SIGNATURES OF SURFACE BUNDLES AND MILNOR WOOD INEQUALITIES

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Abstract. Let $E \to B$ be an aspherical surface bundle over a surface. We use a variant of the Milnor Wood inequality to show that $3|\sigma(E)| \le \chi(E)$.

1. Introduction

The Miaoka inequality for a complex surface X of general type states that

$$c_1^2(X)[X] \le 3c_2(X)[X]$$

where $c_i(X)$ $(i = 1, 2)$ are the Chern classes of X. Equality holds if and only if X is a quotient of the ball.

Now let $\sigma(X)$ be the signature of X. By Hirzebruch's formula, if we denote by $p_1(TX)$ the first Pontrjagin class of the tangent bundle TX of X then

$$3\sigma(X) = p_1(TX)[X].$$

Using $p_1(TX) = c_1^2(X) - 2c_2(X)$ one concludes that

$$3|\sigma(X)| \le \chi(X) \tag{1}$$

where $\chi(X)$ is the Euler characteristic of X.

In this note we will establish that the Miaoka inequality also holds true for surface bundles over surfaces. Such a surface bundle is a smooth 4-manifold E which admits a smooth surjection $\pi : E \to B$ onto a smooth closed oriented surface B of genus $h \ge 0$. Each fibre is required to be a closed oriented surface S_g of genus $g \ge 0$. We are only interested in the case that $h \ge 2$ and $g \ge 2$. Then the manifold E is determined up to diffeomorphism by a homomorphism $\rho : \pi_1(B) \to \Gamma_g$ where Γ_g denotes the mapping class group of S_g.

Kotschick [K98] showed the inequality $2|\sigma(E)| \le \chi(E)$. For surface bundles which admit a complex structure or an Einstein metric, he obtained the bound $3|\sigma(E)| < \chi(E)$ [K98, K99], i.e. the Miaoka inequality (1) holds true.

Admitting a complex structure, however, is restrictive. Baykur [B11] constructed for any numbers $g \ge 4$, $h \ge 9$ infinitely many inequivalent surface bundles over surfaces with fibre genus g and base genus h which do not admit a complex structure with either orientation but have non-zero signature. In contrast, I am not aware of obstructions to the existence of Einstein metrics for surface bundles over surfaces. The best known such obstruction is given by the Hitchin-Thorpe inequality $|3\sigma(X)| \le 2\chi(X)$ for Einstein manifolds which is always valid by Kotschick's result (see [LB99] for more information and for references).

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The goal of this note is to show that the Miaoka inequality (1) holds true for all surface bundles.

Theorem. $3|\sigma(E)| \leq \chi(E)$ *for every surface bundle* E *over a surface.*

Bryan and Donagi [BD02] constructed a surface bundle E_0 over a surface of genus 2, with fibre of genus 91 and

$$\frac{3\sigma(E_0)}{\chi(E_0)} = \frac{8}{15}.$$

According to my knowledge, this is the example with the largest known ratio between signature and Euler characteristic.

Since signature and Euler characteristic are multiplicative under coverings, taking unbranched covers of the base of the example of Bryan and Donagi and pulling back the fibre yields surface bundles E over a base of any genus $h \geq 2$ with fibre genus 91 and ratio $\frac{3\sigma(E)}{\chi(E)} = \frac{8}{15}$. Morita [Mo87] showed that for a suitable cover of the base, there are coverings of the total space which restrict to coverings of the fibre and hence surface bundles E with ratio $\frac{3\sigma(E)}{\chi(E)} = \frac{8}{15}$ exist for an infinite sequence of fibre genera.

The paper [BD02] contains more precise information for surface bundles with even fibre genus. Namely, define

$$h_g(m) = \min\{h \mid \exists \text{ a genus } g \text{ bundle } E \to B \text{ with } \chi(B) = 2-2h \text{ and } \sigma(E) = 4m\}.$$

It is easy to see that the limit

$$H_g = \lim_{m\to\infty} \frac{h_g(m)}{m}$$

exists and is finite. Our main theorem shows that

$$H_g \geq \frac{3}{g-1}.$$

On the other hand, for even g there is an upper bound [BD02]

$$H_g \leq \frac{6}{g-2}.$$

The case of odd fibre genus is less well understood. In [EKKOS02] the authors construct in a systematic way examples which show that $H_g \leq \frac{16}{g-1}$ for all $g \geq 3$.

A *section* of a surface bundle $E \to B$ is a smooth map $f : B \to E$ so that $\pi \circ f = \mathrm{Id}$. In Section 2 we observe that the sharp bound $|[f(B)]^2| \leq |\chi(B)|$ for the self-intersection number $[f(B)]^2$ of a section f established in [BKM11] is a simple consequence of the Milnor Wood inequality.

In Section 3 we give an elementary proof of a result of Korotkin and Zograf [KZ11]. Namely, let $\mathcal{P}$ be the projectivized Hodge bundle over the moduli space of Riemann surfaces of genus g. We calculate the first Chern class $\eta \in H^2(\mathcal{P}, \mathbb{Q})$ of the line bundle defined by the divisor $\mathcal{P}_1$ of projective abelian differentials with at least one zero which is not simple.

As an immediate consequence of the proof of this result we obtain the following well known fact (see [CLM76]). For its formulation, we call the surface bundle $E \to B$ *hyperelliptic* if each fibre can be equipped with the structure of a hyperelliptic Riemann surface depending smoothly on the basepoint. This includes all surface bundles with fibre of genus 2.

Corollary. *The signature of a hyperelliptic surface bundle vanishes.*

In Section 4 we determine explicitly a line bundle with first Chern class $\eta \in H^2(\mathcal{P}, \mathbb{Q})$. For this line bundle, we use obstruction theory to establish a version of the Milnor Wood inequality which yields the above Theorem.

2. Self-intersection numbers of sections

As in the introduction, let $\pi : E \to B$ be a surface bundle over a surface B of genus $h \geq 2$ with fibre S_g of genus $g \geq 2$. In this section we study the self-intersection number $[f(B)]^2$ of a section $f : B \to E$ of E.

Let ν be the tangent bundle of the fibres of E. Then $\nu \to E$ is a complex line bundle whose Chern class $c_1(\nu)$ is defined. We note for later reference

Lemma 2.1. $[f(B)]^2 = c_1(\nu)(f(B))$.

Proof. Since $f(B)$ is a smoothly embedded surface in E, the self-intersection number of $f(B)$ equals the Euler number of the oriented normal bundle of $f(B)$ in E. Now f is a section and consequently this oriented normal bundle is isomorphic to the restriction of ν to $f(B)$. □

Recall from [M58, W71] that a circle bundle $H \to X$ over a manifold X is *flat* if the following holds true. Let $\mathrm{Top}^+(S^1)$ be the group of orientation preserving homeomorphisms of the circle S^1. Then there is a homomorphism $\rho : \pi_1(X) \to \mathrm{Top}^+(S^1)$ such that

$$H = \pi_1(X)\backslash \tilde{X} \times S^1$$

where $\pi_1(X)$ acts on the universal covering $\tilde{X}$ of X as a group of decktransformations, and it acts on S^1 via the homomorphism ρ. The same definition applies if X is a good orbifold and $\pi_1(X)$ is replaced by the orbifold fundamental group. It was shown by Milnor [M58] and Wood [W71] that the Euler number $e(H)$ of a flat circle bundle $H \to B$ over a closed oriented surface B of genus $g \geq 2$ and the Euler characterisitc $\chi(B)$ of B satisfies the inequality

$$|e(H)| \leq |\chi(B)|.$$

This fact can be used to give an elementary proof of the following result of Baykur, Korkmaz and Monden (Proposition 1 of [BKM11]). Our proof is equally valid for smooth sections of Lefschetz fibrations not passing through a singular point.

Proposition 2.2. $|[f(B)]^2| \leq |\chi(B)|$.

Proof. Let $\Gamma_{g,1}$ be the mapping class group of a surface of genus $g \geq 2$ with one marked point. There is an exact sequence [Bi74]

$$0 \to \pi_1(S_g) \to \Gamma_{g,1} \to \Gamma_g \to 0. \tag{2}$$

Let B be a closed oriented surface of genus $h \geq 2$ and let $\pi : E \to B$ be a surface bundle with fibre S_g which admits a section $f : B \to E$. This is equivalent to stating that the homomorphism $\rho : \pi_1(B) \to \Gamma_g$ which defines E lifts to a homomorphism

$$\tilde{\rho} : \pi_1(B) \to \Gamma_{g,1}.$$

Let $\mathcal{T}(S_{g,1})$ be the Teichmüller space of surfaces of genus g with one marked point. Let $\mathcal{C} \to \mathcal{T}(S_{g,1})$ be the universal curve. This is the complex fibre bundle over $\mathcal{T}(S_{g,1})$ whose fibre over x is the Riemann surface x. There is a $\Gamma_{g,1}$-equivariant smooth section $\Phi : \mathcal{T}(S_{g,1}) \to \mathcal{C}$ which maps a point $x \in \mathcal{T}(S_{g,1})$ to the marked point in the fibre over x.

Let $L \to \mathcal{T}(S_{g,1})$ be the pull-back under Φ of the tangent bundle of the fibres and let $H \subset L$ be the circle subbundle of L. The group $\Gamma_{g,1}$ acts on the circle bundle H as a group of bundle automorphisms.

