GEOMETRIC PROPERTIES OF THE MAPPING CLASS GROUP

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1. Introduction

Consider a compact oriented surface S of genus $g \geq 0$ from which $m \geq 0$ points, so-called *punctures*, have been deleted. The *mapping class group* $\mathcal{M}_{g,m}$ of S is the space of isotopy classes of orientation preserving homeomorphisms of S. It is a subgroup of index 2 in the *extended mapping class group* $\mathcal{M}_{g,m}^{\pm}$ defined as the group of isotopy classes of *all* homeomorphisms of S.

The extended mapping class group of the two-sphere S^2 is the group $\mathbb{Z}_2$ generated by the orientation-reversing involution $z \to -z$ (where we identify S^2 with the set $\{z \in \mathbb{R}^3 \mid \|z\| = 1\}$). In the case that S is a *closed* surface different from S^2 it was shown by Dehn, Nielsen and Baer that the extended mapping class group coincides with the group of outer automorphisms of the fundamental group $\pi_1(S)$ of S (see Section 2.9 of [I] for details and references). In particular, the mapping class group of the two-torus equals the group $SL(2,\mathbb{Z})$. Similarly, the extended mapping class group of a surface S with punctures coincides with the group of outer automorphisms of $\pi_1(S)$ which preserve the *peripheral structure*, i.e. the set of conjugacy classes of elements of $\pi_1(S)$ which can be represented by simple closed curves homotopic into a puncture. Thus the mapping class group $\mathcal{M}_{1,1}$ of the once punctured torus coincides with $SL(2,\mathbb{Z})$, and the mapping class group $\mathcal{M}_{0,4}$ of the four punctured sphere has a subgroup of finite index isomorphic to $SL(2,\mathbb{Z})$. We therefore only consider *non-exceptional* surfaces, i.e. we restrict to the case that $3g - 3 + m \geq 2$.

The mapping class groups have been intensively studied in the past. Many of the most important results known to date are described in the beautiful survey of Ivanov [I] which also contains an extensive list of references. The goal of this note is to present open problems about the mapping class groups of geometric nature. Our presentation includes the discussion of geometric results on the mapping class groups which either were obtained after the appearance of Ivanov's survey or can be understood with the present knowledge in a more consistent and unified way.

Our point of view will be the one of geometric group theory. This is possible because the mapping class group $\mathcal{M}_{g,m}$ is finitely generated (it is even finitely presented, see Section 4.3 of [I]). A finite symmetric set of generators $\mathcal{G}$ for $\mathcal{M}_{g,m}$ defines a *word norm* $\| \ \|$ on $\mathcal{M}_{g,m}$ where $\|g\|$ equals the smallest length of a word in the generators $\mathcal{G}$ which represents g. This word norm in turn induces a distance

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function on $\mathcal{M}_{g,m}$ which is invariant under the left action of $\mathcal{M}_{g,m}$ on itself by defining $d(g,h) = \|g^{-1}h\|$. The distance function depends on the choice of $\mathcal{G}$, but its large-scale properties are independent of this choice. Namely, changing the set of generators changes the metric d to an equivalent metric d' which means that the identity $(\mathcal{M}_{g,m}, d) \to (\mathcal{M}_{g,m}, d')$ is a bilipschitz map.

A basic idea of geometric group theory is to relate geometric properties of a finitely generated group Γ to global group theoretic properties of Γ. A fundamental and striking example for such an interplay between geometry and group theory is the following celebrated theorem of Gromov [G]. Define the *growth function* of a finitely generated group Γ as follows. For $R > 0$, let $m(R)$ be the number of elements in Γ whose distance to the identity element with respect to some fixed word norm is at most R. The group is called *of polynomial growth* if there is some $d > 0$ such that $m(R) < dR^d$ for all R. Note that this property is independent of the choice of generators. Gromov shows that a group is of polynomial growth if and only if it is *virtually nilpotent*, i.e. if it contains a nilpotent subgroup of finite index.

On the other hand, a group Γ which acts on a *geodesic* metric space X as a group of isometries inherits from X large-scale geometric properties provided that the action satisfies some discreteness assumptions. The most elementary result along this line is the theorem of Švarc and Milnor (see Chapter I.8 in [BH]) which can be stated as follows. Assume that a countable group Γ acts as a group of isometries on a proper geodesic metric space (X,d). If the action is proper and cocompact then Γ is finitely generated, and for any $x \in X$ the orbit map $g \in \Gamma \to gx \in X$ is a *quasi-isometry*. This means that if we denote by d_Γ any distance on Γ defined by a finite generating set, then there is a number $c > 0$ such that $d_\Gamma(g,h)/c - c \leq d(gx,hx) \leq cd_\Gamma(g,h) + c$ for all $g,h \in \Gamma$.

There are two natural metric graphs on which the mapping class group $\mathcal{M}_{g,m}$ acts by isometries. These graphs are the so-called *curve complex* (or, rather, its one-skeleton) and the *train track complex*. In Chapter 3 we give a description of the curve complex and its most relevant geometric properties. In Chapter 4 we discuss the action of $\mathcal{M}_{g,m}$ on the curve complex and some of its consequences for the structure of $\mathcal{M}_{g,m}$. In Chapter 5 we introduce the complex of train tracks and show how it can be used to study $\mathcal{M}_{g,m}$. In Chapter 2 we collect those properties of train tracks and geodesic laminations which are important for the later chapters. Chapters 3-5 also contain a collection of open problems.

2. Geodesic laminations and train tracks

A *geodesic lamination* for a complete hyperbolic structure of finite volume on S is a compact subset of S which is foliated into simple geodesics. Particular geodesic laminations are simple closed geodesics, i.e. laminations which consist of a single leaf. A geodesic lamination λ is called *minimal* if each of its half-leaves is dense in λ. Thus a simple closed geodesic is a minimal geodesic lamination. A minimal geodesic lamination with more than one leaf has uncountably many leaves. Every geodesic lamination λ is a disjoint union of finitely many minimal components and a finite number of non-compact isolated leaves. Each of the isolated leaves of λ

either is an isolated closed geodesic and hence a minimal component, or it *spirals* about one or two minimal components ([Bo1], Theorem 4.2.8 of [CEG], [O]).

A geodesic lamination λ is *maximal* if all its complementary components are ideal triangles or once punctured monogons. A geodesic lamination is called *complete* if it is maximal and can be approximated in the Hausdorff topology for compact subsets of S by simple closed geodesics. Every minimal geodesic lamination λ is a *sublamination* of a complete geodesic lamination [H1], i.e. there is a complete geodesic lamination which contains λ as a closed subset. In particular, every simple closed geodesic c on S is a sublamination of a complete geodesic lamination. Such a lamination can be constructed as follows. Let P be a geodesic *pants decomposition* for S containing c; this means that P consists of a collection of $3g-3+m$ simple closed pairwise disjoint geodesics, and c is one of these. Then $S-P$ consists of $2g-2+m$ connected components, and each of these components is a *pairs of pants*, i.e. an oriented surface homeomorphic to a thrice punctured sphere. The metric completion of each such pair of pants is a bordered surface with one, two or three boundary circles depending on the number of punctures of S which it contains. For each such pair of pants S_0 choose a maximal collection of simple disjoint geodesics embedded in S_0 which spiral about the boundary circles of its metric completion $\overline{S}_0$. We also require that for every pair c,d of boundary components of S_0 there is a geodesic from the collection which spirals in one direction about c, in the other direction about d. In particular, there is at least one geodesic spiraling from each side of a curve from the collection P. We require that the spiraling directions from both sides of such a pants curve are opposite. The resulting lamination is then complete [H1].

A *measured geodesic lamination* on S is a geodesic lamination λ together with a translation invariant transverse measure supported in λ. Here a transverse measure for λ assigns to every smooth arc c on S with endpoints in the complement of λ and which intersects λ transversely a measure on c supported in $c\cap\lambda$. These measures transform in the natural way under homotopies of c by smooth arcs transverse to λ which move the endpoints of the arc c within fixed complementary components. The support of the measure is the smallest sublamination ν of λ such that the measure on any arc c which does not intersect ν is trivial. This support is necessarily a union of minimal components of λ. As an example, every simple closed geodesic γ naturally carries a transverse *counting measure* which associates to an arc c as above the sum of the Dirac masses at the intersection points between c and γ. If μ is any transverse measure for λ, then for every $a>0$ the same is true for $a\mu$ and hence the group $(0,\infty)$ of positive reals naturally acts on the space $\mathcal{ML}$ of measured geodesic laminations. The space $\mathcal{ML}$ carries a natural topology, the so-called weak*-topology, which locally restricts to the usual weak*-topology for measures on transverse arcs. The action of $(0,\infty)$ is continuous with respect to the weak*-topology. The projectivization of $\mathcal{ML}-\{0\}$ is the space $\mathcal{PML}$ of *projective measured laminations* on S. Equipped with the quotient of the weak*-topology, $\mathcal{PML}$ is homeomorphic to a sphere of dimension $6g-7+2m$; in particular, $\mathcal{PML}$ is compact (for all this, see [FLP] and [PH], in particular Theorem 3.1.4).

