New examples of maximal surfaces

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Abstract: We describe all closed hyperbolic triangle surfaces of a particularly simple type which are maximal, i.e. for which the length of the systole is a local maximum in Teichmüller space. We show that this class of triangle surfaces contains exactly three maximal surfaces. One of these surfaces is the well known Klein's surface, the other two examples are new. [2]

1. Introduction

A *Riemann surface of finite type* is a closed Riemann surface from which a finite number $m \geq 0$ of points, the so-called *punctures*, have been deleted. Closed Riemann surfaces (with no punctures) are topologically determined by their genus. In this note we only consider surfaces of genus $g \geq 2$ with $m \geq 0$ punctures. Such a surface admits a family of complete hyperbolic metrics of finite volume. Each of these metrics corresponds to precisely one complex structure of finite type.

The easiest way to describe all such hyperbolic metrics is to look at the *Teichmüller space* $\mathcal{T}_{g,m}$ of *marked* hyperbolic metrics of finite volume on a surface S_0 of genus g with m punctures. This Teichmüller space is the set of all pairs (f, h) where h is a hyperbolic metric on a surface S and f is the homotopy class of a homeomorphism $F : S_0 \to S$ of S_0 onto S. The *mapping class group* $Map(g, m)$ is the group of all isotopy classes of homeomorphisms of S_0 onto itself. It acts on $\mathcal{T}_{g,m}$ via $\Psi(f, h) = (f \circ \Psi^{-1}, h)$ and identifies those points in $\mathcal{T}_{g,m}$ which correspond to isometric surfaces. With respect to a natural topology, the Teichmüller space $\mathcal{T}_{g,m}$ is homeomorphic to a cell of dimension $6g - 6 + 2m$, and the mapping class group $Map(g, m)$ acts properly discontinuously, but not fixed-point free. The quotient of $\mathcal{T}_{g,m}$ under $Map(g, m)$ is the moduli space of hyperbolic metrics on our surface of genus g with m punctures.

A *systole* of an oriented hyperbolic surface S of finite volume is a simple closed geodesic on S of minimal length. The length of the systole depends on the choice of the hyperbolic metric and defines a $Map(g, m)$-invariant

[1] Partially supported by the Sonderforschungsbereich 256

[2] AMS subject classification: 30F60

continuous function on $\mathcal{T}_{g,m}$. This function is bounded from above on $\mathcal{T}_{g,m}$ by a constant depending on g and m [Bu] (which however tends to infinity as g tends to infinity [BS]), but it is not bounded from below on $\mathcal{T}_{g,m}$. We refer to [S3] for other interesting properties of this function.

Following Schmutz [S1] we call a point in $\mathcal{T}_{g,m}$ a *maximal surface* if the length of the systole has a local maximum at that point. Maximal surfaces always exist, and Schmutz found in [S1] explicit examples.

The goal of this paper is to look for maximal surfaces among all hyperbolic surfaces which admit a particularly simple combinatorial description. For this recall that every *closed* hyperbolic surface S is given by a discrete torsion free subgroup G of the isometry group $PSL(2,\mathbb{R})$ of the hyperbolic plane $\mathbf{H}^2$ which acts cocompactly on $\mathbf{H}^2$. The surface S then simply equals $\mathbf{H}^2/G$. The *Dirichlet fundamental polygon* for G based at a point $y \in \mathbf{H}^2$ is the set D of all points $z \in \mathbf{H}^2$ with the property that $\mathrm{dist}\,(z,y) \leq \mathrm{dist}\,(z,\Psi y)$ for every $\Psi \in G$, where dist is the distance function of the hyperbolic metric. This set is a convex hyperbolic polygon.

For a number $p \geq 5$ define a *fundamental* $2p$*-gon* to be a regular $2p$-gon Ω in the hyperbolic plane $\mathbf{H}^2$ with angles $2\pi/p$ and sides of equal length. Such a $2p$-gon admits a cyclic group Γ of order p of isometries whose elements rotate Ω about a fixed point, with a multiple of $2\pi/p$ as rotation angle. We call the fixed point of the elements of Γ the *center* of Ω. If we draw $2p$ geodesic segments from the center 0 to the vertices of the boundary $\partial\Omega$ of Ω, then these segments decompose Ω into $2p$ equilateral triangles with angle π/p.

We call a closed surface $S = \mathbf{H}^2/G$ a *simple triangle surface* if G admits a fundamental $2p$-gon Ω as the Dirichlet fundamental polygon based at the center of Ω and if moreover G is normalized by the cyclic group Γ. The action of Γ on $\mathbf{H}^2$ then descends to an isometric action on $S = \mathbf{H}^2/G$. We call this group of isometries the *basic group of isometries* of S. The Gauß-Bonnet formula shows that the genus g of S equals $\frac{1}{2}(p-1)$. In particular, the number p is odd.

We number the vertices of $\partial\Omega$ in counter-clockwise order. These vertices are contained in exactly two vertex cycles under the action of Γ. One of these vertex cycles contains the even vertices, the other contains the odd vertices. The triangulation of Ω into $2p$ equilateral triangles with vertices at 0 and on the boundary $\partial\Omega$ of Ω descends to a triangulation of the quotient surface S with 3 vertices. We call this triangulation the *canonical triangulation* of S. If we delete the vertices of the canonical triangulation from the surface S then we obtain a surface of genus g with 3 punctures together with a complex structure of finite type which is invariant under the natural action of the basic group of isometries of S. The unique hyperbolic metric of finite volume which defines this complex structure is again invariant under this group of isometries. In other words, to every simple triangle surface S of genus g corresponds a point S_∞ in the Teichmüller space $\mathcal{T}_{g,3}$ of surfaces of genus g with 3 punctures which we call the *ideal surface* S_∞ associated to S.

The main purpose of this note is to show.

Theorem A:

1) *Among the simple triangle surfaces there are exactly* 3 *which are maximal. They are of genus* 3, 6 *and* 10.

2) *The ideal surface associated to a simple triangle surface S is maximal if and only if S is maximal.*

The maximal surface of genus 3 listed in the above theorem is the well known Klein's surface of genus 3 and appears already in the list of maximal surfaces given by Schmutz in [S1] (compare also the proceedings volume [L] about Klein's surface). The examples of genus 6 and genus 10 are new. We remark that by construction our simple triangle surfaces are indeed triangle surfaces in the usual sense, i.e. their isometry group is a nontrivial finite quotient of a triangle group.

From the proof of Theorem A we obtain additional informations on some of the Teichmüller spaces $\mathcal{T}_{g,0}$. To explain this let $[\gamma]$ be a nontrivial free homotopy class on the closed base surface S_0 of genus g. For every point $(f,h) \in \mathcal{T}_{g,0}$, the class $f[\gamma]$ can be represented by a unique closed geodesic with respect to the hyperbolic metric h. The length of this geodesic defines a continuous function on $\mathcal{T}_{g,0}$. We call this function the *length function* of $[\gamma]$. We show.

Theorem B: *For every $k \geq 2$ and $g = \frac{k}{2}(k+1)$ the Teichmüller space $\mathcal{T}_{g,0}$ can be parametrized by the lengths functions of $6g+3$ free homotopy classes contained in the orbit of a fixed class under a maximal finite subgroup G of $Mod(g,0)$. The group G is a semidirect product of a cyclic group of order $2g+1$ and a cyclic group Σ of order* 3.

We refer to [S2] for a discussion of other interesting parametrizations of $\mathcal{T}_{g,0}$.

The structure of this note is as follows. In Section 2 we look at simple triangle surfaces with additional symmetries. In Section 3 we give a combinatorial description of a family of curves which contains the systoles of every simple triangle surface. Length estimates in Section 4 lead to a complete description of the systoles of a simple triangle surface. This is used in Section 5 to show our theorems.

As a notational convention, we number the vertices of a fundamental $2p$-gon Ω counter-clockwise in consecutive order and we number and orient the edges of Ω in such a way that the edge i as an oriented arc joins the vertex $i-1$ to the vertex i. Moreover we write simply $\mathcal{T}_g$ for the Teichmüller space of marked hyperbolic structures on a closed surface of genus g.

2. Basic properties of simple triangle surfaces

Let $g \geq 2$ and let $p = 2g+1$. There is up to isometry a unique $2p$-gon Ω in the hyperbolic plane $\mathbf{H}^2$ with geodesic sides of equal length and with angles $2\pi/p$. In the introduction we called Ω a *fundamental* $2p$*-gon*. The *center* of Ω is the unique point $z \in \Omega$ which has the same distance to each of the vertices. A fundamental $2p$-gon admits a cyclic group Γ of isometries whose elements rotate Ω about the center with a rotation angle which is a multiple of $2\pi/p$. We view Γ as a cyclic group of isometries of the whole hyperbolic plane $\mathbf{H}^2$.

We call a closed hyperbolic surface S of genus g a *simple triangle surface* if $S = \mathbf{H}^2/G$ where G is a discrete torsion free group $G \subset PSL(2,\mathbb{R})$ of isometries of $\mathbf{H}^2$ which is normalized by the group Γ and which admits Ω as a fundamental polygon (see [M] for basic informations on fundamental polygons). The group G then acts as a group of side pairing transformations for the polygon Ω. This means that for each side a of Ω there is an isometry $\Psi \in G$ which maps a to a second side $\Psi(a) \neq a$ of Ω in such a way that $\Psi(\Omega) \cap \Omega = \Psi a$.

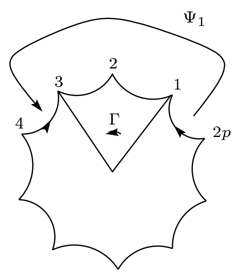
Our first observation is that simple triangle surfaces exist for every genus $g \geq 2$.

Lemma 2.1: *For every $g \geq 2$ there is a simple triangle surface of genus g.*

Proof: Let $p \geq 5$ be an odd number and let Ω be a fundamental $2p$-gon with center $0 \in \mathbf{H}^2$. We have to show that there is a discrete subgroup G of

$PSL(2,\mathbb{R})$ which is normalized by Γ and which admits Ω as a fundamental polygon.

Choose a number $k \in \{2,\dots,p-1\}$ and define a family $\{\Psi_1,\dots,\Psi_p\}$ of isometries of $\mathbf{H}^2$ by requiring that Ψ_j maps the (oriented) edge with odd number $2j+1$ orientation reversing onto the (oriented) edge $2j+2k$ in such a way that $\Psi_j(\Omega)\cap\Omega$ is just the edge $2j+2k$. Then necessarily the vertex $2j$ is mapped to the vertex $2j+2k$, and the vertex $2j+1$ is mapped to the vertex $2j+2k-1$. We claim that these isometries $\{\Psi_1,\dots,\Psi_p\}$ generate a discrete subgroup of $PSL(2,\mathbf{R})$ with fundamental domain Ω if and only if k and $k-1$ are prime to p.



To see this let G be the subgroup of $PSL(2,\mathbb{R})$ generated by $\Psi_1,\dots,\Psi_p$ and assume that G is discrete and torsion free, with fundamental polygon Ω. By the choice of $\Psi_1,\dots,\Psi_p$, the G-orbit of an even (or odd) vertex of Ω intersects Ω only in the set of even (or odd) vertices. Different such vertex cycles project to different points on the surface $S=\mathbf{H}^2/G$. If $m\geq 2$ is the number of points in the vertex cycle of the vertex a, then a neighborhood of the projection $\bar{a}$ of a to S consists of $2m$ equilateral hyperbolic triangles with angle π/p which contain $\bar{a}$ as one of their vertices. Since S is a smooth hyperbolic surface, the angles at $\bar{a}$ of these triangles must add up to 2π. This means that there are precisely 2 vertex cycles for the action of G containing each only even or only odd vertices. By the definition of G this is the case if and only if the number $k\in\{2,\dots,p-1\}$ is prime to p and $k-1$ is prime to p as well. Such a group G is then normalized by the group Γ of rotations of Ω with rotation angle a multiple of 2π.

