## AN $\mathcal{EZ} ext{-STRUCTURE}$ FOR THE MAPPING CLASS GROUP

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ABSTRACT. We construct a boundary for the mapping class group  $\operatorname{Mod}(S)$  of a surface S of finite type. The action of  $\operatorname{Mod}(S)$  on this boundary is minimal, strongly proximal and topologically free. The boundary is the boundary of an  $\mathcal{EZ}$ -structure for  $\operatorname{Mod}(S)$ .

#### 1. Introduction

The mapping class group  $\operatorname{Mod}(S)$  of a closed oriented surface S of genus  $g \geq 0$  from which  $m \geq 0$  points have been removed and so that  $3g-3+m \geq 1$  is the group of isotopy classes of diffeomorphisms of S. The mapping class group is well known to be finitely presented, and it admits explicit torsion free finite index subgroups.

A torsion free finite index subgroup  $\Gamma$  of  $\operatorname{Mod}(S)$  admits a *finite* classifying space. Such a classifying space can be constructed as follows.

Since the Euler characteristic of S is negative, the *Teichmüller space*  $\mathcal{T}(S)$  of S of all marked finite area complete hyperbolic structures on S is defined. By elementary hyperbolic geometry, there exists a number  $\epsilon_0 > 0$  such that any two closed geodesics on a hyperbolic surface of length at most  $\epsilon_0$  are disjoint. The systole systole(X) of a hyperbolic metric X is the length of a shortest closed geodesic. For  $\epsilon < \epsilon_0$  define

$$\mathcal{T}_{\epsilon}(S) = \{X \in \mathcal{T}(S) \mid \text{systole}(X) \geq \epsilon\}.$$

The following is due to Ji and Wolpert [JW10], extending an earlier result of Ivanov [Iv02], see Proposition 3.1 and Theorem 3.9 of [J14] for an explicit statement.

**Theorem 1** (Ji-Wolpert). For sufficiently small  $\epsilon < \epsilon_0$ , the set  $\mathcal{T}_{\epsilon}(S)$  is a manifold with corners which is a deformation retract of  $\mathcal{T}(S)$ . The mapping class group Mod(S) acts on  $\mathcal{T}_{\epsilon}(S)$  properly and cocompactly.

Since  $\mathcal{T}(S)$  is homeomorphic to  $\mathbb{R}^{6g-6+2m}$ , we obtain that  $\mathcal{T}_{\epsilon}(S)$  is contractible, locally contractible and finite dimensional. As torsion free finite index subgroups  $\Gamma$  of Mod(S) act freely on  $\mathcal{T}_{\epsilon}(S)$ , this implies that  $\Gamma \setminus \mathcal{T}_{\epsilon}(S)$  is a finite classifying space for  $\Gamma$ . In particular,  $\Gamma$  is of type F.

Motivated by the construction of the Borel-Serre bordification of an arithmetic group which can be used to compute its *virtual cohomological dimension*, that is, the cohomological dimension of a torsion free finite index subgroup, Harer [Har86]

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initiated the construction of a bordification of  $\mathcal{T}_{\epsilon}(S)$  which computes the virtual cohomological dimension  $\operatorname{vcd}(\operatorname{Mod}(S))$  of the mapping class group. This program was completed by Ivanov [Iv02] and consists in attaching to  $\mathcal{T}_{\epsilon}(S)$  the *curve complex* as an analog of a spherical building. The bordification, which indeed computes the virtual cohomological dimension of  $\operatorname{Mod}(S)$ , has the homotopy type of an infinite wedge of spheres [IJ08] and does not compactify the space  $\mathcal{T}_{\epsilon}(S)$ .

In the setting of hyperbolic groups  $\Gamma$ , it turns out that the *Gromov boundary* of  $\Gamma$  can be used to compute the virtual cohomological dimension [BM91]. This Gromov boundary is the boundary of a compactification of  $\Gamma$  in the following sense.

**Definition 2** (Small boundary). A boundary of a finitely generated group  $\Gamma$  is a compact  $\Gamma$ -space Z with the following properties.

- There exists a topology on  $\Gamma \cup Z$  which restricts to the discrete topology on  $\Gamma$ , to the given topology on Z and is such that  $\Gamma \cup Z$  is compact.
- The left action of  $\Gamma$  on itself extends to the  $\Gamma$ -action on Z.

The boundary is called *small* if the right action of  $\Gamma$  extends to the trivial action of  $\Gamma$  on Z.

The Gromov boundary of a hyperbolic group has additional desirable properties. One of these is captured in the following definition, which is Lemma 1.3 of [B96] and Definition 1.1 of [FL05].

**Definition 3** ( $\mathcal{EZ}$ -structure). An  $\mathcal{EZ}$ -structure for a finitely generated torsion free group  $\Gamma$  consists of a pair  $(\overline{X}, Z)$  of finite dimensional compact metrizable spaces, with Z nowhere dense in  $\overline{X}$ , and the following additional properties.

- (1)  $X = \overline{X} Z$  is contractible and locally contractible.
- (2) For every  $z \in Z$  and every neighborhood U of z in  $\overline{X}$  there exists a neighborhood  $V \subset U$  of z such that the inclusion  $V Z \to U Z$  is null-homotopic.
- (3) X admits a covering space action of  $\Gamma$  with compact quotient.
- (4) The collection of all translates of a compact set in X form a null sequence in  $\overline{X}$ : that is, for every open cover  $\mathcal{U}$  of  $\overline{X}$ , all but finitely many translates are  $\mathcal{U}$ -small.
- (5) The action of  $\Gamma$  on X extends to an action on  $\overline{X}$ .

The significance of an  $\mathcal{EZ}$ -structure  $(\overline{X}, Z)$  for a torsion free group  $\Gamma$  lies in the fact that the Čech cohomology of the space Z computes the cohomological dimension  $\operatorname{cd}(\Gamma)$  of the group, with a dimension shift of one (Theorem 1.7 of [B96]). Furthermore, groups with an  $\mathcal{EZ}$ -structure admit an  $\mathcal{EZ}$ -structure of the form  $(\mathbb{D}^n, \Delta)$  where  $\Delta$  is a closed subset of  $\partial \mathbb{D}^n = S^{n-1}$ , and the Novikov conjecture and the K-theoretic Farell Jones conjecture hold for these groups (Theorem 1.1 and Theorem 1.2 of [FL05]).

An action of a group G on a compact topological space Z is called *minimal* if every G-orbit is dense. It is called *topologically free* if for every  $\varphi \in G - \{1\}$  the fixed point set of  $\varphi$  has empty interior. Furthermore, it is called *strongly proximal* 

if the action of G on the space of Borel probability measures on Z is such that the closure of any orbit contains some Dirac measure. The following is our main result.

**Theorem 4.** There exists a compactification  $\overline{\mathcal{T}}(S)$  of  $\mathcal{T}_{\epsilon}(S)$  with the following properties.

- (1)  $\mathcal{X}(S) = \overline{\mathcal{T}}(S) \setminus \mathcal{T}_{\epsilon}(S)$  is a small boundary for Mod(S).
- (2) The action of Mod(S) on  $\mathcal{X}(S)$  is minimal, strongly proximal and topologically free.
- (3) The pair  $(\overline{\mathcal{T}}(S), \mathcal{X}(S))$  is a  $\mathcal{EZ}$ -structure for Mod(S).

We call the space  $\mathcal{X}(S)$  the geometric boundary of Mod(S).

By the main results of [FL05], Theorem 4 implies the following.

**Corollary 5.** Mod(S) satisfies the Novikov conjecture and the K-theoretic Farell-Jones conjecture.

Both statements in the corollary are known and were earlier established with different methods. Proofs of the Novikov conjecture can be found in [H09, K10, BBF15, BaB19]. The full Farell Jones conjecture for mapping class groups is due to Bartels and Bestvina [BaB19].

An alternative approach to the construction of an  $\mathcal{EZ}$ -structure for the mapping class group, based on hierarchical hyperbolicity, is due to Durham, Minsky and Sisto [DMS25]. Hierarchical hyperbolicity was also used by Durham, Hagen and Sisto [DHS17] to construct a boundary for Mod(S). As a set, this boundary can be identified with the boundary constructed in Theorem 4, however the topology is different. There are open sets in the boundary of [DHS17] which do not contain any open subset of the boundary we construct. Hierarchical hyperbolicity for Mod(S) only appears indirectly in this article, but our construction shares with [DMS25] the strategy to view the mapping class group as a CAT(0)-space on the large scale.

As the virtual cohomological dimension  $\operatorname{vcd}(\operatorname{Mod}(S))$  equals 4g-5 if  $g\geq 2$  and  $m=0,\ 4g-4+m$  if  $g\geq 1$  and m-3 if g=0 [Har86], the covering dimension of the space  $\mathcal{X}(S)$  equals 4g-6 if  $g\geq 2$  and  $m=0,\ 4g-5+m$  if  $g\geq 1$  and m>0, and m-4 if g=0 [B96]. Note that for any torsion free finite index subgroup  $\Gamma$  of  $\operatorname{Mod}(S)$ , the cohomology group  $H^{\operatorname{vcd}(\operatorname{Mod}(S))}(\Gamma,\mathbb{Z}\Gamma)$  identifies with the 2g-2+m-th homology group of the curve complex. Since the curve complex has the homotopy type of an infinite wedge of spheres of dimension 2g-2+m (Theorem 1.4 of [IJ08]), this implies that the top dimensional Čech cohomology group of  $\mathcal{X}(S)$  is also infinite dimensional by Proposition 1.5 of [B96].

Theorem 4 can be viewed as giving some evidence that the asymptotic dimension of Mod(S), which is known to be finite and at most quadratic in the virtual cohomological dimension, in fact equals the virtual cohomological dimension of Mod(S). We refer to [BB19] for a more detailed discussion on this and related questions and results.