The *intersection number* $i(\gamma,\delta)$ between two simple closed curves $\gamma,\delta\in\mathcal{C}(S)$ equals the minimal number of intersection points between representatives of the free homotopy classes of γ, δ. This intersection number extends bilinearly to a continuous pairing for measured geodesic laminations on S.

A *train track* on the surface S is an embedded 1-complex $\tau \subset S$ whose edges (called *branches*) are smooth arcs with well-defined tangent vectors at the endpoints. At any vertex (called a *switch*) the incident edges are mutually tangent. Through each switch there is a path of class C^1 which is embedded in τ and contains the switch in its interior. In particular, the branches which are incident on a fixed switch are divided into "incoming" and "outgoing" branches according to their inward pointing tangent at the switch. Each closed curve component of τ has a unique bivalent switch, and all other switches are at least trivalent. The complementary regions of the train track have negative Euler characteristic, which means that they are different from discs with $0, 1$ or 2 cusps at the boundary and different from annuli and once-punctured discs with no cusps at the boundary. We always identify train tracks which are isotopic. A detailed account on train tracks can be found in [PH] and [M2].

A train track is called *generic* if all switches are at most trivalent. The train track τ is called *transversely recurrent* if every branch b of τ is intersected by an embedded simple closed curve $c = c(b) \subset S$ which intersects τ transversely and is such that $S - \tau - c$ does not contain an embedded *bigon*, i.e. a disc with two corners at the boundary. In this case we say that c *hits* τ *efficiently.*

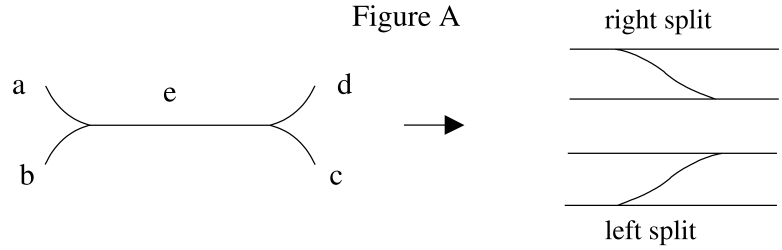
A geodesic lamination or a train track λ is *carried* by a transversely recurrent train track τ if there is a map $F : S \to S$ of class C^1 which is isotopic to the identity and maps λ to τ in such a way that the restriction of its differential dF to every tangent line of λ is non-singular. Note that this makes sense since a train track has a tangent line everywhere. A train track τ is called *complete* if it is generic and transversely recurrent and if it carries a complete geodesic lamination [H1].

If c is a simple closed curve carried by τ with carrying map $F : c \to \tau$ then c defines a *counting measure* μ_c on τ. This counting measure is the non-negative weight function on the branches of τ which associates to an open branch b of τ the number of connected components of $F^{-1}(b)$. A counting measure is an example for a *transverse measure* on τ which is defined to be a nonnegative weight function μ on the branches of τ satisfying the *switch condition*: For every switch s of τ, the sum of the weights over all incoming branches at s is required to coincide with the sum of the weights over all outgoing branches at s. The set $V(\tau)$ of all transverse measures on τ is a closed convex cone in a linear space and hence topologically it is a closed cell. For every transverse measure μ on τ there is a measured geodesic lamination λ and a carrying map $F : \lambda \to \tau$ such that for every branch b of τ, the weight $\mu(b)$ is just the transverse measure of a compact arc transverse to λ which is mapped by F to a single point in the interior of b. A train track is called *recurrent* if it admits a transverse measure which is positive on every branch. For every recurrent train track τ, measures which are positive on every branch define the interior of the convex cone $V(\tau)$. A complete train track τ is recurrent [H1]. An arbitrary train track which is both recurrent and transversely recurrent is called *birecurrent.*

A half-branch $\tilde{b}$ in a generic train track τ incident on a switch v is called *large* if the switch v is trivalent and if every arc $\rho : (-\epsilon, \epsilon) \to \tau$ of class C^1 which passes through v meets the interior of $\tilde{b}$. A branch b in τ is called *large* if each of its two half-branches is large; in this case b is necessarily incident on two distinct switches (for all this, see [PH]).

There is a simple way to modify a complete train track τ to another complete train track. Namely, if e is a large branch of τ then we can perform a right or left *split* of τ at e as shown in Figure A below. The split τ' of a train track τ is carried by τ. If τ is complete and if the complete geodesic lamination λ is carried by τ, then for every large branch e of τ there is a unique choice of a right or left split of τ at e with the property that the split track τ' carries λ, and τ' is complete. In particular, a complete train track τ can always be split at any large branch e to a complete train track τ'; however there may be a choice of a right or left split at e such that the resulting track is not complete any more (compare p.120 in [PH]).

Figure A



In the sequel we denote by $\mathcal{TT}$ the collection of all isotopy classes of complete train tracks on S. A sequence $(\tau_i)_i \subset \mathcal{TT}$ of complete train tracks is called a *splitting sequence* if τ_{i+1} can be obtained from τ_i by a single split at some large branch e. Note that in this case for each i the train track τ_{i+1} is carried by τ_i.

3. The complex of curves

In [Ha], Harvey defined the *complex of curves* $\mathcal{C}(S)$ for S. The vertices of this complex are free homotopy classes of essential simple closed curves on S, i.e. simple closed curves which are not contractible nor homotopic into a puncture. For every fixed choice of a complete hyperbolic metric on S of finite volume, every such free homotopy class can be represented by a unique simple closed geodesic. The simplices in $\mathcal{C}(S)$ are spanned by collections of such curves which can be realized disjointly and hence the dimension of $\mathcal{C}(S)$ equals $3g - 3 + m - 1$ (recall that $3g-3+m$ is the number of curves in a pants decomposition of S). In the sequel we restrict our attention to the one-skeleton of $\mathcal{C}(S)$ which we denote again by $\mathcal{C}(S)$ by abuse of notation. Since $3g - 3 + m \geq 2$ by assumption, $\mathcal{C}(S)$ is a nontrivial graph which moreover is connected [Ha]. However, this graph is locally infinite. Namely, for every simple closed curve α on S the surface $S - \alpha$ which we obtain by cutting S open along α contains at least one connected component which is different from a thrice punctured sphere, and such a component contains infinitely many distinct free homotopy classes of simple closed curves which viewed as curves in S are disjoint from α.

Providing each edge in $\mathcal{C}(S)$ with the standard euclidean metric of diameter 1 equips the complex of curves with the structure of a geodesic metric space. Since $\mathcal{C}(S)$ is not locally finite, this metric space $(\mathcal{C}(S), d)$ is not locally compact. Nevertheless, its geometry can be understood quite explicitly. Namely, for some $\delta > 0$ a geodesic metric space is called *δ-hyperbolic in the sense of Gromov* if it satisfies the *δ-thin triangle condition*: For every geodesic triangle with sides a, b, c the side c is contained in the δ-neighborhood of $a \cup b$. The following important result is due to Masur and Minsky [MM1] (see also [B1] and [H4] for alternate proofs).

Theorem 3.1 [MM1]: *The complex of curves is hyperbolic.*

For some $q > 1$, a *q-quasi-geodesic* in $\mathcal{C}(S)$ is a curve $c : [a, b] \to \mathcal{C}(S)$ which satisfies

$$d(c(s), c(t))/q - q \leq |s - t| \leq q d(c(s), c(t)) + q \quad \text{for all } s, t \in [a, b].$$

Note that a quasi-geodesic does not have to be continuous. Call a curve $c : [0, m] \to \mathcal{C}(S)$ an *unparametrized q-quasigeodesic* if there is some $p > 0$ and a homeomorphism $\rho : [0, p] \to [0, m]$ such that the curve $c \circ \rho : [0, p] \to \mathcal{C}(S)$ is a q-quasi-geodesic. In a hyperbolic geodesic metric space, every q-quasi-geodesic is contained in a tubular neighborhood of fixed radius about any geodesic joining the same endpoints, so the δ-thin triangle condition also holds for triangles whose sides are uniform unparametrized quasi-geodesics (Theorem 1.7 in Chapter III.H of [BH]). Moreover, to understand the coarse geometric structure of $\mathcal{C}(S)$ it is enough to identify for a fixed $q > 1$ a collection of unparametrized q-quasi-geodesics connecting any pair of points in $\mathcal{C}(S)$.