The same argument also shows that for $k\in\{2,\dots,p-1\}$ which is prime to p and such that $k-1$ is prime to p as well the group G induces a simple triangle surface of genus g. Since $p=2g+1$ is odd we can always choose

$k = 2$ to obtain an example. **q.e.d.**

In the above proof we observed that we obtain a simple triangle surface from a fundamental $2p$-gon Ω by identifying the edge 1 with the edge $2k$ for some $k \in \{2, \dots, p-1\}$ if and only if k and $k-1$ are prime to p. We denote by $S(p;k)$ the surface obtained in this way. For fixed $p \geq 5$ this defines a finite non-empty collection of simple triangle surfaces of genus $\frac{1}{2}p - 1$ indexed by the set of all numbers $k \in \{2, \dots, p-1\}$ which are prime to p and such that $k-1$ is prime to p as well. However these surfaces are not necessarily distinct as hyperbolic surfaces. For example, via exchanging the roles of the even and odd vertices of our fundamental $2p$-gon Ω we observe that the surface $S(p;k)$ is isometric to the surface $S(p;p-k+1)$. Thus we may restrict our attention to the case that $k \leq \frac{1}{2}(p+1)$. In the sequel we sometimes identify the surfaces $S(p;k)$ and $S(p;p-k+1)$ without further comment.

Let again Γ be the group of rotations of Ω which descends to a group of isometries on a simple triangle surface S of genus g. The natural Γ-invariant triangulation of Ω into $2p$ equilateral triangles with angle π/p projects to the Γ-invariant canonical triangulation whose 3 vertices $0, A, B$ are just the fixed points for the action of Γ. The quotient S/Γ of S under Γ is a topological 2-sphere. The hyperbolic metric on S projects to a hyperbolic metric on S/Γ with 3 singular points $\hat{A}, \hat{B}, \hat{0}$ which are the projections of the vertices $A, B, 0$ of the canonical triangulation of S. With this metric, S/Γ is isometric to two equilateral hyperbolic triangles with angle π/p glued at their boundaries. This observation is used in the proof of the following.

Lemma 2.2:

1) *Let $p \geq 5$ be an odd number and let $k, m \in \{2, \dots, p-1\}$ be numbers which are prime to p and such that $k-1, m-1$ are prime to p as well. If either $(k-1)m + 1 \equiv 0 \bmod p$ or $(m-1)k + 1 \equiv 0 \bmod p$ then the surfaces $S(p;k)$ and $S(p;m)$ are isometric.*

2) *A simple triangle surface S with basic group Γ of isometries admits a nontrivial group $\Sigma \not\subset \Gamma$ of orientation preserving isometries which normalizes Γ if and only if one of the following holds.*

   i) *$S = S(p;k)$ for some $k \geq 2$ and a divisor $p \geq k+1$ of $k(k-1)+1$. The group Σ is then cyclic of order 3.*

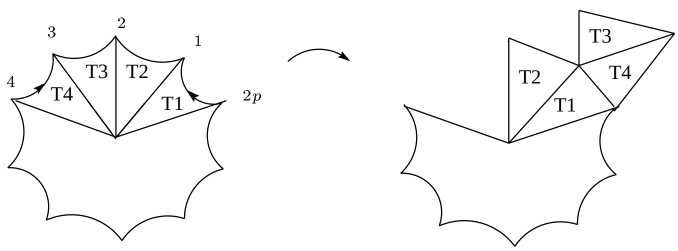
ii) $S = S(p;2)$ *and the group* Σ *is cyclic of order* 2 *and generated by a hyperelliptic involution.*

Proof: Let $p \geq 5$ and let $k \leq p-1$ be such that $k-1$ and k are prime to p. Let Ω be a fundamental $2p$-gon and let $0, A, B$ be the vertices of the canonical triangulation of S. We assume that 0 is the projection of the center of Ω and A is the projection of the odd vertices of the boundary of Ω.

As in the introduction we number the $2p$ edges of Ω in counterclockwise order in such a way that the edge i is adjacent to the vertices $i-1$ and i. Let $T_i \subset S$ be the projection of the triangle in Ω with one vertex at the center of Ω and with the edge i of Ω as the opposite side. The triangles $T_1, \dots, T_{2p}$ are arranged in counterclockwise order around the vertex 0.

There is a different representation of S as a quotient of Ω under a group of side pairing transformations in such a way that the center of Ω projects to the vertex A of the canonical triangulation. Namely, if we cut S open along the geodesic arcs connecting the vertices 0 and B, then the result is a fundamental $2p$-gon which consists again of the triangles $T_1, \dots, T_{2p}$. The arrangement of these triangles around the vertex A is given by a permutation σ of $\{1, \dots, 2p\}$ with $\sigma(1) = 1$, i.e. the counterclockwise order of the triangles around the vertex A is $T_{\sigma(1)}, \dots, T_{\sigma(2p)}$. The parity of $\sigma(i)$ coincides with the parity of i. Moreover for every $i \in \{1, \dots, p\}$ we have $\sigma(2i) = \sigma(2i+1) + 1$ mod $2p$.

The side pairings of Ω which define S in such a way that the center of Ω projects to 0 glue the edge $2k$ to the edge 1 and therefore we have $\sigma(2) = 2k$ and $\sigma(3) = 2k-1$. The basic group Γ of isometries of S permutes the triangles T_i and fixes the vertex A. This implies that σ normalizes the group of permutations of $\{1, \dots, 2p\}$ generated by the permutation $\tau(i) = i+2$ mod $2p$ and hence necessarily $\sigma(2i) = 2i(k-1)+2$.



To obtain our surface S we have to identify the edge $2i-1$ with the edge $2im$ for some $m \in \{2, \dots, p-1\}$ with an orientation reversing isometry. The number m is uniquely determined if we require in addition that the triangles adjacent to odd edges of Ω project once again to the triangles T_{2i-1} ($i = 1, \dots, p$) of the canonical triangulation.

Comparing the arrangement of triangles around 0 and A we conclude that $\sigma(2m) = 2p$. Together with the above this shows that $2m(k-1)+2 \equiv 0 \bmod 2p$ or, equivalently, $m(k-1)+1 \equiv 0 \bmod p$. In other words, if $m, k \geq 2$ are such that $m(k-1)+1 \equiv 0 \bmod p$ then the surfaces $S(p;k)$ and $S(p;m)$ are isometric. This shows the first part of the lemma.

To show the second part of our lemma let S be a simple triangle surface which admits a non-trivial group Σ of orientation preserving isometries normalizing the basic group Γ. Then the action of Σ on S descends to an isometric action on the sphere S/Γ. The sphere S/Γ consists of two equilateral triangles with angle π/p glued at their boundaries. One of these triangles is the projection of the odd triangles of the canonical triangulation of S, the other one is the projection of the even triangles.

Every isometry of S/Γ has to preserve the singular set $\{\hat{A}, \hat{B}, \hat{0}\} \subset S/\Gamma$ of ramification points which consists of the vertices of the two triangles forming S/Γ. The only nontrivial isometry of S/Γ which fixes each of the ramification points $\hat{0}, \hat{A}, \hat{B}$ is the orientation reversing reflection which exchanges the two triangles forming S/Γ. By assumption the elements of Σ preserve the orientation of S and hence of S/Γ, and therefore there are two possibilities:

1) Σ contains an element Ψ which permutes cyclicly the singular points $\hat{A}, \hat{B}, \hat{0}$ of S/Γ and preserves each of the two triangles which form S/Γ.

2) Σ fixes one singular point of S/Γ, permutes the two other ones and exchanges the two triangles which form S/Γ.

Assume that $S = S(p;k)$ admits an isometry Ψ as in 1) above. Then Ψ permutes the triangles of the canonical triangulation, but preserves their parity. If we cut $S = S(p;k)$ open along those edges of the triangles of the canonical triangulation which connect the vertices A and B, then the result is the fundamental $2p$-gon Ω and we obtain our surface from Ω by a side pairing which identifies the edges 1 and $2k$. Since Ψ is an isometry of S which preserves the canonical triangulation, if we cut S open along the edges connecting the vertices $\Psi(A)$ and $\Psi(B)$ then the result is again the polygon Ω, and once again we obtain S from Ω by identifying the edges 1 and $2k$.

This together with the above consideration shows that $k(k-1)+1 \equiv 0 \bmod p$ and therefore p divides $k(k-1)+1$.

Assume now that S admits an isometry Ψ as in 2) above. Then Ψ permutes the triangles of the canonical triangulation and changes their parity with respect to a given counter clockwise numbering around a given vertex. Let $m \leq p-1$ be such that $k(m-1)+1 \equiv 0 \bmod p$. The above considerations imply that necessarily $k = p-m+1$ and hence $(m-1)^2 \equiv 1 \bmod p$ or equivalently $m(m-2) \equiv 0 \bmod p$. Since $m \geq 1$ is prime to p we conclude that either $m=2$ or that p divides $m-2$. But $m \leq p-1$ and therefore only the case $m=2$ is possible.

We are left with showing that the isometry Ψ is a hyperelliptic involution. For this notice that every fixed point of Ψ projects to a fixed point for the induced isometry $\hat{\Psi}$ of S/Γ. The map $\hat{\Psi}$ has precisely two fixed points: A singular point $\hat{0}$ of S/Γ and the midpoint y of the geodesic arc connecting the two other singular points.

There are exactly $p = 2g+1$ preimages of y in S. Since $\Psi^2 = Id$ and since Ψ normalizes Γ, either every preimage or no preimage is fixed by Ψ. The Riemann Hurwitz-formula [F] shows that the second case is impossible. Thus Ψ has exactly $p+1 = 2g+2$ fixed points and is a hyperelliptic involution. **q.e.d.**

Corollary 2.3: *For every $g \geq 2$ there is a hyperelliptic surface of genus g whose full automorphism group is the direct product of a cyclic group of order $2g+1$ and a cyclic group of order 2 generated by a hyperelliptic involution.*

Proof: We showed in Lemma 2.1 that for each $g \geq 2$ there is a simple triangle surface $S(2g+1;2)$. By Lemma 2.2 and its proof, this surface is hyperelliptic and its isometry group is a stated in the corollary. **q.e.d.**

Remark: There are surfaces $S(p;k)$ for $p \notin \{\ell(\ell-1)+1 \mid \ell \geq 2\}$ which admit a cyclic group Σ of isometries of order 3 contained in the normalizer of the basic group Γ. The simplest surface of this kind is the surface $S(19;8)$ of genus $g=9$.

3. Geometric properties of systoles of simple triangle surfaces

This section is devoted to a description of some geometric properties of the systoles on a simple triangle surface $S = S(p;k)$ and its associated ideal surface S_∞. We continue to use the notations from Section 2.

The canonical triangulation of the surface S is invariant under the group Γ of isometries of S, and its vertices $0, A, B$ are fixed points for the action of $\tilde{\Gamma}$. The quotient S/Γ is a topological 2-sphere with a singular hyperbolic metric which is isometric to two equilateral hyperbolic triangles with angles π/p glued at their boundaries. Every closed geodesic on S which does not pass through any of the vertices $A, B, 0$ projects to a closed geodesic on S/Γ. We first observe that this is the case for the projection to S/Γ of a systole on S.

Lemma 3.1: *A systole of S does not pass through a vertex of the canonical triangulation.*

Proof: Let γ be a geodesic in S which passes through one of the vertices of the canonical triangulation, say through the vertex 0. Assume that we obtain S from side pairing transformations of a fundamental $2p$-gon Ω in such a way that the center of Ω projects to the point 0.

The lift of γ to the polygon Ω has to intersect the boundary $\partial\Omega$ of Ω and hence its length is not smaller than twice the distance between the center of Ω and $\partial\Omega$. In particular, if α is any geodesic arc in Ω of minimal length which connects the edge 1 to an edge $r \neq p+1$, then α is necessarily shorter than γ.

Let $k < p$ be such that the side pairings for Ω which define S identify the edge 1 with the edge $2k$. If $2k \neq p+1$ then the above shows that the closed geodesic on S which is the projection of the arc of minimal length in Ω connecting the edges 1 and $2k$ is shorter than γ.

