ERGODIC PROPERTIES OF GIBBS MEASURES ON NILPOTENT COVERS

URSULA HAMENSTÄDT

ABSTRACT. Let $M = \tilde{M}/\Gamma$ be a closed negatively curved manifold with universal covering $\tilde{M}$ and fundamental group Γ. Every Gibbs equilibrium state ν of a Hölder continuous function on the unit tangent bundle T^1M of M projects to a Γ-invariant ergodic measure class $mc(\nu_+)$ on the ideal boundary $\partial\tilde{M}$ of $\tilde{M}$. We show that this measure class is also ergodic under the action of any normal subgroup Γ' of Γ for which the factor group Γ/Γ' is nilpotent.

Let M be a closed Riemannian manifold of negative sectional curvature, with fundamental group Γ. The geodesic flow Φ^t of M is an Anosov flow on the unit tangent bundle T^1M of M. In particular, every Hölder continuous function on T^1M admits a unique *Gibbs equilibrium state* which is a Φ^t-invariant Borel-probability measure on T^1M.

Denote by $\tilde{M}$ the universal covering of M with unit tangent bundle $T^1\tilde{M}$ and *ideal boundary* $\partial\tilde{M}$. The group Γ acts on $T^1\tilde{M}$ and on $\partial\tilde{M}$, and there is a natural Γ-equivariant projection $\pi\colon T^1\tilde{M} \to \partial\tilde{M}$.

The Gibbs equilibrium state ν of a Hölder continuous function on T^1M defines a Γ-invariant ergodic measure class $mc(\nu_+)$ on $\partial\tilde{M}$. If $\tilde{\nu}$ is the lift of ν to $T^1\tilde{M}$, then this measure class is determined by the property that for every Borel subset B of $\partial\tilde{M}$ we have $mc(\nu_+)(B) = 0$ if and only if $\tilde{\nu}(\pi^{-1}(B)) = 0$. We call $mc(\nu_+)$ the *positive disintegration class* of ν.

The *flip* $\mathcal{F}\colon T^1M \to T^1M$ is defined by $\mathcal{F}(v) = -v$ and preserves the class of Gibbs equilibrium states of Hölder continuous functions. The measure ν is invariant under the flip if and only if $mc(\nu_+) = mc((\mathcal{F}\circ\nu)_+)$.

Let Γ' be a normal subgroup of Γ, and consider the regular covering $M' = \tilde{M}/\Gamma'$ of M. The lift ν' of the Gibbs equilibrium state ν to the unit tangent bundle T^1M' of M' is recurrent under the geodesic flow on T^1M' if and only if it is ergodic, and this is the case if and only if the measure class $mc(\nu_+) \times mc((\mathcal{F}\circ\nu)_+)$ on $\partial\tilde{M}\times\partial\tilde{M}$ is ergodic under the diagonal action of Γ' (this fact is well known, but difficult to trace back in the literature).

Rees showed [R] that if the measure ν is flip-invariant and if the factor group $\Gamma_0 = \Gamma/\Gamma'$ is abelian, then the geodesic flow on T^1M' is ergodic if and only if the rank of Γ/Γ' is not bigger than 2 (compare also [LS] and [G]). This is analogous to the fact that Brownian motion on $\mathbf{R}$ and $\mathbf{R}^2$ is recurrent, and Brownian motion on $\mathbf{R}^k$ for $k \geq 3$ is transient.

On the other hand, the exit boundary of Brownian motion on a simply connected manifold $\tilde{M}$ of bounded negative curvature can naturally be identified with the ideal boundary of $\tilde{M}$ equipped with an exit probability measure (see [K] for references). If $\tilde{M}$ covers a compact

2000 *Mathematics Subject Classification.* 37D40.

manifold M, then this exit probability defines the positive disintegration class of a unique Gibbs measure on T^1M [L1]. Ledrappier calls this Gibbs measure the *harmonic measure*. Kaimanovich [K] observed that similar statements hold for exit measures of a large class of Γ-equivariant diffusion processes on $\tilde M$ including Brownian motion of all (not necessarily negatively curved) Γ-invariant Riemannian metrics. This suggests that there should be a dictionary between global properties of exit measures of Γ-equivariant diffusion processes and the corresponding global properties of Gibbs measures on T^1M.

The purpose of this note is to add to this type of results. Let ν be a Gibbs equilibrium state on T^1M. Recall that ergodicity of $mc(\nu_+)\times mc((\mathcal{F}\circ\nu)_+)$ under the diagonal action of a normal subgroup Γ' of Γ implies ergodicity of $mc(\nu_+)$ under Γ', but the reverse is not true. Thus the result of Rees shows that the action of Γ' with respect to the measure class $mc(\nu_+)$ is ergodic if $\Gamma_0=\Gamma/\Gamma'$ is an abelian group of rank at most two, but it does not give any information on this question of ergodicity for more general groups. On the other hand, if the factor group $\Gamma_0=\Gamma/\Gamma'$ is nilpotent and if ν is the harmonic measure for any smooth Riemannian metric on M then it follows from the results of Lyons and Sullivan [LS] that the measure class $mc(\nu_+)$ on $\partial\tilde M$ is ergodic under the action of Γ'. Our goal is to extend this result to arbitrary Gibbs equilibrium states. We show.

