KÄHLER–EINSTEIN METRICS OF NEGATIVE CURVATURE

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Abstract. Given any integer $n \geq 2$, we construct a compact Kähler–Einstein manifold of dimension n of negative sectional curvature which is not covered by the ball.

1. Introduction

An important problem in complex geometry consists in finding compact complex manifolds M admitting a hermitian metric ω with good curvature properties. Formulated as such, the problem is of course vague and there are many ways to make it more precise. In what follows, we will be exclusively interested in Kähler metrics, that is, we will impose that $d\omega = 0$.

Given a compact Kähler manifold (M, ω), there exist several distinct notions of curvature, e.g. the sectional curvature (K_ω), the holomorphic bisectional curvature (HBC_ω), the holomorphic sectional curvature (HSC_ω), the Ricci curvature (Ric_ω) and the scalar curvature (s_ω). Although each of these objects are tensors of different types, it makes sense to talk about (semi)posivity or (semi)negativity of these curvatures. Then we have the following implications

$$\begin{array}{ccccc} K_\omega < 0 & \Longrightarrow & \mathrm{HBC}_\omega < 0 & \Longrightarrow & \mathrm{HSC}_\omega < 0 \\ & & \Downarrow & & \Downarrow \\ & & \mathrm{Ric}_\omega < 0 & \Longrightarrow & s_\omega < 0 \end{array}$$

and similarly with seminegativity or (semi)positivity.

If (M, ω) is a compact Kähler manifold with positive bisectional curvature, a celebrated theorem of Siu and Yau [SY80] implies that M is biholomorphic to the projective space, cf also Mori's theorem [Mo79] in the algebraic setting. In the negative curvature case, that is, if (M, ω) has negative sectional curvature, it was asked by Yau in [Yau82] whether the universal cover $\widetilde{M}$ is biholomorphic to the ball $\mathbb{B}^n$. It turns out that this question has a negative

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answer. Throughout the years several counterexamples have been exhibited, e.g. in dimension two by Mostow and Siu [MS80], in dimension three by Deraux [De05], and in any dimension by Stover and Toledo [ST22].

The examples of Mostow-Siu, of Deraux and of Stover-Toledo have infinite fundamental group. They are either finite branched covers of ball quotients (the examples of Mostow-Siu and of Stover-Toledo), or their universal covers can locally be described as branched covers of the ball (the examples of Deraux). That the manifolds found by Stover-Toledo admit Kähler metrics with negative definite complex curvature operator and hence negative curvature in the sense of Siu [S80] follows from an earlier result of Zheng [Zh96]. As consequence [S80], the manifolds are *holomorphically rigid*: any compact complex manifold which is homotopy equivalent to one of these manifolds is biholomorphic to it. Minemyer [Mi25] equipped these manifolds, called Stover-Toledo manifolds in the sequel, with non-Kähler Riemannian metrics whose Riemannian curvature operator is non-positive.

In a different direction, Mohsen [Moh22] produced examples of *simply connected* Kähler manifolds (M, ω) such that the holomorphic bisectional curvature is negative. These Kähler manifolds are complete intersections (of large codimension) in the projective space endowed with the restriction of the Fubini-Study metric.

This leaves open the question whether there are "canonical" Kähler metrics of negative curvature on compact complex manifolds which are not locally symmetric. More precisely, we ask about the existence of a non-locally symmetric compact Kähler manifold (M, ω) such that

$$K_\omega < 0 \quad \text{and} \quad \operatorname{Ric}\omega = c\,\omega$$

where $c \in \mathbb{R}$ is a (negative) so-called Einstein constant.

Thanks to a celebrated theorem of Aubin [Au78] and Yau [Yau78b], it is known that a compact Kähler manifold M admits a unique normalized Kähler–Einstein metric of negative Ricci curvature, that is, a Kähler metric ω such that $\operatorname{Ric}(\omega) = -\omega$, if and only if the first Chern class of M is negative, in the sense that there exists a Kähler metric in the class $-c_1(M)$. If this cohomological condition is satisfied, the unique Kähler–Einstein metric is constructed indirectly by solving a complex Monge-Ampère equation. However, as it is in general impossible to read off from the latter partial differential equation information on the sectional curvature, the above question is quite delicate. Our main result is the following.

Theorem. *For every $n \geq 2$ there exists a compact complex manifold M of dimension n whose universal covering is not biholomorphic to the ball and which admits a Kähler–Einstein metric of negative sectional curvature.*

Actually one can obtain the following refined statement. For an a priori chosen constant $\epsilon > 0$ and any number $n \geq 2$, there exists a compact Kähler–Einstein manifold $(M_\varepsilon, g_\varepsilon)$ of dimension n and Einstein constant -1 such that the sectional curvature κ of g_ε satisfies

$$\min \kappa \in [-1, -1+\epsilon] \quad \text{and} \quad \max \kappa \in [-\epsilon, 0).$$

In particular, we can find in any given complex dimension n an infinite countable family of Kähler–Einstein manifolds $(M_k, g_k)_{k\in\mathbb{N}}$ of negative curvature whose universal covers $\widetilde{M_k}$ are mutually non biholomorphic. All of these examples are Stover-Toledo manifolds. Furthermore, the Kähler–Einstein metrics in the theorem have very strongly negative curvature tensor in the sense of Siu. We refer to the last paragraph of the article for more information.

An old conjecture (see p.322 of [MS80] for an explicit statement) predicts that a simply connected complete Kähler manifold of negative sectional curvature is biholomorphic to a bounded domain in $\mathbb{C}^n$. This is open even if one requires the metric to be Kähler-Einstein. We believe that our examples hint at the possibility that this conjecture is not true.

Relation to earlier work. The question on the existence of negatively curved Einstein metrics on closed manifolds which do not admit a locally symmetric metric also makes sense in the non-complex setting. The first examples of such metrics are due to Fine and Premoselli [FP20]. They considered suitably chosen branched covers of some real hyperbolic four-manifolds (which in contrast to the complex setting are fairly easy to construct) and were able to show that an explicit negatively curved approximate Einstein metric on the branched cover can be perturbed to a negatively curved Einstein metric. This construction was extended in [HJ24] to any dimension at least four. In the real setting, it is the *existence* of some Einstein metric on a given closed negatively curved manifold of (real) dimension at least four which is difficult to establish. In the complex world, the existence of a Kähler Einstein metric can be read off from complex analytic invariants of the manifold, and the interest lies in the relation between geometric properties of a Kähler (Einstein) metric and complex analytic properties of the manifold.

Strategy of proof. Let $M := \Gamma\backslash B$ be a compact quotient of the unit ball $B \subset \mathbb{C}^n$ by a torsion free uniform arithmetic lattice of simple type admitting a totally geodesic embedded smooth complex hypersurface $D \subset M$. Such lattices $\Gamma \subset \mathrm{PU}(n,1)$ are the starting point for the work of Stover and Toledo (see [ST22]). We fix an integer $d \geq 2$.

Step 1. Produce an orbifold model Kähler–Einstein metric ω_d near D.

Let $B_0 \subset B$ be the totally geodesic complex hypersurface $B_0 := \{z_1 = 0\} \cap B$. Thanks to the theorem of Cheng-Yau [CY80], there exists on B a unique complete Kähler–Einstein metric ω_d which has cone singularities with cone angle $2\pi(1-\frac{1}{d})$ along B_0. In other words, ω_d can be desingularized by taking the ramified cover $(z_1, \underline{z}) \to (z_1^d, \underline{z})$ defined on the weakly pseudoconvex so-called *Thüllen domain* $\Omega_d := \{|z_1|^{2d} + |\underline{z}|^2 < 1\} \subset \mathbb{C}^n$. The metric ω_d is invariant under the automorphisms of B preserving B_0 and hence it descends to $\Gamma_0 \backslash B$ where $\Gamma_0 < \Gamma$ is the stabilizer of B_0 inside Γ, and we have $\Gamma_0 \backslash B_0 = D$. The desingularization of the metric ω_d on $\Gamma_0 \backslash B$ serves as a model for the Kähler Einstein metric near the divisor $D \subset M$ along which a branched covering is taken.

Step 2. Computing the curvature of ω_d*.*

A large part of the article is devoted to analyzing the model orbifold metric ω_d on the ball B, or rather its desingularization on the Thüllen domain Ω_d. Such an investigation was carried out by Bland [Bl86], but his results are not strong enough for our needs. Our approach is completely different and based on the observation that the behavior of ω_d is fully determined by a well-chosen real valued function solving a second order ordinary differential equation, cf Theorem 2.9. This leads to explicit negative bounds for the sectional curvature of ω_d described in Theorem 2.11 and exponential convergence of ω_d to the complex hyperbolic metric ω_B as the distance to B_0 goes to $+\infty$, which is formulated in Theorem 2.13.

Step 3. Gluing ω_d *to the hyperbolic metric.*

One would like to glue ω_d on a tubular neighborhood U of $D \subset M$ to the complex hyperbolic metric ω_B on $M \backslash U$. This is of course always possible, but unless the two metrics match very well in the gluing zone, the resulting metric will no longer have good curvature properties there. Controlling the glued metric requires a large collar size of the divisor in the arithmetic manifold as this will guarantee that the gluing metric is close to the ball metric on the gluing zone. That one can find Stover-Toledo manifolds obtained by a covering branched along a *single connected* divisor with arbitrarily large collar size is shown in Section 3, see Theorem 3.1. It involves among other things of subgroup separability of stabilizers of hyperplanes in arithmetic lattices in $\mathrm{PU}(n, 1)$ of simple type.

Step 4. Deforming to the Kähler–Einstein metric.

As the collar size R of the neighborhood of the divisor D tends to infinity, the glued metric will be arbitrarily close to a Kähler Einstein metric. All of them have uniformly bounded geometry. Using standard tools we find that they can be deformed to Kähler Einstein orbifold metrics with controlled negative curvature provided that R is sufficiently large. The desingularization of these Kähler Einstein orbifolds in covers branched along the singular divisors of the metrics provide the examples in the main Theorem.

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2. Kähler–Einstein metrics on Thüllen domains

For $n \geq 2$ consider $\mathbb{C}^n$ with the standard coordinates $(z_1, \ldots, z_n)$ and euclidean norm $|\,|$. The unit ball B in $\mathbb{C}^n$ is be defined by

$$B = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \mid |z_1|^2 + \sum_{i\geq 2} |z_i|^2 < 1\}.$$

The group of biholomorphic automorphisms of B is the group $\mathrm{PU}(n,1)$. The stabilizer of the divisor $B_0 = \{z_1 = 0\}$ equals

$$\mathrm{Stab}_{\mathrm{PU}(n,1)}(B_0) = P(S^1 \times U(n-1,1)) = \mathrm{U}(n-1,1).$$

The circle group S^1 acts on B by $(e^{i\theta}, (z_1, \ldots, z_n)) \to (e^{i\theta} z_1, \ldots, z_n)$, and it is the subgroup of $\mathrm{Stab}_{\mathrm{PU}(n,1)}(B_0)$ which fixes B_0 pointwise. More concretely, $\mathrm{Stab}_{\mathrm{PU}(n,1)}(B_0)$ is a central extension of $\mathrm{PU}(n-1,1)$, the group of biholomorphic automorphisms of B_0, by the circle group S^1.

For $\alpha \in [1, \infty)$ consider the *Thüllen domain*

$$\Omega = \Omega_\alpha = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_1|^{2\alpha} + \sum_{i\geq 2} |z_i|^2 < 1\}.$$

Clearly we have $\Omega_\alpha = B$ for $\alpha = 1$, and $\Omega_\infty = D \times B_0$, the product of the unit disk D and the ball of dimension $n-1$. For $\alpha < \infty$ the bounded domain $\Omega_\alpha \subset \mathbb{C}^n$ is weakly C^2-pseudoconvex. Moreover, for $\alpha = d \in \mathbb{N}$, the domain Ω_α maps onto the ball $B \subset \mathbb{C}^n$ by the holomorphic map

$$\Phi_d : (z_1, z_2, \ldots, z_n) \to (z_1^d, z_2, \ldots, z_n).$$

The map Φ_d is a covering of degree d, branched along B_0. For arbitrary $\alpha \geq 1$ we can also formally write a map $\Phi_\alpha : \Omega_\alpha \to B$, however it is multi-valued.

The following is due to Naruki [Na68]. It relies on the fact that the coordinate projection $(z_1, \ldots, z_n) \to (z_2, \ldots, z_n)$ is a holomorphic fibration with fiber the disk.

Lemma 2.1 (Naruki)**.** *There is a central extension G of* $\mathrm{PU}(n-1,1)$ *by* S^1 *which acts on* Ω_α *as a group of biholomorphic automorphisms, and complex conjugation* $z \to \bar{z}$ *acts as an antiholomorphic automorphism. Moreover, one has an isomorphism* $G \simeq \mathrm{Aut}(\Omega_\alpha)$.

Although the statement of the lemma is well known, we provide a sketch of a proof to illustrate the nature of the action of G on Ω_α as this will be important in the sequel and is not well documented in the literature. The multi-valued map Φ_α induces a homomorphism $G \to \mathrm{U}(n-1,1)$.

Proof of Lemma 2.1. By the definition of Ω_α, the circle group S^1 of rotations in the z_1-coordinate, defined by

$$(\theta,(z_1,z_2,\dots,z_n)) \to (e^{i\theta}z_1,z_2,\dots,z_n),$$

acts on Ω_α as a group of biholomorphic automorphisms. The map Φ_α maps orbits of S^1 to orbits of S^1, but it does not commute with the S^1-action. More precisely, we have $\Phi_\alpha \circ \theta = \alpha\theta \circ \Phi_\alpha$.

Consider the ball $B_0 = \{z_1 = 0\} \subset B \cap \Omega_\alpha$. If ψ_0 is an automorphism of B_0, then there is a nowhere vanishing holomorphic function θ on B_0 so that ψ_0 extends to an automorphism ψ of B defined by $\psi(z_1,z_2,\dots,z_n) = (\theta(z_2,\dots,z_n)z_1, \psi_0(z_2,\dots,z_n))$. We know that $|\theta|^2(\underline{z}) = \frac{1-|\psi_0(\underline{z})|^2}{1-|\underline{z}|^2}$.

Any choice $\hat{\theta}$ of a root of θ, that is, a holomorphic function $\hat{\theta}$ with $\hat{\theta}^\alpha = \theta$, defines an automorphism $\hat{\psi}$ of Ω_α by

$$\hat{\psi}(z_1,\dots,z_n) = (\hat{\theta}(z_2,\dots,z_n)z_1, \psi_0(z_2,\dots,z_n))$$

so that $\Phi_\alpha \circ \hat{\psi} = \psi \circ \Phi_\alpha$. If $\tilde{\theta}$ is another root, then $\tilde{\theta} = e^{ia}\theta$ where $(e^{ia})^\alpha = 1$. In other words, any two such choices differ by an element of S^1. As a consequence, the stabilizer G of B_0 in the automorphism group of Ω_α surjects onto the automorphism group $\mathrm{PU}(n-1,1)$ of B_0, with kernel S^1, and this surjection induces a homomorphism $G \to \mathrm{U}(n-1,1) = \mathrm{Stab}_{\mathrm{PU}(n,1)}(B_0)$.

At this point we have proved that for any $g \in G$, there exists $h_g \in \mathrm{Stab}_{\mathrm{PU}(n,1)}(B_0)$ such that

$$\Phi_\alpha \circ g = h_g \circ \Phi_\alpha. \tag{1}$$

That complex conjugation is an antiholomorphic automorphism of Ω_α is immediate from the definition.

In order to prove that G is isomorphic to the automorphism group of Ω_α we need to show that any element $g \in \mathrm{Aut}(\Omega_\alpha)$ fixes B_0. This can be seen as follows. Up to composing with the lift to G of a suitable element in $\mathrm{PU}(n-1,1)$, one can assume that g fixes the origin. By Cartan's theorem, g has to act on Ω_α by a linear transformation of $\mathbb{C}^n$. Now g fixes $\partial\Omega_\alpha$, and it also fixes the non strictly pseudoconvex locus of the latter, which is exactly $B_0 \cap \partial\Omega_\alpha \simeq \{\underline{z} \in \mathbb{C}^{n-1}; |\underline{z}| = 1\}$ since $\alpha > 1$. If g is represented by the matrix (a_{ij}) then we must have $\sum_{j=2}^n a_{1j} z_j = 0$ for any $\underline{z} = (z_2, \dots, z_n) \in S^{2n-3}$ hence for any $\underline{z} \in \mathbb{C}^{n-1}$. That is, we have $g(B_0) \subset B_0$, and equality follows. □

Since Ω_α is weakly C^2 pseudoconvex, it follows from the work of Cheng and Yau [CY80] that Ω_α admits a unique complete Kähler–Einstein metric.

