the lack of completeness of the one-sided Lipschitz metric on Outer space leads to a lack of convexity of length functions. Second, Teichmüller theory indicates that no information is lost. Namely, there are various ways to describe the quality of the axis of a pseudo-Anosov element, and one way to do this is to project the axis to a fixed thick part of Teichmüller space and measure the contraction properties of this projection. This is exactly what is done here.

Definition 3.4. For $B > 1$, a positive pair of points

$$([\mu], [\nu|) \in \mathcal{PML}(F_n) \times \mathcal{PML}(F_n) - \Delta$$

is called *B-contracting* if for any pair $\mu, \nu \in \mathcal{ML}(F_n)$ of representatives of $[\mu], [\nu]$ there is some "distinguished" $T \in \mathrm{Min}_\epsilon(\mu + \nu)$ with the following properties.

(1) $\langle T, \mu\rangle / \langle T, \nu\rangle \in [B^{-1}, B]$.
(2) If $\tilde{\mu}, \tilde{\nu} \in \Lambda(T)$ are representatives of $[\mu], [\nu]$ then $\langle S, \tilde{\mu} + \tilde{\nu}\rangle \geq 1/B$ for all $S \in \Sigma(T)$.
(3) Let $\mathcal{B}(T) \subset \Lambda(T)$ be the set of all normalized measured laminations which are up to scaling induced by a basic primitive conjugacy class for a tree $U \in \mathrm{Bal}(\mu, \nu) \cap \mathrm{Thick}_\epsilon(F_n)$. Then $\langle S, \xi\rangle \geq 1/B$ for every $\xi \in \mathcal{B}(T)$ and every tree
$$S \in \Sigma(T) \cap \big(\bigcup_{s \in (-\infty, -B) \cup (B, \infty)} \mathrm{Bal}(e^s \mu, e^{-s} \nu)\big).$$

Remark 3.5. The above definition is symmetric in $[\mu], [\nu]$. Moreover, if $([\mu], [\nu])$ is a B-contracting pair then $([\mu], [\nu])$ is C-contracting for every $C \geq B$,

Remark 3.6. By invariance of the length pairing under the diagonal action of $\mathrm{Out}(F_n)$, B-contracting pairs and their defining data transform correctly under $\mathrm{Out}(F_n)$. Thus if $([\mu], [\nu]) \in \mathcal{PML}(F_n)^2$ is a contracting pair, if $\mu, \nu \in \mathcal{ML}(F_n)$ is a pair of representatives of $[\mu], [\nu]$ and if $T \in \mathrm{Min}_\epsilon(\mu + \nu)$ is a distinguished point for μ, ν which has properties (1),(2),(3) stated in the definition, then for every $\varphi \in \mathrm{Out}(F_n)$ the pair $(\varphi[\mu], \varphi[\nu]) \in \mathcal{PML}(F_n)^2$ is B-contracting, and the tree $\varphi(T)$ is a distinguished point for $\varphi(\mu), \varphi(\nu)$.

Remark 3.7. In Definition 3.4, the tree T could be replaced by any tree in $\mathrm{Min}_\epsilon(\mu + \nu)$ (however at the expense of changing the constant B by a controlled amount). This fact can be derived from the discussion in later sections, but it will not be used. Moreover, singling out a special tree T turns out to be convenient for the proofs.

We call a pair $(\mu, \nu) \in \mathcal{ML}(F_n)^2$ *B-contracting* for some $B > 0$ if the pair $([\mu], [\nu])$ of its projectivizations is B-contracting.

As in the case of lines of minima in Teichmüller space, we use B-contracting pairs to constuct coarsely well defined lines in Outer space. This construction is carried out in detail in Section 4. In the remainder of this section we establish some first properties of B-contracting pairs which shows in particular that the set of B-contracting pairs is not empty and in fact contains many elements which can be identified explicitly.

Proposition 3.8. *For any $B > 0$, the set $\mathcal{A}(B)$ of B-contracting pairs is an $\mathrm{Out}(F_n)$-invariant closed subset of the space of positive pairs in*

$$\mathcal{PML}(F_n) \times \mathcal{PML}(F_n) - \Delta.$$

Proof. By definition, $\mathcal{A}(B)$ is $\mathrm{Out}(F_n)$-invariant.

To show that $\mathcal{A}(B)$ is a closed subset of the set of all positive pairs in $\mathcal{PML}(F_n)\times\mathcal{PML}(F_n) - \Delta$, let $([\mu_i], [\nu_i])$ be a sequence of B-contracting pairs converging to a positive pair $([\mu], [\nu]) \in \mathcal{PML}(F_n) \times \mathcal{PML}(F_n) - \Delta$.

Let $\mu, \nu \in \mathcal{ML}(F_n)$ be preimages of $[\mu], [\nu]$ and choose any sequence $(\mu_i, \nu_i) \in \mathcal{ML}(F_n)^2$ of pairs of preimages of $[\mu_i], [\nu_i]$ which converges to (μ, ν). By Lemma 3.2, the family of functions

$$\mathcal{F} = \{\mu_i + \nu_i, \mu + \nu\}$$

is uniformly proper. Thus if $T_i \in \mathrm{Min}_\epsilon(\mu_i + \nu_i)$ is a point as in the definition of a B-contracting pair then up to passing to a subsequence, we may assume that the sequence (T_i) converges to a point $T \in \mathrm{Thick}_\epsilon(F_n)$. By continuity of the length pairing, we have $T \in \mathrm{Min}_\epsilon(\mu + \nu)$ and moreover $\langle T, \mu\rangle/\langle T, \nu\rangle \in [B^{-1}, B]$.

Our goal is now to show that T has properties (2) and (3) in Definition 3.4.

To see property (2), note that by Lemma 2.11, if $\alpha \in \mathcal{ML}(F_n)$ and if $a_i > 0, a > 0$ is such that $a_i\alpha \in \Lambda(T_i), a\alpha \in \Lambda(T)$ then $a_i \to a$ since $T_i \to T$. In particular, if $\tilde\mu_i, \tilde\nu_i \in \Lambda(T_i)$ are representatives of $[\mu_i], [\nu_i]$ then $\tilde\mu_i \to \tilde\mu \in \Lambda(T)$ and $\tilde\nu_i \to \tilde\nu \in \Lambda(T)$ where $\tilde\mu, \tilde\nu$ are representatives of $[\mu], [\nu]$.

On the other hand, if $S \in cv_0(F_n) \cup \partial cv_0(F_n)$ and if $b_i > 0$ is such that $b_iS \in \Sigma(T_i)$ then Lemma 2.13 shows that $b_i \to b$ where $bS \in \Sigma(T)$. Then

$$1/B \le \langle b_iS, \tilde\mu_i + \tilde\nu_i\rangle \text{ and } \langle b_iS, \tilde\mu_i + \tilde\nu_i\rangle \to \langle bS, \tilde\mu + \tilde\nu\rangle$$

by continuity of the length pairing. This shows property (2) in the definition of a B-contracting pair.

Now let $\xi \in \mathcal{B}(T) \subset \Lambda(T)$ be as in the third part of the definition for the pair (μ, ν). Then there is a tree $U \in \mathrm{Bal}(\mu, \nu) \cap \mathrm{Thick}_\epsilon(F_n)$ such that up to scaling, ξ is induced by a primitive conjugacy class which can be represented by a loop on U/F_n of length at most two. Let

$$S \in \Sigma(T) \cap \mathrm{Bal}(e^s\mu, e^{-s}\nu) \text{ for some } s \in (-\infty, -B) \cap (B, \infty).$$

We have to show that $\langle S, \xi\rangle \ge 1/B$.

To see that this is the case, let $a_i > 0$ be such that $a_i\xi \in \Lambda(T_i)$; then $a_i \to 1$. Let $t_i \in \mathbb{R}$ be such that $U \in \mathrm{Bal}(e^{t_i}\mu_i, e^{-t_i}\nu_i)$; then $t_i \to 0$ $(i \to \infty)$. There is a sequence

$b_i \to 1$ and for every sufficiently large i there is a number $s_i \in (-\infty, B) \cup (B, \infty)$ such that $b_i S \in \Sigma(T_i) \cap \mathrm{Bal}(e^{s_i+t_i}\mu_i, e^{-s_i-t_i}\nu_i)$.

Now $s_i + t_i \to s$ $(i \to \infty)$ and hence $s_i + t_i \in (-\infty, -B) \cup (B, \infty)$ for sufficiently large i. By the third requirement in the definition of a B-contracting pair we have $\langle b_i S, a_i \xi \rangle \geq 1/B$ for all sufficiently large i and hence $\langle S, \xi \rangle \geq 1/B$ by continuity. This completes the proof of the proposition. ☐

Remark 3.9. We have $\mathcal{A}(B) \subset \mathcal{A}(C)$ for $B < C$ and hence $\cup_{B>0}\mathcal{A}(B)$ is a countable union of closed subsets of the set of positive pairs.

Corollary 5.6 and the remark thereafter shows that for $([\mu], [\nu]) \in \mathcal{PML}(F_n)$, being a B-contracting pair for some $B > 0$ is a property of the individual projective measured laminations $[\mu], [\nu]$ rather than of the pair. Once again, Teichmüller theory shows that there are positive pairs which are not contracting. Such a pair can be constructed from a minimal filling measured geodesic lamination on a compact surface S with connected boundary which is not uniquely ergodic. Since we do not need this fact we do not discuss it in more detail here.

The following proposition is the key observation in this paper.

Proposition 3.10. *If* $([\nu_+], [\nu_-]) \in \mathcal{UML}^2$ *is the pair of fixed points of an iwip element of* $\mathrm{Out}(F_n)$ *then* $([\nu_+], [\nu_-])$ *is* B*-contracting for some* $B > 0$.

