

LINES OF MINIMA IN OUTER SPACE

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ABSTRACT. We define lines of minima in the thick part of Outer space for the free group F_n with $n \geq 3$ generators. We show that these lines of minima are contracting for the Lipschitz metric. Every fully irreducible outer automorphism of F_n defines such a line of minima. Now let Γ be a subgroup of the outer automorphism group of F_n which is not virtually abelian. We obtain that if Γ contains at least one fully irreducible element then for every $p \in (1, \infty)$ the second bounded cohomology group $H_b^2(\Gamma, \ell^p(\Gamma))$ is infinite dimensional.

1. INTRODUCTION

There are many resemblances between the extended mapping class group of a closed oriented surface S , i.e the outer automorphism group of the fundamental group of S , and the outer automorphism group $\text{Out}(F_n)$ of a free group of F_n with $n \geq 2$ generators. Most notably, $\text{Out}(F_2)$ is just the *extended mapping class group* $GL(2, \mathbb{Z})$ of a torus. However, while in recent years the attempt to understand the mapping class group via the geometry of spaces on which it acts lead to a considerable gain of knowledge of the mapping class group, so far this approach has not been carried out successfully for $\text{Out}(F_n)$.

The group $\text{Out}(F_n)$ acts properly discontinuously on *Outer space* $\text{CV}(F_n)$. This space consists of metric graphs of volume one with fundamental group F_n and can be viewed as an equivalent of Teichmüller space for a closed surface S of higher genus. Teichmüller space admits several natural and quite well understood metrics which are invariant under the action of the extended mapping class group. The best known such metrics are the *Teichmüller metric* and the *Weil-Petersson metric*. However, up to date there is no good geometric theory of Outer space. Only very recently Francaviglia and Martino [FM08] looked in a systematic way at a natural $\text{Out}(F_n)$ -invariant metric on Outer space and investigated some of its properties. This metric is the symmetrization of a non-symmetric geodesic metric, the so-called *Lipschitz metric*. However, the symmetric metric is not geodesic, and its analog for Teichmüller space, the *Thurston metric*, also turned out to be harder to understand than the Teichmüller metric and the Weil-Petersson metric.

The viewpoint we take in this note is motivated by a slightly different approach to the geometry of Teichmüller space. Namely, *lines of minima* in Teichmüller space for a closed surface S of higher genus were defined and investigated by Kerckhoff [Ke92]. As for geodesics for the Teichmüller metric, such a line of minima is

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determined by two projective measured geodesic laminations which jointly fill up S . A line of minima uniformly fellow-travels its corresponding Teichmüller geodesic provided that this Teichmüller geodesic entirely remains in the *thick* part of Teichmüller space. There is also a very good control in the thin part of Teichmüller space though the uniform fellow traveller property is violated [CRS08].

More precisely, for a number $\epsilon > 0$, the ϵ -thick part of Teichmüller space for a closed oriented surface S of higher genus is the set of all hyperbolic metrics on S whose systole (i.e. the length of a shortest closed geodesic) is at least ϵ . Teichmüller geodesics γ entirely contained in the ϵ -thick part of Teichmüller space $\mathcal{T}(S)$ and hence their corresponding lines of minima have a *uniform contraction property*: If B is a closed metric ball in $\mathcal{T}(S)$ for the Teichmüller metric or the Weil-Petersson metric which is disjoint from γ then the diameter of a shortest distance projection of B into γ is bounded from above by a constant only depending on ϵ [Mi96, BF09].

Our main goal is to define such lines of minima for Outer space. We show that these lines of minima define uniform coarse geodesics for the symmetrized Lipschitz metric, and these coarse geodesics have the uniform contraction property. We also observe that for every fully irreducible element $\varphi \in \text{Out}(F_n)$ there is such a line of minima which is φ -invariant. This recovers a recent result of Algom-Kfir [AK08] who showed that axes for fully irreducible elements as defined by Handel and Mosher [HM06] have the contraction property.

As an immediate application of our discussion and earlier results for isometry groups of proper CAT(0)-spaces [H08] we obtain the following

Corollary. *Let $\Gamma < \text{Out}(F_n)$ be a subgroup which is not virtually abelian and which contains at least one fully irreducible element. Then for every $p > 1$ the second bounded cohomology group $H_b^2(\Gamma, \ell^p(\Gamma))$ is infinite dimensional.*

Earlier Bestvina and Feighn [BF08] showed that a subgroup Γ as in the corollary has nontrivial second bounded cohomology with real coefficients. This is also immediate from our approach.

All constructions in this paper are equally valid for the action of the mapping class group on Teichmüller space. This leads for example to a new proof of the main result of [Mi96] avoiding completely the explicit use of Teichmüller theory. However, in this case our more abstract approach does not have any obvious advantages over the original arguments.

2. MEASURED LAMINATIONS AND TREES

In this section we introduce currents, trees and measured laminations for the free group F_n of rank $n \geq 3$. We single out an $\text{Out}(F_n)$ -invariant subset of the space of measured laminations which is used for the construction of lines of minima in the later sections. We continue to use the notations from the introduction.

The Cayley graph of F_n with respect to a fixed standard symmetric generating set is a regular simplicial tree which can be compactified by adding the *Gromov boundary* ∂F_n . This boundary is a compact totally disconnected topological space

on which F_n acts as a group of homeomorphisms. It does not depend on the generating set up to F_n -equivariant homeomorphism. Every element $w \neq e \in F_n$ acts on ∂F_n with *north-south dynamics*. This means that w fixes precisely two points $a_+, a_- \in \partial F_n$, and for every neighborhood U_+ of a_+ , U_- of a_- there is some $k > 0$ such that $w^k(\partial F_n - U_-) \subset U_+$ and $w^{-k}(\partial F_n - U_+) \subset U_-$.

A *geodesic current* for F_n is a locally finite Borel measure on $\partial F_n \times \partial F_n - \Delta = \partial^2(F_n)$ (where Δ denotes the diagonal in $\partial F_n \times \partial F_n$) which is invariant under the action of F_n and under the flip $\iota : \partial^2(F_n) \rightarrow \partial^2(F_n)$ exchanging the two factors. The space $\text{Curr}(F_n)$ of all geodesic currents equipped with the weak*-topology is a locally compact topological space which can be projectivized to the compact space $\mathcal{P}\text{Curr}(F_n)$ of *projective currents*. The *outer automorphism group* $\text{Out}(F_n)$ of F_n naturally acts on $\text{Curr}(F_n)$ and on $\mathcal{P}\text{Curr}(F_n)$ as a group of homeomorphisms.

If $w \neq e \in F_n$ is any indecomposable element, then the set of all pairs of fixed points in ∂F_n of all elements of F_n which are conjugate to w is a discrete F_n -invariant flip invariant subset of $\partial F_n \times \partial F_n - \Delta$. Thus the sum of the Dirac measures supported at these points defines a geodesic current which we call *dual* to the indecomposable conjugacy class. We define a *weighted dual current* to be a geodesic current which is obtained by multiplying a geodesic current dual to an indecomposable conjugacy class by a positive weight. The set of weighted dual currents is invariant under the action of $\text{Out}(F_n)$.

An element $w \neq e \in F_n$ is *primitive* if it belongs to some basis, i.e. if there is a decomposition of F_n into a free product of the form $\langle w \rangle * H$ where $\langle w \rangle$ is the infinite cyclic subgroup of F_n generated by w and where H is a free subgroup of F_n . A conjugacy class in F_n is *primitive* if one (and hence each) of its elements is primitive. The set of primitive elements is invariant under the action of the full automorphism group $\text{Aut}(F_n)$ of F_n and hence $\text{Out}(F_n)$ naturally acts on the set of all primitive conjugacy classes.

Definition 2.1. The space $\mathcal{ML}(F_n)$ of *measured laminations* is the closure in $\text{Curr}(F_n)$ of the set of all currents which are weighted duals of primitive conjugacy classes.

The projectivization $\mathcal{PML}(F_n)$ of $\mathcal{ML}(F_n)$, equipped with the weak* topology, is compact and invariant under the action of $\text{Out}(F_n)$. Theorem B of [KL07] shows that $\mathcal{PML}(F_n)$ is the unique smallest non-empty closed $\text{Out}(F_n)$ -invariant subset of $\mathcal{P}\text{Curr}(F_n)$.

The following observation is due to Martin (Theorem 36 in [Ma95] which is attributed to Bestvina).

Lemma 2.2. *If $\varphi \in \text{Out}(F_n)$ is fully irreducible then φ acts on $\mathcal{PML}(F_n)$ with north-south dynamics.*

In particular, there is a unique pair of fixed points in $\mathcal{PML}(F_n)$ for the action of a fully irreducible element (in the sequel called an iwip element for short).

A closed F_n -invariant subset of $\partial^2(F_n)$ which is moreover invariant under the flip ι is called a *topological lamination*. The space \mathcal{L} of all topological laminations

can be equipped with the *Chabauty topology*. With respect to this topology, \mathcal{L} is compact. The group $\text{Out}(F_n)$ acts on \mathcal{L} as a group of homeomorphisms. Every nontrivial element $w \neq e \in F_n$ defines a point $[w] \in \mathcal{L}$ which is just the set of all pairs of fixed points of all elements of F_n which are conjugate to w . The set $[w]$ only depends on the conjugacy class of w . Topological laminations of this form are called *rational*. The support of a measured lamination is a topological lamination.

We call a (projective) geodesic current supported in a topological lamination L a (*projective*) *transverse measure* for L . If L is the topological lamination defined by the conjugacy class of a primitive element $w \in F_n$ then L admits a unique projective transverse measure. This measure is just the projective measured lamination dual to the conjugacy class of w . Note that if $L_i \rightarrow L$ in \mathcal{L} and if ζ_i is a projective geodesic current supported in L_i then up to passing to a subsequence, the projective geodesic currents ζ_i converge in $\mathcal{PCurr}(F_n)$ to a projective geodesic current supported in L (we refer to [CHL08b] for a more precise discussion).

Remark: The definition of a topological lamination does not correspond to the definition of a geodesic lamination for closed surfaces. The correct analog of a lamination in the surface case is a closed F_n -invariant subset of $\partial^2(F_n)$ which is contained in the Chabauty closure of those closed subsets of $\partial^2(F_n)$ which consists of pairs of fixed points of elements in a given primitive conjugacy class.

Let $cv(F_n)$ be the space of all minimal free and discrete isometric actions of F_n on \mathbb{R} -trees. Two such actions of F_n on \mathbb{R} -trees T and T' are identified in $cv(F_n)$ if there exists an F_n -equivariant isometry between T and T' . The quotient of a tree $T \in cv(F_n)$ under the action of F_n is a finite metric graph T/F_n without vertices of valence one or two whose fundamental group is marked isomorphic to F_n .

Let $CV(F_n)$ be the space of F_n -trees $T \in cv(F_n)$ whose quotient graphs T/F_n have volume one. Thus $CV(F_n)$ can naturally be identified with the projectivization of $cv(F_n)$, i.e. with *outer space*. The space $CV(F_n)$ admits a natural locally compact topology, and the group $\text{Out}(F_n)$ acts properly discontinuously on $CV(F_n)$.

The *boundary* $\partial cv(F_n)$ of $cv(F_n)$ consists of all minimal *very small* isometric actions of F_n on \mathbb{R} -trees which either are non-simplicial or which are not free. Here an action is very small if and only if every nontrivial arc stabilizer is maximal cyclic and if tripod stabilizers are trivial. Again, any two such actions define the same point in $\partial cv(F_n)$ if there exists an F_n -equivariant isometry between them. Write $\partial CV(F_n)$ to denote the projectivization of $\partial cv(F_n)$. Then $CV(F_n) \cup \partial CV(F_n)$ is a compact $\text{Out}(F_n)$ -space.

For every $T \in \overline{cv(F_n)} = cv(F_n) \cup \partial cv(F_n)$ and every indivisible $w \in F_n$, the *translation length* $\|w\|_T$ for the action of w on T is defined to be the *dilation*

$$\|w\|_T = \inf\{d(x, wx) \mid x \in T\}$$

of the action of w on T . To every $T \in \partial cv(F_n)$ we can associate a topological lamination $L(T)$ of zero-length geodesics [CHL08a] as follows. For every $\epsilon > 0$ define $\Omega_\epsilon(T)$ to be the set of all elements $w \in F_n$ with translation length $\|w\|_T < \epsilon$.