Theorem. *Let $M=\tilde M/\Gamma$ be a closed negatively curved manifold, and let Γ' be a normal subgroup of Γ such that $\Gamma_0=\Gamma/\Gamma'$ is nilpotent. Then the positive disintegration class $mc(\nu_+)$ of a Gibbs equilibrium state of a Hölder continuous function on T^1M is ergodic under the action of Γ'.*

For abelian coverings our theorem was earlier proved by Coudene (I am grateful to the referee for informing me about Coudene's paper). His argument is based on the coding of the geodesic flow by a suspension of a subshift of finite type, while our proof uses diffusion processes and is based on the ideas of Lyons and Sullivan [LS].

We begin with recalling that there is a Φ^t-invariant Hölder continuous decomposition $TT^1M=TW^{su}\oplus TW^{ss}\oplus\mathbb{R}X$ where X is the generator of the geodesic flow and the bundles TW^{su} and TW^{ss} are the tangent bundles of the *strong unstable* and *strong stable foliation* of T^1M, respectively.

Let $f\colon T^1M\to\mathbb{R}$ be a Hölder continuous function of vanishing pressure. The unique Gibbs equilibrium state ν defined by f can be described as follows. There is a family ν^{su} of locally finite Borel measures on strong unstable manifolds, unique up to a universal constant, which are quasiinvariant under the action of the geodesic flow and transform via $\frac{d}{dt}\nu^{su}\circ\Phi^t|_{t=0}=f$. Denote by dt the usual Lebesgue measure on the flow lines of the geodesic flow. Then the measure ν can be represented in the form $d\nu=dt\times d\nu^{su}\times d\nu^{ss}$ with a family ν^{ss} of locally finite Borel measures on strong stable manifolds. The lifts to $T^1\tilde M$ of the measures ν^{su},ν^{ss} project via the projection π to measures on the ideal boundary $\partial\tilde M$ which all define the same Γ-invariant measure class $mc(\nu_+),mc(\nu_-)$. The measure class $mc(\nu_+)$ is the positive disintegration class of ν, and the measure class $mc(\nu_-)$ is the positive disintegration class of the measure $\nu\circ\mathcal{F}$.

Following Ledrappier [L2] we can replace our function f without changing the Gibbs equilibrium state it defines by a function, again denoted by f, which is Hölder continuous, strictly positive (Corollary 2.5 and Lemma 2.2 of [H2] are valid with exactly the same proof for Hölder continuous functions of vanishing pressure whose Gibbs equilibrium state is not invariant under the flip) and smooth with pointwise uniformly bounded derivatives up to third order along the leaves of the *stable foliation* W^s [L2]. Let $P\colon T^1\tilde{M} \to \tilde{M}$ be the canonical projection, and denote the lift of f to $T^1\tilde{M}$ by the same symbol. There is then a unique Hölder continuous function $K\colon \tilde{M} \times \tilde{M} \times \partial\tilde{M} \to (0,\infty)$ such that $K(Pv, P\Phi^t v, \pi(v)) = e^{\int_0^t f(\Phi^s v)ds}$ [H2]. More precisely, for $v \in T^1\tilde{M}$ and $y \in \tilde{M}$ let w be the unique lift of y to the leaf $W^s(v) = \pi^{-1}(\pi(v))$ of the stable foliation on $T^1\tilde{M}$. Write

$$K(Pv, y, \pi(v)) = \lim_{t\to\infty} \int_0^t f(\Phi^s v)ds - \int_0^{t-\tau} f(\Phi^s w)ds$$

where $\tau \in \mathbb{R}$ is such that $w \in \Phi^\tau W^{ss}(v)$ and $W^{ss}(v)$ is the leaf of the strong stable foliation containing v. The function K is invariant under the action of Γ, and $\log K$ is uniformly Hölder continuous. Moreover, by our choice of f, for every fixed $\xi \in \partial\tilde{M}$ and $x \in \tilde{M}$ the function $y \to \log K(x, y, \xi)$ has uniformly bounded derivatives up to third order which depend Hölder continously on ξ.

The measures ν^{su} on strong unstable manifolds project to a Γ-equivariant family $\{\nu^x\}$ of finite Borel measures on $\partial\tilde{M}$ which define $mc(\nu_+)$ and such that for all $x, y \in \tilde{M}$ and all $\xi \in \partial\tilde{M}$ we have $\frac{d\nu^y}{d\nu^x}(\xi) = K(x, y, \xi)$ (Lemma 2.1 of [H2]). We therefore call K the *Radon Nikodym kernel* of the measures $\{\nu^x\}$. Let M_0 be a relative compact fundamental domain for the action of Γ, and denote by λ_0 the Lebesgue measure on M_0. Then $M_0 \times \partial\tilde{M}$ has a natural identification with T^1M and we obtain a finite Borel measure σ on $M_0 \times \partial\tilde{M} \sim T^1M$ by defining $\sigma(A) = \int \nu^x(A)d\lambda_0(x)$.

Let g be the Riemannian metric on $\tilde{M}$. For $\xi \in \partial\tilde{M}$ and any fixed $x \in \tilde{M}$ the g-gradient of the function $y \to \log K(x, y, \xi)$ is a smooth vector field Y_ξ on $\tilde{M}$ which does not depend on x and is uniformly bounded together with its derivatives up to second order. Moreover the assignment $(z, \xi) \to Y_\xi(z)$ is equivariant under the action of Γ and uniformly Hölder continuous. Thus the vector fields Y_ξ define a leafwise smooth globally Hölder continuous section Y of TW^s over T^1M.

