ZOOMING IN AND OUT: THE WORK OF MARYAM MIRZAKHANI FROM THE EYES OF A GEOMETER

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1. Prelude

Let us imagine that we embarked in an attempt to understand the geometry of the universe. How would be proceed?

A perhaps naive idea is as follows. Identify some small region in the universe we are reasonably familiar with. What this means can be argued about, but the amount of information used to gain the level of understanding we feel comfortable with should be controlled. Then we send a spacecraft from our region in some arbitrary direction to collect information on the change of shape with respect to the fixed reference region. As the journey of the spacecraft continues and we collect more information, we may have to aquire more knowledge about the reference region to detect hidden similarities with the pictures taken along the journey. The vague hope is that the geometric interaction of nearby regions on the small scale reveals the laws of nature which control the large scale shape of the universe as recorded in the random sampling.

To develop an understanding of the shape of the universe along such lines requires an intimate understanding of what "shapes" are supposed to represent, how we can describe their interaction, how simultaneously zooming in and out can be done in such a way that pattern formation can be correctly observed.

In my mind, Maryam Mirzakhani succesfully implemented this program for the universe formed by Riemann surfaces and their canonical bundles, ie for the moduli space of Riemann surfaces and the moduli space of abelian differentials. These moduli spaces are well studied mathematical objects which naturally emerge in almost any mathematical context I can think of. Building on the work of many distinguished researchers, Maryam developed a new geometric way to study these spaces which puts earlier seemingly unrelated results into complete harmony and, at the same time, creates many new research directions which should be explored by future generations of mathematicians.

In the sequel I will discuss three of Maryam's results which were obtained using the principle of zooming in and zooming out. Geometry should be

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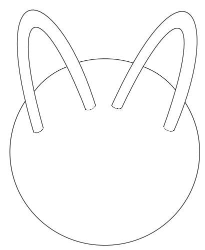
understood in the broad sense of measurement, which includes counting of shapes sorted by sizes etc.

2. Riemann surfaces

An oriented *surface* is two-dimensional oriented manifold and as such a quite simple object, in particular if it is closed (ie compact, without boundary). Viewed as topological or a smooth manifolds, surfaces can easily be classified. The most basic closed surface is the two-sphere, and any other closed surface arises from the two-sphere by attaching a number $g \geq 0$ of so-called *handles*. The number of handles attached is the genus of the surface.

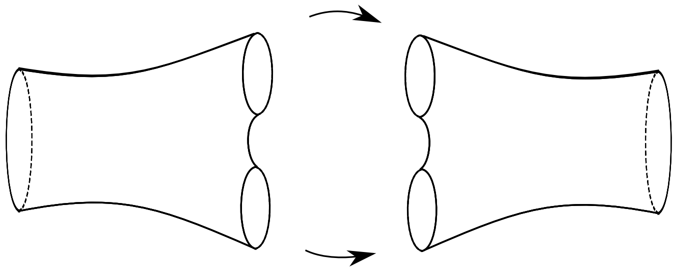


Once this classification is established, seemingly there is not much more to say, with the exception of understanding the different geometric shapes a fixed closed surface S admits. The perhaps best known classical result which restricts the geometric structures on S is the *Gauss Bonnet theorem*. It says that if we denote by K the Gauss curvature of a smooth Riemannian metric on S, then the integral $\frac{1}{2\pi}\int_S K d\mathrm{vol}$ equals the *Euler characteristic* $\chi(S) = 2-2g$ of S, where vol is the volume form of the metric. Furthermore, for $g \geq 2$ there exists a *hyperbolic metric* on S, ie a metric of constant Gauss curvature -1.

The space of all isometry classes of such metrics can be described as follows. First, the set of all *marked* hyperbolic structures on S is defined to be the set of equivalence classes of hyperbolic metrics where two such metrics are identified if they can be transformed into each other by a diffeomorphism of S isotopic to the identity. This so-called *Teichmüller space* $\mathcal{T}(S)$ admits a natural topology which identifies it with an open cell homeomorphic to $\mathbb{R}^{6g-6}$. The group of isotopy classes of diffeomorphisms of S coincides with the quotient of the automorphism group of $\pi_1(S)$ by the subgroup of inner automorphisms. This so-called *mapping class group* $\mathrm{Mod}(S)$ of S acts from the left on $\mathcal{T}(S)$ by precomposition. The space of isometry classes of hyperbolic surfaces is the quotient $\mathrm{Mod}(S)\backslash\mathcal{T}(S)$, called the *moduli space* of S.

