THE GEOMETRY OF BRANCHED COVES OF HYPERBOLIC MANIFOLDS

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Abstract. We discuss covers of closed hyperbolic manifolds branched along a totally geodesic codimension two submanifold Σ and their geometries. We survey what is known about the existence of Einstein metrics on such manifolds. In dimension $n \geq 4$, we show for suitable branch loci Σ, at most one of these branched coverings admits a hyperbolic metric.

1. Introduction

As a consequence of the solution to the geometrization conjecture by Perelmann, any closed manifold of dimension three which admits a negatively curved metric also admits a hyperbolic metric, that is, a metric of constant curvature -1. The existence of hyperbolic metrics on closed surfaces of negative Euler characteristic is a classical consequence of the uniformization theorem. An analogous property is not true any more for closed negatively curved manifolds of dimension at least four.

Gromov and Thurston [GT87] considered cyclic coverings of arithmetic hyperbolic manifolds of simplest type, branched along a null-homologous totally geodesic submanifold of codimension two. They showed that suitable choices of such manifolds, called *Gromov Thurston manifolds* in the sequel, admit metrics whose curvature is arbitrarily close to -1 but which do not admit a hyperbolic metric. The proof of non-existence of a hyperbolic metric on such manifolds is however indirect, that is, it it shown that among an infinite collection of candidate manifolds with pinched curvature, at most finitely many admit hyperbolic metrics.

This leads to the question of the existence of distinguished metrics on Gromov Thurston manifolds. A class of metrics which generalizes constant curvature metrics on 3-manifolds are *Einstein metrics* h, characterized by the property that their Ricci curvature Ric_h satisfies $\mathrm{Ric}_h = ch$ for a constant $c \in \mathbb{R}$.

Part of the following conjecture was phrased as a question in [GT87].

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Conjecture. A nontrivial covering of a closed hyperbolic manifold M of dimension $n \geq 4$, branched along a closed totally geodesic submanifold, admits a unique Einstein metric up to scale, and this metric is not of constant curvature

Progress towards this conjecture is due to Fine and Premoselli [FP20] who constructed examples of negatively curved Einstein metrics on some four-dimensional Gromov Thurston manifolds which are not of constant curvature. That in dimension four such Einstein metrics are unique up to scaling follows from the work of Besson, Courtois and Gallot (see [And10] for an explicit statement). The results in [FP20] were extended in [HJ24a] as follows.

main1 **Theorem 1** (Hamenstädt-Jäckel [HJ24a]). *For any $n \geq 4$, there are Gromov Thurston manifolds of dimension n which admit a negatively curved Einstein metric but no hyperbolic metric.*

The article [GH25] contains an analogous result in the Kähler case.

The fact that the examples of [HJ24a] do not admit a hyperbolic metric used some additional property of the branched covers considered. Namely, the branch locus of the base hyperbolic manifold is contained in a closed totally geodesic hyperplane H which is contained in the fixed point set of an isometric involution on M and is null-homologous in H. We call such a submanifold of y closed hyperbolic manifold M *special* in the sequel. In [HJ24a], it was shown that among the covers of degree a multiple of four, branched along a special submanifold, at most one can admit a hyperbolic metric. The following is an improvement of this result following the arguments in [HJ24a]. For the formulation, denote by $\|M\|$ the *simplicial volume* of the manifold M, defined as the infinimum of the quantity $\sum_i |a_i|$ where $\sum_i a_i\sigma_i$ is a simplicial cycle representing the fundamental class of M. It is known that for hyperbolic manifolds, the simplicical volume equals a constant multiple of the volume for a constant only depending on the dimension. We refer to [Fr17] for more information.

mainthm **Theorem 2.** *Let M be an n-dimensional closed hyperbolic manifold and let $\Sigma \subset M$ be a closed totally geodesic embedded null-homologous submanifold of codimension two. For $d \geq 2$ let M_d be the cyclic covering of M of degree d, branched along Σ.*

(1) If $n = 3$ then M_d admits a hyperbolic metric, and the non-compact manifold $M \smallsetminus \Sigma$ admits a complete hyperbolic metric of finite volume.
(2) If $n \geq 4$ and if Σ is special, then M_d admits a hyperbolic metric for at most one degree d.
(3) For all n, it holds $\|M_d\| > d\|M\|$.

The organization of this article is as follows. In Section 2 we give a proof of the following statement which is well known to the experts but hard to find in the literature: Given a closed oriented n-manifold M, a closed oriented embedded codimension two submanifold $\Sigma \subset M$ and a number $d \geq 2$, there exists a degree d cyclic covering of M, branched along Σ, if and only if the mod d fundamental class of Σ is trivial in $H_{n-2}(M, \mathbb{Z}/d\mathbb{Z})$.

In Section 3 we give a short introduction to hyperbolic cone metrics and their relation to Theorem 2. This is used in Section 4 to study branched covers of hyperbolic 3-manifolds and review some results from [HJ22]. The proof of Theorem 2 is contained in Section 5.

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2. Branched covers

The goal of this section is to introduce cyclic coverings of smooth closed oriented manifolds M branched along a smooth closed oriented embedded submanifold Σ of codimension two.

Definition 2.1. A *cyclic d-fold covering of M branched along* Σ is a manifold M_d which admits a degree d map $\Pi : M_d \to M$ with the following properties.

(1) The restriction of Π to $\Pi^{-1}(M \setminus \Sigma)$ is a degree d regular covering onto $M \setminus \Sigma$, with deck group $\mathbb{Z}/d\mathbb{Z}$.
(2) $\Pi|\Pi^{-1}(\Sigma)$ is a homeomorphism.

We refer to [F57] for more information on more general branched coverings of simplicial complexes.

Let $\nu \to \Sigma$ be the normal bundle of Σ, which is an oriented two-dimensional vector bundle over Σ. The orientation of ν can be used to equip ν with the structure of a complex line bundle. Denote by $e(\nu) \in H^2(\Sigma, \mathbb{Z})$ the Euler class of ν. By Proposition 6.41 of [BT82], this is the pull-back of the *Thom class* of ν by the zero section. Proposition 6.18 of [BT82] shows that the Thom class of ν is the unique class in the second compactly supported cohomology group $H^2_c(\nu, \mathbb{Z})$ which restricts to the generator of the compactly supported second cohomology of the fiber. This characterizing property is all we shall use.