Let g^s be the Riemannian metric on the stable foliation which is the lift of the Riemannian metric on M. For each $v \in T^1M$ the leaf $W^s(v)$ of W^s equipped with this metric is a complete Riemannian manifold of bounded negative curvature which is isometric to a (possibly trivial) quotient of $\tilde{M}$. Denote by Δ the leafwise Laplacian of g^s. Then Δ is a second order leafwise elliptic differential operator on T^1M with Hölder continuous coefficients.

Recall that a Borel measure η on T^1M is *harmonic* for $\Delta + Y$ if for every smooth function φ on T^1M we have $\int(\Delta + Y)(\varphi)d\eta = 0$, and it is called *self-adjoint harmonic* if in addition we have $\int \psi(\Delta + Y)(\varphi)d\eta = \int \varphi(\Delta + Y)(\psi)d\eta$ for smooth functions φ, ψ on T^1M. We showed in [H1] that the above measure σ on T^1M is the only self-adjoint harmonic measure for $\Delta + Y$ on T^1M up to a constant.

Let M' be a regular covering of M with fundamental group Γ' (we allow for the moment that Γ' is trivial). The operator $\Delta + Y$ lifts to a second order differential operator on the unit tangent bundle T^1M' of $M' = \tilde{M}/\Gamma'$ with Hölder continuous coefficients. Thus we may investigate the space of self-adjoint harmonic measures for the operator $\Delta + Y$ on T^1M'. Here we call a measure η on T^1M' *self-adjoint harmonic* for $\Delta + Y$ if η is locally finite and if the above equation $\int \varphi(\Delta + Y)(\psi)d\eta = \int \psi(\Delta + Y)(\varphi)d\eta$ holds for smooth functions φ, ψ on T^1M' with compact support. Let M_0' be a fundamental domain for the action of Γ' on $\tilde{M}$ and identify T^1M' with $M_0' \times \partial\tilde{M}$. Corollary 2.6 of [H1] shows that η is self-adjoint harmonic for $\Delta + Y$ if and only if there is a family $\{\eta^x\}$ $(x \in \tilde{M})$ of finite Borel measures on $\partial\tilde{M}$ with the following properties:

1) $\eta(A) = \int \eta^x(A)d\lambda(x)$ where λ is the Lebesgue measure on $M_0' \subset \tilde{M}$.
2) The measures η^x are equivariant under the action of Γ' and define all the same measure class $mc(\eta)$. Moreover they transform via $\frac{d\eta^y}{d\eta^x}(\xi) = K(x, y, \xi)$.

We call the family $\{\eta^x\}$ of measures as above the *defining family* of η.
If we fix a point $x \in M_0'$ and normalize the measure η in such a way that $\eta^x(\partial\tilde{M}) = 1$ then for every compact subset $C \subset T^1M' \sim M_0' \times \partial\tilde{M}$ the total mass of C is bounded from above and below by a positive constant which does not depend on η but only on K and C. In the sequel we call a self-adjoint harmonic measure η *normalized* if its defining family $\{\eta^y\}$ satisfies $\eta^x(\partial\tilde{M}) = 1$.
Let now $\mathcal{H}$ be the space of normalized self-adjoint harmonic measures for $\Delta + Y$. We equip $\mathcal{H}$ with the weak* topology. Thus a sequence of measures $\eta_i \subset \mathcal{H}$ converges in $\mathcal{H}$ to a measure η if and only if for every continuous function φ on T^1M' with compact support we have $\int \varphi d\eta_i \to \int \varphi d\eta$. Then we have.

Lemma 1. *The space $\mathcal{H}$ is compact and convex.*

Proof: Let $\eta_i \subset \mathcal{H}$ be a sequence of normalized self adjoint harmonic measures for $\Delta + Y$. Using the above uniform boundedness of the measures η_i and a diagonal argument we may assume that there is a locally finite Borel measure η on T^1M' such that for every compact subset $C \subset T^1M'$ the restrictions to C of the measures η_i converge weakly to the restriction of η. This then means that for every continuous function φ on T^1M' with compact support we have $\int \varphi d\eta_i \to \int \varphi d\eta$.
To show that $\eta \in \mathcal{H}$, observe that for smooth functions ψ, φ on T^1M' with compact support we have

$$0 = \int (\psi(\Delta + Y)(\varphi) - \varphi(\Delta + Y)(\psi))d\eta_i \to \int (\psi(\Delta + Y)(\varphi) - \varphi(\Delta + Y)(\psi))d\eta.$$

In other words, $\mathcal{H}$ is compact. Moreover $\mathcal{H}$ is clearly convex as well. □

Recall that every self-adjoint harmonic measure η for $\Delta + Y$ on T^1M' defines a Γ'-invariant measure class $mc(\eta)$ on $\partial\tilde{M}$. We call the measure η *minimal* if this measure class is ergodic under the action of Γ'. The next lemma follows from standard arguments.