The following is an easy consequence of Theorem 4 and Theorem 1.1 and Theorem 1.2 of [FL05]. In its formulation we denote by  $\mathbb{D}^n$  the standard ball in  $\mathbb{R}^n$ .

**Corollary 6.** If  $3g - 3 + m \ge 3$  then there exists a closed subset  $\Delta$  of  $S^{6g-5+2m}$  such that Mod(S) admits an  $\mathcal{EZ}$ -structure of the form  $(\mathbb{D}^{6g-4+2m}, \Delta)$ .

The boundary of the curve graph can be obtained from the subset  $\mathcal{FML}$  of  $\mathcal{PML} = S^{6g-7+2m}$  of measured geodesic laminations with minimal filling support by an equivariant continuous surjective map  $\mathcal{FML} \to \partial \mathcal{CG}(S)$ . This map is however not injective and the following statement requires a proof.

**Corollary 7.** The boundary  $\partial \mathcal{CG}(S)$  of the curve graph of S admits an embedding into a manifold of dimension 6g - 6 + 2m and into  $S^{6g-5+2m}$ .

We next describe the boundary  $\mathcal{X}(S)$  of Mod(S) as a set.

The curve complex  $\mathcal{CG}(S_0)$  of a (not necessarily proper) essential subsurface  $S_0$  of S different from a pair of pants or an annulus is the simplicial complex whose vertices are isotopy classes of simple closed curves and where k such curves span a k-1-simplex if they can be realized disjointly. If  $S_0$  is a four-holed sphere or a one holed torus, then this definition has to be modified by connecting two vertices by an edge if they intersect in the minimal number of points.

The curve complex, equipped with the natural simplicial metric, is a hyperbolic geodesic metric space of infinite diameter [MM99]. Its Gromov boundary  $\partial \mathcal{CG}(S_0)$  is the space of minimal geodesic laminations on  $S_0$  which fill  $S_0$ , that is, which intersect every essential simple closed curve on  $S_0$  transversely. The topology on  $\partial \mathcal{CG}(S_0)$  is the coarse Hausdorff topology. With respect to this topology, a sequence  $\lambda_i$  of minimal filling laminations converges to the lamination  $\lambda$  if and only if the limit of any subsequence which converges in the Hausdorff topology on compact subsets of  $S_0$  contains  $\lambda$  as a sublamination [H06, K99]. The space  $\partial \mathcal{CG}(S_0)$  is separable and metrizable. Define the boundary of the curve complex of an essential annulus  $A \subset S$  with core curve c to consist of two points  $c^+, c^-$ .

If  $S_1, \ldots, S_k$  is a collection of isotopy classes of pairwise disjoint subsurfaces of S, then we can form the join

$$\mathcal{J}(\cup_{i=1}^k S_i) = \partial \mathcal{CG}(S_1) * \cdots * \partial \mathcal{CG}(S_k).$$

It can be viewed as the set of formal sums  $\sum_i a_i \lambda_i$  where  $a_i > 0$ ,  $\sum_i a_i = 1$  and where  $\lambda_i \in \partial \mathcal{CG}(S_i)$  for all i. This join is a separable metrizable topological space. Note that if  $S_{i_1}, \ldots, S_{i_s}$  is a subset of the set of surfaces  $S_1, \ldots, S_k$ , then  $\mathcal{J}(\bigcup_{j=1}^s S_{i_j})$  is naturally a non-empty closed subset of  $\mathcal{J}(\bigcup_{i=1}^k S_i)$  corresponding to formal sums  $\sum_i a_i \lambda_i$  with  $a_i = 0$  for  $i \notin \{i_1, \ldots, i_s\}$ . Define

$$\mathcal{X}(S) = \cup \mathcal{J}(\cup_{i=1}^k S_i)$$

where the union is over all collections of pairwise disjoint essential subsurfaces of S and we use the obvious identification of points which arise in more than one way in this union. Here we view an essential annulus A as an essential subsurface which is disjoint from any subsurface which can be moved off A by an isotopy. Thus  $\mathcal{X}(S)$  is just the set of formal sums  $\sum_i a_i \lambda_i$  where  $a_i > 0$ ,  $\sum_i a_i = 1$ , where  $\lambda_1, \ldots, \lambda_k$  are pairwise disjoint minimal geodesic laminations on S and where each simple closed curve component  $\lambda_i$  is equipped with an additional label +, -. The mapping class group acts naturally on  $\mathcal{X}(S)$  as a set.

The following theorem summarizes some more technical properties of the geometric boundary. For its formulation, let us invoke the Nielsen Thurston classification which states that any nontrivial mapping class has a finite power  $\varphi$  with the following property. There exists a decomposition  $S = S_1 \cup \cdots \cup S_k$  of S into subsurfaces that are preserved by  $\varphi$  and such that for all i < k, the surface  $S_i$  is connected and the restriction of  $\varphi$  to  $S_i$  is pseudo-Anosov if  $S_i$  is not an annulus, and it is a Dehn twist if  $S_i$  is an annulus. The restriction of  $\varphi$  to  $S_k$  is trivial. We call a mapping class with this property a Nielsen Thurston mapping class.

Let  $\varphi$  be a Nielsen Thurston mapping class. For each i < k such that  $S_i$  is not an annulus, the restriction  $\varphi_i$  of  $\varphi$  to  $S_i$  preserves precisely two geodesic laminations  $\xi_i^{\pm}$  which are the attracting and repelling laminations of  $\varphi_i$ . Similarly, for any component  $S_i$  which is an annulus, the two labeled copies  $\xi_i^{\pm}$  of the core curve of the annulus are preserved as well. Thus  $\varphi$  fixes any point of the form  $\sum_i a_i \zeta_i$  where  $\zeta_i$  is one of the laminations  $\xi_i^{\pm}$  if i < k and where  $\zeta_k$  is an arbitrary point of the geometric boundary of the (possibly disconnected) surface  $S_k$ . We call points of this form the *obvious fixed point set*.

An embedding of a topological space X into a topological space Y is an injective map  $f: X \to Y$  which is a homeomorphism onto its image, equipped with the subspace topology.

**Proposition 8.** Let  $\mathcal{X}(S)$  be the geometric boundary of Mod(S).

- (1) For any collection  $S_1, \ldots, S_k$  of pairwise disjoint subsurfaces of S, the inclusion  $\mathcal{J}(\bigcup_{i=1}^k S_i) \to \mathcal{X}(S)$  is an embedding. In particular, the covering dimension of  $\partial \mathcal{CG}(S)$  is at most  $\operatorname{vcd}(\operatorname{Mod}(S)) 1$ .
- (2) The fixed point set for the action of a Nielsen Thurston mapping class  $\varphi$  on  $\mathcal{X}(S)$  is precisely the obvious fixed point set of  $\varphi$ .

That the covering dimension of  $\partial \mathcal{CG}(S)$  is bounded from above by  $\operatorname{vcd}(\operatorname{Mod}(S))$  is due to Gabai (Proposition 16.3 of [Ga14]).

Our construction is valid for the mapping class group of a once punctured torus or a four punctured sphere. In this case the mapping class group is virtually free and, in particular, it is a hyperbolic group whose Gromov boundary is a Cantor set. It is due to Bestvina and Mess [BM91] that a hyperbolic group admits a  $\mathcal{EZ}$ -structure whose boundary is its Gromov boundary. The boundary we find is the Gromov boundary of the group as well.

The construction of the boundary  $\mathcal{X}(S)$  is motivated by the construction of the visual boundary of a CAT(0)-space. Along the way we identify in Section 2 an analog of the familiar Tits boundary of a symmetric space of higher rank.

The advantage of our construction is that the space  $\mathcal{X}(S)$  and its topology as well as the action of the group  $\operatorname{Mod}(S)$  on  $\mathcal{X}(S)$  is completely explicit and can be used among others to study subgroups of  $\operatorname{Mod}(S)$ .

Overview of the article: In the first part of the article, we define a topology on the set  $\mathcal{X}(S)$  and show that this topology extends to  $\mathcal{T}_{\epsilon}(S) \cup \mathcal{X}(S)$  and defines

a compactification of  $\mathcal{T}_{\epsilon}(S)$ . This is carried out with an inductive construction set up in Section 3 and Section 4. We also observe that the action of Mod(S) on  $\mathcal{X}(S)$  is strongly proximal. In Section 2, we introduce the *oriented curve complex* and show that it can be viewed as a Tits type boundary for the mapping class group.

In Section 5, we show that  $\mathcal{X}(S)$  is metrizable. This result depends on the construction of an explicit neighborhood basis of a given point in  $\mathcal{X}(S) \subset \overline{\mathcal{T}}(S)$ . The neighborhood basis is used in Section 7 to construct another neighborhood basis for points in  $\mathcal{X}(S)$  consisting of sets whose intersections with  $\mathcal{T}_{\epsilon}(S)$  are contractible, which is the most involved part of the article. We use the augmented Teichmüller space as a witness of CAT(0) properties to this end. Section 6 is devoted to showing that the covering dimension of  $\mathcal{X}(S)$  is finite.

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## 2. The Tits boundary of Mod(S)

The join  $X_1 * X_2$  of two topological spaces  $X_1, X_2$  is defined to be the quotient  $X_1 \times X_2 \times [0,1]/\sim$  where the equivalence relation  $\sim$  collapses  $X_1 \times X_2 \times \{0\}$  to  $X_1$  and collapses  $X_1 \times X_2 \times \{1\}$  to  $X_2$ . For example, the join  $S_1^0 * S_2^0$  of two 0-spheres is the circle  $S^1$ , thought of as a union of four intervals glued at the endpoints, where each interval has one endpoint in  $S_1^0$  and the second endpoint in  $S_2^0$ . The join of two spaces  $X_1, X_2$  contains an embedded copy of  $X_1, X_2$ .