Namely, for each $x \in \mathcal{T}(S_{g,1})$ use the marked point in x as a basepoint for the fundamental group of S_g. Each $x \in \mathcal{T}(S_{g,1})$ defines a unique marked hyperbolic metric on the closed surface S_g (here we forget the puncture). A nontrivial homotopy class $\alpha \in \pi_1(S_g)$ can be represented by a unique oriented geodesic loop connecting the marked point to itself. The initial unit tangent of this curve is a unit vector in the tangent space of the marked point which depends smoothly on x. Lifting this vector to the vertical unit vector over the point i in the upper half-plane defines an identification of the fibre at x of the circle bundle H with the ideal boundary $\partial\mathbf{H}^2$ of the hyperbolic plane $\mathbf{H}^2$.

The group $\Gamma_{g,1}$ is just the group of orientation preserving automorphisms of $\pi_1(S_g)$. As a consequence, any $\varphi \in \Gamma_{g,1}$ acts as an automorphism on the circle bundle H. Using the identification of the fibres of H with $\partial\mathbf{H}^2 = S^1$ from the previous paragraph, for each choice of a basepoint $x \in \mathcal{T}(S_{g,1})$ the restriction of this action to the $\Gamma_{g,1}$-orbit of x defines a homomorphism of $\Gamma_{g,1}$ into the group $\mathrm{Top}^+(S^1)$ of orientation preserving homeomorphisms of the circle S^1.

The homomorphism $\tilde{\rho} : \pi_1(B) \to \Gamma_{g,1}$ induces a homomorphism $\pi_1(B) \to \mathrm{Top}^+(S^1)$. By the above discussion, the induced flat circle bundle over B is just the circle bundle in the pullback of the tangent bundle of the fibres of E via the section f. By the Milnor Wood inequality [M58, W71], the Euler number of any such circle bundle on B is bounded in absolute value by $2h-2$. This shows the proposition. ☐

Remark: The group $\Gamma_{g,1}$ has a natural identification with the automorphism group of $\pi_1(S_g)$. The proof of Proposition 2.2 shows the following. Let $f : B \to E$ be a section and let $\tilde{\rho} : \pi_1(B) \to \Gamma_{g,1}$ be the corresponding homomorphism. If the image of $\tilde{\rho}$ is contained in the stabilizer of any non-trivial element of $\pi_1(S_g)$ then the self-intersection number of f vanishes. This of course can also easily be seen explicitly.

Theorem 15 of [BKM11] shows that for every $g \geq 2, h \geq 1$ and every integer $k \in [-2h+2, 2h-2]$ there is a surface bundle with fibre S_g and base of genus h which admits a section of self-intersection number k. We complement this result by analyzing self-intersection numbers of sections of the trivial bundle.

Proposition 2.3. *Let $E \to B$ be the trivial surface bundle with fibre genus $g \geq 2$ and base genus h. If $h < g$ then every section of E has self-intersection number zero. If $h \geq g$ then for each integer k with $h \geq |k|(g-1)+1$ there is a section of self-intersection number $2k(g-1)$, and no other self-intersection numbers occur.*

Proof. Let B be a surface of genus $h \geq 2$ and let $E = B \times S_g \to B$ be the trivial surface bundle. This means that the defining homomorphism $\pi_1(B) \to \Gamma_g$ is trivial.

Every section $f : B \to E$ defines a lift of the trivial homomorphism $\pi_1(B) \to \Gamma_g$ into $\Gamma_{g,1}$ and hence by the sequence (2), it defines a homomorphism $\rho : \pi_1(B) \to \pi_1(S_g)$. This homomorphism can be represented by a smooth map $\Phi : B \to S_g$. Let

$d \in \mathbb{Z}$ be the degree of Φ. We claim that the self-intersection number of f equals $d(2-2g)$.

To see that this is the case note that the Euler number of the pull-back $\Phi^* TS_g$ of the tangent bundle of S_g equals $d(2-2g)$. However, by the above discussion, the self-intersection number of the section f coincides with this Euler number.

As the consequence, the self-intersection number of a trivial bundle is a multiple of $2g-2$. The proposition now follows from the fact that for every $k \geq 1$ and every surface Σ of genus $h \geq k(g-1)+1$ there is a map $\Sigma \to S_g$ of degree k. □

3. Signature as an intersection number

Throughout this section let S_g be a closed oriented surface of genus $g \geq 2$. As before, we denote by $\mathcal{T}(S_g)$ the Teichmüller space of S_g, and we let Γ_g be the mapping class group.

Let $E \to B$ be a surface bundle over a surface with fibre S_g. We may assume that E is defined by a smooth map $\varphi : B \to \mathcal{M}(S_g)$ where $\mathcal{M}(S_g) = \mathcal{T}(S_g)/\Gamma_g$ is the moduli space of Riemann surfaces. In particular, each fibre $\pi^{-1}(x)$ has the structure of a Riemann surface which depends smoothly on x.

Let $\mathcal{H} \to \mathcal{M}(S_g)$ be the Hodge bundle whose fibre over a point $x \in \mathcal{M}(S_g)$ is the g-dimensional complex vector space of holomorphic one-forms (or *abelian differentials*) on the Riemann surface x.

For $g \geq 3$, the non-torsion part of $H_2(\mathcal{M}(S_g), \mathbb{Z})$ is of rank one and is generated by the dual of the first Chern class λ of $\mathcal{H}$. The signature $\sigma(E)$ of E equals

$$\sigma(E) = 4\lambda(\varphi[B])$$

where $[B]$ denotes the fundamental cycle of B (see [K98] for references). Or, up to a factor 4, the signature of E is the first Chern class of a flat vector bundle on B.

The Hodge bundle $\mathcal{H}$ naturally extends to the Deligne Mumford compactification $\overline{\mathcal{M}}(S_g)$ of $\mathcal{M}(S_g)$ as a vector bundle over each local manifold cover (see [W83] for details). The fibre over a stable curve consists of all holomorphic one-forms with at most simple poles at the punctures and so that residues at paired punctures are opposite. Therefore the projectivization

$$P : \mathcal{P} \to \mathcal{M}(S_g)$$

of the Hodge bundle extends to $\overline{\mathcal{M}}(S_g)$ as well, and this extension is a compact complex projective variety of dimension $4g-4$.

Let $\mathcal{P}_1 < \mathcal{P}$ be the subspace of all projective holomorphic one-forms with at least one zero which is not simple. Then $\mathcal{P}_1$ is an algebraic subvariety of $\mathcal{P}$ of complex codimension one. More explicitly, the set of all projective abelian differentials with precisely one singularity of order two and $2g-4$ singularities of order one is a smooth complex suborbifold of $\mathcal{P}$ of complex codimension one. Its closure contains the set $\mathcal{P}_2$ of projective differentials with more than one singularity of order two or at least one singularity of order at least three as a subvariety of complex codimension one.

Since $\mathcal{P}$ (or, rather, its extension to $\overline{\mathcal{M}}(S_g)$) is a Poincaré duality space, there is a cohomology class $\eta \in H^2(\mathcal{P}, \mathbb{Q})$ which is Poincaré dual to $\mathcal{P}_1$. A standard spectral sequence argument shows that the rational cohomology ring of $\mathcal{P}$ is the adjoint of the pull-back of the rational cohomology ring of $\mathcal{M}(S_g)$ with the cohomology class ψ of the universal line bundle of the fibre (see [KZ11] for details). As a consequence, we have $\eta = aP^*\lambda + b\psi$ where $a, b \in \mathbb{Q}$.

Korotkin and Zograf [KZ11] calculated the numbers $a, b \in \mathbb{Q}$. They proved

Theorem 3.1. $\eta = 24P^*\lambda - (6g-6)\psi$.

This equality also holds as cohomology classes in $H^2(\overline{\mathcal{M}}(S_g), \mathbb{Q})$ [KZ11].

The goal of this section is to give an elementary proof of Theorem 3.1. The intermediate steps of the proof will be used towards the proof of the main theorem from the introduction. Another proof of Theorem 3.1 using tools from algebraic geometry is due to Dawei Chen [C12].

Let B be a Riemann surface of genus $g \geq 2$ and let $\theta : B \to \mathcal{P}$ be a smooth map. We may assume that $\theta(B)$ intersects the divisor $\mathcal{P}_1$ in isolated smooth points, and that each of these intersection points is transverse.

Let $\pi : E \to B$ be the surface bundle defined by the map $P \circ \theta : B \to \mathcal{M}(S_g)$. For each $x \in B$ let $\delta(x)$ be the set of zeros of the holomorphic one-form $\theta(x)$ on the fibre $\pi^{-1}(x)$ counted with multiplicities. Thus $\delta(x)$ is a point in the space $\pi^{-1}(x)^{(2g-2)}$ of unordered $2g-2$-tuples of points in $\pi^{-1}(x)$.

For $x \in B$ denote by $\Delta(x) \subset \pi^{-1}(x)$ the image of $\delta(x)$, viewed as a subset of $\pi^{-1}(x)$. By assumption, for $x \in B$ there is at most one zero of $\theta(x)$ which is not of multiplicity one, and this zero is of multiplicity two. Therefore $\Delta(x)$ is a subset of $\pi^{-1}(x) \subset E$ which consists of $2g-2$ or $2g-3$ points. Define

$$\Delta = \cup_{x \in B} \Delta(x);$$

then Δ is a closed subset of E.