Lemma 2. *Minimal self-adjoint harmonic measures for $\Delta + Y$ are just the extreme points of $\mathcal{H}$.*

Lemma 3. *Let $\mu \neq \eta \in \mathcal{H}$ with defining families $\{\mu^x\}, \{\eta^x\}$. If η is minimal then there is a sequence of points $\{p_i\}_i$ in M' going to infinity such that $\eta^{p_i}(\partial\tilde{M})/\mu^{p_i}(\partial\tilde{M}) \to 0$.*

Proof: Let σ be the lift to $T^1\tilde{M}$ of the unique self-adjoint harmonic measure for $\Delta + Y$ on T^1M. The defining family $\{\nu^y\}$ for σ on $\partial\tilde{M}$ just equals the family of projections of our Gibbs equilibrium state ν as above. The measures $\{\nu^y\}$ are equivariant under the action of the group Γ, and their total mass is bounded from above and below by a universal constant.
Let $K\colon \tilde{M} \times \tilde{M} \times \partial\tilde{M} \to (0,\infty)$ be the Radon Nikodym kernel of the measures $\{\nu^y\}$. By the choice of our function f inducing the Gibbs state ν there is a constant $\alpha > 0$ such that for every $v \in T^1\tilde{M}$ and every $t > 0$ we have $K(Pv, P\Phi^t v, \pi(v)) \geq e^{2\alpha t}$. Since $\log K$ is uniformly Hölder continuous, there is a number $\varepsilon > 0$ such that for every $x \in \tilde{M}$ and all $v, w \in T^1_x\tilde{M}$ with $\angle(v,w) < \varepsilon$ we have $K(Pv, P\Phi^1 v, \pi(w)) \geq e^{\alpha}$.
For $x \in \tilde{M}$, $v \in T^1_x\tilde{M}$ and $t > 0$ define

$$B_x(\pi(v), e^{-t}) = \{\pi(w) \mid w \in T^1_{P\Phi^t v}\tilde{M}, \angle(w, \Phi^t v) \leq \pi - \varepsilon\} \subset \partial\tilde{M}.$$

For fixed $\xi \in \partial\tilde{M}$, the sets $B_x(\xi, r)$ $(r > 0)$ form a basis of closed neighborhoods of $\xi \in \partial\tilde{M}$ such that $B_x(\xi,r) \subset B_x(\xi,s)$ for $r < s$. The collection of all sets $B_x(\xi,r)$ defines a Vitali relation for every measure from the defining family of a self-adjoint harmonic measure for $\Delta + Y$ (this fact follows from standard measure theory and is discussed in the proof of Proposition 2.8 of [H2]). Moreover there is a constant $c_0 > 0$ such that $\nu^x(B_x(\pi(v), r))K(Pv, P\Phi^{-\log r}v, \pi(v)) \in [c_0^{-1}, c_0]$ for every $x \in \tilde{M}$, every $v \in T^1_x\tilde{M}$ and every $r < 1$ [H2].
Let now $\mu \neq \eta$ be normalized self-adjoint harmonic measures for $\Delta + Y$ and assume that η is minimal. Since μ^y and η^y are finite Borel measures on $\partial\tilde{M}$ we can write $\mu^y = \chi^y\eta^y + \beta^y$ for a nonnegative η^y-measurable function χ^y on $\partial\tilde{M}$ and a Borel measure β^y which is singular to η^y. By absolute continuity of the measures μ^y and our transformation rule the measures β^y transform via $d\beta^y/d\beta^x(\xi) = K(x,y,\xi)$. Thus these measures are the defining family of a self-adjoint harmonic measure β for $\Delta + Y$. Then $\{\chi^y\eta^y\}$ is the defining family of a self-adjoint harmonic measure for $\Delta + Y$ as well. Since the measure η is minimal this implies that the functions χ^y are constant, i.e. we have $\beta \neq 0$ and there is some $\chi \geq 0$ such that $\mu = \chi\eta + \beta$.
If $\{\beta^y\}, \{\eta^y\}$ are the defining families for β, η on $\partial\tilde{M}$, then $\{\chi\eta^y + \beta^y = \mu^y\}$ is the defining family for μ. The measure classes of the measures β^y, η^y on $\partial\tilde{M}$ are singular. Thus for our fixed point $x \in \tilde{M}$ and for β^x-almost every $\xi \in \partial\tilde{M}$ we have
$\lim_{r\searrow 0} \eta^x(B_x(\xi,r))/\mu^x(B_x(\xi,r)) = 0$, moreover $\lim_{r\searrow 0} \nu^x(B_x(\xi,r))/\mu^x(B_x(\xi,r)) = a < \infty$.
Recall that the kernel K is the Radon Nikodym kernel of both the measures $\{\mu^y\}$ and the measures $\{\nu^y\}$. By uniform Hölder continuity of $\log K$ there is a constant $c_1 \geq c_0$ such that

$$K(Pv, P\Phi^t v, \zeta)/K(Pv, P\Phi^t v, \pi(v)) \in [c_1^{-1}, c_1]$$

for all $v \in T^1\tilde{M}$ and all $t > 0$, $\zeta \in B_x(\pi(v), e^{-t})$. This implies that