Theorem 2.2 (Theorem 7.5 of [CY80])**.** *There exists a unique complete Kähler–Einstein metric g_α on Ω_α with Einstein constant $-(2n+2)$. In particular, g_α is invariant under the group G of biholomorphic transformations and under complex conjugation.*

Proof. Since Ω_α is weakly C^2 pseudoconvex, the existence of *some* complete invariant Kähler–Einstein metric ω_α on Ω_α is Theorem 7.5 of [CY80], which however does not state uniqueness explicitly. Uniqueness is a classic consequence of Yau's Schwarz lemma and his generalized maximum principle. Indeed, Theorem 3 in [Yau78a] shows that if ω and ω' are two complete Kähler–Einstein metrics with the same Einstein constant $c < 0$, then the ratio $F := \log\left(\frac{\omega'^n}{\omega^n}\right)$ is globally bounded. Finally, since $dd^c F = -c(\omega' - \omega)$, applying the maximum principle [Yau75] to $\pm F$ yields $F \equiv 0$, hence $\omega' = \omega$.

The invariance of the associated Riemannian metric g_α under the group of holomorphic automorphisms G is a direct consequence of the invariance of ω_α and the fact that the Riemannian metric can be recovered from the Kähler form. Now, if ϕ is the diffeomorphism of Ω_α induced by complex conjugation and J is the complex structure, we have $\phi J = -J\phi$. This implies that J preserves $\phi^* g_\alpha$ and that $\phi^* J = -J$. In particular, we have $\nabla^{\phi^* g_\alpha} J = 0$ so that the positive real $(1,1)$-form associated to $(\phi^* g_\alpha, J)$ (which is nothing but $-\phi^*\omega_\alpha$) is closed; thus it is Kähler–Einstein. By uniqueness, it must coincide with ω_α. This implies that $\phi^* g_\alpha = g_\alpha$. □

Remark 2.3 (Comparison with the Bergman metric)**.** The bounded domain Ω_α can be equipped with the *Bergman metric* h_α. It was proved in Theorem 3 of [AS83] that the holomorphic sectional curvature of the Bergman metric h_α is contained in an interval $[-b^2, -a^2]$ for some $0 < a < b < \infty$ not depending on α. In particular, it follows from Theorem 4.4 of [CY80] that g_α is bi-Lipschitz equivalent to h_α.

The invariant Kähler–Einstein metric g_α on Ω_α with Einstein constant $-(2n+2)$ whose existence was pointed out in Theorem 2.2 was studied by Bland [Bl86] who proved that its sectional curvature is negative. The goal of this section is to improve Bland's result and establish the following explicit description of g_α.

Theorem 2.4. *The complete G-invariant Kähler–Einstein metric g_α on Ω_α has the following properties.*

(1) *The divisor B_0 is totally geodesic.*
(2) *The sectional curvature of g_α is contained in an interval of the form $[-2n-2, -a_\alpha^2]$ for some $0 < a_\alpha \leq 1$.*
(3) *The holomorphic sectional curvature ranges in $[-2n-2, -4]$.*
(4) *For $d \in \mathbb{N}$, it holds $(\Phi_d^* g_1 - g_\alpha)(z) \to 0$, exponentially with the distance of $\Phi_d(z)$ from B_0.*

The last property of the theorem will be made more precise during the course of the proof.

Bland does not establish the asymptotic behavior of the metric transverse to the divisor (part (4) of the above theorem), which is a crucial ingredient in the proof of our main result. This property as well as the explicit description of the curvature does not seem obvious from his formulas.

The remainder of this section is devoted to the proof of Theorem 2.4. Our argument is different from Bland's approach. Its main idea is to reduce the study of the metric to an ordinary differential equation which can be solved fairly explicitly. The proof is spread over four subsections. In the first subsection we collect some properties of arbitrary invariant Kähler metrics on Ω_α, and we use this in the second subsection to obtain some first information on the curvature tensor of such metrics. These results equally hold true for the Bergman metric. In the third subsection we turn to the Kähler–Einstein metric and set up an ordinary differential equation whose solutions describe the metric fairly explicitly as described in the theorem. The curvature computation is contained in the forth subsection.

2.1. **Geometric properties of G-invariant Kähler metrics on Ω_α.** In this subsection we consider an arbitrary complete Kähler metric g on Ω_α which is invariant under the group G and under complex conjugation. Examples we have in mind are the Bergman metric of Ω_α and the invariant Kähler–Einstein metric g_α whose existence was shown in Theorem 2.2. We establish some general geometric properties with the goal to reduce curvature computations to the computation of the curvature of some specific planes in the tangent bundle of Ω_α.

A *standard totally real* plane in Ω_α is the intersection of Ω_α with $\{z \in \Omega_\alpha \mid z_i = 0 \text{ for } i \geq 3 \text{ and } z - \bar{z} = 0\}$. A *totally real plane* in Ω_α is the image of the standard totally real plane under an element of the group G. We have

Lemma 2.5. (1) *The isometry group of g is of cohomogeneity one.*
(2) *The disk $D = \{z_i = 0 \text{ for } i \geq 2\}$ and the standard totally real plane are totally geodesic.*

(3) *The ball $B_0 = \{z_1 = 0\}$ is totally geodesic, and the restriction of g to B_0 is up to a constant factor the complex hyperbolic metric.*

Proof. As the metric g is invariant under the group G and the generic orbit of this group on the ball B and hence on Ω_α by equivariance is of real codimension one, the action of the isometry group of g is of cohomogeneity one showing (1) of the lemma.

Since the disk D is the fixed point set of the holomorphic involution

$$(z_1, z_2, \dots, z_n) \to (z_1, -z_2, \dots, -z_n)$$

which is an element of the group $U(n-1) \subset G$ (the symmetric involution at the point $0 \in B_0$) and hence an isometry for g, the disk D is totally geodesic.

Similarly, the ball B_0 is the fixed point set of the holomorphic reflection

$$(z_1, z_2, \dots, z_n) \to (-z_1, z_2, \dots, z_n) \in S^1$$

and hence it is totally geodesic. Since the restriction of g to B_0 is invariant under G and since G projects to $\mathrm{PU}(n-1,1)$ and hence acts transitively on the unit tangent bundle of B_0 for the complex hyperbolic metric, the restriction of g to B_0 is a multiple of the complex hyperbolic metric which establishes part (3) of the lemma.

Now the subspace $V = \{z_i = 0 \text{ for all } i \geq 3\}$ also is the fixed point set of a holomorphic isometry $(z_1, z_2, z_3, \dots, z_n) \to (z_1, z_2, -z_3, \dots, -z_n)$ of g contained in the group G and hence it is totally geodesic. Furthermore, the set $\{\Im z_i = 0, i \geq 1\}$ is the fixed point set of complex conjugation and hence it is totally geodesic. As the intersection of two totally geodesic subspaces is totally geodesic, the standard real plane is totally geodesic. By invariance, the same then holds true for any of its images under the isometry group of g. This completes the proof. □

Consider a point $z \in D$. The real tangent space of Ω_α at z decomposes as

$$T_z\Omega_\alpha = T_zD \oplus T_zD^\perp$$

where $T_zD^\perp$ is the orthogonal complement of T_zD. Since g is Kähler and T_zD is invariant under the complex structure J, viewed as a tensor field on Ω_α, the same holds true for $T_zD^\perp$.

The group G of biholomorphic transformations of Ω_α preserves the totally geodesic submanifold B_0. Then it also preserves the level sets of the distance function to B_0 for the G-invariant Kähler metric g.

Lemma 2.6. *A level set of the distance function from B_0 is the preimage under Φ_α of a level set of the distance function from B_0 in B equipped with the complex hyperbolic metric g_1. The group G of automorphisms of Ω_α acts transitively on any such level set.*

Proof. The action of G on the preimage under Φ_α of the boundary of a tubular neighborhood of the divisor $\{z_1 = 0\}$ in the ball B is transitive, and an orbit is connected and separates Ω_α into two components, one of which contains B_0. As B_0 can be connected to any point in Ω_α by a minimal geodesic, we conclude that such an orbit equals the boundary $N(r)$ of the tubular neighborhood of radius $r \geq 0$ about B_0. As a consequence, the action of G on $N(r)$ is transitive. □

2.2. **The curvature operator of an invariant Kähler metric.** In this subsection we investigate the curvature tensor R of an arbitrary G-invariant Kähler metric $g = \langle , \rangle$ on Ω_α. It can be viewed as a section of the tensor bundle $\mathrm{Sym}(\Lambda^2 T\Omega_\alpha)$ of symmetric linear maps $\wedge^2 T\Omega_\alpha \to \wedge^2 T\Omega_\alpha$ (all the vector spaces here are viewed as real vector spaces). For $z \in D$ the stabilizer $\mathrm{U}(n-1) \subset G$ of z in the isometry group of g acts on $T_z\Omega_\alpha$ as a group of isometries commuting with the complex structure. This action induces a representation of $\mathrm{U}(n-1)$ on $\wedge^2 T_z\Omega_\alpha$ by linear isometries for the induced metric. The representation decomposes into irreducible components. The curvature tensor R is equivariant under the action of $\mathrm{U}(n-1)$ and hence it preserves the union of all linear subspaces of $\wedge^2 T_z\Omega_\alpha$ belonging to isomorphic irreducible components. This leads to the following statement.

Lemma 2.7. (1) *Let $v_1, v_2 = Jv_1$ be an orthonormal basis of T_zD; then $v_1 \wedge v_2$ is an eigenvector for R.*
(2) *Let $\{v \wedge w \mid v \in T_zD, \text{ and } w \in T_zD^\perp\}$; then $v \wedge w$ is an eigenvector for R. The eigenvalue does not depend on v, w.*
(3) *The subspace $\wedge^2 T_zD^\perp$ is invariant under R.*

Proof. The representation of $\mathrm{U}(n-1)$ on $T_z\Omega_\alpha$ decomposes into irreducible components as follows. The restriction of $\mathrm{U}(n-1)$ to the tangent space T_zD of D is the trivial representation, while the restriction of $\mathrm{U}(n-1)$ to $T_zD^\perp$ is the standard representation of $\mathrm{U}(n-1)$ on a complex vector space of dimension $n-1$. This representation is well known to be irreducible (for example via transitivity of the action of $\mathrm{U}(n-1)$ on the unit sphere in $\mathbb{C}^{n-1}$).

From this information, we can compute the irreducible components of the action of $\mathrm{U}(n-1)$ on $\wedge^2 T_z\Omega_\alpha$. Observe that $\wedge^2 T_z\Omega$ is a direct sum of subspaces

$$\wedge^2 T_z\Omega_\alpha = A_1 \oplus A_2 \oplus A_3$$

where $A_1 = \wedge^2 T_zD$, $A_2 = T_zD \wedge T_zD^\perp$ and $A_3 = \wedge^2 T_zD^\perp$. This decomposition is invariant under the action of $\mathrm{U}(n-1)$ and orthogonal with respect to the inner product induced by g. The real dimension of A_2 equals $2(2n-2)$.

The line A_1 is contained in the fixed point set for the action of $\mathrm{U}(n-1)$, that is, it is contained in a copy of the trivial representation.

For a unit vector $v \in T_zD$, the action of $\mathrm{U}(n-1)$ on the real $2n-2$-dimensional subspace $A_2(v) = \mathrm{span}\{v \wedge w \mid w \in T_zD^\perp\} \subset A_2$ of A_2 can be identified with the standard action of $\mathrm{U}(n-1)$ on $\mathbb{C}^{n-1}$, viewed as a real vector space. Thus $A_2(v)$ is invariant under $\mathrm{U}(n-1)$, and the restriction of the representation to this subspace is irreducible. Now the image of $A_2(v)$ under the complex structure J is the subspace $A_2(Jv)$, and we have $A_2 = A_2(v) \oplus A_2(Jv)$ as $\mathrm{U}(n-1)$-spaces. Thus as an $U(n-1)$-representation, A_2 is a direct sum of two standard representations of $\mathrm{U}(n-1)$ on $\mathbb{C}^{n-1}$.

On the other hand, the representation of $\mathrm{U}(n-1)$ on $\wedge^2 T_zD^\perp$ is the standard representation of $\mathrm{U}(n-1)$ on the exterior product $\wedge^2\mathbb{C}^{n-1}$, where we view $\mathbb{C}^{n-1}$ as a real vector space. The complex structure J acts on $\wedge^2\mathbb{C}^{n-1}$ as an involution. Since $\mathrm{U}(n-1)$ commutes with J, it preserves the eigenspaces $V_\pm$ for J with respect to the eigenvalues ± 1.

The eigenspace V_+ for the eigenvalue one is the kernel of the $\mathbb{R}$-linear map $\Lambda : \wedge^2\mathbb{C}^{n-1} \to \wedge^2_{\mathbb{C}}\mathbb{C}^{n-1}$ obtained by extension of scalars. Here the vector space on the right hand side is the second exterior power of the complex vector space $\mathbb{C}^{n-1}$. The vector space V_+ is spanned by elements of the form $v \wedge Jv = -Jv \wedge v$ for $v \in T_zD^\perp$. Since the center S^1 of $U(n-1)$ which contains the complex structure acts trivially on V_+ but it does not act trivially on the standard representation space $A_2(v)$, there can not be a copy of the standard representation in V_+.

The representation of $\mathrm{U}(n-1)$ on V_- is the representation of $\mathrm{U}(n-1)$ on the complex vector space $\wedge^2_{\mathbb{C}}\mathbb{C}^{n-1}$, viewed as a vector space over $\mathbb{R}$, and hence it irreducible, with highest weight different from the weight of the standard representation. As a consequence, A_2 equals the union of those irreducible components for the $\mathrm{U}(n-1)$-representation on $\wedge^2 T_z\Omega_\alpha$ which are isomorphic to the standard representation of $\mathrm{U}(n-1)$ on $\mathbb{C}^{n-1}$.

Since the curvature tensor R commutes with the action of $\mathrm{U}(n-1)$ on $\wedge^2 T_z\Omega_\alpha$, the vector space A_2 is invariant. But R also commutes with the complex structure J which maps $A_2(v)$ to $A_2(Jv)$ and therefore A_2 is an eigenspace for R. Moreover, R preserves $A_1 \oplus A_3$ since R is symmetric and the decomposition $\wedge^2 T_z\Omega_\alpha = A_2 \oplus (A_1 \oplus A_3)$ is orthogonal.

We use this to establish that for $v \in T_zD$, the vector $v \wedge Jv$ is an eigenvector for R. Namely, as R is a symmetric operator and the decomposition $A = A_2 \oplus (A_1 \oplus A_3)$ is orthogonal, with A_2 invariant under R, if $v \wedge Jv$ is not an eigenvector for R then there are $w_1 \neq w_2 \in T_zD^\perp$ orthogonal so that

$$\langle R(v, Jv)w_1, w_2\rangle \neq 0.$$

However, by the Bianchi identity, we have

$$R(v, Jv)w_1 + R(Jv, w_1)v + R(w_1, v)Jv = 0.$$

But $Jv \wedge w_1 \in A_2, w_1 \wedge v \in A_2$ and $v \wedge w_2 \in A_2$ is orthogonal to $Jv \wedge w_1$ and $w_1 \wedge v$ is orthogonal to $Jv \wedge w_2$. Since A_2 is an eigenspace for R for a fixed real eigenvalue, this implies that $\langle R(Jv, w_1)v, w_2\rangle = 0 = \langle R(w_1, v)Jv, w_2\rangle = 0$ and hence $\langle R(v, Jv)w_1, w_2\rangle = 0$, a contradiction to the assumption that $v \wedge Jv$ is not an eigenvector for R.

As a consequence, the decomposition $A = A_1 \oplus A_2 \oplus A_3$ is invariant under R. Furthermore, A_1 and A_2 are eigenspaces for R. This completes the proof of the lemma. □

Corollary 2.8. *The curvature of g is negative if and only if the following three conditions are satisfied.*

(1) *The Gauss curvature of the disk D is negative.*
(2) *The curvature of the standard totally real plane is negative.*
(3) *For every $z \in D$ there exists a J-invariant plane in $T_z^\perp D$ whose curvature is negative.*

Proof. Clearly the conditions in the corollary are necessary. We only show that they are sufficient if we replace assumption (3) by the following stronger assumption.

(3′) For every $z \in D$ the curvature of every plane in $T_z^\perp D$ is negative.

In the proof of Theorem 2.9 below we shall establish that (3) implies (3′), cf Remark 2.10.

To show that the assumptions (1), (2), (3′) imply negative curvature of g note first that by invariance under the isometry group of Ω_α, it suffices to verify that the curvature is negative at every point $z \in D$. Using the assumptions in the lemma, it suffices to compute the curvature of a plane spanned by $u_1 = v_1 + w_1, u_2 = v_2 + w_2$ with $v_i \in T_zD$ and $w_j \in T_zD^\perp$ and such that $v_1 \neq 0, w_2 \neq 0$. We allow that either $v_2 = 0$ or $w_1 = 0$. We may also assume that $\langle v_1, v_2\rangle = 0$.

Now $u_1 \wedge u_2 = v_1 \wedge v_2 + v_1 \wedge w_2 + w_1 \wedge v_2 + w_1 \wedge w_2$. By Lemma 2.7 and orthogonality of the decomposition of $\wedge^2 T_z\Omega_\alpha$ into eigenspaces for R, we compute

$$\begin{aligned}\langle R(u_1, u_2)u_2, u_1\rangle &= \langle R(u_1 \wedge u_2), u_2 \wedge u_1\rangle \\ &= \langle R(v_1, v_2)v_2, v_1\rangle + \langle R(w_1, w_2)w_2, w_1\rangle \\ &+ \langle R(v_1, w_2)w_2, v_1\rangle + \langle R(w_1, v_2)v_2, w_1\rangle.\end{aligned}$$

By the assumption in the corollary, this is a sum of non-positive terms, with at least one term negative. This completes the proof of the lemma. □

2.3. **An ordinary differential equation for the Kähler–Einstein metric.** From now on we consider the G-invariant Kähler–Einstein metric $g = g_\alpha$ on Ω_α whose existence was shown in Theorem 2.2.