**Example 2.1.** The product of two hyperbolic planes  $\mathbb{H}^2 \times \mathbb{H}^2$  is a complete simply connected Riemannian manifold of non-positive curvature. Its *visual boundary* is the join  $S^1 * S^1$  of two circles that are the Gromov boundaries of the embedded copies of  $\mathbb{H}^2$ . This corresponds to the fact that the projection of any geodesic in  $\mathbb{H}^2 \times \mathbb{H}^2$  to each of the two factors is a geodesic. Note that the join of two circles is homeomorphic to  $S^3$ .

Define the oriented curve complex  $\mathcal{OG}(S)$  of an oriented connected surface S of genus g with m punctures and  $3g-3+m\geq 2$  to be the complex whose vertices are isotopy classes of oriented simple closed curves in S and whose one-skeleton consists of edges (of length 1) connecting two vertices if they can be realized disjointly and are not homotopic up to orientation. Thus any simple closed curve in S defines two distinct vertices in  $\mathcal{OG}(S)$ , and these vertices are not connected by an edge. Furthermore, we require that any collection of  $k\geq 2$  oriented disjoint simple closed curves which are distinct as unoriented curves span a simplex. The union of these simplices defined by a fixed collection of k curves equipped with all combinations of orientations is a sphere of dimension k-1. Note that a point in  $\mathcal{OG}(S)$  can be viewed as a formal linear combination  $\sum_{i=1}^k a_i \lambda_i$  where for some  $k\geq 1$ ,  $\lambda_1,\ldots,\lambda_k$  are pairwise disjoint oriented simple closed curves, where  $a_i>0$  for all i and  $\sum_i a_i=1$ . In other words, a point in the oriented curve complex can be viewed as a point in the join of a finite collection of oriented pairwise disjoint simple closed curves. If S is a once punctured torus or a four punctured sphere, then the

oriented curve complex is defined in the same way except that two oriented curves are connected by an edge if they intersect in the minimal number of points (one for the once punctured torus and two for the four punctured sphere).

Remark 2.2. If we choose the length of the edges of the oriented curve complex to be  $\pi/2$ , then this is consistent with the idea that the oriented curve complex can be thought of as being contained in the Tits boundary of Mod(S), equipped with the angular length metric which identifies each sphere with a sphere of constant curvature one.

A simple closed curve c on S is the core curve of an embedded annulus  $A(c) \subset S$ . The "curve graph"  $\mathcal{CG}(A(c))$  of the annulus A(c) is a graph of isotopy classes of arcs connecting the two boundary components and whose endpoints are allowed to move freely in the complement of a fixed point on each of the two boundary circles. The curve graph of A(c) is a simplicial line. If  $\alpha$  is a given vertex of  $\mathcal{CG}(A(c))$ , then any other isotopy class of arcs can be represented by an arc which is the image of  $\alpha$  by a multipe of a Dehn twist about c. The distinction between a positive and a negative Dehn twist about c only depends on the orientation of c but not on the orientation of c. The choice of an orientation of c can be thought of as a spiraling direction about c for oriented arcs connecting the two boundary components of a and a is a spiral component of a in a is a spiral connection about a in a

In the sequel we denote by  $c^+$  the point in the Gromov boundary of  $\mathcal{CG}(A(c))$  (which consists of two points) which corresponds to an iteration of positive Dehn twists about c, and we denote by  $c^-$  the point in the Gromov boundary of  $\mathcal{CG}(A(c))$  which corresponds to an iteration of negative Dehn twists about c. Write  $\mathcal{J}(c) = \{c^+, c^-\}$ . It will be convenient to think about  $\mathcal{J}(c)$  as a set of two distinct points in the oriented curve complex of S, with the same underlying curve.

If  $S_0$  is a subsurface of S different from a pair of pants or an annulus, then we denote its (non-oriented) curve complex by  $\mathcal{CG}(S_0)$ . The vertices of this complex are isotopy classes of non-peripheral simple closed curves. If  $S_0$  is different from a one-holed torus or a four-holed sphere, then a collection of  $k \geq 2$  such disjoint simple closed curves span a simplex of dimension k-1. If  $S_0$  is a one-holed torus or a four-holed sphere then two simple closed curves are connected by an edge if they intersect transversely in the minimal number of points. The curve complex of  $S_0$  is hyperbolic and hence it has a *Gromov boundary*  $\partial \mathcal{CG}(S_0)$ . As a set, the Gromov boundary  $\partial \mathcal{CG}(S_0)$  is the set of all minimal filling geodesic laminations on  $S_0$ . We refer to [H06] for an account on this result of Klarreich.

There is a natural metrizable topology on the union  $\overline{\mathcal{CG}}(S_0)$  of  $\mathcal{CG}(S_0)$  with its Gromov boundary, called the coarse Hausdorff topology. With respect to this topology, the subspace  $\mathcal{CG}(S_0)$ , equipped with its simplicial topology, is an open dense subset. To define this topology equip the surface  $S_0$  with a hyperbolic metric with geodesic boundary. This choice defines a Hausdorff topology on the space of compact subsets of  $S_0$ . A sequence  $\lambda_i \subset \mathcal{CG}(S_0) \subset \mathcal{CG}(S_0) \cup \partial \mathcal{CG}(S_0)$  of vertices in  $\mathcal{CG}(S_0)$  converges in the coarse Hausdorff topology to  $\lambda \in \partial \mathcal{CG}(S_0)$  if and only if the limit of any converging subsequence of  $\lambda_i$  in the Hausdorff topology on compact subsets of  $S_0$  contains  $\lambda$  as a sublamination [H06]. Define

$$\mathcal{J}(S_0) = \partial \mathcal{CG}(S_0),$$

equipped with the topology as a subset of  $\overline{\mathcal{CG}}(S_0)$ . If  $S_0$  is a pair of pants, then we define  $\mathcal{J}(S_0) = \emptyset$ .

If  $S_1, \ldots, S_k$  are *disjoint* connected subsurfaces of S (we allow that they share boundary components, and annuli about such boundary components may be included in the list), then we define

(1) 
$$\mathcal{J}(\cup_i S_i) = \partial \mathcal{CG}(S_1) * \cdots * \partial \mathcal{CG}(S_k)$$

to be the join of the spaces  $\mathcal{J}(S_i) = \partial \mathcal{CG}(S_i)$ . For example, if  $S_1 \subset S$  is a subsurface which is the complement of a non-separating simple closed curve c, then

$$\mathcal{J}(S_1 \cup A(c)) = \partial \mathcal{CG}(S_1) * \{c^+, c^-\}.$$

A point in  $\mathcal{J}(S_1 \cup \cdots \cup S_k)$  can be viewed as a formal linear combination

$$\xi = \sum_{i} a_i \xi_i$$

where  $\xi_i \in \partial \mathcal{CG}(S_i)$ ,  $a_i \geq 0$  for all i and, furthermore,  $\sum_i a_i = 1$ . The union

$$\operatorname{supp}(\xi) = \bigcup_{a_i > 0} \xi_i$$

is a geodesic lamination with minimal components  $\xi_i$ , and  $\xi$  can be viewed as a weighted (and partially labeled if there are simple closed curve components of  $\xi$  with positive weight) geodesic lamination. For all  $u \leq k$  there is an inclusion  $\mathcal{J}(S_1 \cup \cdots \cup S_u) \subset \mathcal{J}(S_1 \cup \cdots \cup S_k)$  which is a topological embedding.

A collection  $S_1, \ldots, S_k$  of disjoint connected subsurfaces of S is called maximal if  $S - \cup_i S_i = \emptyset$ . By convention, this means that for any boundary component c of one of the surfaces  $S_i$ , the annulus A(c) is contained in the collection. Any collection  $S_1, \ldots, S_\ell$  of disjoint connected subsurfaces of S is contained in a maximal collection of such subsurfaces, however this maximal collection is in general not unique. For example, there is a canonical maximal collection containing  $S_1, \ldots, S_k$  which is comprised of the surfaces  $S_i$ , the annuli A(c) where c runs through all boundary components of  $O_i S_i$  which are not already contained in the list, and all connected components of  $S_i - O_i S_i$ .

Define

(2) 
$$\mathcal{X}(S) = \bigcup \mathcal{J}(S_1 \cup \cdots \cup S_k) / \sim$$

where the union is over all collections of disjoint subsurfaces  $S_1, \ldots, S_k$  of S. The equivalence relation  $\sim$  identifies two points  $\sum_i a_i \xi_i$  and  $\sum_j b_j \zeta_j$  if they coincide as weighted labeled geodesic laminations. Thus a point in  $\mathcal{X}(S)$  is nothing else but a formal sum  $\sum_{i=1}^k a_i \xi_i$  where  $a_i > 0$ ,  $\sum_i a_i = 1$ , where  $\xi_1, \ldots, \xi_k$  are pairwise disjoint minimal geodesic laminations on S and where every simple closed curve component of this collection is in addition equipped with a label  $\pm$ . Note that the oriented curve complex  $\mathcal{OG}(S)$  of S can naturally be identified with the union of the subsets  $\mathcal{J}(A(c_1) \cup \cdots \cup A(c_k))$  of  $\mathcal{X}(S)$ , and its Gromov boundary (which is just the Gromov boundary  $\partial \mathcal{CG}(S)$  of the non-oriented curve complex of S) also is contained in  $\mathcal{X}(S)$ . The mapping class group  $\operatorname{Mod}(S)$  naturally acts on the set  $\mathcal{X}(S)$ .

**Example 2.3.** The definition (2) also makes sense if S is a once punctured torus or a four punctured sphere. In this case there are no non-peripheral subsurfaces of S different form annuli and pairs of pants, and the set  $\mathcal{X}(S)$  is just the union of the Gromov boundary of the curve graph  $\mathcal{CG}(S)$  with a countable set, consisting of all oriented non-peripheral simple closed curves on S. We discuss the case of the once punctured torus in detail, the case of the four punctured sphere is completely analogous.