The complex line bundle $\nu \to \Sigma$ is defined by a classifying map $\Sigma \to \mathbb{C}P^\infty$, so that ν is the pull-back of the tautological bundle $\tau \to \mathbb{C}P^\infty$ under this map. Since the odd dimensional homology of $\mathbb{C}P^\infty$ vanishes, the universal coefficient theorem shows that any class $a \in H^2(\mathbb{C}P^\infty, \mathbb{Z})$ can be identified with a homomorphism $H_2(\mathbb{C}P^\infty, \mathbb{Z}) \to \mathbb{Z}$ and hence for every $d \geq 2$, the mod d reduction of a is the class in $H^2(\mathbb{C}P^\infty, \mathbb{Z}/d\mathbb{Z})$ which arises from composing this homomorphism with the quotient map $\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$. The mod d reduction of the Euler class of ν then is the class $e(\nu)_d \in H^2(\Sigma, \mathbb{Z}/d\mathbb{Z})$ which is the pull-back of the resulting class in $H^2(\mathbb{C}P^\infty, \mathbb{Z}/d\mathbb{Z})$ by the classifying map. Similarly, for every integer $d \geq 2$ the mod d fundamental class of Σ is the generator of the group $H_{n-2}(\Sigma, \mathbb{Z}/d\mathbb{Z})$. Its image under the inclusion $\Sigma \to M$ is a class in $H_{n-2}(M, \mathbb{Z}/d\mathbb{Z})$.

A cyclic cover of M branched along Σ restricts to a cyclic branched cover $\Pi|\Pi^{-1}(U) \to U$ of a tubular neighborhood U of Σ in M, and U can be chosen to be diffeomorphic to the total space of the normal bundle ν. With this identification and up to homotopy, this cyclic cover restricts to a cyclic unbranched cover of the complement of zero in each fiber.

euler **Lemma 2.2.** *There is a d-fold cyclic cover of U branched along Σ if and only if the mod d reduction of the Euler class $e(\nu)_d \in H^2(\Sigma, \mathbb{Z}/d\mathbb{Z})$ vanishes.*

Proof. Let $\nu_0 \subset \nu$ be the complement of the zero section of ν. Let R be a ring with unit and donsider the exact cohomology sequence of the pair (ν, ν_0) given by

{exact} (2.1) $$\cdots \to H^1(\nu_0; R) \to H^2(\nu, \nu_0; R) \to H^2(\nu; R) \to H^2(\nu_0; R) \to \cdots$$

The Thom class $\tau(\nu)$ of ν is a distinguished element of $H^2(\nu, \nu_0; \mathbb{Z})$ which for each $x \in \Sigma$ restricts to the generator of $H^2(\nu_x, \nu_x \smallsetminus 0; \mathbb{Z})$ where ν_x is the oriented fiber of ν over x (Theorem 10.4 of [MS74]). Using the Gysin sequence Theorem 12.2 of [MS74] or Proposition 6.41 of [BT82], the image of $\tau(\nu)$ under the restriction map $H^2(\nu; \nu_0; \mathbb{Z}) \to H^2(\nu, \mathbb{Z})$ is the Euler class $e(\nu)$ of ν via the homotopy equivalence of the total space of ν with Σ.

Now the fundamental group of a d-fold cyclic cover of ν_0 restricting to a d-fold cover of a fiber is the kernel of a homomorphism $\pi_1(\nu_0) \to \mathbb{Z}/d\mathbb{Z}$ which restricts to a surjective homomorphism of the fundamental group $\mathbb{Z}$ of the fiber of ν_0 onto $\mathbb{Z}/d\mathbb{Z}$. By the exact cohomology sequence (2.1, with $R = \mathbb{Z}/d\mathbb{Z}$, such a homomorphism exists if and only if the mod d reduction of the Euler class of ν vanishes. □

For a ring R with unit consider now the exact sequence

$$\cdots \to H^1(M \smallsetminus \Sigma; R) \to H^2(M; M \smallsetminus \Sigma; R) \to H^2(M, R) \to H^2(M \smallsetminus \Sigma; R) \to \cdots$$

By excision, the Thom class of the normal bundle ν defines a nontrivial class in $H^2(M; M \smallsetminus \Sigma; R) = H^2(U, U \smallsetminus \Sigma; R) = H^2(\nu, \nu_0; R)$. It image under the map $H^2(M; M \smallsetminus \Sigma; R) \to H^2(M; R)$ is just the Poincaré dual of Σ by Proposition 6.24 of [BT82]. Passing to the ring $\mathbb{Z}/d\mathbb{Z}$, this class vanishes if and only if the mod d reduction of the Thom class is contained in the image of $H^1(M \smallsetminus \Sigma; \mathbb{Z}/d\mathbb{Z})$. Now every element of $H^1(M \smallsetminus \Sigma; \mathbb{Z}/d\mathbb{Z})$ defines a homomorphism $\pi_1(M \smallsetminus \Sigma) \to \mathbb{Z}/d\mathbb{Z}$, and the kernel of this homomorphism defines a d-fold cyclic covering of $M \smallsetminus \Sigma$. This covering restricts to a cyclic d-fold covering of U and hence defines a cyclic cover of M branched along Σ. Using Poincaré duality, we thus have shown.

prop:branched **Proposition 2.3.** *There exists a cyclic d-fold covering of M branched along Σ if and only if $[\Sigma] = 0 \in H_{n-2}(M, \mathbb{Z}/d\mathbb{Z})$.*

sec:cone

3. Branched covers and hyperbolic cone metrics

Let $\mathbb{H}^n$ be the hyperbolic space and let $\mathbb{H}^{n-2}$ be a totally geodesic subspace of codimension two. The subgroup of the group $PO(n,1)$ of orientation preserving isometries of $\mathbb{H}^n$ which fixes $\mathbb{H}^{n-2}$ pointwise is the circle group S^1 acting on a fiber of the normal bundle as a group of rotations. The fundamental group of $\mathbb{H}^n \smallsetminus \mathbb{H}^{n-2}$ equals the group $\mathbb{Z}$.

If we denote by $\tilde{X}$ the universal covering of $\mathbb{H}^n \smallsetminus \mathbb{H}^{n-2}$, then the abelian group $\mathbb{R}$, which is the universal covering of the circle S^1, acts freely and isometrically on $\tilde{X}$ with respect to the (incomplete) hyperbolic pull-back metric. Thus we can take the quotient X of $\tilde{X}$ under an infinite cyclic subgroup of $\mathbb{R}$. The metric completion $\bar{X}$ of X contains the isometrically embedded subspace $\mathbb{H}^{n-2}$, and a fiber of the normal bundle has a natural identification with a two-dimensional hyperbolic cone with angle $\alpha \in (0,\infty)$. If $\alpha \in (0,2\pi]$ then this cone is obtained as follows.

