AN $\mathcal{EZ}$-STRUCTURE FOR THE MAPPING CLASS GROUP

URSULA HAMENSTÄDT

Abstract. We construct a boundary for the mapping class group $\mathrm{Mod}(S)$ of a surface S of finite type. The action of $\mathrm{Mod}(S)$ on this boundary is minimal, strongly proximal and topologically free. The boundary is the boundary of an $\mathcal{EZ}$-structure for $\mathrm{Mod}(S)$.

1. Introduction

The *mapping class group* $\mathrm{Mod}(S)$ of a closed oriented surface S of genus $g \geq 0$ from which $m \geq 0$ points have been removed and so that $3g-3+m \geq 1$ is the group of isotopy classes of diffeomorphisms of S. The mapping class group is well known to be finitely presented, and it admits explicit torsion free finite index subgroups.

A torsion free finite index subgroup Γ of $\mathrm{Mod}(S)$ admits a *finite* classifying space. Such a classifying space can be constructed as follows.

Since the Euler characteristic of S is negative, the *Teichmüller space* $\mathcal{T}(S)$ of S of all *marked* finite area complete hyperbolic structures on S is defined. By elementary hyperbolic geometry, there exists a number $\epsilon_0 > 0$ such that any two closed geodesics on a hyperbolic surface of length at most ϵ_0 are disjoint. The *systole* $\mathrm{systole}(X)$ of a hyperbolic metric X is the length of a shortest closed geodesic. For $\epsilon < \epsilon_0$ define the *ϵ-thick part* $\mathcal{T}_\epsilon(S)$ of Teichmüller space as

$$\mathcal{T}_\epsilon(S) = \{X \in \mathcal{T}(S) \mid \mathrm{systole}(X) \geq \epsilon\}.$$

The following is due to Ji and Wolpert [JW10], extending an earlier result of Ivanov [Iv02], see Proposition 3.1 and Theorem 3.9 of [J14] for an explicit statement.

Theorem 1 (Ji-Wolpert)**.** *For sufficiently small $\epsilon < \epsilon_0$, the set $\mathcal{T}_\epsilon(S)$ is a manifold with corners which is a deformation retract of $\mathcal{T}(S)$. The mapping class group $\mathrm{Mod}(S)$ acts on $\mathcal{T}_\epsilon(S)$ properly and cocompactly.*

Since $\mathcal{T}(S)$ is homeomorphic to $\mathbb{R}^{6g-6+2m}$, we obtain that $\mathcal{T}_\epsilon(S)$ is contractible, locally contractible and finite dimensional. As torsion free finite index subgroups Γ of $\mathrm{Mod}(S)$ act freely on $\mathcal{T}_\epsilon(S)$, this implies that $\Gamma\backslash\mathcal{T}_\epsilon(S)$ is a finite classifying space for Γ. In particular, Γ is of type F.

Motivated by the construction of the Borel-Serre bordification of an arithmetic group which can be used to compute its *virtual cohomological dimension*, that is,

Date: April 30, 2026.

AMS subject classification: 20F65.

the cohomological dimension of a torsion free finite index subgroup, Harer [Har86] initiated the construction of a bordification of $\mathcal{T}_\epsilon(S)$ which computes the virtual cohomological dimension vcd(Mod(S)) of the mapping class group. This program was completed by Ivanov [Iv02] and consists in attaching to $\mathcal{T}_\epsilon(S)$ the *curve complex* as an analog of a spherical building. The bordification, which indeed computes the virtual cohomological dimension of Mod(S), has the homotopy type of an infinite wedge of spheres [IJ08] and does not compactify the space $\mathcal{T}_\epsilon(S)$.

In the setting of hyperbolic groups Γ, it turns out that the *Gromov boundary* of Γ can be used to compute the virtual cohomological dimension [BM91]. This Gromov boundary is the boundary of a compactification of Γ in the following sense.

Definition 2 (Small boundary)**.** A *boundary* of a finitely generated group Γ is a compact Γ-space Z with the following properties.

- There exists a topology on $\Gamma \cup Z$ which restricts to the discrete topology on Γ, to the given topology on Z and is such that $\Gamma \cup Z$ is compact.
- The left action of Γ on itself extends to the Γ-action on Z.

The boundary is called *small* if the right action of Γ extends to the trivial action of Γ on Z.

The Gromov boundary of a hyperbolic group has additional desirable properties. One of these is captured in the following definition, which is Lemma 1.3 of [B96] and Definition 1.1 of [FL05]. Here the notion $\mathcal{EZ}$-structure stems from the idea of compactifying a classifying space for proper actions of a not necessarily torsion free group Γ. That the Gromov boundary of a torsion free hyperbolic group defines an $\mathcal{EZ}$-structure for the group was established by Bestvina and Mess [BM91].

Definition 3 ($\mathcal{EZ}$-structure)**.** An *$\mathcal{EZ}$-structure* for a finitely generated torsion free group Γ consists of a pair $(\overline{X}, Z)$ of finite dimensional compact metrizable spaces, with Z nowhere dense in $\overline{X}$, and the following additional properties.

(1) $X = \overline{X} - Z$ is contractible and locally contractible.
(2) For every $z \in Z$ and every neighborhood U of z in $\overline{X}$ there exists a neighborhood $V \subset U$ of z such that the inclusion $V - Z \to U - Z$ is null-homotopic.
(3) X admits a covering space action of Γ with compact quotient.
(4) The collection of all translates of a compact set in X form a null sequence in $\overline{X}$: that is, for every open cover $\mathcal{U}$ of $\overline{X}$, all but finitely many translates are *$\mathcal{U}$-small*, which means that they are entirely contained in one of the sets from the covering $\mathcal{U}$.
(5) The action of Γ on X extends to an action on $\overline{X}$.

If Γ is a group with torsion, we define an $\mathcal{EZ}$-structure for Γ as a pair $(\overline{X}, Z)$ as in Definition 3, but where property (3) is replaced by property

(3′) X admits a properly discontinuous action of Γ with compact quotient.

Note that if Γ is torsion free, then $(3')$ is equivalent to (3).

The significance of an $\mathcal{EZ}$-structure $(\overline{X}, Z)$ for a torsion free group Γ lies in the fact that the Čech cohomology of the space Z computes the cohomological dimension $\mathrm{cd}(\Gamma)$ of the group, with a dimension shift of one (Theorem 1.7 of [B96]). Thus an $\mathcal{EZ}$-structure for a finitely generated group which admits a torsion free finite index subgroup computes its virtual cohomological dimension. Furthermore, torsion free groups with an $\mathcal{EZ}$-structure admit an $\mathcal{EZ}$-structure of the form $(\mathbb{D}^n, \Delta)$ where Δ is a closed subset of $\partial \mathbb{D}^n = S^{n-1}$, and the Novikov conjecture holds for these groups (Theorem 1.1 and Theorem 1.2 of [FL05]). Theorem 1.3 of [FL05] shows that it also implies a partial result towards the Farell Jones conjecture.

An action of a group G on a compact topological space Z is called *minimal* if every G-orbit is dense. It is called *topologically free* if for every $\varphi \in G - \{1\}$ the fixed point set of φ has empty interior. Furthermore, it is called *strongly proximal* if the action of G on the space of Borel probability measures on Z is such that the closure of any orbit contains some Dirac measure. Compact G-spaces on which the G-action is minimal and strongly proximal were introduced by Furstenberg and are commonly called *G-boundaries*, using a slightly misleading terminology. They have many applications, in particular towards rigidity questions. For example, the existence of a G-boundary on which the G-action is in addition topological free implies C^*-simplicity of G (which is known for the mapping class group), see [B+17] for this result and a more comprehensive discussion of further applications and references. The Gromov boundary of a non-elementary hyperbolic group has all these properties, but there are many other examples of groups to which these constructions apply, including lattices in semisimple Lie groups of non-compact type. The following is our main result.

Theorem 4. *There exists a compactification $\overline{\mathcal{T}}(S)$ of $\mathcal{T}_\epsilon(S)$ with the following properties.*

(1) *$\mathcal{X}(S) = \overline{\mathcal{T}}(S) \setminus \mathcal{T}_\epsilon(S)$ is a small boundary for* $\mathrm{Mod}(S)$.
(2) *The action of* $\mathrm{Mod}(S)$ *on $\mathcal{X}(S)$ is minimal, strongly proximal and topologically free.*
(3) *The pair $(\overline{\mathcal{T}}(S), \mathcal{X}(S))$ is an $\mathcal{EZ}$-structure for* $\mathrm{Mod}(S)$.

We call the space $\mathcal{X}(S)$ the *geometric boundary* of $\mathrm{Mod}(S)$.

By the main results of [FL05], Theorem 4 implies the following.

Corollary 5. *Any torsion free finite index subgroup of* $\mathrm{Mod}(S)$ *satisfies the Novikov conjecture.*

The statement in the corollary was earlier established with different methods, see [H09, Ki10, BBF15, BaB19]. The Farell Jones conjecture for mapping class groups, which implies the Novikov conjecture but can not be deduced from Theorem 4, is due to Bartels and Bestvina [BaB19].

An alternative approach to the construction of an $\mathcal{EZ}$-structure for the mapping class group, based on *hierarchical hyperbolicity*, is due to Durham, Minsky and Sisto

[DMS25]. They construct an $\mathcal{EZ}$-structure for such a group as an abstract boundary from a property called asymptotically CAT(0) which is extracted from the combinatorial properties of hierarchically hyperbolic spaces. Earlier Durham, Hagen and Sisto [DHS17] constructed a boundary for hierarchially hyperbolic groups in the sense of Definition 2. In the case of the mapping class group, this boundary can be identified with the boundary constructed in Theorem 4 as a set, however the topology is different. There are open sets in the boundary of [DHS17] which do not contain any open subset of the boundary we construct. We do not have information on the relation to the construction in [DMS25]. Note to this end that an $\mathcal{EZ}$-structure for a discrete group is by no means unique. Hierarchical hyperbolicity for $\mathrm{Mod}(S)$ only appears indirectly in this article, but our approach shares with [DMS25] the strategy to view the mapping class group as a CAT(0)-space on the large scale.

As the virtual cohomological dimension $\mathrm{vcd}(\mathrm{Mod}(S))$ equals $4g-5$ if $g \geq 2$ and $m = 0$, $4g-4+m$ if $g \geq 1$ and $m-3$ if $g = 0$ [Har86], the covering dimension of the space $\mathcal{X}(S)$ equals $4g-6$ if $g \geq 2$ and $m = 0$, $4g-5+m$ if $g \geq 1$ and $m > 0$, and $m-4$ if $g = 0$ [B96]. Note that for any torsion free finite index subgroup Γ of $\mathrm{Mod}(S)$, the cohomology group $H^{\mathrm{vcd}(\mathrm{Mod}(S))}(\Gamma, \mathbb{Z}\Gamma)$ identifies with the $2g-2+m$-th homology group of the curve complex. Since the curve complex has the homotopy type of an infinite wedge of spheres of dimension $2g-2+m$ (Theorem 1.4 of [IJ08]), this implies that the top dimensional Čech cohomology group of $\mathcal{X}(S)$ is also infinite dimensional by Proposition 1.5 of [B96].

Theorem 4 can be viewed as giving some evidence that the *asymptotic dimension* of $\mathrm{Mod}(S)$, which is known to be finite and at most quadratic in the virtual cohomological dimension, in fact equals the virtual cohomological dimension of $\mathrm{Mod}(S)$. We refer to [BB19] for a more detailed discussion on this and related questions and results.

The following is a consequence of Theorem 4 and Theorem 1.1 and Theorem 1.2 of [FL05]. In its formulation we denote by $\mathbb{D}^n$ the standard ball in $\mathbb{R}^n$.

Corollary 6. *If* $3g-3+m \geq 3$ *then there exists a closed subset* Δ *of* $S^{6g-5+2m}$ *such that* $\mathrm{Mod}(S)$ *admits an* $\mathcal{EZ}$*-structure of the form* $(\mathbb{D}^{6g-4+2m}, \Delta)$.

Description of the boundary $\mathcal{X}(S)$ of $\mathrm{Mod}(S)$. The curve complex $\mathcal{CG}(S_0)$ of a (not necessarily proper) essential subsurface S_0 of S different from a pair of pants or an annulus is the simplicial complex whose vertices are isotopy classes of simple closed curves and where k such curves span a $k-1$-simplex if they can be realized disjointly. If S_0 is a four-holed sphere or a one holed torus, then this definition has to be modified by connecting two vertices by an edge if they intersect in the minimal number of points.

The curve complex, equipped with the natural simplicial metric, is a hyperbolic geodesic metric space of infinite diameter [MM99]. Its *Gromov boundary* $\partial\mathcal{CG}(S_0)$ is the space of *minimal geodesic laminations* on S_0 which *fill* S_0, that is, which intersect every essential simple closed curve on S_0 transversely. The topology on $\partial\mathcal{CG}(S_0)$ is the *coarse Hausdorff topology.* With respect to this topology, a sequence λ_i of minimal filling laminations converges to the lamination λ if and only if the limit of any subsequence which converges in the Hausdorff topology on compact subsets of S_0 contains λ as a sublamination [H06, Kl22]. The space $\partial\mathcal{CG}(S_0)$ is separable and metrizable. Define the boundary of the curve complex of an essential annulus $A \subset S$ with core curve c to consist of two points c^+, c^-.

If $S_1, \dots, S_k$ is a collection of isotopy classes of pairwise disjoint subsurfaces of S, then we can form the join

$$\mathcal{J}(\cup_{i=1}^k S_i) = \partial\mathcal{CG}(S_1) * \cdots * \partial\mathcal{CG}(S_k).$$

It can be viewed as the set of formal sums $\sum_i a_i\lambda_i$ where $a_i > 0$, $\sum_i a_i = 1$ and where $\lambda_i \in \partial\mathcal{CG}(S_i)$ for all i. This join is a separable metrizable topological space. Note that if $S_{i_1}, \dots, S_{i_s}$ is a subset of the set of surfaces $S_1, \dots, S_k$, then $\mathcal{J}(\cup_{j=1}^s S_{i_j})$ is naturally a non-empty closed subset of $\mathcal{J}(\cup_{i=1}^k S_i)$ corresponding to formal sums $\sum_i a_i\lambda_i$ with $a_i = 0$ for $i \notin \{i_1, \dots, i_s\}$. Define

$$\mathcal{X}(S) = \cup\mathcal{J}(\cup_{i=1}^k S_i)$$

where the union is over all collections of pairwise disjoint essential subsurfaces of S and we use the obvious identification of points which arise in more than one way in this union. Here we view an essential annulus A as an essential subsurface which is disjoint from any subsurface which can be moved off A by an isotopy. Thus $\mathcal{X}(S)$ is just the set of formal sums $\sum_i a_i\lambda_i$ where $a_i > 0, \sum_i a_i = 1$, where $\lambda_1, \dots, \lambda_k$ are pairwise disjoint minimal geodesic laminations on S and where each simple closed curve component λ_i is equipped with an additional label $+, -$. The mapping class group acts naturally on $\mathcal{X}(S)$ as a *set*.

The topology on the space $\mathcal{X}(S)$ can be described as follows. Each of the subspace $\mathcal{J}(\cup_{i=1}^k S_i)$, equipped with its topology as a join of metric spaces, can be thought of as the visual boundary of the product of the hyperbolic metric spaces $\mathcal{CG}(S_i)$, where the coefficients in a sum $\sum_i a_i\lambda_i$ capture the relative speed in the factors with which a geodesic in the product converges to a boundary point. It is embedded in $\mathcal{X}(S)$ as a topological space, which means that the inclusion $\mathcal{J}(\cup_i S_i) \to \mathcal{X}(S)$ is a homeomorphism onto its image, equipped with the subspace topology.

These countably many subspaces are glued together by first performing the obvious identifications and then using the Hausdorff topology on the space of geodesic laminations on S to determine which points in distinct of the spaces $\mathcal{J}(\cup_i S_i)$ are close to each other to construct a neighborhood basis of every point. The idea is that as convergence in the coarse Hausdorff topology to a point $\xi \in \partial\mathcal{CG}(S)$ describes convergence of a sequence in the metric space $\mathcal{CG}(S)$ to ξ, closeness in $\mathcal{X}(S)$ of two points $\sum_i a_i\xi_i$ and $\sum_j b_j\zeta_j$ in two distinct of the subspaces $\mathcal{J}(\cup_i S_i)$ is captured by closeness in the coarse Hausdorff topology of the laminations $\cup_i\xi_i$ and $\cup_j\zeta_j$, with a careful bookkeeping of the role of the coefficients in the joins that mimics what one would observe in a CAT(0)-space containing two distinct convex subspaces of infinite diameter. Note that we use here specific properties of surfaces which have no analog in the context of hierarchically hyperbolic spaces, although it seems possible that the construction of the topology of $\mathcal{X}(S)$ can be carried out in this more general context as well, using combinatorial tools.

The following result summarizes some technical properties of the geometric boundary. For its formulation, let us invoke the Nielsen Thurston classification which states that any nontrivial mapping class has a finite power φ with the following property. There exists a decomposition $S = S_1 \cup \cdots \cup S_k$ of S into subsurfaces that are preserved by φ and such that for all $i < k$, the surface S_i is connected and the restriction of φ to S_i is pseudo-Anosov if S_i is not an annulus, and it is a Dehn twist if S_i is an annulus. The restriction of φ to S_k is trivial. We call a mapping class with this property a *Nielsen Thurston mapping class.*

Let φ be a Nielsen Thurston mapping class. For each $i < k$ such that S_i is not an annulus, the restriction φ_i of φ to S_i preserves precisely two geodesic laminations $\xi_i^\pm$ which are the attracting and repelling laminations of φ_i. Similarly, for any component S_i which is an annulus, the two labeled copies $\xi_i^\pm$ of the core curve of the annulus are preserved as well. Thus φ fixes any point of the form $\sum_i a_i\zeta_i$ where ζ_i is one of the laminations $\xi_i^\pm$ if $i < k$ and where ζ_k is an arbitrary point of the geometric boundary of the (possibly disconnected) surface S_k. We call points of this form the *obvious fixed point set.*

Proposition 7. *Let $\mathcal{X}(S)$ be the geometric boundary of* $\mathrm{Mod}(S)$.

(1) *For any collection $S_1, \dots, S_k$ of pairwise disjoint subsurfaces of S, the inclusion $\mathcal{J}(\cup_{i=1}^k S_i) \to \mathcal{X}(S)$ is an embedding. In particular, the covering dimension of $\partial\mathcal{CG}(S)$ is at most* $\mathrm{vcd}(\mathrm{Mod}(S)) - 1$.
(2) *The fixed point set for the action of a Nielsen Thurston mapping class φ on $\mathcal{X}(S)$ is precisely the obvious fixed point set of φ.*

That the covering dimension of $\partial\mathcal{CG}(S)$ is bounded from above by $\mathrm{vcd}(\mathrm{Mod}(S))$ is due to Gabai (Proposition 16.3 of [Ga14]). In view of the expectation that the asymptotic dimension of the mapping class group equals its virtual cohomological dimension, and that this dimension is captured by the covering dimension of the Gromov boundary of the curve graph paralleling the result of Harer [Har86], we expect that the bound we find in part (1) of Proposition 7 is sharp. We are not aware of any lower bound available in the literature.

In [BB19], Bestvina and Bromberg converted Gabai's upper bound on the covering dimension of $\partial\mathcal{CG}(S)$ into the same upper bound for its capacity dimension. We do not know whether this can also be achieved for the bound we find.

The Gromov boundary of the curve complex of S can be obtained from the space $\mathcal{FML} \subset \mathcal{PML} = S^{6g-7+2m}$ of projective measured geodesic laminations with minimal filling support, equipped with the weak*-topology, by an equivariant continuous surjective map $\mathcal{FML} \to \partial\mathcal{CG}(S)$. This map is however not injective and the following statement, which can be deduced from Theorem 4, Proposition 7 and the results in [FL05], requires a proof.

Corollary 8. *The boundary $\partial\mathcal{CG}(S)$ of the curve complex of S admits an embedding into a manifold of dimension $6g - 6 + 2m$ and into $S^{6g-5+2m}$.*

Our construction is valid for the mapping class group of a once punctured torus or a four punctured sphere. In this case the mapping class group is virtually free and, in particular, it is a hyperbolic group whose Gromov boundary is a Cantor set. The boundary we find is the Gromov boundary of the group.

The construction of the boundary $\mathcal{X}(S)$ is motivated by the construction of the visual boundary of a CAT(0)-space. Along the way we identify in Section 2 an analog of the familiar Tits boundary of a symmetric space of higher rank. Although this has no obvious application, it draws a parallel between the mapping class group and a lattice in a higher rank symmetric space and connects to Harer's partial bordification of $\mathrm{Mod}(S)$.

The advantage of our construction of a geometric boundary $\mathcal{X}(S)$ is that as a topological $\mathrm{Mod}(S)$-space, it is completely explicit and can be used among others to study subgroups of $\mathrm{Mod}(S)$.

Overview of the article. In the first part of the article, contained in Sections 2-3, we define a topology on the *set* $\mathcal{X}(S)$ which is invariant under the action of $\mathrm{Mod}(S)$. We also observe that the action of $\mathrm{Mod}(S)$ on $\mathcal{X}(S)$ is strongly proximal. The more preliminary Section 2 introduces the *oriented curve complex* and shows that it can be viewed as a Tits type boundary for the mapping class group. It seems likely that the constructions and results in this first part carry over to colorable hierarchically hyperbolic groups, however carrying this program out would add an extra layer of notations and combinatorial discussions.

The second part of this article is devoted to showing that this topology extends to $\mathcal{T}_\epsilon(S) \cup \mathcal{X}(S)$ and defines a compactification of $\mathcal{T}_\epsilon(S)$ in such a way that every point in $\mathcal{X}(S)$ has a neighborbood basis consisting of sets whose intersections with $\mathcal{T}_\epsilon(S)$ are contractible. This is carried out in Section 4 and Section 5 and is the most involved part of this article. We use the augmented Teichmüller space as a witness of CAT(0) properties, thus mainly relying on geometry rather than combinatorics. The proof of Theorem 4 and the corollaries is completed in Section 6.

Acknowledgement: I am grateful to Alessandro Sisto for informing me about the article [DMS25]. Special thanks go to the referee for their careful reading and the many excellent suggestions for improving the exposition. This work was partially supported by the Hausdorff Center Bonn. The article was put into the current form while the author visited the Newton Institute in Cambridge during the program Operators, Graphs, Groups in summer 2025.

2. The Tits boundary of $\mathrm{Mod}(S)$

The *join* $X_1 * X_2$ of two topological spaces X_1, X_2 is defined to be the quotient $X_1 \times X_2 \times [0,1]/\sim$ where the equivalence relation $\sim$ collapses $X_1 \times X_2 \times \{0\}$ to X_1 and collapses $X_1 \times X_2 \times \{1\}$ to X_2. For example, the join $S^0_1 * S^0_2$ of two 0-spheres is the circle S^1, thought of as a union of four intervals glued at the endpoints, where each interval has one endpoint in S^0_1 and the second endpoint in S^0_2. The join of two spaces X_1, X_2 contains an embedded copy of X_1, X_2.

Example 2.1. The product of two hyperbolic planes $\mathbb{H}^2\times\mathbb{H}^2$ is a complete simply connected Riemannian manifold of non-positive curvature. Its *visual boundary* is the join $S^1 * S^1$ of two circles that are the Gromov boundaries of the embedded copies $\mathbb{H}^2\times\{y\}$ and $\{x\}\times\mathbb{H}^2$ of $\mathbb{H}^2$. This corresponds to the fact that the projection of any geodesic in $\mathbb{H}^2\times\mathbb{H}^2$ to each of the two factors is a geodesic. Note that the join of two circles is homeomorphic to S^3. ■

Define the *oriented curve complex* $\mathcal{OG}(S)$ of an oriented connected surface S of genus g with m punctures and $3g-3+m\geq 2$ to be the complex whose vertices are isotopy classes of essential oriented simple closed curves in S and whose one-skeleton consists of edges (of length 1) connecting two vertices if they can be realized disjointly and are not homotopic up to orientation. Thus any simple closed curve in S defines two distinct vertices in $\mathcal{OG}(S)$, and these vertices are not connected by an edge. Furthermore, we require that any collection of $k\geq 2$ oriented disjoint simple closed curves which are distinct as unoriented curves span a simplex. The union of these simplices defined by a fixed collection of k curves equipped with all combinations of orientations is a sphere of dimension $k-1$. If $3g-3+m=1$ then we define the oriented curve complex in the same way, but defining edges in the one-skeleton by connecting two vertices if up to homotopy, they intersect transversely in the minimal number of points (one for the once punctured torus and two for the four punctured sphere).

Note that a point in $\mathcal{OG}(S)$ can be viewed as a formal linear combination $\sum_{i=1}^k a_i\lambda_i$ where for some $k\geq 1$, $\lambda_1,\dots,\lambda_k$ are pairwise disjoint oriented simple closed curves, where $a_i>0$ for all i and $\sum_i a_i=1$. In other words, a point in the oriented curve complex can be viewed as a point in the join of a finite collection of oriented pairwise disjoint simple closed curves. If S is a once punctured torus or a four punctured sphere, then the oriented curve complex is defined in the same way except that two oriented curves are connected by an edge if they intersect in the minimal number of points (one for the once punctured torus and two for the four punctured sphere).

Remark 2.2. If we choose the length of the edges of the oriented curve complex to be $\pi/2$, then this is consistent with the idea that the oriented curve complex can be thought of as being contained in the Tits boundary of $\mathrm{Mod}(S)$, equipped with the angular length metric which identifies each sphere with a sphere of constant curvature one. Any such sphere will be viewed as the visual boundary of a *Dehn twist flat*, that is, a free abelian subgroup of $\mathrm{Mod}(S)$ generated by Dehn twists about a collection of disjoint simple closed curves. ■

A simple closed curve c on S is the core curve of an embedded annulus $A(c)\subset S$. Define the "curve graph" $\mathcal{CG}(A(c))$ of the annulus $A(c)$ as a graph of isotopy classes of arcs connecting the two boundary components of $A(c)$ and whose endpoints are allowed to move freely in the complement of a fixed point p_-,p_+ on each of the two boundary circles $\partial_\pm A(c)$. Two such arcs are connected by an edge if they can be realized disjointly in their homotopy class. Although this is not the standard definition, it will be convenient for our purpose. Note that it depends on the choice of the points p_-,p_+. To apply it to arcs with at least one endpoint at p_- or p_+ we deform the arc with a small homotopy to a disjoint arc with no endpoint at p_-,p_+, creating a small ambiguity which is unavoidable when one discusses curve graphs of annuli.