The curve graph of S is the well-known  $Farey\ graph$ . It vertices can be represented by the rational points in the boundary  $\partial \mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$  of the hyperbolic plane. If one represents the edges of the Farey graph by geodesics in  $\mathbb{H}^2$ , then one obtains a tesselation of the hyperbolic plane by ideal triangles which is invariant under the mapping class group  $\mathrm{PSL}(2,\mathbb{Z})$  of S. The boundary  $\partial T$  of the dual tree T of this tesselation is a Cantor set which admits a surjective continuous map onto the boundary  $\partial \mathbb{H}^2$  of the hyperbolic plane. Each irrational point in  $\partial \mathbb{H}^2$  corresponds to a point in the Gromov boundary of  $\mathcal{CG}(S)$  and has precisely one preimage, and the rational points which correspond to the vertices of the curve graph have two preimages.

The vertices of the Farey graph correspond to the fixed points of the parabolic subgroups of  $\operatorname{PSL}(2,\mathbb{Z})$ . With this interpretation, the set  $\mathcal{X}(S)$  can be identified with the Cantor set  $\partial T$  obtained by replacing each rational point in  $\mathbb{R} \cup \{\infty\}$  by a compact interval and removing the interior of the interval. This Cantor set in turn has a natural identification with the Gromov boundary  $\partial T$  of the virtually free group  $\operatorname{PSL}(2,\mathbb{Z})$ . In particular, there is a natural invariant topology on  $\mathcal{X}(S)$  so that with this topology,  $\mathcal{X}(S)$  is a compact  $\operatorname{PSL}(2,\mathbb{Z})$ -space which contains the Gromov boundary  $\partial \mathcal{CG}(S)$  of the curve graph of S as a dense embedded subset. Furthermore, following [BM91], with this topology the set  $\mathcal{X}(S)$  is the boundary of an  $\mathcal{EZ}$ -structure for  $\operatorname{PSL}(2,\mathbb{Z})$ .

**Example 2.4.** Let  $S_1, \ldots, S_k$  be a disjoint union of subsurfaces of S which are different from pairs of pants. Then the join  $\mathcal{X}(S_1) * \cdots * \mathcal{X}(S_k)$  is a subset of  $\mathcal{X}(S)$ .

The oriented curve complex of S is connected, and any non-filling geodesic lamination, that is, a geodesic lamination which is disjoint from some simple closed curve, is disjoint from some vertex of  $\mathcal{OG}(S)$ . Thus if we equip  $\mathcal{X}(S) \setminus \partial \mathcal{CG}(S)$  with the topology of a simplicial complex whose edges are the joins of two disjoint (perhaps labeled) geodesic laminations, then this complex is connected. As a consequence, the set  $\mathcal{X}(S)$  can be equipped with a topology which coincides with the topology of a (non-locally finite) simplicial complex on  $\mathcal{X}(S) \setminus \partial \mathcal{CG}(S)$  and is such that each point in  $\partial \mathcal{CG}(S)$  is isolated. We write  $\mathcal{X}_T(S)$  for  $\mathcal{X}(S)$  equipped with this topology and call  $\mathcal{X}_T(S)$  the Tits boundary of Mod(S) (having the Tits boundary of a CAT(S)0 space as guidance). From this description, we obtain

**Lemma 2.5.** The mapping class group Mod(S) of S acts on  $\mathcal{X}_T(S)$  as a group of simplicial automorphisms.

*Proof.* The mapping class group acts on the oriented curve complex of S as a group of simplicial automorphisms, and this action extends to an action on the space of formal sums of weighted disjoint minimal geodesic laminations preserving weight

and disjointness. Furthermore, it acts on  $\partial \mathcal{CG}(S)$  as a group of transformations. Since the topology on  $\mathcal{X}_T(S)$  is the topology of a disconnected simplicial complex, constructed from the curve complexes of subsurfaces, the lemma follows.

**Remark 2.6.** The Tits boundary of a CAT(0) space X can be viewed as the geometric boundary (that is, the CAT(0) boundary) of X, equipped with a topology which in general is finer than the geometric topology. We shall see in Section 4 that the same holds true for the Tits boundary and the geometric boundary of Mod(S).

# 3. A TOPOLOGY FOR $\mathcal{X}(S)$

The goal of this section is to equip the set  $\mathcal{X}(S)$  with a topology which is coarser than the Tits topology so that for this topology,  $\mathcal{X}(S)$  becomes a compact Mod(S)-space.

Let  $\xi^j = \sum_m a_m^j \xi_m^j$  be a sequence in  $\mathcal{X}(S)$ . We shall impose two requirements for the sequence to converge to a point  $\zeta = \sum_{i=1}^k b_i \zeta_i \in \mathcal{X}(S)$ . Here as before, we assume that  $a_m^j > 0, b_i > 0, \sum_i b_i = 1 = \sum_m a_m^j$  for all j and that furthermore,  $\operatorname{supp}(\xi_i)$ ,  $\operatorname{supp}(\zeta)$  are disjoint unions of minimal components. The three steps needed to construct the topology are contained in three different subsections.

3.1. Convergence to a minimal filling lamination. Recall that the space of geodesic laminations on S is compact with respect to the Hausdorff topology.

Requirement 1: Convergence in the coarse Hausdorff topology Let  $\xi^{\ell_n}$  be any subsequence of the sequence  $\xi^j$  such that the geodesic laminations  $\sup(\xi^{\ell_n})$  converge in the Hausdorff topology to a geodesic lamination  $\beta$ . Then  $\beta$  contains  $\sup(\zeta)$  as a sublamination.

**Example 3.1.** A geodesic lamination c coarsely determines a point in  $\mathcal{CG}(S) \cup \partial \mathcal{CG}(S)$ . Namely, if c is minimal filling, then  $c \in \partial \mathcal{CG}(S)$ . Otherwise c is disjoint from a simple closed curve  $c' \in \mathcal{CG}(S)$ .

By a result of Klarreich [K99] as reported in [H06], a sequence of non-filling geodesic laminations  $c_i$  converges in the coarse Hausdorff topology to a minimal filling geodesic lamination  $\eta$  if and only if the simple closed curves  $c_i' \in \mathcal{CG}(S)$  converge in  $\mathcal{CG}(S) \cup \partial \mathcal{CG}(S)$  to  $\eta \in \partial \mathcal{CG}(S)$ .

**Example 3.2.** Let  $S_1, \ldots, S_k \subset S$  be disjoint subsurfaces. Example 2.4 shows that  $\mathcal{X}(S)$  contains the join  $\mathcal{X}(S_1) * \cdots * \mathcal{X}(S_k)$  as a subset. An element  $\xi \in \mathcal{X}(S_1) * \cdots * \mathcal{X}(S_k)$  can be represented in the form

$$\xi = \sum_{i} a_i \xi_i$$

where  $\xi_i \in \mathcal{X}(S_i)$ , in particular,  $\operatorname{supp}(\xi_i) \subset S_i$ , and  $\sum_i a_i = 1$ . Since the subset of geodesic laminations on S which are supported in  $S_i$  is closed with respect to the Hausdorff topology, this implies that for any topology on  $\mathcal{X}(S)$  which fulfills the first requirement above, the subspace  $\mathcal{X}(S_1) * \cdots * \mathcal{X}(S_k)$  of  $\mathcal{X}(S)$  is closed.

The examples show that the requirement (1) determines completely and geometrically the convergence of a sequence  $\xi_i \subset \mathcal{X}(S)$  to a point  $\xi \in \partial \mathcal{CG}(S) \subset \mathcal{X}(S)$ .

**Example 3.3.** In the case that S is a once punctured torus or a four punctured sphere, then any non-trivial subsurface of S different from a pair of pants is an annulus. This easily implies that the topology of  $\mathcal{X}(S)$  is determined by the requirement (1). Furthermore, it follows from Example 2.3 and the discussion in Example 3.1 that the space  $\mathcal{X}(S)$  is naturally homeomorphic to the Gromov boundary of the hyperbolic group Mod(S).

3.2. **Product spaces.** In this subsection we consider a collection  $S_i$   $(1 \le 1 \le k)$  of pairwise disjoint proper subsurfaces of S. This collection determines the subspace

$$\mathcal{X}(\cup_i S_i) = \mathcal{X}(S_1) * \cdots * \mathcal{X}(S_k) \subset \mathcal{X}(S).$$

Put

$$\mathcal{CG}(\cup_i S_i) = \mathcal{CG}(S_1) \times \cdots \times \mathcal{CG}(S_k).$$

Our goal is to define a topology on the union

$$\mathcal{Y}(\cup_i S_i) = \mathcal{CG}(\cup_i S_i) \cup \partial \mathcal{CG}(S_1) * \cdots * \partial \mathcal{CG}(S_k) = \mathcal{CG}(\cup_i S_i) \cup \mathcal{J}(\cup_i S_i)$$

which will be used in the construction of a topology on  $\mathcal{X}(S)$ .

Our main tool are complete markings of (not necessarily proper) essential subsurfaces  $S_0$  of the surface S. Such a marking consists of a pants decomposition P for  $S_0$  together with a collection of spanning curves. For every component c of P, there exists such a spanning curve which intersects c in the minimal number of points (one or two) and is disjoint from all other pants curves. Two spanning curves may not be disjoint, but we require that the number of their intersection points is bounded from above by a universal constant. Since there are only finitely many topological types of pants decompositions, this can clearly be achieved. There is a natural way to equip the set of all markings on  $S_0$  with the structure of a locally finite connected graph on which the mapping class group  $\text{Mod}(S_0)$  of  $S_0$  acts properly and cocompactly. We refer to [MM00] for more information on this construction.

Choose a marking  $\mu$  on S as a basepoint for the proper cocompact action of Mod(S) on the marking graph. For every subsurface  $S_0$  of S which is distinct from a pair of pants or an annulus, this marking can be used to construct a marking  $\text{pr}_{S_0}(\mu)$  of  $S_0$  as follows.