In the disk model $D = \{z \mid |z| < 1\}$ for the hyperbolic plane, straight line segments starting at the origin are geodesic up to parameterization. Cut D open along the rays $\rho = \{\mathfrak{I} = 0, \mathfrak{R} \geq 0\}$ and $e^{i\alpha}\rho$ and glue the component of the resulting space which is a sector of angle α along the boundary with the rotation $e^{i\alpha}$. The hyperbolic metric on D descends to a hyperbolic cone metric with cone angle α at the image of the vertex 0 of the sector.

We call a metric on a closed manifold which is hyperbolic outside of a closed codimension two submanifold Σ and such that any point $x \in \Sigma$ has a neighborhood which is isometric to a neighborhood of a point in the singular set of a hyperbolic cone metric as described above, with locally constant cone angle, a *hyperbolic cone metric* as well.

Assume now that M is a closed oriented hyperbolic manifold containing a closed totally geodesic oriented submanifold Σ of codimension two which is homologous to zero. By Proposition prop:branched 2.3, for each $d \geq 2$ we then can construct a d-fold cyclic cover $\Pi : M_d \to M$ branched along Σ. The pull-back under Π of the hyperbolic metric on M is a hyperbolic cone metric with cone angle $2d\pi$. This observation is summarized as follows.

conemetric **Lemma 3.1.** *The d-fold cyclic branched covering M_d of M admits a hyperbolic cone metric with conical singularity along Σ and cone angle $2\pi d$.*

For a hyperbolic cone metric, the volume form is defined and gives rise to a volume by integration. The following is immediate from the above construction.

cor:cone **Corollary 3.2.** *The volume of M_d with respect to the hyperbolic cone metric equals $d\mathrm{vol}(M)$.*

Proof. As the branched covering projection $\Pi : M_d \to M$ is a map of degree d which is furthermore an orientation preserving local diffeomorphism on $M_d \smallsetminus \Sigma$, the volume form of the cone metric equals the pull-back of the volume form of M outside of a submanifold of codimension two and this form integrates to $d\mathrm{vol}(M)$. $\square$

We next discuss another viewpoint of the hyperbolic cone metric g_d on M_d. Namely, consider again the hyperbolic space $\mathbb{H}^n$ and a totally geodesic subspace $\mathbb{H}^{n-2}$ of codimension two. The stabilizer $\mathrm{Stab}(\mathbb{H}^{n-2})$ of $\mathbb{H}^n$ in $\mathrm{PO}(n,1)$ which consists of isometries whose restriction to $\mathbb{H}^{n-2}$ is orientation preserving can be identified with $\mathrm{PO}(n-2,1)\times S^1$ where S^1 is the subgroup fixing $\mathbb{H}^{n-2}$ pointwise.

The hyperbolic cone metric on M_d pulls back to a hyperbolic cone metric on the covering $\hat{M}_d$ of M_d with fundamental group $\pi_1(\Sigma)$. The singular locus of this metric is the submanifold Σ. The cyclic deck group Γ of M_d acts on $\hat{M}_d$ as a cyclic group of isometries fixing each point of Σ pointwise. If h is a smooth Riemannian metric on M_d which is invariant under the action of Γ, then h lifts to a smooth metric $\hat{h}$ on $\hat{M}_d$ which is invariant under Γ. Consequently h descends to a cone metric on $\Gamma\backslash\hat{M}_d = M$ with singular locus Σ and cone angle $\frac{2\pi}{d}$. This manifold in turn is just the quotient of $\mathbb{H}^n$ under the action of the fundamental group of Σ.

Thus if one is interested in understanding distinguished metrics on M_d one is led to studying distinguished cone metrics on $\mathbb{H}^n$ with singular locus $\mathbb{H}^{n-2}$ and cone angle $\frac{2\pi}{d}$ which are invariant under the action of the group $S^1\times\mathrm{PO}(n-2,1)$. In order to be the lift of a metric on M_d, it is necessary that the metric glues to a metric on $M_d\smallsetminus\Sigma$. This motivates the study of cone metrics on $\mathbb{H}^n$ which are asymptotic to the hyperbolic metric as the distance from $\mathbb{H}^{n-2}$ tends to infinity.

In the absence of negatively curved locally symmetric metrics, which do not admit deformations as soon as the dimension of the manifold is at least 3 by Mostow rigidity, one has to look for other distinguished classes of metrics which are stable under small deformations and extend the class of locally symmetric metrics. A natural such class is the class of *Einstein metrics* g which are defined by the requirement that the Ricci tensor Ric_g satisfies $\mathrm{Ric}_g = cg$ for a constant $c\in\mathbb{R}$. While by the Bonnet Myers theorem, $c>0$ forces the manifold to be compact, with finite fundamental group, metrics with negative Ricci curvature abound [Lo94] on any closed manifold of dimension at least 3.

For three-dimensional manifolds, the Einstein condition is equivalent to constant curvature [Bes08], but this is not true any more in higher dimension. Einstein metrics on the branched coverings M_d of closed hyperbolic manifolds lift to Einstein metrics on $\hat{M}_d$, and hence one is lead to look for $S^1\times\mathrm{PO}(n-2,1)$-invariant Einstein cone metrics on $\mathbb{H}^n$ with singular locus $\mathbb{H}^{n-2}$ and Einstein constant $-(n-1)$ whose desingularizations serve as models for Einstein metrics on M_d near the submanifold Σ.

The following is due to Fine-Premoselli (Proposition 3.2 of [FP20]). An analogous statement in the Kähler setting was established in [GH25].

Proposition 3.3 (Proposition 3.2 and Lemma 3.3 of [FP20])**.** *Let us consider a totally geodesic subspace $\mathbb{H}^{n-2} \subset \mathbb{H}^n$ and let $\alpha \in (0,\infty)$. Then there exists a $S^1 \times \mathrm{PO}(n-2,1)$-invariant Einstein cone metric g_α on $\mathbb{H}^n$ with Einstein constant $-(n-1)$ and the following properties.*

(1) The singular locus of g_α equals $\mathbb{H}^{n-2}$, and $\mathbb{H}^{n-2}$ is totally geodesic for g_α.
(2) The cone angle of g_α along $\mathbb{H}^{n-2}$ equals α.
(3) The sectional curvature of g_α is negative.
(4) As $d(\gamma,x) \to \infty$, the metric converges smoothly to the hyperbolic metric g.

We have $\alpha < 2\pi$ if and only if the restriction of g_α to $\mathbb{H}^{n-2}$ satisfies $g_\alpha|\mathbb{H}^{n-2} \le g_1|\mathbb{H}^{n-2}$. In particular, for $\alpha \neq 2\pi$, the metric is hyperbolic if and only if $n = 3$.