To see that this definition well encodes the action of the infinite cyclic group of Dehn twists of $A(c)$ equip the annulus $A(c)$ with an orientation inherited from an orientation of S. Given an arc $\alpha \subset A(c)$ with endpoints in $\partial A(c)\setminus\{p_\pm\}$, we can slide the endpoint of α on $\partial_x A(c)$ across p_+ in positive or negative direction, keeping its second endpoint fixed, to create a disjoint arc α' which is homotopic in the above sense to the image of α under a simple Dehn twists T_c about the core curve c. It is connected to α by an edge in $\mathcal{CG}(A)$. A repetition of this construction gives rise to the arc $T_c^2(\alpha)$ (up to homotopy) which is not connected to α by an edge. Thus the curve graph $\mathcal{CG}(A(c))$ is a simplicial line which admits the infinite cyclic group of Dehn twists as a vertex transitive group of translations. The distinction between a positive and a negative Dehn twist about c only depends on the orientation of S but not on the orientation of c. The choice of an orientation of c can be thought of as a spiraling direction about c for oriented arcs connecting the two boundary components of $A(c)$.

In the sequel we denote by c^+ the point in the Gromov boundary of $\mathcal{CG}(A(c))$ (which consists of two points) which corresponds to an iteration of positive Dehn twists about c, and we denote by c^- the point in the Gromov boundary of $\mathcal{CG}(A(c))$ which corresponds to an iteration of negative Dehn twists about c. Write $\mathcal{J}(c) = \{c^+, c^-\}$. It will be convenient to think about $\mathcal{J}(c)$ as a set of two distinct points in the oriented curve complex of S, with the same underlying curve.

If S_0 is a subsurface of S different from a pair of pants or an annulus, then we denote its (non-oriented) curve complex by $\mathcal{CG}(S_0)$. The vertices of this complex are isotopy classes of essential, that is, non-peripheral simple closed curves. If S_0 is different from a one-holed torus or a four-holed sphere, then a collection of $k \geq 2$ such disjoint simple closed curves span a simplex of dimension $k-1$. If S_0 is a one-holed torus or a four-holed sphere then two simple closed curves are connected by an edge if they intersect transversely in the minimal number of points. The curve complex of S_0 is hyperbolic and hence it has a *Gromov boundary* $\partial\mathcal{CG}(S_0)$. As a set, the Gromov boundary $\partial\mathcal{CG}(S_0)$ is the set of all minimal filling geodesic laminations on S_0. We refer to [H06] for an account on this result of Klarreich [Kl22].

There is a natural metrizable topology on the union $\overline{\mathcal{CG}}(S_0)$ of $\mathcal{CG}(S_0)$ with its Gromov boundary, called the *coarse Hausdorff topology*. With respect to this topology, the subspace $\mathcal{CG}(S_0)$, equipped with its simplicial topology, is an open dense subset. To define this topology equip the surface S_0 with a hyperbolic metric with geodesic boundary. This choice defines a Hausdorff topology on the space of compact subsets of S_0. A sequence $\lambda_i \subset \mathcal{CG}(S_0) \subset \mathcal{CG}(S_0) \cup \partial\mathcal{CG}(S_0)$ of vertices in $\mathcal{CG}(S_0)$ converges in the coarse Hausdorff topology to $\lambda \in \partial\mathcal{CG}(S_0)$ if and only if the limit of any converging subsequence of λ_i in the Hausdorff topology on compact subsets of S_0 contains λ as a sublamination [H06]. Define

$$\mathcal{J}(S_0) = \partial\mathcal{CG}(S_0),$$

equipped with the topology as a subset of $\overline{\mathcal{CG}}(S_0)$. If S_0 is a pair of pants, then we define $\mathcal{J}(S_0) = \emptyset$.

If $S_1,\dots,S_k$ are *disjoint* connected subsurfaces of S (we allow that they share boundary components, and annuli about such boundary components may be included in the list), then we define

$$\mathcal{J}(\cup_i S_i)=\partial\mathcal{CG}(S_1)*\cdots*\partial\mathcal{CG}(S_k) \tag{1}$$

to be the join of the spaces $\mathcal{J}(S_i)=\partial\mathcal{CG}(S_i)$. For example, if $S_1\subset S$ is a subsurface which is the complement of a non-separating simple closed curve c, then

$$\mathcal{J}(S_1\cup A(c))=\partial\mathcal{CG}(S_1)*\{c^+,c^-\}.$$

A point in $\mathcal{J}(S_1\cup\cdots\cup S_k)$ can be viewed as a formal linear combination

$$\xi=\sum_i a_i\xi_i$$

where $\xi_i\in\partial\mathcal{CG}(S_i)$, $a_i\geq 0$ for all i and, furthermore, $\sum_i a_i=1$. The union

$$\mathrm{supp}(\xi)=\cup_{a_i>0}\xi_i$$

is a geodesic lamination with minimal components ξ_i, and ξ can be viewed as a weighted (and partially labeled if there are simple closed curve components of ξ with positive weight) geodesic lamination. For all $u\leq k$ there is an inclusion $\mathcal{J}(S_1\cup\cdots\cup S_u)\subset\mathcal{J}(S_1\cup\cdots\cup S_k)$ which is a topological embedding.

A collection $S_1,\dots,S_k$ of disjoint connected subsurfaces of S is called *maximal* if $S-\cup_i S_i=\emptyset$. By convention, this means that for any boundary component c of one of the surfaces S_i, the annulus $A(c)$ is contained in the collection. Any collection $S_1,\dots,S_\ell$ of disjoint connected subsurfaces of S is contained in a maximal collection of such subsurfaces, however this maximal collection is in general not unique. For example, there is a canonical maximal collection containing $S_1,\dots,S_k$ which is comprised of the surfaces S_i, the annuli $A(c)$ where c runs through all boundary components of $\cup_i S_i$ which are not already contained in the list, and all connected components of $S-\cup_i S_i$.

Define

$$\mathcal{X}(S)=\cup\mathcal{J}(S_1\cup\cdots\cup S_k)/\sim \tag{2}$$

where the union is over all collections of disjoint subsurfaces $S_1,\dots,S_k$ of S. The equivalence relation $\sim$ identifies two points $\sum_i a_i\xi_i$ and $\sum_j b_j\zeta_j$ if they coincide as weighted labeled geodesic laminations. Thus a point in $\mathcal{X}(S)$ is nothing else but a formal sum $\sum_{i=1}^k a_i\xi_i$ where $a_i>0,\sum_i a_i=1$, where $\xi_1,\dots,\xi_k$ are pairwise disjoint minimal geodesic laminations on S and where every simple closed curve component of this collection is in addition equipped with a label $\pm$. Note that the oriented curve complex $\mathcal{OG}(S)$ of S can naturally be identified with the union of the subsets $\mathcal{J}(A(c_1)\cup\cdots\cup A(c_k))$ of $\mathcal{X}(S)$, and its Gromov boundary (which is just the Gromov boundary $\partial\mathcal{CG}(S)$ of the non-oriented curve complex of S) also is contained in $\mathcal{X}(S)$. The mapping class group $\mathrm{Mod}(S)$ naturally acts on the set $\mathcal{X}(S)$.

Example 2.3. The definition (2) makes sense if S is a once punctured torus or a four punctured sphere. In this case there are no non-peripheral subsurfaces of S different from annuli and pairs of pants, and the set $\mathcal{X}(S)$ is just the union of the Gromov boundary of the curve graph $\mathcal{CG}(S)$ with a countable set, consisting of all oriented non-peripheral simple closed curves on S. We discuss the case of the once punctured torus in detail, the case of the four punctured sphere is completely analogous.

The curve graph of S is the well-known *Farey graph.* Its vertices can be represented by the rational points in the boundary $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$ of the hyperbolic plane. If one represents the edges of the Farey graph by geodesics in $\mathbb{H}^2$, then one obtains a tesselation of the hyperbolic plane by ideal triangles which is invariant under the mapping class group $\mathrm{PSL}(2,\mathbb{Z})$ of S. The boundary ∂T of the dual tree T of this tesselation is a Cantor set which admits a surjective continuous map onto the boundary $\partial\mathbb{H}^2$ of the hyperbolic plane. Each irrational point in $\partial\mathbb{H}^2$ corresponds to a point in the Gromov boundary of $\mathcal{CG}(S)$ and has precisely one preimage, and the rational points which correspond to the vertices of the curve graph have two preimages.

The vertices of the Farey graph also correspond to the fixed points of the parabolic subgroups of $\mathrm{PSL}(2,\mathbb{Z})$. With this interpretation, the set $\mathcal{X}(S)$ can be identified with the Cantor set ∂T obtained by replacing each rational point in $\mathbb{R} \cup \{\infty\}$ by a compact interval and removing the interior of the interval. This Cantor set in turn has a natural identification with the Gromov boundary ∂T of the virtually free group $\mathrm{PSL}(2,\mathbb{Z})$. In particular, there is a natural invariant topology on $\mathcal{X}(S)$ so that with this topology, $\mathcal{X}(S)$ is a compact $\mathrm{PSL}(2,\mathbb{Z})$-space which contains the Gromov boundary $\partial\mathcal{CG}(S)$ of the curve graph of S as a dense embedded subset. Furthermore, following [BM91], the set $\mathcal{X}(S)$ equipped with this topology is the boundary of an $\mathcal{EZ}$-structure for $\mathrm{PSL}(2,\mathbb{Z})$. ■

Example 2.4. Let $S_1, \dots, S_k$ be a disjoint union of subsurfaces of S which are different from pairs of pants. Then the join $\mathcal{X}(S_1) * \cdots * \mathcal{X}(S_k)$ is a subset of $\mathcal{X}(S)$. ■

The short remainder of this section is not used in the sequel, but was included here to illustrate the similarities and differences of the geometry of the mapping class group as encoded in the boundary $\mathcal{X}(S)$ and classical constructions for Gromov hyperbolic spaces and symmetric spaces of higher rank.

The oriented curve complex of S is connected, and any non-filling geodesic lamination, that is, a geodesic lamination which is disjoint from some simple closed curve, is disjoint from some vertex of $\mathcal{OG}(S)$. Thus if we equip $\mathcal{X}(S) \setminus \partial\mathcal{CG}(S)$ with the topology of a simplicial complex whose edges are the joins of two disjoint (perhaps labeled) geodesic laminations, then this complex is connected. As a consequence, the *set* $\mathcal{X}(S)$ can be equipped with a topology which coincides with the topology of a (non-locally finite) simplicial complex on $\mathcal{X}(S) \setminus \partial\mathcal{CG}(S)$ and is such that each point in $\partial\mathcal{CG}(S)$ is isolated. We write $\mathcal{X}_T(S)$ for $\mathcal{X}(S)$ equipped with this topology and call $\mathcal{X}_T(S)$ the *Tits boundary* of $\mathrm{Mod}(S)$ (having the Tits boundary of a CAT(0) space as guidance). From this description, we obtain

Lemma 2.5. *The mapping class group* $\mathrm{Mod}(S)$ *of* S *acts on* $\mathcal{X}_T(S)$ *as a group of simplicial automorphisms.*

Proof. The mapping class group acts on the oriented curve complex of S as a group of simplicial automorphisms, and this action extends to an action on the space of formal sums of weighted disjoint minimal geodesic laminations preserving weight and disjointness. Furthermore, it acts on $\partial\mathcal{CG}(S)$ as a group of transformations. Since the topology on $\mathcal{X}_T(S)$ is the topology of a disconnected simplicial complex, constructed from the curve complexes of subsurfaces, the lemma follows. □

Remark 2.6. The Tits boundary of a CAT(0) space X can be viewed as the geometric boundary (that is, the CAT(0) boundary) of X, equipped with a topology which in general is finer than the geometric topology. We shall see in Section 4 that the same holds true for the Tits boundary and the geometric boundary of $\mathrm{Mod}(S)$.

3. A topology for $\mathcal{X}(S)$

The goal of this section is to equip the set $\mathcal{X}(S)$ with a topology which is coarser than the Tits topology so that for this topology, $\mathcal{X}(S)$ becomes a compact $\mathrm{Mod}(S)$-space.

Let $\xi^j = \sum_m a_m^j \xi_m^j$ be a sequence in $\mathcal{X}(S)$. We shall impose in three steps a requirement for the sequence to converge to a point $\zeta = \sum_{i=1}^k b_i\zeta_i \in \mathcal{X}(S)$. Here as before, we assume that $a_m^j > 0, b_i > 0, \sum_i b_i = 1 = \sum_m a_m^j$ for all j and that furthermore, $\mathrm{supp}(\xi_i)$, $\mathrm{supp}(\zeta)$ are disjoint unions of minimal components. The three steps used to construct the topology are contained in three different subsections.

3.1. **Convergence to a minimal filling lamination.** Recall that the space of geodesic laminations on S is compact with respect to the Hausdorff topology. We refer to Chapter 3 of [CB88] for the case that the surface is compact. The case that S has finite type but is non-compact follows in exactly the same way as in this case, any simple geodesic is contained in a fixed compact subset of S, see e.g. Section 4.1 of [CEG87]. Equivalently, replacing a puncture p by a boundary component via removing the interior of an embedded disk in S punctured at p represents a geodesic lamination as a lamination on a compact surface with boundary, to which [CB88] can directly be applied.

Requirement 1: *Convergence in the coarse Hausdorff topology*
Let ξ^{ℓ_n} be any subsequence of the sequence ξ^j such that the geodesic laminations $\mathrm{supp}(\xi^{\ell_n})$ converge in the Hausdorff topology to a geodesic lamination β. Then β contains $\mathrm{supp}(\zeta)$ as a sublamination.

Example 3.1. A geodesic lamination c coarsely determines a point in $\mathcal{CG}(S) \cup \partial\mathcal{CG}(S)$. Namely, if c is minimal and filling, then $c \in \partial\mathcal{CG}(S)$. Otherwise c is disjoint from a simple closed curve $c' \in \mathcal{CG}(S)$.

By a result of Klarreich [Kl22] as reported in [H06], a sequence of non-filling geodesic laminations c_i converges in the coarse Hausdorff topology to a minimal filling geodesic lamination η if and only if the simple closed curves $c_i' \in \mathcal{CG}(S)$ converge in $\mathcal{CG}(S) \cup \partial\mathcal{CG}(S)$ to $\eta \in \partial\mathcal{CG}(S)$. ■

Example 3.2. Let $S_1, \dots, S_k \subset S$ be disjoint subsurfaces. Example 2.4 shows that $\mathcal{X}(S)$ contains the join $\mathcal{X}(S_1) * \cdots * \mathcal{X}(S_k)$ as a subset. An element $\xi \in \mathcal{X}(S_1) * \cdots * \mathcal{X}(S_k)$ can be represented in the form

$$\xi = \sum_i a_i \xi_i$$

where $\xi_i \in \mathcal{X}(S_i)$, in particular, $\mathrm{supp}(\xi_i) \subset S_i$, and $\sum_i a_i = 1$. Since the subset of geodesic laminations on S which are supported in S_i is closed with respect to the Hausdorff topology, this implies that for any topology on $\mathcal{X}(S)$ which fulfills the first requirement above, the subspace $\mathcal{X}(S_1) * \cdots * \mathcal{X}(S_k)$ of $\mathcal{X}(S)$ is closed. ■

The examples show that the requirement (1) determines completely and geometrically the convergence of a sequence $\xi_i \subset \mathcal{X}(S)$ to a point $\xi \in \partial\mathcal{CG}(S) \subset \mathcal{X}(S)$.

Example 3.3. In the case that S is a once punctured torus or a four punctured sphere, then any non-trivial subsurface of S different from a pair of pants is an annulus. In particular, any geodesic lamination which is a disjoint union of minimal components is minimal and either fills S or is a simple closed curve. This easily implies that the topology of $\mathcal{X}(S)$ is determined by the requirement (1). Furthermore, it follows from Example 2.3 and the discussion in Example 3.1 that the space $\mathcal{X}(S)$ is naturally homeomorphic to the Gromov boundary of the hyperbolic group $\mathrm{Mod}(S)$. ■

3.2. **Product spaces.** In this subsection we consider a collection S_i $(1 \leq 1 \leq k)$ of pairwise disjoint proper subsurfaces of S. This collection determines the subspace

$$\mathcal{X}(\cup_i S_i) = \mathcal{X}(S_1) * \cdots * \mathcal{X}(S_k) \subset \mathcal{X}(S).$$

Put

$$\mathcal{CG}(\cup_i S_i) = \mathcal{CG}(S_1) \times \cdots \times \mathcal{CG}(S_k).$$

Our goal is to define a topology on the union

$$\mathcal{Y}(\cup_i S_i) = \mathcal{CG}(\cup_i S_i) \cup \partial\mathcal{CG}(S_1) * \cdots * \partial\mathcal{CG}(S_k) = \mathcal{CG}(\cup_i S_i) \cup \mathcal{J}(\cup_i S_i) \tag{3}$$

which will be used in the construction of a topology on $\mathcal{X}(S)$.

The main tool are *complete markings* of (not necessarily proper) essential subsurfaces S_0 of the surface S. Such a marking consists of a pants decomposition P for S_0 together with a collection of *spanning curves*. For every component c of P, there exists such a spanning curve which intersects c in the minimal number of points (one or two) and is disjoint from all other pants curves. Two spanning curves may not be disjoint, but we require that the number of their intersection points is bounded from above by a universal constant. Since there are only finitely many topological types of pants decompositions, this can clearly be achieved. There is a natural way to equip the set of all markings on S_0 with the structure of a locally finite connected graph on which the mapping class group $\mathrm{Mod}(S_0)$ of S_0 acts properly and cocompactly. We refer to [MM00] for more information on this construction.

Choose a marking μ on S as a basepoint for the proper cocompact action of $\mathrm{Mod}(S)$ on the marking graph. For every subsurface S_0 of S which is distinct from a pair of pants or an annulus, this marking can be used to construct a marking $\mathrm{pr}_{S_0}(\mu)$ of S_0 as follows.

There is a coarsely well defined *subsurface projection*

$$\mathrm{pr}_{S_0} : \mathcal{CG}(S) \to \mathcal{CG}(S_0)$$

which associates to a simple closed curve c its intersection $\mathrm{pr}_{S_0}(c) = c \cap S_0$ with S_0 in the following sense. If $c \subset S_0$ then put $\mathrm{pr}_{S_0}(c) = c$, and if c is disjoint from S_0 then put $\mathrm{pr}_{S_0}(c) = \emptyset$. In all other cases, $c \cap S_0$ consists of a collection of pairwise disjoint arcs with endpoints on the boundary of S_0. We then put $\mathrm{pr}_{S_0}(c) = u$ for a simple closed curve u in S_0 which is obtained from one of these intersection arcs by choosing a component of the boundary of a tubular neighborhood of the union of the arc with the boundary components of S_0 containing its endpoints. Informally, we say that the simple closed curve is obtained by surgery on the arc.

Given a marking μ for S, the union of the intersections of the marking curves with S_0 consists of a union of arcs and simple closed curves on S_0 with pairwise uniformly bounded intersection numbers which decompose S_0 into simply connected regions. Hence via deleting some of these arcs and modifying some arcs with a surgery to simple closed curves as described in the previous paragraph, the projection of μ into S_0 coarsely defines a marking $\mathrm{pr}_{S_0}(\mu)$ of S_0 (there is a small abuse of notation here), called the *subsurface projection* of μ [MM00]. Here a coarse definition means that the construction depends on choices, but any two choices give rise to markings which are uniformly close in the marking graph of S_0, independent of the subsurface S_0.

If S_0 is an annulus, then a similar construction applies. In this case a marking consists of the choice of a marked point on each boundary component of S_0 and an embedded arc in S_0 connecting the two distinct boundary components which is disjoint from the marked points. With a bit of care, a subsurface projection is defined for annuli as well. We refer to [MM00] for more information.

By the above discussion, for every subsurface S_0 of S the marking μ coarsely determines a basepoint $x_0 \in \mathcal{CG}(S_0)$ by choosing one of the marking curves (or arcs if S_0 is an annulus) of $\mathrm{pr}_{S_0}(\mu)$. As the intersection number between any two curves (or arcs) of $\mathrm{pr}_{S_0}(\mu)$ is uniformly bounded, the distance in the curve graph of S_0 between x_0 and any other curve from $\mathrm{pr}_{S_0}(\mu)$ or any other marking of S_0 constructed in the same fashion from μ is uniformly bounded.

Let $\mathrm{Min}_\cup(S)$ be the space of geodesic laminations on S which are disjoint unions of minimal components. Using the basepoint x_0 for $\mathcal{CG}(S_0)$, we can extend the subsurface projection pr_{S_0} to all of $\mathrm{Min}_\cup(S)$ as follows. Let $\nu = \cup_i \nu_i \in \mathrm{Min}_\cup(S)$. Then there are three possibilities.

- If the lamination ν is disjoint from S_0 up to homotopy, define $\mathrm{pr}_{S_0}(\nu) = x_0$.
- If there exist components $\nu_1, \dots, \nu_\ell$ of ν which are contained in S_0 then define $\mathrm{pr}_{S_0}(\nu) = \cup_{i=1}^{\ell} \nu_i$.
- If $\nu \cap S_0$ consists of a collection of disjoint simple arcs with endpoints on the boundary of S_0 which coarsely define a point in $\mathcal{CG}(S_0)$ then define $\mathrm{pr}_{S_0}(\nu)$ to be any one of these points.

Note that by the definition, pr_{S_0} is contained in $\mathrm{Min}_{\cup}(S_0)$, and if ν is a disjoint union of simple closed curves, then the same holds true for $\mathrm{pr}_{S_0}(\nu)$.

Let again $S = \cup_{i=1}^k S_i$ be a collection of pairwise disjoint subsurfaces of S. It then follows from the above discussion that a choice μ of a marking of S coarsely determines a basepoint $x = (x_1, \dots, x_k)$ for the product space $\mathcal{CG}(\cup_i S_i)$ consisting of the product of the coarsely well defined basepoints $x_i \in \mathcal{CG}(S_i)$.

Recall from (1) the definition of the sets $\mathcal{J}(\cup_i S_i)$. Since the curve graph $\mathcal{CG}(S_i)$ is a hyperbolic geodesic metric space, for every $p > 1$ and every p-quasi-geodesic ray $\gamma : [0, \infty) \to \mathcal{CG}(S_i)$, there exists a coarsely well defined shortest distance projection $\Pi_\gamma : \mathcal{CG}(S_i) \to \gamma$ which extends to the complement of the endpoint $\gamma(\infty) \in \partial\mathcal{CG}(S_i)$ in $\partial\mathcal{CG}(S_i)$. The following definition should be thought of viewing $\mathcal{Y}(\cup_i S_i)$ as the union of a product of hyperbolic spaces with its visual boundary. As curve complexes are non-proper hyperbolic spaces and we are ultimately interested in the mapping class group, it is necessary to simultaneously work on all subsurfaces, that is, capture what happens with subsurface projections on nested subsurfaces. This is implemented in condition (3) below. In the statement, $p > 1$ is a fixed number which fulfills some constraint that will be determined later.

Definition 3.4. Define a topology on $\mathcal{Y}(\cup_i S_i)$ by the following requirements.

- The product space $\mathcal{CG}(\cup_i S_i)$ is equipped with the product topology and is an open subset of $\mathcal{Y}(\cup_i S_i)$.
- The subspace $\mathcal{J}(\cup_i S_i)$ is equipped with the topology as a join of the Gromov boundaries of the curve graphs of S_i.
- Let $\xi = \sum_i a_i \xi_i \in \partial\mathcal{CG}(S_1) * \cdots * \partial\mathcal{CG}(S_k)$ and after reordering, assume that $a_i > 0$ for all $i \leq \ell$ and $a_i = 0$ for $i > \ell$. A sequence of points $(y_1^j, \dots, y_k^j)_j \subset \mathcal{CG}(\cup_i S_i)$ converges to ξ if the following three conditions are fulfilled.
  1. For each $i \leq \ell$ the components $y_i^j \in \mathcal{CG}(S_i)$ converge as $j \to \infty$ to ξ_i in the coarse Hausdorff topology (and hence they converge in $\mathcal{CG}(S_i) \cup \partial\mathcal{CG}(S_i)$ to ξ_i, see [H06]). In particular, we have $d_{\mathcal{CG}(S_i)}(y_i^j, x_i) \to \infty$ $(j \to \infty)$.
  2. For all $i \leq \ell$ denote by Π_i the shortest distance projection of $\mathcal{CG}(S_i)$ onto a p-quasi-geodesic connecting the basepoint x_i to ξ_i; then
$$\frac{d_{\mathcal{CG}(S_i)}(\Pi_i(y_i^j), x_i)}{d_{\mathcal{CG}(S_1)}(\Pi_1(y_1^j), x_1)} \to \frac{a_i}{a_1} \quad (j \to \infty).$$
  3. Let $i > \ell$ and let $V \subset S_i$ be any subsurface; then
$$\frac{d_{\mathcal{CG}(V)}(\mathrm{pr}_V(y_i^j), \mathrm{pr}_V(\mu))}{d_{\mathcal{CG}(S_1)}(\Pi_1(y_1^j), x_1)} \to 0 \quad (j \to \infty).$$

Lemma 3.5. *The notion of convergence in Definition 3.4 defines a separable topology on $\mathcal{Y}(\cup_i S_i)$ which restricts to the given topology on $\partial\mathcal{CG}(S_1) * \cdots * \partial\mathcal{CG}(S_k)$ and on $\mathcal{CG}(\cup S_i)$. The subspace $\partial\mathcal{CG}(S_1) * \cdots * \partial\mathcal{CG}(S_k)$ is closed in $\mathcal{Y}(\cup_i S_i)$.*

Proof. Define a subset A of $\mathcal{Y}(\cup_i S_i)$ to be *closed* if $A_1 = A \cap \mathcal{CG}(\cup_i S_i)$ is closed, $A_2 = A \cap \partial\mathcal{CG}(S_1) * \cdots * \partial\mathcal{CG}(S_k)$ is closed and if furthermore the following holds true. If $y_i \subset A_1$ is a sequence which converges in the sense of Definition 3.4 to a point $y \in \partial\mathcal{CG}(S_1) * \cdots * \partial\mathcal{CG}(S_k)$, then $y \in A_2$. Note that by definition, the empty set is closed, and the same holds true for the total space.

We have to show that complements of closed sets defined in this way fulfill the axioms of a topology, that is, they are stable under arbitrary unions and finite intersections. Equivalently, the family of closed sets is stable under arbitrary intersections and finite unions. As this holds true for the closed subsets of $\mathcal{CG}(\cup_i S_i)$ and for the closed subsets of $\mathcal{J}(\cup_i S_i) = \partial\mathcal{CG}(S_1) * \cdots * \partial\mathcal{CG}(S_k)$, all we need to observe is that taking arbitrary intersections and finite unions is consistent with the notion of convergence of points in $\mathcal{CG}(\cup_i S_i)$ to points in the join $\partial\mathcal{CG}(S_1) * \cdots * \partial\mathcal{CG}(S_k)$ in the sense of Definition 3.4.