On the other hand, if $2k = p+1$, then we obtain from Lemma 2.2 that the side pairings which define Ω with center at the point A identify the edge 1 with an edge $2m$ for some $m \neq (p+1)/2$. Again we conclude that the arc γ is longer than a systole on S. **q.e.d.**

Let Ω be a fundamental $2p$-gon and let γ be the geodesic arc through the center 0 of Ω which connects the vertex $2p$ to the vertex p. Let Ψ be the reflection in $\mathbf{H}^2$ along γ. Then Ψ leaves Ω invariant and maps a pair of edges of the form $\{2i+1, 2i+2k\}$ to the pair $\{2p-2i, 2p-2i-2k+1\}$ of the same form. In other words, Ψ descends to an orientation reversing isometry of S. The group $\tilde{\Gamma}$ of isometries of S generated by Ψ and the basic group Γ has order $p+1$ and contains the group Γ as a normal subgroup of index 2. The orientation reversing isometry Ψ of S descends to an orientation reversing isometry $\hat{\Psi}$ of order 2 of S/Γ which exchanges the two triangles.

Let $\triangle$ be an equilateral hyperbolic triangle with angle π/p. The triangle $\triangle$ will be viewed as a billiard table. A billiard orbit consists of geodesic arcs inside $\triangle$ which are joined at points of the boundary $\partial\triangle$ according to the rule that the angle of incidence equals the angle of reflection. We view a billiard orbit as unparametrized and unoriented.

A closed geodesic on S/Γ not passing through one of the singular points $\hat{0}, \hat{A}, \hat{B}$ corresponds to a periodic billiard orbit in $\triangle$ of one of the following three types:

a) A periodic billiard orbit with an odd number of collisions with the boundary of $\triangle$, none of them perpendicular.

In the sequel we call such a billiard orbit an *A-orbit*. An A-orbit $\tilde{\gamma}$ admits a lift to a closed geodesic $\hat{\gamma}$ on S/Γ, unique up to reparametrization, which is freely homotopic as a curve on the thrice punctured sphere $S/\Gamma - \{\hat{0}, \hat{A}, \hat{B}\}$ to its image under the isometry $\hat{\Psi}$. Its trace is invariant under $\hat{\Psi}$. The lift of every collision point of the billiard orbit with $\partial\triangle$ is a transversal intersection of $\hat{\gamma}$ with the common boundary of the two triangles forming S/Γ. The length of $\hat{\gamma}$ is twice the length of $\tilde{\gamma}$.

b) A periodic billiard orbit whose trace consists of one piecewise geodesic arc which meets the boundary $\partial\triangle$ orthogonally at its endpoints.

We call such an orbit a *B-orbit* in the sequel. A B-orbit $\tilde{\gamma}$ admits a lift to S/Γ, unique up to reparametrization, which is freely homotopic to the image $\hat{\Psi}(\hat{\gamma}^{-1})$ under $\hat{\Psi}$ of its inverse $\hat{\gamma}^{-1}$. Its trace is invariant under $\hat{\Psi}$ and its length is twice the length of $\tilde{\gamma}$.

c) A periodic billiard orbit with an even number of collisions with the boundary of $\triangle$, none of them perpendicular.

We call such an orbit a *C-orbit*. A C-orbit $\tilde{\gamma}$ admits two different lifts $\hat{\gamma}_1, \hat{\gamma}_2$ to closed geodesics on S/Γ whose traces intersect transversally and whose lengths coincide with the length of the billiard orbit. The geodesic $\hat{\gamma}_2$ is the image of $\hat{\gamma}_1$ under the isometry $\hat{\Psi}$ of S/Γ. Neither the geodesic $\hat{\gamma}_i$ nor its inverse $\hat{\gamma}_i^{-1}$ is freely homotopic to $\hat{\Psi}(\hat{\gamma}_i)$.

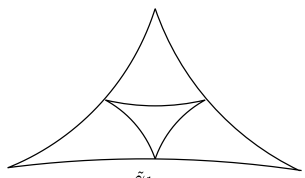
Call a periodic billiard orbit $\tilde{\gamma}$ on $\triangle$ as above *liftable to S* if there is a closed geodesic γ on S whose projection to S/Γ is a lift $\hat{\gamma}$ of $\tilde{\gamma}$ to S/Γ. We then call γ a *lift of $\tilde{\gamma}$ to S*.

The group $\tilde{\Gamma}$ also acts as a group of isometries on the ideal surface S_∞ associated to S. The quotient of S_∞ unter the basic group Γ is the thrice punctured sphere S_∞/Γ with the complete hyperbolic metric of finite volume. The orientation reversing involution $\hat{\Psi}$ acts on S_∞/Γ as the natural reflection which leaves each of the punctures fixed. Every closed geodesic on S_∞ projects to a closed geodesic on S_∞/Γ.

Let $\triangle_\infty$ be an ideal triangle. Once again we can view $\triangle_\infty$ as a billard table. The above definition for billard orbits in $\triangle$ can also be made for billard orbits in $\triangle_\infty$. We call a billard orbit $\tilde{\gamma}$ in $\triangle_\infty$ *liftable* to the ideal surface S_∞ if there is a closed geodesic γ on S_∞ which projects to $\tilde{\gamma}$. In the remainder of this section the ideal triangle, its billard orbits and their lifts to the ideal surface S_∞ are always included in our considerations without further comments. More precisely, even though for simplicity we formulate all our statements only for billard orbits in $\triangle$ and the surface S it is immediately clear from the proofs that they are equally valid for $\triangle_\infty$ and the ideal surface S_∞.

A first example of a liftable billiard orbit is given in the next lemma.

Lemma 3.2: *There is a unique A-orbit $\tilde{\gamma}_1$ in $\triangle$ with 3 collisions with the boundary, and this orbit is liftable. The length of a lift of $\tilde{\gamma}_1$ to S is not bigger than* $6\text{arccosh}\,\frac{3}{2}$.



Proof: Let $S = S(p;k)$ and let Ω be a fundamental $2p$-gon. Connect the midpoint of the edge 1 in Ω with the midpoint of the edge 3 by a simple arc, and connect the midpoint of the edge $2k$ with the midpoint of the edge $2k+2$ by a simple arc. These two arcs together project to a simple closed curve on S which is freely homotopic to a closed geodesic γ on S. The geodesic γ is necessarily a lift of an A-orbit $\tilde{\gamma}_1$ in $\triangle$ of period 3. Notice that there are exactly p lifts of $\tilde{\gamma}_1$, and every such lift intersects exactly 6 other lifts, with each of these intersections consisting of a single point. The length ℓ_1 of a lift of $\tilde{\gamma}_1$ to S is twice the length of $\tilde{\gamma}_1$.

To give a sharp upper bound for ℓ_1 notice that $\ell_1/2$ is just the smallest circumference of a hyperbolic triangle with vertices on the sides of $\triangle$ and hence $\ell_1/2$ is not larger than the smallest circumference of a hyperbolic triangle T_∞ with vertices on the boundary of an ideal triangle. This circumference is the limit as $k \to \infty$ of the circumferences of hyperbolic triangles T_k whose vertices are the midpoints of the sides of an equilateral triangle $\triangle_k$ with angle π/k.

To give a formula for the circumference of T_k let λ_k be the length of the sides of $\triangle_k$, and let ℓ_k be the length of the sides of T_k.

Hyperbolic trigonometry (see [I]) gives $\cosh \frac{\lambda_k}{2} = \frac{\cos \pi/2k}{\sin \pi/k}$ and

$$\cosh \ell_k = (\cosh \frac{\lambda_k}{2})^2 - (\sinh \frac{\lambda_k}{2})^2 \cos \frac{\pi}{k} = \frac{(1-\cos \pi/k)(\cos \pi/2k)^2}{(\sin \pi/k)^2} + \cos \frac{\pi}{k}.$$

This shows that as $k \to \infty$ we have $\cosh \ell_k \to \frac{3}{2}$ and $6\ell_k \to 6 \operatorname{arccosh} \frac{3}{2} \sim 5.775$. This then finishes the proof of our lemma. **q.e.d.**

As an immediate consequence of Lemma 3.2, the length of the systole of a simple triangle surface and its associated ideal surface does not exceed 6arccosh $\frac{3}{2} < 5.8$. In particular, for large genus such triangle surfaces are never globally maximal [BS].

Lemma 3.3: *A lift to S of an A-orbit $\tilde{\gamma}$ which is different from $\tilde{\gamma}_1$ is not a systole.*

Proof: By Lemma 3.2 it suffices to show that the length of every A-orbit $\tilde{\gamma}$ in $\triangle$ is not smaller than the length of the A-orbit $\tilde{\gamma}_1$ from Lemma 3.1, with equality if and only if $\tilde{\gamma} = \tilde{\gamma}_1$.

For this recall from the definition that an A-orbit $\tilde{\gamma}$ is a closed curve in $\triangle$ with an odd number of collisions with the boundary, none of them perpendicular. This implies that for every pair of sides of the boundary of $\triangle$ there is a geodesic arc of $\tilde{\gamma}$ with endpoints on these sides.

Thus we can find three points E_1, E_2, E_3 which lie on the three different sides of the boundary of $\triangle$ and are contained in $\tilde{\gamma}$ in this order with respect to the choice of some fixed orientation and some fixed initial point. Since $\tilde{\gamma}$ is closed, its length is not smaller than the circumference of the triangle T inscribed in $\triangle$ with vertices E_1, E_2, E_3 with equality if and only if $\tilde{\gamma}$ coincides with the boundary of T. However the length of the orbit $\tilde{\gamma}_1$ from Lemma 3.2 is the smallest circumference of any triangle with vertices on the three different sides of $\triangle$. From this the lemma is immediate. **q.e.d.**

B-orbits and C-orbits in $\triangle$ are more difficult to control. For their investigation let S_* be a thrice punctured sphere. We equip S_* with the (noncomplete) hyperbolic metric which we obtain by glueing two equilateral hyperbolic triangles T_1, T_2 with angle π/p along their boundaries. Thus S_* with this metric is just the space $S/\Gamma - \{\hat{0}, \hat{A}, \hat{B}\}$. The sides of T_1, T_2 are geodesics a, b, c in S_* which connect a pair of punctures of S_*. We call a, b, c the *edges* of S_*. Define a curve α in S_* to be *admissible* if α is a closed curve with the additional property that every connected component of an intersection of α with one of the triangles T_i consists of a single geodesic arc in T_i. We call these components the *segments* of α. Thus α is composed of a finite number of geodesic arcs with endpoints on the edges of S_*, and no two consecutive such segments are contained in the same triangle T_i. In the sequel we identify two such curves if they coincide up to an orientation preserving reparametrization.

An *admissible homotopy* of an admissible curve α is a free homotopy of α through admissible curves. We call the admissible curve α on S_* *essential* if α can not be homotoped into one of the punctures. An *admissible subcurve* of α is a connected subarc β of α such that there exists an admissible homotopy of α which deforms β into a closed admissible curve. For every admissible subcurve β of α we can write $\alpha = \beta\gamma$ for an admissible subcurve γ. We say that α is *irreducible* if for every essential admissible subcurve β of α the curve $\gamma = \alpha - \beta$ is not essential. A curve which is not irreducible is called *reducible*. An irreducible essential curve α is called *minimal* if α does not contain any nontrivial essential closed subcurve.

There are two obvious types of minimal closed curves which can be described as follows. The first type consists of curves which are freely homotopic to a lift of the A-orbit $\tilde{\gamma}_1$ from Lemma 3.2. We call such a curve a *minimal curve of type A*. The second type consists of curves which are freely homotopic to a curve of the form $\alpha\beta$ where α and β are simple closed curves in S_* which generate the fundamental group of S_*. Up to orientation there are three different free homotopy classes of such minimal curves which correspond to a choice of two of the three punctures.

Lemma 3.4: *Every minimal admissible closed curve is either a minimal curve of type A or a minimal curve of type B.*

Proof: Let α be a minimal admissible closed curve. If α contains two consecutive geodesic segments with endpoints on the same pair of edges of S_* then α contains a nontrivial non-essential admissible subcurve β and necessarily $\alpha = \beta\gamma$ where γ is non-essential. Since α is essential, β and γ are homotopic to different punctures. The same argument can be applied to any subarc of γ which consists of two consecutive geodesic segments and shows that γ has exactly two segments. This means that α is of type B.

On the other hand, if there are no two consecutive segments of α hitting the same edges of S_* then α is necessarily homotopic to a multiple of the lift of the A-orbit $\tilde{\gamma}_1$ from Lemma 3.2. By minimality, α is of type A. This shows the lemma. **q.e.d.**

Let now α be any irreducible closed curve. A *simplification* of α is an admissible essential subcurve β of α such that α can be written in the form $\alpha = \beta\gamma$ where γ is non-essential. A *minimal model* is a minimal closed curve which can be obtained from α by finitely many simplifications. Clearly every irreducible closed curve has a minimal model which is not necessarily unique.