Call a point $z \in \Delta$ *singular* if it is a double zero of the abelian differential $\theta(\pi(z))$. A point in Δ which is not singular is called *regular*. By the choice of the section θ, the number of singular points is finite.

Lemma 3.2. *The set $\Delta \subset E$ is a 2-cycle in E which defines a homology class $\beta \in H_2(E, \mathbb{Z})$.*

Proof. The zeros of an abelian differential depend smoothly on the differential. Thus if $x \in B$ and if $z \in \Delta(x)$ is a regular point then there is a neighborhood U of z in Δ which is mapped by the projection π homeomorphically onto a neighborhood of x in B. In particular, if $\Sigma \subset \Delta$ is the finite set of singular points then the restriction of the projection $\pi : E \to B$ to $\Delta - \pi^{-1}(\pi(\Sigma))$ is a $2g-2$-sheeted covering onto $B - \pi(\Sigma)$.

Choose a triangulation T of the base surface B so that the projection of each of these singular points $z \in \Sigma$ is a vertex of the triangulation and that moreover a triangle has at most one vertex which is a projection of a singular point.

The restriction of the projection π to $\Delta - \pi^{-1}(\pi(\Sigma))$ is a covering. Thus each of the triangles from the triangulation lifts to exactly $2g-2$ embedded triangles in Δ. The orientation of B determines a compatible orientation for each of these lifted triangles. These oriented triangles define on Δ the structure of a compact oriented 2-dimensional simplicial complex without boundary. Then Δ equipped with this structure is a two-dimensional integral simplicial cycle in E which defines a class $\beta \in H_2(E, \mathbb{Z})$.

Using a standard refinement argument, the homology class β does not depend on any choices made in the above construction. $\square$

The next observation is the key ingredient for the proof of Theorem 3.1. For its formulation we denote by ν^* the cotangent bundle of the fibres of $E \to B$, and we let $c_1(\nu^*) \in H^2(E, \mathbb{Z})$ be its (first) Chern class.

Proposition 3.3. *The homology class $\beta \in H_2(E, \mathbb{Z})$ is Poincaré dual to $c_1(\nu^*)$.*

Proof. We follow p. 141 of [GH78].

Let $\mathcal{F}$ be the sheaf of smooth functions on E whose restrictions to a fibre are holomorphic and not identically zero. Let $\mathcal{F}^*$ be the sheaf of functions in $\mathcal{F}$ which vanish nowhere. A global section of $\mathcal{F}/\mathcal{F}^*$ is given by an open cover $\{U_\alpha\}$ of E and a function $f_\alpha \in \mathcal{F}$ on U_α for each α so that

$$\frac{f_\alpha}{f_\beta} \in \mathcal{F}^*(U_\alpha \cap U_\beta).$$

We first claim that the cycle Δ defines a section of $\mathcal{F}/\mathcal{F}^*$. Namely, choose a cover $\mathcal{U} = \{U_\alpha\}$ of E which consists of open contractible sets with the following additional properties. Each set U_α contains at most one singular point of Δ. There is a function $f_\alpha \in \mathcal{F}$ on U_α such that for each regular point $y \in U_\alpha \cap \Delta$ the restriction of f_α to $\pi^{-1}(\pi(y)) \cap U_\alpha$ has a simple zero at y, and for each singular point $y \in U_\alpha \cap \Delta$ the function $f_\alpha|\pi^{-1}(\pi(y))$ has a double zero at y. Moreover, these are the only zeros of f_α.

By the construction of Δ, such a function f_α exists provided that the sets U_α are sufficiently small. Namely, let z be a local holomorphic coordinate on the fibres depending smoothly on the base. The functions f_α can be chosen as local coefficient functions for the abelian differentials $\theta(x)$ $(x \in B)$ with respect to the differential dz. Any two such sections of $\mathcal{F}/\mathcal{F}^*$ differ by a section of $\mathcal{F}^*$ and hence these sections define a class $\hat{\Delta} \in H^1(E, \mathcal{F}^*)$.

The class $\hat{\Delta} \in H^1(E, \mathcal{F}^*)$ defines a line bundle L on E. Its Chern class $c_1(L) \in H^2(E, \mathbb{Z})$ is the image of $\hat{\Delta}$ in the long exact sequence of sheaf cohomology induced by the exact sequence

$$0 \to \mathbb{Z} \to \mathcal{F} \xrightarrow{\exp} \mathcal{F}^* \to 0.$$

For each $x \in B$, the divisor $\delta(x) \in \pi^{-1}(x)^{(2g-2)}$ is the zero divisor of a holomorphic one-form on $\pi^{-1}(x)$. By naturality of the restriction map, the restriction of the line bundle L to a fibre $\pi^{-1}(x)$ is just the cotangent bundle of $\pi^{-1}(x)$. In other words, we have $L|\pi^{-1}(x) = \nu^*|\pi^{-1}(x)$.

Recall that the set $\Sigma \subset \Delta$ of singular points of Δ is finite, and $\Delta - \Sigma$ is a smoothly embedded subsurface of E. Thus to show that β is Poincaré dual to $c_1(\nu^*)$, it suffices to show the following. Let F be a smooth oriented surface and let $\alpha : F \to E$ be a smooth map which intersects Δ transversely in finitely many regular points. Then the number of such intersection points counted with sign and multiplicity equals the degree of the line bundle $\alpha^*\nu^*$ on F.

However, this can be seen as follows. Let as before ν be the tangent bundle of the fibres of the bundle $E \to B$. For each $z \in E - \Delta$ there is a unique tangent vector $X(z) \in T_z(\pi^{-1}(\pi z)) = \nu_z$ so that $\theta(\pi(z))(X(z)) = 1$. The vector $X(z)$ depends smoothly on $z \in E - \Delta$ and hence the assignment $z \to X(z)$ defines a trivialization of $\nu|E - \Delta$. Dualizing with a smooth Riemannian metric yields a trivialization of ν^* on $E - \Delta$.

Now if F is a closed surface and if $\alpha : F \to E$ is a smooth map which intersects Δ transversely in finitely many regular points, then via modifying α with a homotopy we may assume that the following holds true. Let $u \in F$ be such that $\alpha(u) \in \Delta$. Then α maps a neighborhood of u in F diffeomorphically to a neighborhood of $\alpha(u)$ in the fibre $\pi^{-1}(\pi\alpha(u))$.

The pull-back of ν^* to $\alpha^{-1}(E - \Delta)$ is naturally trivialized. We claim that at each point $z \in \Delta$, the restriction of this trivialization to the oriented fibre $\pi^{-1}(\pi(z))$ has rotation number 1 about z with respect to a trivialization which extends across z. Namely, z is a four-pronged singular point for the singular euclidean metric on the fibre $\pi^{-1}(\pi(z))$ which is defined by the holomorphic one-form $\theta(z)$. Hence the rotation number of the restriction of the trivialization of the tangent bundle ν of the fibre constructed above to a small circle about z in $\pi^{-1}(\pi(z))$ with respect to a local trivialization of ν on $\pi^{-1}(\pi(z))$ which extends across z equals -1. The above claim now follows by duality.

As a consequence, for each $u \in F$ with $\alpha(u) \in \Delta$, the induced trivialization of the pull-back bundle $\alpha^*\nu^*$ on $F - \alpha^{-1}(\Delta)$ has rotation number 1 or -1 at u depending on whether the diffeomorphism of a neighborhood of u in F onto a neighborhood of $\alpha(u)$ in $\pi^{-1}(\pi(u))$ is orientation preserving or orientation reversing. However, this just means that the degree of the line bundle $\alpha^*\nu^*$ on F equals the number of intersection points of $\alpha(F)$ with Δ counted with signs (and multiplicities- by the assumption on α, these multiplicities are all one). In other words, the degree of the bundle $\alpha^*\nu^*$ equals the intersection number $\alpha(F) \cdot \Delta$. □

Denoting again by ν the tangent bundle of the fibres, we obtain as an immediate corollary

Corollary 3.4. $c_1(\nu^*) \wedge c_1(\nu^*)(E) = \beta \cdot \beta = -c_1(\nu)(\beta)$.

Proof. The first equation follows from Poincaré duality for the bundle E, and the second equation is a consequence of the fact that $c_1(\nu) = -c_1(\nu^*)$. □

To calculate this intersection number we use the following more precise description of the cycle Δ.

Lemma 3.5. *Let $\Sigma \subset \Delta$ be the finite set of singular points. Then up to homotopy, there is a closed oriented surface F and a smooth embedding $f : F \to E$ with $f(F) = \Delta$. The map $\pi \circ f : F \to B$ is a cover of degree $2g - 2$ branched at every point in $f^{-1}(\Sigma)$. At each point $z \in \Sigma$ which corresponds to a positive (or negative) intersection point of $\theta(B)$ with $\mathcal{P}_1$, the oriented tangent plane of $f(F)$ equals the oriented tangent plane of the fibre (or the tangent plane of the fibre with the reversed orientation).*

Proof. Choose a complex structure for B. Let $x \in B$ be such that $\theta(x)$ is a positive transverse intersection point with $\mathcal{P}_1$. Then up to changing θ with a homotopy near x we may assume that there is a neighborhood W of x in B such that the restriction of θ to W is holomorphic.