Denote by $L_\epsilon(T)$ the smallest F_n -invariant closed subset of $\partial^2(F_n)$ which contains all pairs of fixed points of each element in $\Omega_\epsilon(T)$. Then

$$L(T) = \bigcap_{\epsilon > 0} L_\epsilon(T)$$

is a nonempty closed F_n -invariant flip invariant subset of $\partial^2 F_n$ which will be called the *zero lamination* of T . Note that two \mathbb{R} -trees $T, T' \in \partial cv(F_n)$ with the same projectivization have the same zero lamination. Thus the zero lamination is defined for points in $\partial CV(F_n)$.

The following result is due to Kapovich and Lustig [KL09a, KL10].

Proposition 2.3. (1) *There is a unique continuous $\text{Out}(F_n)$ -invariant length pairing*

$$\langle \cdot, \cdot \rangle : \overline{cv(F_n)} \times \text{Curr}(F_n) \rightarrow [0, \infty)$$

which satisfies $\langle T, \eta \rangle = \|w\|_T$ for every current η dual to an indivisible conjugacy class w in F_n and for every $T \in \overline{cv(F_n)}$.

(2) *If $T \in \partial cv(F_n)$ then $\langle T, \nu \rangle = 0$ if and only if ν is supported in the zero lamination of T .*

Remark: Kapovich and Lustig call the length pairing as defined above an intersection form.

Define $\mathcal{UML}_0 \subset \mathcal{PML}(F_n)$ to be the set of all projective measured laminations $[\nu]$ with the property that $\langle [T], [\nu] \rangle = 0$ for precisely one projective tree $[T] \in \partial CV(F_n)$. Note that this makes sense without referring to specific representatives of the projective classes. The projective tree $[T]$ is called *dual* to $[\nu]$. We denote by $\mathcal{UT}_0 \subset \partial CV(F_n)$ the set of all projective trees which are dual to points in \mathcal{UML}_0 . By invariance of the length pairing, the sets \mathcal{UML}_0 and \mathcal{UT}_0 are invariant under the action of $\text{Out}(F_n)$. The assignment ω_0 which associates to $[\nu] \in \mathcal{UML}_0$ the projective tree $\omega_0([\nu]) \in \mathcal{UT}_0$ which is dual to $[\nu]$ is $\text{Out}(F_n)$ -equivariant, however it is not injective. Examples for distinct points in \mathcal{UML}_0 with the same image under ω_0 arise from minimal non-uniquely ergodic measured geodesic laminations on a compact surface S with a single boundary component and fundamental group F_n which fill up S .

Let

$$\begin{aligned} \mathcal{UML} = \{ & [\nu] \in \mathcal{UML}_0 \mid \\ & \langle \omega_0[\nu], [\zeta] \rangle = 0 \text{ for } [\zeta] \in \mathcal{PML}(F_n) \text{ only if } [\zeta] = [\nu] \}. \end{aligned}$$

By equivariance, the set \mathcal{UML} is invariant under the action of $\text{Out}(F_n)$. The restriction ω of the map ω_0 to \mathcal{UML} is a bijection onto an $\text{Out}(F_n)$ -invariant subset \mathcal{UT} of \mathcal{UT}_0 .

Every iwip automorphism of F_n acts with north-south dynamics on the boundary $\partial CV(F_n)$ of outer space (Theorem 1.1 of [LL03]). In particular, there are precisely two fixed points for this action. We show that \mathcal{UT} contains all fixed points of iwip elements of $\text{Out}(F_n)$. For the proof of this and for later use, we call an iwip automorphism $\alpha \in \text{Out}(F_n)$ *non-geometric* if α does not admit any periodic conjugacy class in F_n . This is equivalent to stating that no power of α can be

realized as a homeomorphism of a compact surface with fundamental group F_n (Theorem 4.1 of [BH92]).

Lemma 2.4. *Any fixed point of an iwip-automorphism on the boundary $\partial\text{CV}(F_n)$ of outer space is contained in \mathcal{UT} .*

Proof. Let $T \in \partial\text{CV}(F_n)$ be the repelling fixed point for the action of a non-geometric iwip-automorphism α on the boundary of outer space. By Proposition 5.6 of [CHL08b], the zero lamination $L(T)$ of T is *uniquely ergodic*, i.e. it supports a unique projective transverse measure $[\nu]$. This projective transverse measure is a projective measured lamination [Ma95]. In particular, if $[\nu] \in \mathcal{UML}_0$ then we also have $[\nu] \in \mathcal{UML}$ and $T \in \mathcal{UT}$.

Now let $\alpha \in \text{Out}(F_n)$ be a geometric iwip element. By Theorem 4.1 of [BH92], there is a compact connected surface S with connected boundary and fundamental group F_n such that α can be represented by a pseudo-Anosov homeomorphism A of S . The repelling projective measured geodesic lamination $[\nu]$ for A determines up to scale an action of $F_n = \pi_1(S)$ on an \mathbb{R} -tree. The projectivization T of this F_n -tree is just the repelling fixed point for the action of α on $\partial\text{CV}(F_n)$. Moreover, $[\nu]$ is supported in the zero-lamination of T .

The boundary of S defines a conjugacy class in F_n which is invariant under α . Any geodesic current supported in the zero lamination $L(T)$ of T can be written in the form $a\nu + b\zeta$ where ν is a representative of the class $[\nu]$, where ζ is the current dual to the conjugacy class defined by the boundary of S and where $a \geq 0, b \geq 0$. If $b > 0$ then the current $a\nu + b\zeta$ is *not* a measured lamination. (The current ζ is dual to an indecomposable conjugacy class ∂S contained in the commutator subgroup of F_n . Since $\mathcal{ML}(F_n)$ is the closure of the weighted duals of primitive conjugacy classes, this alone does not imply that $\zeta \notin \mathcal{ML}(F_n)$. However, the boundary circle ∂S of S is not contained in the Hausdorff limit of any sequence of non-separating non-boundary-parallel simple closed curves in S which implies that indeed $a\nu + b\zeta \notin \mathcal{ML}(S)$, compare [Ma95]). As a consequence, if $[\nu] \in \mathcal{UML}_0$ then also $[\nu] \in \mathcal{UML}$ and $T \in \mathcal{UT}$.

We are now left with showing that the projective measured lamination $[\nu] \in \mathcal{PML}(F_n)$ defined as above by the repelling fixed point $T \in \partial\text{CV}(F_n)$ of an arbitrary iwip element $\alpha \in \text{Out}(F_n)$ is contained in \mathcal{UML}_0 . For this let $\nu \in \mathcal{ML}(F_n)$ be a measured lamination which represents the projective class $[\nu]$. To show that $\langle T', \nu \rangle > 0$ for $T' \in \partial\text{CV}(F_n) - T$ (by a slight abuse of notation), assume to the contrary that there is some $T' \in \partial\text{CV}(F_n) - T$ with $\langle T', \nu \rangle = 0$. Since $T' \neq T$ by assumption and since by Theorem 1.1 of [LL03] α acts with north-south dynamics on $\partial\text{CV}(F_n)$, we have $\alpha^k T' \rightarrow Q$ ($k \rightarrow \infty$) where Q is the attracting fixed point for the action of α on $\partial\text{CV}(F_n)$.

Choose a continuous section $\Sigma : \partial\text{CV}(F_n) \rightarrow \partial cv(F_n)$; such a section was for example constructed by Skora and White [S89, W91] (or see the definition below). Then $\Sigma(\alpha^k T') \rightarrow \Sigma Q$, moreover we have $\langle \Sigma Q, \nu \rangle > 0$. Hence by continuity of the length function on $\partial cv(F_n) \times \text{Curr}(F_n)$ [KL09a] and by naturality under scaling, we conclude that $\langle \Sigma(\alpha^k T'), \nu \rangle > 0$ and hence $\langle \alpha^k(\Sigma T'), \nu \rangle > 0$ for sufficiently large k .

Then $\langle \Sigma T', \alpha^{-k} \nu \rangle = \langle \alpha^k(\Sigma T'), \nu \rangle > 0$ by invariance of the length function under the action of $\text{Out}(F_n)$ which contradicts the fact that $\alpha \nu = \lambda \nu$ for some $\lambda > 0$. \square

Remark: 1) The proof of Lemma 2.4 implies that any iwip element of $\text{Out}(F_n)$ acts with north-south dynamics on $\mathcal{PM}\mathcal{L}(F_n)$, an unpublished result of Martin [Ma95]. However, we used Proposition 5.6 of [CHL08b] which in turn uses a weak version of Martin's result. We also used the work of Levitt and Lustig [LL03] which appeared after Martin's thesis.

2) In general, there may be points $T \neq T' \in \partial\text{CV}(F_n)$ with the same zero lamination (see [CHL07] for a detailed account on this issue). Lemma 2.4 implies that for the fixed point T of an iwip element, there is no tree $T \neq T' \in \partial\text{CV}(F_n)$ whose zero lamination coincides with the zero lamination of T .

We equip $\mathcal{UM}\mathcal{L}$ with the topology as a subspace of $\mathcal{PM}\mathcal{L}(F_n)$, and we equip \mathcal{UT} with the topology as a subspace of $\partial\text{CV}(F_n)$.

For a tree $T_0 \in \text{CV}(F_n)$ define

$$(1) \quad \Lambda(T_0) = \{\nu \in \mathcal{ML}(F_n) \mid \langle T_0, \nu \rangle = 1\}.$$

The next lemma is immediate from continuity of the length pairing.

Lemma 2.5. $\Lambda(T_0)$ is a continuous section of the fibration $\mathcal{ML}(F_n) \rightarrow \mathcal{PM}\mathcal{L}(F_n)$ depending continuously on T_0 .

For $T_0 \in \text{CV}(F_n)$ there is a dual section

$$(2) \quad \Sigma(T_0) = \{T \in \text{cv}(F_n) \cup \partial\text{cv}(F_n) \mid \max\{\langle T, \nu \rangle \mid \nu \in \Lambda(T_0)\} = 1\}$$

of the fibration $\text{cv}(F_n) \cup \partial\text{cv}(F_n) \rightarrow \text{CV}(F_n) \cup \partial\text{CV}(F_n)$. We use these sections to show

Lemma 2.6. $\mathcal{UM}\mathcal{L}$ and \mathcal{UT} are Borel subsets of $\mathcal{PM}\mathcal{L}(F_n)$ and $\partial\text{CV}(F_n)$, and the map $\omega : \mathcal{UM}\mathcal{L} \rightarrow \mathcal{UT}$ is an $\text{Out}(F_n)$ -equivariant homeomorphism.

Proof. Since we do not need the fact that $\mathcal{UM}\mathcal{L}$ and \mathcal{UT} are Borel sets we omit the proof. The map $\omega : \mathcal{UM}\mathcal{L} \rightarrow \mathcal{UT}$ is clearly $\text{Out}(F_n)$ -equivariant and hence we just have to show that ω is continuous and open.

Let $T_0 \in \text{CV}(F_n)$ be a simplicial F_n -tree with quotient of volume one and let $\Lambda = \Lambda(T_0)$ and $\Sigma = \Sigma(T_0)$ as defined in equation (1,2). Then $\mathcal{UM}\mathcal{L}$ is naturally homeomorphic to a subset Λ_0 of Λ , and \mathcal{UT} is homeomorphic to a subset Σ_0 of Σ . We denote again by ω the induced map $\Lambda_0 \rightarrow \Sigma_0$.

Both $\mathcal{PM}\mathcal{L}(F_n)$ and $\text{CV}(F_n) \cup \partial\text{CV}(F_n)$ are compact and metrizable topological spaces and hence the same holds true for Λ, Σ . Thus to show continuity of ω , it suffices to show that if $(\nu_i) \subset \Lambda_0$ is any sequence converging to some $\nu \in \Lambda_0$ then $\omega(\nu_i) \rightarrow \omega(\nu)$. Via passing to a subsequence, we may assume that $\omega(\nu_i) \rightarrow T$ for some $T \in \Sigma$. Since $\langle \omega(\nu_i), \nu_i \rangle = 0$ for all i , by continuity of the length pairing we have $\langle T, \nu \rangle = 0$ and hence $T = \omega(\nu)$.