Recall that S_* admits a natural orientation reversing isometry $\hat{\Psi}$ which fixes pointwise the edges of S_*. This isometry acts on the space of admissible curves. We have.

Lemma 3.5: *Let α be an irreducible admissible curve which admits a minimal model of type B. Then α is freely homotopic to $\hat{\Psi}(\alpha^{-1})$.*

Proof: Let α be an irreducible admissible closed curve. Assume that α admits a minimal model β of type B. We have to show that $\hat{\Psi}(\alpha^{-1})$ is freely homotopic to α.

By definition of a minimal model, with respect to a suitable numbering of the edges of S_* the curve β can be written in the form $\beta = \beta_1\beta_2\beta_3\beta_4$ where β_1 connects the edge a to the edge b, β_2 connects the edge b to the edge a, β_3 connects a to c and β_4 connects c to a. Notice that β has exactly 4 intersection points with the edges of S_*.

Since β is a minimal model for α, the curve α can be represented in the form $\alpha = \beta_1\alpha_1\beta_2\alpha_2\beta_3\alpha_3\beta_4\alpha_4$ where α_i is an admissible closed curve. By assumption α is irreducible and therefore the curves α_i are non-essential.

We distinguish three cases.

1) The curve $\beta_1\alpha_1\beta_2$ is essential.

Then α_1 consists of an even number of geodesic arcs which connect the edges b and c. Moreover the subcurve $\alpha_2\beta_3\alpha_3\beta_4\alpha_4$ has to be non-essential and therefore $\alpha = \beta_1\alpha_1\beta_2(\beta_3\beta_4)^m$ for some $m \geq 1$. In particular, α is freely homotopic $\hat{\Psi}(\alpha^{-1})$.

2) The curve $\beta_3\alpha_3\beta_4$ is essential.

As above we conclude that then $\alpha = (\beta_1\beta_2)^m\beta_3\alpha_3\beta_4$ and α is freely homotopic to $\hat{\Psi}(\alpha^{-1})$.

3) $\beta_1\alpha_1\beta_2 = (\beta_1\beta_2)^{m_1}$ and $\beta_3\alpha_3\beta_4 = (\beta_3\beta_4)^{m_2}$ for some $m_1, m_2 \geq 1$.

Since the curves α_2 and α_4 are non-essential and have their endpoints on the side a this implies that α can be represented in the form $\alpha = (\beta_1\beta_2)^{\ell_1}(\beta_3\beta_4)^{\ell_2}$ for some $\ell_1, \ell_2 \geq 1$. Once again we conclude that α is homotopic to $\hat{\Psi}(\alpha^{-1})$. **q.e.d.**

Remark: The proof of Lemma 3.5 also shows the following: Let α be an irreducible admissible essential closed curve on S_* which admits a minimal model of type B. Then with respect to a suitable labeling of the edges of S_*, α is freely homotopic to a curve of the form $(\beta_1\beta_2)^k\beta_3\zeta^m\beta_4$ where $k \geq 1$, $m \geq 0$ and β_1 is an arc joining the edge a to the edge b, β_2 connects b to a, β_3 joins b to c, ζ is nonessential and β_4 connects c to a.

Lemma 3.6: *The projection to $S/\Gamma - \{\hat{0}, \hat{A}, \hat{B}\}$ of a systole on a simple triangle surface $S = S(p;k)$ is irreducible.*

Proof: By Lemma 3.2 it suffices to show that the length of every admissible reducible closed curve α in S_* is bigger than twice the length of the A-orbit $\tilde{\gamma}_1$. For this let α be reducible and write $\alpha = \alpha_1\alpha_2$ where the curves α_1, α_2 are essential.

Let β be an irreducible admissible essential subcurve of α_1. If β has a minimal model of type A, then we can cut from β finitely many non-essential closed curves to obtain a shorter curve which is homotopic to two copies of the A-orbit $\tilde{\gamma}_1$ from Lemma 3.2. Since the lift $\hat{\gamma}_1$ of $\tilde{\gamma}_1$ to S/Γ has minimal length in its free homotopy class and since α is homotopic to $\beta\gamma$ for some closed curve γ, the length of α is bigger than the length of the lift $\hat{\gamma}_1$ of $\tilde{\gamma}_1$ to S_*. Thus by Lemma 3.2 α can not lift to a systole on S.

We are left with the case that all minimal models of irreducible subcurves α_1, α_2 of α are of type B. Then we can cut away finitely many closed curves from α which shortens the length of α to end up with a closed curve β of the form $\beta = \beta_1\gamma\beta_2\delta$ where β_1, β_2 are minimal curves of type B and γ, δ are possibly trivial arcs connecting the edges containing the endpoints of β_1, β_2. If γ, δ are not trivial then we can replace $\gamma\beta_2\delta$ by a minimal curve $\gamma\tilde{\beta}_2\delta$ of type B where $\tilde{\beta}_2$ is an admissible subcurve of β_2. In other words, we may as well assume that $\beta = \beta_1\beta_2$.

We distinguish now two cases.

1) The curves β_1, β_2 are homotopic.

Then there are simple closed generators η, ζ of the fundamental group of S_* such that β is freely homotopic to $\eta\zeta\eta\zeta$. In particular there is a closed geodesic ρ on S_* which is freely homotopic to β, whose length is not bigger than the length of β and which is not a prime geodesic. This geodesic is the double of a minimal curve γ of type B. The length of ρ equals twice the length of γ. However, since the length ℓ_1 of the A-orbit $\tilde{\gamma}_1$ from Lemma 3.2 is the minimal length of any closed curve in the triangle $\triangle$ which intersects the three sides of $\triangle$, the length of $\tilde{\gamma}_1$ is strictly smaller than the length of γ. Thus ρ is longer than a lift of $\tilde{\gamma}_1$ and α can not lift to a systole on S.

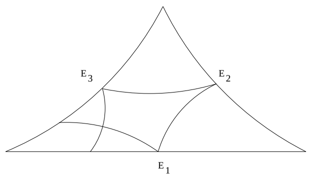
2) The curves β_1, β_2 are not homotopic.

Let $\tilde{\zeta}$ be the B-orbit in $\triangle$ whose lift to $S/\Gamma - \{\hat{A}, \hat{B}, \hat{0}\} = S_*$ is freely homotopic to $\beta_1\beta_2$. The length of $\tilde{\zeta}$ is not bigger than half the length of $\beta_1\beta_2$ and $\tilde{\zeta}$ consists of four arcs $\tilde{\zeta}_1, \tilde{\zeta}_2, \tilde{\zeta}_3, \tilde{\zeta}_4$. The arc $\tilde{\zeta}_1$ meets one of the sides, say the side a, perpendicularly, and $\tilde{\zeta}_4$ meets a different side, say the side b, perpendicularly.

We denote by E_1, E_2, E_3 the endpoints of $\tilde{\zeta}_2$ and $\tilde{\zeta}_3$; they lie on the three different sides of $\triangle$.

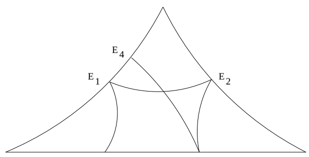
We distinguish once again two cases:

a) The arcs $\tilde{\zeta}_1$ and $\tilde{\zeta}_4$ intersect.



Then the length of $\tilde{\zeta}$ is bigger than the length of the triangle inscribed in $\triangle$ with vertices E_1, E_2, E_3. In particular, the length of $\tilde{\zeta}$ is bigger than the length of the A-orbit $\tilde{\gamma}_1$ from Lemma 3.2.

b) The arcs $\tilde{\zeta}_1$ and $\tilde{\zeta}_4$ do not intersect.



In this case either the arc $\tilde{\zeta}_1$ intersects the arc $\tilde{\zeta}_3$ or the arc $\tilde{\zeta}_4$ intersects the arc $\tilde{\zeta}_2$. Assume that the second case holds.

Let again E_1, E_2 be the endpoints of $\tilde{\zeta}_2$ where E_1 lies on the edge b and let E_4 be the endpoint of the arc $\tilde{\zeta}_4$ on the edge b. Since $\tilde{\zeta}_4$ meets b orthogonally at E_4 and has its second endpoint on the side a, the angle at E_4 of the triangle with vertices E_1, E_4, E_2 is strictly bigger than $\frac{\pi}{2}$. This means that the distance between E_2 and E_4 is smaller than the length of the arc $\tilde{\zeta}_2$ and therefore the length of $\tilde{\zeta}$ is bigger than the circumference of the triangle with vertices E_1, E_2, E_4. In particular, this length is bigger than the length of the A-orbit $\tilde{\gamma}_1$.

This finishes the proof of our lemma. **q.e.d.**

As an immediate corollary of Lemma 3.6 and Lemma 3.5 we obtain.

Corollary 3.7: *A C-orbit in $\triangle$ does not lift to a systole on S.*

4. Length estimates for systoles

In this section we complete the geometric description of the systoles of a simple triangle surface and its associated ideal surface. As a consequence we obtain that a simple triangle surface which is different from one of the three surfaces listed in the introduction is not maximal.

We resume the assumptions and notations from Section 3. Our goal is to describe all B-orbits in the equilateral triangle $\triangle$ with angle π/p or in an ideal triangle $\triangle_\infty$ which lift to a systole on a simple triangle surface S or its associated ideal surface S_∞. For this it is convenient to consider any piecewise geodesic α in $\triangle$ with the following properties:

a) There is a pair e_1, e_2 of sides of $\triangle$ which is connected by at most one subarc of α.

b) If e_3 is the third side of $\triangle$ then the subcurves α_1, α_2 of α which contain all arcs of α joining e_1, e_2 to e_3 are connected and either $\alpha = \alpha_1\alpha_2$ or $\alpha_1\alpha_2$ is not connected.

We call such a curve *irreducible*. A B-orbit $\tilde{\gamma}$ which is irreducible in this sense and with the additional property that there is a pair of sides of $\triangle$ which is not connected by any geodesic segment of $\tilde{\gamma}$ will be called *a B_0-orbit*. An irreducible B-orbit which is not a B_0-orbit will be called a *B_1-orbit*. In the same way we define irreducible B_0-orbits and B_1-orbits in the ideal triangle $\triangle_\infty$.

A *lift* to S/Γ of an irreducible curve α in $\triangle$ is an admissible closed piecewise geodesic in $S/\Gamma - \{\hat{A}, \hat{B}, \hat{0}\}$ whose trace is invariant under the natural isometry $\hat{\Psi}$ of order 2 of S/Γ exchanging the two triangles and which projects to α. Call two irreducible curves α, β in $\triangle$ *homotopic* if there are lifts of β and α to S/Γ which are freely homotopic in $S/\Gamma - \{\hat{A}, \hat{B}, \hat{0}\}$.

The remark after Lemma 3.5 shows that a B-orbit in $\triangle$ is irreducible in the above sense if and only if its lift to $S/\Gamma - \{\hat{0}, \hat{A}, \hat{B}\}$ is irreducible in the sense of Section 3. Thus we obtain from the results in Section 3.

Corollary 4.1: *A B-orbit in $\triangle$ or $\triangle_\infty$ which lifts to a systole on S or S_∞ is irreducible.*

For the description of all B-orbits in $\triangle$ which lift to a systole of a simple triangle surface we use a length comparison argument. Namely, observe that we can talk about homotopic irreducible arcs in nonisometric hyperbolic triangles in an obvious way. We have.

Lemma 4.2: *Let $q > p \geq 5$ and let $\triangle, \triangle'$ be equilateral triangles with angles $\pi/p, \pi/q$ respectively. Let γ, γ' be two homotopic B-orbits in $\triangle, \triangle'$. Then the length of γ is smaller than the length of γ'.*

Proof: For $t < \pi/3$ denote by T_t the equilateral hyperbolic triangle with angle t. Since a B-orbit is the shortest curve in its homotopy class it suffices to show the following: If $t < t_0 < \pi/3$ and if $\gamma \subset T_{t_0}$ is any B-orbit, then every admissible curve in T_t which is homotopic to γ is longer than γ.