Let $y \in \Sigma \subset \Delta$ be the double zero of the abelian differential $\theta(x) \in \mathcal{P}_1$. Note that since θ is holomorphic in a neighborhood U of x and the projection $P : \mathcal{P} \to \mathcal{M}(S_g)$ is holomorphic, the bundle $\pi^{-1}(U)$ is a holomorphic surface bundle. Choose holomorphic coordinates $(u, v) \in \mathbb{C}^2$ near y so that in these coordinates the projection $\pi : E \to B$ is the projection $(u, v) \to v$. We may assume that the coordinates (u, v) are defined on a set of the form $U \times V$ for open discs $U, V \subset \mathbb{C}$ centered at 0, that the singular point y corresponds to the origin 0 and that y is the only singular point in $U \times V$. Moreover, we may assume that for every $0 \neq v \in V$ the intersection $(U \times \{v\}) \cap \Delta$ consists of precisely two points.

Up to a biholomorphic coordinate change, power series expansion of the differential $\theta(v)$ ($v \in V$) on U in the coordinates (u,v) about the point 0 yields

$$\theta(v)(u) = \sum_{n=0}^{\infty} a_n(v)u^n du$$

with holomorphic functions $a_n : V \to \mathbb{C}$ which satisfy $a_0(0) = a_1(0) = 0$ and $a_2(0) = 1$. Thus up to a coordinate change by a biholomorphic map, locally near $(0,0) \in U \times V$ we have

$$\theta(v) = (b(v) + u^2)du$$

for a holomorphic function $b : V \to \mathbb{C}$ which vanishes at $v = 0$. Since the intersection between $\theta(B)$ and $\mathcal{P}_1$ is transverse, we have $b'(0) \neq 0$. This implies that up to another holomorphic coordinate change we may assume that $b(v) = v$ for all $v \in V$.

As a consequence, the zero set of $\theta(v)$ near $v = 0$ is given by the equation $v = -u^2$. But this just means that the image of a neighborhood Z of 0 in $\mathbb{C}$ under the holomorphic map $z \to (z, -z^2) \in U \times V$ is a neighborhood of y in $U \times V \cap \Delta$.

The above discussion shows that there is a disc $D \subset \mathbb{C}$ and a holomorphic embedding $\psi : D \to E$ whose image is a neighborhood of $y = \psi(0)$ in Δ. The projection $\pi \circ \psi$ is locally of the form $z \to -z^2$ and hence it has a branch point at 0. The tangent plane of $\psi(D)$ at z equals the tangent plane of the fibre with the usual orientation.

Similarly, if $\theta(x)$ is a point of negative intersection index then we may assume that θ is antiholomorphic near x. The same argument as before applies and yields the existence of an embedding $\psi : D \to E$ whose image is a neighborhood of $z = \psi(0)$ in Δ and such that the tangent plane of $\psi(D)$ at $\psi(0)$ equals the tangent plane of the fibre with the reversed orientation.

Since the complement of the singular set $\Sigma \subset \Delta$ is clearly a smoothly embedded surface in E which is transverse to the fibres, the surface $\Delta - \Sigma$ can be extended across the singular points by attaching a disc to the complement of a small neighborhood of a singular point in the way described above. Together we obtain a smooth closed surface F and a smooth injective map $f : F \to \Delta$ such that the map $\pi \circ f$ is a branched cover $F \to B$ whose branchpoints are precisely the singular points of Δ. This shows the lemma. □

Lemma 3.5 is used to show

Lemma 3.6. $\theta(B) \cdot \eta = 2\beta \cdot \beta = 2c_1(\nu^*)(\beta)$.

Proof. For simplicity we restrict the discussion to the case that every intersection point of θ with $\mathcal{P}_1$ has positive intersection index. This is for example the case if $\theta : B \to \mathcal{P}$ is holomorphic.

Let again $\Sigma \subset \Delta$ be the set of singular points. By Lemma 3.5 and its proof, there is a closed oriented surface F and a smooth embedding $f : F \to E$ so that $f(F) = \Delta$. The map $\pi \circ f$ is a covering branched at the preimages of the points in Σ. Up to a homotopy we may assume that near each singular point y there are local complex coordinates with y corresponding to the origin such that in these coordinates, the map f is of the form $z \to (z, -z^2)$. The projection π in these coordinates is just the second factor projection.

Since Σ is precisely the set of branch points of the map $\pi \circ f$, the Hurwitz formula shows that the tangent bundle of F can be represented in the form $f^*(\pi|\Delta)^*(TB \otimes (-H))$ where H is the line bundle on F with divisor $f^{-1}(\Sigma)$. Then the normal bundle N of $f(F)$ can be written as $N = \nu \otimes H$. This implies that the self-intersection number in E of the surface $f(F) \subset E$ equals $c_1(f^*\nu)(F) + b$ where

$$b = |\Sigma| = \theta(B) \cdot \eta$$

is the number of branch points of $\pi \circ f$.

By Poincaré duality (see Corollary 3.4), we have

$$c_1(\nu^*)(\beta) = \beta \cdot \beta = c_1(\nu)(\beta) + b = -c_1(\nu^*)(\beta) + b$$

and hence $b = 2c_1(\nu^*)(\beta)$. This completes the proof of the lemma. □

Proof of Theorem 3.1: The first tautological class $\kappa_1 \in H^2(\mathcal{M}(S_g), \mathbb{Q})$ is defined as follows [HaM98]. Let $\mathcal{C} \to \mathcal{M}(S_g)$ be the *universal curve* which is the quotient of the universal curve over Teichüller space $\mathcal{T}(S_g)$ under the action of the mapping class group. Let η be the first Chern class of the relative dualizing sheaf on $\mathcal{C}$. On fibres over smooth points $x \in \mathcal{M}(S_g)$, the relative dualizing sheaf is just the sheaf of sections of the holomorphic cotangent bundle ν^* of the fibres. Then

$$\kappa_1 = \Pi_*(\eta \wedge \eta)$$

where Π_* is the Gysin push-forward map obtained by intergration over the fibre [HaM98]. In particular, for any map $\varphi : B \to \mathcal{M}(S_g)$ which defines the surface bundle $E \to B$ we have

$$\kappa_1(\varphi(B)) = c_1(\nu^*) \wedge c_1(\nu^*)(E). \tag{3}$$

Now $\kappa_1 = 12\lambda$ as classes in $H^2(\mathcal{M}(S_g), \mathbb{Q})$ [HaM98] and hence by Proposition 3.3 and Lemma 3.6, for the proof of Theorem 3.1 we are left with calculating the contribution of the Chern class ψ of the tautological bundle. Since we do not need this is in the sequel, we only sketch the calculation.

First note that in the case $g = 2$, we have $\eta = P^*\lambda - 6\psi = -6\psi$. Namely, the complex dimension of the Hodge bundle equals 2 and hence the fibre of the bundle $\mathcal{P} \to \mathcal{M}(S_g)$ over a point x is just $\mathbb{C}P^1$.

A Weierstrass point on x is a double zero of a holomorphic one-form, and these are the only double zeros of holomorphic one-forms on x. Since x has $6 = 3\chi(S_2)$ Weierstrass points, we conclude that the intersection number of the fibre with the singular cycle $\mathcal{P}_1$ equals 6. Now the Chern class of the tautological bundle on $\mathbb{C}P^1$ equals -1 and hence the claim follows in this case.

If $g \geq 2$ is arbitrary then choose a surface $x \in \mathcal{M}(S_g)$ which admits an unbranched cover of degree d onto a surface $y \in \mathcal{M}(S_2)$. Note that $d = g - 1$. The two-sphere of projective holomorphic one-forms on y pulls back to a sphere of projective holomorphic one-forms on x. The preimage of a regular point is regular, but the preimage of a singular point is a holomorphic differential with d double zeros. With a local deformation we can deform this two-sphere of abelian differentials to a two-sphere of differentials with at most one double zero each without changing the total intersection index. As a consequence, the intersection number with $\mathcal{P}_1$ of this lifted sphere equals $6d = 6g - 6$. This completes the proof of Theorem 3.1. □

As an immediate consequence of the above discussion we can derive the corollary from the introduction. For its formulation, the *hyperelliptic locus* in $\mathcal{M}(S_g)$ is the space of all hyperelliptic complex structures. In the case $g = 2$, this is just the entire moduli space. The hyperelliptic mapping class group is essentially the pure braid group of $2g+2$ points in $\mathbb{C}P^1$, and it is well known [CLM76] that the second $\mathbb{Q}$-cohomology group of braid groups vanishes.

Corollary 3.7. *Let $E \to B$ be a surface bundle defined by a map $\varphi : B \to \mathcal{M}(S_g)$ with image in the hyperelliptic locus. Then $\sigma(E) = 0$.*

Proof. Choose a basepoint $x_0 \in B$ and a Weierstrass point $z_0 \in \pi^{-1}(x_0)$ in the fibre. Since Weierstrass points are distinct, every loop γ in B based at x_0 admits a unique lift to E beginning at z_0 whose image consists of Weierstrass points. The endpoint is another Weierstrass point in $\pi^{-1}(x_0)$ which only depends on the homotopy class of the loop. Thus this construction defines a homomorphism of $\pi_1(B)$ into the permutation group of the $2g+2$ Weierstrass points of $\pi^{-1}(x_0)$.

Let $\Gamma < \pi_1(B)$ be the kernel of this homomorphism and let B_0 be the finite cover of B with fundamental group Γ. Let moreover $E_0 \to B_0$ be the pull-back of E. Then $\sigma(E_0) = p\sigma(E)$ where p is the degree of the covering $B_0 \to B$ and hence it suffices to show that $\sigma(E_0) = 0$.