3. Zooming in: Local from global

For a fixed hyperbolic metric on a closed surface S of genus g, there exists a decomposition of S into geometric pieces, so-called *hyperbolic pairs* *of pants*. Topologically, such a hyperbolic pair of pants is a two-sphere from which the interiors of three disjoint compact disks have been removed. This decomposition is done by cutting S open along a collection of $3g-3$ pairwise disjoint *simple* geodesics. Here a geodesic is simple if its does not have self-intersections.



Clearly, the geometric description of the hyperbolic surface obtained from such a decomposition into hyperbolic pairs of pants depends on the choice of the boundary geodesics. A natural question is how many different such decompositions exist for a fixed hyperbolic metric. The easy answer is: This number is infinite as there are simple closed geodesics on S of arbitrary length, and each simple closed geodesic belongs to such a decomposition. But what about restricting the geometry, ie bounding the length of these geodesics?

A result going back to Huber and Selberg states that the number $c(L)$ of singly covered closed geodesics of length at most L on a hyperbolic surface S is asymptotic to e^L/L as $L \to \infty$. In particular, the growth rate of the number of such geodesics is not only independent of the specific hyperbolic metric on S, but also independent of the genus of S.

However, most of these geodesics have self-intersections, and if we want to understand the geomety of a hyperbolic surface S by decomposing it into pairs of pants, we need to understand simple closed geodesics on S. And, as we vary the hyperbolic metric over moduli space, we may expect that these simple geodesics and their growth rate give information on the geometry of the surface.

In her thesis, Maryam gave an affirmative answer to quite a few questions of this type. Among others she showed (the following result is a special case of Theorem 1.1 of [8])

Theorem 1. *For a closed hyperbolic surface X of genus $g \geq 2$ and $L > 0$ let us denote by $s_X(L)$ the number of distinct geodesic pairs of pants decompositions of X of total length at most L. Then*

$$\lim_{L\to\infty} \frac{s_X(L)}{L^{6g-6+2n}} = n(X)$$

where $n : \mathcal{M}_g \to \mathbb{R}_+$ is a continuous proper function.

Thus as moduli space is non-compact, not only the growth rate of the number of pants decompositions of bounded length depends on the genus of the surface, but it also depends on the surface itself!

Maryam showed this result by first zooming out and then zooming in. The first step is to prove that for fixed L, the locally bounded function $X \to s_X(L)$ can be integrated over moduli space, where this moduli space is equipped with the so-called Weil-Petersson volume form, the volume form of an (incomplete) Kähler metric. She discovers that this integral, denoted here by $p(L)$, is a polynomial in L of degree $6g-6$.

Some pointwise estimates on the growth rate $s_X(L)$ were known earlier. But it requires far-reaching insight and courage to believe that one can basically effectively compute this integral by finding a global relation between these integrals for all $L>0$. Maryam explains this explicitly on p.99 in [8]: For a fixed pants decomposition γ of S, "the crux of the matter is to understand the density of $\mathrm{Mod}_g \cdot \gamma$ in $\mathcal{ML}_g$". Here $\mathcal{ML}_g$ is the space of so-called *measured geodesic laminations* which is homeomorphic to an open cone of dimension $6g-6$, ie an euclidean space with the point 0 removed. It contains *weigted simple multicurves*, ie collections of disjoint closed geodesics equipped with some positive weights, as a dense subset, and it admits an action of the multiplicative group $(0,\infty)$ by scaling.

Each *marked* hyperbolic metric on S defines a continuous length function on $\mathcal{ML}_g$. This function associates to a multicurve the length of its geodesic representative for this metric.

