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ADDENDUM TO NEW EXAMPLES OF MAXIMAL SURFACES

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Consider the Teichmüller space $\mathcal{T}_{g,3}$ of hyperbolic surfaces of genus $g \geq 2$ with three punctures. The *systole function* on $\mathcal{T}_{g,3}$ associates to a surface $S \in \mathcal{T}_{g,3}$ its *systole*, i.e. the length of its shortest closed geodesic. A surface S is called *maximal* if the systole function has a local maximum at S.

Let $S \in \mathcal{T}_{g,3}$ and let $\gamma_1, \dots, \gamma_k$ be the *systoles* of S, i.e. the shortest closed geodesics on S. Each of these systoles is a simple closed curve. Denote by ℓ_{γ_i} the length function of γ_i, viewed as a function on $\mathcal{T}_{g,3}$. Following Theorem 1.2 of Schmutz [S93], the surface S is maximal if and only if the following two conditions hold true.

(1) The differentials of the k length functions $d\ell_{\gamma_i}$ span the cotangent space of $\mathcal{T}_{g,3}$ at S.
(2) The function $X \in \mathcal{T}_{g,3} \to \min\{\ell_{\gamma_i}(X) \mid 1 \leq i \leq k\}$ has a local maximum at S.

Corollary 5.5 of [H01] states that three explicit triangle surfaces $S(7;3)$ in genus 3, $S(13;4)$ in genus 6 and $S(21,5)$ in genus 10 are maximal.

The proof of this result is based on Corollary 5.3 of that article which shows that indeed, property (1) above is fulfilled for these surfaces.

The proof of property (2) is erroneously omitted and is provided here. We use the following

Lemma 0.1. *Let $f_1, \dots, f_k$ be smooth functions on $\mathbb{R}^n$. Assume that $f_i(0) = a > 0$ independent of i and that moreover the convex hull of the differentials $df_i(0)$ of f_i at 0 contains 0 in its interior. Then the function $x \to \min\{f_i(x) \mid i \leq k\}$ has a local maximum at 0.*

Proof. The condition in the lemma is equivalent to stating that for every vector $0 \neq X \in T_0\mathbb{R}^n$ there exists some $i \leq k$ such that $df_i(X) < 0$. As a consequence, $\min\{f_i(x) \mid i \leq k\}$ locally strictly decreases along any ray issuing from 0. The statement of the lemma now follows from continuity. □

Now let $p \geq 5$ be an odd number and let Λ be the linear isometry of $\mathbb{R}^p$ defined in canonical coordinates by $\Lambda(x_1, \dots, x_p) = (x_2, \dots, x_p, x_1)$, and let $\Lambda_3 = \Lambda \times \Lambda \times \Lambda$ be the diagonal isometry of $\mathbb{R}^{3p}$. Let furthermore τ by the linear isometry of $\mathbb{R}^{3p}$ which cyclicly permutes the factors $\mathbb{R}^p$ in the direct decomposition of $\mathbb{R}^{3p}$. Denote by G be the subgroup of the isometry group of $\mathbb{R}^{3g}$ generated by Λ and τ.

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The eigenvalues of Λ are the p-th roots of unity. The eigenspace for the eigenvalue 1 is spanned by $(1,\dots,1)$, and the other generalized eigenspaces are of dimension 2. Let Z be the eigenspace of Λ^3 for the eigenvalue one, and let $Z^\perp$ be its orthogonal complement. The space $Z^\perp$ decomposes into $g=(p-1)/2$ invariant subspaces of dimension 6 each corresponding to the $p-1$ distinct eigenvalues of Λ different from one. The group G acts on the dual $(\mathbb{R}^{3p})^*$ of $\mathbb{R}^{3p}$ as a group of isometries for the obvious invariant inner product.

Lemma 0.2. *Let $f_1,\dots,f_k$ be a G-invariant set of real linear functional on $\mathbb{R}^{3p}$ whose differentials span $Z^\perp$. Then the convex hull of the restriction of f_i to $Z^\perp$ contains 0 in its interior.*

Proof. Let us assume to the contrary that there is a closed halfspace in $(Z^\perp)^*$ such that $df_i \in H$ for all i. Denote by H the intersection of all such half-spaces. By invariance of the set df_i under the group G, the closed convex set $H \subset (Z^\perp)^*$ is invariant under G, furthermore its interior is not empty.

Let $D \subset S^{3p-7}$ be the intersection of H with the unit sphere in $(Z^\perp)^*$. Then D is a compact convex G-invariant subset of S^{3p-7} with non-empty interior which is contained in a hemisphere. As a hemisphere in S^{3p-7} is convex for the round metric, the set D has a unique center of mass v. Now G acts as a group of isometries on $(Z^\perp)^*$ preserving D and hence G preserves the center of mass v. Then v is an eigenvector for the generators of G for the eigenvalue one which is a contradiction. The lemma follows. □

The isometry group of each of the surfaces $S=S(7;3), S(13;4), S(21;7)$ contains a subgroup G of the above form, and by Lemma 5.1 of [H01] there is a natural G-equivariant linear isomorphism of the tangent space of $\mathcal{T}_{g,3}$ at each of these three surfaces onto the vector space $Z^\perp$, where this tangent space is viewed as a G-space. The set of systoles for S is G-invariant, and by Corollary 5.3 of [H01], their length functions satisfy the assumptions in Lemma 0.2. An application of Lemma 0.1 then shows that indeed, the surfaces $S(7;3), S(13;4), S(21;7)$ have property (2) above and are maximal.

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References

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