To obtain such a system of curves we define a map from the set $\mathcal{TT}$ of complete train tracks on S into $\mathcal{C}(S)$. Call a transverse measure μ on a birecurrent train track τ a *vertex cycle* if μ spans an extreme ray in the convex cone $V(\tau)$ of all transverse measures on τ. Up to scaling, every vertex cycle μ is a counting measure of a simple closed curve c which is carried by τ (p. 115 of [MM1]). A simple closed curve which is carried by τ, with carrying map $F : c \to \tau$, defines a vertex cycle on τ if and only if $F(c)$ passes through every branch of τ at most twice, with different orientation (Lemma 2.2 of [H2]). Thus if c is a vertex cycle for τ then its counting measure μ_c satisfies $\mu_c(b) \leq 2$ for every branch b of τ.

In the sequel we mean by a vertex cycle of a complete train track τ an *integral* transverse measure on τ which is the counting measure of a simple closed curve c on S carried by τ and which spans an extreme ray of $V(\tau)$; we also use the notion vertex cycle for the simple closed curve c. Since the number of branches of a complete train track on S only depends on the topological type of S, the number of vertex cycles for a complete train track on S is bounded by a universal constant (see [MM1] and [H2]). Moreover, there is a number $D_0 > 0$ with the property that for every train track $\tau \in \mathcal{TT}$ the distance in $\mathcal{C}(S)$ between any two vertex cycles of τ is at most D_0 (see [MM1] and the discussion following Corollary 2.3 in [H2]).

Define a map $\Phi : \mathcal{TT} \to \mathcal{C}(S)$ by assigning to a train track $\tau \in \mathcal{TT}$ a vertex cycle $\Phi(\tau)$ for τ. By our above discussion, for any two choices Φ, Φ' of such a map we have $d(\Phi(\tau), \Phi'(\tau)) \leq D_0$ for all $\tau \in \mathcal{TT}$. The following result is due to Masur and Minsky ([MM3], see also [H2] for an alternate proof).

Theorem 3.2 **[MM3]:** *There is a number $q > 0$ such that the image under Φ of an arbitrary splitting sequence in $\mathcal{TT}$ is an unparametrized q-quasi-geodesic.*

Theorem 3.2 can be used to construct for any pair α, β of points in $\mathcal{C}(S)$ an unparametrized q-quasi-geodesic connecting α to β. Namely, for a given $\alpha \in \mathcal{C}(S)$ choose a pants decomposition P containing α. Then every $\beta \in \mathcal{C}(S)$ is uniquely determined by the $3g - 3 + m$-tuple of intersection numbers between β and the pants curves of P and a $3g-3+m$-tuple of twist parameters with respect to a fixed system of *spanning arcs*. Such a system of spanning arcs consists of a choice of a point on each component of P and a maximal collection of disjoint simple pairwise not mutually homotopic arcs each embedded in a pair of pants and with endpoints at the distinguished points on the components of P (see [FLP]). Using the pants decomposition and the spanning arcs we can construct explicitly a complete train track τ which admits α as a vertex cycle and carries β (such a construction can be found in Section 2.6 of [PH]); this train track is then the initial point of a splitting sequence which connects a train track admitting α as a vertex cycle to a train track admitting β as a vertex cycle. For a suitable choice of the map Φ, the image under Φ of this splitting sequence is an unparametrized q-quasi-geodesic in $\mathcal{C}(S)$ connecting α to β.

However, it is also possible to construct explicitly for each pair of points $\alpha, \beta \in \mathcal{C}(S)$ a *geodesic* connecting α to β. Namely, building on the results of [MM2], Shackleton recently proved [S].

Theorem 3.3 **[S]:** *There is an algorithm which takes as input two curves $\alpha, \beta \in \mathcal{C}(S)$ and returns a geodesic between α and β.*

A hyperbolic geodesic metric space X admits a *Gromov boundary* which is defined as follows. Fix a point $p \in X$ and for two points $x, y \in X$ define the *Gromov product* $(x, y)_p = \frac{1}{2}(d(x, p) + d(y, p) - d(x, y))$. Call a sequence $(x_i) \subset X$ *admissible* if $(x_i, x_j)_p \to \infty$ $(i, j \to \infty)$. We define two admissible sequences $(x_i), (y_i) \subset X$ to be *equivalent* if $(x_i, y_i)_p \to \infty$. Since X is hyperbolic, this defines indeed an equivalence relation (see the discussion on p. 431 of [BH]). The Gromov boundary ∂X of X is the set of equivalence classes of admissible sequences $(x_i) \subset X$. It carries a natural Hausdorff topology. For the complex of curves, the Gromov boundary was determined by Klarreich [K] (see also [H2]).

For the formulation of Klarreich's result, we say that a geodesic lamination λ *fills up* S if every simple closed geodesic on S intersects λ transversely, i.e. if every complementary component of λ is an ideal polygon or a once punctured ideal polygon with geodesic boundary. For any geodesic lamination λ which fills up S, the number of geodesic laminations which contain λ as a sublamination is bounded by a universal constant only depending on the topological type of the surface S. Namely, each such lamination μ can be obtained from λ by successively subdividing complementary components P of λ which are different from an ideal triangle or a once punctured monogon by adding a simple geodesic line which either connects two non-adjacent cusps of P or goes around a puncture of S. Note that every leaf of μ which is not contained in λ is necessarily isolated in μ.

The space $\mathcal{L}$ of geodesic laminations on S can be equipped with the *Hausdorff topology* for compact subsets of S. With respect to this topology, $\mathcal{L}$ is compact and metrizable. We say that a sequence $(\lambda_i) \subset \mathcal{L}$ *converges in the coarse Hausdorff topology* to a minimal lamination μ which fills up S if every accumulation point of (λ_i) with respect to the Hausdorff topology contains μ as a sublamination. We equip the space $\mathcal{B}$ of minimal geodesic laminations which fill up S with the following topology. A set $A \subset \mathcal{B}$ is closed if and only if for every sequence $(\lambda_i) \subset A$ which converges in the coarse Hausdorff topology to a lamination $\lambda \in \mathcal{B}$ we have $\lambda \in A$. We call this topology on $\mathcal{B}$ the *coarse Hausdorff topology.* Using this terminology, Klarreich's result [K] can be formulated as follows.

Theorem 3.4 [K],[H2]:

(1) *There is a natural homeomorphism Λ of $\mathcal{B}$ equipped with the coarse Hausdorff topology onto the Gromov boundary $\partial\mathcal{C}(S)$ of the complex of curves $\mathcal{C}(S)$ for S.*
(2) *For $\mu \in \mathcal{B}$ a sequence $(c_i) \subset \mathcal{C}(S)$ is admissible and defines the point $\Lambda(\mu) \in \partial\mathcal{C}(S)$ if and only if (c_i) converges in the coarse Hausdorff topology to μ.*

For every hyperbolic geodesic metric space X, the Gromov product $(\,,)_p$ based at a point $p \in X$ can be extended to a product on $X \cup \partial X$ by defining

$$(\xi,\eta)_p = \sup_{(x_i),(y_j)} \lim \inf_{i,j\to\infty} (x_i,y_j)_p$$

where the supremum is taken over all sequences representing the points ξ,η (i.e. if $\xi \in X$ the $x_i = \xi$ for all i). There is a natural topology on $X\cup\partial X$ which restricts to the given topology on X and on ∂X. For any given point $p \in X$ and every $\xi \in \partial X$, the family of *cones* based at p of the form $C_p(\xi,\delta) = \{y \in X\cup\partial X \mid (y,\xi)_p \geq -\log\delta\}$ $(\delta > 0)$ define a neighborhood basis at ξ with respect to this topology.