The homomorphism of $\pi_1(B_0)$ into the permutation group of $2g+2$ points defined above is trivial by construction and hence the bundle $E_0 \to B_0$ admits six pairwise disjoint sections whose images consist of Weierstrass points.

Let $\alpha_1, \dots, \alpha_{2g-2} : B_0 \to E_0$ be these sections. For each $x \in B$ the image of the $(2g+2)$-tuple of Weierstrass point $(\alpha_1(x), \dots, \alpha_{2g+2}(x))$ under the hyperelliptic involution is a $(2g+2)$-tuple $(y_1(x), \dots, y_{2g-2}(x))$ of points in $\mathbb{C}P^1$, well defined up to the action of $\mathrm{SL}(2, \mathbb{C})$. There is a quadratic differential $q(x)$ on $\mathbb{C}P^1$, unique up to a constant, which has a zero of order $2g-2$ at $y_1(x)$ and a single pole at each of the points $y_i(x)$ for $i \geq 2$. The pullback of $q(x)$ to x is a holomorphic quadratic differential on x with a single zero of order $4g-4$. The projective class of this differential depends smoothly on x and hence it defines a section of the bundle of holomorphic quadratic differentials. As a consequence, the cycle defined by $4g-4$ times the section α_1 is Poincaré dual to the first Chern class $c_1((\nu^*)^2)$ of $(\nu^*)^2 = \nu^* \otimes \nu^*$ (see the discussion in the proof of Proposition 3.3). Since the section α_1 is smooth, its self-intersection number equals $c_1(\nu)(\alpha_1)$ and therefore $c_1(\nu^*) \wedge c_1(\nu^*)(E) = \frac{1}{4} c_1((\nu^*)^2) \wedge c_1((\nu^*)^2) = 0$. On the other hand, equation (3) shows that

$$3\sigma(E) = c_1(\nu^*) \wedge c_1(\nu^*)(E)$$

whence the corollary. □

4. The signature as the Chern class of a line bundle

In Section 3 we showed that the cohomology class in $\eta \in H^2(\mathcal{P}, \mathbb{Q})$ which is Poincaré dual to the singular cycle $\mathcal{P}_1$ in the projectivized Hodge bundle $P : \mathcal{P} \to \mathcal{M}(S_g)$ is a sum of the class $24P^*(\lambda)$ and a multiple of the first Chern class of the tautological line bundle on the fibres.

In this section we construct explicitly a line bundle on $\mathcal{P}$ whose first Chern class equals the class $P^*(24\lambda)$. We show that the pullback of this line bundle to a bundle over $\mathcal{P}$ with discrete fibre is the tensor product of $2g-2$ flat line bundles. The theorem of the introduction then follows from a version of the Milnor Wood inequality.

To avoid complications due to surfaces with automorphisms we pass to the moduli space $\hat{\mathcal{M}}(S_g)$ of surfaces with level-3 structure. This means the following. Let Γ_g be the mapping class group of S_g and let $\hat{\Gamma}_g < \Gamma_g$ be the normal subgroup of finite index of all elements which act trivially on the first homology group $H_1(S_g,\mathbb{Z}/3\mathbb{Z})$. Then $\hat{\Gamma}_g$ is torsion free, and the quotient $\hat{\mathcal{M}}(S_g)=\mathcal{T}(S_g)/\hat{\Gamma}_g$ is a smooth algebraic variety with fundamental group $\hat{\Gamma}_g$. The universal curve with level 3-structure is the holomorphic surface bundle

$$\Pi:\mathcal{C}\to\hat{\mathcal{M}}(S_g)$$

whose fibre over x is the Riemann surface x.

Let $\mathcal{C}^{2g-2}\to\hat{\mathcal{M}}(S_g)$ be the bundle whose fibre over x is the product $\Pi^{-1}(x)^{2g-2}$ of $2g-2$ copies of the fibre $\Pi^{-1}(x)$ of the universal curve.

Lemma 4.1. *There are $2g-2$ globally defined component projections $\Pi_i:\mathcal{C}^{2g-2}\to\mathcal{C}$.*

Proof. Let $\tilde{\mathcal{C}}^{2g-2}\to\mathcal{T}(S_g)$ be the bundle over Teichmüller space $\mathcal{T}(S_g)$ whose fibre over x is the product of $2g-2$ copies of the Riemann surface x. Then $\tilde{\mathcal{C}}^{2g-2}$ is diffeomorphic to a product. The subgroup $\hat{\Gamma}_g$ of the mapping class group acts on the bundle $\tilde{\mathcal{C}}^{2g-2}$ by the diagonal action. Therefore the quotient manifold $\mathcal{C}^{2g-2}=\tilde{\mathcal{C}}^{2g-2}/\hat{\Gamma}_g$ has the property claimed in the lemma. □

By Lemma 4.1, the symmetric group Σ_{2g-2} in $2g-2$ elements acts on $\mathcal{C}^{2g-2}$ as a group of bundle automorphisms. The quotient under this action is the bundle $\mathcal{D}\to\hat{\mathcal{M}}(S_g)$ of divisors of degree $2g-2$ on Riemann surfaces $x\in\hat{\mathcal{M}}(S_g)$. Let

$$\alpha:\mathcal{C}^{2g-2}\to\mathcal{D}$$

be the canonical projection.

Denote again by

$$P:\mathcal{P}\to\hat{\mathcal{M}}(S_g)$$

the bundle over $\hat{\mathcal{M}}(S_g)$ whose fibre $\mathcal{P}_x$ over x is the space of projective holomorphic one-forms on x. The set of zeros of a projective abelian differential counted with multiplicities can be viewed as a point in the fibre $\mathcal{D}_x$ of $\mathcal{D}$ over x. Since up to multiplication with a non-zero complex number an abelian differential on a Riemann surface is determined by its set of zeros, this construction defines a bundle embedding $\mathcal{P}\to\mathcal{D}$. Let $\mathcal{P}_0\subset\mathcal{D}$ be the image of this embedding and let

$$\mathcal{Q}=\alpha^{-1}\mathcal{P}_0\subset\mathcal{C}^{2g-2}.$$

Clearly $\mathcal{Q}$ is a closed Σ_{2g-2}-invariant subset of $\mathcal{C}^{2g-2}$.

To avoid having to calculate the contribution of the first Chern class of the tautological bundle on $\mathcal{P}$ let

$$A:\mathcal{S}\to\hat{\mathcal{M}}(S_g)$$

be the sphere bundle in the Hodge bundle over $\hat{\mathcal{M}}(S_g)$. Then $\mathcal{S}$ admits a free cicle action with quotient $\mathcal{P}$. The singular divisor $\mathcal{P}_1$ lifts to a cycle $\mathcal{S}_1\subset\mathcal{S}$. Theorem 3.1 shows that the homology class of this cycle is Poincaré dual to the class $A^*(24\lambda)$.

The pullback $(\alpha|\mathcal{Q})^*\mathcal{S}$ to $\mathcal{Q}$ of the circle bundle $\mathcal{S}\to\mathcal{P}$ is a circle bundle $\mathcal{Z}\to\mathcal{Q}$. The pullback of the singular cycle $\mathcal{S}_1$ of $\mathcal{S}$ is a cycle $\mathcal{Z}_1$ in $\mathcal{Z}$ which defines an integral homology class χ in $\mathcal{Z}$.

By Lemma 4.1, there are $2g-2$ bundle projections $\Pi_i:\mathcal{C}^{2g-2}\to\mathcal{C}$ $(1\leq i\leq 2g-2)$. These projections induce projections $\mathcal{Z}\to\mathcal{C}$ which we denote by the same symbol.

View $\mathcal{S}$ as a circle bundle over $\mathcal{P}_0 \subset \mathcal{D}$. The map

$$\alpha : \mathcal{Z} \to \mathcal{S}$$

maps $\mathcal{Z}_1$ to the singular cycle $\mathcal{S}_1$, and it is branched at $\mathcal{Z}_1$. Therefore the pullback to $\mathcal{Z}$ of a line bundle on $\mathcal{S}$ whose Chern class is Poincaré dual to the homology class defined by the singular cycle $\mathcal{S}_1 \subset \mathcal{S}$ is a line bundle on $\mathcal{Z}$ whose Chern class is Poincaré dual to twice the homology class defined by the singular cycle in $\mathcal{Z}$. We use this fact to show the following

Proposition 4.2. *The first Chern class of the line bundle $L = \otimes_i \Pi_i^* \nu^*$ on $\mathcal{Z}$ equals $(\alpha \circ A)^*(12\lambda)$.*

Proof. Since $H^2(\mathcal{S}, \mathbb{Q}) = \mathbb{Q}$, the Chern class of the line bundle L is a multiple of $\alpha \circ A^*(\lambda)$.

To calculate the multiplicity, let B be a closed surface and let $\theta : B \to \mathcal{P}$ be a holomorphic map. Via changing θ with a homotopy we may assume that $\theta(B)$ intersects the singular cycle $\mathcal{P}_1$ transversely in finitely many points. The index of each intersection point is positive.