Proof. Let $\varphi \in \mathrm{Out}(F_n)$ be an iwip element with pair of fixed points $[\nu_+], [\nu_-] \in \mathcal{UML}$. In particular, $([\nu_+], [\nu_-])$ is a positive pair. Up to exchanging φ and φ^{-1} there are numbers $\lambda_+, \lambda_- > 1$ such that for any representatives ν_+, ν_- of the classes $[\nu_+], [\nu_-]$ we have

$$\varphi\nu_+ = \lambda_+\nu_+, \varphi\nu_- = \lambda_-^{-1}\nu_-.$$

Let $s_0 > 0, a > 0$ be such that $e^{s_0} = a\lambda_+, e^{-s_0} = a\lambda_-^{-1}$. Then

$$\mathcal{F} = \{f_s : T \to \langle T, e^s\nu_+ + e^{-s}\nu_- \rangle \mid s \in [-s_0, s_0]\}$$

is a set of functions on $cv_0(F_n)$ which is compact with respect to the topology of uniform convergence on compact sets. Lemma 3.2 shows that the set

$$C \subset \mathrm{Thick}_\epsilon(F_n)$$

of all minima of the restrictions of all functions from the collection $\mathcal{F}$ to $\mathrm{Thick}_\epsilon(F_n)$ is compact. In particular, there is a number $B_1 > 0$ such that

$$\langle S, e^s\nu_+ \rangle / \langle S, e^{-s}\nu_- \rangle \in [B_1^{-1}, B_1]$$

for all $S \in C$ and all $s \in [-1 - s_0, s_0 + 1]$. This shows that for $B = B_1$ and for any $s \in [-s_0, s_0]$, the first requirement in Definition 3.4 is fullfilled for for $e^s\nu_+, e^{-s}\nu_-$.

To establish property (2), for $T \in \mathrm{Thick}_\epsilon(F_n)$ let $\tilde{\nu}_+(T), \tilde{\nu}_-(T) \in \Lambda(T)$ be the representative of $[\nu_+], [\nu_-]$ contained in $\Lambda(T)$. Then

$$g_T : S \to \langle S, \tilde{\nu}_+(T) + \tilde{\nu}_-(T) \rangle$$

is a function on $cv_0(F_n)$ which depends continuously on T. In particular, the family $\mathcal{G} = \{g_T \mid T \in C\}$ is compact with respect to the compact open topology for continuous functions on $cv_0(F_n) \cup \partial cv_0(F_n)$.

Now $\Sigma(T)$ depends continuously on $T \in cv_0(F_n)$ and therefore

$$\mathcal{T} = \cup_{T \in C} \Sigma(T)$$

is a compact subset of $cv(F_n) \cup \partial cv(F_n)$. The restriction to $\mathcal{T}$ of every $g \in \mathcal{G}$ is positive. This implies that

$$b = \inf\{g_T(S) \mid g_T \in \mathcal{G}, S \in \mathcal{T}\} > 0.$$

As a consequence, for $B = 1/b$ and $s \in [-s_0, s_0]$, property (2) in Definition 3.4 is fullfilled for $e^s \nu_+, e^{-s} \nu_-$

To establish property (3), let $T \in \mathrm{Min}_\epsilon(\nu_+ + \nu_-)$ and let $K \subset \Sigma(T)$ be the compact subset of all trees which are balanced for (ν_+, ν_-) and whose projectivizations are contained in $\overline{[\mathrm{Thick}_\epsilon(F_n)]}$. Let $\Theta(K) \subset \Lambda(T)$ be the closure of the set of all normalized measured laminations which are up to scaling induced by some basic primitive conjugacy class for any tree which is the projectivization of an element of K. By Corollary 2.21, we have $\tilde{\nu}_+, \tilde{\nu}_- \notin \Theta(K)$.

Let $T_+, T_- \in \Sigma(T)$ be dual to $[\nu_+], [\nu_-]$. If $\zeta \in \mathcal{ML}(F_n)$ is any measured lamination then $\langle T_\pm, \zeta \rangle = 0$ only if the projective class of ζ equals $[\nu_\pm]$. By continuity of the length pairing, the set of functions

$$\mathcal{F} = \{T \to \langle T, \zeta \rangle \mid \zeta \in \Theta(K)\}$$

is compact for the compact open topology on the space of continuous functions on $\Sigma(T)$. As a consequence, their values on T_+, T_- are bounded from below by a positive number.

Since $[T_+], [T_-] \in \mathcal{UT}$, the sets

$$U(p) = \{[S] \in \overline{[\mathrm{Thick}_\epsilon(F_n)]} \mid S \in \mathrm{Bal}(e^t \nu_+, e^{-t} \nu_-) \text{ for some } t > p\}$$

$(p > 0)$ form a neighborhood basis for $[T_+]$ in $\overline{[\mathrm{Thick}_\epsilon(F_n)]}$. This implies that there is some $p > 0$ and a number $c > 0$ such that the functions from the set $\mathcal{F}$ are bounded from below by a positive number on $\tilde{U}(p) = \{S \in \Sigma(T) \mid [S] \in U(p)\}$. In the same way we can construct a neighborhood $V(p)$ of $[T_-]$ in $\overline{[\mathrm{Thick}_\epsilon(F_n)]}$ so that the values of the functions from $\mathcal{F}$ on $\tilde{V}(p)$ are bounded from below by a positive number. As a consequence, for $s \in [-s_0, s_0]$, property (3) in Definition 3.4 holds true for $e^2 \nu_+, e^{-s} \nu_-$.

If $s \in \mathbb{R}$ is arbitrary then there is some $m \in \mathbb{Z}$ and some $s_1 \in [0, s_0)$ such that $s = ms_0 + s_1$. By the choice of s_0, the function

$$T \to \langle T, \varphi^m(e^{s_1} \nu_+) + \varphi^m(e^{-s_1} \nu_-) \rangle$$

is a multiple of the function $T \to \langle T, e^s \nu_+ + e^{-s} \nu_- \rangle$. Since φ^m acts on $cv(F_n) \times \mathcal{ML}(F_n)$ diagonally as a homeomorphism preserving the length pairing, the three properties of a contracting pair for $e^s \nu_+, e^{-s} \nu_-$ follow from the corresponding properties for $e^{s_1} \nu_+, e^{-s_1} \nu_-$. We refer to Remark 3.6 which explains how the defining objects of a contracting pair transform under the action of $\mathrm{Out}(F_n)$. $\square$

4. Axes of B-contracting pairs

The goal of this section is to relate B-contracting pairs to the geometry of $\mathrm{Out}(F_n)$. For this we first equip $cv_0(F_n)$ with an $\mathrm{Out}(F_n)$-invariant distance as follows.

For trees $T, T' \in cv_0(F_n)$ let $d_L(T,T')$ be the logarithm of the minimal Lipschitz constant of a marked homotopy equivalence $T/F_n \to T'/F_n$. Then

$$d(T,T') = d_L(T,T') + d_L(T',T)$$

is an $\mathrm{Out}(F_n)$-invariant distance function on $cv_0(F_n)$ inducing the original topology [FM11]. This distance d will be called the *symmetrized Lipschitz distance* in the sequel, and we call the function d_L the *one-sided Lipschitz metric*. The group $\mathrm{Out}(F_n)$ acts properly, isometrically and cocompactly on $\mathrm{Thick}_\epsilon(F_n)$ equipped with the restriction of d. Here $\mathrm{Thick}_\epsilon(F_n)$ is defined as in (4) of Section 2. Unfortunately, the symmetrized Lipschitz metric d is *not* a geodesic metric.

As in Section 2, call a primitive conjugacy class $[w]$ in F_n *basic* for a simplicial tree $T \in \mathrm{Thick}_\epsilon(F_n)$ if $[w]$ can be represented by a loop in T/F_n of length at most two. The following result is due to Tad White (unpublished; a published account can be found in the paper [FM11] of Francaviglia and Martino).

Lemma 4.1. *For $T, T' \in cv_0(F_n)$,*

$$d_L(T,T') = \sup\{\log \frac{\langle T',\alpha\rangle}{\langle T,\alpha\rangle} \mid \alpha \in \mathcal{ML}(F_n)\}.$$

The supremum is attained for a measured lamination α which is induced by a basic primitive conjugacy class for T.

Proof. Let $T, T' \in cv_0(F_n)$. Clearly for every measured lamination $\alpha \in \mathcal{ML}(F_n)$ we have

$$d_L(T,T') \geq \log \frac{\langle T',\alpha\rangle}{\langle T,\alpha\rangle}.$$

On the other hand, Proposition 3.15 of [FM11] states that $d_L(T,T')$ is the minimum of the logarithm of the quotients $\frac{\langle T',\alpha\rangle}{\langle T,\alpha\rangle}$ where α passes through the set of all currents dual to a conjugacy class $[w]$ in F_n of the following form. $[w]$ can be represented by a loop γ in T/F_n which either is simple or defines an embedded bouquet of two circles in T/F_n or defines two disjointly embedded simple closed curves in T/F_n joined by a disjoint embedded arc traveled through twice in opposite direction. In particular, the length of γ is a most two.

It is well known (see p. 197/198 of [M67]) that if γ is an embedded loop in T/F_n then γ represents a primitive conjugacy class in F_n. If $\gamma = \gamma_1\gamma_2$ where γ_1, γ_2 are two embedded loops which intersect in a single point then γ can be obtained from the primitive element γ_1 by a Nielsen move with the primitive element γ_2 and once again, γ is primitive. The third case is completely analogous. ☐

For the following observation, recall from (2) and (3) of Section 2 the definitions of the sets $\Lambda(T)$, $\Sigma(T)$ for a tree $T \in cv_0(F_n)$. Lemma 4.1 implies

Corollary 4.2. *Let $T, S \in cv_0(F_n)$ and let $b > 0$ be such that $bS \in \Sigma(T)$. If $\langle bS, \nu\rangle \geq 1/B$ for some $B > 0$ and some $\nu \in \Lambda(T)$ then*

$$\log\langle S, \nu\rangle \leq d_L(T, S) \leq \log\langle S, \nu\rangle + \log B.$$

Proof. Let $T, S \in cv_0(F_n)$, $b > 0$ be as in the corollary. By Lemma 4.1 and invariance under scaling, we have

$$d_L(T, S) = \sup\{\log\langle S, \alpha\rangle \mid \alpha \in \Lambda(T)\}.$$

This implies the left hand side of the inequality.

If $b > 0$ is such that $bS \in \Sigma(T)$ then $b = e^{-d_L(T,S)}$. Moreover, if $\nu \in \Lambda(T)$ and if $\langle bS, \nu\rangle \geq 1/B$ then $\langle S, \nu\rangle \geq e^{d_L(T,S)}/B$. Taking the logarithm shows the right hand side of the inequality in the corollary. ☐

From a B-contracting pair we now construct a family of "lines" in the ϵ-thick part $\text{Thick}_\epsilon(F_n)$ of $cv_0(F_n)$.

Definition 4.3. An *axis* for a B-contracting pair $([\mu], [\nu])$ is a map

$$\gamma : \mathbb{R} \to \cup_{s\in\mathbb{R}} \text{Min}_\epsilon(e^{s/2}\mu + e^{-s/2}\nu)$$

for some representatives μ, ν of $[\mu], [\nu]$ such that for every $t \in \mathbb{R}$, $\gamma(t)$ is a point in $\text{Min}_\epsilon(e^{t/2}\mu + e^{-t/2}\nu)$ which has the properties (1)-(3) as in Definition 3.4 for the pair of representatives $e^{t/2}\mu, e^{-t/2}\nu \in \mathcal{ML}(F_n)$ of the projective measured laminations $[\mu], [\nu]$.