By Lemma 2.6, for $r > 0$ the level surface $N(r)$ of level r for the distance to the hyperplane B_0 is a real hypersurface in the complex manifold Ω_α which is invariant under the action of the group G, and this action is transitive on $N(r)$.

The maximal J-invariant subbundle $\mathcal{D}$ of $TN(r)$ is a smooth subbundle of $N(r)$ of codimension one. The fiber $\mathcal{D}_z$ of $\mathcal{D}$ at a point $z \in D \cap N(r)$ is invariant under the action of the group $\mathrm{U}(n-1)$, and $\mathrm{U}(n-1)$ acts transitively on the sphere of unit tangent vectors in $\mathcal{D}_z$. Since the action of G preserves $N(r)$ and is transitive on $N(r)$, it follows that G acts transitively on the sphere bundle of unit tangent vectors in $\mathcal{D}$.

Let $\Pi_r : N(r) \to B_0$ be the shortest distance projection. Since by Lemma 2.5, the disk $D = \{z_i = 0 \text{ for } i \geq 2\}$ is totally geodesic and its tangent space at 0 is the orthogonal complement of TB_0, the fiber of Π_r over 0 is an S^1-orbit in D. As the distance to B_0 is G-invariant, the projection Π_r is equivariant with respect to the G-action. Thus the fiber of Π_r over every point $p \in B_0$ is an orbit of the $S^1 \subset G$-action, and the differential of Π_r maps the bundle $\mathcal{D}$ equivariantly onto the tangent bundle of B_0.

Since the action of G on the unit sphere bundle in $\mathcal{D}$ is transitive and the action of G on the unit sphere bundle of TB_0 is transitive as well, there exists a constant $f_\alpha(r) > 0$ so that the restriction of $d\Pi_r$ to any fiber of $\mathcal{D}$ is a homothety of the metric tensors with factor $f_\alpha(r)^{-2}$. Here we equip B_0 with the metric g_0 of constant holomorphic sectional curvature -4 and hence $f_\alpha(0)^{-2} g_0$ is the restriction of the metric g to B_0, where $f_\alpha(0)$ may be different from 1.

This discussion is valid for any $\alpha \geq 1$, and the function f_α depends on α. The following is the main result of this section and our main technical tool.

Theorem 2.9. *For $\alpha \in [1, \infty)$ the function f_α is a solution of the differential equation*

$$\frac{f''}{f} + n\frac{(f')^2}{f^2} + n\frac{1}{f^2} = n + 1 \tag{2}$$

with initial condition $f'_\alpha(0) = 0$ and $f_\alpha(0) \in (\sqrt{\frac{n}{n+1}}, 1]$. The map $\alpha \to f_\alpha(0)$ is a decreasing homeomorphism $[1, \infty) \to (\sqrt{\frac{n}{n+1}}, 1]$.

Note that the solution of (2) for the initial condition $f(0) = 1, f'(0) = 0$ is the function $f(t) = \cosh(t)$ which describes the metric of constant holomorphic sectional curvature -4 on the ball, and the solution with initial condition $f(0) = \sqrt{\frac{n}{n+1}}$, $f'(0) = 0$ is the constant function which can be thought of as belonging to a product metric, corresponding to the case $\alpha = \infty$.

Proof of Theorem 2.9. Let for the moment g be any Kähler metric on Ω_α which is invariant under the group G of biholomorphic automorphisms of Ω_α and under complex conjugation.

The holomorphic disk $D = \{z_i = 0 \text{ for } i \geq 2\}$ is totally geodesic for g, and the same holds true for any of its images under the group G. Thus if we denote by ξ the outer normal field of the distance hypersurface $N(t)$, then as for a point $z \in D$ the vector ξ_z is tangent to D, we have $J\xi_z \in TD \cap TN(t)$. As D is totally geodesic, this implies that $J\xi$ is a principal vector field for the hypersurface $N(t)$. Similarly, since the group G acts transitively on the sphere subbundle of the complex subbundle $\mathcal{D} = (J\xi)^\perp \subset TN(t)$, the bundle $\mathcal{D}$ is a principal bundle for $N(t)$ by equivariance. Put $f = f_\alpha$ for simplicity of notation.

Claim 1: The principal curvature λ of $\mathcal{D}$ equals

$$\lambda = -\frac{d}{dt} f(t)/f(t). \tag{3}$$

Proof of Claim 1: The proof of the claim is standard. Let $\gamma : (-\infty, \infty) \to D$ be a geodesic through $\gamma(0) = 0$ and parameterized by arc length. Choose a one-parameter group φ_s of transvections in $\mathrm{PU}(n-1, 1)$ so that $s \to \varphi_s(0)$ is a geodesic in (B_0, g) parameterized by arc length. Note that this makes sense as $B_0 \subset \Omega_\alpha$ is a totally geodesic hypersurface by Lemma 2.5 and by invariance, the restriction of g to B_0 is a multiple of the standard metric on the ball.

This one-parameter group of transvections defines a one-parameter group in G, denoted by the same symbol φ_s, so that the image of the map $(s, t) \in \mathbb{R}^2 \to \alpha(s, t) = \varphi_s(\gamma(t)) \subset \Omega_\alpha$ is a totally real plane H containing γ. Lemma 2.5 shows that H is totally geodesic, and it is foliated by the geodesics $\varphi_s(\gamma)$. The vector field $Y(t) = \frac{\partial}{\partial s}\varphi_s(\gamma(t))|_{s=0}$ is a normal Jacobi field along γ, and as Y is orthogonal to γ and tangent to H, it is a section of $\mathcal{D}|\gamma$. Thus we have

$$|Y(t)| = f(t)/f(0).$$

Let h be the second fundamental form of the hypersurface $N(t)$ with respect to the outer normal field ξ of $N(t)$. We have to show that

$$h(Y(t), Y(t))/|Y(t)|^2 = -\frac{d}{dt} f(t)/f(t).$$

Namely, we know that

$$h(Y(t),Y(t)) = \langle \nabla_{Y(t)}Y(t),\xi\rangle = \langle \frac{\nabla}{ds}\frac{\partial}{\partial s}\alpha(s,t),\xi\rangle$$

where ∇ denotes the Levi Civita connection of g. Using the fact that $Y(t)\perp \gamma'(t)$ and that $\xi = \frac{\partial}{\partial t}\alpha(s,t)$, we compute

$$\begin{aligned}\langle \frac{\nabla}{ds}\frac{\partial}{\partial s}\alpha(s,t),\xi\rangle &= -\langle \frac{\partial}{\partial s}\alpha(s,t),\frac{\nabla}{ds}\frac{\partial}{\partial t}\alpha(s,t)\rangle \\ &= -\langle \frac{\partial}{\partial s}\alpha(s,t),\frac{\nabla}{dt}\frac{\partial}{\partial s}\alpha(s,t)\rangle = -\frac{1}{2}\frac{d}{dt}|Y(t)|^2\end{aligned}$$

from which the claim follows. ■

Note that the Gauss curvature $K_{\rm tr}(t)$ of the totally geodesic real plane H at $\gamma(t)$ equals

$$K_{\rm tr}(t) = -\frac{d^2}{dt^2}f(t)/f(t). \tag{4}$$

Namely, by the Jacobi equation, this curvature equals the quantity

$$-\langle Y''(t),Y(t)\rangle/|Y(t)|^2.$$

Following p.166 of [KN69], the curvature tensor R_0 of a Kähler manifold of constant holomorphic sectional curvature -4 can pointwise explicitly written only in terms of the metric and the complex structure. Thus it is (formally) defined on any complex vector space with a J-invariant inner product. In particular, it is defined on a fiber of the bundle $\mathcal{D}$. We next compare the restriction of R to $\wedge^2\mathcal{D}$ with R_0.

Claim 2: $R|_{\wedge^2\mathcal{D}} = \frac{1}{f(t)^2}(f'(t)^2+1)R_0$.

Proof of Claim 2. Let $A: TN(t)\to TN(t)$ be the shape operator (or fundamental tensor) of $N(t)$ with respect to ξ, defined by

$$h(X,Y) = \langle AX,Y\rangle = \langle \nabla_X\xi,Y\rangle$$

where as before h denotes the second fundamental form of the hypersurface $N(t)$. Claim 1 yields that $A|\mathcal{D} = \lambda \mathrm{Id} = -(\frac{d}{dt}f(t)/f(t))\mathrm{Id}$ and hence $h|\mathcal{D} = \lambda\langle,\rangle|\mathcal{D}$. Denote by R^t the curvature tensor of $N(t)$ with respect to the restriction of the metric g. If $\owedge$ denotes the Kulkarni Nomizu product, then it follows from the Gauss Codazzi equations that we have

$$R = R^t - \frac{1}{2}h \owedge h.$$

Thus to compute the restriction of the curvature operator R to the invariant subbundle $\wedge^2\mathcal{D}$ it suffices to compute the curvature operator R^t of $N(t)$.

Put $U = J\xi$; then U is the normal field to $\mathcal{D}$ in $TN(t)$. Since g is Kähler we have

$$\nabla_X U = \nabla_X(J\xi) = J(\nabla_X \xi) = JA(X) = PAX$$

where P is the skew-symmetric $(1,1)$-tensor field on M characterized by

$$JX = PX + \langle X, U\rangle \xi.$$

This shows that $PA|\mathcal{D}$ is the fundamental tensor of the bundle $\mathcal{D}$ with respect to the normal field $-U$. Note that $PU = 0, P|\mathcal{D} = J|\mathcal{D}$ and $P\mathcal{D} = \mathcal{D}$. Furthermore, we have

$$\nabla_X Y - \lambda\langle PX, Y\rangle U \in \mathcal{D}$$

for any sections X, Y of $\mathcal{D}$ where as before, λ is the principal curvature of $\mathcal{D}$. In particular, if X, Y are sections of $\mathcal{D}$ then as ∇ is torsion free, we have

$$[X, Y] = Z + 2\lambda\langle JX, Y\rangle U \tag{5}$$

for a section Z of $\mathcal{D}$.

As the map $\Pi_t = \Pi|N(t)$ restricts to a homothety on $\mathcal{D}$, with scaling factor $f^2(t)$ with respect to the metric g_0 on B_0, the map $\Pi_t : N(t) \to (B_0, f(t)^2 g_0)$ is a Riemannian submersion. Thus the formula (5) together with O'Neill's curvature formula for Riemannian submersions shows that we have

$$\begin{aligned}\langle R^t(X,Y)Z, W\rangle = \frac{1}{f(t)^2}&\langle R_0(X,Y)Z, W\rangle + \lambda^2\langle JX, Z\rangle\langle JY, W\rangle \\ &- \lambda^2\langle JY, Z\rangle\langle JX, W\rangle + 2\lambda^2\langle JZ, W\rangle\langle JX, Y\rangle.\end{aligned}$$

On the other hand, we have

$$-\frac{1}{2} h \owedge h(X,Y,Z,W) = \lambda^2\langle X, Z\rangle\langle X, W\rangle - \lambda^2\langle X, W\rangle\langle Y, Z\rangle.$$

and consequently

$$\begin{aligned}\langle R(X,Y)Z, W\rangle =&\frac{1}{f(t)^2}\langle R_0(X,Y)Z, W\rangle + \lambda^2\langle X, Z\rangle\langle Y, W\rangle \\ &- \lambda^2\langle X, W\rangle\langle Y, Z\rangle + \lambda^2\langle JX, Z\rangle\langle JY, W\rangle \\ &- \lambda^2\langle JY, Z\rangle\langle JX, W\rangle + 2\lambda^2\langle JZ, W\rangle\langle JX, Y\rangle.\end{aligned}$$

Following p.166f of [KN69], the above equality shows that

$$\langle R(X,Y)Z, W\rangle = \frac{1}{f(t)^2}\langle R_0(Y,Y)Z, W\rangle + \lambda^2 q$$

where q is the curvature tensor of the complex hyperbolic space with holomorphic sectional curvature -4. As a consequence, the restriction of R to $\wedge^2\mathcal{D}$ equals

$$R\,|_{\wedge^2\mathcal{D}} = (\lambda^2 + \frac{1}{f^2(t)})R_0\,|_{\wedge^2\mathcal{D}}. \tag{6}$$

As $\lambda = f'(t)/f(t)$, we obtain that the multiplicity is given by

$$\frac{1}{f(t)^2}(f'(t)^2+1)$$

which completes the proof of the claim. ∎

By the above computation, the value of the Ricci tensor Ric on a unit tangent vector $X \in \mathcal{D}$ equals

$$-2f''(t)/f(t) - 2n\frac{1}{f^2(t)}(f'(t)^2+1) \tag{7}$$

since $-2n$ is the Ricci curvature of B_0 and $-f''/f$ is the Gauss curvature of a totally real plane in Ω_α. Here we use that the metric $g = \langle , \rangle$ is Kähler and hence if we denote again by ξ the outer normal field of $N(t)$, then we have

$$\langle R(X,\xi)\xi, X\rangle = \langle R(JX, J\xi)J\xi, JX\rangle = \langle R(X, J\xi)J\xi, X\rangle$$

where the last equality follows from $\mathrm{U}(n-1)$-equivariance of R and the invariance of $\mathcal{D}$ under the complex structure J.

The above computations are valid for any Kähler metric on Ω_α which is invariant under the group G and complex conjugation. In particular, it also holds true for the Bergman metric on Ω_α. Let us now assume in addition that the metric g is Kähler–Einstein, with Einstein constant $-(2n+2)$. Then the value of the Ricci tensor of g, applied to a unit tangent vector in $\mathcal{D}$, equals $-(2n+2)$. Inserting this value into the equation (7) is equivalent to the differential equation (2) for the function $f = f_\alpha$ stated in the theorem.

As a consequence, we obtain that the growth function $f = f_\alpha$ for the invariant Kähler–Einstein metric on Ω_α is a solution of the equation (2). This completes the establishment of the differential equation (2) for f_α.

We are left with showing that the initial condition for the solution f_α of the differential equation (2) which determines the metric on Ω_α for $\alpha \geq 1$ is a condition $f_\alpha(0) \in (\sqrt{\frac{n}{n+1}}, 1]$ and $f'_\alpha(0) = 0$, and that the map $\alpha \to f_\alpha(0)$ is a decreasing homeomorphism $[1,\infty) \to (\sqrt{\frac{n}{n+1}}, 1]$.

By invariance of the metric under the reflection in the z_1-coordinate, we know that $f'_\alpha(0) = 0$.

Observe that the metric g_α on Ω_α is completely determined by the function f_α. Namely, $f_\alpha(0)$ determines the restriction of g_α to the divisor B_0. Furthermore, let us consider a standard totally real plane H containing a geodesic line η in B_0 through 0. This plane is foliated by geodesics orthogonal to η, parameterized by arc length with respect to the metric g_α. The function f_α completely determines the metric on H in these coordinates as it determines the length of the tangent vectors orthogonal to the tangents of these geodesics. In particular, it computes for every $t > 0$ the metric on the J-invariant subbundle $\mathcal{D}$ of the tangent bundle of the real hypersurface $G\gamma(t)$ as a multiple of the pull-back of the metric on B_0 under the natural projection.

Now viewing the disk D as the S^1-orbit of the geodesic γ in H through 0 which is orthogonal to B_0, we know that we can also recover the restriction of the metric g_α to the disk D by knowing the curvature of the metric and hence the growth of the lengths of the S^1-orbits.

As a consequence, if $\alpha \neq \beta$ but $f_\alpha(0) = f_\beta(0)$ then there exists an G-equivariant isometry $(\Omega_\alpha, g_\alpha) \to (\Omega_\beta, g_\beta)$ whose restriction to the disk D is a biholomorphic map. By equivariance under the action of the group G this isometry commutes with the complex structure and hence is a biholomorphic map. By Corollary 1 of [AS83], this is impossible.

As a consequence, the map $\alpha \to f_\alpha(0)$ is injective. As f_1 defines the metric on the ball, to complete the proof of the theorem is suffices to show the following statement.

Claim 3. The map $\alpha \mapsto f_\alpha(0)$ is continuous, and $f_\alpha(0) \to \sqrt{\frac{n}{n+1}}$ as $\alpha \to \infty$.

Proof of Claim 3. Put $\Omega_\infty = D \times B_0$ where $D \subset \mathbb{C}$ is the standard unit disk. For $1 \leq \alpha \leq \beta \leq \infty$ let

$$\iota_{\alpha,\beta} : \Omega_\alpha \to \Omega_\beta$$

be the natural G-equivariant inclusion.

Denote as before by g_α the Kähler–Einstein metric on Ω_α with Einstein constant $-(2n+2)$. Let ω_α be the Kähler form associated to g_α. Since ω_α is Kähler-Einstein, one can find a potential φ_α for the metric (that is, $\omega_\alpha = dd^c\varphi_\alpha$) such that

$$\omega_\alpha^n = e^{2(n+1)\varphi_\alpha}\omega_{\mathbb{C}^n}^n, \tag{8}$$

where $\omega_{\mathbb{C}^n}$ is the standard euclidean metric.