There is a coarsely well defined subsurface projection

$$\operatorname{pr}_{S_0}: \mathcal{CG}(S) \to \mathcal{CG}(S_0)$$

which associates to a simple closed curve c its intersection  $\operatorname{pr}_{S_0}(c) = c \cap S_0$  with  $S_0$  in the following sense. If  $c \subset S_0$  then put  $\operatorname{pr}_{S_0}(c) = c$ , and if c is disjoint from  $S_0$  then put  $\operatorname{pr}_{S_0}(c) = \emptyset$ . In all other cases,  $c \cap S_0$  consists of a collection of pairwise disjoint arcs with endpoints on the boundary of  $S_0$ . We then put  $\operatorname{pr}_{S_0}(c) = u$  for a simple closed curve u in  $S_0$  which is obtained from one of these intersection arcs by choosing a component of the boundary of a tubular neighborhood of the union of the arc with the boundary components of  $S_0$  containing its endpoints. Informally, we say that the simple closed curve is obtained by surgery on the arc.

Given a marking  $\mu$  for S, the union of the intersections of the marking curves with  $S_0$  consists of a union of arcs and simple closed curves on  $S_0$  with pairwise uniformly bounded intersection numbers which decompose  $S_0$  into simply connected regions. Hence via deleting some of these arcs and modifying some arcs with a surgery to simple closed curves as described in the previous paragraph, the projection of  $\mu$  into  $S_0$  coarsely defines a marking  $\operatorname{pr}_{S_0}(\mu)$  of  $S_0$  (there is a small abuse of notation here), called the subsurface projection of  $\mu$  [MM00]. Here a coarse definition means that the construction depends on choices, but any two choices give rise to markings which are uniformly close in the marking graph of  $S_0$ , independent of the subsurface  $S_0$ .

If  $S_0$  is an annulus, then a similar construction applies. In this case a marking consists of the choice of a marked point on each boundary component of  $S_0$  and an embedded arc in  $S_0$  connecting the two distinct boundary components which is disjoint from the marked points. With a bit of care, a subsurface projection is defined for annuli as well. We refer to [MM00] for more information.

By the above discussion, for every subsurface  $S_0$  of S the marking  $\mu$  coarsely determines a basepoint for  $\mathcal{CG}(S_0)$  by choosing one of the marking curves (or arcs if  $S_i$  is an annulus) of  $\operatorname{pr}_{S_i}(\mu)$ . As the intersection number between any two curves (or arcs) of  $\operatorname{pr}_{S_i}(\mu)$  is uniformly bounded, the distance in the curve graph of  $S_i$  between  $x_i$  and any other curve from  $\operatorname{pr}_{S_i}(\mu)$  or any other marking of  $S_i$  constructed in the same fashion from  $\mu$  is uniformly bounded.

Let  $\operatorname{Min}_{\cup}(S)$  be the space of geodesic laminations on S which are disjoint unions of minimal components. Using the basepoint  $x_0$  for  $\mathcal{CG}(S_0)$ , we can extend the subsurface projection  $\operatorname{pr}_{S_0}$  to all of  $\operatorname{Min}_{\cup}(S)$  as follows. Let  $\nu = \bigcup_i \nu_i \in \operatorname{Min}_{\cup}(\mathcal{L})$ . Then there are three possibilities.

- If the lamination  $\nu$  is disjoint from  $S_0$  up to homotopy, define  $\operatorname{pr}_{S_0}(\nu) = x_0$ .
- If there exist components  $\nu_1, \ldots, \nu_\ell$  of  $\nu$  which are contained in  $S_0$  then define  $\operatorname{pr}_{S_0}(\nu) = \bigcup_{i=1}^{\ell} \nu_i$ .
- If  $\nu \cap S_0$  consists of a collection of disjoint simple arcs with endpoints on the boundary of  $S_0$  which coarsely define a point in  $\mathcal{CG}(S_0)$  then define  $\operatorname{pr}_{S_0}(\nu)$  to be any one of these points.

Note that by the definition,  $\operatorname{pr}_{S_0}$  is contained in  $\operatorname{Min}_{\cup}(S_0)$ , and if  $\nu$  is a disjoint union of simple closed curves, then the same holds true for  $\operatorname{pr}_{S_0}(\nu)$ .

Let again  $S = \bigcup_{i=1}^k S_i$  be a collection of pairwise disjoint subsurfaces of S. It then follows from the above discussion that a choice  $\mu$  of a marking of S coarsely determines a basepoint  $x = (x_1, \ldots, x_k)$  for the product space  $\mathcal{CG}(\bigcup_i S_i)$  consisting of the product of the coarsely well defined basepoints  $x_i \in \mathcal{CG}(S_i)$ .

Recall from (1) the definition of the sets  $\mathcal{J}(\cup_i S_i)$ . Since the curve graph  $\mathcal{CG}(S_i)$  is a hyperbolic geodesic metric space, for every p > 1 and every p-quasi-geodesic ray  $\gamma : [0, \infty) \to \mathcal{CG}(S_i)$ , there exists a coarsely well defined shortest distance projection  $\Pi_{\gamma} : \mathcal{CG}(S_i) \to \gamma$  which extends to the complement of the endpoint  $\gamma(\infty) \in \partial \mathcal{CG}(S_i)$  in  $\partial \mathcal{CG}(S_i)$ .

**Definition 3.4.** Define a topology on  $\mathcal{Y}(\cup_i S_i)$  by the following requirements.

- The product space  $\mathcal{CG}(\cup_i S_i)$  is equipped with the product topology and is an open subset of  $\mathcal{Y}(\cup_i S_i)$ .
- The subspace  $\mathcal{J}(\cup_i S_i)$  is equipped with the topology as a join of the Gromov boundaries of the curve graphs of  $S_i$ .
- Let  $\xi = \sum_i a_i \xi_i \in \partial \mathcal{CG}(S_1) * \cdots * \partial \mathcal{CG}(S_k)$  and after reordering, assume that  $a_i > 0$  for all  $i \leq \ell$  and  $a_i = 0$  for  $i > \ell$ . A sequence of points  $(y_1^j, \ldots, y_k^j)_j \subset \mathcal{CG}(\cup_i S_i)$  converges to  $\xi$  if the following three conditions are fulfilled.
  - (1) For each  $i \leq \ell$  the components  $y_i^j \in \mathcal{CG}(S_i)$  converge as  $j \to \infty$  to  $\xi_i$  in the coarse Hausdorff topology (and hence they converge in  $\mathcal{CG}(S_i) \cup \partial \mathcal{CG}(S_i)$  to  $\xi_i$ , see [H06]). In particular, we have  $d_{\mathcal{CG}(S_i)}(y_i^j, x_i) \to \infty$   $(j \to \infty)$ .
  - (2) For all  $i \leq \ell$  denote by  $\Pi_i$  the shortest distance projection of  $\mathcal{CG}(S_i)$  onto a p-quasi-geodesic connecting the basepoint  $x_i$  to  $\xi_i$ ; then

$$\frac{d_{\mathcal{CG}(S_i)}(\Pi_i(y_i^j),x_i)}{d_{\mathcal{CG}(S_1)}(\Pi_1(y_1^j),x_1)} \to \frac{a_i}{a_1} \quad (j\to\infty).$$

(3) Let  $i > \ell$  and let  $V \subset S_i$  be any subsurface; then

$$\frac{d_{\mathcal{CG}(V)}(\operatorname{pr}_V(y_i^j),\operatorname{pr}_V(\mu))}{d_{\mathcal{CG}(S_1)}(\Pi_1(y_1^j),x_1)} \to 0 \quad (j\to\infty).$$

**Lemma 3.5.** The notion of convergence in Definition 3.4 defines a topology on  $\mathcal{Y}(\cup_i S_i)$  which restricts to the given topology on  $\partial \mathcal{CG}(S_1) * \cdots * \partial \mathcal{CG}(S_k)$  and on  $\mathcal{CG}(\cup S_i)$ . The subspace  $\partial \mathcal{CG}(S_1) * \cdots * \partial \mathcal{CG}(S_k)$  is closed in  $\mathcal{Y}(\cup_i S_i)$ .

*Proof.* Define a subset A of  $\mathcal{Y}(\cup_i S_i)$  to be *closed* if  $A_1 = A \cap \mathcal{CG}(\cup_i S_i)$  is closed,  $A_2 = A \cap \partial \mathcal{CG}(S_1) * \cdots * \partial \mathcal{CG}(S_k)$  is closed and if furthermore the following holds true. If  $y_i \subset A_1$  is a sequence which converges in the sense described above to a point  $y \in \partial \mathcal{CG}(S_1) * \cdots * \partial \mathcal{CG}(S_k)$ , then  $y \in A_2$ . Note that by definition, the empty set is closed, and the same holds true for the total space.

We have to show that complements of closed sets defined in this way fulfill the axioms of a topology, that is, they are stable under arbitrary unions and finite intersections. Equivalently, the family of closed sets is stable under arbitrary intersections and finite unions. As this holds true for the closed subsets of  $\mathcal{CG}(\cup_i S_i)$  and for the closed subsets of  $\mathcal{J}(\cup_i S_i) = \partial \mathcal{CG}(S_1) * \cdots * \partial \mathcal{CG}(S_k)$ , all we need to observe is that taking arbitrary intersections and finite unions is consistent with the notion of convergence of points in  $\mathcal{CG}(\cup_i S_i)$  to points in the join  $\partial \mathcal{CG}(S_1) * \cdots * \partial \mathcal{CG}(S_k)$  in the sense of Definition 3.4.