Consistency with arbitrary intersections is straightforward. To show consistency with taking finite unions let $B_1, \dots, B_\ell \subset \mathcal{Y}(\cup_i S_i)$ be closed in the above sense. Let $y_j \subset \cup_k(B_k \cap \mathcal{CG}(\cup_i S_i))$ be any sequence which converges to a point in $\mathcal{J}(\cup_i S_i)$ according to the definition of convergence. By passing to a subsequence, we may assume that $y_j \in B_m$ for a fixed $m \leq \ell$ and all j. As B_m is closed and the subsequence also fulfills the requirements for convergence, its limit is contained in $B_m \subset \cup_k B_k$. Hence indeed, the notion of a closed set is consistent with taking finite unions. ☐

3.3. **Projections and a topology on $\mathcal{X}(S)$.** As mentioned in the previous subsection, to capture the geometry of the mapping class group using curve graphs we have to work simultaneously with all subsurfaces, which includes the necessity to study degenerating sequences of mapping classes whose restrictions to a nontrivial subsurface are all homotopic to the identity.

To this end define for a disjoint union $\cup_{i=1}^k S_i$ of subsurfaces of S the *set*

$$\mathcal{Z}(\cup_i S_i) = \cup_I \mathcal{Y}(\cup_{i\in I} S_i) * \mathcal{J}(\cup_{j\in\{1,\dots,k\}\setminus I} S_j) \tag{4}$$

where I runs through all (possibly empty) subsets of the index set $\{1, \dots, k\}$ and we perform the obvious identifications of points which appear in several ways in this union. Namely, for every I, the space $\mathcal{Y}(\cup_{i\in I} S_i) * \mathcal{J}(\cup_{j\in\{1,\dots,k\}\setminus I} S_j)$ contains $\mathcal{J}(\cup_{i=1}^k S_i)$, the join of the boundaries of the curve graphs of the surfaces S_i, and we perform the obvious identification of the points which coincide as formal sums $\sum_i a_i\xi_i$ where $\sum_i a_i = 1$ and $\xi_i \in \partial\mathcal{CG}(S_i)$. No other identifications are made. Note that $\mathcal{Z}(\cup_i S_i)$ contains $\mathcal{Y}(\cup_{i\in I} S_i)$ for all $I \subset \{1, \dots, k\}$.

Lemma 3.6. *There exists a unique topology on $\mathcal{Z}(\cup_i S_i)$ with the property that a set $U \subset \mathcal{Z}(\cup_i S_i)$ is open if and only if its intersection with each of the subspaces $\mathcal{Y}(\cup_{i\in I} S_i) * \mathcal{J}(\cup_{j\in\{1,\dots,k\}\setminus I} S_j)$ is open.*

Proof. For every $I \subset \{1, \dots, k\}$, the set $\mathcal{J}(\cup_i S_i)$ is a closed subspace of

$$\mathcal{Y}(\cup_{i\in I} S_i) * \mathcal{J}(\cup_{j\in\{1,\dots,k\}\setminus I} S_j),$$

equipped with the topology of a join. Thus the topology described in the lemma is just the quotient topology on the quotient of the disjoint union of the spaces $\mathcal{Y}(\cup_{i\in I} S_i) * \mathcal{J}(\cup_{j\in\{1,\dots,k\}\setminus I} S_j)$ by the closed equivalence relation which identifies the points in the subspaces $\mathcal{J}(\cup_i S_i)$. ☐

We next define a projection

$$\mathrm{pr}_{\mathcal{Z}(\cup S_i)} : \mathcal{X}(S) \to \mathcal{Z}(\cup_i S_i)$$

as follows. Let $\xi = \sum_{j=1}^m a_j\xi_j \in \mathcal{X}(S)$ with $a_j > 0$ and $\sum_j a_j = 1$ and write as before $\mathrm{supp}(\xi) = \cup_j \xi_j$. After perhaps a reordering of the components ξ_j, assume that for some $u \leq \min\{k, m\}$ the components $\xi_1, \dots, \xi_u$ fill the subsurfaces $S_1, \dots, S_u$, that is, they define points in $\partial\mathcal{CG}(S_i)$, with the convention of remembering labels of simple closed curves, and that for no $j > u$, the component ξ_j fills any of the surfaces S_i. As the components of $\mathrm{supp}(\xi)$ are disjoint, this implies that if $s, t > u$, if $i \in \{u+1, \dots, k\}$ and if the subsurface projections $\mathrm{pr}_{S_i}(\xi_s), \mathrm{pr}_{S_i}(\xi_t)$ of ξ_s, ξ_t into S_i are not empty, then they are of uniformly bounded distance in $\mathcal{CG}(S_i)$. Recall that this makes sense even if ξ_s, ξ_t are different from simple closed curves. If the lamination $\mathrm{supp}(\xi) = \cup_i \xi_i$ is disjoint from the subsurface S_ℓ, then the projection component is defined to be the basepoint of $\mathcal{CG}(S_\ell)$ constructed from the base marking.

Define

$$\mathrm{pr}_{\mathcal{Z}(\cup S_i)}(\sum_{j=1}^m a_j\xi_j) = \sum_{j=1}^u a_j\xi_j + (1 - \sum_{j=1}^u a_j)(\mathrm{pr}_{\mathcal{CG}(\cup_{i\geq u+1} S_i)}(\cup_{j\geq u+1}\xi_j)) \\ \in \mathcal{CG}(\cup_{i=u+1}^k S_i) * \mathcal{J}(\cup_{i=1}^u S_i).$$

Here the term on the right hand side is understood in the following sense. First, if one of the surfaces S_i $(i \leq u)$ is an annulus then the label of ξ_i is remembered in $\mathrm{pr}_{\mathcal{Z}(\cup_i S_i)}(\sum_j a_j\xi_j)$. Second, for some $\ell \in \{u+1, \dots, k\}$ let us consider the subsurface S_ℓ. If there exists some $s > u$ such that ξ_s intersects S_ℓ, then the component in S_ℓ of the projection $\mathrm{pr}_{\mathcal{CG}(\cup_{i\geq u+1} S_i)}(\cup_{j\geq u+1}\xi_j)$ is a point in $\mathcal{CG}(S_\ell)$. Although this projection depends on choices, it is coarsely well defined, that is, well defined up to a uniformly bounded error. Finally, recall that we take the join of the product of the curve graphs of the subsurfaces S_i for $i \leq u$ and the join of the Gromov boundaries $\partial\mathcal{CG}(S_i)$ for $i \geq u+1$, so the sum $1 - \sum_{j=1}^u a_j$ is treated as a single coefficient.

Requirement 2: A sequence $\xi^j = \sum_m a_m^j \xi_m^j \subset \mathcal{X}(S)$ converges to $\zeta = \sum_{i=1}^k b_i\zeta_i \in \mathcal{X}(S)$ if the following holds true. For each $i \leq k$ let S_i be the subsurface of S filled by ζ_i and put $S_{k+1} = S \setminus \cup_{i=1}^k S_i$; then

$$\mathrm{pr}_{\mathcal{Z}(\cup_{i=1}^{k+1} S_i)}(\xi^j) \to \zeta \text{ in } \mathcal{Z}(\cup_{i=1}^{k+1} S_i) \supset \mathcal{J}(\cup_{i=1}^{k+1} S_i) \supset \mathcal{J}(\cup_{i\leq k} S_i).$$

In other words, for all i, j the subsurface projection of ξ^j to S_i is either a filling lamination or empty or coarsely determines a point in $\mathcal{CG}(S_i)$, and these projections, equipped with the weight inherited from the weights of the components of ξ^j, converge in $\mathcal{Z}(\cup_{i=1}^{k+1} S_i)$ to ζ.

Remark 3.7. It follows from the above description that for this notion of convergence, the following holds true. Let ξ^j be a sequence in $\mathcal{X}(S)$ consisting of minimal geodesic laminations which converges to a point $\zeta = \sum_u b_u\zeta_u$.

(a) The lamination $\mathrm{supp}(\zeta)$ is a sublamination of the limit in the Hausdorff topology of any convergent subsequence of the sequence $\mathrm{supp}(\xi^j)$.

(b) For each j let η^j be a minimal geodesic lamination disjoint from ξ^j (we allow $\eta^j = \xi^j$) and let $s_i \in [0,1]$. Then any limit of a convergent subsequence of the sequence $\nu^j = s_i\xi^j + (1-s_i)\eta^j$ is of the form $s\zeta + (1-s)\eta$ where η is a limit of a subsequence of the sequence η^j and where $s \in [0,1]$.

(c) Requirement (1) in the definition of the topology on $\mathcal{X}(S)$ is a consequence of requirement (2) and was included for added clarity.

(d) If $\xi^j = \sum_{i=1}^{m^j} a_i^j \xi_i^j \subset \mathcal{X}(S)$ is any sequence so that for some fixed $1 \leq m \leq m^j$ it holds $\sum_{i=1}^m a_i^j \to 1$ and if the sequence $\frac{1}{\sum_{i=1}^m a_i^j} \sum_{i=1}^m a_i^j \xi_i^j$ converges to $\zeta \in \mathcal{X}(S)$, then the same holds true for the sequence ξ^j. This makes the topology weak enough for our goal and is motivated by the weak*-topology on the space of measured geodesic laminations. ■

Definition 3.8. A subset $A \subset \mathcal{X}(S)$ is called *closed for the geometric topology of* $\mathcal{X}(S)$ if the following holds true. Let $\xi_i \subset A$ be any sequence which converges to a point $\xi \in \mathcal{X}(S)$ in the sense described by the requirements (2); then $\xi \in A$.

Recall that for any collection $S_1, \dots, S_k$ of pairwise disjoint subsurfaces of S, the space $\mathcal{J}(\cup_{i=1}^k S_i)$ is equipped with a natural topology as a join of the Gromov boundaries of the curve graphs of the subsurfaces S_i. The following statement is the first main step towards the proof of Theorem 4.

Proposition 3.9. (1) *Closed subsets of $\mathcal{X}(S)$ in the sense of Definition 3.8 define a separable Hausdorff topology $\mathcal{O}$ on $\mathcal{X}(S)$.*

(2) *For any collection $S_1, \dots, S_k$ of pairwise disjoint subsurfaces, the natural inclusion $\mathcal{J}(\cup_{i=1}^k S_i) \to (\mathcal{X}(S), \mathcal{O})$ is an embedding.*

Proof. Let $\mathcal{O} \subset \mathcal{X}(S)$ be the family of all subsets of $\mathcal{X}(S)$ whose complement is closed in the above sense. Sets in $\mathcal{O}$ are called *open*. We have to show that $\mathcal{O}$ defines a topology on $\mathcal{X}(S)$.

As the empty set and the entire space $\mathcal{X}(S)$ are open, to show that $\mathcal{O}$ is indeed a topology on $\mathcal{X}(S)$ it suffices to show that arbitrary unions of open sets are open, and that finite intersections of open sets are open as well. Or, equivalently, arbitrary intersections of closed sets are closed, and finite unions of closed sets are closed. This can be established using exactly the same reasoning as in the proof of Lemma 3.5.

Namely, that the collection of closed sets is stable under arbitrary intersections is immediate from the definition. So let $B_1, \dots, B_k$ be closed sets and let $B = \cup_i B_i$. Choose a sequence $\xi_i \subset B$ which converges in the sense of requirement (2) to some point ζ. By passing to a subsequence, we may assume that $\xi_i \in B_\ell$ for some $\ell \leq k$ and all i. But then $\zeta \in B_\ell \subset B$ as B_ℓ is closed which completes the proof that $\mathcal{O}$ is indeed a topology on $\mathcal{O}$.

We show next the second property claimed in the proposition. Thus let $S_1, \dots, S_k$ be a collection of pairwise disjoint subsurfaces of S. Our goal is to show that the inclusion $\mathcal{J}(\cup_{i=1}^k S_i) \to (\mathcal{X}(S), \mathcal{O})$ is an embedding. Since the inclusion is injective, and closed sets in both spaces are defined via convergence of sequences, this is equivalent to stating that a sequence $\xi^j = \sum_i a_i^j \xi_i^j \subset \mathcal{J}(\cup_{i=1}^k S_i)$ converges in

$(\mathcal{X}(S),\mathcal{O})$ to a point $\zeta\in\mathcal{J}(\cup_{i=1}^k S_i)$ if and only if ξ^j converges in $\mathcal{J}(\cup_{i=1}^k S_i)$ to ζ. However, putting $S_{k+1}=S\setminus\cup_i S_i$, this is immediate from the definition of the topology on $\mathcal{Z}(\cup_{i=1}^{k+1}S_i)$ and requirement (2) in the definition of convergence in $\mathcal{X}(S)$ and shows the second part of the proposition.

Since each of the spaces $\mathcal{J}(\cup_{i=1}^k S_i)$ is a finite join of separable metrizable spaces (namely, the Gromov boundary of a curve graph of a subsurface of S) and hence separable metrizable, the second part of the proposition implies that $(\mathcal{X}(S),\mathcal{O})$ is a countable union of (in general not disjoint) separable metrizable spaces and hence is separable.

To show that the topology is Hausdorff let $\xi=\sum_i a_i\xi_i\neq\zeta=\sum_j b_j\zeta_j\in\mathcal{X}(S)$. We have to show that ξ,ζ have disjoint neighborhoods.

If this is not the case, then any neighborhoods U_ξ of ξ and U_ζ of ζ intersect nontrivially. Since $\mathcal{X}(S)$ is separable, and since points in $\mathcal{X}(S)$ are closed by construction, we conclude that there is a sequence $\xi^j\subset\mathcal{X}(S)$ which converges both to ξ,ζ. But for the notion of convergence used to define the topology $\mathcal{O}$, the limit of a converging sequence is unique. Thus $\mathcal{O}$ is indeed Hausdorff which completes the proof the first part of the proposition. □

Example 3.10. i) Let $\varphi\in\mathrm{Mod}(S)$ be a pseudo-Anosov element. Then φ acts as a loxodromic isometry on the curve graph of S, with attracting and repelling fixed points $\nu_+,\nu_-\in\partial\mathcal{CG}(S)$. Let $\mu\in\mathcal{X}(S)$ be any minimal geodesic lamination which is distinct from the repelling fixed point ν_- of φ. Then $\varphi^j\mu\to\nu_+$ $(j\to\infty)$ in the coarse Hausdorff topology and therefore $\varphi^j\mu\to\nu_+$ in $\mathcal{X}(S)$.

ii) Now let us assume that $S_0\subset S$ is a proper connected subsurface different from an annulus and a pair of pants and that $\varphi\in\mathrm{Mod}(S)$ restricts to a pseudo-Anosov mapping class on S_0 and to the trivial mapping class on $S-S_0$. Let $\nu_+,\nu_-\in\partial\mathcal{CG}(S_0)$ be the attracting and repelling geodesic lamination for the action of φ on S_0, respectively. Let furthermore $\mu\neq\nu_-\in\mathcal{X}(S)$ be any *minimal* geodesic lamination on S. Then there are two possibilities. In the first case, μ is supported in $S-S_0$. Then we have $\varphi^j(\mu)=\mu$ for all j. However, if μ intersects S_0, then either $\mu=\nu_-$ or μ intersects ν_- and we have $\varphi^j(\mu)\to\nu_+$ $(j\to\infty)$ in $\mathcal{X}(S)$.

Namely, if μ intersects S_0 then the subsurface projection of μ into any subsurface disjoint from S_0 is a collection of arcs intersecting ∂S_0. In particular, the subsurface projection into any subsurface V of $S-S_0$ is a point of $\mathcal{CG}(V)$. Since φ can be represented by a diffeomorphism which fixes $S-S_0$ pointwise, it acts trivially on $\mathcal{CG}(V)$ which yields the above statement. ■

Corollary 3.11. *$(\mathcal{X}(S),\mathcal{O})$ is a Lindelöf space.*

Proof. We have to show that any open cover of $\mathcal{X}(S)$ has a countable subcover. To this end let $\mathcal{U}$ be such an open cover. List the countably many spaces $\mathcal{J}(\cup_i S_i)$ as $\mathcal{J}_1,\mathcal{J}_2,\dots.$ Since for each i, the space $\mathcal{J}_i$ is separable and metrizable, the restriction of $\mathcal{U}$ to $\mathcal{J}_i$, which is an open covering of $\mathcal{J}_i$, has a countable subcover, say by sets $U_i^1,U_i^2,\dots.$ The standard diagonal argument shows that the union $\mathcal{V}=\cup_{i,j}U_i^j$ consists of countably many sets, and for each i, the sets from $\mathcal{V}$ cover $\mathcal{J}_i$. Since

$\mathcal{X}(S) = \cup_i \mathcal{J}_i$ (as a set), this shows that $\mathcal{V}$ is a countable subcover of the cover $\mathcal{U}$. In other words, $\mathcal{X}(S)$ is a Lindelöf space as claimed. □

Proposition 3.12. *$(\mathcal{X}(S), \mathcal{O})$ is compact.*

Proof. As by Proposition 3.9 and Corollary 3.11, the space $\mathcal{X}(S)$ is a separable Lindelöf space, moreover it is Hausdorff, to show that $\mathcal{X}(S)$ is compact it suffices to show that $\mathcal{X}(S)$ is sequentially compact.

Thus let $\xi^j = \sum_i a_i^j \xi_i^j \subset \mathcal{X}(S)$ be any sequence. We have to construct a convergent subsequence. Since the space of geodesic laminations equipped with the Hausdorff topology is compact, by passing to a subsequence we may assume that the geodesic laminations $\mathrm{supp}(\xi^j) = \cup_i \xi_i^j$ converge in the Hausdorff topology to a geodesic lamination $\hat{\zeta}$ with minimal components $\zeta_1, \dots, \zeta_k$.

For each $i \leq k$ let $S_i \subset S$ be the subsurface of S filled by ζ_i. Assume by passing to a subsequence that

$$\xi^j = \sum_{i=1}^{u} a_i^j \xi_i^j + \sum_{\ell > u} a_\ell^j \xi_\ell^j$$

for all j where for each $i \leq u$, the component ξ_i^j fills S_i and that none of the components ξ_ℓ^j for $\ell > u$ fills any of the surfaces S_i. There is no assumption on the supports of the laminations ξ_ℓ^j for $\ell > u$ (besides of course that they are pairwise disjoint and disjoint from the surfaces S_i for $i \leq u$). By passing to another subsequence, we may assume that for $i \leq u$, the labels $\pm$ of the components ξ_i^j are constant along the sequence, and that the weights $a_i^j \in (0,1]$ of the components ξ_i^j converge to weights $b_i \geq 0$. In particular, the sums $1 - \sum_{i \leq u} a_i^j$ converge to $1 - \sum_{i \leq u} b_i = \kappa$.

Since $\mathrm{supp}(\xi^j) \to \hat{\zeta}$ in the Hausdorff topology, we know that for each $i \leq u$, the laminations ξ_i^j converge in the coarse Hausdorff topology to ζ_i and hence ξ_i^j converges to ζ_i in $\partial\mathcal{CG}(S_i)$ [H06]. Thus if $\kappa = 0$ then by the definition of the topology on $\mathcal{X}(S)$, we know that $\xi^j \to \sum_{i=1}^u b_i \zeta_i$ and we are done. Namely, putting $S_{u+1} = S \setminus \cup_{i=1}^u S_i$, in this case we immediately obtain that $\mathrm{pr}_{\mathcal{Z}(\cup_{i=1}^{u+1} S_i)} \xi^j \to \sum_{i=1}^u b_i \zeta_i$. Thus we are left with the case $\kappa > 0$. Viewing $\xi^j = (\sum_{i \leq u} a_i^j \zeta_i) + (\sum_{i \geq u+1} a_i^j \xi_i^j)$ as points in the join of two subspaces of $\mathcal{X}(S)$, using the above argument it suffices in fact to assume that for no j there exists a component of $\mathrm{supp}(\xi^j)$ which fills any of the subsurfaces S_i.

Then for each i, we can consider the subsurface projection $\mathrm{pr}_{S_i}(\mathrm{supp}(\xi^j))$ of $\mathrm{supp}(\xi^j)$ into the surface S_i. Furthermore, by passing to another subsequence, we may assume that for all j and all $i \leq k$, this subsurface projection is non-empty since the geodesic lamination ζ_i which fills S_i is contained in the limit with respect to the Hausdorff topology of the sequence of laminations $\mathrm{supp}(\xi^j)$. Put differently, we may assume that for each i and all j, the subsurface projection $\mathrm{pr}_{S_i}(\mathrm{supp}(\xi^j))$ of the lamination $\mathrm{supp}(\xi^j)$ into the subsurface S_i is a coarsely well defined point in $\mathcal{CG}(S_i)$. Furthermore, using once more that ζ_i fills S_i and that ζ_i is contained in the Hausdorff limit of the sequence $\mathrm{supp}(\xi^j)$, if we denote by x_i the fixed basepoint in $\mathcal{CG}(S_i)$, then we know that $d_{\mathcal{CG}(S_i)}(\mathrm{pr}_{S_i}(\mathrm{supp}(\xi^j)), x_i) \to \infty$ $(j \to \infty)$.

By passing to another subsequence and reordering indices, we may assume that

$$a_1^j = d_{\mathcal{CG}(S_1)}(\mathrm{pr}_{S_1}(\mathrm{supp}(\xi^j)), x_1) \geq a_i^j = d_{\mathcal{CG}(S_i)}(\mathrm{pr}_{S_i}(\mathrm{supp}(\xi^j)), x_i)$$

for all $i \geq 2$ and all j. Passing to another subsequence, we may assume furthermore that $a_i^j / a_1^j \to a_i \in [0, 1]$ for all $i \geq 2$. Put $a_1 = 1$; then we have $\sum_u a_u \geq 1$ and hence defining $b_i = a_i / \sum_u a_u > 0$, we conclude that $\sum_u b_u = 1$. It now follows from the definition of the topology on $\mathcal{X}(S)$ that $\xi^j \to \sum_i b_i \zeta_i$. This completes the proof that $\mathcal{X}(S)$ is sequentially compact. □

Lemma 3.13. $\mathrm{Mod}(S)$ *acts on* $\mathcal{X}(S)$ *as a group of transformations.*

Proof. Observe first that by construction, $\mathrm{Mod}(S)$ acts on $\mathcal{X}(S)$ as a group of bijections (equivalently, transformations for the discrete topology). Thus it suffices to show that this action is continuous for the topology $\mathcal{O}$.

By the definition of $\mathcal{O}$, for this it suffices to show the following. Let ξ^j be a sequence converging for the topology $\mathcal{O}$ to a point ξ. Then for every $\varphi \in \mathrm{Mod}(S)$, the sequence $\varphi(\xi^j)$ converges to $\varphi(\xi)$. Namely, if this holds true then any $\varphi \in \mathrm{Mod}(S)$ is a closed map, that is, it maps closed subsets of $(\mathcal{X}(S), \mathcal{O})$ to closed sets. Since each φ is a bijection, with inverse φ^{-1}, this implies that the preimage under φ of any open set is open and hence φ is continuous.

We need to check that requirement (2) for convergence is natural for the action of $\varphi \in \mathrm{Mod}(S)$. Now if $S_1, \dots, S_k$ is a partition of S into disjoint subsurfaces, then the same holds true for $\varphi(S_1), \dots, \varphi(S_k)$, and for any geodesic lamination ν, we have $\mathrm{pr}_{\mathcal{Y}(\cup_i \varphi(S_i))}(\varphi(\nu)) = \varphi(\mathrm{pr}_{\mathcal{Y}(\cup_i S_i)}(\nu))$ up to replacing the basepoints y_i of $\mathcal{CG}(\varphi(S_i))$ by $\varphi(x_i)$. Note that φ also naturally acts on orientations of simple closed curves on S as no oriented simple closed curve on S is freely homotopic to its inverse and hence φ acts on labelled simple closed curves. As for all i, we have $d_{\mathcal{CG}(\varphi(S_i))}(\mathrm{pr}_{\varphi(S_i)}(\xi^j), \varphi(x_i)) = d_{\mathcal{CG}(S_i)}(\xi^j, x_i) \to \infty$ $(j \to \infty)$ and the determination of the weights of the limit points are computed using ratios of distances to the basepoint defined by subsurface projections, with the distances tending to infinity along the sequence, we conclude that the requirement (2) in the definition of convergence is fulfilled for $\varphi(\xi^i)$ if it is fulfilled for ξ^i. Thus indeed, $\mathrm{Mod}(S)$ acts on $\mathcal{X}(S)$ as a group of transformations which shows the lemma. □

Definition 3.14. The space $(\mathcal{X}(S), \mathcal{O})$ is called the *geometric boundary* of $\mathrm{Mod}(S)$.

Let us note another naturality property of the geometric boundary of $\mathrm{Mod}(S)$. Namely, if $S_0 \subset S$ is any essential subsuface, then we can construct a geometric boundary $\mathcal{X}(S_0)$ for the mapping class group $\mathrm{Mod}(S_0)$ of isotopy classes of homeomorphisms of S_0 fixing the boundary pointwise. As a set, this is a subset of the geometric boundary of S which includes the Gromov boundary of the curve graph for peripheral annuli. The above construction immediately yields

Corollary 3.15. *If* $S_0 \subset S$ *is any subsurface of* S*, then the geometric boundary of* $\mathrm{Mod}(S_0)$ *is a closed subspace of the geometric boundary of* $\mathrm{Mod}(S)$.

From what we achieved so far, the second part of Proposition 7 is now immediate.

Proof of (2) of Proposition 7. Let $\varphi \in \mathrm{Mod}(S)$ be a Nielsen-Thurston mapping class. Then there are pairwise disjoint subsurfaces $S_1, \dots, S_{k+1}$, some of them may be annuli, so that $S = \cup_i S_i$, that φ preserves S_i for each i and that the following holds true. If $i \leq k$ and if S_i is distinct from an annulus, then S_i also is distinct from a pair of pants and φ acts on S_i as a pseudo-Anosov mapping class preserving each boundary component of S_i. If $i \leq k$ and if S_i is an annulus, then φ acts on S_i as a Dehn (multi)-twist, and the restriction of φ to S_{k+1} is the trivial mapping class.

If $\xi \in \mathcal{X}(S)$ is fixed by φ and not contained in the obvious fixed point set, then $\xi = \sum_j a_j \xi_j$ where either for at least one j, the lamination ξ_j is not supported in any of the subsurfaces S_i, or for at least one j the lamination is supported in one of the surfaces S_i but it is not fixed by $\varphi|S_i$. But then Example 3.10 and the definition of the topology on $\mathcal{X}(S)$ together yield that ξ is not a fixed point for the action of φ on $\mathcal{X}(S)$. □

4. A small boundary for Mod(S)

The purpose of this section is to construct a topology on $\overline{\mathcal{T}}(S) = \mathcal{T}_\epsilon(S) \cup \mathcal{X}(S)$ which restricts to the given topologies on $\mathcal{T}_\epsilon(S)$ and on $\mathcal{X}(S)$ and such that with respect to this topology, $\overline{\mathcal{T}}(S)$ is a compact $\mathrm{Mod}(S)$-space. The construction of this topology is carried out using Teichmüller geometry.