We next derive some uniform estimates for ω_β as β ranges in $[1, +\infty]$. First, since ω_β is Kähler-Einstein, of negative Ricci curvature, Theorem 3 of [Yau78a] shows that there is a universal constant $C > 0$ so that

$$\iota_{\alpha,\infty}^*\omega_\infty^n \leq C\iota_{\alpha,\beta}^*\omega_\beta^n \leq C^2\omega_\alpha^n \tag{9}$$

holds on Ω_α for any $\beta \geq \alpha$.

The Kähler-Einstein metric ω_∞ on $D \times B_0$ is just the product of suitably scaled complex hyperbolic metrics on each factor and hence it has negative holomorphic sectional curvature. Therefore, Theorem 1 of [Roy80] shows that there is a constant $c > 0$ independent of β such that

$$c\,\iota_{\beta,\infty}^*\omega_\infty \leq \omega_\beta \tag{10}$$

holds for any $\beta \geq 1$.

Finally, if ω_B denotes the complex hyperbolic metric on the unit ball $B \subset \mathbb{C}^n$, then Theorem 2 of [Yau78a] shows that there is a constant $c' > 0$ independent of β such that

$$c'\, \Phi_\beta^* \omega_B \leq \omega_\beta \tag{11}$$

where $\Phi_\beta : \Omega_\beta \to B$ is the holomorphic map defined in the beginning of this section. Strictly speaking, Φ_β is multivalued when β is not an integer, but $\Phi_\beta^* \omega_B$ is well-defined. Therefore, pointwise computations can be done by choosing a local branch and one can then apply the maximum principle just as in [Yau78a].

As a consequence of (9) and (10), the following holds true. Let $K \subset \Omega_\alpha$ be a compact set and let $\epsilon > 0$ be sufficiently small that for $|\alpha - \beta| < \epsilon$, we have $K \subset \Omega_\beta$. Then for $|\alpha - \beta| \leq \epsilon/2$, there is a constant C_K independent of β such that

$$C_K^{-1} \omega_{\mathbb{C}^n} \leq \omega_\beta \leq C_K \omega_{\mathbb{C}^n} \quad \text{on } K.$$

Given the complex Monge-Ampère equation (8), a standard bootstrapping argument yields that if $\beta_i \to \alpha$ is any convergent sequence, then by passing to a subsequence, we may assume that the Kähler metrics ω_{β_i} converge uniformly on K to a Kähler metric $\hat{\omega}$ on K. This metric then is Kähler-Einstein, with constant $-2(n+1)$. As Φ_α depends in an analytic fashion on α, we have $\Phi_{\beta_i}^* \omega_B|_K \to \Phi_\alpha^* \omega_B|_K$ and hence (11) shows that $\hat{\omega} \geq c'\, \Phi_\alpha^* \omega_B$. As $K \subset \Omega_\alpha$ was arbitrary, using a diagonal sequence we deduce that $\hat{\omega}$ is defined on all of Ω_α. Since Φ_α is proper, this implies that $\hat{\omega}$ is complete. Theorem 2.2 then yields that $\hat{\omega} = \omega_\alpha$. In particular, the assignment $\alpha \mapsto \omega_\alpha(0)$ is continuous with respect to the usual topology on $[1, +\infty]$ and $\Lambda^2 \mathbb{R}^{2n}$, respectively.

As $f_\alpha(0)$ determines the scaling factor of the restriction of g_α to B_0 with respect to the Kähler–Einstein metric on B_0 with constant $-2(n+1)$, we conclude that the map $\alpha \mapsto f_\alpha(0)$ is continuous. This continuity is also valid for $\alpha = 1$, which corresponds to the ball, and for $\alpha = \infty$ which corresponds to the product $D \times B_0$. As $f_1(0) = 1$ and $f_\infty(0) = \sqrt{\frac{n}{n+1}}$ (the latter value describing the product Kähler–Einstein metric), injectivity of the assignment $\alpha \mapsto f_\alpha(0)$ yields that $f_\alpha(0) \in (\sqrt{\frac{n}{n+1}}, 1]$ for all $\alpha \geq 1$. This completes the proof of the claim. ■ □

Remark 2.10. Claim 2 in the proof of Theorem 2.9, which is valid for any G-invariant Kähler metric, implies the equivalence of assumption (3) in Corollary 2.8 and condition (3′) stated in its proof and hence completes the proof of Corollary 2.8.

2.4. **The curvature of the Kähler–Einstein metric.** The goal of this section is to analyze the solutions of the differential equation (2) and use it to control the curvature of the Kähler–Einstein metric g_α on the domain Ω_α ($\alpha < \infty$) with Einstein constant $-(2n+2)$. The following theorem summarizes the relevant curvature properties.

Theorem 2.11. *Let g_α be the invariant Kähler–Einstein metric on the domain $\Omega_\alpha \subset \mathbb{C}^n$ Then the following holds true.*

(1) *The sectional curvature of a standard totally real plane $H \subset \Omega_\alpha$ is negative and bounded from below by -1.*
(2) *The sectional curvature K_α of the complex disk D is negative and contained in the interval $(-2n-2,-4]$. For every $\epsilon > 0$ there exists a number $C = C(\alpha,\epsilon) > 0$ such that $\left|K_\alpha + 4\right| \le Ce^{-(1-\epsilon)\,d(0,\cdot)}$.*
(3) *The sectional curvature is bounded from above by a negative constant, and bounded from below by $-2n-2$.*

Proof of Theorem 2.11. For convenience, we drop the index α from the notation. By the first part of Theorem 2.9, we know that the invariant Kähler–Einstein metric $g = g_\alpha$ on Ω_α determines a solution $f = f_\alpha$ of the differential equation (2) with initial condition $f(0) \in (\sqrt{\frac{n}{n+1}}, 1]$ and $f'(0) = 0$. It is a direct consequence of the equation that we have $f''(0) > 0$ and hence $f'(t) > 0$ for $t > 0$ sufficiently close to 0. We divide the argument into six claims.

Claim 1: The function $\log f$ is convex, that is, $\frac{d}{dt}\frac{f'}{f} = \frac{d^2}{dt^2}\log f = \frac{f''}{f} - (\frac{f'}{f})^2 \ge 0$.

Proof of Claim 1. The inequality clearly holds true for $t = 0$. Assume to the contrary that there exists a smallest number $\tau > 0$ so that $(f''/f - (f'/f)^2)(\tau) = 0$ and that this quantity is negative for $t \in (\tau, \tau+\delta)$ for some small $\delta > 0$. This means that the value of f'/f is strictly decreasing on (τ,σ) for some $\sigma \in (\tau,\tau+\delta)$.

Since f'/f is non-decreasing on $[0,\tau]$, and $f'(t) > 0$ for sufficiently small $t > 0$, by possibly decreasing δ we may assume that $f' > 0$ on $(\tau-\delta,\tau+\delta)$. Then $(f')^2/f^2$ is also strictly decreasing on (τ,σ) and hence $n+1-(n+1)\frac{(f')^2}{f^2} - \frac{1}{f^2}$ is strictly increasing on $(\tau,\tau+\delta)$. Inserting into the equation (2) yields a contradiction which shows the claim. ■

As a consequence of Claim 1, we have $f''(t) > 0$ for all t. In particular it holds $\frac{f''}{f}(t) > 0$ for all t. Moreover, f' is strictly increasing in t and hence $f' > 0$ on $(0,\infty)$, which yields that f is strictly increasing on $(0,\infty)$ as well.

As the function f'/f is non-decreasing, we can ask for its limit as $t \to \infty$.

Claim 2: It holds $f'/f \to 1$ as $t \to \infty$.

Proof of Claim 2. Inserting the inequality of Claim 1 into the differential equation (2) yields that $f'/f < 1$ on $[0, \infty)$ and hence $\lim_{t\to\infty}(f'/f)(t) = a \in (0, 1]$. As $f'' > 0$, we have $f(t) \to \infty$ $(t \to \infty)$. Thus if $a < 1$ then the equation (2) shows that for all sufficiently large $t > 0$ we have $\frac{f''}{f} > 1 + \epsilon$ for $\epsilon = n(1 - a/2) > 0$. But then for large t the quantity $\frac{d}{dt}\frac{f'}{f}(t)$ is bounded from below by a universal positive constant which contradicts the fact that $f'/f < 1$. ■

Let now $f_1(t) = \cosh(t)$ be the solution of the equation (2) with initial condition $f_1(0) = 1$ and $f_1'(0) = 0$. Assume that $\alpha \neq 1$, that is, $f(0) < 1 = f_1(0)$.

Claim 3: We have $f(t) < f_1(t)$ and $(f'/f)(t) < (f_1'/f_1)(t)$ for all $t > 0$.

Proof of Claim 3. Assume to the contrary that there is a first $\tau > 0$ so that $f(\tau) = f_1(\tau)$. Since log is a monotone function and f, f_1 are positive, we then we have $\frac{d}{dt}\log f(\tau) \geq \frac{d}{dt}\log f_1(\tau)$, that is, $(f'/f)(\tau) \geq (f_1'/f_1)(\tau)$. But if equality holds then $f'(\tau) = f_1'(\tau)$ and hence the initial conditions at τ of the solutions f, f_1 of the equation (2) coincide. Then $f = f_1$ which is impossible. So we have $(f'/f - f_1/f_1)(\tau) > 0$.

The equation (2) shows that $f''(\tau) < f_1''(\tau)$ and hence $\frac{d}{dt}(\frac{f'}{f} - \frac{f_1'}{f_1})|_{t=\tau} < 0$. Thus the function $f'/f - f_1'/f_1$ is decreasing near τ. On the other hand, the initial conditions for f, f_1 at $t = 0$ imply that $f'/f - f_1'/f_1$ is also decreasing near 0. As its value at 0 equals zero and its value at τ is positive, the intermediate value theorem yields that there is some smallest $\sigma \in (0, \tau]$ with $f'/f(\sigma) = f_1'/f_1(\sigma)$. Since $f'/f - f_1'/f_1$ is decreasing near τ, we have $\sigma < \tau$ and hence $f(\sigma) < f_1(\sigma)$ by the choice of τ.

Insertion of this inequality into (2) yields $(f''/f)(\sigma) < (f_1''/f_1)(\sigma)$ and hence $f'/f - f_1'/f_1$ is decreasing near σ. This is a contradiction to the choice of σ. Together we obtain that $f(t) < f_1(t)$ for all t and also $f'/f < f_1'/f_1$. ■

Claim 4: The function $t \to \frac{f''}{f}(t)$ is strictly increasing, and $\frac{f''}{f}(t) \to 1$ as $t \to \infty$.

Proof of Claim 4. The equation (2) shows that

$$\frac{f''}{f} = n + 1 - n(\frac{f'}{f})^2 - \frac{n}{f^2}.$$

Differentiating this equations yields

$$\frac{d}{dt}(\frac{f''}{f}) = -2n(\frac{d}{dt}\frac{f'}{f})(\frac{f'}{f}) + \frac{2n}{f^2}(\frac{f'}{f}). \tag{12}$$

Inserting the initial condition for f shows that the right hand side of equation (12) vanishes at $t=0$. In view of $\frac{1}{f^2(0)} > 1$, dividing by $\frac{f'}{f}$, which is positive for all $t>0$ by Claim 1, and taking the limit as $t \searrow 0$ yields that the right hand side of (12) is positive for small $t>0$.

We use equation (12) to study the critical points of $\frac{f''}{f}$. Let $\tau > 0$ be a first positive critical point. Since $\frac{f'}{f}(\tau) > 0$, equation (12) yields that

$$-2n(\frac{f''}{f}(\tau) - (\frac{f'}{f})^2(\tau)) + \frac{2n}{f^2}(\tau) = 0$$

and hence

$$\frac{1}{f^2}(\tau) = \frac{f''}{f}(\tau) - (\frac{f'}{f})^2(\tau).$$

Insertion of the expression for $\frac{1}{f^2}(\tau)$ into the differential equation (2) shows that $\frac{f''}{f}(\tau) = 1$.

Now by Claim 1 and Claim 2, we have $\frac{f''}{f} \geq (\frac{f'}{f})^2$, moreover $\frac{f'}{f}$ is increasing and converges to 1 as $t\to\infty$. Using once more equation (2), we also have $\frac{f''}{f} \to 1$ $(t\to\infty)$. Thus if there exists a number $t>0$ so that $\frac{f''}{f}(t) > 1$, then the function $\frac{f''}{f}$ assumes a global maximun at a number $t>0$ with $\frac{f''}{f}(t) > 1$. But then t is a critical point for $\frac{f''}{f}$ violating that by the above computation, its value at every critical point is one.

We conclude that $\frac{f''}{f}(t) \leq 1$ for all t, moreover the only critical points in $(0,\infty)$ are global maxima with functional value one. As $\frac{f''}{f}(t) \to 1$ $(t\to\infty)$, we deduce that the function $\frac{f''}{f}$ is non-decreasing. Since it also is analytic, it can not assume the value one as this would imply that the function is constant. Hence $\frac{f''}{f}$ is strictly increasing as predicted in the claim. ■

We can now use what we established to give an explicit description of the curvature of the metric g_α. To this end we need to control the Gaussian curvature $K_\alpha(t)$ of the totally geodesic holomorphic disk D, the curvature $K_{\rm tr}(t)$ of a totally real plane and the curvature of the planes in $\mathcal{D}$. We have

$$K_{\rm tr}(t) = -\frac{f''}{f}$$

by (4) so that Claim 4 yields

$$-1 \leq K_{\rm tr} \leq -(n+1) + \frac{n}{f(0)^2}. \tag{13}$$

By Claim 2 from the proof of Theorem 2.9, we have

$$R|_{\Lambda^2\mathcal{D}} = \frac{1}{f^2}((f')^2+1)R_0$$

where the function $\frac{1}{f^2}((f')^2+1) = -\frac{1}{n}\frac{f''}{f} + \frac{n+1}{n}$ is decreasing by Claim 4. Recall that the sectional curvature of the metric g_0 on the ball B_0 is contained in the interval $[-4,-1]$ since g_0 has holomorphic sectional curvature -4. This implies that for any plane $P \subset \mathcal{D}$, it holds

$$-\frac{4}{f(0)^2} \leq K(P) \leq -1. \tag{14}$$

Finally, since $\operatorname{Ric} g = -2(n+1)g$, we have that

$$K_\alpha = -2(n+1) - 2(n-1)K_{\mathrm{tr}}, \tag{15}$$

hence

$$-2(n+1) \leq K_\alpha \leq -4. \tag{16}$$

By Lemma 2.8, it follows that the curvature of g is negative, and the first three items of the theorem are proved. Moreover, Claim 4 implies that $K_{\mathrm{tr}}(t) \to -1$ and $K_\alpha(t) \to -4$ as $t \to \infty$. Finally, we see that the supremum of the sectional curvature is attained by the totally real planes at a point of B_0. That is,

$$\sup_{z\in\Omega_\alpha} \sup_{\substack{P\subset T_z\Omega_\alpha \\ \text{plane}}} K_{g_\alpha}(P) = -(n+1) + \frac{n}{f_\alpha(0)^2}. \tag{17}$$

Note that as $\alpha \to +\infty$, the right hand side increases to 0.

The following computation yields that the convergence of the curvature tensor to the curvature tensor of a metric on the ball is exponential in t.

Claim 5: For $\epsilon > 0$ there exists a number $C = C(f,\epsilon) > 0$ such that

$$\left|\frac{f'(t)}{f(t)} - 1\right| + \left|\frac{f''(t)}{f(t)} - 1\right| \leq Ce^{-(1-\epsilon)t}.$$

Moreover, for any integer $k \geq 1$, there is a constant $C_k = C(k,f,\epsilon)$ such that

$$\left|\frac{d^k}{dt^k}\Big(\frac{f''}{f}\Big)\right| \leq C_k e^{-(1-\epsilon)t}. \tag{18}$$

Proof of Claim 5. We know that $\frac{f'}{f} \leq \frac{f''}{f} \leq 1$ for all t. On the other hand, we also have $\frac{f''(t)}{f(t)} + n\frac{(f'(t))^2}{f(t)^2} \geq n+1-n\frac{1}{f^2}$. For large t, $\frac{d}{dt}\log f(t) \geq 1-\epsilon$ and hence $n/f^2(t) \leq e^{-(1-\epsilon)t}$. From this we get $\frac{f'(t)}{f(t)} - 1 \geq Ce^{-(1-\epsilon)t}$. The remaining inequality for $\frac{f''(t)}{f(t)} - 1$ is a consequence of the previous one and (2).