We do *not* require the map γ to be continuous.

Note that an axis as in Definition 4.3 depends on choices and hence is by no means unique. For a fixed choice of representatives μ, ν of the classes $[\mu], [\nu]$, the point $\gamma(t)$ of an axis γ is chosen from a fixed compact subset of $\text{Thick}_\epsilon(F_n)$.

A different choice μ_0, ν_0 of representatives of the projective measured laminations $[\mu], [\nu]$ yields the same axes, but with a parametrization which is modified by a translation. Namely, multiplying μ, ν with the same positive scalar does not change $\text{Min}_\epsilon(e^s\mu + e^{-s}\nu)$ for any s. On the other hand, if we replace ν by $a\nu$ for some $a > 0$ then we have

$$\text{Min}_\epsilon(e^s\mu + e^{-s}(a\nu)) = \text{Min}_\epsilon(e^t\mu + e^{-t}\nu)$$

where $t = s - \frac{1}{2}\log a$.

For a number $D > 0$, a *D-coarse geodesic* in a metric space (X, d) is a (possibly non-continuous) map $\gamma : J \to X$ defined on a closed connected subset J of $\mathbb{R}$ such that

$$d(\gamma(s), \gamma(t)) \in [t - s - D, t - s + D] \text{ for all } s, t \in J, s \leq t.$$

In particular, if $s < t < u$ then $d(\gamma(s), \gamma(u)) \geq d(\gamma(s), \gamma(t)) + d(\gamma(t), \gamma(u)) - 3D$. The main goal of this section is to relate axes of B-contracting pairs to the symmetrized Lipschitz metric. The following proposition is more generally true for a map as in Definition 4.3 defined by a positive pair $([\mu], [\nu])$ which only has properties (1) and (2) in Definition 3.4.

Proposition 4.4. *For all $B > 0$ there is a number $\kappa_1(B) > 0$ such that an axis of a B-contracting pair is a κ_1-coarse geodesic for the symmetrized Lipschitz metric.*

Since axes of B-contracting pairs are entirely contained in $\mathrm{Thick}_\epsilon(F_n)$ and since $\mathrm{Out}(F_n)$ acts properly and cocompactly on $\mathrm{Thick}_\epsilon(F_n)$, axes of B-contracting pairs determine a collection of uniform quasi-geodesics for $\mathrm{Out}(F_n)$ which only depend on the length pairing (note that this is true in spite of the fact that the symmetrized Lipschitz metric is not geodesic). This is in analogy to Teichmüller space where the construction of lines of minima only uses convexity of length functions [Ke92].

The proof of Proposition 4.4 is a consequence of two simple observations which are used several times in a sequel.

Lemma 4.5. *Let $(\mu, \nu) \in \mathcal{ML}(F_n)^2$ be a B-contracting pair, let $s \geq 0$ and let $T \in \mathrm{Min}_\epsilon(\mu, \nu), S \in \mathrm{Min}_\epsilon(e^s\mu, e^{-s}\nu)$ be points which have properties (1) and (2) in Definition 3.4. Then*

$$\log(\langle S, \nu\rangle / \langle T, \nu\rangle) \geq d_L(T, S) - 3\log B - \log 2.$$

Proof. Via multiplying μ, ν with a fixed constant (compare the above discussion) we may assume without loss of generality that $\nu \in \Lambda(T)$. By the first property of a B-contracting pair, applied to both T and S, we have

$$\langle T, \mu\rangle \in [B^{-1}, B],\ \langle S, \mu\rangle \leq e^{-2s} B\langle S, \nu\rangle.$$

In particular, if $\tilde{\mu} \in \Lambda(T)$ is the normalization of μ at T then $\langle S, \tilde{\mu}\rangle \leq e^{-2s}B^2\langle S, \nu\rangle$ and hence $\langle S, \nu + \tilde{\mu}\rangle \leq 2B^2\langle S, \nu\rangle$.

Now let $b > 0$ be such that $bS \in \Sigma(T)$. The second property of a B-contracting pair and the above estimate shows that $1/B \leq 2B^2\langle bS, \nu\rangle$ and hence from Corollary 4.2 we infer that

$$d_L(T, S) \leq \log\langle S, \nu\rangle + 3\log B + \log 2$$

as claimed. □

Corollary 4.6. *Let $([\mu], [\nu]) \in \mathcal{PML}(F_n)^2$ be a B-contracting pair. Let μ, ν be representatives of $[\mu], [\nu]$, let $s \geq 0$ and let $T \in \mathrm{Min}_\epsilon(\mu + \nu), S \in \mathrm{Min}_\epsilon(e^s\mu + e^{-s}\nu)$ be points which have the properties (1) and (2) in Definition 3.4. Then*

$$2s - 2\log B \leq d(T, S) \leq 2s + 8\log B + 2\log 2.$$

Proof. Let $T \in \mathrm{Min}_\epsilon(\mu + \nu), S \in \mathrm{Min}_\epsilon(e^s\mu + e^{-s}\nu)$ for some $s \geq 0$ be points as in Definition 3.4. Assume by multiplying μ, ν with a fixed constant that $\nu \in \Lambda(T)$. Lemma 4.5 and Corollary 4.2 show that

$$d_L(T, S) - 3\log B - \log 2 \leq \log\langle S, \nu\rangle \leq d_L(T, S). \tag{7}$$

By the first property in the definition of a B-contracting pair we have

$$\langle T, \mu\rangle \in [B^{-1}, B] \text{ and } \langle S, \nu\rangle / \langle S, \mu\rangle \in [e^{2s}B^{-1}, e^{2s}B]. \tag{8}$$

Another application of Lemma 4.5 with the roles of T, S, μ, ν exchanged yields

$$\begin{aligned} d_L(S, T) - 3\log B - \log 2 &\leq \log(\langle T, \mu\rangle / \langle S, \mu\rangle) \\ &\leq 2s - \log\langle S, \nu\rangle + 2\log B. \end{aligned} \tag{9}$$

Replacing $-\log\langle S,\nu\rangle$ in inequality (9) by the expression in inequality (7) shows that

$$d_L(T,S)+d_L(S,T)\le 2s+8\log B+2\log 2.$$

On the other hand, Lemma 4.1 immediately yields that

$$\log\big(\langle S,\nu\rangle\langle T,\mu\rangle/\langle T,\nu\rangle\langle S,\mu\rangle\big)\le d_L(T,S)+d_L(S,T).$$

Together with the estimate (8) this implies the lower bound for $d_L(T,S)+d_L(S,T)$ stated in the corollary. □

Proof of Proposition 4.4: Let $\kappa_1=\kappa_1(B)=8\log B+2\log 2$. Let $([\mu],[\nu])\in\mathcal{PML}(F_n)$ be a B-contracting pair, let μ_0,ν_0 be representatives of $[\mu],[\nu]$ and let $\gamma:\mathbb{R}\to cv_0(F_n)$ be an axis for $([\mu],[\nu])$ as in Definition 4.3.

Let $t<s$ and write $\mu=e^t\mu_0,\nu=e^{-t}\nu_0$. Then

$$\mathrm{Min}_\epsilon(e^s\mu_0,e^{-s}\nu_0)=\mathrm{Min}_\epsilon(e^{s-t}\mu,e^{-(s-t)}\nu)$$

and therefore Corollary 4.6 applied to μ,ν and $e^{s-t}\mu,e^{-(s-t)}\nu$ shows that

$$2(s-t)-\kappa\le d(\gamma(2t),\gamma(2s))\le 2(s-t)+\kappa$$

as promised. □

The *Hausdorff distance* between two closed (not necessarily compact) sets $A,B\subset\mathrm{Thick}_\epsilon(F_n)$ is the infimum of the numbers $r\in[0,\infty]$ such that A is contained in the r-neighborhood of B and B is contained in the r-neighborhood of A. As an immediate corollary, we obtain for $\kappa_1(B)=8\log B+2\log 2$ the following

Corollary 4.7. *For every B-contracting pair $([\mu],[\nu])$, the Hausdorff distance between any two axes of $([\mu],[\nu])$ is at most $\kappa_1(B)$.*

Proof. Let γ_1,γ_2 be any two axes of a B-contracting pair $([\mu],[\nu])$. Assume without loss of generality that $\gamma_1(0)\in\mathrm{Min}_\epsilon(\mu+\nu)$, $\gamma_2(0)\in\mathrm{Min}_\epsilon(\mu+\nu)$. It suffices to show that $d(\gamma_1(s),\gamma_2(s))\le\kappa_1(B)$ for all $s\in\mathbb{R}$. However, the map $\tilde\gamma$ defined by $\tilde\gamma(s)=\gamma_1(s)$ and $\tilde\gamma(t)=\gamma_2(t)$ for $t\neq s$ is an axis and hence the distance estimate is immediate from Corollary 4.6 and its proof. □

Remark 4.8. In fact, since by Proposition 4.4, axes of B-contracting pairs with the parametrization as in the definition of an axis are uniform coarse geodesics, any two of them are uniform fellow travellers after perhaps changing the parametrization by a translation.

Recall from Lemma 2.15 the definition of the equivariant homeomorphism $\omega:\mathcal{UML}\to\mathcal{UT}$. The next proposition explains that the axis of a B-contracting pair which determines a pair of distinct points in $\partial\mathrm{CV}(F_n)$ indeed connect these points. This is in particular relevant in view of Proposition 3.10 and the work of Handel and Mosher [HM06]. For its formulation we denote as before by $[T]\in\mathrm{CV}(F_n)$ the projective class of the tree $T\in cv_0(F_n)$.

Proposition 4.9. *Let γ be an axis of a B-contracting pair $([\mu],[\nu])\in\mathcal{UML}^2$. Then $\lim_{t\to\infty}[\gamma(t)]=\omega([\mu])$ in $\overline{\mathrm{CV}(F_n)}$.*

Proof. Let $T = \gamma(0)$ and for $s \geq 0$ let $\beta(s) > 0$ be such that $\beta(s)\gamma(s) \in \Sigma(T)$. Lemma 2.19 shows that $\beta(s) \to 0$ $(s \to \infty)$.