But this follows simply from the fact that for $t < t_0$ the triangle T_{t_0} can be isometrically embedded into the triangle T_t (see [I]). More precisely, the center of the triangle T_t is the unique point in T_t which has the same distance to each of the vertices of T_t. There is an (essentially unique) isometric embedding of T_{t_0} into T_t which maps the center of T_{t_0} to the center of T_t and such that each geodesic in T_t which connects the center to one of the vertices passes through a vertex of T_{t_0}. Map T_{t_0} onto T_t by a diffeomorphism which maps each geodesic γ through the center to itself and scales the parametrization by the proportionality factor length$(\gamma \cap T_t)$/length$(\gamma \cap T_{t_0})$. This map strictly increases the length of nontrivial curves in T_{t_0}. From this the lemma is immediate. **q.e.d.**

Let again Ω be a fundamental $2p$-gon, let $k \in [2, (p+1)/2]$ and let $S =$

$S(p;k)$ be a simple triangle surface. The side pairings for Ω which induce the surface S define a collection of p simple closed geodesics on S which are invariant under the action of the basic group Γ. Each of these geodesics is freely homotopic to the projection to S of a geodesic arc in Ω connecting the midpoint of the side $2i+1$ to the midpoint of the side $2i+2k$. Their projection to S/Γ is the lift of an irreducible B_0-orbit $\tilde{\gamma}_0$ which can be described as follows.

a) $\tilde{\gamma}_0$ has one endpoint on the edge opposite to a vertex $\tilde{0}$ which is the only collision point with this edge.

b) There are k collisions with the edge joining $\tilde{0}$ to a second vertex $\tilde{A}$ and $k-1$ collisions with the edge joining $\tilde{0}$ to the third vertex $\tilde{B}$ for some $k \in [2, p/2]$.

We call a B_0-orbit $\tilde{\gamma}$ with properties a) and b) for an arbitrary $k \leq p/2$ a *side pairing orbit*. With this notation, every minimal B_0-orbit is a side pairing orbit. Moreover a side pairing orbit is determined up to isometries of $\triangle$ by the number of its geodesic segments, or, equivalently, by the number of its collision points with the boundary of $\triangle$. For a simple triangle surface S there are at most three different liftable side pairing orbits (compare Section 2).

Using Lemma 4.2 and a comparison argument we can now estimate the length of a large family of irreducible B-orbits.

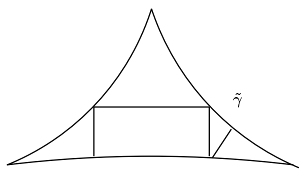
Lemma 4.3: *Let $\tilde{\eta}$ be an irreducible B-orbit. Assume that either*

1. *$\tilde{\eta}$ is a B_1-orbit with at least 5 collisions with the boundary or*

2. *$\tilde{\eta}$ is a B_0-orbit which is not a side pairing orbit and has at least 6 collisions with the boundary.*

Then a lift of $\tilde{\eta}$ to $S/\Gamma - \{\hat{A}, \hat{B}, \hat{0}\}$ is longer than a systole on S.

Proof: By definition, a B_1-orbit contains at least 3 geodesic arcs. Up to isometries of $\triangle$ there is a unique B_1-orbit $\tilde{\gamma}$ consisting of exactly 4 arcs.



This orbit admits a subarc which is homotopic to a side pairing orbit with 3 segments. In particular, if $S = S(p;k)$ admits a liftable side pairing orbit which consists of at most three segments, then this side pairing orbit is homotopic to a proper subarc of $\tilde{\gamma}$ and therefore a lift of $\tilde{\gamma}$ to S/Γ is longer than a systole on S.

Lemma 2.2 shows that for $p \leq 9$ every simple triangle surface of genus $\frac{p-1}{2}$ is isometric to a surface $S(p;m)$ for $m = 2$ or $m = 3$ and hence admits a liftable side pairing orbit which consists of at most 3 segments.

On the other hand, an explicit computation (using Maple or Mathematica) shows that for $p = 11$ the length of $\tilde{\gamma}$ is bigger than $3\text{arccosh}\,\frac{3}{2}$. Thus by Lemma 3.2, Lemma 4.2 and the above, a lift of $\tilde{\gamma}$ to $S/\Gamma - \{\hat{A}, \hat{B}, \hat{0}\}$ is longer than a systole on $S(p;k)$.

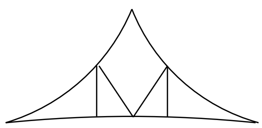
Since every B_1-orbit $\tilde{\eta}$ with at least 5 collisions with the boundary admits a subarc which is homotopic to $\tilde{\gamma}$, our statement for B_1-orbits follows.

Let $\tilde{\eta}$ be a B_0-orbit which is not a side pairing orbit and has at least 6 collisions with the boundary. Denote by C the vertex of $\triangle$ whose adjacent sides are not connected by any subarc of $\tilde{\gamma}$. Then $\tilde{\gamma}$ contains a subarc which consists of two segments and connects the sides adjacent to C. If we replace this arc by a single geodesic segment, then we obtain a shorter curve which contains a subcurve homotopic to the B_1-orbit $\tilde{\gamma}$ above. Thus the statement for B_0-orbits follows once again from the length estimate for $\tilde{\gamma}$. **q.e.d.**

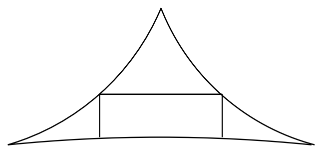
Corollary 4.4: *Every systole on a simple triangle surface is either a lift of the A-orbit* $\tilde{\gamma}_1$ *in* $\triangle$ *or a lift of a side pairing orbit on* $\triangle$.

Proof: By Lemma 4.3, a B-orbit $\tilde{\eta}$ which is not a side pairing orbit can only lift to a systole if either

1) $\tilde{\eta}$ is a B_0-orbit with exactly 5 collisions with the boundary or



2) $\tilde{\eta}$ is a B_1-orbit with exactly 4 collisions with the boundary.



Consider first an orbit $\tilde{\eta}$ as in 1) above. Assume that $\tilde{\eta}$ lifts to a closed geodesic on the surface $S(p;k)$. The lifts of $\tilde{\eta}$ then define piecewise geodesics in the fundamental $2p$-gon Ω.

Choose such a piecewise geodesic η with the property that the center of Ω corresponds to a vertex of $\triangle$ whose adjacent sides are connected by an arc of $\tilde{\eta}$. Then η consists of two components η_1, η_2. After a suitable numbering of the edges of Ω we may assume that η_1 connects the edge 1 to the edge 6 and that η_2 connects the edge $6-2k+1$ to the edge $6-2k+2$ where $k \geq 2$ is such that $S = S(p;k)$.

Since η projects to a closed geodesic on $S(p;k)$ we have $6-4k+3 \equiv 1 \mod 2p$ and therefore $4-2k \equiv 0 \mod p$. Since p is odd and $k \leq p-1$ this is only possible if $k=2$. But then there is a liftable side pairing orbit of $S(p;k)$ which consists of 2 segments and is shorter than $\tilde{\eta}$.

A similar purely combinatorial argument shows that an orbit $\tilde{\eta}$ as in 2) above is not liftable to any simple triangle surface. This shows the lemma.
q.e.d.

Now we are ready to show.

Proposition 4.5 :

1) *For $3 \leq k \leq 5$ the surface $S(k(k-1)+1;k)$ and its associated ideal surface has $3k(k-1)+3$ systoles. These systoles are the images of a single simple closed geodesic under the action of the isometry group of $S(k(k-1)+1;k)$.*

2) *A simple triangle surface S which is not isometric to one of the surfaces $S(k(k-1)+1;k)(3 \leq k \leq 5)$ is not maximal, neither is the ideal surface associated to S.*

Proof: Let $p = 2g+1$ for an arbitrary $g \geq 2$ and let S be a simple triangle surface of genus g.

Recall that there are numbers $k(0), k(A), k(B) \geq 2$ such that the side-pairings of the $2p$-gon Ω defining S with base-point $0, A, B$ identify the edge 1 with the edge $2k(0), 2k(A), 2k(B)$.

Let $k_0 = \min\{k(0), k(A), k(B)\}$ and assume (via renaming) that $k_0 = k(0)$. The projection to S of the geodesic arc $\tilde{\gamma}_0$ in Ω which connects the edge 1 to the edge $2k_0$ and is orthogonal to both edges is then a simple closed geodesic γ_0 in S whose length we denote by ℓ_0.

Corollary 4.4 shows that there are only two possibilities for a systole γ on S.

1) γ is a lift γ_1 of length ℓ_1 of the A-orbit $\tilde{\gamma}_1$ on $\triangle$ of period 3.

2) γ is the image under an isometry of S of the geodesic γ_0 of length ℓ_0.

Consider a surface $S = S(p; k)$ as in Lemma 2.2 which admits a cyclic group Σ of order 3 of isometries normalizing the basic group Γ. If ℓ_0 is smaller than ℓ_1 then S admits $3p = 6g + 3$ systoles which are just the lifts of the unique liftable side pairing orbit for S. We claim that this is the case if and only if $S = S(7; 3)$ or $S = S(13; 4)$ or $S = S(21; 5)$.

To see this recall from Lemma 2.2 that each such surface with these additional symmetries is of the form $S = S(p; k)$ for some $k \geq 3$ and a divisor $p > k$ of $k(k-1)+1$. The unique liftable side pairing orbit for $S(p; k)$ consists of $\min\{k, p-k+1\}$ segments. However, explicit computation shows that a side pairing orbit with 6 segments in an equilateral triangle with angle $\pi/15$ is longer than the upper bound $3\text{arccosh}\,\frac{3}{2}$ for ℓ_1. Together with Lemma 4.2 this shows that if $S(p; k)$ is such that $\ell_0 \leq \ell_1$ then either $p \leq 13$ or $\min\{k, p-k+1\} \leq 5$.

The surfaces $S(7; 3)$ and $S(13; 4)$ are such surfaces $S(p; k)$ with $p \leq 13$. Any further example corresponds to a pair of numbers (p, k) such that $k < p \leq 13$ and that moreover p is a *proper* divisor of $k(k-1)+1$. However the only pairs of this kind are $(13, 10)$ and $(7, 5)$ and we find once again our surfaces $S(13; 10) = S(13; 4)$ and $S(7; 5) = S(7; 3)$.

Next we look for surfaces $S(p; k)$ as above with $\min\{k, p-k+1\} \leq 5$ and such that $p > \min\{k, 14\}$ is a divisor of $k(k-1)+1$. Write $m = p - k$ and assume that $m \leq 4$ and that $p = k + m$ divides
$k(k-1)+1 = (p-m)(p-m-1)+1 = p(p-2m-1)+m(m+1)+1$.
Then p also divides $m(m+1)+1$, and since we assumed that $p \geq 15$ we just obtain the surface $S(21; 17) = S(21; 5)$ as a solution.

In other words, if $\ell_0 \leq \ell_1$ and if $S(p;k)$ admits a cyclic group of order 3 of isometries normalizing the basic group Γ then S is one of the surfaces $S(7;3)$, $S(13;4)$ and $S(21;5)$. Explicit computation now shows that for these surfaces we indeed have $\ell_0 < \ell_1$.

Schmutz observed in [S1] that a closed hyperbolic surface S of genus g can only be maximal if S has at least $6g-5$ systoles. Using this fundamental fact, the proof of our proposition can now be reduced to the above discussion by distinguishing the following 4 cases.

i) $\ell_1 < \ell_0$.

Then only lifts of the A- orbit $\tilde{\gamma}_1$ can be systoles of S. If g is the genus of S then there are $p = 2g+1$ systoles, and S is not maximal.

ii) $S = S(p;2)$ for some $p \geq 5$.

The surface $S(p;2)$ admits a liftable side pairing orbit $\tilde{\gamma}_0$ which consists of 2 segments and hence is shorter than the orbit $\tilde{\gamma}_1$ from Lemma 3.2. Moreover it admits a cyclic group σ of order 2 of isometries which commutes with the basic group Γ. The action of Σ on the sphere S/Γ does not leave the trace of a lift of the side pairing orbit $\tilde{\gamma}_0$ invariant. Thus $S(p;2)$ has exactly $2p = 4g+2$ systoles and can only be maximal if either $g = 2$ or $g = 3$. However an explicit analysis of the surfaces $S(5;2)$ and $S(7;2)$ shows that these surfaces are not maximal (this fact was already established by Schmutz [S1]).

iii) $S \notin \{S(k(k-1)+1;k) \mid k \geq 2\} \cup \{S(p;2) \mid p \geq 5\}$ and $\ell_0 \leq \ell_1$.