As for the last statement of the claim, let us set $u = \log f$. We have $u'' = \frac{f''}{f} - (\frac{f'}{f})^2 = \frac{f''}{f} - u'^2$ and $u'' + (n+1)(u'^2 - 1) + ne^{-2u} = 0$ thanks to (2). From the previous steps, we know that $e^{-u} + |u' - 1| + u'' = O(e^{-(1-\epsilon)t})$. An elementary induction based on the ODE for u and the previous estimates shows that $u^{(k)}$ converges to zero exponentially fast for any $k \geq 2$. Since $u'' = \frac{f''}{f} - u'^2$, the same goes for the derivatives of $\frac{f''}{f}$, hence the claim. ■

From Claim 5, (4) and (15), one deduces the second item in the Theorem, which concludes its proof. □

Remark 2.12. J.F. Lafont and B. Minemyer [LM25] made independent computations to analyze (real) Einstein metrics on Ω_α. Combined with our results, their work leads to an explicit solution of the differential equation (2) with respect to the initial conditions $f(0) \in (\sqrt{\frac{n}{n+1}}, 1], f'(0) = 0$.

2.5. **Comparison with the pull-back of the ball metric.** In Theorem 2.11, we established a precise curvature control for the Kähler–Einstein metric g_α on Ω_α. In particular, it follows from its second part that the curvature tensor of g_α converges exponentially with the distance from the divisor B_0 to the curvature tensor of a metric of constant holomorphic sectional curvature and the same Einstein constant. In this section it will be convenient to normalize this constant to be $\frac{1}{2}(n+1)$ so that the holomorphic sectional curvature for the metric on the ball equals -1.

The pull-back $\Phi_\alpha^* g_1$ by Φ_α of the metric g_1 on the ball B is a metric of constant holomorphic sectional curvature -1 on $\Omega_\alpha \setminus B_0$. The goal of this section is to compare the metrics g_α and $\Phi_\alpha^* g_1$ as the distance $d_\alpha(B_0, \cdot)$ from B_0, measured with respect to the distance function d_α of g_α, tends to infinity. Our findings are summarized in the following result.

Theorem 2.13. *For $k \geq 0$ there exist numbers $a(\alpha, k) > 0, C(\alpha, k) > 0$ such that the metrics g_α and $\Phi_\alpha^* g_1$ satisfy $\|g_\alpha - \Phi_\alpha^* g_1\|_{C^k} \leq C(\alpha, k) e^{-a(\alpha,k)\, d_\alpha(\cdot, B_0)}$ on the complement of the tubular neighborhood of radius one about B_0.*

Remark 2.14. Since the map Φ_α is singular on B_0, the pull-back $\Phi_\alpha^* g_1$ is not a metric on Ω_α, but it is a Kähler–Einstein metric on $\Omega_\alpha - B_0$. Theorem 2.13 says that this metric is arbitrarily close to the metric g_α in the C^k-topology on the complement of a suitable tubular neighborhood of B_0 in Ω_α, measured with respect to the metric g_α. Since any such tubular neighborhood is the preimage under Φ_α of a tubular neighborhood of B_0 in B for the complex hyperbolic metric g_1, we can rephrase the result also in terms of the distance from B_0 with respect to the pull-back $\Phi_\alpha^* g_1$ provided that we restrict measuring distances to the complement of the preimage of the radius one tubular neighborhood of B_0 in B.

Proof of Theorem 2.13. Let ω (resp. ω_1) be the Kähler form associated to g_α (resp. g_1). Put $\widehat{\omega} = \Phi_\alpha^* \omega_1$. The two-form $\widehat{\omega}$ on $\Omega_\alpha - B_0$ defines a Kähler–Einstein metric of constant holomorphic sectional curvature -1. The punctured holomorphic disk $D \setminus \{0\} \subset D = \{z_i = 0 \text{ for } i \geq 2\} \subset \Omega_\alpha$ is totally geodesic for both $\omega, \widehat{\omega}$. By invariance of ω under G (resp. ω_1 under $\mathrm{PU}(n,1)$) and equivariance of the map Φ_α as formulated in (1), the two-forms $\widehat{\omega}^D = \widehat{\omega}|_D$ and $\omega^D = \omega|_D$ are invariant under the circle group S^1 of rotations acting on D.

We begin with showing that the metrics $\omega^D, \widehat{\omega}^D$ are exponentially close with the distance from $0 \in D$, where closeness means pointwise closeness in norm with respect to the metric ω^D, equivalently C^0-closeness. The proof of this statement is carried out in three steps. Throughout we denote for $r > 0$ by $D_r \subset D$ the disk of radius r about 0 for the metric ω^D.

Claim 1. There exists a number $\kappa = \kappa(\alpha) > 0$ so that $\frac{\widehat{\omega}^D}{\omega^D} \in [\kappa, \kappa^{-1}]$ on $D \setminus D_1$.

Proof of Claim 1. Choose a function φ supported in D_1 such that $\widehat{\omega}^D + dd^c\varphi$ is a Kähler metric on D of bounded negative curvature. The existence of such a function is standard, see e.g. [Zh96], its proof will be omitted. Since the curvature of ω^D is also bounded negative, the classical Schwarz Pick lemma (see [Yau78a] for more information) shows that $\frac{\widehat{\omega}^D + dd^c\varphi}{\omega^D} \in [\kappa, \kappa^{-1}]$ for some constant $\kappa = \kappa(\alpha) > 0$. In particular, we have $\frac{\widehat{\omega}^D}{\omega^D} \in [\kappa, \kappa^{-1}]$ on $D - D_1$. ∎

Let $d = d_\alpha$ be the distance function on D for the metric ω^D. To simplify the notations, by modifying $\widehat{\omega}^D$ with a potential supported in D_1 we assume that $\widehat{\omega}^D$ is a complete Kähler metric on D. As we are only interested in estimates outside of D_1 this does not alter our analysis.

Claim 2. There exist numbers $a_1 > 0, C_1 > 0$ so that

$$\omega^D < (1 - C_1 e^{-a_1 d(0,\cdot)})^{-1} \widehat{\omega}^D.$$

Proof of Claim 2. By Claim 1, distances from 0 in D with respect to the distance functions of ω^D and $\widehat{\omega}^D$ are uniformly comparable. This implies that there exists a number $\kappa \in (0,1)$ such that for every $x \in D \setminus D_1$, we have $\kappa^{-1} d(0,x) > \widehat{d}(0,x) > \kappa d(0,x)$. Here $\widehat{d}$ is the distance function of the metric $\widehat{\omega}^D$.

Assume that $d(x,0) > 2\kappa^{-1}$ and let $\widehat{Q}_x \subset D \setminus \{0\}$ be the ball of radius $u = \frac{\kappa}{2} d(0,x) \leq \frac{1}{2}\widehat{d}(0,x)$ about x for the metric $\widehat{\omega}$. Since $\widehat{d}(x,0) > 2$, the curvature of the restriction of the metric $\widehat{\omega}^D$ to $\widehat{Q}_x$ is constant -1. Then up to isometry, the set $\widehat{Q}_x$ is the round disk in $\mathbb{C}$ of euclidean radius $r \in (0,1)$, with x corresponding to the center 0 of the disk, and equipped with the restriction of the Poincaré metric $\frac{4dz\wedge d\bar{z}}{(1-|z|^2)^2}$. The Euclidean radius $r > 0$ of $\widehat{Q}_x$ is computed by the formula $\cosh(u) = \frac{1+r^2}{1-r^2}$.

Denote by $\widehat{\omega}_x$ the standard *complete* Poincaré metric on $\widehat{Q}_x$, obtained as the pull-back of the Poincaré metric $\frac{4dz\wedge d\bar{z}}{(1-|z|^2)^2}$ on the unit disk by the scaling map $z \to \frac{1}{r}z$. This rescaling operation replaces the restriction of the Poincaré metric $\frac{4dz\wedge d\bar{z}}{(1-|z|^2)^2}$ to $\widehat{Q}_x$ by the metric $\frac{4dz\wedge d\bar{z}}{(1-r^{-2}|z|)^2}$ A standard calculation shows that

$$\widehat{\omega}^D(x) \leq \widehat{\omega}_x(x) \leq (1 - Ce^{-\kappa\, d(0,x)/2})^{-1}\widehat{\omega}^D(x) \tag{19}$$

for a universal constant $C > 0$.

Write $\widehat{\omega}_x = e^{2\rho_x}\omega^D|\widehat{Q}_x$ for a function ρ_x on $\widehat{Q}_x$. Since $\widehat{\omega}_x$ is a complete metric on $\widehat{Q}_x$ and by Claim 1, ω^D is bi-Lipschitz equivalent to $\widehat{\omega}^D$, it follows from the construction of $\widehat{\omega}_x$ that ρ_x is a proper function.

Let K_g be the Gauss curvature of ω^D. The curvature of $\widehat{\omega}_x = e^{2\rho_x}\omega^D$ is constant -1 and hence denoting by $\Delta_x = \mathrm{tr}\nabla^2$ the Laplacian of $\widehat{\omega}_x$, we have

$$K_g = e^{2\rho_x}(-1 - \Delta_x(-\rho_x)). \tag{20}$$

Now ρ_x is proper and hence it assumes a minimum at some point $y \in \widehat{Q}_x$. Then it holds $\Delta_x(\rho_x)(y) \geq 0$. On the other hand, by Theorem 2.11, the Gauss curvature K_g of ω^D satisfies $K_g < -1$. Insertion into the equation (20) implies that we have $e^{2\rho_x}(y) \geq 1$. As ρ_x assumes a minimum at y, this then implies that $\rho_x \geq 0$ and hence $\omega^D \leq \widehat{\omega}_x$. The claim now follows from the estimate (19). ■

Claim 3. There exist numbers $a_2 > 0, C_2 > 0$ so that

$$\omega^D > (1 - C_2 e^{-a_2\, d(0,\cdot)})\widehat{\omega}^D.$$

Proof of Claim 3. The proof of the claim follows from reversing the roles of ω^D and $\widehat{\omega}^D$ in the proof of Claim 2. Let $x \in D$ be such that $d(0,x) > 2\kappa^{-1}$ and let Q_x be the metric disk of radius $\frac{1}{2}d(0,x)$ about x for the metric ω^D. By the second part of Theorem 2.11 and the triangle inequality, we have $K_g(Q_x) \subset [-1 - Ce^{-\sigma\, d(0,x)}, -1]$ for some constants $C = C(\alpha) > 0$, $\sigma = \sigma(\alpha) > 0$.

Let $\widehat{Q}_x$ be the ball of radius $u = \frac{1}{2}\kappa d(0,x)$ about x for the metric $\widehat{\omega}$. We know that $\widehat{Q}_x \subset \Omega_x$. Moreover, up to isometry, $\widehat{Q}_x$ is the round disk in $\mathbb{C}$

centered at 0, of euclidean radius $r \in (0,1)$, and equipped with the metric $\frac{4dz\wedge d\bar{z}}{(1-|z|^2)^2}$. Here x corresponds to the center 0 of the euclidean disk, and the radius r is computed by $\cosh(u) = \frac{1+r^2}{1-r^2}$.

Similar to the construction in the proof of Claim 2, replace the restriction of the metric $\widehat{\omega}^D$ to $\widehat{Q}_x$ by an incomplete conformal metric $\widehat{\omega}_x$ which is the pull-back of the Poincaré metric on the unit disk by a scaling map $z \to sz$. Here the scaling parameter $s < 1$ is chosen in such a way that the pull-back metric $\widehat{\omega}_x$ can be written as $\widehat{\omega}_x = e^{2\psi}\widehat{\omega}^D$ where the function ψ satisfies $e^{2\psi} \equiv \kappa^2/4$ on $\partial\widehat{Q}_z$. Explicitly, the parameter s is determined by the equation $(1-s^2r^2)^2 = \frac{\kappa^2}{2(1-r^2)^2}$. As in the proof of Claim 2, we have the estimate

$$e^{2\psi(x)} > 1 - Ce^{-\kappa\, d(z,0)/2} \tag{21}$$

for a universal constant $C > 0$.

Assume from now on that $d(x,0)$ is sufficiently large that $e^{2\psi(x)} \geq \frac{1}{2}$. There exists a smooth function ρ_x on $\widehat{Q}_x$ such that $\widehat{\omega}_x = e^{2\rho_z}\omega^D|\widehat{Q}_x$. Note that by construction, we have $e^{2\rho_x} = e^{2\psi}\frac{\widehat{\omega}^D}{\omega^D}$. As $\widehat{\omega}^D \leq \kappa^{-1}\omega^D$ and $e^{2\psi} \equiv \frac{\kappa^2}{4}$ on $\partial\widehat{Q}_x$, the value of the function $e^{2\rho_x}$ on $\partial\widehat{Q}_x$ is smaller than $\kappa/4$. Moreover by the estimate (21) and the assumption on $d(0,x)$, we have $e^{2\rho_x}(x) \geq \kappa/2$. Thus ρ_z assumes a maximum at an interior point $y \in \widehat{Q}_x$. Denoting by Δ_g the Laplacian for the metric ω^D, it follows $\Delta_g(\rho_x)(y) \leq 0$.

Now the constant curvature -1 of the metric $\widehat{\omega}_x$ can be computed by

$$-1 = e^{-2\rho_x}(K_g - \Delta_g(\rho_x)).$$

Since $K_g(y) \geq -1 - Ce^{-\sigma\, d(0,x)}$, we obtain $e^{-2\rho_x}(y) \geq (1+Ce^{-\sigma\, d(0,x)})^{-1}$ and hence $e^{2\rho_z}(y) \leq 1 + Ce^{-\sigma\, d(0,x)}$. Since y was a maximum for ρ_x, we also have $e^{2\rho_x}(x) \leq 1 + Ce^{-\sigma\, d(0,x)}$. Together with the estimate (21), this completes the proof of the claim. ■

Remark 2.15. There is an alternative, slightly different way to prove Claims 2 and 3 above, which we briefly sketch now. Write $\omega^D = e^f\widehat{\omega}^D$ and set $f(z) = g(t)$ where $t = \ \log|z|^{2\alpha}$. Using the curvature decay of K_α and Claim 1, we see that g satisfies the double sided inequality $1 + C(-t)^\gamma \geq e^{-g}\Big(\alpha^2 e^{-t}(1-e^t)^2 g''(t) + 1\Big) > 1$ for some $C > 0$ and $\gamma \in (0,1)$. Using the maximum principle, one can prove that $g(t) \to 0$ as $t \to 0^-$. Moreover, the inequality above implies that $g(t) + C(-t)^\gamma$ is concave, equal to $+\infty$ (resp. 0) at $t = -\infty$ (resp. $t = 0$), hence it is non-negative. Similarly, $g(t) + t - C(-t)^\gamma$ is convex, equal to $-\infty$ (resp. 0) at $t = -\infty$ (resp. $t = 0$), hence it is nonpositive. This yields the desired estimate $|g(t)| \leq C(-t)^\gamma$ near $t = 0$.

Claim 4. Put $\rho = \frac{1}{2}\log\frac{\omega^D}{\hat{\omega}^D}$. For any integer $k \geq 0$ there are numbers $a_k > 0, C_k > 0$ such that $\|\rho\|_{C^k} < C_k e^{-a_k d(0,\cdot)}$ where the C^k-norm at a point x is taken as the C^k-norm in standard coordinates on the unit ball about x for the metric $\hat{\omega}^D$.

Proof of Claim 4. By Claim 2 and Claim 3, we have

$$(1 - C_1 e^{-a_1 d(0,\cdot)})\omega^D < \hat{\omega}^D < (1 - C_2 e^{-a_2 d(0,\cdot)})^{-1}\omega^D$$

and hence $|\rho| \leq Ce^{-ad(0,\cdot)}$ for some $C > 0, a > 0$ by smoothness of the logarithm near 1. Equation (20) together with the estimates for the Gauss curvature for ω_D then shows that up to increasing C and decreasing a, we have

$$|\Delta(\rho)| = |e^{-2\rho}K_g + 1| \leq Ce^{-ad(0,\cdot)}$$

where Δ is the Laplacian of the constant curvature metric $\hat{\omega}$. Standard Schauder estimates now imply that

$$\|\rho\|_{C^{1,\alpha}} \leq Ce^{-ad(0,\cdot)},$$

for a possibly different constant C. Since $\omega^D = e^{2\rho}\hat{\omega}^D$, this implies that the C^1 norms with respect to covariant derivatives of ω^D and $\hat{\omega}^D$ are uniformly comparable. Now, by (18) and (15), the quantity $K_g + 1$ goes to zero exponentially fast along with all its covariant derivatives with respect to ω^D. Since $\Delta(\rho) = e^{-2\rho}K_g + 1$, we get exponential decay for the $C^{2,\alpha}$ norm of ρ with respect to $\hat{\omega}^D$. We can now iterate the argument by the usual bootstrapping process. ■

We can now conclude the proof of the theorem. Set $\widehat{\omega} := \Phi_\alpha^* \omega_1$ which is a Kähler metric away from B_0. On that locus, one can write

$$\omega = \widehat{\omega} + dd^c\varphi = \widehat{\omega} + \frac{i}{2}\partial\bar{\partial}\varphi$$

where $\varphi := \frac{1}{2n+2}\log\left(\frac{\omega^n}{\widehat{\omega}^n}\right)$. By invariance of ω under G (resp. ω_1 under $\mathrm{PU}(n,1)$) and equivariance of the map Φ_α in the sense of (1), the function φ is G-invariant and hence it is determined by its restriction to the disk D.

The restriction $\varphi^D = \varphi|_D$ of the potential φ to the disk D satisfies

$$\omega^D = \hat{\omega}^D + dd^c\varphi^D.$$

By Claim 4, we know that for any integer $k \geq 0$ the estimate

$$\left|dd^c\varphi^D\right|_{C^k} \leq C_k e^{-a_k\, d(0,\cdot)} \tag{22}$$

holds for some $a_k > 0, C_k > 0$.