Proposition 3.3 allows to glue the singular Einstein metric on a collar neighborhood of Σ in $\pi_1(\Sigma)\backslash\mathbb{H}^n$ to the hyperbolic metric on the complement of the collar neighborhood and desingularize to obtain a smooth metric on M_d which is Einstein on the complement of the gluing region. Theorem 1 is proved by deforming the metric to an Einstein metric using an implicit function theorem. This strategy is also used in [GH25] in the Kähler case, and in [J25] to construct Einstein metrics on manifolds which admit metrics with controlled curvature properties in dimension at most twelve.

The *simplicial volume* $\|M\|$ of a closed oriented manifold M is the infimum of the quantities $\sum|a_i|$ where $\sum_i a_i\sigma_i$ is a singular chain representing the fundamental class of M and where $a_i \in \mathbb{R}$. It is know that $\kappa_n\|M\| = \mathrm{vol}(M)$ for every closed hyperbolic manifold M where κ_n is the volume of a regular ideal hyperbolic tetrahedron. Thus we have.

Corollary 3.4. *The simplicial volume of M_d is strictly larger than $d\|M\|$.*

Proof. Since the natural map $M_d \to M$ is of degree d, we know that $\|M_d\| \ge d\|M\|$. We have to show that this inequality is strict. As the cone metric is a limit of smooth metrics of sectional curvature bounded from below by -1, the cone manifold is an $\mathrm{RCD}(n-1,n)$-space. Thus the rigidity statement is an immediate consequence of Corollary 1.5 of [C⁺⁺24]. □

4. Branched coverings of hyperbolic 3-manifolds

Consider now a closed hyperbolic 3-manifold M and an embedded geodesic multicurve $c \subset M$ which is homologous to zero. By Proposition 2.3, we know that for every $d \ge 2$ there exists a covering $M_d \to M$ of degree d, branched along c. It follows from the discussion in Section 3 that this branched covering manifold admits a hyperbolic cone metric with cone angle $2\pi d > 2\pi$ which can be deformed to a smooth negatively curved metric. We refer to [GT87] for an explicit construction of such a metric which is a warped product metric near the singular set of the cone metric. Thus as a consequence of hyperbolization, there also exists a hyperbolic metric on M_d. Note that this is consistent with the fact that the Einstein cone metric on $\mathbb{H}^n$ with singular locus $\mathbb{H}^{n-2}$ found in Proposition 3.3 is hyperbolic if and only if $n = 3$.

Since M_d is a cyclic branched cover of M, it admits a deck group action by a cyclic group of order d, with quotient M. The fixed point set of a generator ζ of this group is the preimage of the geodesic multicurve c under the covering map. By Mostow rigidity, there exists a cyclic group of order d of isometries of M_d for the hyperbolic metric. This means that there exists an isometry $\zeta^\sharp$ of M_d which is homotopic to ζ and such that $(\zeta^\sharp)^d = \mathrm{Id}$.

As follows from the work [CLW18], in general it is difficult to determine the relation between the fixed point sets of ζ and $\zeta^\sharp$. However, in the specific situation at hand we have the following.

fixed3 **Proposition 4.1** (Proposition 5.8 of [HJ24a]). *The fixed point set* $\mathrm{Fix}(\zeta^\#) \subset X$ *of* $\zeta^\#$ *is (abstractly) diffeomorphic to* $\mathrm{Fix}(\zeta)$. *Moreover,* $\mathrm{Fix}(\zeta^\#)$ *and* $\mathrm{Fix}(\zeta)$ *are freely homotopic inside* X.

As a consequence, the quotient of M_d by the group $\langle\zeta^\sharp\rangle$ generated by $\zeta^\sharp$ is a hyperbolic orbifold, or a cone manifold, which is diffeomorphic to M, with singular locus homotopic to c. The cone angle equals $2\pi/d$.

Question. Is it true that the length of c for the smooth hyperbolic metric on M_d is strictly smaller than the length of c on M?

As a consequence of Corollary 3.4, we know that the volume of the hyperbolic cone metric on M is strictly larger than the volume of the smooth hyperbolic metric.

Since the quotient $\langle\zeta^\sharp\rangle\backslash M_d$ is a hyperbolic cone manifold so that the multicurve c is the singular locus and the cone angle is at most π we can now invoke a result of Kojima [Ko98] and obtain the following.

Theorem 4.2. *The cone metric can be deformed to a complete finite volume hyperbolic metric on* $M \smallsetminus c$.

As a consequence we obtain.

finitevol **Corollary 4.3.** *Let* $c \subset M$ *be any null homologous embedded finite collection of closed geodesics on* M. *Then* $M \smallsetminus c$ *admits a complete hyperbolic metric of finite volume.*

The volume of a hyperbolic 3-manifold can be studied using various tools. To obtain an upper bound one can proceed as follows. Choose an effective triangulation of a closed hyperbolic manifold M. This triangulation can be *straightened* to a triangulation with geodesic edges and totally geodesic facets. The hyperbolic volume of a straightened simplex does not exceed the hyperbolic volume κ_n of an ideal hyperbolic tetrahedron, so the volume of M is at most $\kappa_n|T|$ where $|T|$ denotes the number of maximal simplices of the triangulation.

Lower bounds on the volume of a hyperbolic manifold in terms of topological information are much harder to obtain. The following was shown in [HJ22].

Theorem 4.4. *For every $g \geq 2$ there exists a constant $C(g) > 0$ so that the following holds true. The volume of a closed hyperbolic 3-manifold of Heegaard genus g is at least $c(g)d_{\mathrm{Hempel}}$ where d_{Hempel} denotes the Hempel distance of the Heegaard splitting.*

Here a Heegaard spliting of a closed 3-manifold M of genus g is the decomposition of into two handlebodies of genus g which are glued along the boundary with a diffeomorphism to obtain M. The Hempel distance is an invariant of the gluing diffeomorphism. We refer to [HJ22] for more information.

sec:branched4

5. Branched covers in dimension $n \geq 4$

In this section we consider a degree d cyclic covering $M_d \to M$ of a closed hyperbolic oriented manifold of dimension $n \geq 4$ branched along a closed totally geodesic oriented submanifold Σ of codimension two. We assume that there exists an isometric involution $\iota : M \to M$ whose fixed point set is a compact (possibly disconnected) totally geodesic embedded hyperplane. We also assume that a component H of this hyperplane contains a totally geodesic embedded hyperplane Σ which is homologous to zero in H and consists of at most two connected components. We refer to [FP20, HJ24a] for a discussion of examples.