To show that ω is open, it suffices to show that $\omega^{-1} : \Sigma_0 \rightarrow \Lambda_0$ is continuous. However, this follows from the above argument by continuity of the length pairing. \square

For sufficiently small $\epsilon > 0$, the set $\text{Spine}_\epsilon(F_n)$ of all simplicial trees $T \in \text{CV}(F_n)$ with the additional property that the smallest translation length on T of any element $w \neq e$ of F_n is at least ϵ is a closed connected $\text{Out}(F_n)$ -invariant subset of $\text{CV}(F_n)$ on which $\text{Out}(F_n)$ acts cocompactly. We call this set the ϵ -*spine* of $\text{CV}(F_n)$. (Note however that this is *not* the spine of outer space in the usual sense). Let $\overline{\text{Spine}_\epsilon(F_n)}$ be the closure of $\text{Spine}_\epsilon(F_n)$ in $\text{CV}(F_n) \cup \partial\text{CV}(F_n)$. Then

$$\partial\text{Spine}_\epsilon(F_n) = \overline{\text{Spine}_\epsilon(F_n)} - \text{Spine}_\epsilon(F_n)$$

is a closed $\text{Out}(F_n)$ -invariant subset of $\partial\text{CV}(F_n)$.

The following simple observation will be used several time in the sequel.

Lemma 2.7. *Let $(T_i) \subset \text{Spine}_\epsilon(F_n)$ be an unbounded sequence and for each i let $a_i > 0$ be such that $a_i T_i \in \Sigma(T_0)$. Then $a_i \rightarrow 0$ ($i \rightarrow \infty$).*

Proof. We follow Kapovich and Lustig [KL09a].

Let $(T_i) \subset \text{Spine}_\epsilon(F_n)$ be an unbounded sequence and for each i let $a_i > 0$ be such that $a_i T_i \in \Sigma(T_0)$. Since $\Sigma(T_0)$ is compact, after passing to a subsequence we may assume that $a_i T_i \rightarrow T$ for some $T \in \Sigma(T_0)$. Since (T_i) is unbounded we have $T \in \partial\text{cv}(F_n)$.

If $a_i \not\rightarrow 0$ ($i \rightarrow \infty$) then after passing to a subsequence we may assume that $a_i \geq a > 0$ for all i . Since $T \in \partial\text{cv}(F_n)$ there are nontrivial elements in F_n acting on T with arbitrarily small translation length. This means that there exists a sequence $(g_j) \subset F_n - \{e\}$ with

$$(3) \quad \lim_{j \rightarrow \infty} \|g_j\|_T = 0.$$

However, $T_i \in \text{Spine}_\epsilon(F_n)$ and hence $\|g_j\|_{T_i} \geq \epsilon$ for all i, j . Hence for all i, j we have

$$a_i \|g_j\|_{T_i} \geq a\epsilon.$$

On the other hand, by continuity of the length pairing on $\overline{\text{cv}(F_n)} \times \text{Curr}(F_n)$ we have $a_i \|g_j\|_{T_i} \rightarrow \|g_j\|_T$ ($i \rightarrow \infty$) for all j which contradicts (3) above. Thus indeed $\lim_{i \rightarrow \infty} a_i = 0$. This shows the lemma. \square

For a simplicial tree $T \in \text{Spine}_\epsilon(F_n)$ call a primitive conjugacy class γ in F_n *basic* for T if γ can be represented by a loop in T/F_n of length at most two. For example, if γ can be represented by a loop which travels through each edge of T/F_n at most twice then γ is basic for T . We have

Lemma 2.8. *Let $C \subset \overline{\text{Spine}_\epsilon(F_n)}$ be a closed set. Denote by $V \subset \mathcal{PML}(F_n)$ the set of all projective measured laminations which are dual to a basic primitive conjugacy class for some tree $T \in C \cap \text{Spine}_\epsilon(F_n)$ and let \overline{V} be the closure of V in $\mathcal{PML}(F_n)$. Then $\overline{V} - V$ consists of projective measured laminations which are supported in the zero lamination of a tree $T \in C \cap \partial\text{Spine}_\epsilon(F_n)$.*

Proof. Let $C \subset \overline{\text{Spine}_\epsilon(F_n)}$ be a closed set. Let $V \subset \mathcal{PML}(F_n)$ be the set of all projective measured laminations which are dual to a basic primitive conjugacy class for some tree $T \in C \cap \text{Spine}_\epsilon(F_n)$. Let $([\alpha_i]) \subset V$ be a sequence which converges to some $[\alpha] \in \overline{V}$. For every $i \geq 0$ let $T_i \in C \cap \text{Spine}_\epsilon(F_n)$ be such that $[\alpha_i]$ is dual to a primitive conjugacy class in F_n which is basic for T_i . After passing to a subsequence we may assume that $T_i \rightarrow T_\infty \in C$.

If $T_\infty \in \text{Spine}_\epsilon(F_n)$ then by continuity of the length function and the fact that for every $T \in \text{Spine}_\epsilon(F_n)$ the number of conjugacy classes which can be represented by a loop on T/F_n of length at most 3 is bounded from above by a universal constant, $[\alpha]$ is dual to a primitive conjugacy class which can be represented by a loop in T_∞/F_n of length at most 2 and hence $\alpha \in V$.

Otherwise let $T \in \Sigma(T_0) \cap \partial cv(F_n)$ be a representative of T_∞ . For $i \geq 0$ let $a_i > 0$ be such that

$$a_i T_i \in \Sigma(T_0).$$

Then $a_i T_i \rightarrow T$ ($i \rightarrow \infty$) and since $T_i \in \text{Spine}_\epsilon(F_n)$ for all i , Lemma 2.7 implies that $a_i \rightarrow 0$ ($i \rightarrow \infty$).

Let $\alpha_i \in \Lambda(T_0)$ be the representative of $[\alpha_i]$. Then $\alpha_i = b_i \alpha'_i$ where α'_i is dual to a primitive conjugacy class which is basic for T_i . Now $T_0 \in \text{Spine}_\epsilon(F_n)$ and hence $\langle T_0, \alpha'_i \rangle \geq \epsilon$ for all i . This shows that $b_i \leq 1/\epsilon$ for all i .

By compactness, we may assume that $\alpha_i \rightarrow \alpha \in \Lambda(T_0)$ where α is a representative of the class $[\alpha]$. Since $\langle T_i, \alpha'_i \rangle \leq 2$ by assumption we have $\langle T_i, \alpha_i \rangle \leq 2/\epsilon$ for all i . But $\langle a_i T_i, \alpha_i \rangle \rightarrow \langle T, \alpha \rangle$ and $a_i \rightarrow 0$ ($i \rightarrow \infty$) and therefore $\langle T, \alpha \rangle = 0$ by continuity. This shows that $[\alpha]$ is supported in the zero lamination of T_∞ and completes the proof of the lemma. \square

Corollary 2.9. *Let $C \subset \overline{\text{Spine}_\epsilon(F_n)}$ be a closed set and let $x \in \mathcal{UT} - C$. Then $\omega^{-1}(x) \in \mathcal{UML}$ is not contained in the closure of the set of all projective measured laminations which are dual to a basic primitive conjugacy class for some tree $T \in C$.*

Proof. If $C \subset \overline{\text{Spine}_\epsilon(F_n)}$ is a closed set and if $x \in \mathcal{UT} - C$ then $\omega^{-1}(x)$ is not supported in the zero lamination of any tree $T \in C$. Together with Lemma 2.8, this shows the corollary. \square

3. LINES OF MINIMA

In this section we use the length pairing to construct a family of lines in the ϵ -spine $\text{Spine}_\epsilon(F_n) \subset \text{CV}(F_n)$ of Outer space which are contracting for the Lipschitz distance (see Section 2 and below for definitions).

Definition 3.1. A family \mathcal{F} of nonnegative functions ρ on $\text{CV}(F_n)$ is called *uniformly proper* if for every $c > 0$ there is a compact subset $A(c)$ of $\text{Spine}_\epsilon(F_n)$ such that $\rho^{-1}[0, c] \cap \text{Spine}_\epsilon(F_n) \subset A(c)$ for every $\rho \in \mathcal{F}$.

Call a pair $(\mu, \nu) \in \mathcal{ML}(F_n)^2$ *positive* if the function $T \rightarrow \langle T, \nu + \mu \rangle$ is positive on $cv(F_n) \cup \partial cv(F_n)$. In the following lemma, we identify outer space $CV(F_n)$ with the set of all simplicial F_n -trees with quotient of volume one. For $x \in CV(F_n)$ recall from Section 2 the definition (1,2) of the sets $\Lambda(x)$ and $\Sigma(x)$.

Lemma 3.2. *Let $K \subset \mathcal{ML}(F_n) \times \mathcal{ML}(F_n)$ be a compact set consisting of positive pairs. Then the family of functions $\{\langle \cdot, \mu + \mu' \rangle \mid (\mu, \mu') \in K\}$ on $CV(F_n)$ is uniformly proper.*

Proof. Let $K \subset \mathcal{ML}(F_n) \times \mathcal{ML}(F_n)$ be as in the lemma. Let $T_0 \in \text{Spine}_\epsilon(F_n)$ and let $\Sigma = \Sigma(T_0)$. By assumption, we have $\langle T, \nu + \nu' \rangle > 0$ for every $T \in \Sigma$ and every $(\nu, \nu') \in K$. By continuity of the length pairing and compactness of Σ and K there is then a number $\delta > 0$ such that $\langle T, \mu + \mu' \rangle \geq \delta$ for every $(\mu, \mu') \in K$ and every $T \in \Sigma$.

Let $c > 0$ and let $A(c) = \{T \in \text{Spine}_\epsilon(F_n) \mid \min\{\langle T, \mu + \mu' \rangle \mid (\mu, \mu') \in K\} \leq c\}$. Then $A(c)$ is a closed subset of $\text{Spine}_\epsilon(F_n)$. Our goal is to show that $A(c)$ is compact.

For this assume otherwise. Since $\text{Spine}_\epsilon(F_n)$ is locally compact, there is then an unbounded sequence $(T_i) \subset A(c)$ and for each i there is some $(\mu_i, \mu'_i) \in K$ with

$$(4) \quad \langle T_i, \mu_i + \mu'_i \rangle \leq c.$$

Let $a_i > 0$ be such that $a_i T_i \in \Sigma$. Since Σ is compact, after passing to a subsequence we may assume that

$$\lim_{i \rightarrow \infty} a_i T_i = T$$

in Σ . Moreover, since K is compact, after passing to another subsequence we may assume that $(\mu_i, \mu'_i) \rightarrow (\mu, \mu') \in K$.

Lemma 2.7 shows that $\lim_{i \rightarrow \infty} a_i = 0$. On the other hand, since $\langle T, \mu + \mu' \rangle \geq \delta$, using once more continuity of the length function we infer from (4) that

$$0 = \lim_{i \rightarrow \infty} a_i \langle T_i, \mu_i + \mu'_i \rangle = \lim_{i \rightarrow \infty} \langle a_i T_i, \mu_i + \mu'_i \rangle = \langle T, \mu + \mu' \rangle \geq \delta.$$

This is a contradiction and shows that $\{\langle \cdot, \mu + \mu' \rangle \mid (\mu, \mu') \in K\}$ is indeed uniformly proper. \square

For trees $T, T' \in CV(F_n)$ define $d_L(T, T')$ to be logarithm of the minimal Lipschitz constant of a marked homotopy equivalence $T/F_n \rightarrow T'/F_n$. Then

$$d(T, T') = d_L(T, T') + d_L(T', T)$$

is an $\text{Out}(F_n)$ -invariant distance function on $CV(F_n)$ inducing the original topology [FM08]. The group $\text{Out}(F_n)$ acts properly, isometrically and cocompactly on $\text{Spine}_\epsilon(F_n)$ equipped with the restriction of the metric d .

As in Section 2, call a primitive conjugacy class γ in F_n *basic* for a simplicial tree $T \in \text{Spine}_\epsilon(F_n)$ if γ can be represented by a loop in T/F_n of length at most two. The following result is due to Francaviglia and Martino [FM08].