Consistency with arbitrary intersections is straightforward. To show consistency with taking finite unions let  $B_1, \ldots, B_\ell \subset \mathcal{Y}(\cup_i S_i)$  be closed in the above sense. Let  $y_j \subset \cup_k (B_k \cap \mathcal{CG}(\cup_i S_i))$  be any sequence which converges to a point in  $\mathcal{J}(\cup_i S_i)$  according to the definition of convergence. By passing to a subsequence, we may assume that  $y_j \in B_m$  for a fixed  $m \leq \ell$  and all j. As  $B_m$  is closed and the subsequence also fulfills the requirements for convergence, its limit is contained in  $B_m \subset \cup_k B_k$ . Hence indeed, the notion of a closed set is consistent with taking finite unions.

3.3. Projections and a topology on  $\mathcal{X}(S)$ . For a disjoint union  $\bigcup_{i=1}^k S_i$  of subsurfaces of S define now

$$\mathcal{Z}(\cup_i S_i) = \cup_I \mathcal{Y}(\cup_{i \in I} S_i) * \mathcal{J}(\cup_{i \in \{1, \dots, k\} \setminus I} S_i)$$

where I runs through all (possibly empty) subsets of the index set  $\{1, \ldots, k\}$ . Note that for every I, the space  $\mathcal{Y}(\cup_{i \in I} S_i) * \mathcal{J}(\cup_{j \in \{1, \ldots, k\} \setminus I} \cup S_j)$  contains  $\mathcal{J}(\cup_{i=1}^k S_i)$ , the join of the boundaries of the curve graphs of the surfaces  $S_i$ . Furthermore,  $\mathcal{Z}(\cup_i S_i)$  contains  $\mathcal{Y}(\cup_{i \in I} S_i)$  for all  $I \subset \{1, \ldots, k\}$ .

**Lemma 3.6.** There exists a unique separable Hausdorff topology on  $\mathcal{Z}(\cup_i S_i)$  so that a set  $U \subset \mathcal{Z}(\cup_i S_i)$  is open if and only if its intersection with each of the subspaces  $\mathcal{Y}(\cup_{i \in I} S_i) * \mathcal{J}(\cup_{j \in \{1,...,k\} \setminus I} S_j)$  is open.

*Proof.* For every  $I \subset \{1, \ldots, k\}$ , the set  $\mathcal{J}(\cup_i S_i)$  is a closed subspace of

$$\mathcal{Y}(\cup_{i\in I}S_i) * \mathcal{J}(\cup_{i\in\{1,\ldots,k\}\setminus I}S_i),$$

equipped with the topology of a join. Thus the topology described in the lemma is just the quotient topology on the quotient of the disjoint union of the spaces  $\mathcal{Y}(\cup_{i\in I}S_i)^*\mathcal{J}(\cup_{j\in\{1,...,k\}\setminus I}S_j)$  by the closed equivalence relation which identifies the points in the subspaces  $\mathcal{J}(\cup_i S_i)$ .

We next define a projection

$$\operatorname{pr}_{\mathcal{Z}(\cup S_i)}: \mathcal{X}(S) \to \mathcal{Z}(\cup_i S_i)$$

as follows. Let  $\xi = \sum_{j=1}^m a_j \xi_j \in \mathcal{X}(S)$  with  $a_j > 0$  and  $\sum_j a_j = 1$  and write as before  $\mathrm{supp}(\xi) = \bigcup_j \xi_j$ . After perhaps a reordering of the components  $\xi_j$ , assume that for some  $u \leq \min\{k,m\}$  the components  $\xi_1,\ldots,\xi_u$  fill the subsurfaces  $S_1,\ldots,S_u$ , that is, they define points in  $\partial \mathcal{CG}(S_i)$ , with the convention of remembering labels of simple closed curves, and that for no j > u, the component  $\xi_j$  fills any of the surfaces  $S_i$ . As the components of  $\mathrm{supp}(\xi)$  are disjoint, this implies that if s,t>u, if  $i \in \{u+1,\ldots,k\}$  and if the subsurface projections  $\mathrm{pr}_{S_i}(\xi_s), \mathrm{pr}_{S_i}(\xi_t)$  of  $\xi_s, \xi_t$  into  $S_i$  are not empty, then they are of uniformly bounded distance in  $\mathcal{CG}(S_i)$ . Recall that this makes sense even if  $\xi_s, \xi_t$  are different from simple closed curves.

Define

$$\operatorname{pr}_{\mathcal{Z}(\cup S_{i})}(\sum_{j=1}^{m} a_{j}\xi_{j}) = \sum_{j=1}^{u} a_{i}\xi_{j} + (1 - \sum_{j=1}^{u} a_{j})(\operatorname{pr}_{\mathcal{CG}(\cup_{i \geq u+1}S_{i})}(\cup_{j \geq u+1}\xi_{j}))$$

$$\in \mathcal{CG}(\cup_{i=u+1}^{k}S_{i}) * \mathcal{J}(\cup_{i=1}^{u}S_{i}).$$

Here the term on the right hand side is understood in the following sense. First, if one of the surfaces  $S_i$   $(i \leq u)$  is an annulus then the label of  $\xi_i$  is remembered in  $\operatorname{pr}_{\mathcal{Z}(\cup_i S_i)}(\sum_j a_j \xi_j)$ . Second, for some  $\ell \in \{u+1,\ldots,k\}$  let us consider the subsurface  $S_\ell$ . If there exists some s > u such that  $\xi_s$  intersects  $S_\ell$ , then the component in  $S_\ell$  of the projection  $\operatorname{pr}_{\mathcal{CG}(\cup_{i\geq u+1}S_i)}(\cup_{j\geq u+1}\xi_j)$  is a point in  $\mathcal{CG}(S_\ell)$ . Although this projection depends on choices, it is coarsely well defined, that is, well defined up to a uniformly bounded error. If the lamination  $\operatorname{supp}(\xi) = \cup_i \xi_i$  is disjoint from the subsurface  $S_\ell$ , then the projection component is defined to be the basepoint of  $\mathcal{CG}(S_\ell)$  constructed from the base marking.

**Requirement 2:** A sequence  $\xi^j = \sum_m a_m^j \xi_m^j \subset \mathcal{X}(S)$  converges to  $\zeta = \sum_{i=1}^k b_i \zeta_i \in \mathcal{X}(S)$  if the following holds true. Put  $S_{k+1} = S \setminus \bigcup_{i=1}^k$ ; then

$$\operatorname{pr}_{\mathcal{Z}(\bigcup_{i=1}^{k+1} S_i)}(\xi^{j_s}) \to \zeta \text{ in } \mathcal{Z}(\bigcup_{i=1}^{k+1} S_i) \supset \mathcal{J}(\bigcup_{i=1}^{k+1} S_i) \supset \mathcal{J}(\bigcup_{i\leq k} S_i).$$

**Remark 3.7.** It follows from the above description that for this notion of convergence, the following holds true. Let  $\xi^j$  be a sequence in  $\mathcal{X}(S)$  consisting of minimal geodesic laminations which converges to a point  $\zeta = \sum_u b_u \zeta_u$ .

- (a) The lamination  $\operatorname{supp}(\zeta)$  is a sublamination of the limit in the Hausdorff topology of any convergent subsequence of the sequence  $\operatorname{supp}(\xi^j)$ .
- (b) For each j let  $\eta^j$  be a minimal geodesic lamination disjoint from  $\xi^j$  (we allow  $\eta^j = \xi^j$ ) and let  $s_i \in [0,1]$ . Then any limit of a convergent subsequence of the sequence  $\nu^j = s_i \xi^j + (1-s_i)\eta^j$  is of the form  $s\zeta + (1-s)\eta$  where  $\eta$  is a limit of a subsequence of the sequence  $\eta^j$  and where  $s \in [0,1]$ .

**Definition 3.8.** A subset  $A \subset \mathcal{X}(S)$  is called *closed for the geometric topology of*  $\mathcal{X}(S)$  if the following holds true. Let  $\xi_i \subset A$  be any sequence which converges to a point  $\xi \in \mathcal{X}(S)$  in the sense described by the requirements (1),(2); then  $\xi \in A$ .

An embedding of a topological space X into a topological space Y is an injective map  $f: X \to Y$  which is a homeomorphism onto its image, equipped with the subspace topology. Recall that for any collection  $S_1, \ldots, S_k$  of pairwise disjoint subsurfaces of S, the space  $\mathcal{J}(\bigcup_{i=1}^k S_i)$  is equipped with a natural topology as a join of the Gromov boundaries of the curve graphs of the subsurfaces  $S_i$ . The following statement is the first main step towards the proof of Theorem 4.

**Proposition 3.9.** (1) Closed subsets of  $\mathcal{X}(S)$  in the sense of Definition 3.8 define a separable Hausdorff topology  $\mathcal{O}$  on  $\mathcal{X}(S)$ .

(2) For any collection  $S_1, \ldots, S_k$  of pairwise disjoint subsurfaces, the natural inclusion  $\mathcal{J}(\bigcup_{i=1}^k S_i) \to (\mathcal{X}(S), \mathcal{O})$  is an embedding.

*Proof.* Let  $\mathcal{O} \subset \mathcal{X}(S)$  be the family of all subsets of  $\mathcal{X}(S)$  whose complement is closed in the above sense. Sets in  $\mathcal{O}$  are called *open*. We have to show that  $\mathcal{O}$  defines a topology on  $\mathcal{X}(S)$ .

As the empty set and the entire space  $\mathcal{X}(S)$  are open, to show that  $\mathcal{O}$  is indeed a topology on  $\mathcal{X}(S)$  it suffices to show that arbitrary unions of open sets are open, and that finite intersections of open sets are open as well. Or, equivalently, arbitrary intersections of closed sets are closed, and finite unions of closed sets are closed. This can be established using exactly the same reasoning as in the proof of Lemma 3.5.

Namely, that the collection of closed sets is stable under arbitrary intersections is immediate from the definition. So let  $B_1, \ldots, B_k$  be closed sets and let  $B = \cup_i B_i$ . Choose a sequence  $\xi_i \subset B$  which converges in the sense of requirements (1)-(3) to some point  $\zeta$ . By passing to a subsequence, we may assume that  $\xi_i \in B_\ell$  for some  $\ell \leq k$  and all i. But then  $\zeta \in B_\ell \subset B$  as  $B_\ell$  is closed which completes the proof that  $\mathcal{O}$  is indeed a topology on  $\mathcal{O}$ .