Then if $k_0 = k(0)$ we have $k(A) > k_0, k(B) > k_0$ and therefore there are at most $p = 2g+1$ systoles which are lifts of a side pairing orbit in $\triangle$. If $\ell_0 < \ell_1$ then these are the only systoles. In the case $\ell_1 = \ell_0$ (which does not occur if the genus g of S is 2 or 3) there are $4g+2$ systoles. The surface S is not maximal.

iv) $k \in \{3,4,5\}$ and $S = S(k(k-1)+1;k)$.

Then the length ℓ_0 of γ_0 is smaller than ℓ_1 and there are $3p = 6g+3$ systoles which are the images of the geodesic γ_0 under the isometry group of S. In particular, the cardinality of the quotient of the isometry group of S under the subgroup fixing a given systole equals $6g+3$.

To finish the proof of our proposition we have to investigate the ideal surfaces S_∞ associated to simple triangle surfaces $S(p;k)$. The above considerations are equally valid for these surfaces and show that S_∞ has more than $4g+4$ systoles if and only if p divides $k(k-1)+1$ and if the length ℓ_0 of a lift of a side pairing orbit for S_∞ is not bigger than $6\operatorname{arccosh}\frac{3}{2}$. An explicit computation shows as before that this is the case if and only if S_∞ is associated to one of the surfaces $S(7;3), S(13;4), S(21;5)$. **q.e.d**

5. Proof of the theorem

Using the notation of Lemma 2.2, our goal is to show that the triangle surfaces $S(7;3), S(13;4), S(21;5)$ and their associated ideal surfaces are maximal. Following Schmutz [S1], for this it is enough to show that for each of these surfaces S the Teichmüller space is parametrized in a neighborhood of S by the lengths of those closed geodesics which are freely homotopic to a systole on S.

Let for the moment $p \geq 5$ be an arbitrary odd number and let $k \in \{2,\dots,p-1\}$ be such that k and $k-1$ are prime to p. Write $g=(p-1)/2$. As in the introduction let $\mathcal{T}_{g,3}$ be the Teichmüller space of surfaces of genus g with 3 punctures. Let $S=S(p;k)$ and let S_∞ be the ideal surface associated to S. The basic group Γ of orientation preserving isometries of S acts as a group of isometries on the surface S_∞.

It will be useful to give a geometric description of S_∞. For this let $\triangle_\infty$ be an ideal triangle in $\mathbf{H}^2$ and let $T \subset \triangle_\infty$ be the finite equilateral triangle inscribed in $\triangle_\infty$ which is invariant under all isometries of $\triangle_\infty$. The vertices of T determine a distinguished point on each side of $\triangle_\infty$.

There is a unique way to glue $2p$ copies of $\triangle_\infty$ to a disc A with one puncture in its interior and $2p$ punctures on the boundary in such a way that the glueing maps identify the distinguished points on the sides of $\triangle_\infty$. The boundary of A then consists of $2p$ geodesic lines. Each of the triangles which makes up A contains exactly one of these boundary geodesics. We number the boundary geodesics in counter clockwise order and glue the $2i+1$-th geodesic to the $2i+2k$-th geodesic by an orientation reversing isometry which identifies the distinguished points on these geodesics. The resulting surface is the ideal surface S_∞ associated to S. Notice that S_∞ admits a canonical triangulation into ideal triangles which corresponds to the canonical triangulation of S.

Denote by $0,1,2$ the cusps of S_∞. There are p edges of the canonical triangulation which connect the cusp 0 to the cusp 1. There is a natural counter clockwise ordering of these edges which is induced by the ordering of the ideal triangles around the cusp 0. We denote by α_i^0 the i-th edge with respect to this ordering and orient it in such a way that it goes from 0 to 1. Similarly we define α_i^1 to be the i-th edge of our triangulation with respect to the counter-clockwise ordering around the cusp 1 which goes from the cusp 1 to the cusp 2. Let also α_i^2 be the i-th edge ordered around the cusp 2 which goes from the cusp 2 to the cusp 0.

Each marked surface of genus $g = (p-1)/2$ with three punctures can be triangulated by $2p$ ideal triangles. If we cut the surface open along the edges of this triangulation, then we obtain $2p$ ideal triangles. To get the surface back we glue the triangles along their boundary geodesics in the fixed combinatorial pattern as above. The different points in $\mathcal{T}_{g,3}$ then differ by the way this glueing is arranged.

Namely, for each glueing we have one degree of freedom which corresponds to a left earthquake path along one of the geodesic arcs α_i^j. Using the marking given by the distinguished points on the boundary of an ideal triangle and the induced boundary orientation, the glueings of an ordered pair (T_1, T_2) of (oriented) ideal triangles along a boundary geodesic can be parametrized by a real (left) sliding parameter. The glueing which identifies the distinguished points corresponds to the parameter 0. A glueing where the distinguished point on the boundary geodesic of the triangle T_1 is mapped to the right of the distinguished point on the boundary geodesic of the triangle T_2 corresponds to a positive sliding parameter.

Following Thurston [T], in order to obtain a complete hyperbolic surface from the $3p$ glueings of $2p$ ideal triangles in the above combinatorial way, it is necessary and sufficient that at each of the three punctures of the resulting surface the sum of all the sliding parameters for all geodesics which go to this puncture vanishes. Thus if we denote by $V \subset \mathbf{R}^p$ the linear subspace of all vectors which are orthogonal to the vector $(1, \ldots, 1)$, then there is a natural bijection of $\mathcal{T}_{g,3}$ onto $V^3 = V \times V \times V$ which maps a surface $M \in \mathcal{T}_{g,3}$ to its $3p$-tuple of sliding parameters.

Let now γ_i^0 be the piecewise geodesic in S_∞ which consists of the arc α_i^0 with the orientation reversed and the arc α_{i+k}^0. If we compactify the surface S_∞ by adding a point at each puncture, then the compactified curve γ_i^0 is a simple closed curve on $S = S(p;k)$ which is freely homotopic to the closed geodesic ψ_i^0 on S obtained by projecting a geodesic in a fundamental $2p$-gon Ω which connects the midpoints of the edges $2i+1$ and $2i+2k$. Similarly, let $k(1), k(2) \in \{2, \ldots, p-1\}$ be such that $k(1)(k-1)+1 \equiv 0 \bmod p$ and

$k(k(2)-1)+1 \equiv 0 \mod p$ and denote for $j=1,2$ by γ_i^j the piecewise geodesic which consists of the geodesic α_i^j with the reversed orientation and the geodesic $\alpha_{i+k(j)}^j$. Write also $k(0)=k$.

An earthquake path through S_∞ induced by the curve γ_i^j deforms the surface S_∞ by a family of glueings with sliding parameter $-t$ along α_i^j, sliding parameter t along $\alpha_{i+k(j)}^j$ ($t \in \mathbf{R}$) and sliding parameter 0 otherwise and hence this earthquake path gives rise to a smooth (in fact real analytic) curve in $\mathcal{T}_{g,3}$. From this observation the following is immediate.

Lemma 5.1: *For every surface $M \in \mathcal{T}_{g,3}$ the tangents of the earthquake paths along the curves γ_i^j span the tangent space of $\mathcal{T}_{g,3}$ at M.*

Proof: Let $M \in \mathcal{T}_{g,3}$ and denote by ξ_i^j the tangent at M of the earthquake path along α_i^j. We observed above that there is a linear isomorphism of the vector space V^3 onto the tangent space of $\mathcal{T}_{g,3}$ at M which maps the point $(0_1,\dots,0_p,a_1,\dots,a_p,b_1,\dots,b_p) \in V^3$ to the tangent vector $\sum_{i,j} j_i \xi_i^j$. Since the tangent at M of the earthquake path induced by γ_i^j is just $\xi_{i+k(j)}^j - \xi_i^j$ the lemma follows. **q.e.d.**

There is a natural real analytic submersion P of $\mathcal{T}_{g,3}$ onto $\mathcal{T}_g$ which is equivariant under the action of the basic group Γ. This submersion simply maps a surface of genus g with 3 punctures to the surface obtained by compactifying each puncture with a single point. For every $S \in \mathcal{T}_g$ the fibre of P over S consists of all surfaces in $\mathcal{T}_{g,3}$ which we obtain from S by removing an ordered triple of pairwise distinct points. In particular, the fibre is a real analytic submanifold of $\mathcal{T}_{g,3}$ of dimension 6. We denote by W the 6-dimensional subbundle of the tangent bundle of $\mathcal{T}_{g,3}$ which is the kernel of the differential of P. This bundle has a natural direct decomposition $W = W_0 \oplus W_1 \oplus W_2$ into two-dimensional subbundles W_j. Here the bundle W_j is the tangent bundle of the fibres of the fibration $\mathcal{T}_{g,3} \to \mathcal{T}_{g,2}$ which we obtain by adding for every surface $M \in \mathcal{T}_{g,3}$ a single point at the punture j of M.

For $M \in \mathcal{T}_{g,3}$ the compactifications of the curves $P\gamma_i^j$ are homotopically nontrivial simple closed curves on PM. There is a unique free homotopy class on M which can be represented by a closed curve which does not intersect γ_i^j and whose projection to PM is freely homotopic to the compactification of γ_i^j. We denote by $\tilde{\psi}_i^j$ the unique geodesic on M representing this class. We have.

Lemma 5.2: *Let ξ_i^j, ζ_i^j be the tangent of the earthquake path along $\tilde{\psi}_i^j, \gamma_i^j$.*

Then there are functions $a_i^j : \mathcal{T}_{g,3} \to \mathbf{R}$ *such that* $\zeta_i^j - a_i^j \xi_i^j \in W_j \oplus W_{j+1}$.

Proof: Let $M \in \mathcal{T}_{g,3}$ and for $i \in \{1, \dots, p\}, j = 0, 1, 2$ consider the piecewise geodesic γ_i^j and the geodesic $\tilde{\psi}_i^j$ on M. Since the number of intersections between γ_i^j and $\tilde{\psi}_i^j$ is the minimum of the number of intersections between γ_i^j and any curve which is freely homotopic to $\tilde{\psi}_i^j$, the geodesics $\tilde{\psi}_i^j$ and γ_i^j on M do not intersect. If we cut the surface M open along the curves γ_i^j and $\tilde{\psi}_i^j$ then the interior of one of the connected surfaces with boundary which we obtain in this way, say the surface C, is homeomorphic to an open annulus. One boundary component of C is the curve $\tilde{\psi}_i^j$, the second boundary component has two punctures and consists of the curve γ_i^j.

By construction, the curve $\tilde{\psi}_i^j$ is non-separating and therefore there is a simple closed geodesic η on M which neither intersects γ_i^j nor $\tilde{\psi}_i^j$ and such that after cutting M along η we obtain two bordered surfaces S_1, S_2. The surface S_1 is a surface of genus 1 with one geodesic boundary circle and two punctures in its interior and contains the annulus C bounded by the curves γ_i^j and $\tilde{\psi}_i^j$. The earthquake paths along the piecewise geodesic γ_i^j and the geodesic $\tilde{\psi}_i^j$ leave the hyperbolic length of a closed geodesic σ on M fixed if and only if σ does not have a transversal intersection with $\gamma_i^j, \tilde{\psi}_i^j$. Thus these earthquake paths define deformations of the hyperbolic structure on S_1 leaving the length of the boundary fixed.

The Teichmüller space of marked hyperbolic structures on the bordered torus S_1 with two punctures and a boundary geodesic of fixed length is 6-dimensional. Its tangent bundle contains a 5-dimensional subbundle V which consists of all infinitesimal deformations preserving the modulus of a maximal (twice punctured) ring domain with core curve homotopic to $\tilde{\psi}_i^j$.

We claim that this 5-dimensional subbundle V contains the tangents of the earthquake paths along the geodesic $\tilde{\psi}_i^j$ and along the piecewise geodesic γ_i^j.