The Gromov boundary ∂X of every Gromov hyperbolic geodesic metric space X carries a natural distance function δ defining its topology with the property that there are numbers $c > 0, \kappa > 0$ only depending on the hyperbolicity constant such that $ce^{-\kappa(\xi,\zeta)_p} \leq \delta(\xi,\zeta) \leq e^{-\kappa(\xi,\zeta)_p}$ for all $\xi,\zeta \in \partial X$ [GH]. If X is proper, then the metric δ is complete and $(\partial X,\delta)$ is compact. However, the metric space $\mathcal{C}(S)$ is not proper, and the metric δ on its boundary is not complete. Thus unlike in the case of proper hyperbolic metric spaces, (metrically) diverging sequences of points in $\mathcal{C}(S)$ may not have any accumulation point in $\partial\mathcal{C}(S)$.

Problem 1: Determine the metric completion of the Gromov boundary of $\mathcal{C}(S)$ and relate this metric completion to the geometry of $\mathcal{C}(S)$.

There is yet another way to construct unparametrized uniform quasi-geodesics in $\mathcal{C}(S)$. Namely, *Teichmüller space* for S is the space $\mathcal{T}_{g,m}$ of marked isometry classes of complete hyperbolic metrics on S of finite volume. The Teichmüller space can naturally be identified with a domain in $\mathbb{C}^{3g-3+m}$ (see Chapter 6 of [IT]).

By a classical result of Bers (see [Bu]), there is a number $\chi > 0$ such that for every complete hyperbolic metric h on S there is a pants decomposition for S consisting of simple closed h-geodesics of length at most χ. Moreover, the diameter in $\mathcal{C}(S)$ of the set of simple closed curves on S of h-length at most χ is bounded from above by a universal constant $D > 0$. Thus we can define a map $\Psi : \mathcal{T}_{g,m} \to \mathcal{C}(S)$ by associating to a marked hyperbolic metric h a simple closed curve of h-length at most χ. For any two such maps Ψ, Ψ' we then have $\sup_{h\in\mathcal{T}_{g,m}} d(\Psi(h), \Psi'(h)) \leq D$.

The *Teichmüller metric* on $\mathcal{T}_{g,m}$ is a complete Finsler metric which is just the *Kobayashi metric* on the domain in $\mathbb{C}^{3g-3+m}$ representing $\mathcal{T}_{g,m}$ (see [IT]). Through any two distinct points in $\mathcal{T}_{g,m}$ passes a unique Teichmüller geodesic. Each such geodesic line in $\mathcal{T}_{g,m}$ is uniquely determined by its endpoints in the *Thurston boundary* of $\mathcal{T}_{g,m}$ which is just the space $\mathcal{PML}$ of projective measured laminations on S. The supports of the two measured laminations on S defining the endpoints of the geodesic together fill up the surface S, i.e. every simple closed curve on S intersects at least one of the two laminations transversely. These laminations then define a *holomorphic quadratic differential* (we refer to Section 4 of [Ke] for a discussion of this fact). The following result is implicitly contained in the paper [MM1] by Masur and Minsky; an explicit proof using a result of Rafi [R] can be found in Section 4 of [H4].

Theorem 3.5 [**MM1**]: *There is a universal constant $\tilde{q} > 0$ such that the image under Ψ of every Teichmüller geodesic is an unparametrized $\tilde{q}$-quasi-geodesic in $\mathcal{C}(S)$.*

The mapping class group acts properly discontinuously on $\mathcal{T}_{g,m}$, viewed as a domain in $\mathbb{C}^{3g-3+m}$, as a group of biholomorphic automorphisms. The quotient of $\mathcal{T}_{g,m}$ under this action is the *moduli space* Mod(S) of S, a non-compact complex orbifold. A geodesic lamination λ on S is called *uniquely ergodic* if it supports up to scaling a *unique* transverse measure. Masur [Mas] showed that the endpoint in $\mathcal{PML}$ of a Teichmüller ray which projects to a *compact* subset of moduli space is uniquely ergodic.

Problem 2: For a fixed number $R > 0$, is there is compact subset $K(R)$ of moduli space containing the projection of every Teichmüller geodesic γ which satisfies $d(\Psi(\gamma(s)), \Psi(\gamma(t))) \geq |s - t|/R - R$ for all $s, t \in \mathbb{R}$? Conversely, is there for a given compact set K in Mod(S) a number $R = R(K) > 0$ such that $d(\Psi(\gamma(s)), \Psi(\gamma(t))) \geq |s - t|/R - R$ for every Teichmüller geodesic which projects into K? Analyze the images in $\mathcal{C}(S)$ of geodesics in $\mathcal{T}_{g,m}$ determined by minimal geodesic laminations which fill up S and are *not* uniquely ergodic.

4. The action of $\mathcal{M}_{g,m}$ on the complex of curves

The relevance of the geometry of the complex of curves for the understanding of the geometry of the mapping class group comes from the obvious fact that the extended mapping class group $\mathcal{M}_{g,m}^{\pm}$ of S acts on the complex of curves as a group of simplicial automorphisms and hence isometries. Even more is true: If S is not a torus with 2 punctures then the extended mapping class group is *precisely* the group of isometries of $\mathcal{C}(S)$ (see Chapter 8 of [I] for references and a sketch of the proof). Thus the mapping class group inherits geometric properties from the complex of curves provided that the action satisfies some properness assumption. In this section we discuss a result of Bowditch [B2] who proved that this is indeed the case, and we derive some consequences for the group structure of $\mathcal{M}_{g,m}$.

To begin with, recall that the action of the mapping class group on $\mathcal{C}(S)$ is essentially transitive.

Lemma 4.1:

(1) *There are only finitely many orbits for the action of* $\mathcal{M}_{g,m}$ *on* $\mathcal{C}(S)$.
(2) *For any pair* $\alpha, \beta \in \mathcal{C}(S)$ *there is some* $h \in \mathcal{M}_{g,m}$ *with* $d(h\alpha, \beta) \le 2$.

The *limit set* of a group Γ of isometries of a Gromov hyperbolic metric space X is the set of accumulation points in ∂X of a fixed Γ-orbit Γx on X. This limit set does not depend on x. The group Γ naturally acts as a group of homeomorphisms on the boundary ∂X, and the limit set $\Lambda(\Gamma)$ is invariant under this action. Moreover, if $\Lambda(\Gamma)$ contains at least 3 points then $\Lambda(\Gamma)$ is uncountable and it is the smallest nontrivial closed Γ-invariant subset of ∂X. This means in particular that for every $\xi \in \Lambda(\Gamma)$ the Γ-orbit of ξ is dense in $\Lambda(\Gamma)$.

The following is immediate from Lemma 4.1.

Corollary 4.2: *The limit set of the action of* $\mathcal{M}_{g,m}$ *on* $\mathcal{C}(S)$ *equals the whole boundary* $\partial\mathcal{C}(S)$*. In particular, every* $\mathcal{M}_{g,m}$*-orbit on* $\partial\mathcal{C}(S)$ *is dense.*

Proof: By Lemma 4.1, the action of $\mathcal{M}_{g,m}$ on $\mathcal{C}(S)$ is essentially transitive. Thus for every fixed $\alpha \in \mathcal{C}(S)$ and every admissible sequence $(c_i) \subset \mathcal{C}(S)$ converging in $\mathcal{C}(S) \cup \partial\mathcal{C}(S)$ to some $\xi \in \partial\mathcal{C}(S)$ there is a sequence $(\varphi_i) \subset \mathcal{M}_{g,m}$ with $d(\varphi_i(\alpha), c_i) \le 2$. Then $(\varphi_i(\alpha))$ converges to ξ, i.e. ξ is contained in the limit set of $\mathcal{M}_{g,m}$. This shows the corollary. □

A *simple Dehn twist* about a simple closed essential curve c in S is an element of $\mathcal{M}_{g,m}$ which can be represented in the following form. Let $A \subset S$ be an embedded closed annulus with smooth boundary and core curve c. There is a diffeomorphism φ of A which preserves the boundary pointwise and maps an arc $\alpha : [0,1] \to A$ connecting two points on the two different boundary components and intersecting c in a single point $\alpha(t)$ to an arc with the same endpoints which is homotopic to the composition $\alpha[0,t] * c * \alpha[t,1]$. The homeomorphism $\tilde{\varphi}$ of S whose restriction to

A equals φ and whose restriction to $S - A$ is the identity then represents a simple Dehn twist about c. Such a simple Dehn twist generates an infinite cyclic subgroup of $\mathcal{M}_{g,m}$ which just equals the center of the stabilizer of c in $\mathcal{M}_{g,m}$. It follows that the stabilizer of c in $\mathcal{M}_{g,m}$ is the direct product of the infinite cyclic group of Dehn twists about c and the mapping class group of the surface $S - c$.