By Proposition 2.3, for every $d \geq 2$ there exists a cover M_d of M of degree d, branched along Σ. We assume that M_d admits a hyperbolic metric and seek to obtain a geometric understanding of this metric.

fixedpointset

5.1. **Fixed point sets of isometries.** Since Σ is homologous to zero in H, it bounds a submanifold $H_0 \subset H$. Put $H_1 = H \setminus H_0$.

Fix a number $d \geq 2$. The d-fold covering X of M branched along the totally geodesic submanifold $\Sigma \subset H \subset M$ can be realized as follows. Let M_{cut} be obtained from M by cutting along H_0, that is, M_{cut} is the metric completion of $M - H_0$. Thus M_{cut} is a compact (topological) manifold whose boundary consists of two copies H_0^- and H_0^+ of H_0 intersecting in Σ. The manifold X is obtained by gluing d copies $M_{\mathrm{cut}}^1, \dots, M_{\mathrm{cut}}^d$ of M_{cut} along the boundary, so that the copy of H_0^+ in M_{cut}^i is glued to the copy of H_0^- in M_{cut}^{i+1} (where the superscripts i are taken $\mathrm{mod}\, d$). We denote by $H_0^{d,i} \subset X$ the resulting manifold with boundary which is homeomorphic to H_0.

Let $\iota = \iota_M : M \to M$ be the isometric involution whose fixed point set contains $H \subseteq \mathrm{Fix}(\iota)$. Since locally near H, ι_M acts as a reflection in H, it exchanges the two components of $U \setminus H$ where U is a tubular neighborhood of H in M. Thus ι_M acts as an involution $\iota_{\mathrm{cut}} : M_{\mathrm{cut}} \to M_{\mathrm{cut}}$ which exchanges H_0^+ and H_0^- and fixes $W = \mathrm{Fix}(\iota_M) \setminus H_0 \supseteq H_1$.

As a consequence, ι_M induces an involution $\iota : X \to X$ with the property that $\iota(M^i_{\rm cut}) = M^{d+2-i}_{\rm cut}$ and so that the restrictions $\iota|M^i_{\rm cut} : M^i_{\rm cut} \to M^{d+2-i}_{\rm cut}$ are identified with $\iota_{\rm cut} : M_{\rm cut} \to M_{\rm cut}$ (superscripts are again taken mod d).

If the degree d of the covering is odd, then the fixed point set of this involution of X is the union of the copy W^1 of W in $M^1_{\rm cut}$ and of $H_0^{d,(d-1)/2}$, glued $H^1_1 \subset W^1$ along Σ. If d is even then the fixed point set of ι consists of the two copies of W in $M^1_{\rm cut}$ and $M^{d/2+1}_{\rm cut}$ glued along Σ.

Let ζ be a generator of the cyclic deck group of $X \to M$. It cyclically permutes the copies $M^1_{\rm cut}, \dots, M^d_{\rm cut}$ of $M_{\rm cut}$ in X. If the degree d is even define $j = \zeta \circ \iota$ (read from right to left), and for odd degree define $j = \iota$. The following is now immediate from the construction.

factfixed **Fact 5.1.**
- If d is even, then the fixed point set of j in X is the union $\mathrm{Fix}(j) = H_0^{d,1} \cup H_0^{d,1+d/2}$, and the manifolds $H_0^{d,1}$ and $H_0^{d,1+d/2}$ of H_0 are glued along Σ.
- If d is odd then the fixed point set of j in X is $\mathrm{Fix}(j) = W^1 \cup H_0^{d,(d-1)/2}$, and it is homeomorphic to $\mathrm{Fix}(\iota_M)$.

The fixed point set of each of the involutions $\zeta^i \circ j \circ \zeta^{-i}$ $(i = 0, \dots, d-1)$ is the embedded submanifold $\zeta^i(\mathrm{Fix}(j))$ of X. Their union cuts X up into the d copies of $M_{\rm cut}$ if d is even, and into d copies of $M \smallsetminus H$ if d is odd. We call any diffeomorphism of X contained in the finite group of diffeomorphisms of X generated by j and ζ an *admissible* diffeomorphism of X.

By Mostow rigidity, any homotopy self-equivalence σ of X is homotopic to a unique isometry $\sigma^\#$ of X. Furthermore, by uniqueness, the map

$$\mathrm{Homeo}(X) \to \mathrm{Isom}(X),\ \sigma \mapsto \sigma^\#$$

which associates to a homeomorphism the unique isometry homotopic to it is a group homomorphism. The following result was established in [HJ24a].

fixed **Proposition 5.2.** *Let ϕ be an admissible diffeomorphism of X and let $\phi^\#$ be the isometry of X homotopic to ϕ. Then the fixed point set $\mathrm{Fix}(\phi^\#) \subset X$ of $\phi^\#$ is (abstractly) diffeomorphic to $\mathrm{Fix}(\phi)$. Moreover, $\mathrm{Fix}(\phi^\#)$ and $\mathrm{Fix}(\phi)$ are freely homotopic inside X.*

Let j be the involution of X described in Fact 5.1, and ζ be the generator of the deck group of $X \to M$ which cyclically permutes the copies $M^1_{\rm cut}, \dots, M^d_{\rm cut}$ of $M_{\rm cut}$ in X.

From now on we always denote by F the component of $\mathrm{Fix}(j)$ containing Σ and by $F^\#$ the homotopic component of $\mathrm{Fix}(j^\#)$ whose existence was shown in Proposition 5.2. By Proposition 5.2 and Mostow rigidity for closed hyperbolic manifolds of dimension $n-1 \geq 3$, there exists an isometry $\psi : F \to F^\#$ which maps Σ to the fixed point set $\Sigma^\#$ of $\zeta^\#$. Furthermore, with a homotopy we may identify Σ and $\Sigma^\#$ in X. For each $i = 0, \dots, d-1$, the map $(\zeta^\#)^i \circ \psi \circ \zeta^{-i}$ maps $\zeta^i(F)$ isometrically onto $(\zeta^\#)^i(F^\#)$.

After possibly changing the hyperbolic metric of X with an isotopy, we may assume that for each connected component Σ_0 of Σ we have $\Sigma_0 \cap \Sigma_0^{\#} \neq \varnothing$, where $\Sigma_0^{\#} = \psi(\Sigma_0)$. So, for each component, we can fix a basepoint $x_0 \in \Sigma_0 \cap \Sigma_0^{\#}$, and we may assume without loss of generality that $\psi(x_0) = x_0$. We call such a basepoint *preferred*. Due to Proposition 5.2, we may also assume that

$$\pi_1(\Sigma_0, x_0) = \pi_1(\Sigma_0^{\#}, x_0) \quad \text{and} \quad \pi_1(F, x_0) = \pi_1(F^{\#}, x_0).$$

In the sequel, the fundamental group $\pi_1(X, x_0)$ will always be represented with respect to a fixed choice x_0 of preferred basepoint.