Lemma 3.3.

$$d_L(T, T') = \sup\{\log \frac{\langle T', \alpha \rangle}{\langle T, \alpha \rangle} \mid \alpha \in \mathcal{ML}(F_n)\}.$$

The supremum is attained for a measured lamination α which is dual to a basic primitive conjugacy class for T .

Proof. Clearly for every measured lamination $\alpha \in \mathcal{ML}(F_n)$ we have

$$d_L(T, T') \geq \log \frac{\langle T', \alpha \rangle}{\langle T, \alpha \rangle}.$$

On the other hand, Proposition 3.15 of [FM08] states that $d_L(T, T')$ is the minimum of the logarithm of the quotients $\frac{\langle T', \alpha \rangle}{\langle T, \alpha \rangle}$ where α passes through those currents which are dual to loops γ in T/F_n of one of the following kinds. Either γ is simple or γ defines an embedded bouquet of two circles in T/F_n or γ defines two embedded simple closed curves in T/F_n joined by an embedded arc traveled through twice in opposite direction. In particular, the length of γ is at most two.

It is well known (see p. 197/198 of [M67]) that if γ is an embedded loop in T/F_n then γ defines a primitive conjugacy class. If $\gamma = \gamma_1\gamma_2$ where γ_1, γ_2 are two embedded loops which intersect in a single point then γ can be obtained from the primitive element γ_1 by a Nielsen move with γ_2 and once again, γ is primitive. The third case is completely analogous. \square

Lemma 3.3 shows that for $x \in \text{CV}(F_n)$, the set $\Sigma(x)$ is precisely the set of all trees $T \in \text{cv}(F_n) \cup \partial\text{cv}(F_n)$ such that the optimal Lipschitz constant for a marked homotopy equivalence $x/F_n \rightarrow T/F_n$ equals one. Thus as an immediate corollary we obtain

Corollary 3.4. *Let $x, y \in \text{CV}(F_n)$ and let $b > 0$ be such that $by \in \Sigma(x)$. If $\langle by, \nu \rangle \geq 1/B$ for some $B > 0$ and some $\nu \in \Lambda(x)$ then*

$$\log \langle y, \nu \rangle \leq d_L(x, y) \leq \log \langle y, \nu \rangle + \log B.$$

A pair of projective measured laminations $([\mu], [\nu]) \in \mathcal{PML}(F_n) \times \mathcal{PML}(F_n) - \Delta$ is called *positive* if for any representatives μ, ν of $[\mu], [\nu]$ the function $\langle \cdot, \mu + \nu \rangle$ on $\text{CV}(F_n) \cup \partial\text{CV}(F_n)$ is positive. A tree $T \in \text{cv}(F_n) \cup \partial\text{cv}(F_n)$ is called *balanced* for a positive pair $(\mu, \nu) \in \mathcal{ML}(F_n)^2$ if $\langle T, \mu \rangle = \langle T, \nu \rangle$. Note that this only depends on the projective class of T . The set

$$\text{Bal}(\mu, \nu) \subset \text{CV}(F_n) \cup \partial\text{CV}(F_n)$$

of projectivizations of all balanced trees for (μ, ν) is a closed subset of $\text{CV}(F_n) \cup \partial\text{CV}(F_n)$ which is disjoint from the set of trees on which either μ or ν vanishes. Let moreover

$$\text{Min}(\mu + \nu) \subset \text{Spine}_\epsilon(F_n)$$

be the set of all points for which the restriction of the function $T \rightarrow \langle T, \mu + \nu \rangle$ to $\text{Spine}_\epsilon(F_n)$ assumes a minimum.

Definition 3.5. For $B > 0$, a positive pair of points

$$([\mu], [\nu]) \in \mathcal{PML}(F_n) \times \mathcal{PML}(F_n) - \Delta$$

is called B -contracting if for any pair $\mu, \nu \in \mathcal{ML}(F_n)$ of representatives of $[\mu], [\nu]$ there is some $x \in \text{Min}(\mu + \nu)$ with the following properties.

- (1) $\langle x, \mu \rangle / \langle x, \nu \rangle \in [B^{-1}, B]$.
- (2) If $y \in \text{Spine}_\epsilon(F_n)$ is such that $d(x, y) > B$ then $\langle y, \mu + \nu \rangle / \langle x, \mu + \nu \rangle \geq 2$.
- (3) If $s \in [-1, 1]$ then there exists a point $y \in \text{Min}(e^s \mu + e^{-s} \nu)$ such that $d(x, y) \leq B$.
- (4) If $\tilde{\mu}, \tilde{\nu} \in \Lambda(x)$ are representatives of $[\mu], [\nu]$ then $\langle T, \tilde{\mu} + \tilde{\nu} \rangle \geq 1/B$ for all $T \in \Sigma(x)$.
- (5) Let $\mathcal{B}(x) \subset \Lambda(x)$ be the set of all normalized measured laminations which are weighted duals of a basic primitive conjugacy class for a tree $T \in \text{Bal}(\mu, \nu) \cap \text{Spine}_\epsilon(F_n)$. Then $\xi(T) \geq 1/B$ for every $\xi \in \mathcal{B}(x)$ and every tree

$$T \in \Sigma(x) \cap \left(\bigcup_{s \in (-\infty, -B) \cup (B, \infty)} \text{Bal}(e^s \mu, e^{-s} \nu) \right).$$

Note that if $([\mu], [\nu])$ is a B -contracting pair then $([\mu], [\nu])$ is C -contracting for every $C \geq B$. We call a pair $(\mu, \nu) \in \mathcal{ML}(F_n)^2$ B -contracting for some $B > 0$ if the pair $([\mu], [\nu])$ of its projectivizations is B -contracting.

Let $\mathcal{A}(B) \subset \mathcal{PML}(F_n) \times \mathcal{PML}(F_n) - \Delta$ be the set of B -contracting pairs. We have

Proposition 3.6. *The set $\mathcal{A}(B)$ of B -contracting pairs is an $\text{Out}(F_n)$ -invariant closed subset of the space of positive pairs in $\mathcal{PML}(F_n) \times \mathcal{PML}(F_n) - \Delta$.*

Proof. By definition, $\mathcal{A}(B)$ is $\text{Out}(F_n)$ -invariant.

To show that $\mathcal{A}(B)$ is a closed subset of the set of all positive pairs in $\mathcal{PML}(F_n) \times \mathcal{PML}(F_n) - \Delta$, let $([\mu_i], [\nu_i])$ be a sequence of B -contracting pairs converging to a positive pair $([\mu], [\nu]) \in \mathcal{PML}(F_n) \times \mathcal{PML}(F_n) - \Delta$. Let $\mu, \nu \in \mathcal{ML}(F_n)$ be preimages of $[\mu], [\nu]$ and choose any sequence $(\mu_i, \nu_i) \in \mathcal{ML}(F_n)^2$ of pairs of preimages of $[\mu_i], [\nu_i]$ which converges to (μ, ν) . By Lemma 3.2, the family of functions $\mathcal{F} = \{\mu_i + \nu_i, \mu + \nu\}$ is uniformly proper. Thus if $x_i \in \text{Min}(\mu_i + \nu_i)$ is a point as in the definition of a B -contracting pair then up to passing to a subsequence, we may assume that the sequence (x_i) converges to a point $x \in \text{Spine}_\epsilon(F_n)$. By continuity of the length pairing, we have $x \in \text{Min}(\mu + \nu)$ and moreover $\langle x, \mu \rangle / \langle x, \nu \rangle \in [B^{-1}, B]$.

Let $y \in \text{Spine}_\epsilon(F_n)$ be such that $d(x, y) > B$. Then $d(x_i, y) > B$ for all sufficiently large i and therefore

$$\langle y, \mu_i + \nu_i \rangle / \langle x_i, \mu_i + \nu_i \rangle \geq 2$$

by the choice of x_i . By continuity, we then obtain $\langle y, \mu + \nu \rangle / \langle x, \mu + \nu \rangle \geq 2$ as well. This shows that the point x has properties 1) and 2) in Definition 3.5 for the pair (μ, ν) .

The third property in Definition 3.5 follows in the same way. Namely, by assumption, for every $s \in [-1, 1]$ and for $i > 0$ there exists a point $y_i \in \text{Min}(e^s \mu_i, e^{-s} \nu_i)$

with $d(y_i, x_i) \leq B$. By passing to a subsequence we may assume that $y_i \rightarrow y \in \text{Spine}_\epsilon(F_n)$. Then $d(x, y) \leq B$ by continuity, moreover $y \in \text{Min}(e^s \mu + e^{-s} \nu)$.

For the fourth property, note that by continuity of the length pairing, if $\alpha \in \mathcal{ML}(F_n)$ and if $a_i > 0, a > 0$ is such that $a_i \alpha \in \Lambda(x_i), a \alpha \in \Lambda(x)$ then $a_i \rightarrow a$ since $x_i \rightarrow x$. In particular, if $\tilde{\mu}_i, \tilde{\nu}_i \in \Lambda(x_i)$ are representatives of $[\mu_i], [\nu_i]$ then $\tilde{\mu}_i \rightarrow \tilde{\mu} \in \Lambda(x)$ and $\tilde{\nu}_i \rightarrow \tilde{\nu} \in \Lambda(x)$ where $\tilde{\mu}, \tilde{\nu}$ are representatives of $[\mu], [\nu]$. On the other hand, if $T \in cv(F_n) \cup \partial cv(F_n)$ and if $b_i > 0$ is such that $b_i T \in \Sigma(x_i)$ then $b_i \rightarrow b$ where $bT \in \Sigma(x)$. Then

$$1/B \leq \langle b_i T, \tilde{\mu}_i + \tilde{\nu}_i \rangle \rightarrow \langle bT, \tilde{\mu} + \tilde{\nu} \rangle$$

by continuity of the length pairing. This shows the fourth property in the definition of a B -contracting pair.

Now let $\alpha \in \mathcal{B}(x) \subset \Lambda(x)$. Then there is a tree $T \in \text{Bal}(\mu, \nu) \cap \text{Spine}_\epsilon(F_n)$ such that α is a weighted dual of a primitive conjugacy class which can be represented by a loop on T/F_n of length at most two. Let

$$S \in \Sigma(x) \cap \text{Bal}(e^s \mu, e^{-s} \nu) \text{ for some } s \in (-\infty, -B) \cap (B, \infty).$$

We have to show that $\langle S, \alpha \rangle \geq 1/B$. To see that this is the case, let $t_i \in \mathbb{R}$ be such that $T \in \text{Bal}(e^{t_i} \mu_i, e^{-t_i} \nu_i)$; then $t_i \rightarrow 0$ ($i \rightarrow \infty$). There is a sequence $b_i \rightarrow 1$ and for every sufficiently large i there is a number $s_i \in (-\infty, B) \cup (B, \infty)$ such that $b_i S \in \Sigma(x_i) \cap \text{Bal}(e^{s_i + t_i} \mu_i, e^{-s_i - t_i} \nu_i)$. Let $a_i > 0$ be such that $a_i \alpha \in \Lambda(x_i)$; then $a_i \rightarrow 1$. By the fifth requirement in the definition of a B -contracting pair we have $\langle b_i S, a_i \alpha \rangle \geq 1/B$ for all sufficiently large i and hence $\langle S, \alpha \rangle \geq 1/B$ by continuity. This completes the proof of the proposition. \square

The following proposition is the key observation of this note.

Proposition 3.7. *If $([\nu_+], [\nu_-]) \in \mathcal{UMC}^2$ is the pair of fixed points of an iwip element of $\text{Out}(F_n)$ then $([\nu_+], [\nu_-])$ is B -contracting for some $B > 0$.*

Proof. Let $\varphi \in \text{Out}(F_n)$ be an iwip element with pair of fixed points $[\nu_+], [\nu_-] \in \mathcal{UMC}$. In particular, $([\nu_+], [\nu_-])$ is a positive pair. Up to exchanging φ and φ^{-1} there are numbers $\lambda^+, \lambda^- > 1$ such that for any representatives ν_+, ν_- of the classes $[\nu_+], [\nu_-]$ we have

$$\varphi \nu_+ = \lambda^+ \nu_+, \varphi \nu_- = \lambda_-^{-1} \nu_-.$$

Let $s_0 > 0, a > 0$ be such that $e^{s_0} = a \lambda^+, e^{-s_0} = a \lambda_-^{-1}$. Then

$$\mathcal{F} = \{f_s : T \rightarrow \langle T, e^s \nu_+ + e^{-s} \nu_- \rangle \mid s \in [-1 - s_0, s_0 + 1]\}$$

is a set of functions on $CV(F_n)$ which is compact with respect to the topology of uniform convergence on compact sets. Lemma 3.2 shows that the set $C \subset \text{Spine}_\epsilon(F_n)$ of all minima of the restrictions of all functions from the collection \mathcal{F} to $\text{Spine}_\epsilon(F_n)$ is compact. In particular, there is a number $B_1 > 0$ such that

$$\langle y, e^s \nu_+ \rangle / \langle y, e^{-s} \nu_- \rangle \in [B_1^{-1}, B_1]$$

for all $y \in C$ and all $s \in [-1 - s_0, s_0 + 1]$. This shows that the first requirement in Definition 3.5 holds true for $e^s \nu_+, e^{-s} \nu_-$ ($s \in [-s_0 - 1, s_0 + 1]$) with $B = B_1$.