As we saw so far, the action of $\mathcal{M}_{g,m}$ on $\mathcal{C}(S)$ is essentially transitive, and for every $\alpha \in \mathcal{C}(S)$ the stabilizer of α in $\mathcal{M}_{g,m}$ contains the mapping class group of the surface $S - \alpha$ as a subgroup of infinite index. Now if $\beta \in \mathcal{C}(S)$ is such that $d(\alpha, \beta) \geq 3$ then $\beta - \alpha$ consists of a collection of simple arcs which decompose $S - \alpha$ into discs and once punctured discs. Since β does not have self-intersections, the number of free homotopy classes relative to α of such components of $\beta - \alpha$ is bounded from above by a universal constant. However, the number of arcs in each free homotopy class is invariant under the action of the stabilizer of α in $\mathcal{M}_{g,m}$, indicating that this action is by no means transitive on the set of curves whose distance to α is a fixed constant. Indeed, Bowditch [B2] showed that the action of $\mathcal{M}_{g,m}$ on pairs of points in $\mathcal{C}(S)$ of sufficiently large distance is proper in a metric sense.

To explain his result, call an isometric action of a group Γ on a hyperbolic metric space X *acylindrical* if for every $m > 0$ there are numbers $R = R(m) > 0, c = c(m) > 0$ with the following property. Let $y, z \in X$ be such that $d(y, z) \geq R$; then there are at most $c(x, m)$ elements $\varphi \in \Gamma$ with $d(\varphi(y), y) \leq m, d(\varphi(x), x) \leq m$. This is a weak notion of properness in a metric sense for a group of isometries of a hyperbolic geodesic metric space. Using an indirect argument via the geometry and topology of 3-manifolds, Bowditch shows (Theorem 1.3 of [B2]).

Theorem 4.3 [B2]: *The action of $\mathcal{M}_{g,m}$ on $\mathcal{C}(S)$ is acylindrical.*

By Thurston's classification of elements of the mapping class group (see [FLP] and [CB]), $\mathcal{M}_{g,m}$ can be divided into three disjoint subsets. The first set contains all *periodic* elements $\varphi \in \mathcal{M}_{g,m}$, i.e. elements for which there is some $k > 1$ with $\varphi^k = Id$. Every orbit of the action on $\mathcal{C}(S)$ of the cyclic group generated by φ is finite and hence *bounded*. The second set contains all *reducible* elements in $\mathcal{M}_{g,m}$ which are not periodic. Such a reducible non-periodic element φ of $\mathcal{M}_{g,m}$ preserves a non-trivial *multi-curve*, i.e. a collection of pairwise disjoint mutually not freely homotopic simple closed essential curves on S. Then there is some $m \geq 1$ such that φ^m fixes an element of $\mathcal{C}(S)$ and once again, the orbits of the action on $\mathcal{C}(S)$ of the cyclic group generated by φ are bounded. The third set contains the so-called *pseudo-Anosov* elements. A pseudo-Anosov mapping class φ acts on $\mathcal{C}(S)$ as a *hyperbolic* isometry. The action of φ on the boundary $\partial\mathcal{C}(S)$ of $\mathcal{C}(S)$ has *north-south dynamics* with respect to a pair $\xi \neq \zeta$ of fixed points. By this we mean the following (compare the discussion in [H3]).

(1) For every neighborhood U of ξ and every neighborhood V of ζ there is some $m > 0$ such that $\varphi^m(\partial\mathcal{C}(S) - V) \subset U$ and $\varphi^{-m}(\partial\mathcal{C}(S) - U) \subset V$.
(2) There is a closed subset D of $\partial\mathcal{C}(S) - \{\xi, \zeta\}$ such that $\cup_i \varphi^i D = \partial\mathcal{C}(S) - \{\xi, \zeta\}$.

There is a number $p > 0$ only depending on the hyperbolicity constant for $\mathcal{C}(S)$ such that every pseudo-Anosov element preserves a p-quasi-geodesic connecting the two fixed points of φ on $\partial\mathcal{C}(S)$. Even more is true. Bowditch (Theorem 1.4 in [B2]) showed that for a pseudo-Anosov mapping class φ there is some $m > 0$ such that φ^m preserves a geodesic in $\mathcal{C}(S)$. Moreover, the *stable length* $\|\varphi\|$ of φ is positive (and moreover rational and bounded from below by a positive constant) where $\|\varphi\| = \lim_{n\to\infty} d(x,\varphi^n x)/n$ for an arbitrary point $x \in \mathcal{C}(S)$ (note that this limit always exists and is independent of x).

Let $\varphi \in \mathcal{M}_{g,m}$ be a pseudo-Anosov element; this element determines a point $a(\varphi)$ in the complement of the diagonal Δ of $\partial\mathcal{C}(S)\times\partial\mathcal{C}(S)$. By Bowditch's result, the set of points $a(\hat\varphi)$ where $\hat\varphi$ ranges over all elements of $\mathcal{M}_{g,m}$ representing the conjugacy class of φ is a discrete subset of $\partial\mathcal{C}(S)\times\partial\mathcal{C}(S)-\Delta$ (this also follows from [BF]). Imitating a construction for Riemannian manifolds of bounded negative curvature and hyperbolic groups (see [Bo2]), define a *geodesic current* for $\mathcal{M}_{g,m}$ to be a locally finite $\mathcal{M}_{g,m}$-invariant Borel measure on $\partial\mathcal{C}(S)\times\partial\mathcal{C}(S)-\Delta$. The sum of Dirac masses at the pairs of points corresponding to a pseudo-Anosov element of $\mathcal{M}_{g,m}$ is a such a geodesic current.

Problem 3: Describe the space of geodesic currents for $\mathcal{M}_{g,m}$. Is the set of weighted sums of Dirac masses at the pairs of fixed points of pseudo-Anosov elements dense? Is there a geodesic current μ which is *absolutely continuous*, i.e. such that there is a $\mathcal{M}_{g,m}$-invariant measure class μ_0 on $\partial\mathcal{C}(S)$ with the property that for every Borel subset A of $\partial\mathcal{C}(S)$ we have $\mu_0(A)=0$ if and only if $\mu(A\times\partial\mathcal{C}(S)-\Delta)=0$? Is there a distinguished absolutely continuous current such that the invariant measure class on $\partial\mathcal{C}(S)$ is determined by a Hausdorff measure with respect to one of the distance functions δ on $\partial\mathcal{C}(S)$?

A subgroup Γ of $\mathcal{M}_{g,m}$ is called *elementary* if its limit set contains at most 2 points. The next lemma follows from the work of McCarthy (compare [MP]).

Lemma 4.4 [MP]: *Let Γ be an elementary subgroup of $\mathcal{M}_{g,m}$. Then either Γ is virtually abelian or Γ contains a subgroup of finite index which stabilizes a nontrivial subsurface of S.*

Proof: Our lemma relies on the following observation. Call a subgroup Γ of $\mathcal{M}_{g,m}$ *reducible* if there is a non-empty finite Γ-invariant family of disjoint simple closed curves on S. A reducible subgroup Γ of $\mathcal{M}_{g,m}$ has a finite orbit on $\mathcal{C}(S)$ and therefore its limit set is trivial. A subgroup Γ which is neither finite nor reducible contains a pseudo-Anosov element φ (this is claimed in Lemma 2.8 of [MP]) and hence its limit set contains at least the fixed points $\alpha\neq\beta$ of the action of φ on $\partial\mathcal{C}(S)$. Thus if Γ is elementary then the limit set of Γ coincides with the set $\{\alpha,\beta\}$ and therefore every element $\psi\in\Gamma$ preserves $\{\alpha,\beta\}$. Since the action of $\mathcal{M}_{g,m}$ on $\mathcal{C}(S)$ is weakly acylindrical the cyclic group generated by φ is of finite index in the subgroup of $\mathcal{M}_{g,m}$ which preserves α,β (this is claimed in Lemma 9.1 of [MC] and also follows from the results in [BF]). As a consequence, an elementary subgroup of $\mathcal{M}_{g,m}$ either is finite or reducible or virtually abelian. $\square$

Let Γ be a countable group and let V be a continuous Banach module for Γ. This means that V is a Banach space and that there is a representation of Γ into the group of linear isometries of V. We are only interested in the case when $V = \mathbb{R}$ with the trivial Γ-action or $V = \ell^p(\Gamma)$ for some $p \in [1, \infty)$ with the standard left action of Γ. The second bounded cohomology group $H_b^2(\Gamma, V)$ of Γ with coefficients in V is defined as the second cohomology group of the complex

$$0 \to L^\infty(\Gamma, V)^\Gamma \xrightarrow{d} L^\infty(\Gamma^2, V)^\Gamma \xrightarrow{d} \dots$$

with the usual homogeneous coboundary operator d and the twisted action of Γ. For $V = \mathbb{R}$ there is a natural homomorphism of $H_b^2(\Gamma, \mathbb{R})$ into the ordinary second cohomology group $H^2(\Gamma, \mathbb{R})$ of Γ which in general is neither injective nor surjective. Since the action of $\mathcal{M}_{g,m}$ on $\mathcal{C}(S)$ is weakly acylindrical we obtain the following [BF], [H3].