Choose representatives $\mu, \nu \in \Lambda(T)$ of $[\mu], [\nu]$; then $\langle \beta(s)\gamma(s), \mu + \nu\rangle \leq 2$. On the other hand, by the definition of an axis and Property (1) in Definition 3.4, we have

$$\langle \gamma(s), e^{s/2}\mu\rangle / \langle \gamma(s), e^{-s/2}\nu\rangle \in [B^{-1}, B].$$

We conclude that $\lim_{s\to\infty}\langle \beta(s)\gamma(s), \mu\rangle = 0$. As a consequence, the support of $[\mu]$ is contained in the zero lamination of every tree $[S] \in \partial\mathrm{CV}(F_n)$ which is an accumulation point of a sequence $[\gamma(s_i)]$ where $s_i \to \infty$. Since $[\mu] \in \mathcal{UML}$, a point in $\partial\mathrm{CV}(F_n)$ whose zero lamination contains the support of $[\mu]$ is unique. The proposition follows. □

5. Axes of contracting pairs are contracting

The goal of this section is to show that an axis of a B-contracting pair is contracting for the Lipschitz distance. We continue to use all assumptions and notations from the previous sections.

Recall in particular the definition of an axis $\gamma : \mathbb{R} \to \mathrm{Thick}_\epsilon(F_n)$ of a B-contracting pair $([\mu], [\nu]) \in \mathcal{PML}^2$. It is given by the choice of representatives $\mu, \nu \in \mathcal{ML}$ of the projective classes $[\mu], [\nu] \in \mathcal{PML}$ and for each $t \in \mathbb{R}$ by a point in $\mathrm{Min}_\epsilon(e^{t/2}\mu + e^{-t/2}\nu)$ with properties (1)-(3) from Definition 3.4. We call this point the *distinguished point* for the pair (μ, ν) in $\mathrm{Min}_\epsilon(e^{t/2}\mu + e^{-t/2}\nu)$. When no confusion is possible we simply talk about the distinguished point or the distinguished point in $\mathrm{Min}_\epsilon(e^{t/2}\mu + e^{-t/2}\nu)$.

Let γ be an axis of a B-contracting pair $([\mu], [\nu])$. There is a coarse projection $\Pi_\gamma : cv_0(F_n) \to \gamma(\mathbb{R})$ as follows. For $T \in cv_0(F_n)$ choose representatives μ, ν of the classes $[\mu], [\nu]$ such that $T \in \mathrm{Bal}(\mu, \nu)$. Note that the measured laminations μ, ν are unique up to a common rescaling. Associate to T the distinguished point $\Pi_\gamma(T) \in \mathrm{Min}_\epsilon(\mu + \nu) \cap \gamma(\mathbb{R})$.

Recall that the map $\gamma : \mathbb{R} \to \mathrm{Thick}_\epsilon(F_n)$ may not be continuous, and in general, the coarse projection Π_γ is discontinuous as well. However, it associates to a tree T a unique point $\Pi_\gamma(T) \in \gamma(\mathbb{R})$. It follows from Corollary 4.6, applied to the special case $s = 0$, that there is a number $\kappa > 0$ only depending on B such that for any other choice γ' of an axis of $([\mu], [\nu])$ and any $T \in cv_0(F_n)$ we have $d(\Pi_\gamma(T), \Pi_{\gamma'}(T)) \leq \kappa$ where as before, d denotes the symmetrized Lipschitz metric on $cv_0(F_n)$.

The following useful fact gives a first idea about the nature of the map Π.

Lemma 5.1. $d(\Pi_\gamma(\gamma(t)), \gamma(t)) \leq 10 \log B + 2 \log 2$ *for all* $t \in \mathbb{R}$.

Proof. Let $\mu, \nu \in \mathcal{ML}$ be representatives of the projective measured laminations which are used to define the axis γ. By property (1) in Definition 3.4, we have

$$\langle \gamma(t), e^{t/2}\mu\rangle / \langle \gamma(t), e^{-t/2}\nu\rangle \in [B^{-1}, B]$$

for all $t \in \mathbb{R}$. Now by definition, if $U = \Pi_\gamma(\gamma(t))$ then $U = \gamma(s)$ where $s \in \mathbb{R}$ is such that

$$\langle \gamma(t), e^{s/2}\mu \rangle = \langle \gamma(t), e^{-s/2}\nu \rangle.$$

Then $|t - s| \le \log B$ and hence the lemma follows from Corollary 4.6. □

In the statement of the following proposition, we use both the one-sided Lipschitz metric d_L as defined in Section 4 (see in particular Lemma 4.1 for the property which is most important for our purpose) and its symmetrization d which is a metric, i.e. which satisfies the triangle inequality. We call the metric d simply the *Lipschitz metric* in the sequel and use the notation *one-sided Lipschitz metric* for d_L.

Proposition 5.2. *For every $B > 0$ there is a number $\kappa_2 = \kappa_2(B) > 0$ with the following property. Let $([\mu], [\nu])$ be a B-contracting pair, let γ be an axis for $([\mu], [\nu])$ and let $T \in \mathrm{Thick}_\epsilon(F_n)$.*

(1) *If $S \in cv_0(F_n)$ is such that $d(\Pi_\gamma(T), \Pi_\gamma(S)) \ge \kappa_2$ then*
$$d_L(T, S) \ge d_L(T, \Pi_\gamma(T)) + d_L(\Pi_\gamma(T), \Pi_\gamma(S)) + d_L(\Pi_\gamma(S), S) - \kappa_2.$$
(2) *If $S \in \mathrm{Thick}_\epsilon(F_n)$ is such that $d(\Pi_\gamma(T), \Pi_\gamma(S)) \ge \kappa_2$ then*
$$d(T, S) \ge d(T, \Pi_\gamma(T)) + d(\Pi_\gamma(T), \Pi_\gamma(S)) + d(\Pi_\gamma(S), S) - \kappa_2.$$
(3) *If $S \in \gamma(\mathbb{R})$ is such that $d(T, S) \le \inf_t d(T, \gamma(t)) + 1$ then $d(S, \Pi_\gamma(T)) \le \kappa_2$.*

Proof. Let $([\mu], [\nu]) \in \mathcal{PML}(F_n)^2$ be a B-contracting pair and let $\mu, \nu \in \mathcal{ML}(F_n)$ be representatives of $[\mu], [\nu]$. Let γ be an axis of $([\mu], [\nu])$ with $\gamma(s) \in \mathrm{Min}_\epsilon(e^{s/2}\mu + e^{-s/2}\nu)$ and write $\Pi = \Pi_\gamma$.

Let $T \in \mathrm{Thick}_\epsilon(F_n)$ and assume that $T \in \mathrm{Bal}(\mu, \nu)$, i.e. that $\Pi(T) = \gamma(0)$. Let $S \in cv_0(F_n)$ be such that

$$d(\Pi(T), \Pi(S)) \ge 2B + 8 \log B + 2 \log 2.$$

Let $s \in \mathbb{R}$ be such that $S \in \mathrm{Bal}(e^{s/2}\mu, e^{-s/2}\nu)$. By the definition of the projection Π we have $\Pi(S) = \gamma(s)$. By perhaps exchanging $[\mu]$ and $[\nu]$ we may assume that $s \ge 0$.

Corollary 4.6 shows that

$$s - 2 \log B \le d(\gamma(0), \gamma(s)) \le s + 8 \log B + 2 \log 2$$

and hence since we assumed that $d(\Pi(T), \Pi(S)) = d(\gamma(0), \gamma(s)) \ge 2B + 8 \log B + 2 \log 2$ we conclude that $s \ge 2B$.

Let α be a cycle of maximal dilatation for a marked homotopy equivalence $T \to \Pi(T)$ with the smallest Lipschitz constant. By Lemma 4.1, we may assume that α is basic for T.

Let $\xi \in \Lambda(\Pi(T))$ be induced by α up to scaling (here as before, $\Lambda(\Pi(T))$ is defined as in (2) in Section 2). By property (3) in the definition of a B-contracting pair, the hypotheses of Corollary 4.2 are satisfied and give

$$\log\langle S, \xi \rangle \ge d_L(\Pi(T), S) - \log B.$$

On the other hand, $d_L(T,\Pi(T)) = -\log\langle T,\xi\rangle$ and therefore

$$d_L(T,S) \geq \log\big(\langle S,\xi\rangle/\langle T,\xi\rangle\big) \geq d_L(T,\Pi(T)) + d_L(\Pi(T),S) - \log B. \tag{10}$$

Next we claim that there is a universal constant $C > 0$ so that

$$d_L(\Pi(T),S) \geq d_L(\Pi(T),\Pi(S)) + d_L(\Pi(S),S) - C. \tag{11}$$

Together with the estimate (10), this shows the first statement in the proposition.

To establish inequality (11), assume without loss of generality that $\nu \in \Lambda(\Pi(T))$. Then we obtain from the left inequality of Corollary 4.2 that

$$d_L(\Pi(T),S) \geq \log\langle S,\nu\rangle, \tag{12}$$

on the other hand also

$$\log\langle S,\nu\rangle = \log\Big(\frac{\langle S,\nu\rangle}{\langle \Pi(S),\nu\rangle}\Big) + \log\langle \Pi(S),\nu\rangle = \log\langle \Pi(S),\nu\rangle + \log\langle S,\tilde\nu\rangle \tag{13}$$

where $\tilde\nu = \nu/\langle \Pi(S),\nu\rangle \in \Lambda(\Pi(S))$.

Now $s \geq 0$ and hence Lemma 4.5 shows that

$$\log\langle \Pi(S),\nu\rangle \geq d_L(\Pi(T),\Pi(S)) - 3\log B - \log 2. \tag{14}$$

By the inequalities (12),(13),(14) we are left with showing that

$$\log\langle S,\tilde\nu\rangle \geq d_L(\Pi(S),S) - \tilde C$$

for a universal constant $\tilde C > 0$.

To this end let $\tilde\mu \in \Lambda(\Pi(S))$ be the normalization of μ at $\Pi(S)$. By property (1) in Definition 3.4 and the definition of the map Π, we have $\langle S,\tilde\nu\rangle \geq \langle S,\tilde\mu\rangle/B$.

Property (2) in Definition 3.4 and Corollary 4.2 imply that

$$\log\langle S,\tilde\nu\rangle \geq \log\langle S,\tilde\nu+\tilde\mu\rangle/2B \geq d_L(\Pi(S),S) - 2\log B. \tag{15}$$

This is what we wanted to show.

Exchanging the role of T,S and adding the resulting inequalities yields the second part of the proposition (with an adjustment of the additive error) provided that in addition we have $S \in \mathrm{Thick}_\epsilon(F_n)$.