But $\varphi^D = \varphi|_D$ and the restriction of φ to any G-orbit is constant. Such an orbit is a distance hyperplane from the totally geodesic subspace B_0 and hence at distance uniformly bounded away from B_0, it is a real hyperplane whose second fundamental form is uniformly bounded. Thus near any point x

of sufficiently large distance from B_0, there are real coordinates $(x_1, \dots, x_{2n})$ of uniformly bounded derivatives of all order so that $x_1 = d(\cdot, B_0)$ and that the function φ only depends on x_1. Furthermore, if $x \in D$ then these coordinates can be chosen so that the disk D is the set $x_3 = \cdots = x_{2n} = 0$, and if $x \notin D$ then these coordinates are obtained from coordinates about a point $\hat{x} \in D$ by precomposing with an element of $PU(n-1,1)$. Hence the C^k-norm of $dd^c\varphi$ at a point $x \in \Omega_\alpha$ with respect to $\widehat{\omega}$ is bounded from above by a constant multiple of the C^k-norm of $dd^c\varphi^D$ at a point $x' \in D$ with respect to $\hat{\omega}^D$, where x' satisfies with $d(0,x') = d(B_0, x)$. The theorem now follows from (22). ☐

3. The construction of Stover and Toledo

Consider a compact arithmetic complex hyperbolic orbifold $\hat{M}$ of simplest type and dimension $n \geq 2$. Such an orbifold is given as follows, see e.g. [McR07, § 2.2] or [St07]. There is a purely imaginary quadratic extension L of a totally real number field F with $[F : \mathbb{Q}] \geq 2$, and there is a Hermitian quadratic form

$$q(z, \bar{z}) = -a_0 z_0 \bar{z}_0 + \sum_{i=1}^{n} a_i z_i \bar{z}_i$$

with coefficients in the ring O_F of integers in F, anisotropic over $\mathbb{Q}$, so that $\hat{M} = \hat{\Gamma}\backslash\mathbb{CH}^n$ where $\hat{\Gamma} = \mathrm{GL}(n+1, \mathcal{O}_L) \cap U(q)$. The latter sits as a lattice inside $\mathrm{PU}(n,1)$, and its action on $\mathbb{CH}^n$ has finite stabilizers.

The involution

$$\iota : (z_0, z_1, \dots, z_n) \to (z_0, \dots, z_{n-1}, -z_n)$$

preserves q and hence lies in $\hat{\Gamma}$. Since $\hat{\Gamma}$ is residually finite, there exists a finite index subgroup $\hat{\Gamma}' \lhd \hat{\Gamma}$ such that $\iota \notin \hat{\Gamma}'$. Since $\hat{\Gamma}'$ is normal, it is preserved by the action of ι by conjugation. By construction, ι descends to a non-trivial holomorphic involution of $\hat{M}' = \hat{\Gamma}'\backslash\mathbb{CH}^n$. Its fixed point set is a totally geodesic suborbifold $\hat{H}' \subset \hat{M}'$ of codimension one which is a compact arithmetic complex hyperbolic orbifold of simplest type in its own right. In general, this orbifold is neither embedded in $\hat{M}'$ nor connected. As a consequence of (23) below, one can actually choose $\hat{\Gamma}'$ to be one of the congruence subgroups $\hat{\Gamma}_\ell$ defined in the next paragraph.

An ideal $\mathcal{I}_L$ in the ring $\mathcal{O}_L$ of integers of L determines a *congruence subgroup* Γ of $\hat{\Gamma}$, defined as the kernel of the homomorphism $\hat{\Gamma} = \mathrm{GL}(n+1, \mathcal{O}_L) \cap U(q) \to \mathrm{GL}(n+1, \mathcal{O}_L/\mathcal{I}_L)$ induced by the quotient projection $\mathcal{O}_L \to \mathcal{O}_L/\mathcal{I}_L$. By construction, Γ also is invariant under the involution ι. By choosing a *sufficiently deep* congruence subgroup we may assume that $M = \Gamma\backslash\mathbb{CH}^n$ is a manifold, called a *congruence manifold* in the sequel. By perhaps passing to a further congruence cover we may also assume that the preimage H of $\hat{H}$ is a compact embedded totally geodesic submanifold of M.

Using the pair (M, H) as a starting point, called a *standard congruence pair* in the sequel, the goal of this section is to establish the following improvement of one of the main results of [ST22]. For its formulation, recall that the *normal injectivity radius* of a totally geodesic hypersurface $H \subset M$ is the supremum of all numbers $R > 0$ so that the tubular neighborhood of radius R about H is diffeomorphic to a disk bundle over H.

Theorem 3.1. *Let (M, H) be a standard congruence pair. Then for all $d \geq 2, R > 0$ there exists a congruence cover $\Pi : M' \to M$ such that for any component $H' \subset M'$ of the preimage of H, the following holds true.*

(1) *The normal injectivity radius of H' is at least R.*
(2) *There exists a finite étale cover $\Pi' : N' \to M'$ and a degree d cyclic cover $N \to N'$, totally branched along $\Pi'^{-1}(H')$.*

Remark 3.2. In other words, if we denote by p the composition $N \to N' \to M'$, we get a tower of finite covers

$$N \xrightarrow{p} M' \xrightarrow{\Pi} M$$

such that

(1) Π is étale.
(2) p is a branched cover which ramifies at order d over the chosen connected component $H' \subset M'$ of $\Pi^{-1}(H)$, i.e. $p^*H' = d\, p^{-1}(H')$ as Cartier divisors. Moreover, p is étale over $M' \setminus H'$. The degree of p may be very large (in a non-explicit way) and $p^{-1}(H')$ may be disconnected.
(3) H' has normal injectivity radius at least R.

The proof of Theorem 3.1 proceeds in two steps. In a first step we use the results of [Be00] to verify that for a given standard congruence pair (M, H) and a number $R > 0$ we can find a congruence cover $M' \to M$ with the property that the normal injectivity radius of one (and hence any) component H' of the preimage of H is larger than R. In a second step we then invoke the main result of [ST22] to complete the proof.

The ideals $\mathcal{I}_L \subset \mathcal{O}_L$ can be equipped with a norm $|\mathcal{I}_L|$ which is just the number of elements in the quotient ring $\mathcal{O}_L/\mathcal{I}_L$. For any $i \geq 1$ there are only finitely many ideals of norm at most i. Each of these ideals can be factorized into prime ideals. We denote by $\hat{\Gamma}_\ell$ the congruence subgroup defined by $\mathcal{O}_L/I_\ell(L)$ where $I_\ell(L)$ is the product of all prime ideals of norm less than ℓ.

This is a normal subgroup of $\hat{\Gamma}$ of finite index, and the groups $\hat{\Gamma}_\ell$ are nested and exhaustive, which means that

$$\forall \ell, \quad \hat{\Gamma}_\ell \rhd \hat{\Gamma}_{\ell+1}, \quad \text{and} \quad \bigcap_\ell \hat{\Gamma}_\ell = \mathrm{Id}. \tag{23}$$

In other words, these groups form a *congruence tower*, and for any congruence subgroup Γ of $\hat{\Gamma}$, there exists some ℓ so that $\hat{\Gamma}_\ell \lhd \Gamma$. The fundamental group of the congruence cover in Theorem 3.1 will be a congruence subgroup from the fixed tower, but many other choices will do as well.

In slight deviation from the previous notations, let H be a compact *connected* embedded totally geodesic hypersurface in the congruence manifold M. The preimage of H in the universal covering $\mathbb{CH}^n$ of M consists of a countable collection $\mathcal{H}$ of pairwise disjoint totally geodesic hyperplanes which are transitively permuted by the deck group $\Gamma = \pi_1(M)$ of M. The following is well known.

Lemma 3.3. *The stabilizer* $\mathrm{Stab}(H_0) := \mathrm{Stab}_\Gamma(H_0)$ *of a component* $H_0 \in \mathcal{H}$ *is a malnormal subgroup of* Γ.

Proof. If $\varphi \in \Gamma \backslash \mathrm{Stab}(H_0)$ then $\varphi(H_0)$ is a component of $\mathcal{H}$ which is stabilized by $\varphi \mathrm{Stab}(H_0)\varphi^{-1}$. As φ does not stabilize H_0, this component is distinct and hence disjoint from H_0. If $\psi \in \mathrm{Stab}(H_0) \cap \varphi\mathrm{Stab}(H_0)\varphi^{-1}$ then ψ also preserves the unique shortest geodesic arc connecting H_0 to $\varphi(H_0)$ and hence it fixes it midpoint. Since Γ acts freely on $\mathbb{CH}^n$, this implies that $\psi = \mathrm{Id}$, from which the lemma is immediate. □

As a consequence, the components of $\mathcal{H}$ are in bijection with the cosets $\Gamma/\mathrm{Stab}(H_0)$.

Each of the groups $\Gamma_i = \hat{\Gamma}_i \cap \Gamma \lhd \Gamma$ acts on $\mathcal{H}$ as a group of permutations, but if $\Gamma_i \neq \Gamma$ then in general, the action is not transitive any more. Indeed, the Γ_i-orbit of a component of $\mathcal{H}$ is a Γ_i-invariant subset of $\mathcal{H}$, and these sets are transitively permuted by the action of the group $\Gamma_i \backslash \Gamma$; the latter being realized as the deck group of the covering $M_i \to M$, where $M_i := \mathbb{CH}^n/\Gamma_i$.

Proposition 3.4. *For the congruence tower* $\Gamma \rhd \Gamma_1 \rhd \Gamma_2 \rhd \cdots$, *the distance in* $\mathbb{CH}^n$ *between distinct hyperplanes in the same* Γ_i*-orbit for the action on* $\mathcal{H}$ *tends to infinity as* $i \to \infty$.

Proof. The stabilizer of H_0 in Γ_i is the congruence subgroup $\mathrm{Stab}(H_0) \cap \Gamma_i$ of $\mathrm{Stab}(H_0)$, in particular it is a normal subgroup of $\mathrm{Stab}(H_0)$. By Lemma 3.3, the intersection $\mathrm{Stab}(H_0) \cap \Gamma_i$ is a malnormal subgroup of Γ_i.

The group Γ acts transitively by conjugation on the conjugates of $\mathrm{Stab}(H_0)$ in Γ, and the orbit space for this action has a natural identification with

$\mathcal{H}$. The group Γ also acts transitively by conjugation on the conjugates of $\mathrm{Stab}(H_0)\cap\Gamma_i$. Since $\Gamma_i\cap\mathrm{Stab}(H_0)$ is a normal subgroup of $\mathrm{Stab}(H_0)$, the stabilizer of $\mathrm{Stab}(H_0)\cap\Gamma_i$ for this action also equals $\mathrm{Stab}(H_0)$. In other words, the orbit space for the action of Γ on the conjugates of $\Gamma_i\cap\mathrm{Stab}(H_0)$ also can be viewed as the space of right cosets $\Gamma/\mathrm{Stab}(H_0)$.

For $\psi\in\Gamma$, the stabilizer of $\psi(H_0)$ in Γ_i is $\Gamma_i\cap\psi\mathrm{Stab}(H_0)\psi^{-1}=\psi(\Gamma_i\cap\mathrm{Stab}(H_0))\psi^{-1}$. As a consequence, the space of Γ_i-orbits for the action of Γ_i on $\mathcal{H}$ can be identified with the double coset space $\Gamma_i\backslash\Gamma/\mathrm{Stab}(H_0)$.

The action of Γ on $\mathbb{CH}^n$ is proper and cocompact, and the same holds true for the action of $\mathrm{Stab}(H_0)$ on H_0. Thus if we denote by d the distance in $\mathbb{CH}^n$, then the following two properties hold. First, given any $H'\in\mathcal{H}$ distinct from H_0, the distance $d(H_0,H')$ is positive, achieved by a geodesic arc such that the starting point is contained in a fixed fundamental domain for the action of $\mathrm{Stab}(H_0)$ on H_0. Second, up to conjugation with elements of $\mathrm{Stab}(H_0)$, for any fixed number $R>0$ there are only finitely many $\varphi\in\Gamma$ so that $d(\varphi(H_0),H_0)\leq R$. Equivalently, there are only finitely many components $H'\in\mathcal{H}$ so that $d(H_0,H')\leq R$ and that the starting point in H_0 of the shortest geodesic arc connecting H_0 to H' is contained in a fixed fundamental domain for the action of $\mathrm{Stab}(H_0)$ on H_0. Any other $\hat{H}\in\mathcal{H}$ so that $d(H_0,\hat{H})\leq R$ is contained in the $\mathrm{Stab}(H_0)$-orbit of these finitely many elements of $\mathcal{H}$.

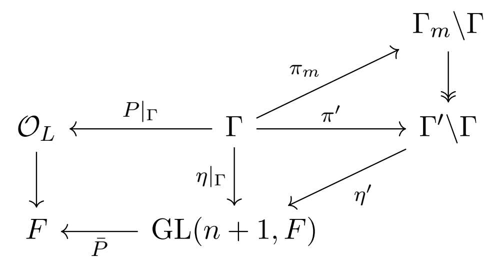
Let $\varphi_1,\dots,\varphi_k$ be representatives of these finitely many $\mathrm{Stab}(H_0)$-conjugacy classes, chosen as representatives of minimal translation length in their class (it will be convenient to set $\varphi_0=\mathrm{Id}$). For each i let $\pi_i:\Gamma\to\Gamma_i\backslash\Gamma$ be the quotient homomorphism. We have to show that

$$\exists m\geq 1;\ \forall\ell=1,\dots,k,\quad \pi_m(\varphi_\ell)\notin\pi_m(\mathrm{Stab}(H_0)). \tag{24}$$

Indeed, what we want to show is that there exists $m\geq 1$ such that one has $d(H_0,\eta H_0)>R$ for any $\eta\in\Gamma_m$ not belonging to $\Gamma_m\cap\mathrm{Stab(H_0)}$. By the definition of the φ_ℓ, this is equivalent to having $\eta(H_0)\neq\varphi_\ell(H_0)$ for all $\ell=1,\dots,k$ and all $\eta\in\Gamma_m$. In order words, $\varphi_\ell\notin\Gamma_m\cdot\mathrm{Stab}(H_0)$, i.e. for any $\gamma\in\mathrm{Stab}(H_0)$, the Γ_m-orbits $\Gamma_m\cdot\varphi_\ell$ and $\Gamma_m\cdot\gamma$ are disjoint. But this is exactly saying that $\pi_m(\varphi_\ell)\notin\pi_m(\mathrm{Stab}(H_0))$.

To see that (24) holds, we follow [Be00] (Lemme principal). First of all, we observe that if $\pi_{m_0}(\varphi_\ell)\notin\pi_{m_0}(\mathrm{Stab}(H_0))$ for *some* ℓ and some $m_0\geq 1$, then it holds for that ℓ and for *any* $m\geq m_0$. Therefore it suffices to check that (24) holds for any given $\ell\in\{1,\dots,k\}$. We fix such an ℓ until the end of the proof. Now, since (the real form of) $\mathrm{GL}(n,\mathbb{C})$ is not Zariski dense in (the real form of) $\mathrm{GL}(n+1,\mathbb{C})$, for each ℓ there exists a polynomial P on $GL(n+1,\mathbb{C})$ vanishing identically on $\mathrm{GL}(n,\mathbb{C})$ and such that $P(\varphi_\ell)\neq 0$. By the explicit form of Γ, we may assume that the coefficients of this polynomial are contained in the ring of integers $\mathcal{O}_L$ of the field L. Namely, using the embedding of $\mathrm{GL}(n, \mathbb{C})$ into $\mathrm{GL}(n+1, \mathbb{C})$ as the group of matrices (a_{ij}) in block form, with $a_{1j} = a_{j1} = 0$ for $j > 1$, a matrix in $\mathrm{GL}(n+1, \mathcal{O}_L) \setminus \mathrm{GL}(n, \mathcal{O}_L)$ has some nontrivial coefficients among the entries $\{a_{1j}, a_{j1}\}$ and hence one can construct such a polynomial explicitly.

By the proof of Lemma 1 in [Be00] or explicit considerations, there exists a prime ideal $\mathcal{J}_\ell \subset \mathcal{O}_L$ such that for the quotient map $\eta : \mathrm{GL}(n+1, \mathcal{O}_L) \to \mathrm{GL}(n+1, \mathcal{O}_L/\mathcal{J}_\ell)$ it holds $\bar{P}(\eta(\varphi_\ell)) \neq 0$ where $\bar{P}$ is the polynomial with coefficients in the field $\mathcal{O}_L/\mathcal{J}_\ell$ obtained by applying the morphism $\mathcal{O}_L \to \mathcal{O}_L/\mathcal{J}_\ell$ to the coefficients of P. Set $F := \mathcal{O}_L/\mathcal{J}_\ell$ and $\Gamma' = \ker(\eta|_\Gamma)$. Then $\Gamma' \lhd \Gamma$ has finite index and by construction there exists some m so that $\Gamma_m \lhd \Gamma'$. Therefore we have the following commutative diagram



and $\eta|_\Gamma$ factors through π' by definition of Γ'. Fix an arbitrary element $\gamma \in \mathrm{Stab}(H_0)$. Since $\mathrm{Stab}(H_0) \subset \mathrm{GL}(n, \mathbb{C})$, we have $P(\gamma) = 0$. Therefore we also have $\bar{P}(\eta(\gamma)) = 0$. Moreover, $\bar{P}(\eta(\varphi_\ell)) \neq 0$ by construction of P and F. This implies that $\eta(\gamma) \neq \eta(\varphi_\ell)$, hence $\pi'(\gamma) \neq \pi'(\varphi_\ell)$ since $\eta|_\Gamma$ factors through π'. In particular, one must have $\pi_m(\gamma) \neq \pi_m(\varphi_\ell)$. Since $\gamma \in \mathrm{Stab}(H_0)$ is arbitrary, (24) follows and the proposition is proved. □

Resuming the notations from Theorem 3.1, we obtain.