Theorem 4.5 [BF], [H3]: *Let $\Gamma < \mathcal{M}_{g,m}$ be any subgroup. If Γ is not virtually abelian then for every $p \in [1, \infty)$ the second bounded cohomology groups $H_b^2(\Gamma, \ell^p(\Gamma)), H_b^2(\Gamma, \mathbb{R})$ are infinitely generated.*

As a corollary, one obtains the following super-rigidity theorem for mapping class groups which was earlier shown by Farb and Masur [FM] building on the work of Kaimanovich and Masur [KM].

Corollary 4.6 [FM], [BF]: *Let G be a semi-simple Lie group without compact factors, with finite center and of rank at least 2. Let $\Gamma < G$ be an irreducible lattice and let $\rho : \Gamma \to \mathcal{M}_{g,m}$ be a homomorphism; then $\rho(\Gamma)$ is finite.*

Proof: Burger and Monod [BM] observed that the second bounded cohomology group of an irreducible lattice in a semi-simple Lie group of higher rank as in the statement of the corollary is finite dimensional. On the other hand, by Margulis' normal subgroup theorem [Ma], for every homomorphism $\rho : \Gamma \to \mathcal{M}_{g,m}$ either the kernel of ρ is finite or the image of ρ is finite. If the kernel of ρ is finite then $\rho(\Gamma)$ is a subgroup of $\mathcal{M}_{g,m}$ which admits Γ as a finite extension. Since the second bounded cohomology group of a countable group coincides with the second bounded cohomology group of any finite extension, the second bounded cohomology group of $\rho(\Gamma)$ is finite dimensional. But $\rho(\Gamma)$ is not virtually abelian and hence this contradicts Theorem 4.5. □

Theorem 4.3 and Theorem 4.5 can be viewed as structure theorems for subgroups of the mapping class group describing a *rank 1-phenomenon* (see also [FLM] for other results along this line). It indicates that a finitely generated infinite group whose geometry is incompatible with the geometry of a hyperbolic space (in a suitable sense) can not be a subgroup of $\mathcal{M}_{g,m}$.

However, the mapping class group has many interesting subgroups, for example free subgroups consisting of pseudo-Anosov elements. We conclude this section with a description of some families of subgroups with particularly simple geometric properties.

Let $\varphi, \eta \in \mathcal{M}_{g,m}$ be pseudo-Anosov elements. Then φ, η act as hyperbolic isometries on $\mathcal{C}(S)$, and they act with north-south dynamics on $\partial\mathcal{C}(S)$. Assume that the fixed point sets of φ, η on $\partial\mathcal{C}(S)$ are disjoint. By the classical ping-pong argument (see Chapter III.Γ.3 in [BH]), there are numbers $\ell > 0, k > 0$ such that the subgroup Γ of $\mathcal{M}_{g,m}$ generated by φ^ℓ, η^k is free and consists of pseudo-Anosov elements. We call such a group a *Schottky-group*. Clearly $\mathcal{M}_{g,m}$ contains infinitely many conjugacy classes of Schottky groups. These Schottky groups are *convex cocompact groups* in the sense of [FMo].

Now define a finitely generated subgroup Γ of $\mathcal{M}_{g,m}$ to be *convex cocompact* if for one (and hence every) $\alpha \in \mathcal{C}(S)$ the orbit map $\varphi \in \Gamma \to \varphi\alpha \in \mathcal{C}(S)$ is a quasi-isometry where Γ is equipped with the distance function defined by the word norm of a fixed symmetric set of generators. A convex cocompact subgroup of $\mathcal{M}_{g,m}$ is necessarily word hyperbolic. Schottky groups in $\mathcal{M}_{g,m}$ are convex cocompact in this sense.

Problem 4: Does the above definition of a convex cocompact subgroup of $\mathcal{M}_{g,m}$ coincide with the definition of Farb and Mosher in [FMo]? Is the natural extension of such a group Γ by the fundamental group $\pi_1(S)$ of S word hyperbolic? Is there a convex cocompact subgroup of $\mathcal{M}_{g,m}$ which is isomorphic to the fundamental group of a *closed* surface of genus at least 2?

A particular interesting class of subgroups of $\mathcal{M}_{g,m}$ arise from *Veech surfaces.* These surfaces are the projections to moduli space of the stabilizer of a *complex geodesic* in Teichmüller space (which is a maximal embedded complex disc in the Teichmüller space viewed as a bounded domain in $\mathbb{C}^{3g-3+m}$) with the additional property that this stabilizer is a lattice in $PSL(2,\mathbb{R})$. Veech surfaces are surfaces of finite type with isolated singularities embedded in moduli space; they are never closed [V]. Thus their corresponding subgroup of $\mathcal{M}_{g,m}$ contains a free group of finite index with a distinguished family of conjugacy classes corresponding to the cusps of the curve. Veech surfaces have many beautiful algebraic and geometric properties (see e.g. [McM1], [McM2]). Elementary constructions of such surfaces and their coresponding subgroups of $\mathcal{M}_{g,m}$ are for example discussed in [L]. Veech surfaces can also be used to construct explicit subgroups of mapping class groups with prescribed geometric properties. A particularly beautiful result along this line was recently obtained by Leininger and Reid [LR].

Theorem 4.7 [LR]: *For every $g \geq 2$ there exists a subgroup of $\mathcal{M}_{g,0}$ which is isomorphic to the fundamental group of a closed surface of genus $2g$ and such that all but one conjugacy class of its elements (up to powers) is pseudo-Anosov.*

Problem 5: Develop a theory of geometrically finite subgroups of $\mathcal{M}_{g,m}$ which include the groups defined by Veech surfaces.

5. The train track complex

In Section 4 we indicated that it is possible to derive many large-scale geometric properties of the mapping class group from the fact that it admits an acylindrical action on a hyperbolic geodesic metric space. On the other hand, the mapping class group $\mathcal{M}_{g,m}$ of a non-exceptional surface is *not* hyperbolic except in the case when the surface is a twice punctured torus [BFa]. To get more precise informations on the geometry of the mapping class group we introduce now a geometric model.

Define a graph whose vertices are the isotopy classes of complete train tracks on S by connecting two such train tracks τ, τ' by a (directed) edge if τ' is obtained from τ by a single split at a large branch e. We call this graph the *train track complex*; it is locally finite and hence locally compact. The mapping class group acts on $\mathcal{TT}$ as a group of simplicial isometries. We have.

Proposition 5.1 [H1]: *$\mathcal{TT}$ is connected, and $\mathcal{M}_{g,m}$ acts on $\mathcal{TT}$ properly and cocompactly.*

As an immediate consequence of Proposition 5.1 we observe that the train track complex is $\mathcal{M}_{g,m}$-equivariantly quasi-isometric to the mapping class group and hence can be viewed as a geometric model for $\mathcal{M}_{g,m}$. The usefulness of this model comes from the fact that $\mathcal{TT}$ admits a natural family of uniform quasi-geodesics. Namely, we have [M3], [H1].

Proposition 5.2: *Splitting sequences in $\mathcal{TT}$ are uniform quasi-geodesics.*

A finitely generated subgroup A of a finitely generate group Γ is called *undistorted* in Γ if the inclusion $\iota : A \to \Gamma$ satisfies $d(\iota g, \iota h) \geq c d(g, h)$ for some $c > 0$ and all $g, h \in \Gamma$ (note that the reverse estimate $d(\iota g, \iota h) \leq C d(g, h)$ for a constant $C > 0$ is always satisfied). As an immediate consequence of Proposition 5.2 we conclude (see [BFa] which contains a proof of the first part of the corollary).