To show the third part of the proposition, let $S \in \gamma(\mathbb{R})$ be such that $d(T,S) \leq \inf_t d(T,\gamma(t))+1$. Lemma 5.1 shows that $d(S,\Pi(S)) \leq 12\log B + 2\log 2$. Inequality (2) in the statement of the proposition together with the fact that $\Pi(T) \in \gamma(\mathbb{R})$ and hence $d(T,\Pi(T)) \geq d(T,S) - 1$ implies that $d(\Pi(T),S) \leq C$ where $C > 0$ is a universal constant. The proposition is proven. □

As an immediate consequence, we obtain the following contraction property (compare [BFu09]). For pairs of fixed points of iwip elements, a version of it was established in [AK08].

Corollary 5.3. *For every $B > 0$ there is a number $\kappa_3 = \kappa_3(B) > 0$ with the following property. Let $([\mu],[\nu])$ be a B-contracting pair, let γ be an axis for $([\mu],[\nu])$ and let $T \in \mathrm{Thick}_\epsilon(F_n)$.*

(1) *Let* $r < \inf\{d_L(T,\gamma(t)) \mid t \in \mathbb{R}\}$ *and let* $K = \{S \in cv_0(F_n) \mid d_L(T,S) \leq r\}$. *Then the diameter of* $\Pi_\gamma(K)$ *with respect to the Lipschitz metric does not exceed* κ_3.
(2) *Let* $r < \inf\{d(T,\gamma(t)) \mid t \in \mathbb{R}\}$ *and let* $K' = \{S \in \mathrm{Thick}_\epsilon(F_n) \mid d(T,S) \leq r\}$. *Then the diameter of* $\Pi_\gamma(K)$ *with respect to the Lipschitz metric does not exceed* κ_3.

Proof. The first part of the corollary follows from the first part of Proposition 5.2 together with the fact that $d_L(T,\Pi(T)) \geq \inf_t d_L(T,\gamma(t))$. The second part follows in the same way. □

As in Lemma 3.5 of [BFu09], we obtain as a corollary

Corollary 5.4. *For all* $B > 0, D \geq 0$ *there is a number* $\kappa_4 = \kappa_4(B,D) > 0$ *with the following property. Let* $\gamma : \mathbb{R} \to \mathrm{Thick}_\epsilon(F_n)$ *be an axis of a* B*-contracting pair. Let* $S \in \gamma(\mathbb{R}), T \in \mathrm{Thick}_\epsilon(F_n)$ *and let* $\rho : [0,m] \to \mathrm{Thick}_\epsilon(F_n)$ *be a* D*-coarse geodesic for the Lipschitz metric connecting* T *to* S*. Then* ρ *passes through the* κ_4*-neighborhood of* $\Pi_\gamma(T)$.

Proof. Let $\kappa_2 > 0$ be as in Proposition 3.4. Observe first that there is a number $\chi > \kappa_2$ with the following property. Let $\gamma : \mathbb{R} \to \mathrm{Thick}_\epsilon(F_n)$ be an axis of a B-contracting pair $([\mu],[\nu])$ and let $T \in \mathrm{Thick}_\epsilon(F_n)$. Let $\Pi = \Pi_\gamma$; then

$$d(T,\gamma(s)) \geq d(T,\Pi(T)) + d(\Pi(T),\gamma(s)) - \chi \tag{16}$$

for every $s \in \gamma(\mathbb{R})$. Namely, if $d(\Pi_\gamma(0),\gamma(s)) \geq \kappa_2 + 12\log B + 2\log 2$ then this is immediate from Proposition 3.4 and Lemma 5.1. Otherwise it is simply the triangle inequality.

Let $\rho : [0,m] \to \mathrm{Thick}_\epsilon(F_n)$ be a D-coarse geodesic connecting a point $T = \rho(0) \in \mathrm{Thick}_\epsilon(F_n)$ to a point $\rho(m) \in \gamma(\mathbb{R})$. Let $k > 0$ be the supremum of all numbers $p > 0$ such that the closed $p+1$-ball about T for the metric d is disjoint from $\gamma(\mathbb{R})$. We may assume that $k > 0$. Since ρ is a D-coarse geodesic, we have

$$k + 3\chi \leq d(\rho(0),\rho(k+D+3\chi)) \leq k + 2D + 3\chi.$$

Moreover, the third part of Proposition 3.4 shows that $d(\rho(0),\Pi\rho(0)) \leq k + \chi$.

If $d(\Pi\rho(0),\Pi\rho(k+D+3\chi)) \leq \chi$ then Proposition 5.2 shows that

$$\begin{aligned} d(\rho(0),\rho(m)) &\geq d(\rho(0),\rho(k+D+3\chi)) + d(\rho(k+D+3\chi),\rho(m)) - D \\ &\geq k + 3\chi + d(\rho(k+D+3\chi),\Pi\rho(k+D+3\chi)) \\ &\quad + d(\Pi\rho(0)),\rho(m)) - 2\chi \\ &\geq d(\rho(0),\Pi\rho(0)) + d(\Pi\rho(0),\rho(m)) + \chi \end{aligned}$$

which is impossible.

However, if $d(\Pi(\rho(0)),\Pi(\rho(k+D+4\kappa_2))\geq\chi$ then the triangle inequality, the second part of Proposition 5.2 and the estimate (16) show that

$$\begin{aligned}d(\rho(0),\rho(m)) &\geq d(\rho(0),\rho(k+D+3\chi))+d(\rho(k+D+3\chi),\rho(m))-D\\ &\geq d(\rho(0),\Pi\rho(0))+d(\Pi\rho(0),\Pi\rho(k+D+3\chi))\\ &+2d(\Pi\rho(k+D+3\chi),\rho(k+D+3\chi))+d(\Pi\rho(k+D+3\chi),\rho(m))-2\chi-D\end{aligned}$$

which is only possible if $d(\Pi\rho(k+D+4\kappa_2),\rho(k+D+4\kappa_2)$ is uniformly bounded. Using once more the estimate (16), then $d(\Pi\rho(0),\Pi\rho(k+D+4\kappa_2)$ is uniformly bounded as well. The corollary follows. ☐

Corollary 5.4 can be used to construct from distinct B-contracting pairs new pairs which are C-contracting for some $C>B$. To this end, recall that for a B-contracting pair $([\mu],[\nu])$, for representatives μ,ν of $[\mu],[\nu]$ and for an axis γ of $([\mu],[\nu])$ such that $\gamma(0)\in\mathrm{Min}_\epsilon(\mu+\nu)$ as in the definition of an axis, the projection $\Pi_\gamma: cv_0(F_n)\to\gamma(\mathbb{R})$ maps $T\in\mathrm{Bal}(\mu,\nu)$ to $\gamma(0)$.

If $[\mu]\in\mathcal{UML}$ then $\langle[T],[\mu]\rangle=0$ for exactly one projective tree $[T]\in\partial\mathrm{CV}(F_n)$ (the tree $\omega([\mu])$ with the notations from Lemma 2.15). As a consequence, if $[\mu],[\nu]\in\mathcal{UML}$ then the projection Π_γ extends to a projection $cv_0(F_n)\cup\partial\mathrm{CV}(F_n)-\{\omega([\mu]),\omega([\nu])\}\to\gamma(\mathbb{R})$.

Lemma 5.5. *For every $B>0$ there is a number $C(B)>0$ with the following property. Let $([\mu],[\nu])\in\mathcal{UML}$ be a B-contracting pair with axis γ. Let $[\zeta]\in\mathcal{UML}-\{[\mu],[\nu]\}$ and let $T=\Pi_\gamma(\omega[\zeta])$. Let $\mu,\nu,\zeta\in\Lambda(T)$ be representatives of $[\mu],[\nu],[\zeta]$. Then for $s>B$ the Hausdorff distance between $\mathrm{Min}_\epsilon(e^s\mu+e^{-s}\nu)$ and $\mathrm{Min}_\epsilon(e^s\mu+e^{-s}\zeta)$ is at most $C(B)$.*

Proof. Let $\Pi=\Pi_\gamma$ and let $T=\Pi(\omega[\zeta])$ be as in the lemma. Let $\mu,\nu,\zeta\in\Lambda(T)$ be representatives of $[\mu],[\nu],[\zeta]$. Note that $T\in\mathrm{Bal}(\mu,\nu)$. The subset of $\partial\mathrm{CV}(F_n)$ of all projective trees which are contained in the closure of $[\mathrm{Thick}_\epsilon(F_n)]$ is closed, non-empty and $\mathrm{Out}(F_n)$-invariant. Corollary 2.17 then shows that this set contains $\mathcal{UT}$. In particular, the projective tree $\omega([\zeta])\in\mathrm{Bal}(\mu,\nu)$ is a limit of a sequence $(T_i)\in\mathrm{Thick}_\epsilon(F_n)\cap\mathrm{Bal}(\mu,\nu)$. Lemma 2.20 then shows that $[\zeta]$ is a limit of a sequence of projective measured laminations which are induced by basic primitive conjugacy classes for the trees T_i.

Let $\zeta\in\Lambda(T)$ be a representative of $[\zeta]$. The third property in the definition of a B-contracting pair, applied to $([\mu],[\nu])$, and continuity then show that for every tree $S\in\Sigma(T)\cap\bigcup_{t\in(B,\infty)}\mathrm{Bal}(e^t\mu,e^{-t}\nu)$ we have $\langle S,\zeta\rangle\geq 1/B$. This implies that

$$\log\langle S,\zeta\rangle\geq d_L(T,S)-\log B \text{ for } S\in cv_0(F_n)\cap\bigcup_{t\in(B,\infty)}\mathrm{Bal}(e^t\mu,e^{-t}\nu). \tag{17}$$

Together with Lemma 4.5 we conclude that there is a number $\beta_0=\beta_0(B)>0$ such that

$$|\log\langle S,\zeta\rangle-\log\langle S,\nu\rangle|\leq\beta_0$$

for all $s>B$ and every $S\in\mathrm{Min}(e^s\mu+e^{-s}\nu)$.

As a consequence, there is a number $\beta_1 = \beta_1(B) > 0$ such that for $S \in \mathrm{Min}(e^s\mu + e^{-s}\nu)$ we have

$$\langle S, e^s\mu + e^{-s}\nu\rangle / \langle S, e^s\mu + e^{-s}\zeta\rangle \in [\beta_1^{-1}, \beta_1]. \tag{18}$$

On the other hand, using once more Lemma 4.5 and the definition of a B-contracting pair, there is a number $\beta_2 = \beta_2(B) > 0$ such that

$$\langle U, e^s\mu + e^{-s}\zeta\rangle / \langle S, e^s\mu + e^{-s}\nu\rangle > \beta_1^2$$

whenever $t \geq s$ and $U \in \mathrm{Bal}(e^t\mu, e^{-t}\zeta) \cap \mathrm{Thick}_\epsilon(F_n)$, $S \in \mathrm{Min}(e^s\mu + e^{-s}\nu)$ and $d_L(S, U) \geq \beta_2$.