Corollary 3.5. *For every $R > 0$ there exists a number $j(R) > 0$ so that for $j > j(R)$ the normal injectivity radius of a component of the preimage of H in $M_j = \Gamma_j \backslash \mathbb{CH}^n$ is at least R.*

Proof. Let $H_j \subset M_j$ be a component of the preimage of H and assume that the normal injectivity radius of H_j in M_j is smaller than ρ for some $\rho > 0$. By definition, there is then a geodesic arc $\gamma \subset M_j$ with endpoints on H_j and of length at most 2ρ which is not homotopic with fixed endpoints into H_j. A lift of γ to the universal covering is a geodesic of length at most 2ρ which connects two distinct lifts of H_j to $\mathbb{CH}^n$. As these lifts are components of $\mathcal{H}$ in the same orbit under the action of the fundamental group of M_j, it follows from Proposition 3.4 that $j \leq j(2\rho)$ for a number $j(2\rho) > 0$ only depending on ρ. This is what we wanted to show. □

To complete the proof of Theorem 3.1, we use the following main result of [ST22] as the key ingredient. It is a concatenation of Corollary 2.15, Theorem 2.16 and Theorem 3.4 in *loc. cit.*

Theorem 3.6 ([ST22]). *Let M be any congruence manifold and let $\eta \in H^{1,1}(M;\mathbb{C})$ be the Poincaré dual of a compact connected embedded totally geodesic hypersurface $H \subset M$. Then*

(1) *There is a congruence cover $\Pi : M' \to M$ so that $\Pi^*\eta$ is contained in the image of the cup product map $\wedge^2 H^1(M';\mathbb{Q}) \to H^2(M';\mathbb{Q})$.*
(2) *For any integer $d \geq 2$, there is a further finite étale cover $\Pi' : M'' \to M'$ such that $(\Pi \circ \Pi')^*\eta$ is divisible by d in $H^2(M'',\mathbb{Z})$. In particular, there is a finite cyclic cover $N \to M''$ of degree d totally ramified over $(\Pi \circ \Pi')^{-1}(H)$.*

Proof of Theorem 3.1. Given a number $R > 0$, Corollary 3.5 shows that there exists a number $m_0 > 0$ so that for any $m \geq m_0$ the normal injectivity radius of any component of the preimage of H under the cover $M_m \to M$ is at least R.

Let H' be a component of the preimage of H in M_{m_0}. By Theorem 3.6, by passing to a further congruence cover $\Pi_1 : M_{m_1} \to M_{m_0}$, one can guarantee the existence of a finite ramified cover (of large, non-explicit degree) $N \to M_{m_1}$ which ramifies at order exactly d along $\Pi_1^{-1}(H')$. This proves Theorem 3.1. □

The manifolds from our main theorem are the branched coverings constructed in Theorem 3.1. That these manifolds are not ball quotients was established in [ST22] (see the proof of Theorem 1.5 of [ST22]).

Theorem 3.7 (Theorem 1.5 of [ST22]). *A covering of a compact ball quotient, branched along a smooth embedded totally geodesic submanifold, is not a quotient of the ball.*

4. Analysis of the Kähler–Einstein cone metric

Let $M = \Gamma\backslash B$ be a compact ball quotient of complex dimension n where Γ is a torsion free arithmetic lattice in $\mathrm{PU}(n,1)$ of simple type. We assume that M contains a smooth totally geodesic embedded subvariety $D \subset M$ of codimension one.

4.1. **Ampleness of the adjoint divisor.** The following observation is well known, see for example [ST22]. As we shall use some more specific information, we provide the proof.

Lemma 4.1. *The normal bundle* $N_D = \mathcal{O}_M(D)|_D$ *satisfies*

$$c_1(N_D) = -\frac{1}{n}c_1(K_D).$$

Proof. On the ball B, the complex hyperbolic metric $-\frac{i}{4}\partial\bar\partial \log(1-|z|^2)$ descends to a Kähler metric ω_B on M which satisfies $\mathrm{Ric}\,\omega_B = -2(n+1)\,\omega_B$. In particular, we have

$$c_1(K_M) = 2(n+1)\,[\omega] \quad \text{in } H^2(M,\mathbb{R}).$$

Now, one can assume without loss of generality that a connected component of the inverse image of D in the universal cover B of M is given by the equation $z_1 = 0$. Performing the same computation as before, one sees that $\mathrm{Ric}\,\omega_B|_D = -n\,\omega_B|_D$. In particular, we have

$$c_1(K_M)|_D = \frac{n+1}{n}\,c_1(K_D). \tag{25}$$

Combining (25) with the adjunction formula $(K_M + D)|_D \simeq K_D$, we get

$$c_1(D)|_D = -\frac{1}{n}\,c_1(K_D). \tag{26}$$

which proves the lemma. □

Remark 4.2. It follows from the lemma above that the normal bundle of D is negative. By Satz 8 in §3 of [Gr62], it is possible to find a surjective holomorphic map $\pi : M \to M^*$ where M^* is a compact normal analytic space, π contracts D to a point and π is an isomorphism when restricted to the complement of D. Then the singularities of M^* are never log canonical. Indeed, assuming that K_{M^*} is $\mathbb{Q}$-Cartier, then we would have $K_M = \pi^* K_{M^*} + bD$ for some $b \in \mathbb{Q}$. Restricting the formula to D and using (25)-(26), we infer that $b = -(n+1) < -1$.

Lemma 4.3. *Let* $a \in [0,\infty)$. *The* $\mathbb{R}$*-line bundle* $K_M + aD$ *is ample if and only if* $a < n+1$.

Proof. Let us first observe that $K_M + aD$ is big for any $a \geq 0$, as a sum of an ample divisor and an effective divisor. Moreover, the non-Kähler locus (or augmented base locus) of $K_M + aD$ is clearly included in D.

Next, (25) and (26) yield

$$(K_M + aD)|_D \equiv \frac{n+1-a}{n}\,K_D$$

and the latter is ample if and only if $a < n+1$. In particular, the same holds for the restriction of $K_M + aD$ to any irreducible component of its non-Kähler locus. The conclusion of the lemma now follows from Theorem 3.17 (iii) in [Bo04]. □

4.2. **Kähler–Einstein cone metrics.** Given a real number $a \in (0,1)$, one says that a Kähler metric ω on $M \setminus D$ has cone singularities with cone angle $2\pi(1-a)$ along D if given any local holomorphic system of coordinates $(z_1,\dots,z_n)$ on an open set $U \subset M$ such that $D \cap U = \{z_1 = 0\}$, the Kähler metric $\omega|_{U\setminus D}$ is quasi-isometric to the model metric

$$\omega_a := \frac{idz_1 \wedge d\bar z_1}{|z_1|^{2a}} + \sum_{j=2}^n idz_j \wedge d\bar z_j.$$

That is, there exists $C > 0$ such that we have

$$C^{-1}\omega_a \le \omega \le C\omega_a \quad \text{on} \ \ U \setminus D.$$

Let us now fix $a \in (0,1)$. Since $K_M + aD$ is ample by Lemma 4.3, it follows from [GP16] (see also [Br13, JMR16]) that there exists a unique Kähler metric $\omega_{\mathrm{KE},a}$ on $M \setminus D$ such that

- $\mathrm{Ric}\,\omega_{\mathrm{KE},a} = -2(n+1)\,\omega_{\mathrm{KE},a}$,
- $\omega_{\mathrm{KE},a}$ has cone singularities with cone angle $2\pi(1-a)$ along D.

Moreover, when a is of the form $a = 1 - \frac{1}{m}$ for some integer $m \ge 2$, then it is well-known that $\omega_{\mathrm{KE},a}$ is an orbifold Kähler metric, see e.g. [Fa19]. What we mean is the following. Given a local chart $U \simeq \Delta^n$ as before, consider the branched cover $p : \Delta^n \to U$ given by $p(z_1,\dots,z_n) = (z_1^m, z_2,\dots,z_n)$. Then $p^*(\omega_{\mathrm{KE},a}|_{U\setminus D})$ extends to a smooth Kähler metric on the whole Δ^n.

4.3. **Cut-off functions.** Let us first work on the ball $B \subset \mathbb{C}^n$ endowed with its Bergman metric ω_B and consider the lower dimensional ball $B_0 := B \cap \{z_1 = 0\}$. We can normalize ω_B such that

$$\omega_B = -\frac{i}{4}\partial\bar\partial \log(1-|z|^2) = \frac{1}{4(1-|z|^2)^2}\sum_{1\le j,k\le n}\big((1-|z|^2)\delta_{jk}+\bar z_j z_k\big) idz_j \wedge d\bar z_k.$$

With this normalization, ω_B has constant holomorphic holomorphic sectional (resp. bisectional) curvature -4 (resp. -2), its sectional curvatures lie in $[-4,-1]$ and we have

$$\mathrm{Ric}\,\omega_B := -\frac{i}{2}\partial\bar\partial \log\det \omega_B^n = -2(n+1)\omega_B.$$

Given a point $p \in B$ with coordinates $(z_1,\dots,z_n)$, we let $d(p, B_0)$ be the geodesic distance to B_0 with respect to g_B.

Fact: For $p = (z_1, w) \in B$, we have $\cosh^2(d(p,B_0)) = \frac{1-|w|^2}{1-|w|^2-|z_1|^2}$.

For a justification, observe that the function $p \mapsto d(p, B_0)$ is invariant under $S^1 \times \mathrm{PU}(n-1,1)$. Hence there is no loss of generality assuming that $p = (x_1, x_2, 0, \dots, 0)$ with $x_i \in [0,1)$, $x_1 = |z_1|$ and $x_2 = |w|$. Since $\{z_i = 0 \text{ for } i \geq 3\}$ is totally geodesic in B, and the same holds true for the totally real plane $\{(r,s) \mid r, s \in \mathbb{R}, r^2 + s^2 < 1\} \subset B \cap \mathbb{C}^2$, it then suffices to compute the distance between $(x_1, x_2) \in B \cap \mathbb{C}^2$ and a point $(0, r) \in B \cap \mathbb{C}^2$ with $r \in [0,1)$. The computation on p.15 of [Pa03] shows that

$$\cosh^2\Big(d(p,(0,r))\Big) = \frac{(1 - r x_2)^2}{(1 - x_1^2 - x_2^2)(1 - r^2)}$$

which is minimized at $r = x_2$. Note that [Pa03] uses a curvature normalization which is different from ours. This is the desired formula.

Set $u := \frac{1-|w|^2}{1-|w|^2-|z_1|^2} = 1 + \frac{|z_1|^2}{1-|z|^2}$ so that $d(\cdot, B_0) = \log(\sqrt{u} + \sqrt{u-1})$ satisfies $\frac{1}{2}\log u \leq d(\cdot, B_0) \leq \frac{1}{2}\log(4u)$. It is easy to check that $\log u$ is smooth and it has uniformly bounded covariant derivatives at any order. Next, let $\xi : \mathbb{R} \to [0,1]$ be a smooth, non-increasing function such that $\xi \equiv 1$ on $(-\infty, 1]$ and $\xi \equiv 0$ on $[\frac{3}{2}, +\infty)$. For any $R > 1$, we set

$$\widetilde{\chi}_R := \xi\Big(\frac{2\log u}{R}\Big) = \xi\left(\frac{2}{R}\log\left(1 + \frac{|z_1|^2}{1-|z|^2}\right)\right). \tag{27}$$

The latter function is smooth and satisfies $\widetilde{\chi}_R \equiv 1$ on $\big\{d(\cdot, B_0) \leq \frac{R}{4}\big\}$ and $\widetilde{\chi}_R \equiv 0$ on $\big\{d(\cdot, B_0) \geq \frac{R}{2}\big\}$ as long as $R \geq 6$, thanks to the above fact. Moreover, by the chain rule we see that for any integer $k \geq 0$, there exists a universal constant $C = C(k) > 0$ independent of R satisfying

$$|\nabla^k \widetilde{\chi}_R|_{\omega_B} \leq \frac{C(k)}{R^k}. \tag{28}$$

Finally, we see that by construction, the function $\widetilde{\chi}_R$ on B is invariant under $S^1 \times \mathrm{PU}(n-1,1)$.

Let us now go back to our compact ball quotient $M = \Gamma \backslash B$ with its embedded totally geodesic smooth connected hypersurface $D \subset M$. From now on, we fix a number $d \geq 2$. By Theorem 3.1 (see also Remark 3.2), there exists a tower of finite covers

$$\begin{array}{ccc} & & M_{d,R} \\ & & \big\downarrow{\scriptstyle p_{d,R}} \\ M & \xleftarrow[\Pi_R]{} & M_R \end{array}$$

where Π_R is étale, $p_{d,R}$ ramifies at order d along a fixed connected component D_R of $\Pi_R^{-1}(D)$ and is étale elsewhere and $D_R \subset M_R$ has normal injectivity radius at least R. In terms of canonical bundles, we have

$$K_{M_{d,R}} = p_{d,R}^* \left(K_{M_R} + (1 - \frac{1}{d}) D_R \right).$$

Remark 4.4. It is important to keep in mind that as R grows, we have no control on the growth of $\deg(p_{d,R})$ so that $p_{d,R}^{-1}(D_R)$ will have a large number of connected components. In what follows, we will perform the analysis directly on M_R and only rely on the existence of $M_{d,R}$ in a qualitative way to desingularize the Kähler–Einstein metric associated to the pair $(M_R, (1 - \frac{1}{d})D_R)$.

Let us now reformulate the defining property of M_R. We fix one connected component V of the preimage of the connected smooth divisor $D_R \simeq D$ in the universal cover B of M_R and we let $\Gamma_0 := \mathrm{Stab}_\Gamma(V)$ be the stabilizer of V in Γ_R. We have $\Gamma_0 \backslash V = D$. Without loss of generality, one can assume that $V = B_0 = (z_1 = 0)$. Since the collar size of D_R in M_R is at least R, the tubular neighborhood of radius R about D_R equals the projection of the tubular neighborhood of radius R about B_0 by the action of Γ_0. Thus there is a holomorphic, isometric embedding

$$j_R : \{x \in M_R; d(x, D_R) < R\} \longrightarrow \Gamma_0 \backslash \{p \in B, d(p, B_0) < R\}$$

with respect to the complex hyperbolic metric. As explained above, the cut-off function $\widetilde{\chi}_R$ defined in (27) is invariant under the stabilizer of B_0 hence it makes sense to define

$$\chi_R := j_R^* \widetilde{\chi}_R.$$

4.4. **The glued metric.** Recall from Section 2.3 that the domain

$$\Omega_d := \{|z_1|^{2d} + \sum_{i=2}^n |z_i|^2 < 1\}$$

has a Kähler–Einstein metric with Ricci constant $-2(n+1)$ which is invariant under $G = \mathrm{Aut}(\Omega_d)$ and therefore descends to a Kähler–Einstein metric on B with *cone singularities* of angle $2\pi(1 - \frac{1}{d})$ along B_0, invariant under Γ_0. We denote by ω_d the induced metric on $\Gamma_0 \backslash B$; by abuse of notation we will also denote by ω_d its pull back to $\{x \in M_R; d(x, D_R) < R\}$ via j_R.

Let us define $U_R := \{x \in M_R; d(x, D_R) < R\} \subset M_R$ on which the function χ_R is well-defined and compactly supported. We introduce the smooth function $F = \frac{1}{2(n+1)} \log \frac{\omega_B^n}{\omega_d^n}$ on $U_R \setminus D_R$ and set

$$\omega_R := \omega_d + dd^c\Big((1 - \chi_R)F\Big) = \omega_B - dd^c(\chi_R F).$$

This current is a priori only defined on U_R. However, since ω_B and ω_d are both Kähler–Einstein metrics with the same Einstein constant on $U_R \setminus D_R$, an elementary computation shows that ω_R coincides with ω_B on $U_R \setminus U_{\frac{R}{2}} \subset \{\chi_R = 0\}$ and hence we can extend ω_R to the whole M_R by setting $\omega_R := \omega_B$ on $M_R \setminus U_R$. It is not difficult to show that $\omega_R \in \frac{1}{2(n+1)} c_1(K_{M_R} + (1-\frac{1}{d})D_R)$. Moreover, ω_R coincides with ω_d on $U_{\frac{R}{4}}$. It remains to analyze the behavior of ω_R on the gluing region $U_{\frac{R}{2}} \setminus U_{\frac{R}{4}}$.