$$\nu^{P\Phi^t v}(B_x(\pi(v), e^{-t}))\mu^x(B_x(\pi(v), e^{-t}))/\nu^x(B_x(\pi(v), e^{-t}))\mu^{P\Phi^t v}(B_x(\pi(v), e^{-t})) \in [c_1^{-2}, c_1^2]$$

and that similarly

$$\eta^{P\Phi^t v}(B_x(\pi(v), e^{-t}))\mu^x(B_x(\pi(v), e^{-t}))/\eta^x(B_x(\pi(v), e^{-t}))\mu^{P\Phi^t v}(B_x(\pi(v), e^{-t})) \in [c_1^{-2}, c_1^2]$$

for all $v \in T^1_x\tilde{M}$ and all $t > 0$. In particular, for every vector $v \in T^1_x\tilde{M}$ whose projection to $\partial\tilde{M}$ is typical for the measure β^x the sequence $\rho_j = \mu^{P\Phi^j v}(\partial\tilde{M})$ is bounded from below by a positive constant $b > 0$.
Let $v \in T^1_x\tilde{M}$ be such a vector which projects to a typical point for β^x and for $t > 0$ write $x_t = P\Phi^t v$. By the above we can find a number $j_0 > 0$ and a sequence $\{k_j\} \subset [j_0, \infty)$ going to infinity such that $\mu^{x_{k_j}}(\partial\tilde{M}) \geq \max\{\mu^{x_i}(\partial\tilde{M}) \mid j_0 \leq i \leq k_j\}/2$.
Write $\pi(v) = \xi$. By our above estimate and our choice of the sets $B_x(\xi, r)$ we have

$$\eta^{x_j}(A) \leq e^{-\alpha}\eta^{x_{j-1}}(A) \quad \text{ for every subset } A \text{ of } \partial\tilde{M} - B_x(\xi,\, e^{-j}).$$

For $M > 0$ choose a number $j_1 \geq j_0$ such that

$$\mu^x B_x(\xi, r) \geq M c_1^2 \eta^x B_x(\xi, r)$$

for all $r < e^{-j_1}$. Using once again the results of [H2], there is a number $j_2 > j_1$ such that $K(x, x_t, \zeta) < b/M$ whenever $t > j_2$ and $\zeta \in \partial\tilde{M} - B_x(\xi, e^{-j_1})$. This then implies that $\eta^{x_t}(\partial\tilde{M} - B_x(\xi, e^{-j_1})) < b/M$ for all $t \geq j_2$. For $k_j > j_2$ we then obtain that

$$\begin{aligned}
\frac{\eta^{x_{k_j}}(\partial\tilde{M})}{\mu^{x_{k_j}}(\partial\tilde{M})} \leq & \frac{\eta^{x_{k_j}}(B_x(\xi, e^{-k_j}))}{\mu^{x_{k_j}}(B_x(\xi, e^{-k_j}))} + \sum_{\ell=j_1}^{k_j-1} \frac{\eta^{x_{k_j}}(B_x(\xi, e^{-\ell}) - B_x(\xi, e^{-(\ell+1)}))}{\mu^{x_{k_j}}(\partial\tilde{M})} + \\
& + \frac{\eta^{x_{k_j}}(\partial\tilde{M} - B_x(\xi, e^{-j_1}))}{\mu^{x_{k_j}}(\partial\tilde{M})} \\
\leq & \frac{1}{M} + 2\sum_{\ell=j_1}^{k_j-1} \frac{\eta^{x_{k_j}}(B_x(\xi, e^{-\ell}) - B_x(\xi, e^{-(\ell+1)}))}{\bar{\mu}^{x_\ell}(\partial\tilde{M})} + \frac{1}{M} \\
\leq & 2\sum_{\ell=j_1}^{k_j-1} e^{-(k_j-\ell)\alpha}\frac{\eta^{x_\ell}(B_x(\xi, e^{-\ell}))}{\mu^{x_\ell}(B_x(\xi, e^{-\ell}))} + \frac{2}{M} \leq 2\sum_{\ell=j_1}^{\infty} e^{-\ell\alpha}\frac{1}{M} + \frac{2}{M}.
\end{aligned}$$

This is the required estimate. **q.e.d.**

Now we can show our theorem for abelian covers of our closed manifold $M = \tilde{M}/\Gamma$. Since every normal subgroup Γ' of Γ with abelian factor group Γ/Γ' contains the commutator group $[\Gamma, \Gamma]$ it is enough to formulate our result for $\Gamma' = [\Gamma, \Gamma]$.

Proposition 4. *Let $M = \tilde{M}/\Gamma$ be a closed negatively curved manifold. If ν is the Gibbs equilibrium state of a Hölder continuous function on T^1M, then the positive disintegration class $mc(\nu_+)$ of ν on $\partial\tilde{M}$ is ergodic under the action of* $[\Gamma, \Gamma]$.

Proof: Write $M' = \tilde{M}/[\Gamma, \Gamma]$. Using the above notations, let $\mathcal{H}$ be the space of normalized harmonic measures for the operator $\Delta + Y$ on T^1M'. The normalized lift σ to T^1M' of the unique harmonic measure for $\Delta + Y$ on T^1M is contained in $\mathcal{H}$. The statement of our proposition is then equivalent to saying that σ is minimal.
We use now an idea from [LS]. For a measure $\mu \in \mathcal{H}$ and $y \in M'$ define $h_\mu(y) = \mu^y(\partial\tilde{M})$. We showed in Lemma 3 that whenever $\eta \neq \mu \in \mathcal{H}$ and η is minimal, then 0 is the infimum of the ratio h_η/h_μ.