Let $B_2 > 0$ be the diameter of the set C with respect to the distance d . By construction, for $s \in [-s_0, s_0]$ the third property in the definition of a B -contracting pair is satisfied for $e^s \nu_+, e^{-s} \nu_-$ with $B = B_2$.

Using once more Lemma 3.2, there is a number $B_3 > 0$ such that for every $s \in [-s_0, s_0]$, every $x \in C$ and every $y \in \text{Spine}_\epsilon(F_n)$ with $d(y, C) \geq B_3$ we have $\langle y, e^s \nu_+ + e^{-s} \nu_- \rangle / \langle x, e^s \nu_+ + e^{-s} \nu_- \rangle \geq 2$. Then the second property in Definition 3.5 holds true with $B = B_2 + B_3$ and for $e^s \nu_+, e^{-s} \nu_-$ where $s \in [-s_0, s_0]$.

To establish the fourth requirement, denote for $x \in \text{Spine}_\epsilon(F_n)$ by $\tilde{\nu}_+(x), \tilde{\nu}_-(x) \in \Lambda(x)$ the representative of $[\nu_+], [\nu_-]$ contained in $\Lambda(x)$. Then

$$g(x) : T \rightarrow \langle T, \tilde{\nu}_+(x) + \tilde{\nu}_-(x) \rangle$$

is a function on $\text{CV}(F_n)$ which depends continuously on x . In particular, the family $\mathcal{G} = \{g(x) \mid x \in C\}$ is compact with respect to the compact open topology for continuous functions on $cv(F_n) \cup \partial cv(F_n)$.

Now $\Sigma(x)$ depends continuously on $x \in \text{CV}(F_n)$ and therefore

$$\mathcal{T} = \cup_{x \in C} \Sigma(x)$$

is a compact subset of $cv(F_n) \cup \partial cv(F_n)$. The restriction to \mathcal{T} of every $g \in \mathcal{G}$ is positive. This implies that

$$b = \inf\{g(T) \mid g \in \mathcal{G}, T \in \mathcal{T}\} > 0.$$

As a consequence, the fourth requirement holds true for $e^s \nu_+, e^{-s} \nu_-$ ($s \in [-s_0, s_0]$) and $B = 1/b$.

We are left with establishing the fifth requirement. For this let $x \in \text{Min}(\nu_+ + \nu_-)$ and let $K \subset \Sigma(x)$ be the compact subset of all trees which are balanced for (ν_+, ν_-) and whose projectivizations are contained in $\text{Spine}_\epsilon(F_n)$. Let $\mathcal{B}(x) \subset \Lambda(x)$ be the closure of the set of all normalized measured laminations which are weighted duals of some basic primitive conjugacy class for any tree which is the projectivization of an element of K . By Corollary 2.9, we have $\tilde{\nu}_+(x), \tilde{\nu}_-(x) \notin \mathcal{B}(x)$.

Let $T_+, T_- \in \Sigma(x)$ be dual to $[\nu_+], [\nu_-]$. If $\zeta \in \mathcal{ML}(F_n)$ is any measured lamination then $\langle T_\pm, \zeta \rangle = 0$ only if the projective class of ζ equals $[\nu_\pm]$. Hence by continuity and compactness, the evaluations on T_+, T_- of the functions from $\mathcal{B}(x)$ are bounded from below by a positive number.

Since $[T_+], [T_-] \in \mathcal{UT}$, the sets

$$U(s) = \{T \in \overline{\text{Spine}_\epsilon(F_n)} \mid T \in \text{Bal}(e^t \nu_+, e^{-t} \nu_-) \text{ for some } t > s\}$$

($s > 0$) form a neighborhood basis for $[T_+]$ in $\overline{\text{Spine}_\epsilon(F_n)}$. This implies that there is some $s > 0$ such that the functions from the set $\mathcal{B}(x)$ are bounded from below on $U(s)$, and a similar statement holds true for $[T_-]$. This establishes the fifth property for $s \in [-s_0, s_0]$.

If $s \in \mathbb{R}$ is arbitrary then there is some $m \in \mathbb{Z}$ and some $s_1 \in [0, s_0)$ such that $s = ms_0 + s_1$. By the choice of s_0 , the function

$$T \rightarrow \langle T, \varphi^m(e^{s_1} \nu_+) + \varphi^m(e^{-s_1} \nu_-) \rangle$$

is a multiple of the function $T \rightarrow \langle T, e^s \nu_+ + e^{-s} \nu_- \rangle$. Since φ^m acts on $(\text{Out}(F_n), d)$ as an isometry, the five properties of a contracting pair for $e^s \nu_+, e^{-s} \nu_-$ follow from the corresponding properties for $e^{s_1} \nu_+, e^{-s_1} \nu_-$. \square

For a B -contracting pair $([\mu], [\nu])$ define the *axis* of $([\mu], [\nu])$ by

$$A = A([\mu], [\nu]) = \cup_{s \in \mathbb{R}} \text{Min}(e^s \mu + e^{-s} \nu).$$

The following simple observation will be used several times in the sequel.

Lemma 3.8. *Let $(\mu, \nu) \in \mathcal{ML}(F_n)^2$ be a B -contracting pair, let $s > 0$ and let $x \in \text{Min}(\mu, \nu), y \in \text{Min}(e^s \mu, e^{-s} \nu)$. Then*

$$\log(\langle y, \nu \rangle / \langle x, \nu \rangle) \geq d_L(x, y) - 3 \log B - \log 2.$$

Proof. Assume without loss of generality that $\nu \in \Lambda(x)$. By the first property of a B -contracting pair, applied to both x and y , we have

$$\langle x, \mu \rangle \in [B^{-1}, B], \langle y, \mu \rangle \leq e^{-2s} B \langle y, \nu \rangle.$$

In particular, if $\tilde{\mu} \in \Lambda(x)$ is the normalization of μ at x then $\langle y, \tilde{\mu} \rangle \leq e^{-2s} B^2 \langle y, \nu \rangle$ and hence $\langle y, \nu + \tilde{\mu} \rangle \leq 2B^2 \langle y, \nu \rangle$. From Corollary 3.4 and the fourth property of a B -contracting pair we then infer that

$$d_L(x, y) \leq \log \langle y, \nu \rangle + 3 \log B + \log 2$$

as claimed. \square

Corollary 3.9. *Let $([\mu], [\nu]) \in \mathcal{PML}(F_n)^2$ be a B -contracting pair. Let μ, ν be representatives of $[\mu], [\nu]$, let $s > 0$ and let $x \in \text{Min}(\mu, \nu), y \in \text{Min}(e^s \mu, e^{-s} \nu)$. Then*

$$2s - 2 \log B \leq d(x, y) \leq 2s + 8 \log B + 2 \log 2.$$

Proof. Let $x \in \text{Min}(\mu, \nu), y \in \text{Min}(e^s \mu, e^{-s} \nu)$ for some $s \geq 0$ and assume that $\nu \in \Lambda(x)$. Lemma 3.3 and Lemma 3.8 show that

$$(5) \quad d_L(x, y) - 3 \log B - \log 2 \leq \log \langle y, \nu \rangle \leq d_L(x, y).$$

By the first property in the definition of a B -contracting pair we have $\langle x, \mu \rangle \in [B^{-1}, B]$ and $\langle y, \nu \rangle / \langle y, \mu \rangle \in [e^{2s} B^{-1}, e^{2s} B]$. Another application of Lemma 3.8 with the roles of x, y, μ, ν exchanged then yields

$$(6) \quad d_L(y, x) - 3 \log B - \log 2 \leq \log(\langle x, \mu \rangle / \langle y, \mu \rangle) \leq 2s - \log \langle y, \nu \rangle + 2 \log B$$

Replacing $-\log \langle y, \nu \rangle$ in inequality (6) by the expression in inequality (5) shows that

$$d_L(x, y) + d_L(y, x) \leq 2s + 8 \log 3 + 2 \log 2.$$

On the other hand, Lemma 3.3 immediately yields that

$$\log(\langle y, \nu \rangle \langle x, \mu \rangle / \langle x, \nu \rangle \langle y, \mu \rangle) \leq d_L(x, y) + d_L(y, x)$$

which implies the lower bound for $d_L(x, y) + d_L(y, x)$ stated in the corollary. \square

For a number $D > 0$, a D -coarse geodesic in a metric space (X, d) is a map $\gamma : [0, m] \rightarrow X$ such that

$$d(\gamma(s), \gamma(t)) \in [t - s - D, t - s + D] \text{ for all } 0 \leq s \leq t \leq m.$$

In particular, if $s < t < u$ then $d(\gamma(s), \gamma(u)) \geq d(\gamma(s), \gamma(t)) + d(\gamma(t), \gamma(u)) - 3D$. As an immediate consequence of Corollary 3.9 we obtain

Corollary 3.10. *For all $B > 0$ there is a number $\kappa_1 = \kappa_1(B) > 0$ with the following property. Let A be the axis of a B -contracting pair. Let $x, y \in A$; then there is a κ_1 -coarse geodesic for the metric d which connects x to y and which is contained in A .*

Proof. Let $([\mu], [\nu])$ be a B -contracting pair. For $s > 0$ and for $x \in \text{Min}(\mu, \nu)$, $y \in \text{Min}(e^s \mu, e^{-s} \nu)$ define a map $\gamma : [0, 2s] \rightarrow A$ by $\gamma(0) = x$, $\gamma(2s) = y$ and by associating to $t \in (0, 2s)$ a point $\gamma(t) \in \text{Min}(e^{t/2} \mu, e^{-t/2} \nu)$. By Corollary 3.9, γ is a $8 \log B + 2 \log 2$ -coarse geodesic. \square

For a B -contracting pair $([\mu], [\nu])$ there is a coarsely well defined projection $\Pi : \text{CV}(F_n) \rightarrow A([\mu], [\nu])$ which is given as follows. For $T \in \text{CV}(F_n)$ choose representatives μ, ν of the classes $[\mu], [\nu]$ such that $T \in \text{Bal}(\mu, \nu)$. Associate to T any point $\Pi(T) \in \text{Min}(\mu + \nu)$ with the properties stated in Definition 3.5.

For pairs of fixed points of iwip elements, a version of the following statement was established in [AK08].

Proposition 3.11. *For every $B > 0$ there is a number $\chi = \chi(B) > 0$ with the following property. Let $([\mu], [\nu])$ be a B -contracting pair and let $T \in \text{Spine}_\epsilon(F_n)$. If K is any closed d -ball about T which is disjoint from the axis $A = A([\mu], [\nu])$ then the diameter of $\Pi(K)$ does not exceed χ . Moreover, if $y \in A$ is a point for which $d(T, y)$ is minimal then $d(y, \Pi(T)) \leq \chi$.*

Proof. Let $([\mu], [\nu]) \in \mathcal{PM}\mathcal{L}(F_n)^2$ be a B -contracting pair and let $\mu, \nu \in \mathcal{ML}(F_n)$ be representatives of $[\mu], [\nu]$. Let A be the axis of $([\mu], [\nu])$ and for $s \in \mathbb{R}$ write $A(s) = \text{Min}(e^s \mu + e^{-s} \nu)$.