Let $Y \subset \mathcal{D}$ be the image of $\theta(B)$ under the embedding $\mathcal{P} \to \mathcal{D}$. Let $\hat{Y} \subset \mathcal{C}^{2g-2}$ be the preimage of Y in $\mathcal{C}^{2g-2}$. The map $\hat{Y} \to Y$ is a covering branched at the singular set $\theta(B) \cap \mathcal{P}_1$.

Let $\pi : E \to B$ be the surface bundle defined by $P \circ \theta$ and let $\Delta \subset E$ be the cycle defined in Lemma 3.2. By Proposition 3.3, the intersection number $\theta(B) \cdot \mathcal{P}_1$ equals $2c_1(\nu^*)(\Delta)$.

By Lemma 3.5, up to changing Δ with a homotopy we may assume that there is a smooth surface F and a smooth map $\psi : F \to \Delta$ with the following properties.

(1) The composition $\pi \circ \psi : F \to B$ is a covering of degree $2g - 2$ branched at each point in the singular set $\Sigma \subset \Delta$.
(2) $\psi : F \to E$ is an immersion with singular points precisely at the points in Σ. Near each singular point, in suitable complex coordinates the map $\pi \circ \psi$ is of the form $z \to z^2$.
(3) ψ has a tangent plane everywhere, and this tangent plane is transverse to the tangent bundle ν of the fibres.

The third property just means that there is a continuous section of the pullback of the bundle of real oriented 2-planes in TE which restricts to the tangent bundle of $\psi(F)$ at regular points.

Since $\hat{Y} \to Y$ is a covering branched at the intersection points of $\theta(B)$ with $\mathcal{P}_1$, there is a smooth surface $\hat{F}$, a smooth map $\hat{\psi} : \hat{F} \to \hat{Y}$ and a branched covering $\hat{F} \to F$ whose composition with ψ is just the composition of the map $\hat{\psi}$ with the projection $\hat{Y} \to Y$. The restriction of $\hat{\psi}$ to the preimage of the regular set of points with no two coordinates equal is a covering of degree $2g - 2$.

The tensor product bundle $L = \otimes_i \Pi_i^* \nu^*$ is defined on $\hat{Y}$. We have to show that

$$c_1(\hat{\psi}^* \nu^*)(\hat{F}) = c_1(\otimes \Pi_i^* \nu^*)(\hat{Y}). \tag{4}$$

Namely, by naturality of Chern classes under pullback and by the discussion in Section 3, the value on the left hand side of this equation just equals $(\alpha \circ A)^*(12\lambda)(\hat{Y})$.

To show the identity (4), choose a smooth connection Θ for the line bundle ν^* on E which is flat near the critical set Σ of Δ. Let κ be the curvature form of this connection. Then

$$c_1(\hat{\psi}^*\nu^*)(\hat{F}) = \int_{\hat{F}} \hat{\psi}^*\kappa(\hat{F}).$$

The connection Θ defines a connection on each of the line bundles $\Pi_i^*\nu^*$, with curvature form $\kappa_i = \Pi_i^*\kappa$. Then $\sum_i \kappa_i$ is the curvature form of a connection for $\otimes\Pi_i^*\nu^*$. Now by construction, we have

$$\int_{\hat{F}} \psi^*\kappa = \int_{\hat{Y}} \sum_i \kappa_i$$

which shows the proposition. □

Let $\mathrm{Top}^+(S^1)$ be the group of orientation preserving homeomorphisms of the circle S^1. Recall that a circle bundle H over a space $\mathcal{Z}$ is flat if it can be represented in the following form. Let $\tilde{\mathcal{Z}}$ be the universal covering of $\mathcal{Z}$. Then

$$H = \pi_1(\mathcal{Z})\backslash\tilde{\mathcal{Z}} \times S^1$$

where $\pi_1(\mathcal{Z})$ acts as the group of decktransformations on $\tilde{\mathcal{Z}}$, and it acts on S^1 through a homomorphism $\pi_1(\mathcal{Z}) \to \mathrm{Top}^+(S^1)$.

Let $H_i \to \mathcal{Z}$ be the circle subbundle of $L_i = \Pi_i^*\nu^*|\mathcal{Z}$ $(i = 1, \dots, 2g-2)$. The next observation follows from the discussion in the proof of Proposition 2.2.

Lemma 4.3. *For each i the circle bundle H_i on $\mathcal{Z}$ is flat.*

Proof. Note first that the pullback of a flat circle bundle by a smooth map is flat. Now H_i is a restriction of a pullback of the circle subbundle V of the unit cotangent bundle of the fibres of the universal curve $\mathcal{C} \to \hat{\mathcal{M}}(S_g)$. Hence it suffices to show that the circle bundle $V \to \mathcal{C}$ is flat.

The universal covering $\tilde{\mathcal{C}}$ of $\mathcal{C}$ is a bundle over Teichmüller space $\mathcal{T}(S_g)$ whose fibre over a point x is the hyperbolic plane. By a result of Bers, this bundle can be identified with the Teichmüller space $\mathcal{T}(S_{g,1})$ of surfaces of genus g with one marked point. The mapping class group $\Gamma_{g,1}$ of surfaces of genus g with one marked point acts on $\tilde{\mathcal{C}}$ by fibrewise isometries. In particular, it preserves the circle subbundle of the tangent bundle of the fibre.

As in the proof of Proposition 2.2, this implies that the circle subbundle W of the tangent bundle of the fibres of $\mathcal{C}$ is flat. Namely, a point in $\tilde{\mathcal{C}}$ is a marked Riemann surface together with a marked point. The marking and the marked point can be used as in the proof of Proposition 2.2 to identify the fibre of W at that point with the boundary of the hyperbolic plane. The action of $\Gamma_{g,1}$ on this boundary as described in the proof of Proposition 2.2 defines a homomorphism $\Gamma_{g,1} \to \mathrm{Top}^+(S^1)$ which induces W.

By duality, the circle subbundle V of the cotangent bundle ν^* of the fibres is flat as well. The lemma follows. □

For the proof of the theorem from the introduction, we will use a variant of the Milnor Wood inequality which we are going to explain now.

Equip the group $\mathrm{Top}^+(S^1)$ with the topology of uniform convergence. There is a retraction $\hat{r} : \mathrm{Top}^+(S^1) \to S^1$ which is defined as follows. Write $S^1 = [0,1]/\sim$ where $\sim$ identifies the points 0 and 1. View the image of 0 as a basepoint in S^1 and put

$$\hat{r}(f) = \int_0^1 (f(t) - t)dt.$$

In particular, the fundamental group of $\mathrm{Top}^+(S^1)$ is infinite cyclic [W71].

Let $\mathrm{Top}^+(S^1)^{2g-2}$ be the direct product of $2g-2$ copies of the group $\mathrm{Top}^+(S^1)$. Then $\mathrm{Top}^+(S^1)^{2g-2}$ is a subgroup of the group $\mathrm{Top}^+((S^1)^{2g-2})$ of orientation preserving homeomorphisms of the torus $(S^1)^{2g-2}$. The retraction $\hat{r}$ extends to a retraction $r : \mathrm{Top}^+(S^1)^{2g-2} \to (S^1)^{2g-2}$.

The symmetric group Σ_{2g-2} in $2g-2$ elements acts on $(S^1)^{2g-2}$ from the left as a group of orientation preserving homeomorphisms. The subgroup $\mathrm{Top}^+(S^1)^{2g-2}$ of $\mathrm{Top}^+((S^1)^{2g-2})$ is invariant under the action of Σ_{2g-2} by conjugation. Thus $\mathrm{Top}^+(S^1)^{2g-2}$ is a normal subgroup of the group $\mathcal{L} < \mathrm{Top}^+((S^1)^{2g-2})$ generated by $\mathrm{Top}^+(S^1)^{2g-2}$ and Σ_{2g-2}. More precisely, there is a split exact sequence

$$0 \to \mathrm{Top}^+(S^1)^{2g-2} \to \mathcal{L} \to \Sigma_{2g-2} \to 0.$$

The retraction r is equivariant with respect to the Σ_{2g-2}-action on $\mathrm{Top}^+(S^1)^{2g-2}$ by conjugation and on $(S^1)^{2g-2}$ by homeomorphisms. In particular, the fundamental group of the quotient space $\Sigma_{2g-2}\backslash\mathrm{Top}^+(S^1)^{2g-2}$ equals the fundamental group of the quotient space $\Sigma_{2g-2}\backslash(S^1)^{2g-2}$. We have

Lemma 4.4. *$\pi_1(\Sigma_{2g-2}\backslash(S^1)^{2g-2}) = \mathbb{Z}$.*

Proof. $\Sigma_{2g-2}\backslash(S^1)^{2g-2}$ admits a retraction onto S^1 (see [K] for a short proof and references). □

The product of the circle subbundles H_i of the line bundles L_i is a bundle $H \to \mathcal{Z}$ with fibre $S^1 \times \cdots \times S^1$. Lemma 4.3 implies that the bundle H can be represented in the form

$$H = (\prod \rho_i)\backslash\tilde{\mathcal{Z}} \times (S^1 \times \cdots \times S^1)$$

where for each i, $\rho_i : \pi_1(\mathcal{Z}) \to \mathrm{Top}^+(S^1)$ is a homomorphism.