To see this let ζ be the unique simple geodesic arc in S_1 which meets the boundary geodesic η perpendicularly and which neither intersects $\tilde{\psi}_i^j$ nor γ_i^j. Let $\bar{S}_1$ be the compactification of S_1 which we obtain by simply adding one point at each puncture. If we cut $\bar{S}_1$ open along ζ, then we obtain a standard ring domain A normalized by the fixed choice of a height, say the height 1, with core curve homotopic to $\tilde{\psi}_i^j$ and whose modulus is maximal among all ring domains with this property [St]. The boundary of A consists of two circles which contain each a copy of the arc ζ as well as a nontrivial component of the boundary geodesic η. We mark the arc on each boundary component which corresponds to the arc ζ. The surface $\bar{S}_1$ is obtained by glueing the two marked arcs on the two boundary components with the restriction of a complex linear map of the complex plane.

The compactification of γ_i^j is a closed curve in the ring domain A which is freely homotopic to the core curve. If we cut A open along this curve then by uniformization we obtain two standard ring domains A_1, A_2 with one common boundary circle. The earthquake path induced by γ_i^j consists in cutting A along the common boundary circle of A_1, A_2 and glueing the ring domains A_1, A_2 back with a new boundary identification. This procedure does not change the lengths of the arcs η or ζ nor the modulus of the annulus A. In other words, the tangent of this earthquake path is contained in V. The same argument applies to the earthquake path induced by the geodesic $\tilde{\psi}_i^j$. We conclude that this earthquake path induces a nontrivial infinitesimal deformation of the conformal structure on the compactification of our bordered punctured torus which leaves the modulus of a maximal ring domain with core curve homotopic to $\tilde{\psi}_i^j$ fixed. In particular, the tangent of this earthquake path is contained in V but not in the kernel of the differential of the natural map which assigns to a twice punctured bordered torus its compactification.

As a conclusion, the tangent at M of the earthquake path induced by γ_i^j can be written in the form $a_i^j \xi_i^j + \eta_i^j$ where ξ_i^j is the tangent of the earthquake path along $\tilde{\psi}_i^j$, $a_i^j \in \mathbf{R}$ and η_i^j is contained in the bundle $W_j \oplus W_{j+1}$. This shows the lemma. **q.e.d**

Let now $k \geq 3$ and consider again the ideal surface S_∞ associated to the simple triangle surface $S = S(k(k-1)+1; k)$. Using the above notation, for $m = jp + i$ ($j \in \{0, 1, 2\}, i < p$) write $\tilde{\psi}_m = \tilde{\psi}_i^j$. For $M \in \mathcal{T}_{g,3}$ and $m \in \{1, \ldots, 3p\}$ denote by $\ell_M(\tilde{\psi}_m)$ the length of the closed geodesic $\tilde{\psi}_m$ on M. The functions $M \in \mathcal{T}_{g,3} \to \ell_M(\tilde{\psi}_m)$ are real analytic [K]. This means that we obtain a real analytic map Ψ_∞ of $\mathcal{T}_{g,3}$ into $\mathbf{R}^{3p}$ by mapping a surface M to $\Psi_\infty(M) = (\ell_M(\tilde{\psi}_1), \ldots, \ell_M(\tilde{\psi}_{3p}))$. From Lemma 5.1 and Lemma 5.2 we conclude.

Corollary 5.3: *The map Ψ_∞ is of maximal rank differentiable at S_∞.*

Proof: Following Wolpert [W], the tangent of the earthquake path along $\tilde{\psi}_i^j$ is dual with respect to the Weil Petersen Kähler form to the differential of the length function of $\tilde{\psi}_i^j$ on $\mathcal{T}_{g,3}$. Thus to show the corollary it is enough to show that the tangent space of $\mathcal{T}_{g,3}$ at S_∞ is spanned by the tangents ξ_i^j of the earthquake paths along the curves $\tilde{\psi}_i^j$.

Let G be the group of isometries of S_∞ which is generated by the basic group Γ and the group Σ of order 3 contained in the normalizer of Γ. The group G acts on the Teichmüller space $\mathcal{T}_{g,3}$ as a group of automorphism which fixes the surface S_∞.

Let Λ be the linear isometry of $\mathbf{R}^p$ defined in canonical coordinates by $\Lambda(x_1, \ldots, x_p) = (x_2, \ldots, x_p, x_1)$; then $\Lambda \times \Lambda \times \Lambda = \Lambda_3$ is a linear isometry of $\mathbf{R}^{3p}$. If J_1 is the canonical generator of the normal cyclic subgroup Γ of G then we have $\Psi_\infty(J_1 M) = \Lambda_3 \Psi_\infty(M)$.

Let τ be the linear isometry of $\mathbf{R}^{3p} = \mathbf{R}^p \times \mathbf{R}^p \times \mathbf{R}^p$ which cyclicly permutes the factors $\mathbf{R}^p$ in the direct decomposition of $\mathbf{R}^{3p}$. There is a permutation σ of $\{1, \ldots, p\}$ of order $p-1$ with diagonal extension σ_3 to $\mathbf{R}^{3p}$ such that the canonical generator J_2 of the cyclic subgroup Σ acts by $\Psi_\infty J_2(M) = \sigma_3 \circ \tau(\Psi_\infty M)$.

The eigenvalues of the linear isometry Λ are the p-th roots of unity. The eigenspace for the eigenvalue 1 is spanned by $(1, \ldots, 1)$ and the other generalized eigenspaces are of dimension 2. The map $\sigma_3 \circ \tau$ permutes the generalized eigenspaces of the diagonal extension Λ_3 which correspond to eigenvalues different from 1 and acts as a cyclic group of permutations on the eigenspace Z of Λ^3 with respect to the eigenvalue 1. The orthogonal complement $Z^\perp$ of Z in $\mathbf{R}^{3p}$ decomposes into g irreducible invariant subspaces of dimension 6 each.

The surface S_∞ is a fixed point for the action of G. By Lemma 5.1, the tangent space of $\mathcal{T}_{g,3}$ at S_∞ as a G-space is isomorphic to $Z^\perp$, where the differential of J_1 acts as the map Λ_3 and the differential of J_2 as $\sigma_3 \circ \tau$. The 6-dimensional tangent space W at S_∞ of the fibre of the fibration $P : \mathcal{T}_{g,3} \to \mathcal{T}_g$ is invariant under the action of G and for reasons of dimension necessarily irreducible.

Let as before ξ_i^j, ζ_i^j be the tangent at S_∞ of the earthquake path along $\tilde{\psi}_i^j, \gamma_i^j$.

Denote by L the linear map which maps ζ_i^j to ξ_i^j. Then L is G-equivariant and by Lemma 5.2 its kernel is contained in the G-invariant space W. Since W is irreducible under G the kernel of L is either trivial or coincides with W.

We have to show that the latter does not hold. For this we have to find a tangent vector $X \in W$ such that $LX \neq 0$.

Consider the unit disc D in the complex plane with boundary circle S^1 and hyperbolic metric. Let D_∞ be the disc with the point 0 deleted. It carries a unique complete hyperbolic metric for which the puncture is a standard cusp.

This metric admits an isometric circle action which induces the standard parametrization of the boundary circle $S^1 = [0, 2\pi)$.

Let Ω_0, Ω be the regular ideal hyperbolic $2p$-gon in D_∞, D whose set $\mathcal{P}$ of vertices consists of the points $j\pi/p$ ($j = 1, \dots, 2p$). These $2p$-gons admit a cyclic group of order $2p$ of isometries, and Ω_0 hence is isometric to the once punctured polygon which we obtain by cutting S_∞ along the geodesics of the canonical triangulation joining the cusps 1 and 2.

For an interior point x of $\tilde{\Omega}$ consider the polygon $\Omega_x = \tilde{\Omega} - \{x\}$ with one puncture at x. The punctured polygon Ω_x carries a hyperbolic metric of finite volume such that the boundary consists of $2p$ geodesic lines, and it is naturally triangulated into $2p$ ideal triangles.

Let γ be a hyperbolic geodesic in D through $\gamma(0) = 0$. For every $t \in \mathbf{R}$ there is a unique hyperbolic isometry Ψ_t of D which fixes the endpoints of γ and maps $\gamma(t)$ to 0. The image under Ψ_t of the punctured polygon $\Omega_{\gamma(t)}$ is an ideal hyperbolic polygon with puncture at 0 and whose vertices on S^1 are the points in $\Psi_t\mathcal{P}$. The punctured polygon $\Psi_t\Omega_{\gamma(t)}$ can be obtained from Ω_0 by an earthquake deformation along the geodesics which joins 0 to the vertices of Ω_0 as follows.

Consider an ordered triple (a, b, c) of 3 pairwise distinct points on the boundary circle S^1 of D_∞ arranged in counter clockwise order. These points determine an ideal quadrangle Q which decomposes Q into 2 ideal hyperbolic triangles embedded in D_∞ which have one vertex at 0. Let T_1 be the triangle with vertices a, b, and let T_2 be the triangle with vertices b, c. If the euclidean distance between a and b is smaller than the distance between b and c then the glueing map which gives the quadrangle Q back from the triangles T_1 and T_2 maps the distinguished point of the boundary geodesic of T_1 to the right of the distinguished point on the boundary geodesic of T_2 with respect to the boundary orientation of T_2. In other words, with our above notation the glueing corresponds to a positive sliding parameter.

The derivative of the restriction of Ψ_t to S^1 has a maximum at its repelling fix point z_1 and a minimum at its attracting fix point z_2. It is strictly monotonous on each of the two components of $S^1 - \{z_1, z_2\}$. Let (z_1, z_2) be the component which corresponds to an open interval in $[0, 2\pi)$ with left endpoint z_1. The above analysis shows that the deformation of the polygon Ω_0 which defines $\Psi_t\Omega_{\gamma(t)}$ has a negative sliding parameter for every geodesic which joins 0 to a point in $\mathcal{P} \cap (z_1, z_2)$. The sliding parameter is positive for all geodesics which join 0 to a point in $\mathcal{P} \cap (z_2, z_1)$.

Choose now γ in such a way that its forward endpoint equals $k\pi/2p$ and that its backward endpoint equals $k\pi/2p+\pi$. Let ρ be the reflection of $\tilde{\Omega}$ along γ. This reflection induces an orientation reversing isometry of D_∞ which commutes with the above deformation of Ω_0 along γ. Denote by β_i the geodesic which connects the center 0 to $(k+i)\pi/2p$ $(1\leq i\leq 2p)$ and let ν_i be the tangent of the earthquake path along β_i. By symmetry, the tangent at $t=0$ of our deformation of Ω_0 along γ can be written in the form $\sum a_i\nu_i$ where $a_i<0$ and $a_{i-p}=-a_i$ for $i=1,\dots,p-1$.

Consider now the geodesic $\tilde{\psi}_1^0$. It intersects γ perpendicularly and has $2k-2\geq 2$ additional intersections with the geodesics β_i. For $i\in\{1,\dots,k-1\}$ denote by δ_i the oriented angle of the intersection of $\tilde{\psi}_1^0$ with the geodesic β_i, where we write $\delta_i=\pi/2$ if the geodesics β_i and γ do not intersect. By invariance under ρ we have $\delta_{2p-i}-\pi/2=-(\delta_i-\pi/2)$.

Following Kerckhoff (see [K]), the derivative at $t=0$ of the length of $\tilde{\psi}_1^0$ under our deformation of Ω_0 equals up to a positive constant the sum $\sum a_i\cos\delta_i$. But $0>\cos\delta_i=-\cos\delta_{2p-i}$ for $1\leq i\leq k-1$ and $\cos\delta_i=0$ otherwise and therefore the derivative of the length of $\tilde{\psi}_1^0$ under our above deformation of Ω_0 does not vanish. In other words, the map L does not vanish on W. This finishes the proof of the corollary. **q.e.d.**

Let now again $p\geq 5$ be arbitrary and write $g=(p-1)/2$. Using the above notation, for $M\in\mathcal{T}_{g,3}$ let ψ_i^j be the closed geodesic on the surface PM which is freely homotopic to the compactification of the curve γ_i^j. For $S\in\mathcal{T}_g$ let $\ell_S(\psi_i^j)$ be the length of ψ_i^j. We then obtain a real analytic map Ψ of $\mathcal{T}_g$ into $\mathbf{R}^{3p}$ by mapping S to $\Psi(S)=(\ell_S(\psi_1^0),\dots,\ell_S(\psi_p^2))$.