Let $T \in \Pi^{-1}(A(0))$; then $T \in \text{Bal}(\mu, \nu)$. Let $x \in \text{CV}(F_n)$ be such that $d(\Pi(T), \Pi(x)) \geq 2B + 8 \log B + 2 \log 2$. Corollary 3.9 shows that $x \in \text{Bal}(e^s \mu, e^{-s} \nu)$ for some $|s| \geq B$. Let γ be a cycle of maximal dilatation for an optimal map $T \rightarrow \Pi(T)$, i.e. a marked homotopy equivalence with the smallest Lipschitz constant. By the results of Francaviglia and Martino [FM08], we may assume that γ is basic for T . Let $\xi \in \Lambda(\Pi(T))$ be the normalized weighted dual of γ . By the fifth property in the definition of a B -contracting pair and by Corollary 3.4, we have

$$\log \langle x, \xi \rangle \geq d_L(\Pi(T), x) - \log B.$$

On the other hand, $d_L(T, \Pi(T)) = -\log \langle T, \xi \rangle$ and therefore

$$(7) \quad d_L(T, x) \geq \log \langle x, \xi \rangle / \langle T, \xi \rangle \geq d_L(T, \Pi(T)) + d_L(\Pi(T), x) - \log B.$$

As a consequence, if $y \in A = A([\mu], [\nu])$ is a minimum for the restriction to A of the function $d_L(T, \cdot)$ then $d_L(\Pi(T), y) \leq 2B + 8 \log B + 2 \log 2 = \chi_0$.

Next we claim that there is a universal constant $C > 0$ such that

$$(8) \quad d_L(\Pi(T), x) \geq d_L(\Pi(T), \Pi(x)) + d_L(\Pi(x), x) - C.$$

For this assume without loss of generality that $s \geq B$ and that $\nu \in \Lambda(\Pi(T))$. Then

$$(9) \quad d_L(\Pi(T), x) \geq \log\langle x, \nu \rangle,$$

on the other hand also

$$(10) \quad \log\langle x, \nu \rangle = \log\langle \Pi(x), \nu \rangle + \log\langle x, \tilde{\nu} \rangle$$

where $\tilde{\nu} = \nu / \langle \Pi(x), \nu \rangle \in \Lambda(\Pi(x))$. Now inequality (5) yields that

$$(11) \quad \log\langle \Pi(x), \nu \rangle \geq d_L(\Pi(T), \Pi(x)) - 3 \log B - \log 2.$$

Moreover, the fourth property in the definition of a B -contracting pair and Corollary 3.4 show that if $\tilde{\mu} \in \Lambda(\Pi(x))$ is the normalization of μ then

$$(12) \quad \log\langle x, \tilde{\nu} \rangle = \log\langle x, \tilde{\nu} + \tilde{\mu} \rangle / 2 \geq d_L(\Pi(x), x) - \log 2B.$$

Inequality (8) now follows from the estimates (9,10,11,12).

If we replace the point $\Pi(T)$ in inequality (8) by an arbitrary $y \in A$ and replace x by T then we conclude that if $y \in A$ is a minimum for the restriction to A of the function $d_L(\cdot, T)$ then $d_L(\Pi(T), y) \leq \chi_1$ where $\chi_1 > \chi_0$ is a universal constant.

The estimates (7) and (8) together show that

$$(13) \quad d_L(T, x) \geq d_L(T, \Pi(T)) + d_L(\Pi(T), \Pi(x)) + d_L(\Pi(x), x) - \chi_3$$

where $\chi_3 > 0$ is a universal constant (in fact $\chi_3 = C + \log B$ will do) provided that $d(\Pi(x), \Pi(T)) \geq \chi_0$. Now exchanging the role of T, x and adding the resulting inequality to inequality (13) shows that inequality (13) holds true for the distance d as well (with an adjustment of χ_3). In particular, we obtain that $d(y, \Pi(T)) \leq \chi_2$ whenever $T \in \text{Spine}_\epsilon(F_n)$ and $y \in A$ is a minimum for the function $d(\cdot, T)$ and where $\chi_2 > 0$ is a universal constant. From this the proposition follows. \square

Remark: Proposition 3.11 is also valid for the left Lipschitz metric.

As in [BF08], we obtain as a corollary

Corollary 3.12. *For all $B > 0, D \geq 0$ there is a number $\kappa_2 = \kappa_2(B, D) > 0$ with the following property. Let A be the axis of a B -contracting pair. Let $y \in A, x \in \text{Spine}_\epsilon(F_n)$ and let $\gamma : [0, m] \rightarrow \text{CV}(F_n)$ be a D -coarse geodesic connecting x to y . Then γ passes through the κ_2 -neighborhood of $\Pi(x)$.*

Proof. Let $\gamma : [0, m] \rightarrow \text{CV}(F_n)$ be a D -coarse geodesic connecting a point $x \in \text{Spine}_\epsilon(F_n)$ to a point y on the axis A of a B -contracting pair. If $d(\gamma(0), A) \leq 1$ then Proposition 3.11 shows that $d(\gamma(0), \Pi(\gamma(0))) \leq \chi + 1$ and there is nothing to show.

Thus let $k > 0$ be the supremum of all numbers $s > 0$ such that the closed $k + 1$ -ball about x is disjoint from A . Proposition 3.11 shows that $d(\Pi(\gamma(0)), \Pi(\gamma(k))) \leq \chi$. Since γ is a D -coarse geodesic, using once more Proposition 3.11 we have

$$(14) \quad \begin{aligned} d(\gamma(0), y) &\geq d(\gamma(0), \gamma(k)) + d(\gamma(k), y) - 3D \\ &\geq d(\gamma(0), \Pi(\gamma(0))) + d(\gamma(k), y) - 3D - 2\chi - 1. \end{aligned}$$

On the other hand, if $d(\gamma(k), \Pi(\gamma(k))) \geq 3D + 6\chi + 3$ then we also have

$$\begin{aligned} d(\gamma(k), y) &\geq d(\gamma(k), \Pi(\gamma(k))) + d(\Pi(\gamma(k)), y) - 2\chi - 1 \\ &\geq d(\Pi(\gamma(k)), y) + 3D + 4\chi + 2 \geq d(\Pi(\gamma(0)), y) + 3D + 3\chi + 2. \end{aligned}$$

Together with the triangle inequality, this contradicts the estimate (14) and completes the proof of the corollary. \square

4. IWIP SUBGROUPS OF $\text{Out}(F_n)$

In this section, we use the results obtained so far to shed some light on subgroups of $\text{Out}(F_n)$ which consist of iwip elements. We begin with an observation of Kapovich and Lustig [KL09b]. For its formulation, we call a pair (B_1, B_2) of disjoint closed subsets of $\text{Spine}_\epsilon(F_n)$ *positive* if the following holds true. Let $K_i \subset \mathcal{PML}(F_n)$ be the set of all projective measured laminations which are contained in the closure of the projective measured laminations dual to basic primitive conjugacy classes for trees in the set $B_i \cap \text{Spine}_\epsilon(F_n)$. Then $([\mu_1], [\mu_2])$ is a positive pair for all $[\mu_i] \in K_i$.

Lemma 4.1. *Let $(B_1, B_2) \subset \text{Spine}_\epsilon(F_n)$ be a positive pair of closed disjoint sets. Let $\varphi \in \text{Out}(F_n)$ be such that*

$$\varphi(\text{Spine}_\epsilon(F_n) - B_2) \subset B_1 \text{ and } \varphi^{-1}(\text{Spine}_\epsilon(F_n) - B_1) \subset B_2.$$

Then φ is iwip, with attracting fixed point in B_1 and repelling fixed point in B_2 .

Proof. Let $\varphi \in \text{Out}(F_n)$ be as in the lemma. Then the subgroup of $\text{Out}(F_n)$ generated by φ is infinite, and $\varphi(B_1)$ is contained in the interior of B_1 . Moreover, we have $\varphi^{-1}(B_1) \cup B_2 = \text{Spine}_\epsilon(F_n)$.

For $i = 1, 2$ let $K_i \subset \mathcal{PML}(F_n)$ be the closure of the set of all projective measured laminations which are dual to basic primitive conjugacy classes for trees in B_i ($i = 1, 2$). Then $\varphi^{-1}(K_1) \cup K_2$ is a closed non-empty $\text{Out}(F_n)$ -invariant subset of $\mathcal{PML}(F_n)$ which contains all projective measured laminations which are dual to basic primitive conjugacy classes for trees in $\text{Spine}_\epsilon(F_n)$. Now the closure of the set of all projective measured laminations which are dual to basic primitive conjugacy classes for trees in $\text{Spine}_\epsilon(F_n)$ is a closed non-empty $\text{Out}(F_n)$ -invariant subset of $\mathcal{PML}(F_n)$ and hence $\mathcal{PML}(F_n) = \varphi^{-1}K_1 \cup K_2$ by minimality [KL07].

Since $\varphi(\mathcal{PML}(F_n) - K_2) \subset K_1$ and $\varphi^{-1}(\mathcal{PML}(F_n) - K_2) \subset K_2$, we conclude that if

$$A_1 = \bigcap_i \varphi^i K_1 \neq \emptyset, \quad A_2 = \bigcap_i \varphi^{-i} K_2 \neq \emptyset$$

then every fixed point for the action of φ on $\mathcal{PML}(F_n)$ is contained in $A_1 \cup A_2$. Moreover, each of the sets A_1, A_2 contains at least one such fixed point. If $[\nu_+] \in A_1, [\nu_-] \in A_2$ are such fixed points then $([\nu_+], [\nu_-]) \in \mathcal{PML}(F_n)^2$ is a positive pair.

Let $\nu_+, \nu_- \in \mathcal{ML}(F_n)$ be representatives of $[\nu_+], [\nu_-]$. We claim that up to replacing φ by φ^{-1} there are numbers $\lambda_+ > 1, \lambda_- > 1$ such that $\varphi(\nu_+) = \lambda_+ \nu_+$ and $\varphi^{-1}(\nu_-) = \lambda_- \nu_-$.

Namely, otherwise up to exchanging φ and φ^{-1} , we have $\varphi(\nu_+ + \nu_-) \leq \nu_+ + \nu_-$ (as functions on $\text{CV}(F_n)$). Thus if $x \in \text{Spine}_\epsilon(F_n)$ is arbitrary and if ν_+, ν_- are normalized in such a way that $\nu_+ \in \Lambda(x), \nu_- \in \Lambda(x)$ then for every $k \geq 0$ we have

$$\langle \varphi^{-k} x, \nu_+ + \nu_- \rangle = \langle x, \varphi^k(\nu_+ + \nu_-) \rangle \leq 2.$$

However, by Lemma 3.2, the function $\langle \cdot, \nu_+ + \nu_- \rangle$ is proper which contradicts the fact that $\text{Out}(F_n)$ acts property discontinuously on $\text{Spine}_\epsilon(F_n)$ and that the order of φ is infinite by assumption.

Now if φ is not an iwip and if φ is non-geometric (i.e. φ does not preserve any non-trivial conjugacy class) then by Proposition 7.1 of [KL09b] there are measured laminations $\nu_+, \nu_- \in \mathcal{ML}(F_n)$ with

$$\varphi \nu_+ = \lambda_+ \nu_+, \varphi^{-1} \nu_- = \lambda_- \nu_-$$

for some $\lambda_+, \lambda_- > 1$ and such that the pair (ν_+, ν_-) is *not* positive. This contradicts the above discussion.

If φ is geometric and reducible then Thurston's classification of surface homeomorphisms implies that either there is a fixed point for the action of φ on $\mathcal{ML}(F_n)$ which is excluded by the above discussion, or there is a pair of points $(\mu, \nu) \in \mathcal{ML}(F_n)$ which is not positive and there is a number $\lambda > 1$ such that $\varphi \mu = \lambda \mu, \varphi^{-1} \nu = \lambda \nu$ which again is impossible. \square

Let $M \subset \partial \text{CV}(F_n)$ be the closure of the set \mathcal{UT} as defined in Section 2. Since the action of $\text{Out}(F_n)$ on $\mathcal{PML}(F_n)$ is minimal [KL07], Lemma 2.6 shows that the action of $\text{Out}(F_n)$ on M is minimal as well. In particular, we have $M \subset \partial \text{Spine}_\epsilon(F_n)$ for all sufficiently small ϵ . As in [H09] we can use Lemma 4.1 to show

Corollary 4.2. *The set of pairs of fixed points of iwip elements is dense in $M \times M$.*

Proof. Let $V_1, V_2 \subset M$ be open disjoint sets. Since \mathcal{UT} is dense in M , by making V_1, V_2 smaller we may assume that if $([\mu], [\nu])$ is any pair of projective measured laminations so that μ is supported in the zero lamination of a tree in V_1 and ν is supported in the zero lamination of a tree in V_2 then the pair $([\mu], [\nu])$ is positive. By continuity of the length pairing and by Lemma 2.8 we may moreover assume that there are compact disjoint neighborhoods $B_1, B_2 \subset \overline{\text{Spine}_\epsilon(F_n)}$ of V_1, V_2 such that the sets $B_1 \cap \text{Spine}_\epsilon(F_n), B_2 \cap \text{Spine}_\epsilon(F_n)$ satisfy the assumptions in Lemma 4.1. We may also assume that there is an iwip-element φ whose fixed points a, b are contained in $M - B_1 - B_2$.