The symmetric group Σ_{2g-2} acts from the left on $\mathcal{Z}$ as a group of homeomorphisms. Chooose a basepoint $z_0 \in \mathcal{Z}$ which is fixed by Σ_{2g-2}. Such a basepoint corresponds to a tuple of points in the fibre with all coordinates equal. It exists since there are abelian differentials with a single zero. With respect to this choice of basepoint, the group Σ_{2g-2} acts as a group of automorphisms on $\pi_1(\mathcal{Z})$. In other words, there is an extension

$$\mathcal{J} = \pi_1(\mathcal{Z}) \ltimes \Sigma_{2g-2}$$

of the group $\pi_1(\mathcal{Z})$ by Σ_{2g-2}. The group multiplication in $\mathcal{J}$ is given by

$$(\alpha_1, \sigma_1) \cdot (\alpha_2, \sigma_2) = (\alpha_1\sigma_1(\alpha_2), \sigma_1\sigma_2). \tag{5}$$

The group $\mathcal{J}$ acts on $\tilde{\mathcal{Z}}$, and the quotient of $\tilde{\mathcal{Z}}$ under this action is the bundle $\mathcal{S} \to \hat{\mathcal{M}}(S_g)$. Note that the action of $\mathcal{J}$ on $\tilde{\mathcal{Z}}$ is not free.

Recall the extension

$$\mathcal{L} = \mathrm{Top}^+(S^1)^{2g-2} \ltimes \Sigma_{2g-2}$$

of the group $\mathrm{Top}^+(S^1)^{2g-2}$. We may assume that the multiplication rule is given by (5). The homomorphism $\rho = \prod \rho_i : \pi_1(\mathcal{Z}) \to \mathrm{Top}^+(S^1)^{2g-2}$ extends to a homomorphism $\rho_{\mathcal{J}} : \mathcal{J} \to \mathcal{L}$ and hence we can form the quotient space

$$\mathcal{R}_0 = \mathcal{J}\backslash\tilde{\mathcal{Z}} \times (S^1)^{2g-2}.$$

The group Σ_{2g-2} also acts on $(S^1)^{2g-2}$ from the right. Thus we can define the double coset space

$$\mathcal{R} = \mathcal{R}_0/\Sigma_{2g-2} = (\mathcal{J}\backslash\tilde{\mathcal{Z}} \times (S^1)^{2g-2})/\Sigma_{2g-2}.$$

Lemma 4.5. *$\mathcal{R}$ is a fibre bundle over $\mathcal{S}$ with fibre $(S^1)^{2g-2}/\Sigma_{2g-2}$.*

Proof. Since the action of Σ_{2g-2} on $\mathcal{Z}$ and on $(S^1)^{2g-2}$ is by permutations of the factors, the space $\mathcal{R}_0 = \mathcal{J}\backslash\tilde{\mathcal{Z}} \times (S^1)^{2g-2}$ projects onto $\mathcal{S} = \Sigma_{2g-2}\backslash\mathcal{Z}$.

The left action of the group Σ_{2g-2} preserves the regular set $\mathcal{Z}-\mathcal{Z}_1$. The restiction of the Σ_{2g-2}-action to $\mathcal{Z} - \mathcal{Z}_1$ is free. As a consequence, the preimage of $\mathcal{S} - \mathcal{S}_1$ under the projection $\mathcal{R}_0 \to \mathcal{S}$ is a fibration with fibre $(S^1)^{2g-2}$. The right action of Σ_{2g-2} on the fibre $(S^1)^{2g-2}$ induces a fibre preserving action on this $(S^1)^{2g-2}$-bundle. The quotient under this action is a fibre bundle over $\mathcal{S} - \mathcal{S}_1$ with fibre $(S^1)^{2g-2}/\Sigma_{2g-2}$.

This reasoning also applies to the singular set $\mathcal{Z}_1$. The singular set is stratified into strata defined by the configurations of equal coordinates. The action of Σ_{2g-2} on $\mathcal{Z}_1$ from the left preserves this stratification. The induced action on strata is not effective. The fibre of the induced bundle over a stratum is the quotient of the space $(S^1)^{2g-2}$ under the kernel of the homomorphisms of Σ_{2g-2} into the group of homeomorphisms of the stratum.

As a consequence, $\mathcal{R}_0$ can be decomposed into strata. Each of the strata is a fibre bundle over a stratum in $\mathcal{S}$. The fibre of each such bundle is a quotient of the generic fibre $(S^1)^{2g-2}$ under a subgroup of Σ_{2g-2}. The right action of Σ_{2g-2} preserves the strata and projects the fibre to $(S^1)^{2g-2}/\Sigma_{2g-2}$. This shows the lemma. □

We use Lemma 4.5 to establish the following

Corollary 4.6. *Let K be a compact two-dimensional cell complex and let $\psi : K \to \mathcal{S}$ be a smooth map. Then $12A^*(\lambda)(\psi(K))$ is the first rational obstruction to the existence of a cross section of the bundle $\psi^*\mathcal{R}$.*

Proof. The first obstruction to the existence of a cross section of the bundle $\mathcal{R}$ over $\mathcal{S}$ is a cohomology class

$$\zeta \in H^2(\mathcal{S}, \pi_1((S^1)^{2g-2}/\Sigma_{2g-2})) = H^2(\mathcal{S}, \mathbb{Z}).$$

Tensoring with $\mathbb{Q}$ yields a class in $H^2(\mathcal{S}, \mathbb{Q})$. Since $H^2(\mathcal{S}, \mathbb{Q})$ is one-dimensional, this is a multiple of the class λ.

To calculate the multiplicity it suffices to evaluate this class on the image of a compact connected two-dimensional cell complex K under a map $\psi : K \to \mathcal{S}$ which factors through a map $\hat{\psi} : K \to \mathcal{Z}$. However, in this case the evaluation is just the first obstruction to the existence of a cross section of the quotient of the bundle $H \to \mathcal{Z}$ under the right action of Σ_{2g-2}, and this obstruction is just the first Chern class of the line bundle L. The claim now follows from Proposition 4.2. □

We are left with estimating the obstruction in Corollary 4.6. For this choose once and for all a basepoint $x_0 \in S^1$. Let $O_0 \subset (S^1)^{2g-2}/\Sigma$ be the projection of the set of all points in $(S^1)^{2g-2}$ which contain at least one coordinate equal to x_0. Let $O_1 \subset O_0$ be the projection of the set of all points which have precisely one coordinate equal to x_0. Then O_1 is a submanifold of $(S^1)^{2g-2}/\Sigma_{2g-2}$ of real codimension one. Locally it divides $(S^1)^{2g-2}/\Sigma_{2g-2}$ into two sides, one to the right and the other to the left of O_1. In the preimage $(S^1)^{2g-2}$, the sides are determined as follows. If the first component of $y \in (S^1)^{2g-2}$ equals x_0, then a point z with components $z_i = y_i$ for $i \geq 2$ and so that the first coordinate is close to x_0 and to the right of x_0 with respect to the orientation of S^1 projects to a point to the right of O_1.

To a path $\gamma : [0,1] \to (S^1)^{2g-2}/\Sigma_{2g-2}$ we associate an index $\mu(\gamma, x_0)$ as follows. Call a point $t \in [0,1]$ a *crossing* if $\gamma(t) \in O_1$. The crossing is *transverse* if γ intersects the submanifold O_1 transversely at $\gamma(t)$. We say that a crossing is *positive* if $\gamma(s)$ is to the right of O_1 for all $s < t$ sufficiently close to t, and we call the crossing *negative* otherwise. There is an obvious modification of this definition at the endpoints of γ.

Assume first that the path γ meets $O_0 - O_1$ at most at its endpoints, and it intersects O_1 transversely in finitely many points. Let $a \in \mathbb{Z}$ be the number of crossings of γ at interior points of $(0,1)$, counted with sign, and let $b \in \frac{1}{2}\mathbb{Z}$ be half of the contributions of the endpoints, counted with sign and multiplicities. Define $\mu(\gamma, x_0) = a + b$.

Two such paths which are homotopic with fixed endpoints have the same index. Hence this definition defines an index for an arbitrary path by modifying this path with a homotopy to a path which is only intersect $O_0 - O_1$ at its endpoints and so that all crossings of O_1 are transverse.

Choose the point $(x_0, \ldots, x_0)$ as the basepoint for the fundamental group of $(S^1)^{2g-2}/\Sigma_{2g-2}$.

Lemma 4.7. *The index $\mu(\gamma, x_0)$ restricts to an isomorphism $\pi_1(S^1)^{2g-2}/\Sigma_{2g-2} \to \mathbb{Z}$.*

Proof. By construction, $\mu(\cdot, x_0)$ clearly defines a homomorphism

$$\pi_1((S^1)^{2g-2}/\Sigma_{2g-2}, (x_0, \ldots, x_0)) \to \mathbb{Z}.$$

Thus we only have to show that μ maps a generator of the fundamental group to ± 1. The projection of a loop in $(S^1)^{2g-2}$ with $2g-3$ fixed coordinates which moves the remaining component one full rotation about S^1 is such a generator. It is mapped by μ to ± 1. □

Let $\widetilde{\mathrm{Top}}^+(S^1)^{2g-2}$ be the universal covering of $\mathrm{Top}^+(S^1)^{2g-2}$. An element in $\widetilde{\mathrm{Top}}^+(S^1)^{2g-2}$ is the homotopy class of a path in $\mathrm{Top}^+(S^1)^{2g-2}$ issuing from the identity. The image of the point $(x_0, \ldots, x_0)$ under such a path is a path in $(S^1)^{2g-2}$.