Corollary 5.3:

(1) *Any free abelian subgroup of $\mathcal{M}_{g,m}$ is undistorted in $\mathcal{M}_{g,m}$.*
(2) *Let $S' \subset S$ be a non-trivial connected subsurface. Then the mapping class group of S' as a subgroup of S is undistorted.*

Proof: We begin with the proof of the second part of our lemma. Namely, let $S' \subset S$ be a subsurface bounded by some simple closed pairwise disjoint curves. Choose a pants decomposition P which contains this system of curves. For every geodesic lamination λ on the surface S' there is a train track $\tau \in \mathcal{TT}$ which carries λ and is adapted to P. As before, the mapping class group of S' is quasi-isometric to the subgraph G of $\mathcal{TT}$ which contains precisely all train tracks of this form and with a fixed intersection with $S - S'$. Since splitting sequences in $\mathcal{TT}$ which do not contain any split at a large branch which is *not* contained in S' are uniform quasi-geodesics in $\mathcal{TT}$ and define uniform quasi-geodesics in the mapping class group of

S', the mapping class group of S' is an undistorted subgroup of $\mathcal{M}_{g,m}$ provided that the collection of pairs of vertices in the graph G which can be connected by a splitting sequence is κ-dense in $G \times G$ for some $\kappa > 0$. However, it was shown in [H1] that this is indeed the case.

To show the first part of the lemma it is enough to show that every infinite cyclic subgroup of $\mathcal{M}_{g,m}$ is undistorted in $\mathcal{M}_{g,m}$. For this let $\varphi \in \mathcal{M}_{g,m}$ be an element of infinite order. We may assume that φ either is a pseudo-Anosov element or it its reducible. In the first case φ preserves a pair of transverse minimal laminations which fill up S. The action of φ on the curve complex $\mathcal{C}(S)$ is hyperbolic and the stable length $\lim_{i\to\infty} \frac{1}{i} d(\varphi^i x, x)$ is positive. Since $\mathcal{TT}$ is quasi-isometric to $\mathcal{M}_{g,m}$ and the map $\Phi : \mathcal{TT} \to \mathcal{C}(S)$ introduced in Section 3 is uniformly Lipschitz we conclude that the cyclic group generated by φ is undistorted in $\mathcal{M}_{g,m}$.

If φ is reducible and not a Dehn twist then there is some $k > 0$ such that φ^k fixes a non-trivial subsurface of S and generates an undistorted subgroup of the mapping class group of this subsurface. Together with the second part of the corollary we conclude that the infinite cyclic subgroup of $\mathcal{M}_{g,m}$ generated by φ is undistorted. However, if φ is a Dehn-twist along a simple closed curve α then there is a splitting sequence $(\tau_i) \subset \mathcal{TT}$ issuing from a train track τ_0 which is adapted to a pants decomposition P containing α and such that $\varphi(\tau_{2i-2}) = \tau_{2i}$. Since splitting sequences are uniformly quasi-geodesic, our claim follows. □

The *Torelli group* is the subgroup of $\mathcal{M}_{g,m}$ of all elements which act trivially on the first homology group of the surface S. For a closed surface of genus $g \geq 3$, the Torelli group is finitely generated [J]; however, this is not true for $g = 2$ [MCM].

Problem 6: For a closed surface of genus $g \geq 3$, is the Torelli subgroup of $\mathcal{M}_{g,m}$ undistorted? More generally, find a *distorted* finitely generated subgroup of $\mathcal{M}_{g,m}$.

The splitting sequences on the complex of train tracks can be used to investigate the large-scale geometric behavior of the mapping class group. Note that such a splitting sequence (τ_i) is determined by an initial train track and for each i by a choice of a splitting move among a uniformly bounded number of possibilities which transforms the train track τ_i to the train track τ_{i+1}. In other words, it is possible to treat splitting sequences and hence uniform quasi-geodesics in $\mathcal{M}_{g,m}$ in an algorithmic way.

Algorithmic calculations in a finitely generated group Γ are very intimately related to two basis decision problems which go back to Dehn and can be formulated as follows (see Chapter III.Γ.1 in [BH]).

Word problem: A word w in a fixed system of generators for Γ is given. One is required to find a method to decide in a finite number of steps whether or not this word represents the identity in Γ.

Conjugacy problem: Two elements $g, h \in \Gamma$ are given. A method is sought to decide in a finite number of steps whether or not the elements g, h are conjugate, i.e. whether there is some $u \in \Gamma$ such that $h = ugu^{-1}$.

In the last decade of the twentith century, Epstein, Cannon, Holt, Levy, Paterson, Thurston [E] formulated a property for finitely generated groups which ensures that these problems can be solved in controlled time. Namely, a *biautomatic structure* for a finitely generated group Γ consists of a finite *alphabet* A, a (not necessarily injective) map $\pi : A \to \Gamma$ and a *regular language* L over the alphabet A with the following properties. The set $\pi(A)$ generates Γ, and there is an inversion $\iota : A \to A$ (i.e. $\iota^2 = \mathrm{Id}$) with $\pi(\iota a) = \pi(a)^{-1}$ for all $a \in A$. In particular, $\pi(A)$ is a symmetric set of generators for Γ. Via concatenation, every word w in the alphabet A is mapped by π to a word in the generators $\pi(A)$ of Γ and hence it defines an element $\pi(w) \in \Gamma$. We require that the restriction of the map π to the set of all words from the language L maps L onto Γ. For all $x, y \in A$ and each word $w \in L$ of length $k \geq 0$, the word xwy defines via the projection π a path $s : [0, k+2] \to \Gamma$. By assumption, there is a word $w' \in L$ of length $\ell > 0$ with $\pi(w') = \pi(xwy)$. Let $s' : [0, \ell] \to \Gamma$ be the corresponding path in Γ; we require that the distance in Γ between $s(i)$ and $s'(i)$ is bounded by a universal constant which neither depends on i nor on the choice of x, y, w, w'.

Extending earlier work of Mosher [M1], in [H1] the complex of train tracks and its algorithmic properties are used to show.

Theorem 5.4 [H1]: *The mapping class group of a non-exceptional surface of finite type admits a biautomatic structure.*

Using the results of [E] one obtains as an immediate corollary.

Corollary 5.5: Let $\mathcal{G}$ be a finite symmetric set of generators of $\mathcal{M}_{g,m}$ and let $\mathcal{F}(\mathcal{G})$ be the free group generated by $\mathcal{G}$.

(1) There is a constant $\kappa_1 > 0$ such that a word w in $\mathcal{G}$ represents the identity in $\mathcal{M}_{g,m}$ if and only if in the free group $\mathcal{F}(\mathcal{G})$ we have $w = \prod_{i=1}^n x_i r_i x_i^{-1}$ where $n \leq \kappa_1 |w|^2$, r_i is a word in $\mathcal{G}$ of length at most κ_1 which represents the identity, and $|x_i| \leq \kappa_1 |w|$. Thus the word problem for $\mathcal{M}_{g,m}$ is solvable in quadratic time.
(2) There is a constant $\kappa_2 > 0$ such that words $u, v \in \mathcal{F}(\mathcal{G})$ represent conjugate elements of Γ if and only if there is a word $w \in \mathcal{F}(\mathcal{G})$ of length at most $\kappa_2^{\max\{|u|,|v|\}}$ with $w^{-1}uw = v$ in $\mathcal{M}_{g,m}$. Consequently the conjugacy problem for $\mathcal{M}_{g,m}$ is solvable in exponential time.

Mosher [M1] showed that the mapping class group admits an *automatic* structure, and from this it is possible to deduce many of the known properties of mapping class groups including the first part of Corollary 5.5. The fact that the conjugacy problem is solvable for $\mathcal{M}_{g,m}$ was first shown by Hemion [He].

Call a finitely generated group Γ *linear* if it admits an injective homomorphism into $GL(n, \mathbb{C})$ for some $n > 0$. Consider for a moment the mapping class group of a *closed* surface of genus $g \geq 2$. This mapping class group contains a free abelian subgroup of dimension $3g-3$ which is generated by Dehn twists about the curves of a pants decomposition for S. Since the Zariski closure of an abelian subgroup of a linear algebraic group is abelian, the existence of such a free abelian subgroup of $\mathcal{M}_{g,0}$ of dimension $3g-3$ can be used to show that there is no injective homomorphism of $\mathcal{M}_{g,0}$ into $GL(n,\mathbb{C})$ for $n < 2\sqrt{g-1}$ [FML].

On the other hand, Krammer [Kr] recently showed that the *braid groups* are linear. There are many known similarities between the braid groups and the mapping class groups (see for example [Bi]). However, the following problem is open.

Problem 7: Is the mapping class group linear?