Since $\mathrm{Out}(F_n)$ acts on $\mathrm{Thick}_\epsilon(F_n)$ cocompactly, there is a number $\beta_3 = \beta_3(B) > 0$ such that $d_L(U, Z) \geq \beta_2$ for all $U, Z \in \mathrm{Thick}_\epsilon(F_n)$ with $d(U, Z) \geq \beta_3$. This implies that $U \notin \mathrm{Min}(e^s\mu + e^{-s}\zeta)$ if $U \in \mathrm{Bal}(e^t\mu, e^{-t}\zeta) \cap \mathrm{Thick}_\epsilon(F_n)$ is such that $d(U, \mathrm{Min}(e^s\mu + e^{-s}\nu)) \geq \kappa_3$ where $\kappa_3 > 0$ only depends on B.

By the definition of a B-contracting pair the diameter of $\mathrm{Min}(e^s\mu + e^{-s}\nu)$ is uniformly bounded independent of s. The lemma follows. ☐

As a consequence we obtain

Corollary 5.6. *Let* $([\mu_1], [\nu_1]), ([\mu_2], [\nu_2]) \in \mathcal{UML}^2$ *be* B*-contracting pairs. If* $[\mu_1] \neq [\mu_2]$ *then* $([\mu_1], [\mu_2])$ *is* C*-contracting for some* $C > 0$.

Proof. Since the pair $([\mu_1], [\mu_2])$ is positive by assumption, for any representatives μ_1, μ_2 and all $s < t$ the set $\cup_{u=s}^t \mathrm{Min}_\epsilon(e^u\mu_1 + e^{-u}\mu_2)$ is compact.

Now let γ_i be an axis of $([\mu_i], [\nu_i])$ $(i = 1, 2)$ and let $T = \Pi_{\gamma_1}(\omega[\mu_2])$. Choose representatives $\mu_1, \nu_1, \mu_2 \in \Lambda(T)$ of $[\mu_1], [\nu_1], [\mu_2]$. By Lemma 5.5, for $s > B$ the Hausdorff distance between $\mathrm{Min}_\epsilon(e^s\mu_1 + e^{-s}\nu_1)$ and $\mathrm{Min}_\epsilon(e^s\mu_1 + e^{-s}\mu_2)$ is at most $C(B)$. Moreover, if $t > B$ and if $S \in \mathrm{Bal}(e^t\mu_1, e^{-t}\nu_1)$ then $S \in \mathrm{Bal}(e^s\mu_1, e^{-s}\mu_2)$ for some s so that $|s - t|$ is bounded by a constant only depending on B.

Apply this reasoning to the line γ_2 and $[\mu_1]$. If $S = \Pi_{\gamma_2}(\omega[\mu_1])$ and if s_0 is such that $S \in \mathrm{Min}_\epsilon(e^{s_0}\mu_2 + e^{-s_0}\nu_2)$ then we can compare balancing and projections for the half-line of γ_2 which begins at S and corresponds to $s \to \infty$.

Since the non-local property (3) in Definition 5.2 only depends on the position of balancing points relative to the minimal sets of a positive pair, the pair $([\mu_1], [\mu_2])$ satisfies the requirements Definition 5.2 for some $C > 0$. ☐

Remark 5.7. In general, the number $C > 0$ in Corollary 5.6 depends on $[\mu_1], [\mu_2]$ and not only on B. This can be seen as follows.

Let S be an oriented surface with connected boundary and fundamental group F_n. The Teichmüller space of S embeds quasi-isometrically into $cv_0(F_n)$. Let φ be a pseudo-Anosov mapping class; then the φ-invariant Teichmüller geodesic is coarsely an axis for φ viewed as in iwip element in $\mathrm{Out}(F_n)$. In particular, its pair of fixed points $([\mu], [\nu])$ in $\mathcal{PML}(F_n)$ is a B-contracting pair for some $B > 0$. Let α be a Dehn twist about a simple closed curve on S. Then for every $k > 0$, the pair $(\alpha^k[\mu], \alpha^k[\nu])$ is the pair of fixed points for $\alpha^k\varphi\alpha^{-k}$ and hence it is a B-contracting pair for the same B. However, as $k \to \infty$, the Teichmüller geodesic connecting $\alpha^k[\nu]$ to $[\mu]$ has longer and longer subsegments which are contained in the thin part of Teichmüller space. These segments, however, are compact. This implies that as $k \to \infty$, the smallest possible contraction constant C_k for the pair $([\mu], \alpha^k[\nu])$ tends to infinity.

6. Iwip subgroups of $\mathrm{Out}(F_n)$

In this section we use the results obtained so far to shed some light on subgroups of $\mathrm{Out}(F_n)$ which consist of iwip elements. We begin with an observation of Kapovich and Lustig [KL10a]. For its formulation, we call a pair (B_1, B_2) of disjoint closed subsets of $\mathrm{Thick}_\epsilon(F_n) \subset cv_0(F_n)$ *positive* if the following holds true. Let $K_i \subset \mathcal{PML}(F_n)$ be the closure of the set of all projective measured laminations which are induced by basic primitive conjugacy classes for trees in the set B_i. Then $([\mu_1], [\mu_2])$ is a positive pair for all $[\mu_i] \in K_i$ $(i = 1, 2)$.

The proof of the second part of the following lemma was communicated to me by Martin Lustig.

Lemma 6.1. *Let $(B_1, B_2) \subset \mathrm{Thick}_\epsilon(F_n)^2$ be a positive pair of closed disjoint sets. Let $\varphi \in \mathrm{Out}(F_n)$ be such that*

$$\varphi(\mathrm{Thick}_\epsilon(F_n) - B_2) \subset B_1 \text{ and } \varphi^{-1}(\mathrm{Thick}_\epsilon(F_n) - B_1) \subset B_2.$$

Then φ is iwip, with attracting fixed point in the closure of the projectivization $[B_1]$ of B_1 and repelling fixed point in the closure of the projectivization $[B_2]$ of B_2.

Proof. Let $\varphi \in \mathrm{Out}(F_n)$ be as in the lemma. Then $\varphi^k T \in B_1$ for every tree $T \in \mathrm{Thick}_\epsilon(F_n) - (B_1 \cup B_2)$ and all $k \geq 1$ and hence the subgroup of $\mathrm{Out}(F_n)$ generated by φ is infinite. Moreover, $\varphi(B_1)$ is contained in the interior of B_1, and we have $\varphi^{-1}(B_1) \cup B_2 = \mathrm{Thick}_\epsilon(F_n)$.

For $i = 1, 2$ let $K_i \subset \mathcal{PML}(F_n)$ be the closure of the set of all projective measured laminations which are induced by basic primitive conjugacy classes for trees in B_i. Then $\varphi^{-1}(K_1) \cup K_2$ is a closed non-empty subset of $\mathcal{PML}(F_n)$ containing all projective measured laminations which are induced by basic primitive conjugacy classes for trees in $\varphi^{-1}(B_1) \cup B_2 = \mathrm{Thick}_\epsilon(F_n)$. Now the closure of the set of all projective measured laminations which are induced by basic primitive conjugacy classes for trees in $\mathrm{Thick}_\epsilon(F_n)$ is a closed non-empty $\mathrm{Out}(F_n)$-invariant subset of $\mathcal{PML}(F_n)$ and hence $\mathcal{PML}(F_n) = \varphi^{-1}(K_1) \cup K_2$ by minimality [KL07]. This implies in particular that $\varphi(\mathcal{PML}(F_n) - K_2) \subset K_1$ and similarly $\varphi^{-1}(\mathcal{PML}(F_n) - K_1) \subset K_2$.

As a consequence, if we define

$$A_1 = \cap_i \varphi^i K_1,\ A_2 = \cap_i \varphi^{-i} K_2$$

then A_i is the intersection of a nested sequence of non-empty compact sets and hence $A_i \neq \emptyset$. Moreover, every periodic point for the action of φ on $\mathcal{PML}(F_n)$ is contained in $A_1 \cup A_2$. Since both sets A_i are compact and φ-invariant, each of the sets A_1, A_2 contains at least one periodic point. If $[\nu_+] \in A_1, [\nu_-] \in A_2$ are such periodic points then $([\nu_+], [\nu_-]) \in \mathcal{PML}(F_n)^2$ is a positive pair by definition of a positive pair (B_1, B_2).

By replacing φ by φ^k for some $k \geq 1$ we may assume that $[\nu_+], [\nu_-]$ are fixed points for φ. Let $\nu_+, \nu_- \in \mathcal{ML}(F_n)$ be representatives of $[\nu_+], [\nu_-]$. We claim that up to replacing φ by φ^{-1} there are numbers $\lambda_+ > 1, \lambda_- > 1$ such that $\varphi(\nu_+) = \lambda_+\nu_+$ and $\varphi^{-1}(\nu_-) = \lambda_-\nu_-$.

Namely, if up to exchanging φ and φ^{-1} we have $\varphi(\nu_+) = \lambda_+\nu_+$, $\varphi(\nu_-) = \lambda_-\nu_-$ for some $\lambda_+ \leq 1, \lambda_- \leq 1$ then $\varphi(\nu_+ + \nu_-) \leq \nu_+ + \nu_-$ (as functions on $cv_0(F_n)$). Thus if $T \in \text{Thick}_\epsilon(F_n)$ is arbitrary and if ν_+, ν_- are normalized in such a way that $\nu_+ \in \Lambda(T), \nu_- \in \Lambda(T)$ then for every $k \geq 0$ we have

$$\langle \varphi^{-k}T, \nu_+ + \nu_- \rangle = \langle T, \varphi^k(\nu_+ + \nu_-) \rangle \leq 2.$$

However, by Lemma 3.2, the function $\langle \cdot, \nu_+ + \nu_- \rangle$ on $\text{Thick}_\epsilon(F_n)$ is proper which contradicts the fact that $\text{Out}(F_n)$ acts property discontinuously on $\text{Thick}_\epsilon(F_n)$ and that the order of φ is infinite.