Since $\Gamma/[\Gamma,\Gamma]=\Gamma_0$ is abelian, the translation of M by an element Ξ in Γ_0 only moves points a bounded amount. From Lemma 3 and transition properties of defining families we therefore conclude that for every minimal harmonic measure $\eta\in\mathcal{H}$ and every $\Xi\in\Gamma_0$ we have $\eta\circ\Xi=c(\Xi)\eta$ for some $c(\Xi)>0$. Then $\Xi\to c(\Xi)$ is a multiplicative character on Γ_0.

We claim that if $\eta\neq\sigma$ then this character is nontrivial. To see this simply recall from Lemma 3 that there is a sequence $\{p_i\}\subset\tilde M$ such that $h_\eta(p_i)/h_\sigma(p_i)\to 0$. Since h_σ is bounded from above by a positive constant this means that $h_\eta(p_i)\to 0$ and hence necessarily the character is nontrivial. In particular, h_η is unbounded.

For $\Psi\in\Gamma$ denote by $[\Psi]$ the homology class it defines. Then there is a cohomology class $\alpha\in H^1(M,\mathbb{R})$ such that the character c ist given by $c(\Psi)=e^{\alpha[\Psi]}$. In other words, c can be viewed as a function on Γ.

For $Id\neq\Psi\in\Gamma$ and $\rho>0$ the shadow $S(\Psi,\rho)$ of the ball $B(\Psi x,\rho)$ of radius ρ about Ψx is the set of all endpoints of geodesic rays which begin at x and meet the ball $B(\Psi x,\rho)$. It is well known that there is a number $\varepsilon=\varepsilon(\rho)>0$ not depending on Ψ with the following property: If $v\in T^1_x\tilde M$ is such that the geodesic ray with initial velocity v meets Ψx, then the shadow $S(\Psi,\rho)$ contains the set $\{\pi(w)\mid w\in T^1_{\Psi x}\tilde M,\angle(\Phi^{\operatorname{dist}(x,\Psi x)}v,w)<\varepsilon\}$ and is contained in the set $\{\pi(w)\mid w\in T^1_{\Psi x}\tilde M,\angle(\Phi^{\operatorname{dist}(x,\Psi x)}v,w)<\pi-\varepsilon\}$.

For our fixed point $x\in\tilde M$ define a function ζ_x on $\tilde M$ by assigning for $v\in T_x{}^1\tilde M$ and $t>0$ the value $\zeta_x(P\Phi^t v)=K(x,P\Phi^t v,\pi(v))^{-1}$ to $P\Phi^t v$. By the properties of the measures $\eta^y(y\in\tilde M)$ we conclude that there is a constant $c(\rho)>0$ such that
$\eta^x(S(\Psi,\rho))/e^{\alpha[\Psi]}\,\zeta_x(\Psi x)\in[c(\rho)^{-1},c(\rho)]$.

For $s>0$ consider now the series

$$P_{\alpha,s}(x)=\sum_{\Psi\in\Gamma}e^{\alpha[\Psi]}\zeta_x(\Psi x)e^{-s\operatorname{dist}(x,\Psi x)}.$$

We claim that this series converges and that moreover for $s\searrow 0$ the measures

$$\sum_{\Psi\in\Gamma}e^{\alpha[\Psi]}\zeta_x(\Psi x)e^{-s\operatorname{dist}(x,\Psi x)}\,\delta_{\Psi x}\,\frac{1}{P_{\alpha,s}(x)}$$

converge weakly on $\tilde M\cup\partial\tilde M$ to the measure η^x on $\partial\tilde M$.

For this choose a number $D>0$ which is larger than the diameter of M. For every $R>0$ the shadows

$$\{S(\Psi,D)\mid\Psi\in\Gamma,\ R<\operatorname{dist}(x,\Psi x)<R+D\}$$

cover $\partial\tilde M$. Moreover there is a number $k>0$ only depending on D such that every point in $\partial\tilde M$ is contained in at most k different of the sets $S(\Psi,D)$. From this it is immediate that for every $R>0$ the value of the truncated series

$$\sum_{R<\operatorname{dist}(x,\Psi x)<R+D}e^{\alpha[\Psi]}\zeta_x(\Psi x)$$

is bounded from above and below by a universal constant. This then implies our claim.

Applying this to the trivial cohomology class which corresponds to the defining family $\{\nu^y\}$ of the unique selfadjoint harmonic measure for $\Delta + Y$ on T^1M we conclude that for $s > 0$ the Poincaré series

$$P_{0,s}(x) = \sum_{\Psi \in \Gamma} \zeta_x(\Psi x)\, e^{-s\,\mathrm{dist}(x,\Psi x)}$$

converges, and it diverges for $s \leq 0$. As $s \searrow 0$ the measures

$$\sum_{\Psi \in \Gamma} \zeta_x(\Psi x)\, e^{-s\,\mathrm{dist}(x,\Psi x)} \delta_{\Psi x} \frac{1}{P_{0,s}(x)}$$