In what follows, we will denote by $C(k)$ a constant that depends on a given integer $k \in \mathbb{N}$ (and implictly on n and d) but not on the parameter R. The actual value of $C(k)$ may change from line to line but it subject to the constraints recalled above. From Theorem 2.13, there exists for any integer $k \geq 0$ a positive number $a = a(d,k)$ such that we have the following decay

$$|\nabla^k(\omega_d - \omega_B)|_{\omega_B} \leq C(k)e^{-aR} \quad \text{on } U_{\frac{R}{2}} \setminus U_{\frac{R}{4}}.$$

In particular, the covariant derivatives of F decay in $O(e^{-aR})$. Since the covariant derivatives of χ_R are bounded (actually they decay polynomially in R, cf (28)) it follows that

$$|\nabla^k(dd^c((1-\chi_R)F))|_{\omega_B} \leq C(k)e^{-aR} \quad \text{on } U_{\frac{R}{2}} \setminus U_{\frac{R}{4}}.$$

Putting everything together, one obtains the following identity

$$|\nabla^k(\omega_R - \omega_B)|_{\omega_B} \leq C(k)e^{-aR} \quad \text{on } M_R \setminus U_{\frac{R}{4}}. \tag{29}$$

In particular, it follows from the third item of Theorem 2.11 that for R large enough, the sectional curvature of ω_R bounded above by a negative independent of R. More precisely, (29) and (17) imply that for R large enough (depending on d), we have

$$\sup_{x \in M_R \setminus D_R} \sup_{\substack{P \subset T_x M_R \\ \text{plane}}} K_{g_R}(P) = -(n+1) + \frac{n}{f_d(0)^2} \tag{30}$$

since the sectional curvatures of ω_B lie in $[-4,-1]$ and $-(n+1) + \frac{n}{f_d(0)^2} \in (-1,0)$

Let us now analyze the Ricci potential of ω_R. From the definition of ω_R, it is straightforward to deduce that

$$\operatorname{Ric}\omega_R + 2(n+1)\omega_R = 2(n+1)dd^c h_R + \big(1 - \frac{1}{d}\big)[D_R]$$

on M_R where

$$h_R := -\frac{1}{2(n+1)} \log \frac{\omega_R^n}{\omega_B^n} - \chi_R F$$

is smooth function on M_R satisfying

$$h_R \equiv 0 \quad \text{on} \quad (M_R \setminus U_{\frac{R}{2}}) \cup U_{\frac{R}{4}}$$

as well as

$$|\nabla^k h_R|_{\omega_R} \leq C(k)e^{-aR} \quad \text{on } U_{\frac{R}{2}} \setminus U_{\frac{R}{4}}. \tag{31}$$

In particular, we have

$$|\mathrm{Ric}\,\omega_R + 2(n+1)\omega_R|_{\omega_R} = O(e^{-aR}).$$

4.5. **Curvature of the Kähler–Einstein cone metric on M_R.** Thanks to Section 4.2, there exists a unique Kähler–Einstein metric $\widehat{\omega}_R$ on M_R with cone angle $2\pi(1-\frac{1}{d})$ along D_R and Einstein constant $-2(n+1)$. Note that since we only picked one component D_R of $\Pi_R^{-1}(D)$, the metric $\widehat{\omega}_R$ is *not* the pullback by Π_R of the Kähler–Einstein metric for the pair $(M,(1-\frac{1}{d})D)$. The forms ω_R and $\widehat{\omega}_R$ are orbifold Kähler metrics, that is, they are genuine Kähler metrics on $M_R \setminus D_R$, and their pullbacks by $\Phi_d : \Omega_d \to B$ (after first pulling back to the universal cover $\widetilde{U}_R \subset B$) is smooth. Equivalently, both pullbacks

$$\widehat{\omega}_{d,R} := p_{d,R}^*\widehat{\omega}_R \quad \text{and} \quad \omega_{d,R} := p_{d,R}^*\omega_R \tag{32}$$

are genuine Kähler metrics on $M_{d,R}$. Since ω_R and $\widehat{\omega}_R$ both belong to the cohomology class $\frac{1}{2(n+1)}c_1(K_{M_R} + \big(1-\frac{1}{d}\big)D_R)$, one can uniquely write $\widehat{\omega}_R = \omega_R + dd^c\varphi_R$ where φ_R solves the Monge-Ampère equation

$$(\omega_R + dd^c\varphi_R)^n = e^{2(n+1)(\varphi_R+h_R)}\omega_R^n. \tag{33}$$

Let us now derive some uniform estimates (as R varies) on $\widehat{\omega}_R$ and φ_R. First, since the holomorphic bisectional curvature of ω_R is bounded above by a negative constant independent of R, Theorem 2 of [Yau78a] shows that

$$\widehat{\omega}_R \geq C^{-1}\omega_R.$$

Here and in what follows, C is a positive constant independent of R which may vary from line to line. Next, since ω_R and $\widehat{\omega}_R$ have Ricci curvature bounded below (say by $-2(n+2)$) we can apply Theorem 3 of [Yau78a] to conclude that the volume elements of both metrics are uniformly comparable. Given the above estimate, this implies that one has an estimate of the form

$$C\omega_R \geq \widehat{\omega}_R \geq C^{-1}\omega_R. \tag{34}$$

Consider the orbifold smooth function $\varphi_R + h_R$ from the identity (33). At a point x_R where it attains its maximum, its Hessian is nonpositive hence $\widehat{\omega}_R(x_R) \leq \omega_R(x_R) - dd^c h_R(x_R)$. To be precise, one works in the branched cover $M_{d,R}$ where the objects become smooth, and then one can descend the estimates which *do not depend* on the cover $p_{d,R}$. Since $|dd^c h_R|_{\omega_R} = O(e^{-aR})$, we infer from the Monge-Ampère equation satisfied by φ_R that $(\varphi_R + h_R)(x_R) \leq Ce^{-aR}$ hence the same holds on the whole M_R. One can similarly use the minimum principle to see that $\varphi_R + h_R \geq -Ce^{-aR}$. By (31), we obtain

$$\sup_{M_R} |\varphi_R| \leq Ce^{-aR}. \tag{35}$$

The remaining task is to improve this C^0 decay to order four decay on φ_R which will guarantee that the curvature of $\widehat{\omega}_R$ is close to that of ω_R, hence it is negative too. For $k \in \mathbb{N}$, and f a smooth orbifold function on M_R, we set $\|f\|_{C^k(M_R)} := \sup_{M_R} \sum_{j=0}^{k} |\nabla^k f|_{\omega_R}$. We will show that $\|\varphi_R\|_{C^5(M_R)}$ gets arbitrarily small if R is chosen large enough. It is convenient to assume that R is integer valued. Let $x_R \in M_R$ be such that $\|\varphi_R\|_{C^5(M_R)} = \sum_{j=0}^{5} |\nabla^5 \varphi_R(x_R)|_{\omega_R}$. Up to extracting subsequences, we only have to consider the following two possibilities.

Case 1. $\limsup_{R\to+\infty} d(x_R, D_R) < +\infty$.

Let us choose a constant $L > 0$ such that $d_{\omega_R}(x_R, D_R) \leq L$. Using j_R, one can embed $\{d_{\omega_R}(\cdot; D_R) \leq 3L\}$ in $\Gamma_0\backslash B$ for R large enough. Let σ_R be the composition $\Omega_d \overset{\Phi_d}{\to} B \to \Gamma_0\backslash B$. It satisfies $\sigma_R^*\omega_R = \omega_d$. Given the structure of the automorphism group G of the pair $(\Omega_d, (z_1 = 0))$ one can find a point $p_R \in \Omega_d$ such that $d_{\omega_d}(p_R, 0) \leq L$ and $\sigma_R(p_R) = x_R$.

From now on, we work on $B_{\omega_d}(0, 3L) \subset \Omega_d$ and define $\widetilde{\varphi}_R := \sigma_R^*\varphi_R$. We can pull back the Monge-Ampère equation (33) there. Since we have the Laplacian estimate (34), one can appeal to Evans-Krylov theorem and Schauder estimates to get uniform estimates for the C^6 norm of $\widetilde{\varphi}_R$ on $B_{\omega_d}(0, 2L)$ with respect to ω_d. In particular, up to extracting again, we can assume that $\widetilde{\varphi}_R$ converges in C^5 on a slightly smaller ball as $R \to +\infty$. By uniqueness of the limit, we see from (35) that $\widetilde{\varphi}_R$ converges to zero in C^5 on that set. Given the choice of x_R and since $p_R \in \bar{B}_{\omega_d}(0, L)$ it follows that

$$\|\varphi_R\|_{C^5(M_R)} = \sum_{j=0}^{5} |\nabla^j \widetilde{\varphi}_R(p_R)|_{\omega_d} \underset{R\to+\infty}{\longrightarrow} 0.$$

Case 2. $\liminf_{R\to+\infty} d(x_R, D_R) = +\infty$.

For every integer $k \geq 0$, we have

$$\sup_{B_{\omega_R}(x_R,1)} |\nabla^k(\omega_R - \omega_B)|_{\omega_B} \underset{R\to+\infty}{\longrightarrow} 0$$

thanks to (29) and the fourth item in Theorem 2.11. Now we pull back our objects to the universal cover $\pi_R : B \to M_R$. Let $p_R \in B$ such that $\pi_R(p_R) = x_R$. By transitivity of the automorphism group of (B, ω_B), we can find $\mu_R \in \mathrm{Aut}(B, \omega_B)$ such that $\mu_R(0) = p_R$. Let us now consider $\sigma_R := \pi_R \circ \mu_R$ and $\widetilde{\varphi}_R := \sigma_R^*(\varphi_R|_{B(x_R,1)})$. We have $\sigma_R(0) = x_R$ and

$\sup_{B_{\omega_B}(0,1)} |\nabla^k(\sigma_R^*\omega_R - \omega_B)| \to 0$ for any integer $k \geq 0$. Similarly to the previous step, we can pull back the Monge-Ampère equation (33) by σ_R. Since we have the Laplacian estimate (34), one can appeal to Evans-Krylov theorem and Schauder estimates to get uniform C^6 estimates for $\widetilde{\varphi}_R$ on $B_{\omega_B}(0, \frac{3}{4})$ with respect to ω_B. Up to extracting again, we can assume that $\widetilde{\varphi}_R$ converges in C^5 on $B_{\omega_B}(0, \frac{1}{2})$ as $R \to +\infty$. By uniqueness of the limit, we see from (35) that $\widetilde{\varphi}_R$ converges to zero in C^5 on that set. It follows that

$$\|\varphi_R\|_{C^5(M_R)} \leq 2\sum_{j=0}^{5} |\nabla^j \widetilde{\varphi}_R(0)|_{\omega_B} \underset{R\to+\infty}{\longrightarrow} 0.$$

In conclusion, we have showed that

$$\limsup_{R\to+\infty} \|\varphi_R\|_{C^5(M_R)} = 0,$$

hence

$$\lim_{R\to+\infty} \sup_{M_{d,R}} \sum_{j=0}^{3} |\nabla^j(\widehat{\omega}_{d,R} - \omega_{d,R})|_{\omega_{p,R}} = 0, \tag{36}$$

where $\widehat{\omega}_{d,R}$ and $\omega_{d,R}$ are defined in (32).

Proof of the main theorem. We can now complete the proof of the theorem announced in the introduction.

The forms $\widehat{\omega}_{d,R}$ and $\omega_{d,R}$ are genuine Kähler metrics on $M_{d,R}$ which are asymptotically close in the sense of (36) as $R \to +\infty$. Since the sectional curvature of the Kähler metric $\omega_{d,R}$ on $M_{d,R}$ belongs to some interval $[-b^2, -a^2]$ for some numbers $0 < a < b$ independent of R by Theorem 2.11, it follows that the sectional curvature of the Kähler–Einstein metric $\widehat{\omega}_{d,R}$ satisfies the same property as long as R is chosen large enough. This proves the theorem. □

Infinite family of examples. One can say more, as claimed in the lines below the theorem in the introduction.

Set $k_d := (n+1) - \frac{n}{f_d(0)^2}$ and $\varepsilon_d := \frac{1}{2}(k_d - k_{d+1})$ which is positive and goes to 0 as $d \to +\infty$. Given (30), one can for any fixed d choose $R = R(d, \varepsilon_d)$ large enough so that

$$\left| \sup_{M_{d,R}} K_{\widehat{g}_{d,R}} - k_d \right| \leq \varepsilon_d.$$

It follows that the quantity

$$\sup_{M_{d,R_d}} K_{\widehat{g}_{d,R_d}}$$

is strictly increasing with d. In particular, given two integers $d, d' \geq 2$, the universal covers of $(M_{d,R_d}, \widehat{\omega}_{d,R_d})$ and $(M_{d',R_{d'}}, \widehat{\omega}_{d',R_{d'}})$ are not isometric unless $d = d'$. By uniqueness of the complete Kähler–Einstein metric on $\widetilde{M}_{d,R_d}$, this implies that $\widetilde{M}_{d,R_d}$ and $\widetilde{M}_{d',R_{d'}}$ are not biholomorphic when $d \neq d'$.

Very strong negativity. Let us recall the notion of very strong negativity introduced by Siu [S80]. Let (M, ω) be a Kähler manifold written locally $\omega = \frac{i}{2}\sum_{i,j} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$. The curvature tensor is given by $R_{i\bar{j}k\bar{\ell}} = -g_{i\bar{j},k\bar{\ell}} + g^{s\bar{t}} g_{s\bar{j},k} g_{i\bar{t},\bar{\ell}}$. We say that the curvature tensor of (M, ω) is very strongly negative if

$$\sum_{i,j,k,\ell} R_{i\bar{j}k\bar{\ell}} \xi^{i\bar{j}} \overline{\xi^{\ell\bar{k}}}$$

is negative for arbitrary complex numbers $\xi^{i\bar{j}}$ such that $\xi^{i\bar{j}} \neq 0$ for at least one pair of indices (i, j). If M is compact, it is equivalent to the existence of $c > 0$ such that $\sum_{i,j,k,\ell} R_{i\bar{j}k\bar{\ell}} \xi^{i\bar{j}} \overline{\xi^{\ell\bar{k}}} \leq -c|\xi|^2_\omega$ for any local holomorphic section ξ of $T_M \otimes T_M$.

Because of the twist of indices in the above negativity condition, the curvature tensor of (M, ω) is negative if and only if the holomorphic cotangent bundle Ω_M equipped with the hermitian metric induced by ω is Nakano positive. Using an other terminology, it can be rephrased by saying that the holomorphic tangent bundle T_M is dual Nakano negative with respect to the hermitian metric induced by ω.

Let $\alpha = \frac{i}{2}\sum_{i,j} h_{i\bar{j}} dz_i \wedge d\bar{z}_j$ be a real $(1,1)$-form and let $H_{i\bar{j}k\bar{\ell}} = -(h_{i\bar{j}} h_{k\bar{\ell}} + h_{i\bar{\ell}} h_{k\bar{j}})$ be the $(0,4)$ tensor induced by α (or h). If α is positive (resp. semipositive), then H is very strongly negative (resp. strongly seminegative). Indeed, one can assume that $h_{i\bar{j}} = \lambda_i \delta_{i\bar{j}}$ for some $\lambda_i > 0$ (resp. $\lambda_i \geq 0$) and then $-\sum_{i,j,k,\ell} H_{i\bar{j}k\bar{\ell}} \xi^{i\bar{j}} \overline{\xi^{\ell\bar{k}}} = |\sum_i \lambda_i \xi^{i\bar{i}}|^2 + \sum_{i,j} \lambda_i \lambda_j |\xi^{i\bar{j}}|^2$. This applies to the curvature tensor of the ball endowed with the Bergman metric and shows that the latter has very strongly negative curvature tensor. Similarly, if f is a real function, then the tensor $-f_i f_{\bar{j}} f_k f_{\bar{\ell}}$ is very strongly seminegative since $f_i f_{\bar{j}} f_k f_{\bar{\ell}} \xi^{i\bar{j}} \overline{\xi^{\ell\bar{k}}} = |\sum_{i,j} f_i f_{\bar{j}} \xi^{i\bar{j}}|^2$.

This discussion applies to the curvature of the Kähler-Einstein metric ω_α on Ω_α as it was showed in Theorem 2 of [Bl86] that its curvature tensor $R_{i\bar{j}k\bar{\ell}}$ can be decomposed as a sum of terms

$$R_{i\bar{j}k\bar{\ell}} = -A(g_{i\bar{j}} g_{k\bar{\ell}} + g_{i\bar{\ell}} g_{k\bar{j}}) - B(\psi_{i\bar{j}} \psi_{k\bar{\ell}} + \psi_{i\bar{\ell}} \psi_{k\bar{j}}) - C\tau_i \tau_{\bar{j}} \tau_k \tau_{\bar{\ell}}$$

where A, B, C are semipositive functions such that $A \geq \frac{2}{n\alpha+1}$, $\psi := \log|z_1|^2 - \frac{1}{\alpha}\log(1 - |z'|^2)$ is plurisubharmonic and $\tau = e^\psi$.

Since ω_B and ω_d have very strongly negative curvature tensor, it follows from (36) that the Kähler-Einstein metric $\widehat{\omega}_{d,R}$ shares the same property as long as R is chosen large enough.

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