Theorem B from the introduction is an immediate consequence of the following.

Lemma 5.4: *The map Ψ is of maximal rank differentiable and injective.*

Proof: Let again ξ_i^j be the tangent of the earthquake path along the closed geodesic ψ_i^j. By the results of Wolpert [W] it suffices to show that the tangent space of $\mathcal{T}_g$ at any point S is spanned by the vectors ξ_i^j.

An arbitrary choice of three points in the complement of the curves ψ_i^j on S defines a surface $M\in\mathcal{T}_{g,3}$. The earthquake path in $\mathcal{T}_g$ induced by ψ_i^j naturally lifts to a path in $\mathcal{T}_{g,3}$. The consideration in the proof of Lemma 5.2 shows that this lift is (up to parametrization and up to possibly moving the punctures) just the earthquake path in S_∞ along $\tilde{\psi}_i^j\in M$. This implies by Lemma 5.2 and Lemma 5.1 that the tangent space of $\mathcal{T}_g$ at M is spanned by the vectors $a_i^j\xi_i^j$ and shows that Ψ is of maximal rank differentiable. Since the earthquake paths along the curves γ_i^j parametrize $\mathcal{T}_{g,3}$ the map Ψ is moreover injective. **q.e.d.**

The next corollary is an immediate consequence of Lemma 5.3, Lemma 5.4, Proposition 4.8 and the results of Schmutz in [S1].

Corollary 5.5: *The surfaces* $S(7;3), S(13;4), S(21;5)$ *and their associated ideal surfaces are maximal.*

We conclude the paper with some remarks about the relation between our triangulation and the structure of the Thurston boundary of Teichmüller space.

Consider for the moment an arbitrary closed surface S. A *geodesic current* for S is a locally finite Borel-measure on the space of unoriented geodesics in the hyperbolic plane $\mathbf{H}^2$ which is invariant under the action of the fundamental group $\pi_1(S)$ of S. The space $\mathcal{C}$ of geodesic currents for S only depends on the topological type of S. There is a bilinear form i on $\mathcal{C}$, the so called *intersection form*, which is continuous with respect to the weak*-topology on $\mathcal{C}$. The subset $\mathcal{L}$ of $\mathcal{C}$ of all geodesic currents μ with vanishing self-intersection $i(\mu, \mu) = 0$ is the space of *measured geodesic laminations* and is homeomorphic to $\mathbf{R}^{6g-6}$ [B].

Let $\mathcal{PC}$ and $\mathcal{PL}$ be the projectivization of the space of nonzero geodesic currents and laminations. There is a natural continuous embedding J of the Teichmüller space $\mathcal{T}_g$ into $\mathcal{PC}$ by mapping $M \in \mathcal{T}_g$ to the projectivization $[\lambda_M]$ of its Lebesgue-Liouville current λ_M. The closure of $J(\mathcal{T}_g)$ in $\mathcal{PC}$ is just $J(\mathcal{T}_g) \cup \mathcal{PL}$ [B].

Every simple closed geodesic ψ on S can naturally be viewed as a measured geodesic lamination and hence induces a linear functional on $\mathcal{C}$ via $\mu \to i(\psi, \mu)$. If λ_M is the Lebesgue-Liouville current of a point $M \in \mathcal{T}_g$ in Teichmüller space then $i(\lambda_M, \psi) = \ell_M(\psi)$ is just the M-length of ψ [B]. In particular, the map $M \in \mathcal{T}_g \to i(\lambda_M, \psi)$ is real analytic.

Recall that a collection $\psi_1, \dots, \psi_k$ of simple closed curves on S *fills up* if every geodesic on S intersects one of the curves ψ_i transversely. This is equivalent to saying that the complement of $\{\psi_1, \dots, \psi_k\}$ in S consists of a finite collection of connected simply connected regions. If $\psi_1, \dots, \psi_k$ fills up then for every measured geodesic lamination $\mu \in \mathcal{L}$ the vector $(i(\psi_1, \mu), \dots, i(\psi_k, \mu)) \in \mathbf{R}^k$ does not vanish. Thus if we denote by $P\mathbf{R}^k$ the real projective space of all lines in $\mathbf{R}^k$ and for $0 \neq x \in \mathbf{R}^k$ by $[x] \in P\mathbf{R}^k$ the line in $\mathbf{R}^k$ through x then the map $A : M \in \mathcal{T}_g \to [\ell_M(\psi_1), \ldots, \ell_M(\psi_k)] \in P\mathbf{R}^k$ extends continuously to the Thurston compactification $\mathcal{PL}$ of $\mathcal{T}_g$ by mapping the projective class $[\mu]$ of $\mu \in \mathcal{L}$ to $A([\mu]) = [i(\psi_1, \mu), \ldots, i(\psi_k, \mu)]$. A family $(\psi_1, \ldots, \psi_k)$ of simple closed curves on S is called *parametrizing* for $\mathcal{PL}$ if the map $[\mu] \in \mathcal{PL} \to A([\mu]) = [i(\psi_1, \mu), \ldots, i(\psi_k, \mu)] \in P\mathbf{R}^k$ is an embedding.

It is also possible to define geodesic currents and measured geodesic laminations for hyperbolic surfaces with cusps. By definition, a measured geodesic lamination of such a surface M with cusps is a compact subset of M which is foliated by geodesics and equipped with a transverse invariant measure.

Let now $p \geq 5$ and let $k \in \{2, \ldots, p-1\}$ be such that k and $k-1$ are prime to p. Denote by S_∞ the ideal surface associated to the triangle surface $S(k;p)$ and let γ_i^j the piecewise geodesics as in Lemma 5.1. If ψ is any closed geodesic in S_∞ then ψ does not disappear in the cusps of S_∞ and hence ψ intersects each of the geodesics γ_i^j transversely in a finite number of points. We denote by $i(\psi, \gamma_i^j)$ the number of intersections of ψ with γ_i^j. Since measured laminations on S_∞ have compact support, intersection of closed geodesics with one of the curves γ_i^j extends to a continuous convex-linear functional $i(\gamma_i^j, \cdot)$ on the space $\mathcal{L}_\infty$ of measured geodesic laminations on S_∞.

We have:

Lemma 5.6: *The map* $\mu \in \mathcal{L}_\infty \to A(\mu) =$

$$(i(\gamma_1^0, \mu), \ldots, i(\gamma_p^0, \mu), i(\gamma_1^1, \mu), \ldots, i(\gamma_p^1, \mu), i(\gamma_q^2, \mu), \ldots, i(\gamma_p^2, \mu))$$

is an embedding.

Proof: It suffices to show that every closed geodesic ψ is determined by $A(\psi)$. For this consider again the edges α_i^j of the canonical triangulation of S_∞. It follows immediately from our construction that $A(\psi)$ determines uniquely the tuple

$$C(\psi) = (i(\alpha_1^0, \psi), \ldots, i(\alpha_p^0, \psi), i(\alpha_1^1, \psi), \ldots, i(\alpha_p^1, \psi), i(\alpha_1^2, \psi), \ldots, i(\alpha_p^2, \psi))$$

(compare the proof of Lemma 5.1). Thus it is enough to show that we can reconstruct ψ from $C(\psi)$.

The arcs α_j^i define a triangulation of S_∞ into $2p$ triangles with vertices at the cusps and such that each arc is the side of exactly two triangles. Let ψ be any closed geodesic on S_∞ and let T be a triangle from the triangulation with sides $\beta_1, \beta_2, \beta_3$. Write $j_i = i(\beta_i, \psi)$ and assume that $j_1 \geq j_2 \geq j_3$. Since T is contractible in the compactification of S_∞, the total intersection number $j_1 + j_2 + j_3$ of ψ with the boundary of T is even and hence $j_2 + j_3 - j_1$ is even as well. Moreover we have $j_1 \leq j_2 + j_3$. Draw $\frac{1}{2}(j_2 + j_3 - j_1)$ simple arcs connecting the sides β_2 and β_3, $j_2 - \frac{1}{2}(j_2 + j_3 - j_1)$ simple arcs connecting the sides β_1 and β_2, $j_3 - \frac{1}{2}(j_2 + j_3 - j_1)$ simple arcs connecting the sides β_1 and β_3 in such a way that all these arcs are disjoint. The configuration of these arcs in T is determined up to isotopy by $j_1 \geq j_2 \geq j_3$. But this means that ψ is uniquely determined by $C(\psi)$ (compare the discussion in [FLP]) and hence the lemma follows. **q.e.d.**

Recall that a closed curve ψ on S_∞ is *cusp-parallel* if ψ is homotopic to a multiple of a circle surrounding one of the cusps of S_∞. This is equivalent to saying that the infimum of the lengths of all curves in S_∞ which are freely homotopic to ψ is zero (notice that by abuse of notation we call a contractible curve cusp-parallel as well). A closed curve ψ on S_∞ is freely homotopic to a closed geodesic if and only if ψ is not cusp-parallel.

We define now an equivalence relation on the set of all closed curves on S_∞ as follows: Let $\psi, \eta\colon [0,1] \to S_\infty$ be parametrized closed curves. Call ψ, η *equivalent* if there is a subdivision $0 < t_1 < \cdots < t_k < 1$ of $[0,1]$ and for each i there is a cusp-parallel loop γ_i through $\psi(t_i)$ such that η is freely homotopic to $\psi|_{[0,t_1]} \cup \gamma_1 \cdots \cup \gamma_k \cup \psi|_{[t_k,1]}$.

This is easily seen to be an equivalence relation. The equivalence classes of this relation are in $1-1$–correspondence to the free homotopy classes of closed curves on the surface S. We denote the class of ψ by $[\psi]$. For a closed curve ψ on S_∞ and for $i \in \{1, \ldots, p\}$, $j = 0, 1, 2$ define $\mathcal{J}(\psi, \gamma_i^j)$ to be the infimum of the number of intersections with γ_i^j of all curves η equivalent to ψ.

Let ψ_i^j be the closed geodesic on the surface S which is freely homotopic to the compactification of γ_i^j viewed as a curve on S. For every closed geodesic η on S which is different from a multiple of ψ_i^j the number of intersection points between η and ψ_i^j is the infimum $i(\eta, \psi_i^j)$ of the number of intersection points between all curves freely homotopic to η, ψ_i^j.

We have:

Lemma 5.7: *$\mathcal{J}(\zeta, \gamma_i^j) = i([\zeta], \psi_i^j)$ for every closed curve ζ on S_∞.*

Proof: For every closed curve ζ on S_∞ there is an equivalent curve η such that $\mathcal{J}(\zeta, \gamma_i^j)$ equals the number of intersection points of η with γ_i^j. Now if we compactify S_∞ by adding a point at each cusp, then we obtain a surface M of genus g and η and ζ are freely homotopic on M, γ_i^j is freely homotopic to the curve ψ_i^j. But this means that $\mathcal{J}(\zeta, \gamma_i^j) \geq i([\zeta], \psi_i^j)$.

On the other hand, if ζ is any closed curve on S with a minimal number of intersections with ψ_i^j in its free homotopy class, then we can remove from S three points which do not lie on ζ and such that two of these points lie on ψ_i^j. If we call the resulting surface S_∞ then ζ defines a closed curve ζ_∞ on S_∞, and $i(\zeta, \psi_i^j)$ equals the number of intersection points between ζ_∞ and γ_i^j (where γ_i^j is given as before). This then shows that $\mathcal{J}(\zeta_\infty, \gamma_i^j) \leq i(\zeta, \psi_i^j) = i([\zeta_\infty], \psi_i^j)$ **q.e.d.**

As an immediate consequence of Lemma 5.6 and Lemma 5.7 we obtain.

Corollary 5.8: *The curves ψ_i^j on S are parametrizing for $\mathcal{PL}$. In particular, for every $g \geq 2$ there is a family of $6g+3$ free homotopy classes on a closed surface of genus g which is parametrizing for $\mathcal{PL}$.*

Remark: From [FLP] one immediately obtains a family of $9g-9$ closed curves on a closed surface of genus g which is parametrizing for $\mathcal{PL}$. To my knowledge, the minimal number of simple closed curves with this property is not known.

Acknowledgment: This work was motivated by computer experiments using a computer program written by Roman Koch and was completed while I visited the IHES. I thank the institute for its hospitality.

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