Thus up to replacing φ by φ^{-1} we may assume the following. If $[\nu_+] \in A_1$ (or $[\nu_-] \in A_2$) is any fixed point for φ and if ν_+ (or ν_-) is a representative of $[\nu_+]$ (or $[\nu_-]$) then $\varphi\nu_\pm = \lambda_\pm\nu_\pm$ where $\lambda_+ > 1, \lambda_- < 1$. Recall that every fixed point for the action of φ on $\mathcal{PML}$ is contained in $A_1 \cup A_2$.

Now assume to the contrary that φ has a power which is reducible. For simplicity, assume that φ is reducible itself. Then up to conjugation, φ preserves a proper free factor U of F_n.

We follow the proof of Proposition 6.1 of [KL10a]. Namely, choose a train track representative for φ. Assume first that there is more than one stratum. We may assume that the lowest stratum represents a proper free factor of F_n.

If the transition matrix for the restriction of φ to this stratum is irreducible then by the argument in the proof of Proposition 6.1 of [KL10a], the map φ admits two invariant measured laminations ν_-, ν_+ carried by the lowest stratum, one contracting and one expanding. Moreover, there is an invariant projective tree $[T]$ obtained from the top stratum, with all lower strata collapsed to become elliptic. By construction, the intersection number between a representative T of $[T]$ and ν_-, ν_+ vanishes.

In the case that the transition matrix of the bottom stratum is the identity there is a primitive element which is fixed which violates the assumption on φ.

If there is a single stratum then there are again two cases. In the first case, the expanding lamination is contained in a free factor in which case the argument in the proof of Proposition 6.1 of [KL10a] applies. Otherwise there is an invariant proper free factor which is elliptic in the tree. However, as before, in this case there is a primitive element which is fixed.

Together we obtain a contradiction to the above discussion. □

As in Section 2, let M be the closure of $\mathcal{UT}$ in $\partial\text{CV}(F_n)$. By Lemma 2.17, the action of $\text{Out}(F_n)$ on M is minimal. Since for sufficiently small $\epsilon > 0$ the set

$$\partial[\text{Thick}_\epsilon(F_n)] = \overline{[\text{Thick}_\epsilon(F_n)]} - [\text{Thick}_\epsilon(F_n)]$$

is a closed non-empty subset of $\partial\mathrm{CV}(F_n)$ which is invariant under the action of $\mathrm{Out}(F_n)$, we have $M \subset \overline{[\mathrm{Thick}_\epsilon(F_n)]}$. As in [H09] we can use Lemma 6.1 to show

Corollary 6.2. *The set of pairs of fixed points of iwip elements is dense in $M \times M$.*

Proof. Let $V_1, V_2 \subset M$ be open disjoint sets. We have to show that there is an iwip element with attracting fixed point in V_1 and repelling fixed point in V_2.

Since $\mathcal{UT}$ is dense in M, by making V_1, V_2 smaller we may assume that if $([\mu],[\nu])$ is any pair of projective measured laminations so that $[\mu]$ is supported in the zero lamination of a tree in V_1 and $[\nu]$ is supported in the zero lamination of a tree in V_2 then the pair $([\mu],[\nu])$ is positive. By continuity of the length pairing and by Lemma 2.20 we may moreover assume that there are compact disjoint neighborhoods $B_1, B_2 \subset \overline{[\mathrm{Thick}_\epsilon(F_n)]}$ of V_1, V_2 with the following property. The sets $B_1 \cap [\mathrm{Thick}_\epsilon(F_n)]$, $B_2 \cap [\mathrm{Thick}_\epsilon(F_n)]$ are projectivizations of sets $\tilde{B}_i \subset \mathrm{Thick}_\epsilon(F_n)$, and $(\tilde{B}_1, \tilde{B}_2)$ is a positive pair of closed disjoint sets as defined in the beginning of this section. We also may assume that there is an iwip-element φ whose fixed points a, b are contained in $M - B_1 - B_2$.

Since $V_1 \subset M$ is open and the action of $\mathrm{Out}(F_n)$ on M is minimal and preserves the set of fixed points of iwip elements, there is an iwip element $u \in \mathrm{Out}(F_n)$ with attracting fixed point in V_1. The stabilizer in $\mathrm{Out}(F_n)$ of a fixed point of an iwip element is virtually cyclic [BFH97] and therefore we may assume that the repelling fixed point of u is distinct from a, b. Now u acts with north-south dynamics on $\partial CV(F_n)$ and hence up to perhaps replacing u by a nontrivial power we may assume that $u\{a,b\} \subset V_1$. Then $v = u\varphi u^{-1}$ is an iwip element with both fixed points in V_1. Similarly, there is an iwip element w with both fixed points in V_2.

Via replacing v, w by sufficiently high powers we may assume that $v(M - V_1) \subset V_1, v^{-1}(M - V_1) \subset V_1$ and $w(M - V_2) \subset V_2, w^{-1}(M - V_2) \subset V_2$. Then $wv(M - V_1) \subset V_2, v^{-1}w^{-1}(M - V_2) \subset V_1$. Moreover, by perhaps replacing v, w by a suitable power we may assume that the assumptions in Lemma 6.1 are satisfied for wv. Then wv is an iwip whose pair of fixed points is contained in $V_1 \times V_2$. □

We also obtain information on subgroups Γ of $\mathrm{Out}(F_n)$ which contain at least one iwip element. For this call iwip elements $\alpha, \beta \in \mathrm{Out}(F_n)$ *independent* if the fixed point sets for the action of α, β on $\partial\mathrm{CV}(F_n)$ do not coincide. By Proposition 2.16 of [BFH97], the stabilizer in $\mathrm{Out}(F_n)$ of a fixed point of an iwip element is virtually cyclic and hence this means that the fixed point sets of α, β are in fact disjoint.

Proposition 6.3. *Let $\Gamma < \mathrm{Out}(F_n)$ be a subgroup which contains an iwip element. If Γ is not virtually cyclic then there are two independent iwip elements $\alpha, \beta \in \Gamma$ with the following properties.*

1. *The subgroup G of Γ generated by α, β is free and consists of iwip elements.*
2. *There are infinitely many elements $u_i \in G$ $(i > 0)$ with fixed points $a_i, b_i \in \mathcal{UT}$ such that for all i the $\mathrm{Out}(F_n)$-orbit of $(a_i, b_i) \in \mathcal{UT} \times \mathcal{UT} - \Delta$ is distinct from the orbit of (b_j, a_j) $(j > 0)$ or (a_j, b_j) $(j \neq i)$.*

Proof. Let $\Gamma < \mathrm{Out}(F_n)$ be a subgroup which contains an iwip element α. Let $[T_+], [T_-]$ be the fixed points of α in $\partial\mathrm{CV}(F_n)$. If the set $\{[T_+], [T_-]\}$ is invariant under the action of Γ then by Theorem 2.14 of [BFH97], the group Γ is virtually cyclic.

Thus we may assume that there is some $\gamma \in \Gamma$ with $\gamma[T_+] \in \mathcal{UT} - \{[T_+], [T_-]\}$. Then $\gamma[T_+], \gamma[T_-]$ are the fixed points of the iwip element $\zeta = \gamma \circ \alpha \circ \gamma^{-1}$. By Proposition 2.16 of [BFH97], the fixed point sets of α, ζ in $\partial\mathrm{CV}(F_n)$ are disjoint, and α, ζ act with north-south dynamics on the compact space $M \subset \partial\mathrm{CV}(F_n)$.

The usual ping-pong lemma, applied to the action of α, ζ on M, implies that for sufficiently large $k > 0, \ell > 0$ the subgroup G of Γ generated by α^k, ζ^ℓ is free. Lemma 6.1 shows that we may assume that this group consists of iwip automorphisms. In particular, each non-trivial element of G acts with north-south-dynamics on M, with fixed points contained in $\mathcal{UT}$. In the case that α, ζ are non-geometric this is a consequence of the main result of [KL10a].

To show the second part of the proposition we have to find infinitely many elements in G which are mutually not conjugate in $\mathrm{Out}(F_n)$ and not conjugate to their inverses. For this we follow the argument in the proof of Proposition 5.7 of [H09]. Namely, let $\Delta \subset M \times M$ be the diagonal. Let $(a_+, a_-) \in M \times M - \Delta$ be the pair of fixed points of $\alpha \in G$. The $\mathrm{Out}(F_n)$-orbit of (a_+, a_-) is a closed subset of $M \times M - \Delta$ (Theorem 5.3 of [BFH97]). Therefore by the ping-pong construction, there is an independent iwip element $\beta \in G$ which is not conjugate to α in $\mathrm{Out}(F_n)$.

Let $(a_+, a_-), (b_+, b_-) \in M \times M - \Delta$ be the pairs of fixed points for the action of α, β on $M \subset \partial\mathrm{CV}(F_n)$. Since α, β are not conjugate in $\mathrm{Out}(F_n)$, the $\mathrm{Out}(F_n)$-orbits of (a_+, a_-) and (b_+, b_-) are distinct. This implies that there are open neighborhoods U_+, U_- of a_+, a_- and V_+, V_- of b_+, b_- such that the $\mathrm{Out}(F_n)$-orbit of (a_+, a_-) does not intersect $V_+ \times V_-$ and that the $\mathrm{Out}(F_n)$-orbit of (b_+, b_-) does not intersect $U_+ \times U_-$. Via replacing α, β by suitable powers we may assume that

$$\alpha(M - \overline{U}_-) \subset U_+,\ \alpha^{-1}(M - \overline{U}_+) \subset U_-,\ \beta(M - \overline{V}_-) \subset V_+ \text{ and } \beta^{-1}(M - \overline{V}_+) \subset V_-.$$

For numbers $n, m, k, \ell > 2$ consider the element

$$f = f_{nmk\ell} = \alpha^n \beta^m \alpha^k \beta^{-\ell} \in G.$$

It satisfies $f(\overline{U}_+) \subset U_+, f^{-1}(\overline{V}_+) \subset V_+$ and hence the attracting fixed point of f is contained in U_+ and its repelling fixed point is contained in V_+.

Since $n > 2$, the conjugate $f_1 = \beta^{-1} f \beta$ satisfies $f_1(\overline{U}_+) \subset U_+$ and $f_1^{-1}(\overline{U}_-) \subset U_-$, i.e. its attracting fixed point is contained in U_+ and its repelling fixed point is contained in U_-. Furthermore, since $m > 2$, its conjugate $f_2 = \beta^{-1}\alpha^{-n} f \alpha^n \beta$ has its attracting fixed point in V_+ and its repelling fixed point in V_-, and its conjugate $f_3 = \beta^{-1} f \beta$ has its attracting fixed point in V_- and its repelling fixed point in V_+.