Although by Proposition 5.2, the cyclic group generated by $\zeta^{\#}$ acts freely on $X \setminus \Sigma^{\#}$ and the manifold $F^{\#}$ is homotopic to F, this does not necessarily imply that $\zeta^{\#}(F^{\#}) \cap F^{\#} = \Sigma^{\#}$. The following lemma takes care of this issue and was also established in [HJ24a].

intersection **Lemma 5.3.** *(1) The differential of $\zeta^{\#}$ acts on the normal bundle of $\Sigma^{\#}$ by a rotation with angle $2\pi/d$.*
(2) We have $F^{\#} \cap \zeta^{\#}(F^{\#}) = \Sigma^{\#}$.

Let N be the compact hyperbolic manifold with totally geodesic boundary ∂N which is obtained by cutting M open along H, that is, N is the metric completion of $M - H$. If H is non-separating, then N is connected, otherwise N has two connected components. The boundary ∂N of N is totally geodesic and consists of two copies of H containing one copy of Σ each. The main tool towards Theorem 2 is the following result.

prop:boundaryrigidity **Proposition 5.4.** *If the cyclic d-fold branched cover X of M admits a hyperbolic metric, then there exists a hyperbolic cone manifold $N^{\#}$ satisfying the following properties:*

(i) $N^{\#}$ is homotopy equivalent to N.
(ii) The boundary $\partial N^{\#}$ of $N^{\#}$ is path isometric to ∂N.
(iii) The singular locus of $N^{\#}$ consists of the two copies of Σ in $\partial N^{\#}$. The cone angle at each of these copies equals $2\pi/d$.

kerckhoffstorm **Remark 5.5.** As the dimension of N is at least four, by a result of Kerckhoff and Storm [KS12, Theorem 2.5] there is no continuous deformation of the convex cocompact hyperbolic manifold $\hat{N}$ within the space of convex cocompact hyperbolic manifolds.

This does not reduce Theorem 2 to Proposition 5.4 as the result of Kerckhoff and Storm does not rule out that there are isolated faithful convex cocompact representations of Γ which are not conjugate to ρ.

even

5.2. **The proof of Proposition 5.4.** In this subsection we assume as before that X admits a hyperbolic metric. Let j be the involution of X described in Fact 5.1, and ζ be the generator of the deck group of $X \to M$ which cyclically permutes the copies $M^1_{\mathrm{cut}}, \dots, M^d_{\mathrm{cut}}$ of M_{cut} in X. The union $\cup_i \zeta^i(F)$ of the corresponding components of the fixed point sets of the involutions $\zeta^i \circ j \circ \zeta^{-i}$ cut X up into d copies of the manifold N from Proposition 5.4.

Using these conventions, Proposition 5.2 shows that the fixed point set of each of the isometric involutions $(\zeta^\#)^i \circ j^\# \circ (\zeta^\#)^{-i}$ homotopic to $\zeta^i \circ j \circ \zeta^{-i}$ has a component $(\zeta^\#)^i(F^\#)$ which is a hyperplane isometric to the component $\zeta^i(F)$ of the fixed point set of $\zeta^i \circ j \circ \zeta^{-i}$ and contains Σ. We shall show that the union $\cup_i (\zeta^\#)^i(F^\#)$ of these submanifolds of X cut X into d sectors homotopy equivalent to $M_{\rm cut}$ if d is even, and homotopy equivalent to N if d is odd. This yields Proposition 5.4 if d is odd. If d is even then such a sector contains an embedded isometric copy of H_1 which intersects the boundary of $M^\#_{\rm cut}$ in $\Sigma^\#$. Cutting $M^\#_{\rm cut}$ along this copy of H_1 then yields a hyperbolic cone manifold with the properties stated in Proposition 5.4.

From now on we always denote by F the component of $\mathrm{Fix}(j)$ containing Σ and by $F^\#$ the homotopic component of $\mathrm{Fix}(j^\#)$ whose existence was shown in Proposition 5.2.

For each $i = 0, \dots, d-1$, Proposition 5.2 yields an isometry $\phi_i : \zeta^i(F) \xrightarrow{\cong} (\zeta^\#)^i(F^\#)$ such that the maps

$$\mathrm{incl}_{\zeta^i(F)} : \zeta^i(F) \hookrightarrow X \quad \text{and} \quad \mathrm{incl}_{(\zeta^\#)^i(F^\#)} \circ \phi_i : \zeta^i(F) \xrightarrow{\cong} (\zeta^\#)^i(F^\#) \hookrightarrow X$$

are homotopic. As $(\zeta^\#)^i(F^\#) \subseteq X$ is totally geodesic, $\phi_i(\Sigma) \subseteq X$ is a closed totally geodesic submanifold of codimension two which is in the same free homotopy class as Σ. Since each free homotopy class can contain at most one totally geodesic representative, this implies that $\phi_0(\Sigma) = \dots = \phi_{d-1}(\Sigma)$. We define $\Sigma^\# \subset X$ to be this closed totally geodesic submanifold of codimenson two. So, by construction, $\Sigma^\# \subseteq (\zeta^\#)^i(F^\#)$ for all $i = 0, \dots, d-1$.

After possibly changing the hyperbolic metric of X with an isotopy, we may assume that for each connected component Σ_0 of Σ we have $\Sigma_0 \cap \Sigma_0^\# \neq \emptyset$, where $\Sigma_0^\# = \phi_0(\Sigma)$. So, for each component, we can fix a basepoint $x_0 \in \Sigma_0 \cap \Sigma_0^\#$, and we may assume without loss of generality that $\phi_0(x_0) = x_0$. We call such a basepoint *preferred*. Due to Proposition 5.2, we may also assume that

$$\pi_1(\Sigma_0, x_0) = \pi_1(\Sigma_0^\#, x_0) \quad \text{and} \quad \pi_1(F, x_0) = \pi_1(F^\#, x_0).$$

In the sequel, the fundamental group $\pi_1(X, x_0)$ will always be represented with respect to a fixed choice x_0 of preferred basepoint. We can now prove Proposition 5.4.

Proof of Proposition 5.4. By construction, the subspace $F \cup \zeta(F)$ of X separates X. If d is even, then by the definition of the map j, the complement $X - (F \cup \zeta(F))$ contains two connected components whose closures are homeomorphic to $M_{\rm cut}$. If d is odd then it contains one connected component whose closure is homeomorphic to $M_{\rm cut}$. In both cases, let Z be the closure of such a component. Its boundary consists of two copies of H_0 glued along Σ.