A locally compact group Γ is said to satisfy the *Haagerup approximation property* or is *a-T-menable* if there exists a continuous, isometric action α of Γ on some affine Hilbert space $\mathcal{H}$ which is metrically proper. This means that for all bounded subsets B of $\mathcal{H}$, the set $\{g \in \Gamma \mid \alpha(g)B \cap B \neq \emptyset\}$ is relatively compact in Γ. There are other equivalent characterizations of this property (see [CCJJV]) which can be viewed as a strong negation of the (perhaps more widely know) property (T) of Kazhdan. The class of a-T-menable groups contains for example all amenable groups, Coxeter groups and the isometry groups of real and complex hyperbolic spaces. It is also known that for a-T-menable groups the Baum-Connes conjecture holds (see [CCJJV]).

Problem 8: Is the mapping class group a-T-menable?

6. References

[BF] M. Bestvina, K. Fujiwara, *Bounded cohomology of subgroups of mapping class groups*, Geom. Topol. 6 (2002), 69-89.

[Bi] J. Birman, *Braids, links and mapping class groups*, Ann. Math. Stud. 82, Princeton Univ. Press 1975.

[Bo1] F. Bonahon, *Geodesic laminations on surfaces*, in "Laminations and foliations in dynamics, geometry and topology" (Stony Brook, NY, 1998), 1–37, Contemp. Math., 269, Amer. Math. Soc., Providence, RI, 2001.

[Bo2] F. Bonahon, *Geodesic currents on hyperbolic groups*, in "Arboretal group theory", Math. Sci. Res. Inst. Publ. 19, 143–168, Springer 1991.

[B1] B. Bowditch, *Intersection numbers and the hyperbolicity of the curve complex*, preprint 2002.

[B2] B. Bowditch, *Tight geodesics in the curve complex*, preprint 2003.

[BH] M. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*, Springer Grundlehren 319, Springer, Berlin 1999.

[BFa] J. Brock, B. Farb, *Curvature and rank of Teichmüller space*, preprint 2002.

[BM] M. Burger, N. Monod, *Bounded cohomology of lattices in higher rank Lie groups*, J. Eur. Math. Soc. 1 (1999), 199–235.

[Bu] P. Buser, *Geometry and spectra of compact Riemann surfaces*, Birkhäuser, Boston 1992.

[CEG] R. Canary, D. Epstein, P. Green, *Notes on notes of Thurston*, in "Analytical and geometric aspects of hyperbolic space", edited by D. Epstein, London Math. Soc. Lecture Notes 111, Cambridge University Press, Cambridge 1987.

[CB] A. Casson with S. Bleiler, *Automorphisms of surfaces after Nielsen and Thurston*, Cambridge University Press, Cambridge 1988.

[CCJJV] P. A. Cherix, M. Cowling, P. Jolissaint, P. Julg, A. Valette, *Groups with the Haagerup property*, Progress in Mathematics 197, Birkhäuser, Basel 2001.

[E] D. A. Epstein, with J. Cannon, D. Holt, S. Levy, M. Paterson, W. Thurston, *Word processing in groups*, Jones and Bartlett Publ., Boston 1992.

[FLM] B. Farb, A. Lubotzky, Y. Minsky, *Rank one phenomena for mapping class groups*, Duke Math. J. 106 (2001), 581–597.

[FM] B. Farb, H. Masur, *Superrigidity and mapping class groups*, Topology 37 (1998), 1169–1176.

[FMo] B. Farb, L. Mosher, *Convex cocompact subgroups of mapping class groups*, Geom. Topol. 6 (2002), 91-152.

[FLP] A. Fathi, F. Laudenbach, V. Poénaru, *Travaux de Thurston sur les surfaces,* Astérisque 1991.

[GH] E. Ghys, P. de la Harpe, *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progr. Math. 83, Birkhäuser, Boston 1990.

[G] M. Gromov, *Groups of polynomial growth and expanding maps*, Publ. Math. IHES 53 (1981), 53–78.

[H1] U. Hamenstädt, *Train tracks and mapping class groups I*, preprint 2004, available at http://www.math.uni-bonn.de/~ursula

[H2] U. Hamenstädt, *Train tracks and the Gromov boundary of the complex of curves*, to appear in Proc. Newton Inst.; math.GT/0409611.

[H3] U. Hamenstädt, *Bounded cohomology and isometry groups of hyperbolic spaces*, preprint 2004; available at http://www.math.uni-bonn.de/~ursula

[H4] U. Hamenstädt, *Train tracks and the geometry of the complex of curves*, preprint 2005, available at http://www.math.uni-bonn.de/~ursula

[Ha] W. J. Harvey, *Boundary structure of the modular group*, in "Riemann Surfaces and Related topics: Proceedings of the 1978 Stony Brook Conference" edited by I. Kra and B. Maskit, Ann. Math. Stud. 97, Princeton, 1981.

[He] G. Hemion, *On the classification of homeomorphisms of* 2*-manifolds and the classification of* 3*-manifolds*, Acta Math. 142 (1979), 123-155.

[IT] Y. Imayoshi, M. Taniguchi, *An introduction to Teichmüller spaces*, Springer, Tokyo 1992/1999.

[I] N. V. Ivanov, *Mapping class groups*, Chapter 12 in "Handbook of Geometric Topology", edited by R.J. Daverman and R.B. Sher, Elsevier Science (2002), 523–633.

[J] D. Johnson, *The structure of the Torelli group: A finite set of generators for* $\mathcal{T}$, Ann. of Math. 118 (1983), 423-442.

[KM] V. Kaimanovich, H. Masur, *The Poisson boundary of the mapping class group*, Invent. Math. 125 (1996), 221–264.

[K] E. Klarreich, *The boundary at infinity of the curve complex and the relative Teichmüller space,* preprint 1999.

[Kr] D. Krammer, *Braid groups are linear*, Ann. of Math. 155 (2002), 131–156.

[L] C. Leininger, *On groups generated by two positive multi-twists: Teichmüller curves and Lehmer's number*, Geom. Topol. 8 (2004), 1301-1359.

[LR] C. Leininger, A. Reid, *A combination theorem for Veech subgroups of the mapping class group*, math.GT/0410041.

[L] G. Levitt, *Foliations and laminations on hyperbolic surfaces,* Topology 22 (1983), 119–135.

[Ma] G. Margulis, *Discrete subgroups of semisimple Lie groups*, Springer Ergebnisse in Math. (3) 17, Springer Berlin Heidelberg 1991.

[Mas] H. Masur, *Interval exchange transformations and measured foliations*, Ann. of Math. 115 (1982), 169–200.

[MM1] H. Masur, Y. Minsky, *Geometry of the complex of curves I: Hyperbolicity*, Invent. Math. 138 (1999), 103–149.

[MM2] H. Masur, Y. Minsky, *Geometry of the complex of curves II: Hierarchical structure*, GAFA 10 (2000), 902–974.

[MM3] H. Masur, Y. Minsky, *Quasiconvexity in the curve complex*, preprint 2003.

[MC] J. McCarthy, *A "Tits alternative" for subgroups of surface mapping class groups*, Trans. Amer. Math. Soc. 291 (1985), 583–612.

[MP] J. McCarthy, A. Papadopoulos, *Dynamics on Thurston's sphere of projective measured foliations*, Comm. Math. Helv. 64 (1989), 133–166.

[MCM] D. McCullough, A. Miller, *The genus 2 Torelli group is not finitely generated*, Topology Appl. 22 (1986), 43-49.

[McM1] C. McMullen, *Billiards and Teichmüller curves on Hilbert modular surfaces*, J. AMS 16 (2003), 857-885.

[McM2] C. McMullen, *Dynamics of* $SL_2(\mathbb{R})$ *over moduli space in genus 2*, preprint.

[M1] L. Mosher, *Mapping class groups are automatic*, Ann. Math. 142 (1995), 303-384.

[M2] L. Mosher, *Train track expansions of measured foliations*, unpublished manuscript.

[M3] L. Mosher, in preparation.

[O] J. P. Otal, *Le Théorème d'hyperbolisation pour les variétés fibrées de dimension 3*, Astérisque 235, Soc. Math. Fr. 1996.

[PH] R. Penner with J. Harer, *Combinatorics of train tracks*, Ann. Math. Studies 125, Princeton University Press, Princeton 1992.

[R] K. Rafi, *A characterization of short curves of a Teichmüller geodesic*, Geom. Top. 9 (2005), 179–202.

[S] K. Shackleton, *Tightness and computing distances in the curve complex*, math.GT/0412078

[V] W. Veech, *Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards*, Invent. Math. 97 (1989), 553-583.