As a consequence, f is conjugate to both an element with fixed points in $U_+ \times U_-$ as well as to an element with fixed points in $V_+ \times V_-$. This implies that f is not conjugate to either α or β. Moreover, since α and β can not both be conjugate to β^{-1}, by eventually adjusting the size of V_+, V_- we may assume that f is not conjugate to β^{-1}.

We claim that via perhaps increasing the values of n, ℓ we can achieve that $f_{nmk\ell}$ is not conjugate to f^{-1}. Namely, as $n \to \infty$, the fixed points of the conjugate $\alpha^{-n} f_{(2n)mkl} \alpha^n = \alpha^n \beta^m \alpha^k \beta^{-\ell} \alpha^n$ of $f_{(2n)mkl}$ converge to the fixed points of α. Similarly, the fixed points of the conjugate $\beta^{-\ell} f^{-1}_{nmk(2l)} \beta^\ell = \beta^\ell \alpha^{-k} \beta^{-m} \alpha^{-n} \beta^\ell$ of $f^{-1}_{nmk(2\ell)}$ converge as $\ell \to \infty$ to the fixed points of β. Thus after possibly conjugating with α, β, if $f_{nmk\ell}$ is conjugate in $\mathrm{Out}(F_n)$ to $f^{-1}_{nmk\ell}$ for all n, ℓ then there is a sequence of pairwise distinct elements $g_i \in \mathrm{Out}(F_n)$ which map a fixed compact subset K of $\mathrm{Thick}_\epsilon(F_n)$ into a fixed compact subset W of $\mathrm{Thick}_\epsilon(F_n)$ and such that $g_i(a, b) \to (b_+, b_-)$. Since $\mathrm{Out}(F_n)$ acts properly discontinuously on $\mathrm{Thick}_\epsilon(F_n)$ this is impossible.

Inductively we can construct in this way a sequence of elements of G with the properties stated in the second part of the proposition. □

Remark 6.4. In analogy to [FM02] we can define a *convex cocompact subgroup* of $\mathrm{Out}(F_n)$ to be a hyperbolic group $\Gamma < \mathrm{Out}(F_n)$ with the following property. There is a number $B > 0$ and there is a Γ-equivariant embedding $\rho : \partial\Gamma \to \mathcal{UML}$ such that for all $\xi \neq \zeta \in \partial\Gamma$ the pair $(\rho(\xi), \rho(\zeta))$ is B-contracting.

Using the results in this note, it is not hard to see that for any two independent iwip elements $\alpha, \beta \in \mathrm{Out}(F_n)$ and for all sufficiently large $k > 0$, the subgroup of $\mathrm{Out}(F_n)$ generated by α^k, β^k is free, consists of iwip elements and is convex cocompact in this sense. However we do not pursue the development of a theory of convex cocompact subgroups of $\mathrm{Out}(F_n)$ here and propose this as an open problem.

About four years after this work was carried out, Bestvina and Feighn [BF11] and Handel and Mosher [HM11] showed that there are two analogues of a curve graph for $\mathrm{Out}(F_n)$ which are hyperbolic. Thus as in the case of mapping class groups, one can define a convex cocompact subgroup of $\mathrm{Out}(F_n)$ to be a subgroup so that the orbit map on one of these graphs is a quasi-isometry. I think it is interesting to explore how this is related to the definition suggested above. We conjecture that our definition is equivalent to a quasi-isometric orbit map for the free factor graph considered in [BF11].

Example 6.5. The mapping class group $\mathrm{Mod}(S)$ of a surface of genus $g \geq 1$ with one puncture is the subgroup of $\mathrm{Out}(F_{2g})$ preserving the conjugacy class of the puncture. If α, β are two independent pseudo-Anosov elements in $\mathrm{Mod}(S)$ then there is some $k > 0$ such that the subgroup Γ of $\mathrm{Out}(F_{2g})$ generated by α^k, β^k satisfies the assumptions in Proposition 6.3. Note that we can also assume that Γ is a Schottky group in $\mathrm{Mod}(S)$ in the sense of [FM02].

7. Second bounded cohomology

This section is devoted to the proof of the theorem from the introduction. We continue to use the assumptions and notations from Sections 2-6. We use the construction in Section 2 and Section 6 of [H11]. To this end we first formulate a general sufficient condition for infinite dimensional second bounded cohomology $H_b^2(\Gamma, \ell^p(\Gamma))$ for a discrete group Γ of isometries of a proper locally path connected metric space. This condition was established in [H11] for groups acting properly on CAT(0)-spaces. However, the CAT(0)-property is never used. The statement in the generality needed for the main application is as follows.

Theorem 7.1. *Let X be a proper locally path connected metric space and let Γ be a countable group of isometries acting properly discontinuously on X. Assume that the following conditions are satsfied.*

1. *X admits a compactification by adding a boundary ∂X. The isometric action of Γ extends to an action on ∂X by homeomorphisms.*
2. *There is a free subgroup G of Γ with two generators so that each element $e \neq g \in G$ acts on ∂X with north-south dynamics. The stabilizer of a pair of fixed points in ∂X for each element in G is virtually cyclic.*
3. *There is some $g \in G$ so that the pair $(a, b) \in \partial X \times \partial X$ of fixed points of g can be connected by a g-invariant B-contracting coarse geodesic $\gamma : \mathbb{R} \to X$, i.e. such that $\gamma(t) \to a$ $(t \to \infty)$ and $\gamma(t) \to b$ $(t \to -\infty)$. The Hausdorff distance between any two such coarse geodesics is at most B.*
4. *There is some $h \neq g \in G$ and there is a fixed point $b \in \partial X$ for g such that the stabilizer in Γ of the pair (b, hb) is trivial.*

Then $H_b^2(\Gamma, \ell^p(\Gamma))$ is infinite dimensional for every $p \in (1, \infty)$.

Passing from geodesics in [H11] to coarse geodesics as needed for the application to $\mathrm{Out}(F_n)$ uses Corollary 5.4 which is an immediate consequence of the B-contraction property in Proposition 3.4, but does not use any property specific to the situation at hand. Moreover, suitable versions of Lemma 5.5 and Corollary 5.6 are also used whose general version can be established with the arguments given in Section 4.

Now let $\Gamma < \mathrm{Out}(F_n)$ be a subgroup which is not virtually abelian and contains an iwip element. Our goal is to apply Theorem 7.1 to the action of Γ on $X = \mathrm{Thick}_\epsilon(F_n)$. We define ∂X to be the complement of the projectivization $[X]$ of X in the closure of $[\mathrm{Thick}_\epsilon(F_n)]$ in $\mathrm{CV}(F_n) \cup \partial\mathrm{CV}(F_n)$. The group Γ acts on X properly discontinuously, and this action extends to an action on ∂X. The existence of a subgroup $G < \Gamma$ with properties (2) and (3) above was shown in Proposition 6.3 and Proposition 5.2. The second part of property 3) follows from Lemma 5.5.

Let $g \in G$ be an element with attracting fixed point $[\mu] \in \mathcal{UML}$, repelling fixed point $[\nu] \in \mathcal{UML}$ and such that the $\mathrm{Out}(F_n)$-orbit of $([\mu], [\nu])$ is distinct from the $\mathrm{Out}(F_n)$-orbit of $([\nu], [\mu])$. The existence of such an element is guaranteed by Proposition 6.3 and Lemma 2.15.

Lemma 7.2. *There is some $h \in G$ such that the stabilizer of the pair $(h[\nu], [\nu])$ in $\mathrm{Out}(F_n)$ is trivial.*

Proof. Using the notation from Lemma 2.15, by Theorem 2.14 of [BFH97] the stabilizer of the projective tree $\omega([\nu]) \in \mathcal{UT}$ is virtually cyclic. Since G is not virtually cyclic, there is some $h \in G$ such that the stabilizer in $\mathrm{Out}(F_n)$ of the pair $(h\omega([\nu]), \omega([\nu]))$ is trivial. Together with Lemma 2.15, this shows the lemma. □

As a consequence, the action of Γ on $\mathrm{Thick}_\epsilon(F_n)$ has all the properties stated in Theorem 7.1 and hence the main theorem from the introduction holds true for Γ.

Nonetheless we discuss a few more details of the proof of Theorem 7.1 for subgroups Γ of $\mathrm{Out}(F_n)$. Namely, let $\mathcal{A}([\nu])$ be the union of the ordered pairs of distinct points in $\Gamma[\nu]$ with the Γ-translates of $([\mu],[\nu]),([\nu],[\mu])$. The group Γ naturally acts on $\mathcal{A}([\nu])$ from the left. Moreover, $\mathcal{A}([\nu])$ is contained in $\mathcal{UML}^2$ and hence it can be identified with a subset of the set of pairs of points in the boundary $\partial\mathrm{CV}(F_n)$ of Outer space. For $X = \mathrm{Thick}_\epsilon(F_n)$, the group Γ acts on $\mathcal{A}([\nu]) \times X$. For a suitable choice of a point $T \in X$, the group Γ acts freely on $\mathcal{A}([\nu]) \times \Gamma T$. In particular, a Γ-orbit under this action can be identified with Γ.

The goal is now to construct a distance δ on $V = \mathcal{A}([\nu]) \times \Gamma T$ which is invariant under the action of Γ and under the action of the flip ι exchanging the two components of the pair in $\mathcal{A}([\nu])$. The construction is done in such a way that functions which are anti-invariant under ι and Hölder continuous with respect to this distance give rise to bounded $\ell^p(\Gamma)$-valued two-cocycles for Γ, i.e. Γ-invariant bounded maps $\Gamma \times \Gamma \times \Gamma \to \ell^p(\Gamma)$ which satisfy the cocycle equation. For specific choices of these maps, these cocycles are then shown to define nontrivial bounded cohomology classes.

If there is a number $B > 0$ such that for all $g, u \in \Gamma$ with $g[\nu] \neq u[\nu]$ the pair $(g[\nu], u[\nu])$ is B-contracting then coarse geodesic triangles with endpoints in $G[\nu] \subset \mathcal{UML}$ are uniformly thin and the construction of a distance on $\mathcal{A}([\nu]) \times X$ parallels the construction of the family of Gromov distances on the boundary of a hyperbolic geodesic metric space. However, in general the pair $(g[\nu], u[\nu])$ is C-contracting for a number $C > 0$ depending on the pair, and only segments of an axis near is endpoints is B'-contracting for a number B' which is controlled by B. The main idea in [H11] is to truncate the coarse geodesics to achieve our goal.

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