Remark 4.1. In [DHS17], there is a construction of a topology on $\mathrm{Mod}(S) \cup \mathcal{X}(S)$ (where however the topology on $\mathcal{X}(S)$ differs from the one we introduced) using the combinatorics of hierarchical hyperbolic spaces and such that with respect to this topology, $\mathrm{Mod}(S) \cup \mathcal{X}(S)$ is compact. Instead our construction uses geometric tools relying on Teichmüller theory which are not available for hierarchically hyperbolic spaces, with the aim at capturing features of $\mathrm{Mod}(S)$ which resemble properties of a CAT(0) group. It is possible that there is a relation of our construction to the one in [DHS17]. A verification would require a translation of our construction into the context of hierarchical hyperbolicity, and it is unclear whether this can be done, see however [DMS25].

4.1. The thick part of Teichmüller space.

In this section we consider a connected surface S of finite type different from a sphere with at most three punctures. The Teichmüller space $\mathcal{T}(S)$ is the space of all complete finite area hyperbolic metrics on S. By the collar lemma for hyperbolic surfaces, there exists a number $\epsilon_0 > 0$ not depending on S with the following property. For any hyperbolic metric on S, any two closed geodesics γ_1, γ_2 of length $\ell(\gamma_1), \ell(\gamma_2) \leq \epsilon_0$ are disjoint.

Let $\mathrm{syst} : \mathcal{T}(S) \to (0, \infty)$ be the *systole function* which associates to a point in $\mathcal{T}(S)$ its systole, that is, the shortest length of an essential closed curve (closed geodesic) on S. For $\epsilon \leq \epsilon_0$ define the *ϵ-thick part* $\mathcal{T}_\epsilon(S)$ of the Teichmüller space $\mathcal{T}(S)$ of marked hyperbolic metrics on S by

$$\mathcal{T}_\epsilon(S) = \{X \in \mathcal{T}(S) \mid \ \mathrm{syst}(X) \geq \epsilon\}.$$

The following statement is well known. We refer to Proposition 1.1 of [JW10] for an explicit account.

Theorem 4.2. *For $\epsilon < \epsilon_0$, the following holds.*

(1) *The subspace $\mathcal{T}_\epsilon(S) \subset \mathcal{T}(S)$ is non-empty, closed, connected and stable under* $\mathrm{Mod}(S)$. *Its quotient under the action of* $\mathrm{Mod}(S)$ *is compact.*
(2) *$\mathcal{T}_\epsilon(S)$ is a real-analytic manifold with corners and hence admits a* $\mathrm{Mod}(S)$*-invariant triangulation such that* $\mathrm{Mod}(S)\backslash\mathcal{T}_\epsilon(S)$ *is a finite CW-complex.*

As a consequence, $\mathcal{T}_\epsilon(S)$ is a topological manifold with boundary and interior $\mathring{\mathcal{T}}_\epsilon(S) = \{X \mid \mathrm{syst}(X) > \epsilon\} \subset \mathcal{T}_\epsilon(S)$.

There is a coarsely well defined map

$$\Upsilon : \mathcal{T}(S) \to \mathcal{CG}(S)$$

which maps a marked hyperbolic metric to a closed non-contractible curve of minimal length. Coarsely well defined means that the map depends on choices, but the images of a point $X \in \mathcal{T}(S)$ for any two choices of such a map are of distance at most two.

Call a map $\Psi : \mathcal{T}(S) \to \mathcal{T}(S)$ *coarsely Υ-invariant* if $d(\Upsilon(\Psi(X)), \Upsilon(X)) \leq 2$ for all X. The following is due to Ivanov [Iv02] if one replaces the mapping class group by a torsion free subgroup. The full version is Theorem 1.2 of [JW10], see also Theorem 3.9 of [J14].

Theorem 4.3 (Ji-Wolpert)**.** *For $\epsilon < \epsilon_0/3$ there is a* $\mathrm{Mod}(S)$*-equivariant coarsely Υ-invariant deformation retraction $\mathcal{T}(S) \to \mathcal{T}_\epsilon(S)$.*

The deformation retraction is constructed as follows. First Ji and Wolpert construct a $\mathrm{Mod}(S)$-invariant continuous uniquely integrable vector field V on $\mathcal{T}(S)$ with the following properties (p.9 of [JW10]).

(a) $V(\mathrm{syst}) = 1$ on $\{\mathrm{syst} \leq 2\epsilon\}$ and
(b) V vanishes on $\{\mathrm{syst} \geq 3\epsilon\}$.

The deformation retraction is then given by the time ϵ-map of the flow defined by V. Note however that the image of $\mathcal{T}(S)$ under this map is the interior of $\mathcal{T}_\epsilon(S)$. Since the time ϵ map of a continuous flow is a homeomorphism, we obtain the following statement as an immediate consequence.

Corollary 4.4. *For every $\epsilon < \epsilon_0/3$ there is a* $\mathrm{Mod}(S)$*-equivariant homeomorphism $\Lambda_\epsilon : \mathcal{T}(S) \to \mathring{\mathcal{T}}_\epsilon(S)$.*

For our purpose, the difficulty arises that we need to construct contractible subsets of $\mathcal{T}_\epsilon(S)$ and not of its interior. But the closure of a contractible open set in a smooth manifold may not be contractible. The following construction will allow us to address this issue.

Define the *small closure* $\overline{A}_{\mathrm{small}}$ of a subset A of $\mathring{\mathcal{T}}_\epsilon(S)$ to be the union of A with the set of all points $z \in \partial\mathcal{T}_\epsilon(S)$ so that z has a neighborhood U in $\mathcal{T}_\epsilon(S)$ with

$U \cap (\mathcal{T}_\epsilon(S) \setminus \partial\mathcal{T}_\epsilon(S)) \subset A$. Note that $\overline{A}_{\rm small} \setminus A$ is an *open* subset of $\partial\mathcal{T}_\epsilon(S)$. More precisely, we have.

Lemma 4.5. (1) *The small closure in $\mathcal{T}_\epsilon(S)$ of an open subset of $\mathring{\mathcal{T}}_\epsilon(S)$ is open in $\mathcal{T}_\epsilon(S)$.*
(2) *If $U \subset \mathcal{T}_\epsilon(S)$ is open, then $U \subset \overline{U \cap \mathring{\mathcal{T}}_\epsilon(S)}_{\rm small}$.*

Proof. If $U \subset \mathring{\mathcal{T}}_\epsilon(S)$ is open, then as $\mathring{\mathcal{T}}_\epsilon(S) \subset \mathcal{T}_\epsilon(S)$ is open, a point $x \in U \subset \overline{U}_{\rm small}$ has a neighborhood in $\mathcal{T}_\epsilon(S)$ which is contained in U.

On the other hand, if $x \in \overline{U}_{\rm small} \setminus U$ then it follows from the definition of $\overline{U}_{\rm small}$ that x has a neighborhood in $\mathcal{T}_\epsilon(S)$ entirely contained in $\overline{U}_{\rm small}$. This shows the first part of the lemma.

The second part of the lemma is immediate from the definitions. □

Lemma 4.6. *The small closure of a contractible subset of $\mathcal{T}_\epsilon(S)$ is contractible.*

Proof. It suffices to deformation retract the small closure $\overline{A}_{\rm small}$ of a contractible subset A of $\mathcal{T}_\epsilon(S)$ into A. The composition of this deformation retraction with a deformation retraction of A to a point then shows that $\overline{A}_{\rm small}$ is contractible.

Since $\mathcal{T}_\epsilon(S) \subset \mathcal{T}(S)$ is a manifold with corners, for $z \in \overline{A}_{\rm small} \setminus A$ there is an open neighborhood U_z of z in $\overline{A}_{\rm small} \cap \mathcal{T}_\epsilon(S) \setminus \mathcal{T}_{2\epsilon}(S)$ which is homeomorphic to the set $B_0 = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_i x_i^2 < 1, x_1 \geq 0\}$, with z corresponding to 0, and such that $B_0 \cap \{x_1 > 0\} \subset A$. Let again V be the vector field on $\mathcal{T}(S)$ with properties (a),(b) as preceding Corollary 4.4 and let $\psi : B_0 \to [0,1]$ be a smooth function with $\psi > 0$ on $\sum_i x_i^2 < 1$ and $\psi \equiv 0$ on $\sum_i x_i^2 = 1$. Via the identification of B_0 with the neighborhood U_z of z in $\overline{A}_{\rm small}$ and constant extension to zero outside this neighborhood, we can view ψ as a continuous function on $\mathcal{T}_\epsilon(S)$ which is supported in the closure $\overline{U}_z$ of U_z. The vector field ψV then defines a flow supported in $\overline{U}_z$ (which is just a local continuous time change of the flow of V) which deformation retracts $\overline{U}_z$ into $\overline{U}_z \setminus (U_z \cap (\overline{A}_{\rm small} \setminus A))$. Its time one map is a homeomorphism onto its image.

As a consequence, for every $z \in \overline{A}_{\rm small} \setminus A$ there is a deformation retraction of $\overline{A}_{\rm small}$ which moves a neighborhood of z in $\overline{A}_{\rm small}$ into A and such that the intersection of the resulting set with $\partial\mathcal{T}_\epsilon(S)$ is contained in the complement of a neighborhood of z in the intersection of $\overline{A}_{\rm small}$ with $\partial\mathcal{T}_\epsilon(S)$.

Since $\overline{A}_{\rm small} \setminus A$ is an open subset of the topological manifold $\partial\mathcal{T}_\epsilon(S)$ and hence it is a topological manifold, we can find a countable covering $\mathcal{W}$ of $\overline{A}_{\rm small} \setminus A$ so that each set $W_i \in \mathcal{W}$ is the intersection with $\overline{A}_{\rm small} \setminus A$ of an open set U_i of $\mathcal{T}_\epsilon(S)$ as described above. Let $\{\zeta_i \mid i\}$ be a partition of unity subordinate to the covering of $\cup_i U_i \supset \overline{A}_{\rm small} \setminus A$ by the sets U_i. Using the above notations but writing ψ_i for the function ψ on $U_i \sim B_0$, for each i the vector field $\zeta_i \psi_i V$ defines a local flow which moves $W_i \cap \{\zeta_i > 0\}$ into A. Then the flow of the vector field $\sum_i \zeta_i \psi_i V$ is well defined, and the image of $\overline{A}_{\rm small} \setminus A$ under its time map is contained in A. This is what we wanted to show. □

4.2. **A topology on $\mathcal{T}_\epsilon(S) \cup \mathcal{X}(S)$.** In this subsection we allow the surface S to be disconnected. All components of S may have punctures, but they have empty boundary unless the component is an annulus. The construction of a topology on $\overline{\mathcal{T}}(S) = \mathcal{T}_\epsilon(S) \cup \mathcal{X}(S)$ is similar to the construction of the topology on $\mathcal{X}(S)$. We begin with having a short look at an annulus.

Example 4.7. In the case S is an annulus, then we have $\mathcal{T}(S) = \mathbb{R}$, $\mathcal{X}(S) = \{+,-\}$. If we equip $\overline{\mathcal{T}}(S)$ with the topology of the compactification of $\mathbb{R}$ which is homeomorphic to a compact interval and is obtained by attaching two points $\pm\infty$, then this construction defines an $\mathcal{EZ}$-structure for the infinite cyclic group of Dehn twists along the core curve of the annulus. ■

If $S = \sqcup_i S_i$ then the Teichmüller space $\mathcal{T}(S) = \prod \mathcal{T}(S_i)$ of S is the product of the Teichmüller spaces of the components of S, and we have $\mathcal{T}_\epsilon(S) = \prod \mathcal{T}_\epsilon(S_i)$. There exists a constant $\rho = \rho(S) > \epsilon_0$, a so-called *Bers constant*, such that any marked hyperbolic surface $X \in \mathcal{T}(S)$ admits a pants decomposition by simple closed curves of X-length at most ρ [Bu92]. If $X \in \mathcal{T}_\epsilon(S)$, then by possibly enlarging ρ, we may in fact assume that X admits a marking $\mu(X)$ consisting of simple closed curves of length at most ρ. We call such a marking *short* for X. Namely, it is a consequence of the collar lemma [Bu92] that for any surface $X \in \mathcal{T}_\epsilon(S_i)$ there exists a compact subset of X whose diameter is bounded from above by a constant only depending on the topological type of S_i and ϵ which contains every simple closed curve on S_i. From this fact it is easy to construct from a pants deomposition whose components have length smaller than a Bers constant an explicit short marking.

Using again the collar lemma [Bu92], the geometric intersection number between any two simple closed curves on S of X-length at most ρ is bounded from above by a universal constant. In particular, the marking $\mu(X)$ defines a subset of uniformly bounded diameter in $\mathcal{CG}(S)$ (see [MM99] for more information).

The marking curves from the marking $\mu(X)$ decompose S into disks. Thus for every proper essential not necessarily connected subsurface S_0 of S, the subsurface projections of the marking curves from $\mu(X)$ decompose S into disks and hence they coarsely define a marking $\mathrm{pr}_{S_0}(\mu(X))$ of S_0. Here as before, the marking depends on choices, but any two markings obtained in this way are of uniformly bounded distance in the marking graph.

The following definition is geared at overcoming some purely technical difficulties in the construction of an $\mathcal{EZ}$-structure for $\mathrm{Mod}(S)$.

Definition 4.8. A topology on $\overline{\mathcal{T}}(S) = \mathcal{T}_\epsilon(S) \cup \mathcal{X}(S)$ is called *nice* if it restricts to the given topologies on $\mathcal{T}_\epsilon(S)$ and on $\mathcal{X}(S)$, with $\mathcal{T}_\epsilon(S) \subset \overline{\mathcal{T}}(S)$ open and dense, and if every point $\xi \in \mathcal{X}(S)$ has a neighborhood basis consisting of sets U_ξ so that $U_\xi \cap \mathring{\mathcal{T}}_\epsilon(S)$ is open and contractible.

We are now ready to state a more precise version of the third part of Theorem 4. In its formulation, we do not require S to be connected.

Theorem 4.9. *For a surface S of finite type there exists a topology on $\overline{\mathcal{T}}(S)$ with the following properties.*

(1) *The topology is nice.*
(2) *Let* $\xi=\sum_{i=1}^k a_i\xi_i\in\mathcal{X}(S)$ *and for each* $i\leq k$ *let* S_i *be the surface filled by* ξ_i*. Put* $S_{k+1}=S\setminus\cup_{i=1}^k S_i$*. A sequence* $X^j\subset\mathcal{T}_\epsilon(S)$ *converges to* ξ *if and only if* $(\mathrm{pr}_{S_1}(\mu(X^j)),\dots,\mathrm{pr}_{S_{k+1}}(\mu(X^j)))\to\xi$ *in* $\mathcal{Y}(\cup_{i=1}^{k+1}S_i)$*.*
(3) *The pair* $(\overline{\mathcal{T}}(S),\mathcal{X}(S))$ *is an* $\mathcal{EZ}$*-structure for* $\mathrm{Mod}(S)$*.*

The remainder of this article is devoted to the proof of Theorem 4.9.

Example 4.10. If S is a once punctured torus or a four-holed sphere, then we saw in Example 3.3 that $\mathcal{X}(S)$ has a natural identification with the Gromov boundary $\partial\mathrm{Mod}(S)$ of the mapping class group of S. Recall that $\mathcal{X}(S)$ is a disjoint union of the set $\partial\mathcal{CG}(S)$ of minimal filling geodesic laminations with the set of all labeled simple closed curves.

Since $\mathrm{Mod}(S)$ is virtually free and hence a hyperbolic group which acts properly and cocompactly on $\mathcal{T}_\epsilon(S)$, the space $\overline{\mathcal{T}}(S)$ has a natural topology which is inherited from the topology of the union of $\mathrm{Mod}(S)$ with its Gromov boundary. The restrictions of this topology to the subsets $\mathcal{T}_\epsilon(S)$ and $\mathcal{X}(S)$ are the given topologies.

Any proper essential subsurface of S either is a three-holed sphere (which does not play a role in our discussion) or an annulus. Let $A\subset S$ be such an annulus. We claim that the topology of $\overline{\mathcal{T}}(S)$ is consistent with the topology of the compactification $\mathbb{R}\cup\{\pm\infty\}$ of the Teichmüller space $\mathcal{T}(A)$ in the sense of property (2) in Theorem 4.9.

Namely, let c be the core curve of A and let $c_+\in\mathcal{X}(S)$ be the curve c equipped with a label. Denote by $\langle T_c\rangle$ the infinite cyclic group of Dehn twists about c and assume that c_+ corresponds to the limit point of the sequence T_c^k as $k\to\infty$. Note that $\langle T_c\rangle$ is a quasi-convex subgroup of $\mathrm{Mod}(S)$. Let $X\in\mathcal{T}_\epsilon(S)$ be an arbitrary point. With respect to the topology of $\mathrm{Mod}(S)\cup\partial\mathrm{Mod}(S)$ as the union of a hyperbolic group with its Gromov boundary, with $\mathrm{Mod}(S)$ identified with the orbit of X, a sequence of points $X_i=\varphi_i(X)\subset\mathcal{T}_\epsilon(S)$ for $\varphi_i\in\mathrm{Mod}(S)$ converges to $c_+\in\mathcal{X}(S)$ if the shortest distance projections of the elements φ_i into the quasi-convex infinite cyclic subgroup $\langle T_c\rangle$ converge to c_+. Translated into properties of the subsurface projections of points in the Farey graph as explained in Example 2.3, this just means that the topology on $\overline{\mathcal{T}}(S)$ fulfills property (2) in Theorem 4.9.

We can also check that the topology is nice. Namely, recall that $\mathcal{T}_\epsilon(S)$ can be identified with the complement in the hyperbolic plane $\mathbb{H}^2$ of a $\mathrm{Mod}(S)$ invariant countable collection of horoballs whose closures are pairwise disjoint. The horoballs are based at the rational points of $\partial\mathbb{H}^2$ and are fixed by an infinite cyclic subgroup of $\mathrm{Mod}(S)$ of parabolic isometries.

Let $H\subset\mathbb{H}^2$ be such a horoball, with boundary ∂H, fixed by the parabolic group $G\subset\mathrm{Mod}(S)$. This is a convex subset of $\mathbb{H}^2$. Let $\eta:\mathbb{R}\to\partial H$ be a parameterization of ∂H by arc length. The geodesics in $\mathbb{H}^2$ which are asymptotic to the fixed point of G in $\partial\mathbb{H}^2$ foliate $\mathbb{H}^2$ and determine a shortest distance projection $P:\mathbb{H}^2\setminus\mathring{H}\to\partial H$. For each integer $m\geq 1$ the set $U_m=P^{-1}(\eta(m,\infty))\setminus\eta(m,\infty)$ is contractible and intersects $\mathring{\mathcal{T}}_\epsilon(S)$ in a contractible open set whose small closure is a neighborhood of the labeled point $\eta(\infty)=c_+$. These neighborhoods define a countable neighborhood basis of c_+ which are small closures of contractible open subsets of $\mathring{\mathcal{T}}_\epsilon(S)$.

Alternatively, let $V_m \supset U_m$ be the union of all leaves of the foliation which pass through $\eta(m,\infty)$. Clearly V_m is contractible. The small closures of the images of the sets V_m under the homeomorphism $\mathcal{T}(S) \to \mathring{\mathcal{T}}_\epsilon(S)$ then define another neighborhood basis of c_+ in the above topology of $\overline{\mathcal{T}}(S)$ consisting of small closures of contractible open subsets of $\mathring{\mathcal{T}}_\epsilon(S)$.

As neighborhood bases of minimal filling laminations will be discussed in detail in a more general context, we omit the discussion here. ■

Proposition 4.11. *There exists a Hausdorff topology $\mathcal{O}_0$ on $\overline{\mathcal{T}}(S) = \mathcal{T}_\epsilon(S) \cup \mathcal{X}(S)$ with the property that a set $A \subset \overline{\mathcal{T}}(S)$ is closed for $\mathcal{O}_0$ if and only if the following holds true.*

1. *$A \cap \mathcal{T}_\epsilon(S)$ is closed in $\mathcal{T}_\epsilon(S)$, and $A \cap \mathcal{X}(S)$ is closed in $\mathcal{X}(S)$.*
2. *If $X_j \subset A \cap \mathcal{T}_\epsilon(S)$ is a sequence which converges to $\xi \in \mathcal{X}(S)$ in the sense of (2) of Theorem 4.9, then $\xi \in A$.*

Proof. The proof is analogous to the proof of Lemma 3.5. By the Hausdorff property of $\mathcal{T}_\epsilon(S)$ and Lemma 3.5, note first that any limit of a convergent sequence $X_j \subset \mathcal{T}_\epsilon(S)$ is unique.

To show that the notion of convergence defines a topology on $\overline{\mathcal{T}}(S)$ with the property that $\mathcal{T}_\epsilon(S) \subset \overline{\mathcal{T}}(S)$ is open and a set $A \subset \overline{\mathcal{T}}(S)$ is closed if $A \cap \mathcal{T}_\epsilon(S)$ and $A \cap \mathcal{X}(S)$ are closed and if A contains the limit of any sequence $X_j \subset \mathcal{T}(S)$ which converges to a point in $\mathcal{X}(S)$, it suffices to verify that the empty set and the entire space are closed, and the same holds true for finite unions and arbitrary intersections of closed sets. The verification that this is satisfied is identical to the argument used in the proof of Lemma 3.5.

Finally we have to show that the topology thus constructed is Hausdorff. Since $\mathcal{X}(S) \subset \overline{\mathcal{T}}(S)$ is a closed Hausdorff space and $\mathcal{T}_\epsilon(S) \subset \overline{\mathcal{T}}(S)$ is an open Hausdorff subspace of $\overline{\mathcal{T}}(S)$, all we need to show is that two points $\xi \neq \eta \in \mathcal{X}(S)$ have disjoint neighborhoods. Now ξ, η have disjoint neighborhoods in $\mathcal{X}(S)$ and hence since $\overline{\mathcal{T}}(S)$ is separable, it suffices to show that the limit of any sequence $X_i \subset \mathcal{T}_\epsilon(S)$ converging to a point in $\mathcal{X}(S)$ is unique. But this was established in the beginning of this proof. □

Assume from now on that $\overline{\mathcal{T}}(S)$ is equipped with the topology defined in Proposition 4.11. We have to verify that this topology satisfies the properties stated in Theorem 4.9.

Example 4.12. If $S = \sqcup_{i=1}^k S_i$ where for each i, S_i is an annulus, then $\mathcal{T}(S) = \mathbb{R}^k$, viewed as a CAT(0) cube complex with a proper isometric action of $\mathbb{Z}^k$. The compactification of $\mathbb{R}^k$ by attaching the visual boundary S^{k-1} defines an $\mathcal{EZ}$-structure for $\mathbb{Z}^k$ with the properties stated in Theorem 4.9.

Now $\mathcal{X}(S)$ is the join of k sets consisting of two points each and hence $\mathcal{X}(S)$ is a sphere of dimension $k-1$. By Proposition 4.11, the topology on $\mathcal{T}_\epsilon(S) \cup \mathcal{X}(S)$ is determined by the properties stated in Theorem 4.9 and hence $\overline{\mathcal{T}}(S)$ equals the compactification of $\mathbb{R}^k$ by adding the visual sphere. In particular, $\overline{\mathcal{T}}(S)$ has all properties stated in Theorem 4.9. ∎

Proposition 4.13. *The topological space* $(\overline{\mathcal{T}}(S), \mathcal{O}_0)$ *has the following properties.*

(1) $\overline{\mathcal{T}}(S)$ *is compact and separable.*
(2) *The mapping class group acts on* $\overline{\mathcal{T}}(S)$ *as a group of transformations.*

Proof. $\overline{\mathcal{T}}(S)$ is clearly separable since this holds true for $\mathcal{X}(S)$ and $\mathcal{T}_\epsilon(S)$. By Proposition 4.11, it is a Hausdorff space.

To show that $\overline{\mathcal{T}}(S)$ is compact, note that since $\mathcal{X}(S)$ is compact and $\mathcal{T}_\epsilon(S)$ is a Lindelöf space, the space $\overline{\mathcal{T}}(S)$ is Lindelöf. Since $\overline{\mathcal{T}}(S)$ also is Hausdorff, it suffices to show that $\overline{\mathcal{T}}(S)$ is sequentially compact, and this follows if we can show that any sequence $X_i \subset \mathcal{T}_\epsilon(S)$ has a convergent subsequence in $\overline{\mathcal{T}}(S)$.

If the sequence has a bounded subsequence in $\mathcal{T}_\epsilon(S)$ with respect to a fixed basepoint $X \in \mathcal{T}_\epsilon(S)$, then as $\mathcal{T}_\epsilon(S)$ is proper, we can extract a converging subsequence. Thus it suffices to show the following

Claim: Any unbounded sequence $X^i \subset \mathcal{T}_\epsilon(S)$ admits a subsequence which converges in $\overline{\mathcal{T}}(S)$ to a point $\xi \in \mathcal{X}(S)$.

Proof of the claim: The proof of the claim is essentially identical with the proof of Proposition 3.12.

Since the space of geodesic laminations on S equipped with the Hausdorff topology is compact, by extracting a subsequence we may assume that the sets of all simple closed curves contained in the marking $\mu(X^i)$ converge in the Hausdorff topology to a finite union $\cup_{j=1}^{\ell} \beta_j$ of (not necessarily minimal) geodesic laminations. Note that as some of the curves in $\mu(X^i)$ may intersect, these laminations are not necessarily disjoint, that is, $\cup_{j=1}^{\ell} \beta_j$ may not be a lamination in its own right. However, since the number of components of $\mu(X^i)$ is uniformly bounded, the same holds true for the number of limit laminations.

Let $\zeta_1, \dots, \zeta_s$ be the collection of all minimal components of the laminations β_u which are distinct from simple closed curves. The number of such components is uniformly bounded. Each of the laminations ζ_j fills a subsurface S_j of S which is different from an annulus or a pair of pants. Thus ζ_j is a point in the Gromov boundary of the curve graph $\mathcal{CG}(S_j)$ of S_j.

Now for $j \leq s$, a sequence c_j^i of simple closed curves on the surface S_j converges to ζ_j in $\mathcal{CG}(S_j) \cup \partial \mathcal{CG}(S_j)$ if and only if their geodesic representatives for some fixed hyperbolic metric on S_j converge to ζ_j in the coarse Hausdorff topology. As the diameter of the subsurface projection of $\mu(X^i)$ to S_j is bounded independent of i, hyperbolicity of $\mathcal{CG}(S_j)$ implies that the subsurface projection to S_j of any of the curves in $\mu(X^i)$ which intersects S_j converges in the coarse Hausdorff topology to

ζ_j. As a consequence, none of the limits in the Hausdorff topology of any sequence of components of $\mu(X^i)$ can intersect ζ_j.