We show next the second property claimed in the proposition. Thus let  $S_1,\ldots,S_k$  be a collection of pairwise disjoint subsurfaces of S. Our goal is to show that the inclusion  $\mathcal{J}(\cup_{i=1}^k S_i) \to (\mathcal{X}(S), \mathcal{O})$  is an embedding. Since the inclusion is injective, and  $\mathcal{J}(\cup_{i=1}^k S_i)$  is a separable Hausdorff space, for this it suffices to show that the inclusion is continuous and its image is locally closed. This is equivalent to stating that a sequence  $\xi^j = \sum_i a_i^j \xi_i^j \subset \mathcal{J}(\cup_{i=1}^k S_i)$  converges in  $(\mathcal{X}(S), \mathcal{O})$  to a point  $\zeta \in \mathcal{J}(\cup_{i=1}^k S_i)$  if and only if  $\xi^j$  converges in  $\mathcal{J}(\cup_{i=1}^k S_i)$  to  $\zeta$ . However, putting  $S_{k+1} = S \setminus \cup_i S_i$ , this is immediate from the definition of the topology on  $\mathcal{Z}(\cup_{i=1}^{k+1} S_i)$  and the second requirement in the definition of convergence in  $\mathcal{X}(S)$  and shows the second part of the proposition.

Since each of the spaces  $\mathcal{J}(\cup_{i=1}^k S_i)$  is a finite join of separable metrizable spaces (namely, the Gromov boundary of a curve graph of a subsurface of S) and hence separable metrizable, the second part of the proposition implies that  $(\mathcal{X}(S), \mathcal{O})$  is a countable union of (in general not disjoint) separable metrizable spaces and hence is separable.

To show that the topology is Hausdorff let  $\xi = \sum_i a_i \xi_i \neq \zeta = \sum_j b_j \zeta_j \in \mathcal{X}(S)$ . We have to show that  $\xi, \zeta$  have disjoint neighborhoods.

If this is not the case, then any neighborhoods  $U_{\xi}$  of  $\xi$  and  $U_{\zeta}$  of  $\zeta$  intersect nontrivially. Since  $\mathcal{X}(S)$  is separable, and since points in  $\mathcal{X}(S)$  are closed by construction, we conclude that there is a sequence  $\xi^{j} \subset \mathcal{X}(S)$  which converges both to  $\xi, \zeta$ . But for the notion of convergence used to define the topology  $\mathcal{O}$ , the limit of a converging sequence is unique. Thus  $\mathcal{O}$  is indeed Hausdorff which completes the proof the first part of the proposition.

**Example 3.10.** i) Let  $\varphi \in \operatorname{Mod}(S)$  be a pseudo-Anosov element. Then  $\varphi$  acts as a loxodromic isometry on the curve graph of S, with attracting and repelling fixed points  $\nu_+, \nu_- \in \partial \mathcal{CG}(S)$ . Let  $\mu \in \mathcal{X}(S)$  be any minimal geodesic lamination which is distinct from the repelling fixed point  $\nu_-$  of  $\varphi$ . Then  $\varphi^j \mu \to \nu_+$   $(j \to \infty)$  in the coarse Hausdorff topology and therefore  $\varphi^j \mu \to \nu_+$  in  $\mathcal{X}(S)$ .

ii) Now let us assume that  $S_0 \subset S$  is a proper connected subsurface different from an annulus and a pair of pants and that  $\varphi \in \operatorname{Mod}(S)$  restricts to a pseudo-Anosov mapping class on  $S_0$  and to the trivial mapping class on  $S-S_0$ . Let  $\nu_+ \in \partial \mathcal{CG}(S_0)$  be the attracting geodesic lamination for the action of  $\varphi$  on  $S_0$ . Let furthermore  $\mu \neq \nu_- \in \mathcal{X}(S)$  be any minimal geodesic lamination on S which is different from the repelling fixed point  $\nu_-$  for the action of  $\varphi$  on  $\mathcal{CG}(S_0)$ . Then there are two possibilities. In the first case,  $\mu$  is supported in  $S-S_0$ . Then we have  $\varphi^j(\mu) = \mu$  for all j. However, if  $\mu$  intersects  $S_0$ , then either  $\mu = \nu_{\pm}$  or  $\mu$  intersects  $\nu_+$  and we have  $\varphi^j(\mu) \to \nu_+$  ( $j \to \infty$ ) in  $\mathcal{X}(S)$ .

Namely, if  $\mu$  intersects  $S_0$  then the subsurface projection of  $\mu$  into any subsurface disjoint from  $S_0$  is a collection of arcs intersecting  $\partial S_0$ . In particular, the subsurface projection into any subsurface V of  $S-S_0$  is a point of  $\mathcal{CG}(V)$ . Since  $\varphi$  can be represented by a diffeomorphism which fixes  $S-S_0$  pointwise, it acts trivially on  $\mathcal{CG}(V)$  which yields the above statement.

Corollary 3.11.  $(\mathcal{X}(S), \mathcal{O})$  is a Lindelöf space.

Proof. We have to show that any open cover of  $\mathcal{X}(S)$  has a countable subcover. To this end let  $\mathcal{U}$  be such an open cover. List the countably many spaces  $\mathcal{J}(\cup_i S_i)$  as  $\mathcal{J}_1, \mathcal{J}_2, \ldots$  Since for each i, the space  $\mathcal{J}_i$  is separable and metrizable, the restriction of  $\mathcal{U}$  to  $\mathcal{J}_i$ , which is an open covering of  $\mathcal{J}_i$ , has a countable subcover, say by sets  $U_i^1, U_i^2, \ldots$  The standard diagonal argument shows that the union  $\mathcal{V} = \cup_{i,j} U_i^j$  consists of countably many sets, and for each i, the sets from  $\mathcal{V}$  cover  $\mathcal{J}_i$ . Since  $\mathcal{X}(S) = \cup_i \mathcal{J}_i$  (as a set), this shows that  $\mathcal{V}$  is a countable subcover of the cover  $\mathcal{U}$ . In other words,  $\mathcal{X}(S)$  is a Lindelöf space as claimed.

**Proposition 3.12.**  $(\mathcal{X}(S), \mathcal{O})$  is compact.

*Proof.* As by Corollary 3.11, the space  $\mathcal{X}(S)$  is a separable Lindelöf space, moreover it is Hausdorff by Proposition 3.9, to show that  $\mathcal{X}(S)$  is compact it suffices to show that  $\mathcal{X}(S)$  is sequentially compact.

Thus let  $\xi^j = \sum_i a_i^j \xi_i^j \subset \mathcal{X}(S)$  be any sequence. We have to construct a convergent subsequence. Since the space of geodesic laminations equipped with the Hausdorff topology is compact, by passing to a subsequence we may assume that the geodesic laminations  $\sup(\xi^j) = \bigcup_i \xi_i^j$  converge in the Hausdorff topology to a geodesic lamination  $\hat{\zeta}$  with minimal components  $\zeta_1, \ldots, \zeta_k$ .

For each  $i \leq k$  let  $S_i \subset S$  be the subsurface of S filled by  $\zeta_i$ . Assume by passing to a subsequence that

$$\xi^{j} = \sum_{i=1}^{u} a_{i}^{j} \xi_{i}^{\pm} + \sum_{\ell > u} a_{\ell}^{j} \xi_{\ell}^{j}$$

for all j where for each  $i \leq u$ , the component  $\xi_i^j$  fills  $S_i$  and that none of the components  $\xi_s^j$  for s > u fills any of the surfaces  $S_i$ . By passing to another subsequence, we may assume that for  $i \leq u$ , the labels  $\pm$  of the components  $\zeta_i$  are constant along the sequence, and that the weights  $a_i^j \in (0,1]$  of the components  $\zeta_i$  converge to weights  $b_i \geq 0$ . In particular, the sums  $1 - \sum_{i \leq u} a_i^j$  converge to  $1 - \sum_{i \leq u} b_i = \kappa$ .

Since  $\operatorname{supp}(\xi^j) \to \hat{\zeta}$  in the Hausdorff topology, we know that for each  $i \leq u$ , the laminations  $\xi_i^j$  converge in the coarse Hausdorff topology to  $\zeta_i$  and hence  $\xi_i^j$  converges to  $\zeta_i$  in  $\partial \mathcal{CG}(S_i)$ . Thus if  $\kappa = 0$  then by the definition of the topology on  $\mathcal{X}(S)$ , we know that  $\xi^j \to \sum_{i=1}^u b_i \zeta_i$  and we are done. Thus we are left with the case  $\kappa > 0$ . Moreover, viewing  $\xi^j = (\sum_{i \leq u} a_i^j \zeta_i) + (\sum_{i \geq u+1} a_i^j \xi_i^j)$  as points in the join of two subspaces of  $\mathcal{X}(S)$ , using the above argument it now suffices to assume that for no j there exists a component of  $\operatorname{supp}(\xi^j)$  which fills any of the subsurfaces  $S_i$ .

Then for each i, we can consider the subsurface projection  $\operatorname{pr}_{S_i}(\operatorname{supp}(\xi^j))$  of  $\operatorname{supp}(\xi^j)$  into the surface  $S_i$ . Furthermore, by passing to another subsequence, we may assume that for all j and all  $i \leq k$ , this subsurface projection is non-empty since the geodesic lamination  $\zeta_i$  which fills  $S_i$  is contained in the limit with respect to the Hausdorff topology of the sequence of laminations  $\operatorname{supp}(\xi^j)$ . Put differently, we may assume that for each i and all j, the subsurface projection  $\operatorname{pr}_{S_i}(\operatorname{supp}(\xi^j))$  of the lamination  $\operatorname{supp}(\xi^j)$  into the subsurface  $S_i$  is a coarsely well defined point in  $\mathcal{CG}(S_i)$ . Furthermore, using once more that  $\zeta_i$  fills  $S_i$  and that  $\zeta_i$  is contained in

the Hausdorff limit of the sequence  $\operatorname{supp}(\xi^j)$ , if we denote by  $x_i$  the fixed basepoint in  $\mathcal{CG}(S_i)$ , then we know that  $d_{\mathcal{CG}(S_i)}(\operatorname{pr}_{S_i}(\operatorname{supp}(\xi^j)), x_i) \to \infty \ (j \to \infty)$ .