Since V_1 is open and the action of $\text{Out}(F_n)$ on M is minimal, there is an iwip element $u \in \text{Out}(F_n)$ with attracting fixed point in V_1 . Since the stabilizer in $\text{Out}(F_n)$ of a fixed point of an iwip element is virtually cyclic [BFH97], we may assume that the repelling fixed point of u is distinct from a, b . Since u acts with north-south dynamics on $\partial \text{CV}(F_n)$, up to perhaps changing u by a nontrivial power we may assume that $u\{a, b\} \subset V_1$. Then $v = u\varphi u^{-1}$ is an iwip element with both

fixed points in V_1 . Similarly, there is an iwip element w with both fixed points in V_2 .

Via replacing v, w by sufficiently high powers we may assume that $v(M - V_1) \subset V_1, v^{-1}(M - V_1) \subset V_1$ and $w(M - V_2) \subset V_2, w^{-1}(M - V_2) \subset V_2$. Then $wv(M - V_1) \subset V_2, v^{-1}w^{-1}(M - V_2) \subset V_1$. Moreover, by perhaps replacing v, w by a suitable power we may assume that the assumptions in Lemma 4.1 are satisfied for wv . Then wv is an iwip whose pair of fixed points is contained in $V_1 \times V_2$. \square

We also obtain information on subgroups Γ of $\text{Out}(F_n)$ which contain at least one iwip element. For this call iwip elements $\alpha, \beta \in \text{Out}(F_n)$ *independent* if the fixed point sets for the action of α, β on $\partial\text{CV}(F_n)$ do not coincide.

Proposition 4.3. *Let $\Gamma < \text{Out}(F_n)$ be a subgroup which contains an iwip element. If Γ is not virtually cyclic then there are two independent iwip elements $\alpha, \beta \in \Gamma$ with the following properties.*

- (1) *The subgroup G of Γ generated by α, β is free and consists of iwip elements.*
- (2) *There are infinitely many elements $u_i \in G$ ($i > 0$) with fixed points $a_i, b_i \in \mathcal{UT}$ such that for all i the $\text{Out}(F_n)$ -orbit of $(a_i, b_i) \in \mathcal{UT} \times \mathcal{UT} - \Delta$ is distinct from the orbit of (b_j, a_j) ($j > 0$) or (a_j, b_j) ($j \neq i$).*

Proof. Let $\Gamma < \text{Out}(F_n)$ be a subgroup which contains an iwip element α . Let $[T_+], [T_-]$ be the fixed points of α in $\partial\text{CV}(F_n)$. If the set $\{[T_+], [T_-]\}$ is invariant under the action of Γ then by Theorem 2.14 of [BFH97], the group Γ is virtually cyclic.

Thus we may assume that there is some $\gamma \in \Gamma$ with $\gamma[T_+] \in \mathcal{UT} - \{[T_+], [T_-]\}$. Then $\gamma[T_+], \gamma[T_-]$ are the fixed points of the iwip element $\beta = \gamma \circ \alpha \circ \gamma^{-1}$. By Proposition 2.16 of [BFH97], the iwip elements β and α are independent. Then the fixed point sets of α, β in $\partial\text{CV}(F_n)$ are disjoint (Proposition 2.16 of [BFH97]), and α, β act with north-south dynamics on the compact space $M \subset \partial\text{CV}(F_n)$.

The usual ping-pong lemma, applied to the action of α, β on M , implies that for sufficiently large $k > 0, \ell > 0$ the subgroup G of Γ generated by α^k, β^ℓ is free. By the main result of [KL09b], if α, β are non-geometric then we may assume that this group consists entirely of non-geometric iwip automorphisms. In particular, each non-trivial element acts with north-south-dynamics on M , with fixed points contained in \mathcal{UT} . If α, β are arbitrary then the claim follows from Lemma 4.1.

To show the second part of the proposition we have to find infinitely many elements in G which are mutually not conjugate in $\text{Out}(F_n)$ and not conjugate to their inverses. For this we follow the argument in the proof of Proposition 5.7 of [H09]. Namely, let $\alpha, \beta \in G$ be independent iwip elements. Let $\Delta \subset M \times M$ be the diagonal. Then the $\text{Out}(F_n)$ -orbit of the pair of fixed points $(a_+, a_-), (b_+, b_-) \in M \times M - \Delta$ for the action of α, β on $M \subset \partial\text{CV}(F_n)$ is a closed subset of $M \times M - \Delta$ (Theorem 5.3 of [BFH97]). Since α, β are not conjugate in $\text{Out}(F_n)$ the $\text{Out}(F_n)$ -orbits of (a_+, a_-) and (b_+, b_-) are distinct. This implies that there are open neighborhoods U_+, U_- of a_+, a_- and V_+, V_- of b_+, b_- such that the $\text{Out}(F_n)$ -orbit of (a_+, a_-) does not intersect $V_+ \times V_-$ and that the $\text{Out}(F_n)$ -orbit of (b_+, b_-)

does not intersect $U_+ \times U_-$. Via replacing α, β by suitable powers we may assume that

$$\alpha(M - \bar{U}_-) \subset U_+, \alpha^{-1}(M - \bar{U}_+) \subset U_-, \beta(M - \bar{V}_-) \subset V_+ \text{ and } \beta^{-1}(M - \bar{V}_+) \subset V_-.$$

For numbers $n, m, k, \ell > 2$ consider the element

$$f = f_{nmk\ell} = \alpha^n \beta^m \alpha^k \beta^{-\ell} \in G.$$

It satisfies $f(\bar{U}_+) \subset U_+, f^{-1}(\bar{V}_+) \subset V_+$ and hence the attracting fixed point of f is contained in U_+ and its repelling fixed point is contained in V_+ .

Since $n > 2$, the conjugate $f_1 = \beta^{-1}f\beta$ satisfies $f_1(\bar{U}_+) \subset U_+$ and $f_1^{-1}(\bar{U}_-) \subset U_-$, i.e. its attracting fixed point is contained in U_+ and its repelling fixed point is contained in U_- . Furthermore, since $m > 2$, its conjugate $f_2 = \beta^{-1}\alpha^{-n}f\alpha^n\beta$ has its attracting fixed point in V_+ and its repelling fixed point in V_- , and its conjugate $f_3 = \beta^{-1}f\beta$ has its attracting fixed point in V_- and its repelling fixed point in V_+ .

As a consequence, f is conjugate to both an element with fixed points in $U_+ \times U_-$ as well as to an element with fixed points in $V_+ \times V_-$. This implies that f is not conjugate to either α or β . Moreover, since α and β can not both be conjugate to β^{-1} , by eventually adjusting the size of V_+, V_- we may assume that f is not conjugate to β^{-1} .

We claim that via perhaps increasing the values of n, ℓ we can achieve that $f_{nmk\ell}$ is not conjugate to f^{-1} . Namely, as $n \rightarrow \infty$, the fixed points of the conjugate $\alpha^{-n}f_{(2n)mkl}\alpha^n = \alpha^n\beta^m\alpha^k\beta^{-\ell}\alpha^n$ of $f_{(2n)mkl}$ converge to the fixed points of α . Similarly, the fixed points of the conjugate $\beta^{-\ell}f_{nmk(2\ell)}^{-1}\beta^\ell = \beta^\ell\alpha^{-k}\beta^{-m}\alpha^{-n}\beta^\ell$ of $f_{nmk(2\ell)}^{-1}$ converge as $\ell \rightarrow \infty$ to the fixed points of β . Thus after possibly conjugating with α, β , if $f_{nmk\ell}$ is conjugate in $\text{Out}(F_n)$ to $f_{nmk\ell}^{-1}$ for all n, ℓ then there is a sequence of pairwise distinct elements $g_i \in \text{Out}(F_n)$ which map a fixed compact subset K of $\text{Spine}(F_n)$ into a fixed compact subset W of $\text{Spine}(F_n)$ and such that $g_i(a, b) \rightarrow (b_+, b_-)$. Since $\text{Out}(F_n)$ acts properly discontinuously on $\text{Spine}(F_n)$ this is impossible.

Inductively we can construct in this way a sequence of elements of G with the properties stated in the second part of the proposition. \square

5. SECOND BOUNDED COHOMOLOGY

As a consequence of the results in the previous sections, we can apply the procedure from [H08] to show that for every subgroup Γ of $\text{Out}(F_n)$ which is not virtually abelian and which contains an iwip element, the second bounded cohomology group $H_b^2(\Gamma, \ell^p(\Gamma))$ ($p > 1$) is infinite dimensional.

Namely, the construction from [H08] only uses the following two properties.

- (1) There is an $\text{Out}(F_n)$ -invariant closed subset $\mathcal{A}(B)$ of positive pairs of points in $\mathcal{PML}(F_n)$ which have the contraction properties stated in Proposition 3.11 of Section 3.

- (2) There is a free subgroup G of Γ with two generators and the properties stated in Proposition 4.3.

We complete this note with a few more details of the construction. Recall that the axis $A([\mu], [\nu])$ of a B -contracting pair is defined to be the union $\cup_s \text{Min}(e^s \mu + e^{-s} \nu)$ where μ, ν are any representatives of $[\mu], [\nu]$. Recall also the definition of the coarsely well defined projection $\Pi_{([\mu], [\nu])} : \text{CV}(F_n) \rightarrow A([\mu], [\nu])$. For representatives μ, ν of $[\mu], [\nu]$, this projection associates to a tree $T \in \text{Bal}(\mu, \nu)$ a point $\Pi_{([\mu], [\nu])}(T) \in \text{Min}(\mu, \nu)$. If $[\mu], [\nu] \in \mathcal{UMC}$ then the projection $\Pi_{([\mu], [\nu])}$ extends to a coarsely well defined projection $\text{CV}(F_n) \cup \partial \text{CV}(F_n) - \{\omega([\mu]), \omega([\nu])\}$ (see Section 2 and Section 3 for definitions).

By Lemma 3.2, an axis $A([\mu], [\nu])$ can be defined for any positive pair $([\mu], [\nu])$, and there is also a coarsely well defined projection $\text{CV}(F_n) \rightarrow A([\mu], [\nu])$. Note however that in general we can not expect that for two distinct choices Π, Π' of such a projection, the distance $d(\Pi(x), \Pi'(x))$ is bounded from above independent of $x \in \text{CV}(F_n)$.

The *Hausdorff distance* between two closed (not necessarily compact) sets $A, B \subset \text{Spine}_\epsilon(F_n)$ is the infimum of the numbers $r \in [0, \infty]$ such that A is contained in the r -neighborhood of B and B is contained in the r -neighborhood of A . A *ray* in an axis $A([\mu], [\nu])$ is defined to be a set of the form $\cup_{s \geq 0} \text{Min}(e^s \mu + e^{-s} \nu)$ where μ, ν are some representatives of $[\mu], [\nu]$. If $x \in \text{Min}(\mu + \nu)$ then we also write $A([\mu], x, [\nu])$ to denote this ray.