By a similar argument, if ζ_j is a closed curve component, then we can consider the subsurface projections of a component of $\mu(X^i)$ to an annulus $A(\zeta_j)$ with core curve ζ_j. Up to passing to a further subsequence, we may assume that these projections are either bounded along the sequence, or converge to one of the two boundary components of the curve graph of $A(\zeta_j)$. In the first case call ζ_j *unlabeled*. In the second case, label ζ_j with the corresponding point in the Gromov boundary of the curve graph of $A(\zeta_j)$ and note by the reasoning used in the previous paragraph, no labeled simple closed curve component ζ_j can be intersected by another component ζ_ℓ.

By reordering, let $\zeta_1, \dots, \zeta_k$ be the components of the limit laminations β_u which either are distinct from simple closed curves or which are labeled simple closed curves. We claim that $k \geq 1$, that is, that there is at least one lamination with this property. Namely, if c is an unlabeled simple closed curve, represented by a closed geodesic for the base surface X, and if with respect to the Hausdorff topology on compact subsets of X a limit of the sequence $\mu(X^i)$ contains c as an unlabeled component, then no component of a limit of the sequence $\mu(X^i)$ in the Hausdorff topology can spiral about c and hence c is a component of $\mu(X^i)$ for all but finitely many i. If $k = 0$ then this holds true for any limit point of the sequence $\mu(X^i)$ in the Hausdorff topology. But $\mu(X^i)$ is a marking of S for all i and hence decomposes S into disks and once punctured disks and consequently the sequence $\mu(X^i)$ is bounded. But this contradicts the assumption that the sequence $X^i \subset \mathcal{T}_\epsilon(S)$ is an unbounded sequence.

By what we showed so far, $\hat{\zeta} = \cup_{j=1}^k \zeta_k$ is a geodesic lamination. Furthermore, if S_j is the subsurface of S filled by ζ_j, then $d_{\mathcal{CG}(S_j)}(\mathrm{pr}_{S_j}(\mu(X^i)), x_j) \to \infty$ where as before, $x_j \in \mathcal{CG}(S_j)$ is a fixed basepoint for $\mathcal{CG}(S_j)$.

If $\hat{\zeta}$ is minimal and fills S then $X_i \to \hat{\zeta} \in \overline{\mathcal{T}}(S)$ and we are done. Otherwise we argue as in the proof of Proposition 3.12. Namely, put $S_{k+1} = S \setminus \cup_i S_i$. Then the projections $\mathrm{pr}_{S_{k+1}}(\mu(X_i))$ define a bounded subset of the curve graph $\mathcal{CG}(S_{k+1})$, and the same holds true for projections $\mathrm{pr}_V(\mu(X_i))$ for any proper subsurface V of S_{k+1}. But then the construction in the proof of Proposition 3.12 produces a subsequence of the sequence $\mu(X^i)$ with the property that $(\mathrm{pr}_{S_1}(\mu(X^i)), \dots, \mathrm{pr}_{S_{k+1}}(\mu(X^i)))$ converges in $\mathcal{Y}(\cup_{i=1}^{k+1} S_i)$ to $\xi = \sum_i a_i \zeta_i$ for $a_i \geq 0$, $\sum_i a_i = 1$. By the requirement (2), we then have $X_i \to \xi \in \overline{\mathcal{T}}(S)$ which completes the proof of the claim. ∎

To summarize, we showed that $\overline{\mathcal{T}}(S)$ is sequentially compact Hausdorff Lindelöf space and hence it is compact.

We are left with showing that $\mathrm{Mod}(S)$ acts on $\overline{\mathcal{T}}(S)$ as a group of transformations. However, as $\mathrm{Mod}(S)$ acts on $\mathcal{T}_\epsilon(S)$ and on $\mathcal{X}(S)$ as a group of transformations, and it maps subsurfaces of S to subsurfaces of the same topological type, moreover the definition of convergence which determines the topology $\mathcal{O}_0$ is natural with respect to the action of $\mathrm{Mod}(S)$ on subsurfaces and subsurface projections, this is indeed the case. The proposition is proven. □

Theorem 4.14. *$\mathcal{X}(S)$ is a small boundary for* $\mathrm{Mod}(S)$*. A pseudo-Anosov mapping class acts on $\mathcal{X}(S)$ with north-south dynamics. In particular, the action of* $\mathrm{Mod}(S)$ *on $\mathcal{X}(S)$ is strongly proximal.*

Proof. We showed so far that $\mathcal{X}(S)$ defines a boundary of $\mathcal{T}_\epsilon(S)$ and hence of $\mathrm{Mod}(S)$ since $\mathrm{Mod}(S)$ acts properly and cocompactly on $\mathcal{T}_\epsilon(S)$. Furthermore, by Example 3.10, a pseudo-Anosov element acts on $\mathcal{X}(S)$ with north-south dynamics and hence the action of $\mathrm{Mod}(S)$ on $\mathcal{X}(S)$ is strongly proximal.

We are left with showing that the right action of $\mathrm{Mod}(S)$ induces the identity. However, this action just consists of a change of basepoint. As a sequence of points of uniformly bounded distance from a convergent sequence converges to the same point, this yields the statement of the theorem. □

5. Neighborhood bases

The main goal of this section is to construct for a point in $\mathcal{X}(S) \subset \overline{\mathcal{T}}(S)$ an explicit neighborhood basis in $\overline{\mathcal{T}}(S)$ consisting of small closures of open contractible subsets of $\mathring{\mathcal{T}}_\epsilon(S)$. Here by a contractible subset of $\mathcal{T}_\epsilon(S)$ we mean a subset V which is a contractible space with respect to the subspace topology. The construction is the most involved part of the proof of Theorem 4. It is carried out in three steps, each of which is contained in a separate subsection.

5.1. **A neighborhood basis for minimal filling laminations.**

In this subsection we prove the following result for a connected finite type surface S.

Proposition 5.1. *Every point $\xi \in \partial\mathcal{CG}(S) \subset \mathcal{X}(S)$ has a countable neighborhood basis in $\overline{\mathcal{T}}(S)$ consisting of sets whose intersections with $\mathcal{T}_\epsilon(S)$ are small closures of contractible open subsets of $\mathring{\mathcal{T}}_\epsilon(S)$ and hence are contractible.*

The proof of Proposition 5.1 introduces the ideas used in the general case, but it is technically easier. We begin with summarizing some tools used to relate Teichmüller theory to combinatorial geometry.

By [MM99], for any surface V of finite type there is a number $p > 0$ only depending on the complexity of V such that the image under the map Υ of a Teichmüller geodesic $\gamma : \mathbb{R} \to \mathcal{T}(S)$ is an *unparameterized p-quasi-geodesic* in $\mathcal{CG}(V)$. This means the following. There is an increasing homeomorphism $\sigma : (a, b) \subset \mathbb{R} \to \mathbb{R}$ such that the map $\Upsilon \circ \gamma \circ \sigma : (a, b) \to \mathcal{CG}(S)$ is a p-quasi-geodesic. This quasi-geodesic may be bounded, one-sided infinite or two-sided infinite. A sufficient but not necessary condition for being one-sided infinite in the positive direction is that the geodesic recurs in the positive direction to the thick part $\mathcal{T}_\epsilon(S)$ for arbitrarily large times. Since the directions of Teichmüller geodesic rays with this property are dense in the cotangent bundle of Teichmüller space and the map Υ is coarsely Lipschitz with respect to the Teichmüller metric on $\mathcal{T}(S)$ and the metric on $\mathcal{CG}(S)$, up to increasing p, any geodesic segment $\alpha : [0, n] \to \mathcal{CG}(S)$ can be extended to a p-quasi-geodesic ray $\alpha : [0, \infty) \to \mathcal{CG}(S)$. Note that this is *not* true for geodesics in $\mathcal{CG}(S)$ [BM15].

Let $\xi \in \partial\mathcal{CG}(S)$. Choose a basepoint $X_0 \in \mathcal{T}_\epsilon(S)$ and let c be a pants curve from $\mu(X_0)$. For $j, \ell \geq 0$ define

$$W(\xi, j, \ell) \subset \mathcal{T}_\epsilon(S)$$

to be the set of all hyperbolic metrics $X \in \mathcal{T}_\epsilon(S)$ with the following properties.

(1) $d_{\mathcal{CG}(S)}(\mu(X), c) \geq j$.
(2) A geodesic in $\mathcal{CG}(S)$ connecting c to $\mu(X)$ can be extended to a p-quasi-geodesic in $\mathcal{CG}(S)$ whose endpoint is contained in the ball of radius $e^{-\ell}$ about ξ in $\partial\mathcal{CG}(S)$, where the metric on $\partial\mathcal{CG}(S)$ is the Gromov distance d_c constructed from the basepoint c.

Lemma 5.2. *For each $\xi \in \partial\mathcal{CG}(S)$ the closures of the sets $W(\xi, j, \ell)$ in $\overline{\mathcal{T}}(S)$ define a neighborhood basis of ξ in $\overline{\mathcal{T}}(S)$.*

Proof. We show first that for each ξ, j, ℓ the closure of $W(\xi, j, \ell)$ in $\overline{\mathcal{T}}(S)$ is a neighborhood of ξ. Since $\mathcal{T}_\epsilon(S)$ is dense in $\overline{\mathcal{T}}(S)$ and by Proposition 4.13, $\overline{\mathcal{T}}(S)$ is a compact separable Hausdorff space, it suffices to show the following. Let $(X_m) \subset \mathcal{T}_\epsilon(S)$ be a sequence converging in $\overline{\mathcal{T}}(S)$ to ξ; then $X_m \in W(\xi, j, \ell)$ for all sufficiently large m.

Now as $\xi \in \partial\mathcal{CG}(S)$, by the definition of the topology of $\overline{\mathcal{T}}(S)$, we know that $\mu(X_m) \to \xi$ in $\mathcal{CG}(S) \cup \partial\mathcal{CG}(S)$. Together with hyperbolicity of $\mathcal{CG}(S)$ and the definition of the topology on the union of a hyperbolic geodesic metric space with its Gromov boundary, this immediately implies that $X_m \in W(\xi, j, \ell)$ for all sufficiently large m.

A similar argument also shows that the sets $W(\xi, j, \ell)$ define a neighborhood basis of ξ. Namely, note that the sets $W(\xi, j, \ell)$ are *nested*: If $j' > j$, then $W(\xi, j', \ell) \subset W(\xi, j, \ell)$, and if $\ell' > \ell$ then $W(\xi, j, \ell') \subset W(\xi, j, \ell)$. Thus since $\mathcal{X}(S)$ is a compact Hausdorff space, to show that the closures $\overline{W(\xi, j, \ell)}$ in $\overline{\mathcal{T}}(S)$ of the sets $W(\xi, j, \ell)$ define a neighborhood basis of ξ in $\overline{\mathcal{T}}(S)$, it suffices to show that $\cap_{j>0,\ell>0}\overline{W(\xi, j, \ell)} = \{\xi\}$.

To see that this is indeed the case note first that $\xi \in \overline{W(\xi, j, \ell)}$ for all j, ℓ and hence as these sets are compact, the point ξ also is contained in the intersection of these sets. Furthermore, the following holds true.

For each j let $X_j \in W(\xi, j, j)$; then the distance of $\mu(X_j)$ to the base curve c in $\mathcal{CG}(S)$ tends to infinity with j. This implies that the sequence X_j can not have a convergent subsequence in $\mathcal{T}_\epsilon(S)$. Thus by compactness of $\overline{\mathcal{T}}(S)$, we observe that $\cap_j W(\xi, j, j) \subset \mathcal{X}(S)$. But for each j, the set $\mu(X_j)$ is contained in a p-quasi-geodesic connecting the basepoint to a point in the e^{-j}-ball about ξ in $(\partial\mathcal{CG}(S), d_c)$ and therefore $X_j \to \xi$ in $\overline{\mathcal{T}}(S)$. Since $X_j \in W(\xi, j, j)$ was arbitrary, this implies that $\cap_j \overline{W(\xi, j, j)} = \{\xi\}$ as claimed. ☐

The following consequence of Lemma 5.2 will be used in Section 6.

Corollary 5.3. *There are countably many open sets $U_i \subset \overline{\mathcal{T}}(S)$ which contain a neighborhood basis for every $\xi \in \partial\mathcal{CG}(S)$.*

Proof. Since $\partial\mathcal{CG}(S)$ is separable, we can choose a countable dense subset $\{\xi_i \mid i\} \subset \partial\mathcal{CG}(S)$. Let d_c be the Gromov metric on $\partial\mathcal{CG}(S)$ with respect to the basepoint c. Then the open d_c-balls $B_{i,j}$ of radius e^{-j} about the points ξ_i $(i, j > 0)$ define a basis of the topology of $\partial\mathcal{CG}(S)$.

By Lemma 5.2 and the definitions, the interiors of the countably many closed subsets $\overline{W(\xi_i, j, \ell)}$ of $\overline{\mathcal{T}}(S)$ have the properties stated in the corollary. $\square$

Any minimal filling geodesic lamination ξ decomposes S into a union of ideal polygons. Each of these polygons which is not an ideal triangle can be subdivided by adding isolated leaves which connect two non-adjacent cusps of the polygon. The various ways to subdivide these polygons determine a finite collection $\xi^0, \dots, \xi^k$ of distinct geodesic laminations which contain ξ as a sublamination. Assume that $\xi^0 = \xi$, that is, ξ^0 is the unique lamination among the laminations ξ^j which does not contain any isolated leaf.

Let d_H be the Hausdorff metric on the space of compact subsets of a fixed hyperbolic surface $X \in \mathcal{T}_\epsilon(S)$. Denote as before by $\mathrm{Min}_\cup(S)$ the space of geodesic laminations on X which are unions of disjoint minimal components. Equivalently, the only isolated leaves of a geodesic lamination in $\mathrm{Min}_\cup(S)$ are simple closed curves. As before, let $\mathrm{supp} : \mathcal{X}(S) \to \mathrm{Min}_\cup(S)$ be the map which associates to a point $\zeta = \sum_i a_i\zeta_i$ $(a_i > 0)$ the support $\mathrm{supp}(\zeta) = \cup_i\zeta_i$. We have

Lemma 5.4. *For $i > 0$ let*

$$U_i = \cup_j\{\beta \in \mathrm{Min}_\cup(S) \mid d_H(\beta, \xi^j) \le 1/i\}$$

and write $V_i = \{\zeta \in \mathcal{X}(S) \mid \mathrm{supp}(\zeta) \in U_i\}$. Then the sets V_i form a neighborhood basis of $\xi \in \partial\mathcal{CG}(S) \subset \mathcal{X}(S)$ in $\mathcal{X}(S)$.

Proof. Clearly $\xi = \xi^0 \in V_i$ for all i. We first show that for each i the set V_i is a neighborhood of ξ. For this it suffices to show that for every sequence $\zeta_\ell \subset \mathcal{X}(S)$ converging to ξ and any i, we have $\zeta_\ell \in V_i$ for all but finitely many ℓ.

By the first requirement for convergence in the definition of the topology on $\mathcal{X}(S)$, we know that $\mathrm{supp}(\zeta_\ell)$ converges in the *coarse* Hausdorff topology to $\xi_0 = \mathrm{supp}(\xi)$. By compactness of the space of geodesic laminations of S with respect to the Hausdorff topology, by passing to a subsequence we may assume that the sequence $\mathrm{supp}(\zeta_\ell)$ converges in the Hausdorff topology to a geodesic lamination ζ. Then ζ contains ξ^0 as a sublamination and hence $\zeta = \xi^s$ for some $s \le k$. By definition, this implies that $\mathrm{supp}(\zeta_\ell) \in U_i$ and hence $\zeta_\ell \in V_i$ for all sufficiently large ℓ as predicted. Thus indeed, each of the sets V_i is a neighborhood of ξ.

To show that the sets V_i form a neighborhood basis for ξ, note that $V_{i+1} \subset V_i$ and hence it suffices to show that $\cap_i V_i = \{\xi\}$. However, this is immediate from the definitions and the fact that the preimage of $\mathrm{supp}(\xi)$ under the support map supp which associates to $\zeta \in \mathcal{X}(S)$ its support consists of the single point ξ. $\square$

A *measured geodesic lamination* on the surface S is a geodesic lamination together with a transverse invariant measure. The space $\mathcal{ML}$ of measured geodesic laminations is equipped with the weak* topology. The quotient of $\mathcal{ML}$ under the natural action of $(0,\infty)$ by scaling is the space $\mathcal{PML}$ of *projective measured geodesic laminations*. This space is homeomorphic to the sphere $S^{6g-7+2m}$. To put Lemma 5.4 into proper context and for later use, we relate the subset $\partial\mathcal{CG}(S) \subset \mathcal{X}(S) \subset \overline{\mathcal{T}}(S)$ to the space $\mathcal{PML}$.

To this end we use a more geometric view on $\mathcal{PML}$. Fix again a point $X \in \mathcal{T}_\epsilon(S)$. The cotangent space $T^*_X\mathcal{T}(S)$ of Teichmüller space at X can be identified with the space of measured geodesic laminations on S. Or, equivalently, by the Hubbard Masur theorem (the main theorem of [HM79], expressed in the language of measured foliations), every measured geodesic lamination ν on S is the *vertical measured geodesic lamination* of a unique marked quadratic differential $q(\nu)$ for the complex structure on S defined by X. With this identification, we can associate to $\nu \in \mathcal{ML}$ the point $\gamma_\nu(1)$ where $\gamma_\nu : [0,\infty) \to \mathcal{T}(S)$ is the Teichmüller geodesic starting at X whose initial (co)-velocity equals $q(\nu)$. This construction defines the *Teichmüller exponential map* $\exp_X : \mathcal{ML} \cup \{0\} \to \mathcal{T}(S)$ at X which is a homeomorphism.

Every nontrivial quadratic differential on X defines a singular euclidean metric on X. Associating to ν the area $\mathrm{area}(q(\nu))$ of the metric defined by $q(\nu)$ defines a positive $\mathbb{R}_+$-homogeneous function on $\mathcal{ML}$ depending on X. Associating to $[\nu] \in \mathcal{PML}$ the unique measured lamination $\rho([\nu])$ with $\mathrm{area}(q(\rho[\nu])) = 1$ then defines a section $\rho : \mathcal{PML} \to \mathcal{ML}$. In this way we can identify $\mathcal{PML}$ with the sphere of unit directions for the Teichmüller metric at X.

The *support* $\mathrm{supp}(\nu)$ of a measured geodesic lamination ν is a point in the space $\mathrm{Min}_\cup(S)$. Each of its components is equipped with a transverse invariant measure and hence it is a measured geodesic lamination in its own right.

Let as before $p > 1$ be a control constant with the following properties.

- The image under the map Υ of any Teichmüller geodesic is an unparameterized p-quasi-geodesic in $\mathcal{CG}(S)$ (see [MM99]).
- Every geodesic segment in $\mathcal{CG}(S)$ can be extended to a p-quasi-geodesic ray.

Consider a minimal filling geodesic lamination $\xi \in \partial\mathcal{CG}(S)$. Let $P(\xi) \subset \mathcal{PML}$ be the set of all projective measured geodesic laminations which are supported in ξ. This is a non-empty compact polytope of dimension $\le 3g-3+m$ whose extreme points are the ergodic projective transverse measures supported in ξ. In particular, $P(\xi)$ is compact and contractible. Since $P(\xi)$ is contractible and since $\mathcal{PML}$ is homeomorphic to a sphere of dimension $6g-7+2m$, we can find a descending chain $V_1 \supset V_2 \supset \cdots$ of closed contractible neighborhoods of $P(\xi)$, each of which is homeomorphic to a closed ball, such that $V_{i+1} \subset \mathring{V}_i$ and $\cap_j V_j = P(\xi)$.

In the sequel we use the terminology *small closure* $\bar{A}_{\mathrm{small}}$ *in* $\overline{\mathcal{T}}(S)$ of a set $A \subset \mathcal{T}_\epsilon(S)$ to denote the union of the small closure of A in $\mathcal{T}_\epsilon(S)$ with the intersection with $\mathcal{X}(S)$ of the closure $\bar{A}$ of A in $\overline{\mathcal{T}}(S)$. Thus for any set $A \subset \mathcal{T}_\epsilon(S)$, $\bar{A}_{\mathrm{small}} \cap \mathcal{X}(S)$ is closed, but $\bar{A}_{\mathrm{small}} \cap \mathcal{T}_\epsilon(S)$ may be open.

Lemma 5.5. *Let $V_1 \supset V_2 \supset \cdots$ be a descending chain of closed contractible neighborhoods of $P(\xi)$ in $\mathcal{PML}$, each of which is homeomorphic to a closed ball, with $\cap_i V_i = P(\xi)$. Let $\Lambda_\epsilon : \mathcal{T}(S) \to \mathring{\mathcal{T}}_\epsilon(S)$ be the homeomorphism from Proposition 4.4 and let $\exp_X : \mathcal{ML}(S) \to \mathcal{T}(S)$ be the Teichmüller exponential map at X. Then for each $j > 0$, the small closure $\overline{Z(i,j)}_{\text{small}}$ in $\overline{\mathcal{T}}(S)$ of the open set*

$$Z(i,j) = \Lambda_\epsilon\{\exp_X(t\rho[\nu]) \mid t > j, [\nu] \in \mathring{V}_i\}$$

is a neighborhood of ξ, and neighborhoods of this form define a neighborhood basis of ξ.

Proof. We divide the proof of the lemma into two claims.

Claim 1: For all i, j, the small closure $\overline{Z(i,j)}_{\text{small}}$ of $Z(i,j)$ in $\overline{\mathcal{T}}(S)$ is a neighborhood of ξ.

Proof of Claim 1: Let $\mathcal{FML} \subset \mathcal{PML}$ be the subset of all projective measured geodesic laminations whose support is a minimal geodesic lamination which fills up S. By Lemma 3.2 of [H09], the support map $F : \mathcal{FML} \to \partial\mathcal{C}(S)$ which associates to a point in $\mathcal{FML}$ its support is continuous and closed. Thus the image $F(\mathcal{FML} \setminus \mathring{V}_i)$ is a closed subset of $\partial\mathcal{CG}(S)$ which does not contain ξ. As a consequence, if we consider the Gromov metric on $\partial\mathcal{CG}(S)$ based at $\Upsilon(X)$, then there exists a number $T(i) > 0$ so that the ball of radius $e^{-T(i)}$ about ξ with respect to this metric is disjoint from $F(\mathcal{FML} \setminus \mathring{V}_i)$.

By the choice of the control constant $p > 1$ and hyperbolicity, there exists a number $\tau(i) > T(i)$ with the following property. Let $[\nu] \in \mathcal{FML} \setminus \mathring{V}_i$; then the endpoint of a p-quasi-geodesic ray in $\mathcal{CG}(S)$ which starts at the basepoint $\Upsilon(X)$ and which passes through a point on the p-quasi-geodesic $\Upsilon\{\exp_X(t\rho[\nu]) \mid t \geq 0\}$ of distance at least $\tau(i)$ to $\Upsilon(X)$ is not contained in the ball of radius $e^{-T(i)/2}$ about ξ.

Since the homeomorphism $\Lambda_\epsilon : \mathcal{T}(S) \to \mathring{\mathcal{T}}_\epsilon(S)$ is coarsely Υ-invariant, the map $\exp_X : \mathcal{ML} \cup \{0\} \to \mathcal{T}(S)$ is a homeomorphism and $\mathcal{FML} \subset \mathcal{PML}$ is dense, it follows that for all i, j there exists ℓ so that the set $W(\xi, \ell, \ell) \subset \mathcal{T}_\epsilon(S)$ constructed in Lemma 5.2 is contained in the closure of the set $\Lambda_\epsilon(\{\exp_X(t\rho[\nu]) \mid t \geq j, [\nu] \in \mathring{V}_i\})$. Thus by Lemma 5.2, $\overline{Z(i,j)}_{\text{small}}$ is a neighborhood of ξ in $\overline{\mathcal{T}}(S)$. ∎

The proof of the lemma is completed once we established the following. In its formulation, $\overline{Z(i,j)}$ is the closure of $Z(i,j)$ in $\overline{\mathcal{T}}(S)$. We do not have to take the small closure here.

Claim 2: Let W be a neighborhood of ξ in $\overline{\mathcal{T}}(S)$; then there exists some i, j so that $\overline{Z(i,j)} \subset W$.

Proof of Claim 2: By Claim 1, each of the sets $\overline{Z(i,j)}$ is a neighborhood of ξ and hence contains ξ. Furthermore, these neighborhoods are nested: If $i_1 \leq i_2$ and $j_1 \leq j_2$ then $\overline{Z(i_1,j_1)} \supset \overline{Z(i_2,j_2)}$. Thus since the sets $\overline{Z(i,j)}$ are moreover closed and hence compact, it suffices to show that $\cap_{i,j}\overline{Z(i,j)} = \{\xi\}$.

Since the Teichmüller exponential map $\exp_X$ at X is a homeomorphism, we clearly have $\cap_{i,j}\overline{Z(i,j)} \subset \mathcal{X}(S)$. On the other hand, the map $\Upsilon : \mathcal{T}(S) \to \mathcal{CG}(S)$ is coarsely Lipschitz, and for $\nu \in P(\xi)$, the p-quasigeodesic $t \to \Upsilon(\exp(t\rho[\nu]))$ has infinite diameter. This implies that for any $k > 0$ there are numbers $i(k) > 0, m(k) > 0$ so that for all $[\eta] \in V_{i(k)}$, the diameter of the image under Υ of the Teichmüller geodesic segment $\exp_X([0, m(k)]\rho[\eta])$ is at least k. As a consequence, if $X_i \in \overline{Z(i,i)}$ for each i, then by compactness of $\overline{\mathcal{T}(S)}$, up to passing to a subsequence the sequence X_i converges to a point $\zeta \in \mathcal{X}(S) \cap \partial\mathcal{CG}(S)$. That this point has to coincide with ξ is an immediate consequence of the discussion in the proof of Claim 1 above. This completes the proof of the claim. ■ □

Lemma 5.6. *The sets $\overline{Z(i,j)}_{\rm small} \cap \mathcal{T}_\epsilon(S)$ are contractible.*

Proof. Since for each i the set $\mathring{V}_i$ is a contractible subset of the space of projective measured geodesic laminations, identified with the unit sphere in the cotangent space of $\mathcal{T}(S)$ at X, the set

$$H(i,j) = \cup_{[\nu]\in \mathring{V}_i}\{\exp_X(t\rho[\nu]) \mid t > j\} \subset \mathcal{T}(S)$$

is open and contractible since it is homeomorphic to $\mathring{V}_i \times (j, \infty)$. This uses the fact that the Teichmüller exponential map at X is a homeomorphism of $T^*_X\mathcal{T}(S)$ onto $\mathcal{T}(S)$.