converge weakly on $\tilde{M} \cup \partial\tilde{M}$ to the measure ν^x on $\partial\tilde{M}$.
Let now ω be a closed 1-form on M which represents the (de Rham) cohomology class α and let $b \in \mathbb{R}$ be the pressure of the function $f + \omega$. Then for every $D > 0$ and every $R > D$ the value of the truncated series

$$\sum_{R < \mathrm{dist}(x,\Psi x) < R+D} e^{\alpha[\Psi]} \zeta_x(\Psi x) e^{-b\,\mathrm{dist}(x,\Psi x)}$$

is bounded from above and below by a universal constant. The measures

$$\sum_{\Psi \in \Gamma} e^{\alpha[\Psi]} \zeta_x(\Psi x) e^{-(s+b)\,\mathrm{dist}(x,\Psi x)} \delta_{\Psi x} \frac{1}{P_{0,s}(x)}$$

converge as $s \searrow 0$ weakly on $\tilde{M} \cup \partial\tilde{M}$ to a Borel-probability measure on $\partial\tilde{M}$ which defines the positive disintegration class of the unique Gibbs equilibrium state of the function $f + \omega$. Together with the above consideration this implies that necessarily $b = 0$ and that the measure class of η^x equals the positive disintegration class of the unique Gibbs equilibrium state of $f + \omega$. In particular, this measure class is invariant and ergodic under the action of the full group Γ.
As a consequence, let $\mathcal{C}$ be the collection of all multiplicative characters χ on Γ_0 with the property that the pressure of the function $f + \chi$ vanishes and that moreover the positive disintegration class of its unique Gibbs equilibrium state is ergodic under the action of $[\Gamma, \Gamma]$. Every $\chi \in \mathcal{C}$ induces an ergodic measure $\eta(\chi) \in \mathcal{H}$ with defining family $\{\eta(\chi)^y\}$. The measure ν^x can uniquely be represented in the form $\nu^x = \int_{\mathcal{C}} \eta(\chi)^x d\zeta(\chi)$ for a probability measure ζ on $\mathcal{C}$. This decomposition is nontrivial if and only if the positive disintegration class of the Gibbs measure ν which equals the measure class of the measures ν^x is *not* ergodic under the action of $[\Gamma, \Gamma]$. But then the measure ν itself can be decomposed in a nontrivial way into a generalized sum of Gibbs equilibrium states on T^1M which are different from ν. Since the Gibbs equilibrium state of every Hölder continuous function on T^1M is ergodic under the geodesic flow this is impossible and finishes the proof of our proposition. □

Remark: The proof of Proposition 4 shows the following: Consider the space $\mathcal{H}$ of all normalized self-adjoint harmonic measures for the operator $\Delta + Y$ on T^1M' where $M' = \tilde{M}/[\Gamma, \Gamma]$ is the homology covering of M. Each of these measures determines a $[\Gamma, \Gamma]$-invariant measure class on $\partial\tilde{M}$. The extemal points of $\mathcal{H}$ are those measures for which the

induced measure class on $\partial\tilde{M}$ is ergodic under the action of $[\Gamma,\Gamma]$. Such ergodic measures are in a one-to-one correspondance with cohomology classes $\alpha = [\omega] \in H^1(M,\mathbb{R})$ for which the pressure of $f+\omega$ vanishes. In particular, each of these measure classes on $\partial\tilde{M}$ is in fact invariant under the full group Γ.
There is an analogous result of Babillot and Ledrappier for transverse invariant measures for the strong stable foliation on the homology covering M' of a closed negatively curved manifold M [BL1], [BL2] (the paper [B] of Babillot contains a very nice description of the special case of surfaces of constant curvature) which motivated this work. Namely, for each cohomology class $[\omega] \in H^1(M,\mathbb{R})$ there is a unique $c>0$ such that the pressure of the function $c+\omega$ vanishes. The Gibbs equilibrium state of each such function then lifts to a transverse invariant ergodic measure for the strong stable foliation on T^1M'. As in our case above, the measures induced by nontrivial cohomology classes project to Borel measures on the compact quotient T^1M which however are *not* invariant any more for the strong stable foliation. It seems to be an open question whether *every* invariant measure for the strong stable foliation on T^1M' is of the above form.

We are left with reducing the case of nilpotent covers to the case of abelian covers. For this we follow again Lyons and Sullivan [LS].

Proposition 5. *Let $M=\tilde{M}/\Gamma$ be a closed negatively curved manifold, and let $\Gamma' \subset \Gamma$ be a normal subgroup such that Γ/Γ' is nilpotent. Then the positive disintegration class $mc(\nu_+)$ on $\partial\tilde{M}$ of every Gibbs equilibrium state of a Hölder continuous function on $\partial\tilde{M}$ is ergodic under the action of Γ'.*