The mapping class group $\mathrm{Mod}(S)$ acts transitively on the multicurves which define a pants decomposition of a fixed combinatorial type. If one equips each component of such a multicurve with the weight $T>0$, then these weighted point masses define a $\mathrm{Mod}(S)$-invariant Radon measure μ_T on $\mathcal{ML}_g$. Maryam observes that dividing by T^{6g-6}, the total mass of the set of laminations of length at most one for a fixed marked hyperbolic structure determines the value of $s_X(T)$. Furthermore, the measures μ_T converge as $T\to\infty$ to a $\mathrm{Mod}(S)$-invariant Radon measure λ on $\mathcal{ML}_g$ which has earlier been constructed by Thurston. Thus zooming in again to the unit length ball defined by a fixed hyperbolic metric, she obtains the asymptotics of the function $s_X(L)$ as $L\to\infty$.

Maryam's result indicates that the action of the mapping class group on $\mathcal{ML}_g$ has properties which resemble a more familiar setting in homogeneous dynamics: The group $SL(2,\mathbb{Z})$ acts by linear transformations on the puncture plane $\mathbb{R}^2-\{0\}$. This action preserves the Radon measure which is just the sum of the Dirac masses on the standard integral grid $\mathbb{Z}^2$ or on any dilation of this grid, and it also preserves the Lebesgue measure of $\mathbb{R}^2$. These measures are all ergodic under the action of $SL(2,\mathbb{R})$, ie there are no invariant Borel subsets B of $\mathbb{R}^2$ of positive measure so that the measure of $\mathbb{R}^2-B$ is positive as well, and in fact, they are the only ergodic $SL(2,\mathbb{Z})$-invariant Radon measures up to scale, with λ being the only non-wandering measure.

It turns out that the dynamics of the action of $\mathrm{Mod}(S)$ on $\mathcal{ML}_g$ has very similar properties: In joint work with Lindenstrauss [9], Maryam proved that the Thurston measure λ is the only non-wandering invariant Radon measure on $\mathcal{ML}_g$ up to scale (a different proof of this result can be found in [7]).

4. Hyperbolicity

Maryam's thesis was built on the interplay between geometric information on individual hyperbolic surfaces and the geometry of moduli space, and it culminated in a counting result for simple closed multicurves on hyperbolic surfaces. But in which way does moduli space resemble the Riemann surfaces it is made of?

It is classical that moduli space admits a natural complete geometric structure in its own right, the so-called *Teichmüller metric*. Although this metric is not Riemannian, nor is it non-positively curved in any reasonable sense, it shares many global features of a hyperbolic metric on a surface. In particular, any two points in Teichmüller space can be connected by a unique geodesic for the pullback of this metric, and periodic geodesics in moduli space are in bijection with conjugacy classes of so-called pseudo-Anosov mapping classes. These periodic geodesics can be sorted and counted according to their length.

In joint work with Alex Eskin [2] and with Eskin and Rafi [5], Maryam uses random sampling and local information on surfaces to obtain a precise counting results for periodic geodesics in moduli space. Starting point for this endeavor is the (classical) fact that the cotangent space of moduli space at a surface X can naturally be identified with the vector space of holomorphic quadratic differentials for X. More precisely, the hyperbolic metric of X determines a complex structure, and a holomorphic quadratic differential for this structure is a holomorphic section of the bundle $(K^*)^2$ (here K^* is the cotangent bundle).

The geometric beauty of this observation lies in the fact that each non-trivial quadratic differential defines a new geometric structure on the surface S in the same conformal class as the hyperbolic structure. This geometric structure is a flat (ie Euclidean) metric with finitely many cone points of cone angle an integral multiple of π. As one varies over the fibre of the cotangent bundle at a given point, these flat metrics vary as well, and the space of all such flat metrics reflects the geometric shape of the hyperbolic metric.

Even more is true: Each such flat metric comes equipped with a pair of preferred orthogonal directions, the so-called horizontal and the vertical direction, which are defined on the complement of the cone points Σ of the differentials and which are just the line bundles on $S-\Sigma$ on which the evaluation of the differential is positive or negative, respectively. These directions define orthogonal foliations of the surface away from the cone points. Furthermore, there is a natural one-parameter group Φ^t of transformations of such flat surfaces which consists in scaling the horizontal direction with the scaling factor $e^{t/2}$ and the vertical direction with the scaling factor $e^{-t/2}$. These transformations clearly preserve the surface area of the flat metric. Two such flat surfaces define the same cotangent vector in moduli space if they can be transformed into each other with finitely many cut- and paste operations.