But $Z(i,j)$ is the image of $H(i,j)$ under the homeomorphism $\Lambda_\epsilon : \mathcal{T}(S) \to \mathring{\mathcal{T}}_\epsilon(S)$ and hence $Z(i,j)$ is contractible. Then by Lemma 4.6, the small closure of $Z(i,j)$ in $\mathcal{T}_\epsilon(S)$ is contractible as well. □

Proof of Proposition 5.1. By Lemma 5.5, the small closure $\overline{Z(i,j)}_{\rm small}$ of $Z(i,j)$ in $\overline{\mathcal{T}}(S)$ is a neighborhood of ξ. Lemma 5.6 shows that its intersection with $\mathcal{T}_\epsilon(S)$ is contractible. Using again Lemma 5.5, the countably many such sets define a countable neighborhood basis of ξ in $\overline{\mathcal{T}}(S)$ whence the proposition. □

5.2. **Neighborhoods of minimal filling laminations for disconnected surfaces.** In this section we consider a disjoint union $S = \sqcup_{i=1}^k S_i$ of finitely many connected surfaces of finite type. Our goal is to construct for any point in

$$\mathcal{E} = \partial\mathcal{CG}(S_1) * \cdots * \partial\mathcal{CG}(S_k) \subset \mathcal{X}(S)$$

a neighborhood basis in $\overline{\mathcal{T}}(S)$ consisting of sets whose intersections with $\mathcal{T}_\epsilon(S)$ are small closures of open contractible subsets of $\mathring{\mathcal{T}}_\epsilon(S)$.

Remark 5.7. In [Ti11], it was shown that if two groups Γ_1, Γ_2 admit $\mathcal{EZ}$-structures (X_1, Z_1) and (X_2, Z_2), then the direct product $\Gamma_1 \times \Gamma_2$ admits an $\mathcal{EZ}$-structure consisting of a compactification of the product $(X_1 \setminus Z_1) \times (X_2 \setminus Z_2)$ by adding the join $Z_1 * Z_2$. Unfortunately, we can not use this result directly in an inductive step as we need more precise information for the proof of Theorem 4.9. The argument in [Ti11] is based on the definition of an $\mathcal{EZ}$-structure using deformation retractions of neighborhoods of boundary points [B96] and a subtle combination of these deformation retractions which leads to deformation retractions of neighborhoods of points in $Z_1 * Z_2$ in the space $X_1 \times X_2 \cup Z_1 * Z_2$. This construction is not very well suited for essentially working with non-proper spaces. ■

The set $\mathcal{E}$ is the set of formal sums $\sum_i a_i\xi_i$ where $\xi_i \in \partial\mathcal{CG}(S_i)$ and $a_i \geq 0, \sum_i a_i = 1$. Choose a basepoint $X = (X_1, \dots, X_k) \in \mathcal{T}_\epsilon(S) = \prod \mathcal{T}_\epsilon(S_i)$ and let $x_i \in \mathcal{CG}(S_i)$ be a component of a short marking $\mu(X_i)$ for X_i. Following Section 5.1, we begin with constructing a neighborhood basis in $\overline{\mathcal{T}}(S)$ for each $\xi \in \mathcal{E}$, and in a second step we improve this basis to a basis consisting of small closures of contractible open subsets of $\mathring{\mathcal{T}}_\epsilon(S)$.

Thus let $\xi = \sum_i a_i\xi_i$ and assume by reordering that there exists some $\ell \leq k$ such that $a_i > 0$ if and only if $1 \leq i \leq \ell$. For $j, \ell, m > 0$ define

$$W(\xi, j, \ell, m) \subset \mathcal{T}_\epsilon(S)$$

to be the set of all hyperbolic metrics $X = (X_1, \dots, X_k)$ on S with the following properties. In its formulation, we view a short marking $\mu(Y)$ of a hyperbolic metric $Y \in \mathcal{T}_\epsilon(S_i)$ as a bounded subset of $\mathcal{CG}(S_i)$, and we continue to use the notation pr_V to denote the subsurface projection into a subsurface V.

(1) $d_{\mathcal{CG}(S_i)}(\mu(X_i), x_i) \geq j$ for $1 \leq i \leq \ell$.
(2) $\frac{d_{\mathcal{CG}(S_i)}(\mu(X_i),x_i)}{d_{\mathcal{CG}(S_1)}(\mu(X_1),x_1)} \in [(1-\frac{1}{m})\frac{a_i}{a_1}, (1+\frac{1}{m})\frac{a_i}{a_1}]$ for all $1 \leq i \leq \ell$.
(3) For any subsurface V of $\cup_{i>\ell}S_i$, it holds
$$d_{\mathcal{CG}(V)}(\mathrm{pr}_V(\mu(X)), \mathrm{pr}_V(\mu(X^0))) < \frac{1}{m} d_{\mathcal{CG}(S_1)}(\mathrm{pr}_{S_1}(\mu(X)), x_1).$$
(4) For each $i \leq \ell$ a geodesic in $\mathcal{CG}(S_i)$ connecting x_i to $\mathrm{pr}_{S_i}(\mu(X))$ can be extended to a p-quasi-geodesic in $\mathcal{CG}(S_i)$ whose endpoint is contained in the ball of radius $e^{-\ell}$ about ξ_i in $\partial\mathcal{CG}(S_i)$, where the metric on $\partial\mathcal{CG}(S_i)$ is the Gromov distance d_{x_i} constructed from the basepoint x_i.

The following statement is completely analogous to Lemma 5.2, and the proof of Lemma 5.2 carries over without modification and will be omitted.

Lemma 5.8. *For each $\xi \in \mathcal{E}$ the closures of the sets $W(\xi, j, \ell, m)$ in $\overline{\mathcal{T}}(S)$ define a neighborhood basis of ξ in $\overline{\mathcal{T}}(S)$.*

Using Lemma 5.8, we obtain the following analog of Corollary 5.3 which will be used in Section 6.

Corollary 5.9. *There are countably many open sets $U_j \subset \overline{\mathcal{T}}(S)$ which contain a neighborhood basis for every $\xi \in \mathcal{E}$.*

Proof. For each $i \leq k$ choose a countable dense subset ξ_i^j of $\partial\mathcal{CG}(S_i)$. Then the set $\{\sum_i a_i\xi_i^{j_i} \mid a_i \in \mathbb{Q}, \sum_i a_i = 1, j_i > 0\}$ is countable and dense in $\mathcal{J}(\cup_i S_i)$ by the definition of the join.

It then suffices to show that the closures in $\overline{\mathcal{T}}(S)$ of the countably many sets $W(\sum_i a_i\xi_i^{j_i}, j, \ell, m)$ contain a neighborhood basis for any $\xi \in \mathcal{E}$. That this holds true follows from the definition of the topology on $\overline{\mathcal{T}}(S)$ as in the proof of Corollary 5.3. □

Recall from Section 5.1 that for each i the choice of a basepoint $X_i \in \mathcal{T}_\epsilon(S_i)$ determines a section $\rho_i : \mathcal{PML}(S_i) \to \mathcal{ML}(S_i)$. Assume that the basepoint $x_i \in \mathcal{CG}(S_i)$ is a component of the pants decomposition of $\mu(X_i)$.

For simplicity of notation, call a function $f : \mathbb{R} \to \mathbb{R}$ *coarsely non-decreasing*, with control constant $q > 0$, if we have $f(t) \geq f(s) - q$ for all $s \leq t$. Then for every projective measured geodesic lamination $[\nu_i]$ on S_i the function

$$t \to d_{\mathcal{CG}(S_i)}(\Upsilon(\exp_{X_i}(t\rho_i[\nu_i])), x_i)$$

is coarsely non-decreasing, with control constant only depending on the complexity of S_i [MM99]. The following was shown in [H09].

Lemma 5.10. *There exists a continuous function*

$$\delta_{x_i} : \mathcal{T}(S_i) \to [0, \infty)$$

which is at uniformly bounded distance from the function $Y_i \to d_{\mathcal{CG}(S_i)}(\Upsilon(Y_i), x_i)$.

To construct open contractible subsets of $\prod \mathring{\mathcal{T}}_\epsilon(S_i)$ whose small closures define neighborhoods of $\sum_i a_i \xi_i$ in $\overline{\mathcal{T}}(\cup_i S_i)$, we shall control the speed of progress in the curve graph of each of the surfaces S_i. To this end note that by Lemma 5.10, for every Teichmüller geodesic $\gamma : \mathbb{R} \to \mathcal{T}(S_i)$ starting at the fixed basepoint X_i, the function $t \to \delta_{x_i}(\gamma(t))$ is coarsely non-decreasing and continuous. We use this to construct a new parameterization of a Teichmüller geodesic starting from X_i which encapsulates its progress in the curve graph. The construction is based on the following elementary observation. Here the distance between two functions $f, g : J \subset \mathbb{R} \to \mathbb{R}$ is defined as $\|f - g\| = \sup\{|f(t) - g(t)| \mid t\}$.

Lemma 5.11. *Let* $f : \mathbb{R}^n \to [0, \infty)$ *be a continuous function whose restriction to each ray* $t \to tx$ $(x \in S^{n-1} \subset \mathbb{R}^n)$ *is coarsely non-decreasing, with fixed control constant* $q > 0$*. Then*

$$u = \inf\{g \mid g \geq f, g \text{ continuous, non-decreasing on rays}\}$$

is non-decreasing on rays, continuous and at distance at most q *from* f*.*

Proof. For $x \in S^{n-1}$ and $t \geq 0$ put

$$u(tx) = \max\{f(sx) \mid s \leq t\}.$$

This makes sense since f is continuous. By definition, u is non-decreasing on rays, $u \geq f$ and $u - f \leq q$ as f is coarsely non-decreasing.

Since f is continuous, it is also immediate that u is continuous. This shows the lemma. □

Let f_i be the function on $T^*_{X_i}\mathcal{T}(S_i) \sim \mathbb{R}^{m_i}$ constructed in Lemma 5.11 from the function $\delta_{x_i} \circ \exp_{X_i}$. For each $[\nu] \in \mathcal{PML}(S_i)$ the restriction of the function f_i to the ray $t\rho_i[\nu]$ $(t \in (0, \infty))$ is non-decreasing, but it may be constant on arbitrarily large intervals. However, by replacing f_i by $f_i + \alpha_i$ where $\alpha_i(t\rho_i[\nu]) = a(t)$ for a smooth strictly increasing function $[0, \infty) \to [0, 1)$, we may assume that the function f_i has the following properties.

(1) The function $f_i : T^*_{X_i}\mathcal{T}(S_i) \to [0,\infty)$ is continuous and strictly increasing on rays starting at 0.
(2) $\sup|f_i - \delta_{x_i} \circ \exp_{X_i}| \leq q+1$.

In particular, if $f_i|\{t\rho_i[\nu] \mid t \in (0,\infty)\}$ is unbounded, then $f_i|\{t\rho_i[\nu] \mid t\}$ is a homeomorphism onto $[0,\infty)$.

Put $\tau[\nu] = \sup\{f_i(t\rho_i[\nu]) \mid t\}$. Note that $\tau[\nu] = \infty$ if the support of the geodesic lamination $[\nu]$ on S_i fills S_i.

Since f_i is continuous and its restriction to each ray $\{t\rho[\nu] \mid t \geq 0\}$ is a homeomorphism onto $[0,\tau[\nu])$, it can be inverted. We then can define a function $g_{[\nu]}$ on $[0,\tau[\nu])$ by

$$g_{[\nu]}(t) = \{s \mid f_i(s\rho[\nu]) = t\}.$$

Using this function, we obtain a parameterization $t \to \hat{\gamma}_{[\nu]}(t)$ of the Teichmüller geodesic $t \to \exp_{X_i}(t\rho[\nu])$ on the interval $[0,\tau[\nu])$ by defining

$$\hat{\gamma}_{[\nu]}(t) = \exp_{X_i}(g_{[\nu]}(t)\rho[\nu]). \tag{5}$$

With this definition, we know that $|d_{\mathcal{CG}(S_i)}(x_i,\hat{\gamma}_{[\nu]}(t)) - t| \leq b$ where $b > 0$ is a universal constant not depending on t or i.

By reordering of components, consider $\xi = \sum_{i=1}^{\ell} a_i\xi_i \in \mathcal{X}(\cup_i S_i)$ where $1 \leq \ell \leq k$, $a_i > 0$ and $\xi_i \in \partial\mathcal{CG}(S_i)$ for all i. For $1 \leq i \leq \ell$ we apply the construction which was carried out in Lemma 5.5: let $V_i^1 \supset V_i^2 \supset \cdots$ be a closed descending chain of contractible neighborhoods of the polytope $P(\xi_i)$ of projective measured geodesic laminations supported in ξ_i in the sphere $\mathcal{PML}(S_i)$ of projective measured geodesic laminations on S_i. If S_i is an annulus, then by convention, $\mathcal{PML}(S_i)$ consists of two points. We assume that each of the sets V_i^j is homeomorphic to a closed ball and that for each $j \geq 1$ there exists a deformation retraction $R_i^j : V_i^j \to V_i^{j+1}$ which maps $V_i^j \setminus V_i^{j+1}$ into $V_i^{j+1} \setminus V_i^{j+2}$ (note that by definition, it equals the identity on V_i^{j+1}. We also may assume that there exists an increasing sequence $m(j) \to \infty$ so that for every $i \leq \ell$ and every $[\nu] \in V_i^j$ the following properties are satisfied.

(1) $\tau[\nu] \geq 2m(j)$.
(2) If the support of $\zeta \in V_i^j$ is minimal and fills, and if c is a shortest distance projection of $\mathrm{supp}(\zeta) \in \partial\mathcal{CG}(S_i)$ into a p-quasi-geodesic connecting the basepoint x_i to ξ_i, then $d_{\mathcal{CG}(S_i)}(c,x_i) \geq 2m(j)$.

Recall to this end that $\tau[\nu] = \infty$ for every $\nu \in P(\xi_i)$ since ξ_i is minimal and filling by assumption, and that a shortest distance projection of $\mathcal{CG}(S_i)$ into any p-quasigeodesic connecting x_i to ξ_i extends to $\partial\mathcal{CG}(S_i) \setminus \xi_i$.

For a pair of points $X, Y \in \mathcal{T}(S_i)$ define

$$\hat{d}_{\mathcal{T}}(X,Y) = \max\{d_{\mathcal{CG}(V)}(\mathrm{pr}_V(\mu(X)), \mathrm{pr}_V(\mu(Y))) \mid V\}$$

where the maximum is over all subsurfaces V of S_i and $\mu(X), \mu(Y)$ are short markings.

Theorem 5.12 (Theorem B of [R14])**.** *For any Teichmüller geodesic $\gamma : [0,\infty) \to \mathcal{T}(S_i)$, the function $t \to \hat{d}_{\mathcal{T}}(\gamma(0),\gamma(t))$ is coarsely non-decreasing, with control constant not depending on γ.*

Proof. By Theorem B of [R14], there is a number $p > 0$ only depending on the complexity of S such that for every subsurface V of S, the image under the map $\mathrm{pr}_V \circ \mu$ of a Teichmüller geodesic $\gamma : \mathbb{R} \to \mathcal{T}(S)$ is an unparameterized p-quasi-geodesic in $\mathcal{CG}(V)$. This quasi-geodesic may be bounded, one-sided infinite or two-sided infinite. Since $\mathcal{CG}(V)$ is a hyperbolic geodesic metric space, this implies that the path $t \to \mathrm{pr}_V \circ \mu \circ \gamma(t)$ coarsely does not backtrack: There exists a universal constant $q > 0$ not depending on the subsurface V such that for $0 \le s \le t$, it holds

$$d_{\mathcal{CG}(V)}(\mathrm{pr}_V(\mu(\gamma(0))), \mathrm{pr}_V(\mu(\gamma(t)))) \ge d_{\mathcal{CG}(V)}(\mathrm{pr}_V(\mu(\gamma(0))), \mathrm{pr}_V(\mu(\gamma(s)))) - q.$$

As the projections $\mathrm{pr}_V(\mu(X))$ only coarsely determine a point in the curve graph of V, the distances in this formula are only coarsely well defined, but this does not affect the validity of the estimate.

As a consequence, for every subsurface V of S_i and every Teichmüller geodesic $\gamma : [0,\infty) \to \mathcal{T}(S_i)$ the function

$$t \to d_{\mathcal{CG}(V)}(\mathrm{pr}_V(\mu(\gamma(0))), \mathrm{pr}_V(\mu(\gamma(t))))$$

is coarsely non-decreasing, with control constant q not depending on V. Then the same holds true for $\hat{d}_{\mathcal{T}}$. ☐

The following proposition is the technically most involved part of the proof of our main theorem. In its formulation, we denote by $\Lambda_i = \Lambda_{\epsilon,i} : \mathcal{T}(S_i) \to \mathring{\mathcal{T}}_\epsilon(S_i)$ a homeomorphism as constructed in Proposition 4.4. Then $\Lambda_\epsilon = \prod \Lambda_{\epsilon,i}$ is a homeomorphism of $\prod \mathcal{T}(S_i)$ onto $\prod \mathring{\mathcal{T}}_\epsilon(S_i)$.

Proposition 5.13. *Assume that $\xi = \sum_{i=1}^k a_i\xi_i \in \mathcal{X}(S)$ is such that $\sum_i a_i = 1$, that $a_i > 0$ precisely if $i \le \ell$ and that $\xi_i \in \partial\mathcal{CG}(S_i)$ for all i. For integers $j, n \ge 1$ and for $\delta > 0$ there is an open subset $E(j,n,\delta)$ of $\prod \mathcal{T}(S_i)$ with the following property.*

1. *$E(j,n,\delta)$ is contractible.*
2. *$E(j,n,\delta) \subset E(j',n',\delta')$ for $j \ge j', n \ge n', \delta \le \delta'$.*
3. *The small closures of the sets $\Lambda_\epsilon E(j,n,\delta)$ ($j \ge 1, n \ge 1, \delta > 0$) in $\overline{\mathcal{T}}(\cup_i S_i)$ define a neighborhood basis of ξ.*

Proof. For $i \le \ell$ let V_i^j be as above. For each $[\nu_i] \in V_i^j$ choose a parameterization of the geodesic $t \to \exp_{X_i}(t\sigma[\nu_i])$ on $[0,\tau[\nu_i])$ as constructed in equation (5) above. Note that by the choice of the constants $m(j)$, the domain of definition of this parameterization contains the interval $[0,m(j)]$, and the restriction of this parameterization to $[0,m(j)]$ depends continuously on $[\nu_i]$. Denote by $\hat{\gamma}_{[\nu_i]} : [0,\tau[\nu_i]) \to \mathcal{T}(S_i)$ this parameterization.

Theorem 5.12 shows that for any Teichmüller geodesic $\gamma : [0,\infty) \to \mathcal{T}(S_i)$, the function $t \to \hat{d}_{\mathcal{T}}(\gamma(0),\gamma(t))$ is coarsely non-decreasing, with fixed control constant $q > 0$. Put $\tilde{d}_{\mathcal{T}}(\gamma(0),\gamma(t)) = \sup_{s \le t} \hat{d}_{\mathcal{T}}(\gamma(0),\gamma(s))$. By uniqueness of Teichmüller geodesics between any pair of points, this defines a function $\mathcal{T}(S_i) \times \mathcal{T}(S_i) \to [0,\infty)$

which however may not be symmetric. For any Teichmüller geodesic γ, the function $t \to \tilde{d}_{\mathcal{T}}(\gamma(0), \gamma(t))$ is non-decreasing.

For an ℓ-tuple $(j_1, \dots, j_\ell) \in \mathbb{N}^\ell$ put $m(j_1, \dots, j_\ell) = \min\{m(j_i) \mid i\}$. For $i \leq \ell$ and $j \geq 1$ put $\hat{V}_i^j = V_i^j \setminus V_i^{j+1}$. For $([\nu_1], \dots, [\nu_\ell]) \in \hat{V}_1^{j_1} \times \cdots \times \hat{V}_\ell^{j_\ell}$ and $\delta > 0$, $n < m(j_1 - 1, \dots, j_\ell - 1)/2$ define

$$\begin{aligned} F(n, \delta, [\nu_1], \dots, [\nu_\ell]) = \{(\hat{\gamma}_{[\nu_1]}(t_1), \hat{\gamma}_{[\nu_2]}(t_2), \dots, \hat{\gamma}_{[\nu_\ell]}(t_\ell), z_{\ell+1}, \dots, z_k) \in \prod \mathcal{T}(S_i) \mid \\ t_i \geq n, \quad |t_i/t_1 - a_i/a_1| \leq \delta \text{ if } t_i < m(j_1 - 1, \dots, j_\ell - 1) \text{ for } i \leq \ell, \\ \tilde{d}_{\mathcal{T}(S_i)}(X_i, z_i) < \delta t_1 \text{ for } i \geq \ell + 1\}. \end{aligned}$$

It is important to note that the constraints on the relation between the points in the components of a tuple in the set $F(n, \delta, [\nu_1], \dots, [\nu_\ell])$ become stronger as the components $[\nu_i]$ get closed to $P(\xi_i)$ as measured by the nested sequence V_i^j, and that the constraint is determined by the component in the tuple which is furthest away from $P(\xi_i)$.

Claim 1: The set $E(j, n, \delta) = \cup_{[\nu_i] \in V_i^j} F(n, \delta, [\nu_1], \dots, [\nu_\ell])$ is contractible for every $n \leq m(j)$.

Proof of Claim 1. Note first that if $(z_1, \dots, z_\ell, z_{\ell+1}, \dots, z_k) \in E(j, n, \delta)$ then the same holds true for $(z_1, \dots, z_\ell, z'_{\ell+1}, \dots, z'_k)$ for any z'_i which is contained in the Teichmüller geodesic connecting X_i to z_i and all $i \geq \ell + 1$. Thus retracting component wise the last $k - \ell$ components z_i to the basepoint X_i $(i \geq \ell + 1)$ along the unique Teichmüller geodesic connecting X_i to z_i and keeping the remaining components fixed defines a retraction of $E(j, n, \delta)$ to $E(j, n, \delta) \cap \{(z_1, \dots, z_k) \mid z_i = X_i \text{ for } \ell + 1 \leq i \leq k\}$. In particular, in the remainder of the construction, it suffices to assume that $\ell = k$.

Next observe that $E(j + 1, n, \delta) \subset E(j, n, \delta)$ for all j, n, δ. We construct a homotopy of $E(j, n, \delta)$ into its subset $E(j + 1, n, \delta)$ as follows.

The set

$$S(m(j-1)) = \{(t_1, \dots, t_k) \in [n, \infty)^k \mid |t_i/t_1 - a_i/a_1| \leq \delta \text{ if } t_i < m(j-1) \, \forall i\}$$

admits a deformation retraction onto its subset

$$S(m(j)) = \{(t_1, \dots, t_k) \in [n, \infty)^k \mid |t_i/t_1 - a_i/a_1| \leq \delta \text{ if } t_i < m(j) \, \forall i\}.$$

Namely, define a homotopy $h : [0, 1] \times [n, \infty) \to [n, \infty)$ by

$$h(u, t) = \begin{cases} \min\{t(1 - u + u(m(j)/m(j-1))), m(j)\} \text{ if } t < m(j) \\ t \text{ if } t \geq m(j). \end{cases}$$

Then for any $t_1, t_2 \in [n, m(j - 1))$ and u with $h(u, t_1) < m(j), h(u, t_2) < m(j)$ we have $h(u, t_1)/h(u, t_2) = t_1/t_2$. As a consequence, the map $(u, (t_1, \dots, t_k)) \to (h(u, t_1), \dots, h(u, t_k))$ preserves $S(m(j - 1))$, and it defines a homotopy of $S(m(j - 1))$ into $S(m(j))$.

Composing this deformation of the domain $S(m(j - 1))$ into $S(m(j))$ with the map

$$(t_1, \dots, t_k) \to (\hat{\gamma}_{[\nu_1]}(t_1), \dots, \hat{\gamma}_{[\nu_k]}(t_k))$$

defines a homotopy of $E(j,n,\delta)$ into its subset

$$\Xi = E(j,n,\delta)\cap\{(\hat\gamma_{[\nu_1]}(t_1),\dots\hat\gamma_{[\nu_k]}(t_k)) \mid |t_i/t_1 - a_i/a_1|\le\delta \text{ if } t_i<m(j)\,\forall i\}.$$

The deformation retractions $R_i^j:[0,1]\times V_i^j\to V_i^j$ of V_i^j onto $R_i^j(V_i^j\times\{1\}) = V_i^{j+1}$ induce a deformation retraction

$$R^j:[0,1]\times V_1^j\times\cdots\times V_k^j\to V_1^j\times\cdots\times V_k^j$$

onto $V_1^{j+1}\times\cdots\times V_k^{j+1}$ by applying the deformation retractions R_i^j component wise. Since for each i, the image of $V_i^j\setminus V_i^{j+1}$ is contained in $V_i^{j+1}\setminus V_i^{j+2}$, and R_i^j equals the identity on V_i^{j+1}, we obtain a deformation retraction of Ξ onto its subset $E(j+1,n,\delta)$ by defining

$$(s,(\gamma_{[\nu_1]}(t_1),\dots,\gamma_{[\nu_k]}(t_k)))\to(\gamma_{R_1^j(s,[\nu_1])}(t_1),\dots,\gamma_{R_k^j(s,[\nu_k])}(t_k)).$$

The composition of these two homotopies yields a homotopy of $E(j,n,\delta)$ into $E(j+1,n,\delta)$.

Now $\cap_j E(j,n,\delta) = \cup_{[\nu_i]\in P(\xi_i)}F(n,\delta,[\nu_1],\dots,[\nu_k])$, and as $P(\xi_i)$ is contractible for all i, this set is contractible as well. Concretely, for tuples $([\nu_1],\dots,[\nu_k])\in\prod P(\xi_i)$, we have $m(j)=\infty$ and the constraint on the time parameters t_i in the definition of the set $F(n,\delta,[\nu_1],\dots,[\nu_k])$ holds for all $t_i\ge n$. Since each of the sets $P(\xi_i)$ are contractible, we can define a contraction of $\cap_j E(j,n,\delta)$ onto $F(n,\delta,[\nu_1],\dots,[\nu_k])$ for a fixed choice of $[\nu_i]\in P(\xi_i)$ and all i, keeping time parameters t_i fixed. We are then left with contracting the time parameters, which is easily possible by the definition of the time constraint. This completes the proof of the claim. ■

So far we constructed from a tuple of contractible neighborhoods V_i^j $(i=1,\dots,k)$ and numbers $j>0,\delta>0$ a contractible subset $E(j,n,\delta)$ of $\mathcal{T}(S)=\prod\mathcal{T}(S_i)$. We aim at using these sets to construct contractible neighborhoods of ξ in $\overline{\mathcal{T}}(\cup_i S_i)$.

Claim 2: For fixed (j,n,δ), if $X^u\subset\prod\mathcal{T}_\epsilon(S_i)$ is a sequence converging to ξ, then $X^u\in\overline{\Lambda_\epsilon E(j,n,\delta)}$ for large enough u.

Proof of Claim 2: Let $X^u=(X_1^u,\dots,X_k^u)\subset\prod\mathcal{T}_\epsilon(S_i)$ be a sequence converging to ξ. Let $j>0,n>0,\delta>0$ be fixed. We show first that $X^u\in E(j,n,\delta)$ for large enough u.