Now let $\tilde{\alpha} \in \widetilde{\mathrm{Top}}^+(S^1)^{2g-2}$ and let $\zeta : [0,1] \to \mathrm{Top}^+(S^1)^{2g-2}$ be a path issuing from the identity which represents $\tilde{\alpha}$. Then $t \to \zeta(t)(x_0, \ldots, x_0)$ is a path in $(S^1)^{2g-2}$ which projects to a path in $(S^1)^{2g-2}/\Sigma_{2g-2}$. Another representative of $\tilde{\alpha}$ projects to a homotopic path in $(S^1)^{2g-2}/\Sigma_{2g-2}$. Thus the index $\mu(\tilde{\alpha}, x_0)$ of this projected path only depends on $\tilde{\alpha}$ but not on the representative ζ.

The action of the group Σ_{2g-2} on $(S^1)^{2g-2}$ determines an extension of the group $\widetilde{\mathrm{Top}}^+(S^1)^{2g-2}$ as a semi-direct product

$$\tilde{\mathcal{L}} = \widetilde{\mathrm{Top}}(S^1)^{2g-2} \ltimes \Sigma_{2g-2}.$$

By invariance, we define an index $\mu(\tilde{\alpha}, x_0)$ for any $\tilde{\alpha} \in \tilde{\mathcal{L}}$. In other words, this construction defines a function

$$\mu : \tilde{\mathcal{L}} \to \mathbb{Z}.$$

The following is fairly immediate from the definitions (compare [W71] for a similar reasoning).

Lemma 4.8. *(1)* $\mu(\gamma^{-1},x_0)=-\mu(\gamma,x_0)$.
(2) $|\mu(\gamma_1\circ\gamma_2,x_0)-\mu(\gamma_1,x_0)-\mu(\gamma_2,x_0)|\leq g-1$ *for a concatenation of elements* $\gamma_1,\gamma_2\in\tilde{\mathcal{L}}$.

Proof. The first part of the lemma is immediate from the definitions.

To show the second part, let $\gamma_1,\gamma_2\in\tilde{\mathcal{L}}$ and let $\tilde\gamma_i$ be paths in $\mathrm{Top}(S^1)^{2g-2}$ which define the first component of γ_i in the decomposition of $\gamma_i\in\widetilde{\mathrm{Top}}^+(S^1)^{2g-2}\ltimes\Sigma_{2g-2}$. Write $\zeta_i(t)=\tilde\gamma_i(t)(x_0,\dots,x_0)$ $(i=1,2)$ and $\xi(t)=\gamma_2\circ\gamma_1(x_0,\dots,x_0)$. Then the image of $(x_0;\dots,x_0)$ under the path ξ is obtained as follows. First let $(y_1,\dots,y_{2g-2})$ be the image of $(x_0,\dots,x_0)$ under the path $\tilde\gamma_1$. After a permutation of the components which is determined by the second component of γ_i, move $(y_1,\dots,y_{2g-2})$ by the path $\tilde\gamma_2$ and permute once more some components.

We claim that for any path $\beta\in\mathrm{Top}^+(S^1)^{2g-2}$, the indices of the paths $t\to\beta(t)(x_0,\dots,x_0)$ and $t\to\beta(t)(y_1,\dots,y_{2g-2})$ differ by at most $g-1$.

To this end consider first the case of one single factor S^1. Let y be the endpoint of the path $\zeta:t\to\gamma_2(t)x_0$ and assume that $y\neq x_0$. Via modifying γ_1 by a homotopy we may assume that all crossing of x_0 of the path ζ have the same sign. Assume for simplicity that these crossings are all positive. Let $m=\mu(\gamma_1,x_0)$. Note that $m-1/2\in\mathbb{Z}$ if $\gamma_1(x_0)\neq x_0$.

Let $t_1\in(0,1]$ be the first crossing time of the path $t\to\gamma_1(t)x_0$ through x_0. Since γ_1 is a path of orientation preserving homeomorphisms, there must be some $s_1\in(0,t_1)$ which is a crossing of y through x_0, and this crossing is positive. Inductively one concludes that the number of crossings of y either equals $m-1/2$ or $m+1/2$. If $\gamma_1(x_0)=x_0$ then the number of crossings equals m. Thus the absolute value of the difference between the indices of the paths issuing from x_0 and y is at most $1/2$.

This reasoning extends without changes to the case of an arbitrary number of factors and implies the lemma. ☐

Now let B be a closed surface of genus $h\geq 2$. Then B is a cell complex which can be obtained from a $4h$-gon by the standard side identifications. The edges of this $4h$-gon define standard generators α_i,β_i of $\pi_1(B)$ $(i=1,\dots,h)$ with the relation

$$[\alpha_1,\beta_1]\dots[\alpha_h,\beta_h]=0.$$

Let $\psi:B\to\mathcal{S}$ be a smooth map. For each i choose lifts of α_i,β_i to $\pi_1(\mathcal{Z})\ltimes\Sigma_{2g-2}=\mathcal{J}$. Via the homomorphism $\rho_{\mathcal{J}}$ these lifts define elements in

$$\mathcal{L}=\mathrm{Top}^+(S^1)^{2g-2}\ltimes\Sigma_{2g-2}$$

which can be lifted to elements $\hat\alpha_i,\hat\beta_i\in\tilde{\mathcal{L}}$. Each of these elements determines a path in $(S^1)^{2g-2}/\Sigma_{2g-2}$ to which we can associate its index.

The following is immediate from Lemma 4.8.

Corollary 4.9.
$$|\mu(\hat\alpha_1\circ\hat\beta_1\circ\dots\circ\hat\alpha_h^{-2}\circ\hat\beta_h^{-1})|\leq 2h.$$

From this observation, the theorem from the introduction easily follows. Namely, let again B be a closed surface of genus $h\geq 2$ and let $E\to B$ be a surface bundle over B with fibre S_g. Since the index of the subgroup $\hat\Gamma_g$ of Γ_g is finite, up to passing to a finite cover of B and pulling back the bundle E which does not change the ratio $3\sigma(E)/\chi(E)$, we may assume that E is induced by a smooth map $\varphi : B \to \hat{\mathcal{M}}(S_g)$.

Since $\mathcal{S} \to \hat{\mathcal{M}}(S_g)$ is a sphere bundle with fibre of real dimension $2g-1$, the map φ can be lifted to a smooth map $\psi : B \to \mathcal{S}$. Via modifying ψ with a homotopy we may assume that $\psi(B)$ meets the singular cycle $\mathcal{S}_1$ transversely at isolated points. The composition $A \circ \psi : B \to \hat{\mathcal{M}}(S_g)$ of ψ with the projection $A : \mathcal{S} \to \hat{\mathcal{M}}(S_g)$ defines the surface bundle $E \to B$.

The first obstruction to the existence of a cross section of the bundle $\psi^*\mathcal{R}$ is just the index of the concatenated paths. This obstruction equals $12\lambda(\varphi(B)) = \kappa_1(\varphi(B))$. The above calculation then shows that

$$3\sigma(E) \leq |(2g-2)2h| \tag{6}$$

where as before, $E \to B$ is the surface bundle over B defined by the map ψ.

Now the signature and Euler characteristic are multiplicative under unbranched covers of B. An application of inequality (6) to unbranched covers of B of larger and larger degree implies the statement of the theorem from the introduction.

For $n \in \mathbb{Z}$ and $g \geq 3$ let

$$h_g(n) = \min\{h \mid \text{ there is an } S_g - \text{bundle } E \to S_h \text{ with } \sigma(E) = 4n\}.$$

Our main theorem shows that

$$h_g(n) \geq \frac{3|n|}{g-1} + 1.$$

In [EKKOS02] the upper bound

$$\lim_{n\to\infty} \frac{h_g(n)}{n} \leq \frac{16}{g-1}$$

was established. Bryan and Donagi [BD02] constructed for every $g \geq 2, n \geq 2$ a complex surface bundle E over a surface with

$$\frac{3\sigma(E)}{\chi(E)} = \frac{n^2-1}{n(gn-1)}.$$

In particular, for $g = 2, n = 3$ one obtains a surface bundle $E \to B$ with $\frac{3\sigma(E)}{\chi(E)} = \frac{8}{15} > \frac{1}{2}$. The fibre genus of this surface bundle equals 91.

To summarize, to date there is a gap (roughly of a factor 2) between an upper bound for the ratio $\frac{|\sigma(E)|}{\chi(E)}$ and the ratio of known examples. This can be seen in the context of surface group representations as follows.

The action of the mapping class group Γ_g on the first homology of the surface S_g defines a surjective representation $\Lambda : \Gamma_g \to Sp(2g, \mathbb{Z})$. The signature of a bundle $E \to B$ defined by a homomorphism $\varphi : \pi_1(B) \to \Gamma_g$ then equals four times the Tolodo invariant of the representation $\Lambda \circ \varphi : \pi_1(B) \to Sg(2g, \mathbb{R})$.

Now the Toledo invariant of a representation of the fundamenatal group of a surface of genus h into $Sp(2g, \mathbb{R})$ is bounded in absolute value by $(h-1)g$. This bound is sharp, and a representation which assumes this bound is called maximal. The main result of this note shows that the Toledo invariant for a representation $\pi_1(B) \to Sp(2g, \mathbb{R})$ which factors through the mapping class group is bounded in absolute value by $\frac{1}{3}(g-1)(h-1)$.

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