The thus defined flow Φ^t on the unit area subbundle of the cotangent bundle of moduli space can be analyzed by zooming out: The local change of shape along a flow line can easily be described. A main result of Eskin and Mirzakhani [2] is the following

Theorem 2. *As $R \to \infty$, the number of periodic trajectories of the flow Φ^t of length as most R is asymptotic to $e^{(6g-6)R}/(6g-6)R$.*

This result was later extended in joint work with Eskin and Rafi with a much more sophisticated argument to so-called *strata* in the moduli space of abelian and quadratic differentials.

The idea of proof consists in zooming out in random directions and averaging the random samples. Namely, it was noted earlier that the flow Φ^t has properties resembling the properties of a hyperbolic flow as long as the trajectories remain in a fixed compact set. What this means precisely was clarified by Maryam in joint work with Atreya, Bufetov and Eskin [1]. But the moduli space is very notably non-compact, and the metric near the cusp resemples a sup metric which can be thought of as a metric of infinite *positive* curvature. There are many recent results which make this idea precise.

Now moduli space is the quotient of Teichmüller space $\mathcal{T}(S)$ under the action of the mapping class group. Moving in Teichmüller space in a random direction can be thought of as looking at a randomly chosen finite set of points in some large ball in $\mathcal{T}(S)$ and trying to understand the change of shape of a hyperbolic or flat surface sampled on these finitely many points. A careful analysis of this sampling process reveals a bias of such random samples to return to the so-called thick part of Teichmüller space. This is defined to be the preimage of a fixed compact subset of moduli space. Knowing furthermore hyperbolic behavior of the flow Φ^t on the cotangent bundle of the thick part allows Eskin and Mirzakhani to show that the growth rate of periodic orbits which are disjoint from some fixed compact subset of moduli space is strictly smaller than the growth rate of all orbits. That this is sufficient for the above counting result can be established by an adaptation of a classical strategy of Margulis.

5. Hidden symmetries and homogeneous dynamics

The singular euclidean metric on the surface S defined by a quadratic differential is given by a collection of charts on the complement of the finite number of singular points with values in the complex plane. Chart transitions are translations or compositions of translations with the reflection $z \to -z$. The differential is the square of a holomorphic one-form, or, equivalently, a holomorphic section of the cotangent bundle K^* of S, if all chart transitions can be chosen to be translations. The differentials is then called *abelian*. Postcomposition of charts then defines a right action of the group $SL(2,\mathbb{R})$ on the moduli space of abelian or quadratic differentials. The action of the diagonal group is just the Teichmüller flow Φ^t.

The $SL(2,\mathbb{R})$-action preserves the so-called *strata* which consists of differentials with the same number and order of singular points. Strata are complex sub-orbifolds in the moduli space of holomorphic one-forms on Riemann surfaces.

Let P be the set of singular points of an abelian differential ω. Then ω determines by integration over a smooth path a class in $H^1(S,P;\mathbb{C})$. In fact, this class determines ω locally uniquely in the following sense. Integration over a fixed basis of $H_1(S,P;\mathbb{Z})$ yields so-called period coordinates for the stratum. These period coordinates equip the stratum with an $SL(2,\mathbb{R})$-invariant affine structure, ie local charts with affine chart transitions. These chart transitions are moreover volume preserving and hence define a Lebesgue measure on the stratum. This so-called *Masur-Veech measure* is finite and invariant under the action of $SL(2,\mathbb{R})$.

Orbits of the $SL(2,\mathbb{R})$-action on the moduli space of area one abelian differentials can be identified with the unit tangent bundle of so-called *Teichmüller disks* which are totally geodesic immersed hyperbolic disks in moduli space. The analog of periodic Teichmüller geodesics are *closed* $SL(2,\mathbb{R})$-orbits. Such orbits project to finite volume immersed hyperbolic surfaces in moduli space. It is known that such closed orbits are dense in any stratum. But how can one globally understand the dynamics of the $SL(2,\mathbb{R})$-action?