For $i\le\ell$ let $[\nu_i]\in\mathcal{PML}(S_i)$ be such that $\mathrm{supp}([\nu_i])=\xi_i$. Let $\hat\gamma_i:[0,\infty)\to\mathcal{CG}(S_i)$ be the p-quasi-geodesic constructed as a reparameterization of the Teichmüller geodesic $\gamma_i(t)=\Upsilon(\exp_{X_i}t[\nu_i])$ as before and let $\Pi_i:\mathcal{CG}(S_i)\to\hat\gamma_i$ be a shortest distance projection. We then have

$$d_{\mathcal{CG}(S_i)}(\Pi_i(\Upsilon(X_i)),\Pi_i(\Upsilon(X_i^u)))/d_{\mathcal{CG}(S_1)}(\Pi_1(\Upsilon(X_1)),\Pi_1(\Upsilon(X_1^u)))\to a_i/a_1. \tag{6}$$

Furthermore, for $i\ge\ell+1$ it holds

$$\hat d_{\mathcal{T}}(X_i,X_i^u)/\min_{i\le\ell}d_{\mathcal{CG}(S_i)}(\Pi_i(\Upsilon(X_i)),\Pi_i(\Upsilon(X_1^u)))\to 0. \tag{7}$$

Let $[\eta_i^u]\in\mathcal{PML}(S_i)$ and $t_i^u\geq 0$ be such that $X_i^u=\hat\gamma_{[\eta_i^u]}(t_i^u)$. Then for all $i\leq\ell$, we have $t_i^u\to\infty$ $(u\to\infty)$, moreover by Lemma 5.5 and its proof, it holds $[\eta_i^u]\to P(\xi_i)$ $(u\to\infty)$. Thus for large enough u and all $i\leq\ell$, we have $[\eta_i^u]\in V_j^i$. As $t_i^u\to\infty$ $(u\to\infty)$ for all i, equation (7) shows that $\hat d_{\mathcal T}(X_i,X_i^u)<\min_{m\leq\ell}\delta t_m^u/2$ for sufficiently large u and all $i\geq\ell+1$. By the definition of the set $E(j,n,\delta)$, this implies that $X^u\in E(j,n,\delta)$ for large enough u if and only if this holds true for $(X_1^u,\dots,X_\ell^u,X_{i+1},\dots,X_k)$ (here as before, X_i is the basepoint). Consequently it follows as in the beginning of this proof that it suffices to assume that $a_i>0$ for all $i\leq k$, in other words, that $\ell=k$.

Now invoking the condition (6) and the fact that $t_i^u\to\infty$ for all i, for all sufficiently large u we have $t_i^u>2n$ and

$$|d_{\mathcal{CG}(S_i)}(\Pi_i(\Upsilon(X_i)),\Pi_i(\Upsilon(X_i^u)))/d_{\mathcal{CG}(S_1)}(\Pi_1(\Upsilon(X_1)),\Pi_1(\Upsilon(X_1^u)))-a_i/a_1|<\delta/2$$

for all i which together with the definitions implies that $X_u\in E(j,n,\delta)$.

We are left with observing that in fact $X^u\in\overline{\Lambda_\epsilon(E(j,n,\delta))}$. However, the map $\Lambda_\epsilon=\Lambda_{1,\epsilon}\times\cdots\times\Lambda_{k,\epsilon}$ is coarsely Υ-invariant for each i. As the defining properties of the sets $E(j,n,\delta)$ only depend on distances in the curve graph of the surfaces S_i, we conclude that $X^u\in\overline{\Lambda_\epsilon(E(j,n,\delta))}$ for large u. ■

Claim 2 shows that each of the sets $\overline{\Lambda_\epsilon E(j,n,\delta)}$ is the intersection with $\mathcal T_\epsilon(S)$ of a neighborhood of ξ. By the definition of the *small* closure of a subset of $\mathcal T(S)$, the small closures of the sets $\Lambda_\epsilon(E(j,n,\delta))$ are then neighborhoods of ξ as well.

To complete the proof of the proposition it remains to show that the small closures $\overline{\Lambda_\epsilon E(j,n,\delta)}_{\text{small}}$ of the sets $\Lambda_\epsilon E(j,n,\delta)$ form a neighborhood basis of ξ in $\overline{\mathcal T}(\cup_i S_i)$. By Proposition 4.11 and Proposition 4.13, the space $\overline{\mathcal T}(\cup_i S_i)$ is a compact Hausdorff space. Thus the complement of an open neighborhood V of ξ is compact and does not contain ξ and hence since the sets $E(j,n,\delta)$ are nested, we only have to show that the intersection $\cap_{j,n,\delta}\overline{\Lambda_\epsilon E(j,n,\delta)}\supset\cap_{j,n,\delta}\overline{\Lambda_\epsilon E(j,n,\delta)}_{\text{small}}=\{\xi\}$. As ξ clearly is contained in this intersection, it suffices to show that it is unique with this property.

To see that this is the case recall from the end of the proof of Claim 1 that $\cap_j E(j,n,\delta)=\cup_{[\nu_i]\in P(\xi_i)}F(n,\delta,[\nu_1],\dots,[\nu_k])$. Furthermore, using again the definitions, the intersection $\cap_n\overline{\Lambda_\epsilon(\cup_{[\nu_i]\in P(\xi_i)}F(n,\delta,[\nu_1],\dots,[\nu_k]))}$ is contained in $\mathcal X(S)$, and finally $\cap_\delta\cap_n\overline{\Lambda_\epsilon(\cup_{[\nu_i]\in P(\xi_i)}F(n,\delta,[\nu_1],\dots,[\nu_k]))}=\{\xi\}$ which is what we wanted to show. □

The following corollary summarizes the results of this subsection.

Corollary 5.14. *Each $\xi=\sum_i a_i\xi_i\in\mathcal E$ admits a countable neighborhood basis in $\overline{\mathcal T}(\cup_i S_i)$ consisting of small closures of open contractible subsets of $\prod\mathring{\mathcal T}_\epsilon(S_i)$.*

Proof. By Proposition 5.13, for each $\xi=\sum_i a_i\xi_i$ with $\xi_i\in\partial\mathcal{CG}(S_i)$ the countably many sets $E(j,n,\frac{1}{m})$ $(j,\ell,m\in\mathbb N)$ are open and contractible, and the small closures of their images under Λ_ϵ define a neighborhood basis of ξ in $\overline{\mathcal T}(S)$ whose intersections with $\mathcal{T}_\epsilon(S)$ are small closures of contractible open as stated in the corollary. $\square$

5.3. **Neighborhoods of arbitrary points.** In this section we complete the proof of the first part of Theorem 4.9. The difficulty is as follows. In Section 5.2, we considered a disjoint union $S_1, \dots, S_k$ of subsurfaces of S. Denote by S_i^* the surface obtained from S_i by replacing each boundary component by a puncture. We constructed for any point $\sum_i a_i\xi_i \in \mathcal{J}(\cup_i S_i) = \mathcal{J}(\cup_i S_i^*)$ (the notations are as in Section 2) a neighborhood basis of ξ in $\overline{\mathcal{T}}(\cup_i S_i^*) = \prod \mathcal{T}_\epsilon(S_i^*) \cup \mathcal{X}(S_1^*) * \cdots * \mathcal{X}(S_k^*)$ consisting of open sets whose intersections with $\prod \mathcal{T}_\epsilon(S_i^*)$ are contractible. From the description of the topology on $\overline{\mathcal{T}}(S)$ in Section 4, we also know how to construct from this neighborhood basis a neighborhood basis of ξ viewed as an element of $\mathcal{X}(S) \subset \overline{\mathcal{T}}(S)$. However, there is no straightforward mechanism which can be applied to guarantee that the intersections with $\mathcal{T}_\epsilon(S)$ of the sets from this neighborhood basis can be chosen to be contractible.

Example 4.10 illustrates both this difficulty and its solution. Namely, in the case of the once punctured torus S, when we want to construct a contractible neighborhood basis for a fixed point of a Dehn twist, viewed as a point in the Gromov boundary of the hyperbolic group $\mathrm{Mod}(S)$, then this task is not well adapted to the geometry of Teichmüller space, which equals the hyperbolic plane. But we can think of Teichmüller space as foliated by geodesics which are forward asymptotic to this fixed point, viewed as a boundary point of the hyperbolic plane, and use this foliation to construct the desired neighborhood basis. We shall use an extension of this strategy to arbitrary Teichmüller spaces to complete the proof of Theorem 4.9.

Define the *complexity* $\kappa(S)$ of a connected surface of genus $g \geq 0$ with $m \geq 0$ holes (which can be boundary components or punctures) as

$$\kappa(S) = 3g - 3 + m$$

if S is not a sphere with two holes, that is, an annulus. If S is an annulus then define $\kappa(S) = 0$. If $S = \sqcup_{i=1}^m S_i$ is a disjoint union of connected surfaces S_i then define $\kappa(S) = \sum_i \kappa(S_i)$.

We proceed by induction on the complexity $\kappa(S)$ of the possibly disconnected surface S. Example 4.12 contains the case $\kappa(S) = 0$, so assume that part (1) of Theorem 4.9 has been established for all surfaces of complexity at most $k-1$ for some $k-1 \geq 0$. Let S be a possibly disconnected surface of complexity $\kappa(S) = k$. By Section 5.1 and Section 5.2, we are left with constructing neighborhood bases for points $\xi = \sum_{i=1}^k a_i\xi_i \in \mathcal{X}(S)$ where each ξ_i fills a proper subsurface S_i of S (which may be a connected component of S) and that furthermore there exists at least one i such that ξ_i does not fill a connected component of S. In particular, $\mathrm{supp}(\xi)$ fills a (possibly disconnected) subsurface of S, and the boundary of this subsurface is non-empty and contains at least one non-peripheral simple closed curve $c \subset S$.

Let c be such a simple closed curve. Then $S_c = S \setminus c$ is a (possibly disconnected) surface of complexity $k-1$ and (with a small abuse of notation) we can write $S = S_c \sqcup A_c$ where A_c is the annulus with core curve c. We then can view ξ as an element in the geometric boundary of the disconnected surface $S_c^* \sqcup A_c$ where again

S_c^* denotes the finite type surface obtained from S_c by replacing each boundary component by a puncture. Since the complexity of S_c^* is at most $k-1$, by the induction hypothesis, the first part of Theorem 4.9 holds true for $S_c^* \sqcup A_c$.

The infinite cyclic group generated by the left Dehn twist T_c about c equals the mapping class group of A_c. The stabilizer $\mathrm{Stab}(c)$ of c in the mapping class group $\mathrm{Mod}(S)$ fits into the exact sequence

$$1 \to \langle T_c \rangle \to \mathrm{Stab}(c) \to \mathrm{Mod}(S_c^*) \to 1, \tag{8}$$

however this sequence does not split in general.

Let S' be the connected component of S containing c. We are interested in analyzing the geometric relation between $\mathcal{T}(S')$ and $\mathcal{T}(S' \setminus c)$. To make the notations less cumbersome, let us assume for the moment that S is connected, that is, in what follows, we tacitly replace S' by S in the notation and resume the case of a disconnected surface later on.

Consider the *augmented Teichmüller space* $\mathcal{T}^{\rm aug}(S)$ of S [Wo03, Ya04]. This is a stratified space whose open stratum of maximal dimension equals the Teichmüller space $\mathcal{T}(S)$. For each multi-curve β on S there exists a stratum $\mathcal{S}(\beta)$ which equals the Teichmüller space of the *noded* surface $(S \setminus \beta)^*$ obtained from $S \setminus \beta$ by replacing each boundary component by a puncture (and remembering that the punctures are identified at the nodes). This Teichmüller space is a direct product of Teichmüller spaces, one for each component of $S \setminus \beta$. The strata in the boundary of $\mathcal{S}(\beta)$ correspond to multi-curves containing β as a subset.

To an essential (that is, non-peripheral) simple closed curve c on the finite area hyperbolic surface $Y \in \mathcal{T}(S)$ we can associate not only its hyperbolic length $\ell_Y(c)$ but also its *extremal length* $e_Y(c)$. By a result of Maskit [Ma85], if $\ell_Y(c)$ is small then $e_Y(c)$ is uniformly proportional to $\ell_Y(c)$: it holds $\ell_Y(c)/e_Y(c) \to \pi$ locally uniformly in Y as $\ell_Y(c) \to 0$, with precise global ratio bound in the region in $\mathcal{T}(S)$ where $\ell_Y(c)$ is sufficiently small. We do not need to know the precise definition of $e_Y(c)$, all what matters for our purpose is how it can be computed. Namely, by the Hubbard Masur theorem [HM79], for each $Y \in \mathcal{T}(S)$ there exists a unique holomorphic *quadratic differential* $q_c(Y)$ on Y, that is, a meromorphic section of $K_Y \otimes K_Y$ which defines a finite area singular flat metric on Y with the following property.

The marked Riemann surface Y is glued from an Euclidean cylinder C of area one, with core curve freely homotopic to c, of circumference $w(Y)$ and height $h(Y)$, by pairwise identifying arcs of the same euclidean length in the boundary of C. Note that the cylinder C is foliated by closed Euclidian geodesics which we call *horizontal* in the sequel, referring to standard coordinates in the Euclidian plane. The extremal length of c is then computed as $e_Y(c) = w(Y)/h(Y)$. The quadratic differential determined by this singular flat metric is called a *one-cylinder Strebel differential*, with core curve c, and it represents a point in the cotangent space of $\mathcal{T}(S)$ at Y. The map which associates to $Y \in \mathcal{T}(S)$ the one cylinder Strebel differential $q_c(Y)$ with core curve c is an analytic section q_c of the bundle of area one quadratic differentials over $\mathcal{T}(S)$. As [HM79] only covers the case when S

is compact but we have to work with arbitrary finite type surfaces, we refer to Theorem 21.1 of [St84] for a precise reference with proof.

For small $\delta < \epsilon/2\pi$ consider the set $N(c)$ of points $Y \in \mathcal{T}(S)$ so that the extremal length of c for the conformal structure defined by Y is at most δ. By [Ma85] we may assume that the hyperbolic length of c for points in $N(c)$ is smaller than ϵ. The set $N(c)$ can be thought of as a tubular neighborhood of the stratum $\mathcal{S}(c)$ of noded Riemann surfaces in the augmented Teichmüller space $\mathcal{T}^{\rm aug}(S)$. Its boundary $\partial N(c)$ is invariant under the action of the infinite cyclic group of Dehn twists $\langle T_c \rangle$ about c. Namely, the action of $\mathrm{Stab}(c) \subset \mathrm{Mod}(S)$ preserves the extremal length of c.

The *horocycle flow* h_t acts on $q_c(\mathcal{T}(S))$ as follows. For each $Y \in \mathcal{T}(S)$, the flow line $t \to h_t(q_c(Y))$ is obtained by shearing (that is, twisting) the one-cylinder differential $q_c(Y)$ with constant speed along its core curve, where the speed parameter is chosen so that $h_1(q_c(Y)) = q_c(T_c(Y))$ [St84]. As σ is an embedding, this flow descends to a twist flow on $\mathcal{T}(S)$ by defining (with a small abuse of notation) $h_t(Y) = q_c^{-1}(h_t(q_c(Y)))$.

To be more explicit, as is familiar for twisting a hyperbolic metric on S along a simple closed geodesic, one can think of this shearing operation as cutting S open along a horizontal closed geodesic in the interior of the flat cylinder and gluing both sides back with a twist whose size is prescribed by the time parameter t. As this twisting operation preserves height and circumference of the cylinder, this flow indeed preserves $q_c(\partial N(c))$ and hence projects to a flow on $\partial N(c)$. The flow lines foliate $\partial N(c)$ and are invariant under the infinite cyclic group $\langle T_c \rangle$. Note that the horocycle flow preserves the *criticial graph* of the differential, which is the marked metric graph obtained as the image of the boundary of the flat cylinder under the gluing operation which recovers the Riemann surface Y. If S is closed then the critical graph is compact, otherwise it can be compactified as a marked metric graph by adding the punctures. The diameter of this compactification equals at most twice the circumference of the flat cylinder. It is connected if c is non-separating, and it is disconnected if c is separating.

We use this and the fundamental result of Strebel (Theorem 23.5 of [St84], we refer to Section 2 of [Zv02] for a precise account on what we need) in the next lemma. Recall that the choice of a base marking of S coarsely determines for each $Y \in \mathcal{T}(S)$ a *twist parameter* $\tau(Y, c) \in \mathbb{Z}$ about c, unique up to an error of ± 1, which is just the subsurface projection of $\mu(Y)$ into an annulus with core curve c.

Lemma 5.15. *There is a continuous surjective map* $\Pi : \partial N(c) \to \mathcal{S}(c)$ *and a continuous map* $\sigma : \mathcal{S}(c) \to \partial N(c)$ *with the following properties.*

1. *For each* $Y \in \partial N(c)$*, the preimage* $\Pi^{-1}(\Pi(Y))$ *equals the flow line* $t \to h_t(Y)$ *of the horocycle flow.*
2. $\Pi \circ \sigma = \mathrm{Id}$.
3. *There exists a constant* $b > 0$ *so that* $\tau(\sigma(Y), c) \in [-b, b]$ *for all* $Y \in \mathcal{S}(c)$.

Proof. We begin with defining the projection Π. Let $Y \in \partial N(c)$ and consider the one cylinder Strebel differential $q_c(Y)$ with core curve c. Its critical graph $\mathcal{G}(Y)$

is an embedded graph in S, equipped with the restriction of the flat metric. Thus $\mathcal{G}(Y)$ is a metric ribbon graph with two faces, that is, the boundary of a small tubular neighborhood of $\mathcal{G}(Y)$ in S has two components which correspond to the two boundary components of the cylinder which determines $q_c(Y)$.

By Proposition 2.3 of [Zv02], this ribbon graph determines uniquely a meromorphic quadratic differential $q_c(\Pi(Y))$ on a marked Riemann surface $\Pi(Y) \in \mathcal{S}(c)$ with one node corresponding to the closed curve c. The differential has a double pole at each of the two punctures corresponding to the node and perhaps a simple pole at a puncture of the surface $\Pi(Y)$ but no other pole. The residues at the two double poles are negative real numbers and equal to the negative of the circumference of the cylinder defining $q_c(Y)$. In particular, by the definition of $\partial N(c)$, these residues do not depend on Y. Proposition 2.3 of [Zv02] shows that the map $\Pi : \partial N(c) \to \mathcal{S}(c)$ is continuous. As $\Pi(Y)$ is constructed geometrically by attaching a one-sided infinite cylinder to the each face of the critical graph of the one cylinder Strebel differential $q_c(Y)$, viewed as a marked metric ribbon graph, clearly $\Pi^{-1}(\Pi(Y))$ contains the orbit of Y under the horocycle flow h_t, and its equals this orbit as the orbit is uniquely determined by the (marked) critical graph of the Strebel differential. This shows the first part of the lemma.

We next observe that the map Π is surjective. Let $r > 0$ be the circumference of the cylinders of the Strebel differentials for points in $\partial N(c)$. By Theorem 23.5 of [St84], given a noded Riemann surface $Z \in \mathcal{S}(c)$, and viewing the node as a pair z_1, z_2 of punctures, there is a unique meromorphic quadratic differential on Z with two double poles with residue $-r$ at the punctures z_1, z_2 and such that if we denote by D_i the disk domain formed by the closed trajectories surrounding z_i $(i = 1, 2)$ then $Z = \cup_i \overline{D_i}$. Note to this end that none of the components of $S \setminus c$ are twice punctured spheres, hence Theorem 23.5 of [St84] can be applied. The differential may have a simple pole at some punctures of $S \setminus c$ different from the node. But then the critical graph of the differential is a marked metric ribbon graph, and to this marked metric ribbon graph one can attach a cylinder of height $1/r$ to obtain a Riemann surface in $\mathcal{T}(S)$, equipped with a Strebel differential with core curve c, and this Riemann surface is contained in $\partial N(c)$ by construction. The resulting surface is not unique, but any two choices of such a surface are contained in the same orbit of the horocycle flow as their critical graphs are marked isometric, and this flow line is mapped to Z by the map Π.

The construction in the previous paragraph associates to $Z \in \mathcal{S}(c)$ a flow line for h_t in $\partial N(c)$ in a continuously varying fashion. To promote this construction to a map σ with properties (2),(3) in the lemma, note that by continuity, for each point $z \in \mathcal{S}(c)$ we can find a neighborhood U_z of z and a local section $\sigma_z : \mathcal{S}(c) \to \partial N(c)$ for the projection Π so that $\tau(\sigma_z(y), c) \in [-m, m]$ for all $y \in U_z$ and some fixed constant $m > 0$. This makes sense since the ambiguity in the definition of the function τ is at most an additive constant one, moreover τ is coarsely continuous in the sense that values of τ at nearby points only differ by a universal additive constant. Using a partition of unity and the fact that the fiber of Π is contractible, these local sections can be patched together to a global section with the properties in the lemma. □

A section σ as in Lemma 5.15 is an embedding of $\mathcal{S}(c)$ into $\partial N(c)$. This embedding can be used to construct a homeomorphism $\Sigma : \mathcal{S}(c) \times \mathbb{R} \to \partial N(c)$ which is equivariant with respect to the action of $\mathbb{Z}$ on $\mathbb{R}$ by translation and the action of the infinite cyclic group $\langle T_c \rangle$ of Dehn twists about c on $\partial N(c)$. As $\mathcal{S}(c) \times \mathbb{R}$ is just the product of the Teichmüller space $\mathcal{T}(S_c^*)$ and the Teichmüller space of the annulus with core curve c, we can view $\partial N(c)$ also in this way.

The *Teichmüller geodesic flow* φ_t acts on the area one Strebel differentials with core curve c, that is, on the section q_c, by scaling the circumference of the cylinder which determines $Y \in \mathcal{T}(S)$ by $e^{t/2}$ and its height by $e^{-t/2}$. Thus $\mathcal{T}(S)$ is foliated by Teichmüller geodesics whose co-velocities are such differentials. By the above discussion, the foliation is invariant under the subgroup $\mathrm{Stab}(c)$ of $\mathrm{Mod}(S)$. Moreover, if $\gamma : \mathbb{R} \to \mathcal{T}(S)$ is such a Teichmüller geodesic, then the extremal length of c along γ is strictly increasing and hence the geodesic intersects $\partial N(c)$ transversely in a single point. The Teichmüller geodesic γ_Y through a point $\gamma_Y(0) = Y \in \partial N(c)$ converges as $t \to -\infty$ to $\Pi(Y)$ in the augmented Teichmüller space, as can easily be seen from rescaling of the Strebel differential to keep the critical graph, viewed as a metric ribbon graph, constant.

The following lemma is a consequence of the article [R14].

Lemma 5.16. *There exists a number $D > 0$ with the following property. Let $Y \in \partial N(c)$; then for any not necessarily proper subsurface V of $S_c = S \setminus c$ we have $\mathrm{diam}(\mathrm{pr}_V(\gamma_Y(-\infty, \infty))) \le D$.*

Proof. Let $t \to q(t)$ be the cotangent line of the geodesic γ_Y. In the flat metric defined by $q(t)$, the surface S_c is degenerate, that is, the critical graph is a deformation retraction of S_c. Any simple closed curve α in S_c is then homotopic to a closed edge path in the critical graph G of the differential, and a closed edge path of minimal length is unique up to parameterization and is the geodesic representing the free homotopy class of α for the locally CAT(0)-metric $q(t)$.

The singular flat metric of $q(t)$ is obtained from the singular flat metric of $q(0)$ by multiplying the horizontal length, that is, the circumference of the cylinder, with $e^{t/2}$, and the vertical length, that is, the height, with $e^{-t/2}$. As a consequence, if for an essential subsurface V of S_c we define the size $\mathrm{size}_{q(t)}(V)$ of V with respect to the metric $q(t)$ as the $q(t)$-length of the shortest essential closed curve in V and if for a non-peripheral simple closed curve $\alpha \subset S_c$ we denote by $\ell_{q(t)}(\alpha)$ the length of its $q(t)$-geodesic representative, then

$$\log \frac{\mathrm{size}_{q(t)}(V)}{\ell_{q(t)}(\alpha)}$$

does not depend on t. From Theorem 3.1 of [R14], one deduces that the extremal length of any non-peripheral simple closed curve in S_c along the Teichmüller geodesic $t \to \gamma_Y(t)$ is bounded from below by a universal positive constant. The same holds true for the extremal length of c along the ray $\gamma_Y[0, \infty)$, which is exponentially increasing along the ray.

Together with the results from Section 5 of [R14], one deduces that for any (not necessarily proper) subsurface V of S_c the *active interval* for V is empty along

$\gamma_Y(-\infty,\infty)$. Theorem A of [R14] then shows that for each such V, the diameter of the projection $\{\mathrm{pr}_V(\mu(\gamma_Y(t))) \mid t \in [0,\infty)\}$ is bounded from above by a universal constant. This is what we wanted to show. $\square$

Let us move now back to the more general situation when the surface S may be disconnected and we apply the above discussion to the connected component S' of S containing the simple closed curve c. Since $\mathcal{T}(S)$ is the product of the Teichmüller spaces of the components of S, the foliation of $\mathcal{T}(S')$ induces a foliation of $\mathcal{T}(S)$ into geodesics which are constant on the component different from S'.

Recall that the Teichmüller space of the annulus A_c is naturally identified with the real line $\mathbb{R}$. Start with a countable family $\mathcal{V} = \{V_i \mid i\}$ of open contractible subsets of $\mathring{\mathcal{T}}_\epsilon(S_c^*) \times \mathbb{R}$ whose small closures define a neighborhood basis of ξ in $\overline{\mathcal{T}}(S_c^* \sqcup A_c) = \mathcal{T}_\epsilon(S_c^*) \times \mathbb{R} \cup \mathcal{X}(S_c^*) * \mathcal{X}(A_c)$. Such a neighborhood basis exists since by the induction hypothesis, part (1) of Theorem 4.9 holds true for $S_c \sqcup A_c$. Let $\Lambda_\epsilon : \mathcal{T}(S_c^*) \to \mathring{\mathcal{T}}_\epsilon(S_c^*)$ be the coarsely Υ-invariant homeomorphism from Corollary 4.4. By induction, we may in fact assume that V_i is the image of an open and contractible subset of $\mathcal{T}(S_c^*) \times \mathbb{R}$ under the map $\Lambda_\epsilon \times \mathrm{Id}$. Denote by $W_i \subset \partial N(c)$ the image of this set under the identification of $\partial N(c)$ with $\mathcal{T}(S_c^*) \times \mathbb{R}$ using the section σ.