By Lemma 5.3, there exists a corresponding component $M^{\#}_{\text{cut}}$ of $X-(F^{\#}\cup\zeta^{\#}(F^{\#}))$. The boundary of its closure $Z^{\#}$ is connected and consists of two copies of H_0 meeting along Σ with an angle $2\pi/d$. Identifying Σ and $\Sigma^{\#}$ as before and choosing a basepoint $x\in\Sigma$, we claim that $\pi_1(Z,x)=\pi_1(Z^{\#},x)$.

Namely, as $\pi_1((\zeta^{\#})^i(F^{\#}),x_0)=\pi_1(\zeta^i(F),x_0)$ for all $i=0,\cdots,d-1$, it holds that $\pi_1(\partial Z,x)=\pi_1(\partial Z^{\#},x)$. As ∂Z is a separating hypersurface in X homotopic to $Z^{\#}$, by the theorem of Seifert-van Kampen, we know that

$$\pi_1(X,x)=\pi_1(Z,x)*_{\pi_1(\partial Z,x)}\pi_1(X-Z,x)=\pi_1(Z^{\#},x)*_{\pi_1(\partial Z,x)}\pi_1(X-Z^{\#},x).$$

It then follows from the normal form for amalgamated products [LS01, p.186] that $\pi_1(Z^{\#},x)$ is isomorphic to either $\pi_1(Z,x)$ or to $\pi_1(X,Z,x)$.

If $d=2$ then $\pi_1(Z,x)$ is isomorphic to $\pi_1(X-Z,x)$ and the claim is clear. If $d\geq 3$ then note that $\zeta_*=\zeta^{\#}_*$ maps $\pi_1(Z,x)$ to a proper subgroup of $\pi_1(X-Z,x)$, and it maps $\pi_1(X-Z,x)$ to a proper supergroup of $\pi_1(Z,x)$. Furthermore, it maps $\pi_1(Z^{\#},x)$ to a proper subgroup of $\pi_1(X-Z^{\#},x)$ and it maps $\pi_1(X-Z^{\#},x)$ to a proper supergroup of $\pi_1(Z^{\#},x)$. Thus we have $\pi_1(Z,x)=\pi_1(Z^{\#},x)$ as claimed.

Note that if d is odd, then the component $Z^{\#}$ contains a totally geodesic hypersurface isometric to H_1 which intersects the boundary of $Z^{\#}$ along Σ. There exists an isometric involution of $M^{\#}_{\text{cut}}$ which exchanges the two copies of H_0 in its boundary and hence H_1 meets the boundary of $M^{\#}_{\text{cut}}$ with an angle of π/d. We refer to Fact 5.1 for more information. Cutting $Z^{\#}$ open along this hypersurface then yields a cone manifold with the properties stated in Proposition 5.4.

Now assume that the covering degree d is even. Note that the roles of the hypersurfaces H_0 and H_1 can be exchanged and hence there exists a second involution j_0 of X whose fixed point set is the union $H_1^1\cup H_1^{1+d/2}$ of the copies of H_1 in M^1_{cut} and $M^{1+d/2}_{\text{cut}}$, glued along Σ, as fixed point set. Denote by $j_0^{\#}$ the isometric involutions of X freely homotopic to j_0. Note that we j_0 is admissible. Thus $\text{Fix}(j_0^{\#})$ is a separating totally geodesic hyperplane in X which is isometric to two copies of H_1 glued along Σ. This hyperplane contains $\Sigma^{\#}$ as $\Sigma^{\#}$ is the unique submanifold of X homotopic to Σ which contains each closed geodesic in the free homotopy class of an element of $\pi_1(\Sigma)$. Furthermore, $j_0^{\#}$ fixes $\Sigma^{\#}$ pointwise, and it exchanges the two components of $X\smallsetminus\text{Fix}(j_0^{\#})$.

By Mostow rigidity, we have

$$j_0^{\#}=j^{\#}\circ\zeta^{\#}.$$

In particular, for $x\in\text{Fix}((\zeta^{\#})^{-1}\circ j^{\#}\circ\zeta^{\#})$ it holds $j_0^{\#}(x)=\zeta^{\#}(x)$. As a consequence, $j_0^{\#}$ induces an isometric involution of $M^{\#}_{\text{cut}}$ which exchanges the two components of $\partial M^{\#}_{\text{cut}}\smallsetminus\Sigma^{\#}$. As $j_0^{\#}$ acts as a reflection along a connected separating hyperplane in X containing $\Sigma^{\#}$, the restriction of $j_0^{\#}$ acts as a reflection along a totally geodesic embedded hyperplane $H_1^\#$ which intersects $\partial M_{\rm cut}^\#$ in $\Sigma^\#$. In particular, as $j_0^\#$ is an isometry, the hyperplane $H_1^\#$ meets a component of $\partial M_{\rm cut}^\# \setminus \Sigma^\#$ along $\Sigma^\#$ with an angle of π/d. Thus cutting $M_{\rm cut}^\#$ open along $H_1^\#$ yields a hyperbolic manifold $N^\#$ with piecewise totally geodesic boundary with properties (ii) and (iii) of Proposition 5.4.

That $N^\#$ is homotopy equivalent to $M - H$ follows from the fact that both are aspherical manifolds with boundary and isomorphic fundamental groups. □

5.3. **Proof of the main Theorem.** We showed so far that the existence of a hyperbolic metric on the d-fold covering X of M branched along Σ gives rise to a convex cocompact hyperbolic manifold N_d with two boundary components, each of which is path isometric to the hypersurface H. The manifold is singular along $\Sigma \subset H$, with cone angle (or bending angle) π/d, and it is homotopy equivalent to the hyperbolic manifold N with totally geodesic boundary obtained by cutting M open along H. Note that N is connected if and only if the hypersurface H is non-separating. By a result of Kerckhoff and Storm [KS12], we know that such a hyperbolic cone manifold N_d can not be obtained from a deformation of N, or, more precisely, by a deformation of the representation $\rho : \pi_1(N) \to {\rm O}^+(n,1)$ which defines the unique complete hyperbolic manifold $\hat{N}$ with convex core N.

Construct a new manifold W by gluing $2d$ copies N_d^i $(i = 1, \dots, 2d)$ of N_d along the boundary as follows. Let $\partial N_d^\pm$ be the two distinct boundary components of N, and let $(\partial N_d^i)^\pm$ be the corresponding boundary components of N_d^i. Each of these components contains a copy of Σ which decomposes the component into two connected components $(H_{0,d}^i)^\pm, (H_{1,d}^i)^\pm$. For each odd $i \le 2d$ identify $(H_{0,d}^i)^\pm$ with $(H_{0,d}^{i+1})^\pm$, and for even $i \le 2d$ identify $(H_{1,d}^i)^\pm$ with $(H_{1,d}^{i+1})^\pm$. As the cone angle of $\partial N_d^\pm$ along Σ equals π/d, the hyperbolic metrics on the bordered manifolds N_d^i induce a smooth hyperbolic metric on W.