Proof: Let Z be the center of $\Gamma_0=\Gamma/\Gamma'$, and let η be a minimal harmonic measure for $\Delta+Y$, defined by the Γ'-equivariant family of measures $\{\eta^x\}$. As before, define a function h_η on $M'=\tilde{M}/\Gamma$ by $h_\eta(y)=\eta^y(\partial\tilde{M})$. Lemma 3 and the proof of Proposition 4 imply that there is a character c on Z such that $h_\eta(\Psi y)=c(\Psi)h_\eta(y)$ for all $\Psi\in Z$, all $y\in M'$.
The lift to $\tilde{M}$ of the function h_η is a generalized convex combination of smooth positive functions on $\tilde{M}$ whose logarithm together with its derivatives up to third order is pointwise uniformly bounded. Thus the logarithm of h_η and its derivatives up to third order is pointwise uniformly bounded as well. Following Lyons and Sullivan ([LS] p. 306) we can take a mean of the 1-form dh_η/h_η over the ameanable group Γ_0 to obtain a closed Γ_0-invariant one-form on M'. This one-form projects to a closed one-form ω on M and therefore it defines a cohomology class in $H^1(M,\mathbb{R})$. The periods of ω around closed loops give an extension of the character c to a character c' on Γ_0 (see Proposition 3 of [LS]).
Let now $k\geq 1$ be the degree of nilpotency of Γ_0. Every character on a nilpotent group vanishes on the commutator subgroup. In particular, since the character c is defined on the entire center Z we conclude that either $k=1$ and Γ_0 is abelian or Γ_0 contains a nontrivial normal subgroup $Z_1=Z\cap[\Gamma_0,\Gamma_0]$ such that h is invariant under Z_1 and that moreover the factor group Γ_0/Z_1 is nilpotent of degree $k-1$. Since our minimal harmonic measure for $\Delta+Y$ on T^1M' was arbitrary we conclude that for *every* harmonic measure η for $\Delta+Y$ the function $h_\eta(y)=\eta^y(\partial\tilde{M})$ is invariant under the action of Z_1 and projects to a function on $M''=M'/Z_1$.

We can now use Lemma 3 for the thus obtained functions on M'' to conclude that each of these functions defines a character $\tilde{c}$ on the center of the factor group Γ_0/Z_1. Proceeding inductively on the degree of nilpotency of Γ_0 we find that for every minimal harmonic measure $\eta \in \mathcal{H}$ there is a character c defined on all of Γ_0 such that the function $h_\eta(y) = \eta^y(\partial\tilde{M})$ satisfies $h_\eta \circ \Psi = c(\Psi)h_\eta$ $(\Psi \in \Gamma_0)$. Every such character defines a cohomology class in $H^1(M, \mathbf{R})$. But then we can use the arguments in the proof of Propostion 4 to deduce that each of these measures is a lift of a harmonic measure for $\Delta + Y$ on the unit tangent bundle of the abelian covering $M'/[\Gamma_0, \Gamma_0]$ of M. From this and Proposition 4 the proposition follows. □

Acknowledgement: This work answers a question posed by V. Kaimanovich during a workshop at the Newton Institute in Cambridge. I am grateful to the Institute for its hospitality, and I am grateful to V. Kaimanovich for inspiring discussions on the subject of the paper.

References

[B] M. Babillot, Géodesiques et horocycles sur le revêtement d'homologie d'une surface hyperbolique, *Séminaire de théorie spectrale et géometrie,* Institut Fourier, Grenoble 1995-96, 89-104.

[BL1] M. Babillot, F. Ledrappier, Lalley's theorem for periodic orbits of hyperbolic flows, *Erg. Th. & Dyn. Sys.* 18 (1998), 17-39.

[BL2] M. Babillot, F. Ledrappier, Geodesic paths and horocycle flow on abelian covers, *Lie groups and ergodic theory* (Mumbai 1996), 1–32, Tata Inst. Fund. Res. Stud. Math. 14, Bombay 1998.

[C] Y. Coudene, Cocycles and stable foliations of Axiom A flows, *Erg. Th. & Dyn. Sys.* 21 (2001).

[G] Y. Guivarch, Propriétés ergodiques, en mesure infinie, de certains systèmes dynamiques fibrés, *Erg. Th. & Dyn. Sys. 7* (1987), 433-453.

[H1] U. Hamenstädt, Harmonic measures for compact negatively curved manifolds, *Acta Math. 178* (1997), 39–107.

[H2] U. Hamenstädt, Cocycles, Hausdorff measures and Cross Ratios, *Erg. Th. & Dyn. Sys. 17* (1997), 1061–1081.

[K] V.A. Kaimanovich, Ergodic theory of harmonic invariant measures for the geodesic flow on hyperbolic spaces, *Journal für die reine und angewandte Mathematik 455* (1994), 57–103.

[L1] F. Ledrappier, Ergodic properties of the Brownian motion on covers of compact negatively curved manifolds, *Bol. Soc. Bras. Mat.* 19 (1988), 115-140.

[L2] F. Ledrappier, Central limit theorem in negative curvature, *Ann. Probab. 23* (1995), 1219–1233.

[LS] T. Lyons, D. Sullivan, Function theory, random paths and covering spaces, *J. Diff. Geom. 19* (1984), 299-323.

[R] M. Rees, Checking ergodicity of some geodesic flows with infinite Gibbs measure, *Erg. Th. & Dyn. Sys. 1* (1981), 107-133.

[S] D. Sullivan, The density at infinity of a discrete group of hyperbolic isometries, *Inst. Hautes Études Sci. Publ. Math 50* (1979), 171-202.

Mathematisches Institut der Universität Bonn
Beringstrasse 1
53115 Bonn, Germany e-mail: ursula@math.uni-bonn.de

The author was partially supported by the Sonderforschungsbereich SFB 256.