Proposition 5.17. *Put $E_i = \{\gamma_Y(-\infty,\infty) \mid Y \in W_i \subset \partial N(c)\}$; then the sets $\Lambda_\epsilon(E_i) \subset \mathring{\mathcal{T}}_\epsilon(S)$ are open and contractible, and their small closures in $\overline{\mathcal{T}}(S)$ define a neighborhood basis of ξ in $\overline{\mathcal{T}}(S)$.*

Proof. Since the Teichmüller geodesics determined by the one-cylinder Strebel differentials with core curve c foliate $\mathcal{T}(S)$ and $\partial N(c)$ is transverse to these geodesics, the set E_i admits a deformation retraction onto W_i. Thus since the sets W_i are contractible, the same holds true for the sets E_i and for the sets $U_i = \Lambda_\epsilon(E_i)$.

We have to show that the small closures of the sets $U_i \subset \mathring{\mathcal{T}}_\epsilon(S)$ in $\overline{\mathcal{T}}(S)$ define a neighborhood basis of ξ in $\overline{\mathcal{T}}(S)$. As $\cap_i \overline{U}_i = \{\xi\}$ since this holds true for $\cap_i \overline{V_i}$, this is the case if for any sequence $X_u \subset \mathring{\mathcal{T}}_\epsilon(S)$ converging to ξ, all but finitely many X_u are contained in U_i.

By Lemma 5.16, for each $Y \in \partial N(c)$ and any subsurface V of S_c', the diameter of the subsurface projection $\mathrm{pr}_V(\mu(\gamma_Y(-\infty,\infty)))$ is uniformly bounded, independent of Y. By the definition of the topology on $\overline{\mathcal{T}}(S)$, coarse Υ-invariance of the projection Λ_ϵ and the choice of the neighborhoods U_i, for sufficiently large u the subsurface projections of short markings of X_u into all of the subsurfaces of $S \setminus c$ are close to the projections for points in V_i. But by the above, the sets U_i are constructed precisely in such a way that subsurface projections into subsurfaces of $S \setminus c$ are close to subsurface projections for points in U_i. As a consequence, for large enough u we have $X_u \in U_i$.

To summarize, for each i the small closure $\overline{U}_{i,\mathrm{small}}$ in $\overline{\mathcal{T}}(S)$ of the set U_i is indeed a neighborhood of ξ in $\mathcal{T}(S)$. Hence the sets $\overline{U}_{i,\mathrm{small}}$ define a neighborhood basis of ξ in $\overline{\mathcal{T}}(S)$ as claimed in the proposition. $\square$

6. Proof of the main Theorem

The goal of this section is to complete the proof of Theorem 4.9 and of the corollaries.

Proposition 6.1. *$\overline{\mathcal{T}}(S)$ is metrizable.*

Proof. By Uryson's theorem, a second countable Hausdorff space is metrizable. As by Proposition 4.13 the space $\overline{\mathcal{T}}(S)$ is Hausdorff, it suffices to show that $\overline{\mathcal{T}}(S)$ is second countable. Since $\mathcal{T}_\epsilon(S)$ is second countable, for this it suffices to show that there are countably many open sets in $\overline{\mathcal{T}}(S)$ which contain a neighborhood basis for every $\xi \in \mathcal{X}(S)$.

Now $\mathcal{X}(S)$ is a countable union of sets of the form $\mathcal{J}(\cup_{i=1}^k S_i)$ where $S_1, \dots, S_k$ is a collection of pairwise disjoint subsurfaces of S. Since there only countably many such collections, it suffices to find for a given *maximal* collection $S_1, \dots, S_k$ of disjoint subsurfaces of S a countable collection of open sets in $\overline{\mathcal{T}}(S)$ which contain a neighborhood basis for each $\xi \in \mathcal{J}(\cup_{i=1}^k S_i)$.

For each i let S_i^* be the surface obtained from S_i by replacing each boundary component by a puncture if S_i is different from an annulus, and put $S_i^* = S_i$ if S_i is an annulus. For each i consider the marking graph $\mathcal{M}(S_i^*)$ of S_i^*, which can be thought of as the marking graph of S_i but forgetting information on the boundary components. There is a coarsely well defined projection $\mathrm{pr}_{S_i} : \mathcal{T}_\epsilon(S) \to \mathcal{M}(S_i^*)$ which associates to $Y \in \mathcal{T}_\epsilon(S)$ the projection of a short marking for Y to a marking on S_i^*. Put

$$\mathrm{pr}_{\cup_i S_i}(Y) = (\mathrm{pr}_{S_1}(Y), \dots, \mathrm{pr}_{S_k}(Y)) \in \prod \mathcal{M}(S_i).$$

Corollary 5.9 shows that there is a countable collection $\{W(\sum_i a_i \xi_i^{j_i}, j, \ell, m) \mid a_i, j, \ell, m\} = \{U_j \mid j\}$ of subsets of $\mathcal{T}_\epsilon(\cup_i S_i^*)$ which contain a neighborhood basis for every $\xi \in \mathcal{J}(\cup_i S_i) = \mathcal{J}(\cup_i S_i^*) \subset \mathcal{X}(\cup_i S_i^*) \subset \mathcal{X}(S)$, and points in these sets are uniquely determined by their short markings. Thus using the projection $\mathrm{pr}_{\cup_i S_i}$, these sets determine countably many subsets of $\mathcal{T}_\epsilon(S)$.

If we denote by $V_j \subset \mathcal{T}_\epsilon(S)$ the set determined in this way by the subset U_j of $\mathcal{T}_\epsilon(\cup_i S_i)$, then by the definition of the topology on $\overline{\mathcal{T}}(S)$, the closures of the sets V_j in $\overline{\mathcal{T}}(S)$ are neighborhoods of points of $\mathcal{J}(\cup_i S_i)$ in $\overline{\mathcal{T}}(S)$, and hence the same holds true for their interiors. As the collection of subsurfaces S_i of S was chosen to be maximal and the sets U_j contain a neighborhood basis for every $\xi \in \mathcal{J}(\cup_i S_i)$, the sets V_j then determine a neighborhood basis in $\overline{\mathcal{T}}(S)$ for the points in $\mathcal{J}(\cup_i S_i)$. This completes the proof of the proposition. □

As a consequence of Proposition 6.1, the pair $(\overline{\mathcal{T}}(S), \mathcal{X}(S))$ is a pair of compact metrizable spaces, with $\mathcal{X}(S)$ nowhere dense in $\overline{\mathcal{T}}(S)$.

Recall that the *covering dimension* of a topological space X is the minimum of the numbers $n \geq 0$ so that the following holds true. Any open cover $\mathcal{U}$ of X has a refinement $\mathcal{V}$ so that a point in X is contained in at most $n+1$ of the sets $V \in \mathcal{V}$. With this terminology, the covering dimension of $\mathbb{R}^n$ is n, and hence the covering dimension of any subset of $\mathbb{R}^n$ equipped with the subspace topology is at most n. In particular, the covering dimension of $\mathcal{T}(S)$ equals $6g-6+2m$.

The following result relies on work of Gabai [Ga14], see also [BB19].

Proposition 6.2. *The covering dimension of $\mathcal{X}(S)$ is finite*

Proof. If S is an annulus then $\mathcal{X}(S)$ consists of two points and there is nothing to show. Consider next a four-holed sphere or a one-holed torus S. By Example 2.3 and Example 3.3, the geometric boundary as a topological space is homeomorphic to the Gromov boundary of the hyperbolic group $\mathrm{PSL}(2,\mathbb{Z})$. Since the group $\mathrm{PSL}(2,\mathbb{Z})$ is virtually free, the boundary $\mathcal{X}(S)$ of S is a Cantor set, which has covering dimension zero.

Let X and Y be compact spaces with covering dimensions m,n. We claim that the covering dimension of the join $X*Y$ is at most $m+n+1$. To see that this is the case recall that $X*Y$ is the quotient of $X\times Y\times[0,1]$ under an equivalence relation $\sim$ which is only nontrivial on $X\times Y\times\{0\}$ and $X\times Y\times\{1\}$. The projection $X\times Y\times[0,1]\to X\times Y\times[0,1]/\sim$ maps $X\times Y\times\{0\}$ to $X\times\{0\}$ and maps $X\times Y\times\{1\}$ to $Y\times\{1\}$. Thus we have $X*Y=X\cup Y\cup C$ where $X\subset X*Y$ is the closed set which is the quotient of $X\times Y\times\{0\}$, $Y\subset X*Y$ is the closed set which is the quotient of $X\times Y\times\{1\}$, and the set C is homeomorphic to $X\times Y\times(0,1)$.

By Alexandrov's definition of dimension (see Theorem 3.4 of [Dr18]), we have $\dim(A\times B)\leq\dim(A)+\dim(B)$ and hence $\dim(C)\leq\dim(X)+\dim(Y)+1$. The compact space $X*Y$ is the union of the closed subset $X\cup Y$ with C and hence the theorem of Menger and Uryson (see Theorem 3.1 of [Dr18]) shows that $\dim(X*Y)=\max\{\dim(X\sqcup Y),\dim(C)\}\leq\dim(X)+\dim(Y)+1=m+n+1$ as claimed.

Since for a disconnected surface $S=\sqcup_{i=1}^k S_i$ it holds $\mathcal{X}(S)=\mathcal{X}(S_1)*\cdots*\mathcal{X}(S_k)$, it suffices to show the proposition for connected surfaces. Assume by induction that the proposition was established for all connected surfaces of complexity at most $k-1\geq 1$. Let S be a connected surface of complexity k. We have $\mathcal{X}(S)=\partial\mathcal{CG}(S)\sqcup\mathcal{Y}$ (disjoint union) where $\mathcal{Y}=\cup\mathcal{X}(S_1)*\cdots*\mathcal{X}(S_p)$ and the union in the definition of $\mathcal{Y}$ is over all disjoint collections of proper subsurfaces $S_1,\dots,S_p$ of S. The union $\mathcal{Y}$ is not disjoint.

The number of disjoint surfaces in one of the joins appearing in the definition of $\mathcal{Y}$ is uniformly bounded in terms of k. Thus by the induction hypothesis and the above dimension estimate for joins, applied inductively, there exists a number $n>0$ which bounds from above the covering dimension of each of the sets $\mathcal{X}(S_1)*\cdots*\mathcal{X}(S_p)$. Example 3.2 shows that as subsets of $\mathcal{X}(S)$, the sets $\mathcal{X}(S_1)*\cdots*\mathcal{X}(S_p)$ are closed and hence compact. As a consequence, the subspace $\mathcal{Y}$ of $\mathcal{X}(S)$, equipped with the induced topology, is a σ-compact Hausdorff space as it is a countable union of compact spaces.

If $K\subset\mathcal{Y}$ is compact, then K can be represented as a countable union of the compact spaces $K\cap\mathcal{X}(S_1)*\cdots*\mathcal{X}(S_p)$. Then the countable union theorem Theorem 3.2 of [Dr18] shows that $\dim(K)=\sup\{\dim(K\cap\mathcal{X}(S_1)*\cdots*\mathcal{X}(S_p))\}$ where the supremum is over all disjoint unions of proper subsurfaces of S. By the above estimate for the dimension of the spaces $\mathcal{X}(S_1) * \cdots * \mathcal{X}(S_p)$, we have

$$\dim(K \cap \mathcal{X}(S_1) * \cdots * \mathcal{X}(S_p)) \leq n$$

for any such intersection. But then the dimension of $\mathcal{Y}$ is at most n (see p.316 of [Mu14] for a sketch of a proof).

Following [Ga14], if g is the genus of S and if m is the number of its punctures, then the covering dimension of $\partial\mathcal{CG}(S)$ is at most $4g-5+2m$. Then by the Uryson-Menger formula (see Theorem 3.3 of [Dr18]), the dimension of the compactum $\mathcal{X}(S)$ is at most

$$\dim(\mathcal{X}(S)) = \dim(\partial\mathcal{CG}(S)) + \dim(\mathcal{Y}) + 1$$

and hence it is finite. □

As a consequence, we obtain

Corollary 6.3. *The pair $(\overline{\mathcal{T}}(S), \mathcal{X}(S))$ is a pair of spaces of finite dimension.*

Proof. By Proposition 6.2, the dimension of $\mathcal{X}(S)$ is finite. As the compactum $\overline{\mathcal{T}}(S) = \mathcal{T}_\epsilon(S) \cup \mathcal{X}(S)$ is a union of two subspaces of finite dimension, with $\mathcal{X}(S) \subset \overline{\mathcal{T}}(S)$ closed, we have

$$\dim(\overline{\mathcal{T}}(S)) = \max\{\dim(\mathcal{T}_\epsilon(S)), \dim(\mathcal{X}(S))\} < \infty.$$

□

The action of $\mathrm{Mod}(S)$ on $\overline{\mathcal{T}}(S)$ is $\mathcal{U}$-small if for any open covering $\mathcal{U}$ of the compact space $\overline{\mathcal{T}}(S)$ and for any compact subset $K \subset \mathcal{T}_\epsilon(S)$, the image of K under all but finitely many elements $\psi \in \mathrm{Mod}(S)$ is completely contained in one of the sets from the covering $\mathcal{U}$ and hence in the intersection of one of these sets with $\mathcal{T}_\epsilon(S)$. Informally, the images of any compact subset of $\mathcal{T}_\epsilon(S)$ only accumulate near the boundary $\mathcal{X}(S) \subset \overline{\mathcal{T}}(S)$.

Proposition 6.4. *The action of* $\mathrm{Mod}(S)$ *on $\mathcal{T}(S)$ is $\mathcal{U}$-small for every open covering $\mathcal{U}$ of $\overline{\mathcal{T}}(S)$.*

Proof. Let $\mathcal{U}$ be an open covering of $\overline{\mathcal{T}}(S)$. By compactness, we may extract a finite subcovering, so we may assume that $\mathcal{U}$ is in fact finite, that is, we have $\mathcal{U} = \cup_{0\leq i\leq m} U_i$ for some open sets $U_i \subset \overline{\mathcal{T}}(S)$.

Arguing by contradiction, assume that there exists a compact set $K \subset \mathcal{T}_\epsilon(S)$ and infinitely many distinct elements $\varphi_j \in \mathrm{Mod}(S)$ such that $\varphi_j K \not\subset U_i$ for any $i \leq m$ and all j.

Let $X \in K$. Since the action of $\mathrm{Mod}(S)$ on $\mathcal{T}_\epsilon(S)$ is properly discontinuous and $\overline{\mathcal{T}}(S)$ is compact, we conclude that up to passing to a subsequence, the sequence $\varphi_j X$ converges in $\overline{\mathcal{T}}(S)$ to a point $\xi \in \mathcal{X}(S)$. Now recall that as the action of $\mathrm{Mod}(S)$ is isometric for the Teichmüller metric and $K \subset \mathcal{T}_\epsilon(S)$ is compact, all diameters of subsurface projections for short markings of all points in $\varphi_j(K)$ are uniformly bounded, independent of j. Then it follows from the definition of the topology on $\overline{\mathcal{T}}(S)$ that $\varphi_j K \to \xi$. Hence if $p \leq m$ is such that $\xi \in U_p$ the $\varphi_j K \subset U_p$

for all sufficiently large j. This is a contradiction which completes the proof of the proposition. □

Proof of Theorem 4.9. Let S be any (possibly disconnected) surface of finite type.

By Proposition 4.11, there exists a topology on $\overline{\mathcal{T}}(S) = \mathcal{T}_\epsilon(S) \cup \mathcal{X}(S)$ with property (2) in the statement of Theorem 4.9. By Proposition 4.13, with respect to this topology, the space $\overline{\mathcal{T}}(S)$ is compact and separable, and $\mathrm{Mod}(S)$ acts on $\overline{\mathcal{T}}(S)$ as a group of transformations. Moreover, by construction, $\mathcal{X}(S)$ is a closed subset of $\overline{\mathcal{T}}(S)$, and $\mathcal{T}_\epsilon(S)$ is open and dense.

Proposition 6.1 shows that $\overline{\mathcal{T}}(S)$, equipped with this topology, is metrizable, and by Corollary 6.3, the pair $(\overline{\mathcal{T}}(S), \mathcal{X}(S))$ is a pair of spaces of finite dimension. Thus $(\overline{\mathcal{T}}(S), \mathcal{X}(S))$ is a pair of finite dimensional compact metrizable spaces, with $\mathcal{X}(S)$ nowhere dense in $\overline{\mathcal{T}}(S)$.

We next verify that the pair $(\overline{\mathcal{T}}(S), \mathcal{X}(S))$ is an $\mathcal{E}\mathcal{Z}$-structure for $\mathrm{Mod}(S)$. Theorem 4.2 and Theorem 4.3 confirm that $\mathcal{T}_\epsilon(S)$ is contractible and locally contractible, which is requirement (1) in Definition 2. By Proposition 5.17, every point $\xi \in \mathcal{X}(S)$ admits a neighborhood basis consisting of sets whose intersections with $\mathcal{T}_\epsilon(S)$ are contractible, whence requirement (2) in Definition 2. That the action of $\mathrm{Mod}(S)$ on $\mathcal{T}_\epsilon(S)$ is properly discontinuous and cocompact is well known and does not require an additional proof, so requirement (3) holds true. Furthermore, by Proposition 6.4, the action of $\mathrm{Mod}(S)$ on $\overline{\mathcal{T}}(S)$ is $\mathcal{U}$-small, verifying requirement (4). That the action of $\mathrm{Mod}(S)$ extends to $\overline{\mathcal{T}}(S)$ was shown in Proposition 4.13. As a consequence, $(\overline{\mathcal{T}}(S), \mathcal{X}(S))$ is indeed an $\mathcal{E}\mathcal{Z}$-structure for $\mathrm{Mod}(S)$.

To complete the proof of Theorem 4.9, it now suffices to verify that the topology on $\overline{\mathcal{T}}(S)$ is nice. If S is connected, then this follows from Proposition 5.1 and Proposition 5.17. If S is disconnected, then it is a consequence of Proposition 5.13 and Proposition 5.17. □

From now on, we only consider connected surfaces S. The following corollary completes the proof of part (1) of Proposition 7.

Corollary 6.5. *We have* $\dim(\partial\mathcal{C}\mathcal{G}(S)) \leq \mathrm{vcd}(\mathrm{Mod}(S)) - 1$.

Proof. Since $\mathcal{X}(S)$ is a $\mathcal{Z}$-set for a torsion free finite index subgroup Γ of $\mathrm{Mod}(S)$, the cohomological dimension of $\mathcal{X}(S)$ equals $\mathrm{vcd}(\mathrm{Mod}(S)) - 1$ [B96]. Furthermore, this dimension also equals the covering dimension of $\mathcal{X}(S)$ [B96].

Now as $\partial\mathcal{C}\mathcal{G}(S)$ is embedded in $\mathcal{X}(S)$, it is equipped with the subspace topology. This means that any open covering of $\partial\mathcal{C}\mathcal{G}(S)$ is the restriction of an open covering of $\mathcal{X}(S)$. Such a covering then has a $\mathrm{vcd}(\mathrm{Mod}(S)) - 1$-finite refinement and hence the same holds true for the refinement of the original cover of $\partial\mathcal{C}\mathcal{G}(S)$. □

The following conjecture is taken from [BB19]. We believe that the results in this work support this conjecture.

Conjecture. For any surface S of finite type, $\mathrm{asdim}(\mathrm{Mod}(S)) = \mathrm{vcd}(\mathrm{Mod}(S))$.

We are left with showing Corollary 6 and Corollary 8 from the introduction.

Proof of Corollary 6. By Theorem 4, $\mathrm{Mod}(S)$ admits an $\mathcal{EZ}$-structure $(\overline{X}, Z)$ where $X = \overline{X} \setminus Z$ is a manifold with boundary of dimension $6g - 6 + 2m$. Assume that $6g - 6 + 2m \geq 5$. By Lemma 2.3 of [FL05], there exists a new $\mathcal{EZ}$-structure for $\mathrm{Mod}(S)$ obtained by doubling X along the boundary. By Proposition 2.1 of [FL05], this structure is of the form $(\overline{Y}, Z)$ where $\overline{Y}$ is a manifold with boundary of dimension $6g - 5 + 2m$.

Proposition 2.2 of [FL05] then shows that another application of this construction to the pair $(\overline{Y}, Z)$ results in an $\mathcal{EZ}$-structure given by the unit ball in $\mathbb{R}^{6g-4+2m}$ and a subset Z of its boundary, the sphere of dimension $6g - 5 + 2m$. This is what we wanted to show. □

Proof of Corollary 8. The corollary follows from the fact that $\partial\mathcal{CG}(S)$ is embedded in $\mathcal{X}(S)$, and by Corollary 6 and its proof, $\mathcal{X}(S)$ embeds into a manifold of dimension $6g - 6 + 2m$ and into the sphere $S^{6g-5+2m}$. □

Question. What is the smallest dimension n so that $\mathrm{Mod}(S)$ admits an $\mathcal{EZ}$ structure on a pair $(\mathbb{D}^n, \Delta)$ where Δ is a subset of S^{n-1}?

References

[BaB19] A. Bartels, M. Bestvina, *The Farell Jones conjecture for mapping class groups*, Invent. Math. 2015 (2019), 651–712.

[BM91] M. Bestvina, and J. Mess, *The boundary of negatively curved groups*, J. Amer. Math. Soc. 4 (1991), 469–481.

[B96] M. Bestvina, *Local homology properties of boundaries of groups*, Michigan Math. J. 1996, 123–141.

[BB19] M. Bestvina, K. Bromberg, *On the asymptotic dimension of the curve complex*, Geometry & Topology 23 (2019), 2227–2276.

[BBF15] M. Bestvina, K. Bromberg, K. Fujiwara, *Constructing group actions on quasi-trees and applications to mapping class groups*, Publ. Math. Inst. Hautes Etudes Sci. 122 (2015), 1–64.

[BM15] J. Birman, W. Menasco, *The curve complex has dead ends*, Geom. Dedicata 177 (2015), 71–74.

[B+17] E. Breuillard, M. Kalantar, M. Kennedy, N. Ozawa, *C^*-simplicity and the unique trace property for discrete groups*, Publ. Math. Inst. Hautes Études Sci. 126 (2017), 35–71.

[Bu92] P. Buser, *Geometry and spectra of compact Riemann surfaces*, Birkhäuser, Boston 1992.

[CEG87] R. Canary, D. Epstein, P. Green, *Notes on notes of Thurston*, in "Analytical and geometric aspects of hyperbolic space", edited by D. Epstein, London Math. Soc. Lecture Notes 111, Cambridge University Press, Cambridge 1987.

[CB88] A. Casson with S. Bleiler, *Automorphisms of surfaces after Nielsen and Thurston*, Cambridge University Press, Cambridge 1988.

[Dr18] A.N. Dranishnikov, *Dimension of compact metric spaces*, Indagnationes math. 29 (2018). 429–449.

[DHS17] M. Durham, M. Hagen, A. Sisto, *Boundaries and automorphisms of hierarchically hyperbolic spaces*, Geometry & Topology 21 (2017), 3659–3758.

[DMS25] M. Durham, M. Minsky, A. Sisto, *Asymptotically* CAT(0)*-metrics, Z-structures and the Farrell-Jones conjecture*, arXiv:2504.17048.

[FL05] F.T. Farrell, J.F. Lafont, *EZ-structures and topological applications*, Comment. Math. Helv. 80 (2005), 103–121.

[Ga14] D. Gabai, *On the topology of ending lamination space*, Geom. & Top. 18 (2014), 2683–2745.

[H06] U. Hamenstädt, *Train tracks and the Gromov boundary of the complex of curves*, in "Spaces of Kleinian groups" (Y. Minsky, M. Sakuma, C. Series, eds.), London Math. Soc. Lec. Notes 329 (2006), 187–207.

[H09] U. Hamenstädt, *Geometry of the mapping class groups I: Boundary amenability*, Invent. Math. 175 (2009), 545–609.

[Har86] J. Harer, *The virtual cohomological dimension of the mapping class group of an oriented surface*, Invent. Math. 84 (1986), 157-176.

[HM79] J. Hubbard, H. Masur, *Quadratic differentials and foliations*, Acta Math. 142 (1979), 221–274.

[Iv90] N.V. Ivanov, *Attaching corners to Teichmüller space*, Algebra i Analiz 1 (1989), 115–143; English translation: Leningrad Math. J. 1 (1990), 1177–1205.

[Iv02] N. V. Ivanov, *Mapping class groups*, Chapter 12 in Handbook of Geometric Topology (Editors R.J. Daverman and R.B. Sher), Elsevier Science (2002), 523-633.

[IJ08] N.V. Ivanov, L. Ji, *Infinite topology of curve complexes and non-Poincaré duality of Teichmüller modular groups*, L'Enseign. Math. 54 (2008), 381–395.

[J14] L. Ji, *Well-rounded equivariant deformation retracts of Teichmüller space*, L'Enseign. Math. 60 (2014), 109–129.

[JW10] L. Ji, S. Wolpert, *A cofinite universal space for proper actions for mapping class groups*, in M. Bonk, J. Gilman, H. Masur, Y. Minsky and M. Wolf (eds.), In the tradition of Ahlfors-Bers, V. Contemporary Math 510. Amer. Math. Soc., Providence, RI, 2010, 151–163.

[Ki10] Y. Kida, *Measure equivalence rigidity of the mapping class group*, Ann. of Math. 171 (2010), 1851–1901.

[Kl22] E. Klarreich, *The boundary at infinity of the curve complex and the relative Teichmüller space*, Groups, Geom. Dyn. 16 (2022), 705–723.

[Ma85] B. Maskit, *Comparison of hyperbolic length and extremal length*, Ann. Aca. Sci. Fenn. Ser. A. I, Math. 10 (1985), 381–386.

[MM99] H. Masur, Y. Minsky, *Geometry of the complex of curves I: Hyperbolicity*, Invent. Math. 138 (1999), 103-149.

[MM00] H. Masur, Y. Minsky, *Geometry of the complex of curves II: Hierarchical structure*, Geom. Funct. Anal. 10 (2000), 902-974.

[Mu14] J. Munkres, *Topology*, Parson New International Edition, Pearson Educated Limited 2014.

[R14] K. Rafi, *Hyperbolicity in Teichmüller space*, Geom. & Topol. 18 (2014), 3025–3053.

[St84] K. Strebel, *Quadratic differentials*, Erg. Math. Grenzgeb. (3), 5, Springer, Berlin 1984.

[Ti11] C. Tirel, *$\mathcal{Z}$-structures on product groups*, Algebr. & Geom. Topol. 11 (2011), 2587–2625.

[Wo03] S. Wolpert, *Geometry of the Weil-Petersson completion of Teichmüller space*, Surv. Diff. Geom. 8, International Press (2003), 357–393.

[Ya04] S. Yamada, *Weil-Petersson completion of Teichmnüller spaces and mapping class group actions*, Math. Res. Lett. 11 (2004), 327–344.

[Zv02] D. Zvonkine, *Strebel differentials on stable curves and Kontsevich's proof of Witten's conjecture*, arXiv:math/0209013, v2, 2018.

MATHEMATISCHES INSTITUT DER UNIVERSITÄT BONN
ENDENICHER ALLEE 60,
D-53115 BONN, GERMANY

email: ursula@math.uni-bonn.de