Lemma 5.6. *The manifold W is a two-sheeted unbranched covering of X.*

In fact, if the hypersurface H in M is non-separating, then the same holds true for the component F containing Σ of the fixed point set of the lift ι to X of the involution ι_M specified in Fact 5.1. Then W is the two-sheeted covering of X so that the preimage of F consists of two components which separate W. If $H \subset M$ is separating, then N consists of two connected components, and W consists of two copies of X.

Proof. The case that H is separating is clear from the above remark. Thus assume that H is non-separating. Then the same holds true for the hypersurface F. Cut X open along F. The resulting manifold Q is connected and has two boundary components $\partial Q^-, \partial Q^+$ which are homeomorphic and path isometric to H. The metric is singular along the two copies of the totally geodesic submanifold $\Sigma^{\#}$ in the two boundary components of Q.

Glue a second copy $\hat{Q}$ of Q to Q along the boundary in such a way that the boundary component $\partial\hat{Q}^-$ is glued to the boundary component ∂Q^+, and the boundary component $\partial\hat{Q}^+$ is glued to the boundary component ∂Q^-. The resulting manifold is equipped with a smooth hyperbolic metric, and admits an obvious two sheeted unbranched covering onto X. The lemma follows. □

We are now in a position to present the proof of our main result.

Proof of the main Theorem. We divide the proof into two claims.

Claim 1: *Among the branched coverings of M of even degree $d \in 2\mathbb{N}$, at most one can be homeomorphic to a hyperbolic manifold.*

Proof of Claim 1: We argue by contradiction and we assume that there are distinct multiples of $d_1 \neq d_2 \in 2\mathbb{N}$ such that the cyclic d_i-fold branched cover $X^{(d_i)}$ admits a smooth hyperbolic metric for $i = 1, 2$. Then, for each $i = 1, 2$, Proposition 5.4 (see the end of the proof for an explicit statement) implies that there exists a hyperbolic cone manifold $M_{\text{cut}}^{2\pi/d_i}$ with totally geodesic boundary $\partial M_{\text{cut}}^{2\pi/d_i}$ homeomorphic and path isometric to ∂M_{cut}, with singular set isometric to Σ, cone angle $2\pi/d_i$ along Σ, and $\pi_1(M_{\text{cut}}^{2\pi/d_i}) = \pi_1(M_{\text{cut}})$.

Note that $\frac{d_1}{2}\frac{2\pi}{d_1} + \frac{d_2}{2}\frac{2\pi}{d_2} = 2\pi$. Therefore, we can glue $d_1/2$ copies of $M_{\text{cut}}^{2\pi/d_1}$ and $d_2/2$ copies of $M_{\text{cut}}^{2\pi/d_2}$ in cyclic order along the components of $\partial M_{\text{cut}}^{2\pi/d_i} \smallsetminus \Sigma$ to a smooth hyperbolic manifold Y. An application of the Seifert–van Kampen theorem shows that the fundamental group of Y is isomorphic to the fundamental group of the $(d_1+d_2)/2$-fold cyclic cover X of M branched along Σ. In particular, this fundamental group admits a finite group of automorphisms generated by an element ζ_* of order $(d_1 + d_2)/2$ and an involution j_* corresponding to the automorphisms induced by the homeomorphisms ζ and j of X (notations are as before). By the proof of Proposition 5.4, for each $i = 0, \ldots, (d_1 + d_2)/2 - 1$, the fixed point group of $\zeta_*^i \circ j_* \circ \zeta_*^{-i}$ is the fundamental group of an embedded codimension one submanifold F_i that, by construction of the hyperbolic metric on Y, is already totally geodesic. Moreover, for some i the totally geodesic submanifolds F_i and F_{i+1} intersect with angle $2\pi/d_1$, while for other i they intersect with angle $2\pi/d_2$.

By Mostow rigidity, there exist isometries $\zeta^{\#}, j^{\#}$ of the hyperbolic manifold Y of order $(d_1 + d_2)/2$ and 2, respectively, that induce the outer automorphism given by ζ_* and j_*. By Lemma 5.3, the fixed point set of $\zeta^{\#}$ is a codimension two totally geodesic submanifold $\Sigma^{\#}$ freely homotopic to Σ, and thus $\Sigma^{\#} = \Sigma$ since Σ is already totally geodesic in Y. Similarly, by Proposition 5.4, the fixed point set $(\zeta^{\#})^i(F^{\#})$ of the involution $(\zeta^{\#})^i \circ j^{\#} \circ (\zeta^{\#})^{-i}$ is a totally geodesic hyperplane freely homotopic to the manifold F_i satisfying $\pi_1(F_i) = \text{Fix}(\zeta_*^i \circ j_* \circ \zeta_*^{-i})$, and thus $(\zeta^{\#})^i(F^{\#}) = F_i$ since F_i is already hyperbolic. However, as $\zeta^{\#}$ acts by rotation with a fixed angle in the normal bundle of Σ, the intersection angle of $(\zeta^{\#})^i(F^{\#})$ and $(\zeta^{\#})^{i+1}(F^{\#})$ is the same for all i. But this contradicts the fact that, by construction, the intersection angle of F_i with F_{i+1} varies between $2\pi/d_1$ and $2\pi/d_2$, completing the proof of the claim. ■

Claim 2: *No branched covering of M of odd degree $d \geq 3$ can be homeomorphic to a hyperbolic manifold.*

Proof of Claim 2: Assume that there exists a covering X of M branched along Σ of odd degree d which admits a hyperbolic metric. By Proposition 5.4, there exists a hyperbolic cone manifold N_d homotopy equivalent to the manifold N with cone angle π/d along the copies of Σ in each boundary component of N_d. Glue d copies of N_d to the manifold $N = M \setminus H$ along the boundary as described in Lemma 5.6. Note that this is possible because d is odd. The resulting manifold is homotopy equivalent to a double unbranched covering of the branched covering X of M of degree $\frac{d+1}{2}$ as in Lemma 5.6, and it is equipped with a smooth hyperbolic metric.

On the other hand, as $W \to X$ is a two-sheeted unbranched covering, the hyperbolic metric on X lifts to a hyperbolic metric on W. However, it follows precisely as in the proof of Claim 1 that this leads to a contradiction. ■ □

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