In homogenous dynamics, the celebrated solution of a conjecture of Raghunathan by Marina Ratner (who sadly passed away one week before the death of Maryam Mirzakhani) states the following (this is a special case of her more general result).

Consider a simple Lie group G of non-compact type and a lattice $\Gamma < G$. Let furthermore $H < G$ be a closed subgroup which is generated by unipotent elements. The group H acts by right translation on $\Gamma\backslash G$. Then for every H-invariant Borel probability measure μ on $\Gamma\backslash G$ there exists a closed subgroup $L < G$ with the following properties.

(1) $\Gamma \cap L$ is a lattice in L.
(2) μ is the projection of the Haar measure on L to $\Gamma \cap L\backslash L$.

Furthermore, the closure of each H-orbit on $\Gamma\backslash G$ equals the closed orbit of a Lie subgroup $L < G$ which intersects Γ in a lattice.

Thus Ratner's theorem gives a complete classification of orbit closures on quotients $\Gamma\backslash G$ as well as of invariant measures.

Can one formulate a conjecture which translates Ratner's theorem into the framework of the action of $SL(2,\mathbb{R})$ on a stratum of area one abelian differentials? Remembering that an orbit closure in $\Gamma\backslash G$ is algebraic, an ad-hoc guess could be that such an orbit closure is an affine suborbifold, ie cut out by polynomial equations in affine coordinate charts. But the affine structure on strata of moduli space is obtained by taking periods of holomorphic one-forms over relative homology classes, with an integral structure that arises from integral homology, and a priori, this is not in harmony with algebraic structures obtained from global algebraic geometric information.

The amazing rigidity result of Eskin and Mirzakhani [3] states as a special case: Every $SL(2, \mathbb{R})$-invariant Borel probability measure on the moduli space of area one abelian differentials is an affine measure of an affine invariant manifold. Together with Mohammadi, they also show that every orbit closure is an affine invariant manifold. By a result of Filip [6], this result can be translated into an algebraic geometric statement for moduli space: The quotient of an affine invariant manifold by the natural circle group of rotations which multiplies an abelian differential by a complex number of absolute value one is a projective subvarity of the projective bundle over moduli space whose fibre at a point X equals the projectivization of the vector space of holomorphic one-forms on X.

But how can one prove this result? The answer is: by an amazing construction which simultaneously zooms in and out. Random sampling is used to reveal the global geometry. One of the main tools are random walks, and vast and sophisticated knowledge about such random walks which provides information on deviation, stationary measures, drifts and structural insights related to the idea of proximality enters into the argument.

It requires the amazing insight, the unsurpassed optimism and the enormous technical strength of Maryam Mirzakhani to carry out this program, using along the way tools from areas of mathematics as diverse as geometry, ergodic theory, homogeneous dynamics, algebraic groups, random walks, algebraic geometry and mathematical physics. Her vision is the door to a wonderland where very concrete but seeminly unrelated mathematical structures combine to a composition of extreme complexity and breathtaking beauty, yet which is made from simple tunes and is orchestrated in perfect harmony. There is no coda, but there is an unspoken invitation to everyone to extract his or her favorite line and explore its variations and ramifications.

I met Maryam for the first time at the occasion of a workshop in Chicago (probably in 2002). Curtis McMullen introduced her to me during lunch, and, as mathematicians do, the small lunch group chatted about math. She fully participated in the discussion, making interesting comments and answered a question that came up. Right after giving the answer, she realized that what she had said was not quite right, and she corrected herself with a laughter, mockingly embarrassed about her inaccuracy.

During our encounters in later years, I found many times again this laughter, reflecting her professional ambition, passion and optimism which enabled her to adjust and fine-tune her fantastic vision until the next step had emerged in complete clarity and in an esthetically pleasing way. Looking backwards, I am very grateful that I had the priviledge to know her in person, early on.

With her creativity, persistence and courage, Maryam Mirzakhani made a contribution to mathematics which I believe will be remembered and further developed by many future generations of mathematicians. Her memory should in particular serve as inspiration and role model for all young women interested in pursuing a career in mathematics.

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