We begin with the following

Lemma 5.1. *Let $B > 0$ and let $([\mu_1], [\nu_1]), ([\mu_2], [\nu_2]) \in \mathcal{UMC}^2$ be B -contracting pairs. If $[\mu_1] \neq [\mu_2]$ then there is a number $C = C(B) > B$ only depending on B with the following property. There is a ray $A([\mu_1], y, [\nu_1]) \subset A([\mu_1], [\mu_2])$ whose Hausdorff distance to $A([\mu_1], \Pi_{([\mu_1], [\nu_1])}(\omega[\mu_2]), [\nu_1])$ is at most $C(B)$.*

Proof. We may assume that $[\mu_2] \neq [\nu_1]$. By assumption, $[\mu_1] \neq [\mu_2]$ and hence $([\mu_1], [\mu_2]) \in \mathcal{UMC}^2$ is a positive pair. Let $T = \omega([\mu_2]) \in \mathcal{UT}$ be the tree which is dual to $[\mu_2]$. Assume that μ_1, ν_1 are representatives of $[\mu_1], [\nu_1]$ such that $T \in \text{Bal}(\mu_1, \nu_1)$.

Let $\Pi = \Pi_{([\mu_1], [\nu_1])}$, let $x = \Pi(\omega[\mu_2])$ and let $\mu_1, \mu_2 \in \Lambda(x)$ be representatives of $[\mu_1], [\mu_2]$. We claim that for $s > B$ the Hausdorff distance between $\text{Min}(e^s \mu_1 + e^{-s} \mu_2)$ and $\text{Min}(e^s \mu_1 + e^{-s} \nu_1)$ is bounded from above by a universal constant $C(B) > 0$ only depending on B .

Namely, by Lemma 2.8, by continuity and by the fifth property in the definition of a B -contracting pair, applied to $([\mu_1], [\nu_1])$, for every tree $y \in \Sigma(x) \cap \bigcup_{t \in (B, \infty)} \text{Bal}(e^t \mu_1, e^{-t} \nu_1)$ we have $\langle y, \mu_2 \rangle \geq 1/B$. This implies that

$$(15) \quad \log \langle y, \mu_2 \rangle \geq d_L(x, y) - \log B \text{ for } y \in \text{CV}(F_n) \cap \bigcup_{t \in (B, \infty)} \text{Bal}(e^t \mu_1, e^{-t} \nu_1).$$

By the discussion in Section 3, there is a number $\kappa_0 = \kappa_0(B) > 0$ such that

$$|d_L(x, y) - s| \leq \kappa_0$$

for every $y \in \text{Min}(e^s \mu_1 + e^{-s} \nu_1)$. As a consequence, there is a number $\kappa_1 = \kappa_1(B) > 0$ such that

$$(16) \quad \langle y, e^s \mu_1 + e^{-s} \nu_2 \rangle \leq \kappa_1 \text{ for } y \in \text{Min}(e^s \mu_1 + e^{-s} \nu_1).$$

Moreover, by inequalities (8) and (15) and the definition of a B -contracting pair, there is a number $\kappa_2 = \kappa_2(B) > 0$ such that $\langle y, e^s \mu_1 + e^{-s} \nu_2 \rangle > \kappa_1$ whenever $t \geq s$ and $y \in \text{Bal}(e^t \mu_1, e^{-t} \nu_2) \cap \text{Spine}_\epsilon(F_n)$ are such that $d_L(\text{Min}(e^s \mu_1, e^{-s} \nu_1), y) \geq \kappa_2$. Since $\text{Out}(F_n)$ acts on $\text{Spine}_\epsilon(F_n)$ cocompactly, there is a number $\kappa_3 = \kappa_3(B) > 0$ such that $d_L(u, z) \geq \kappa_2$ for all $u, z \in \text{Spine}_\epsilon(F_n)$ with $d(u, z) \geq \kappa_3$. This implies that $y \notin \text{Min}(e^s \mu_1 + e^{-s} \nu_2)$ if $y \in \text{Bal}(e^t \mu_1, e^{-t} \nu_2) \cap \text{Spine}_\epsilon(F_n)$ is such that $d(y, \text{Min}(e^s \mu_1 + e^{-s} \nu_1)) \geq \kappa_3$ where $\kappa_3 > 0$ only depends on B .

The claim now follows from another application of the same argument (but with the roles of μ_1 and μ_2 exchanged) for points $y \in \text{Bal}(e^t \mu_1, e^{-t} \nu_2)$ with $t \in [B, s]$.

Since by the definition of a B -contracting pair the diameter of $\cup_{s \in [0, B]} \text{Min}(e^s \mu_1 + e^{-s} \nu_1)$ is uniformly bounded, the discussion above shows that the Hausdorff distance between $\cup_{s > B} \text{Min}(e^s \mu_1 + e^{-s} \nu_2)$ and $\cup_{s \geq 0} \text{Min}(e^s \mu_1 + e^{-s} \nu_2)$ is uniformly bounded. \square

Corollary 5.2. *Let $B > 0$ and let $([\mu_1], [\nu_1]), ([\mu_2], [\nu_2]) \in \mathcal{UM}\mathcal{L}^2$ be B -contracting pairs. If $[\mu_1] \neq [\mu_2]$ then there is a number $C > 0$ such that $A([\mu_1], [\mu_2])$ is contained in the C -neighborhood of $A([\mu_1], [\nu_1]) \cup A([\mu_2], [\nu_2])$.*

Proof. Since the pair $([\mu_1], [\mu_2])$ is positive by assumption, for any representatives μ_1, μ_2 and all $s < t$ the set $\cup_{u=s}^t \text{Min}(e^u \mu_1 + e^{-u} \mu_2)$ is compact. Thus the corollary follows from Lemma 5.1. \square

For a number $B > 0$, define a positive pair $([\mu], [\nu]) \in \mathcal{PM}\mathcal{L}(F_n)$ to be *geometrically B -contracting* if for every $x \in \text{Spine}_\epsilon(F_n)$ the following holds true. Let Q be a ball about x of radius $R > 0$ which does not intersect $A([\mu], [\nu])$ and let Q_0 be the connected component of Q containing x (note that since the Lipschitz metric is not geodesic, it is unclear whether large metric balls are connected). Then the diameter of $\Pi_{([\mu], [\nu])}(Q_0)$ does not exceed B . We call a positive pair *geometrically contracting* if it is geometrically B -contracting for some $B > 0$.

Corollary 5.3. *Let $([\mu_1], [\nu_1]) \in \mathcal{UM}\mathcal{L}^2$, $([\mu_2], [\nu_2]) \in \mathcal{UM}\mathcal{L}^2$ be B -contracting pairs. If $[\mu_1] \neq [\mu_2]$ then the pair $([\mu_1], [\mu_2])$ is geometrically contracting.*

Proof. Lemma 5.1 and Lemma 3.11 show that there are geometrically contracting rays $A([\mu_1], y_1, [\mu_2])$, $A([\mu_2], y_2, [\mu_1])$. This means that there is a number $C > 0$ with the following property. If $x \in \text{Spine}_\epsilon(F_n)$ and if Q_0 is the connected component containing x of a metric ball about x which is disjoint from $A([\mu_1], [\mu_2])$ then the diameter of the intersection of $\Pi_{([\mu_1], [\mu_2])}(Q_0)$ with each of these two rays is at most C .

The corollary now follows from continuity of the length pairing and from Lemma 3.2. Namely, let μ_1, μ_2 be any representatives of $[\mu_1], [\mu_2]$ and let $s > 0$. If $x_1 \in Q_0 \cap \text{Bal}(\mu_1, \mu_2), x_2 \in Q_0 \cap \text{Bal}(e^s \mu_1, e^{-s} \mu_2)$ then for every $u \in [0, s]$ there is some $z \in Q_0 \cap \text{Bal}(e^u \mu_1, e^{-u} \mu_2)$. \square

Now let $\Gamma < \text{Out}(F_n)$ be any subgroup which is not virtually abelian and which contains at least one iwip element. By Proposition 4.3, there is an iwip-element g with fixed points $[\mu], [\nu] \in \mathcal{UMC}$ so that the $\text{Out}(F_n)$ -orbits of $([\mu], [\nu])$ and $([\nu], [\mu])$ are distinct. By Theorem 2.14 of [BFH97], the stabilizer of $[\nu]$ in $\text{Out}(F_n)$ is virtually cyclic. Since Γ is not virtually cyclic by assumption, this means that there is some $h \in \Gamma$ such that the stabilizer of the pair $(h[\nu], [\nu])$ is trivial.

By Proposition 3.7 and invariance under the action of $\text{Out}(F_n)$, the pairs $([\mu], [\nu])$ and $(h[\mu], h[\nu])$ are B -contracting for some $B > 0$. By Corollary 5.3, for $u \neq v \in \Gamma$ with $u[\nu] \neq v[\nu]$ the pair of points $(u[\nu], v[\nu])$ is geometrically contracting. In particular, Corollary 5.2 shows that there is a number $C > 0$ such that $A(h[\nu], [\nu])$ is contained in the C -neighborhood of $A(h[\nu], h[\mu]) \cup A([\nu], [\mu])$.

Define a *truncated axis* of a positive pair $([\mu], [\nu])$ to be a set of the form $A([\mu], x, [\nu]) \cup A([\nu], y, [\mu])$ for some $x, y \in A([\mu], [\nu])$. Note that with this definition, an axis is a truncated axis as well. By Lemma 5.1, for $u \neq v \in \Gamma$ such that $u[\nu] \neq v[\nu]$ there is some $x \in A(u[\nu], v[\nu])$ such that the Hausdorff distance between the ray $A(u[\nu], x, v[\nu])$ and some ray $A(u[\nu], y, uh^{-1}[\nu])$ is at most $2C$. Choose x in such a way that the ray $A(u[\nu], x, v[\nu])$ is as large as possible with this property. Using the same procedure for the geometrically contracting pairs $(v[\nu], u[\nu])$ and $(v[\nu]h^{-1}, v[\nu])$ defines a truncated axis $A_t(u[\nu], v[\nu])$.

The above construction associates to any ordered pair of points $(\sigma, \eta) \in \Gamma[\nu] \times \Gamma[\nu] - \Delta$ a truncated axis $A_t(\sigma, \eta) \subset A(\sigma, \eta)$. The assignment $\rho : (\sigma, \eta) \rightarrow A_t(\sigma, \eta)$ is invariant under exchange of σ and η , and it is invariant under the action of Γ .

Let $\mathcal{A}([\nu])$ be the union of the ordered pairs of distinct points in $\Gamma[\nu]$ with the Γ -translates of $([\mu], [\nu]), ([\nu], [\mu])$. The group Γ naturally acts on $\mathcal{A}([\nu])$ from the left. For $x \in \text{Spine}_e(F_n)$ define a distance δ_x on $\mathcal{A}([\nu])$ as follows.

As in [H08], for $([y], [z]) \neq ([v], [w]) \in \mathcal{A}([\nu])$ consider the two "endpoints" of a subray of the truncated axis $A_t([y], [z])$, say of the subray ending at $[y]$. Then these endpoints are $[y]$ and $\text{Min}(y, z)$ for suitably chosen representatives y, z of $[y], [z]$ so that the ray equals $\cup_{s \geq 0} \text{Min}(e^s y, e^{-s} z)$. Choose similarly representatives (v, w) of $[v], [w]$. Let $s_1, s_2 \in \mathbb{R}$ be such that $x \in \text{Bal}(e^{s_1} x, e^{-s_1} y)$ and $y \in \text{Bal}(e^{s_2} x, e^{-s_2} y)$. If one of the numbers s_1, s_2 are smaller than 0 then define $\tau_x^1((y), [z]), ([v], [w])) = 0$. Otherwise let $\tau_x^1((y), [z]), ([v], [w]))$ be the length of the maximal connected subinterval of $[0, \infty)$ so that the Hausdorff distance of the subsets of the axes $A([y], [z]), A([v], [w])$ defined by these intervals does not exceed $2C(B)$. Define similarly τ_x^2 for the other two subrays of the truncated axes and let $\tau_x = \max\{\tau_x^1, \tau_x^2\}$. By the contraction property, these functions satisfy the ultrametric inequality. Therefore these functions can be used to define a distance δ_x on $\mathcal{A}([\nu])$ (see [H08] for details).

The metrics δ_x as x varies through $\text{Spine}_\epsilon(F_n)$ can be used exactly as in the proof of Proposition 6.1 of [H08] to construct nontrivial bounded cohomology classes for Γ with coefficients in $\ell^p(\Gamma)$ for every $p > 1$ (with the above discussion, the proof can be copied word by word, with “geodesic” replaced by “axis”).

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