By passing to another subsequence and reordering indices, we may assume that

$$a_1^j = d_{\mathcal{CG}(S_1)}(\operatorname{pr}_{S_1}(\operatorname{supp}(\xi^j)), x_1) \ge a_i^j = d_{\mathcal{CG}(S_i)}(\operatorname{pr}_{S_i}(\operatorname{supp}(\xi^j)), x_i)$$

for all  $i \geq 2$  and all j. Passing to another subsequence, we may assume furthermore that  $a_i^j/a_1^j \to a_i \in [0,1]$  for all  $i \geq 2$ . Put  $a_1 = 1$ ; then we have  $\sum_u a_u \geq 1$  and hence defining  $b_i = a_i/\sum_u a_u > 0$ , we conclude that  $\sum_u b_u = 1$ . It now follows from the definition of the topology on  $\mathcal{X}(S)$  that  $\xi^j \to \sum_i b_i \zeta_i$ . This completes the proof that  $\mathcal{X}(S)$  is sequentially compact.

**Lemma 3.13.** Mod(S) acts on  $\mathcal{X}(S)$  as a group of transformations.

*Proof.* Observe first that by construction, Mod(S) acts on  $\mathcal{X}(S)$  as a group of bijections (equivalently, transformations for the discrete topology). Thus it suffices to show that this action is continuous for the topology  $\mathcal{O}$ .

By the definition of  $\mathcal{O}$ , for this it suffices to show the following. Let  $\xi^j$  be a sequence converging for the topology  $\mathcal{O}$  to a point  $\xi$ . Then for every  $\varphi \in \operatorname{Mod}(S)$ , the sequence  $\varphi(\xi^j)$  converges to  $\varphi(\xi)$ .

That the first defining requirement for convergence is passed on to the image sequence follows from continuity of the action of  $\varphi$  on the space of geodesic laminations, equipped with the Hausdorff topology.

For the second requirement, if  $S_1, \ldots, S_k$  is a partition of S into disjoint subsurfaces, then the same holds true for  $\varphi(S_1), \ldots, \varphi(S_k)$ , and for any geodesic lamination  $\nu$ , we have  $\operatorname{pr}_{\mathcal{Y}(\cup_i \varphi(S_i))}(\varphi(\nu)) = \varphi(\operatorname{pr}_{\mathcal{Y}(\cup_i S_i)}(\nu))$  up to replacing the basepoints  $y_i$  of  $\mathcal{CG}(\varphi(S_i))$  by  $\varphi(x_i)$ . Note that  $\varphi$  also naturally acts on orientations of simple closed curves on S as no oriented simple closed curve on S is freely homotopic to its inverse and hence  $\varphi$  acts on labelled simple closed curves. As for all i, we have  $d_{\mathcal{CG}(\varphi(S_i))}(\operatorname{pr}_{\varphi(S_i)}(\xi^j), \varphi(x_i)) = d_{\mathcal{CG}(S_i)}(\xi^j, x_i) \to \infty \ (j \to \infty)$  and the determination of the weights of the limit points are computed using ratios of distances to the basepoint defined by subsurface projections, with the distances tending to infinity along the sequence, we conclude that the second requirement in the definition of convergence is fulfilled for  $\varphi(\xi^i)$  if it is fulfilled for  $\xi^i$ . The same reasoning also applies to the third requirement. Thus indeed,  $\operatorname{Mod}(S)$  acts on  $\mathcal{X}(S)$  as a group of transformations and shows the lemma.

**Definition 3.14.** The space  $(\mathcal{X}(S), \mathcal{O})$  is called the *geometric boundary* of Mod(S).

Let us note another naturality property of the geometric boundary of  $\operatorname{Mod}(S)$ . Namely, if  $S_0 \subset S$  is any essential subsuface, then we can construct a geometric boundary  $\mathcal{X}(S_0)$  for the mapping class group  $\operatorname{Mod}(S_0)$  of isotopy classes of homeomorphisms of  $S_0$  fixing the boundary pointwise. As a set, this is a subset of the geometric boundary of S which includes the Gromov boundary of the curve graph for peripheral annuli. The above construction immediately yields

**Corollary 3.15.** If  $S_0 \subset S$  is any subsurface of S, then the geometric boundary of  $Mod(S_0)$  is a closed subspace of the geometric boundary of Mod(S).

### 4. A SMALL BOUNDARY FOR Mod(S)

Define the complexity  $\kappa(S)$  of a connected surface of genus  $g \geq 0$  with  $m \geq 0$  holes (which can be boundary components or punctures) as

$$\kappa(S) = 3g - 3 + m$$

if S is not a sphere with two holes, e.g. an annulus. If S is an annulus then define  $\kappa(S) = 0$ . If  $S = \bigsqcup_{i=1}^m S_i$  is a disjoint union of connected surfaces  $S_i$  then define  $\kappa(S) = \sum_i \kappa(S_i)$ .

The purpose of this section is to set up an inductive procedure over the complexity of the surface S to construct a topology on  $\overline{\mathcal{T}}(S) = \mathcal{T}_{\epsilon}(S) \cup \mathcal{X}(S)$  which restricts to the given topologies on  $\mathcal{T}_{\epsilon}(S)$  and on  $\mathcal{X}(S)$  and such that with respect to this topology,  $\overline{\mathcal{T}}(S)$  is a compact Mod(S)-space. The construction of this topology is carried out using Teichmüller geometry.

Remark 4.1. In [DHS17], there is a construction of a topology on  $Mod(S) \cup \mathcal{X}(S)$  (where however the topology on  $\mathcal{X}(S)$  differs from the one we introduced) using the combinatorics of hierarchical hyperbolic spaces and such that with respect to this topology,  $Mod(S) \cup \mathcal{X}(S)$  is compact. It is possible that the notion of consistence we introduce below is related to the construction in [DHS17], but it is geared at capturing features of Mod(S) which resemble properties of a CAT(0) group and is not well adapted to hierarchical hyperbolicity, see however [DMS25].

4.1. The thick part of Teichmüller space. By the collar lemma for hyperbolic surfaces, there exists a number  $\epsilon_0 > 0$  with the following property. For any closed hyperbolic surface  $\Sigma_g$  of genus  $g \geq 2$ , any two closed geodesics  $\gamma_1, \gamma_2$  on  $\Sigma_g$  of length  $\ell(\gamma_1), \ell(\gamma_2) \leq \epsilon_0$  are disjoint.

Let syst:  $\mathcal{T}(S) \to (0, \infty)$  be the *systole function* which associates to a point in  $\mathcal{T}(S)$  its systole, that is, the shortest length of a closed geodesic on S. For  $\epsilon \leq \epsilon_0$  define the  $\epsilon$ -thick part  $\mathcal{T}_{\epsilon}(S)$  of the Teichmüller space  $\mathcal{T}(S)$  of marked hyperbolic metrics on S by

$$\mathcal{T}_{\epsilon}(S) = \{ X \in \mathcal{T}(S) \mid \operatorname{syst}(X) \ge \epsilon \}.$$

The following statement is well known. We refer to Proposition 1.1 of [JW10] for an explicit account.

**Theorem 4.2.** For  $\epsilon < \epsilon_0$ , the following holds.

- (1) The subspace  $\mathcal{T}_{\epsilon}(S) \subset \mathcal{T}(S)$  is non-empty, closed, connected and stable under  $\operatorname{Mod}(S)$ . Its quotient under the action of  $\operatorname{Mod}(S)$  is compact.
- (2)  $\mathcal{T}_{\epsilon}(S)$  is a real-analytic manifold with corners and hence admits a  $\operatorname{Mod}(S)$ invariant triangulation such that  $\operatorname{Mod}(S) \setminus \mathcal{T}_{\epsilon}(S)$  is a finite CW-complex.

As a consequence,  $\mathcal{T}_{\epsilon}(S)$  is a topological manifold with boundary and interior  $\mathring{\mathcal{T}}_{\epsilon}(S) = \{X \mid \operatorname{syst}(X) > \epsilon\} \subset \mathcal{T}_{\epsilon}(S)$ .

There is a coarsely well defined map

$$\Upsilon: \mathcal{T}(S) \to \mathcal{CG}(S)$$

which maps a marked hyperbolic metric to a closed non-contractible curve of minimal length. Coarsely well defined means that the map depends on choices, but the images of a point  $X \in \mathcal{T}(S)$  for any two choices of such a map are of distance at most two.

Call a map  $\Psi: \mathcal{T}(S) \to \mathcal{T}(S)$  coarsely  $\Upsilon$ -invariant if  $d(\Upsilon(\Psi(X)), \Upsilon(X)) \leq 2$  for all X. The following is due to Ivanov [Iv02] if one replaces the mapping class group by a torsion free subgroup. The full version is Theorem 1.2 of [JW10], see also Theorem 3.9 of [J14].

**Theorem 4.3** (Ji-Wolpert). For  $\epsilon < \epsilon_0/3$  there is a Mod(S)-equivariant coarsely  $\Upsilon$ -invariant deformation retraction  $\mathcal{T}(S) \to \mathcal{T}_{\epsilon}(S)$ .

The deformation retraction is constructed as follows. First Ji and Wolpert construct a Mod(S)-invariant continuous uniquely integrable vector field V on  $\mathcal{T}(S)$  with the following properties (p.9 of [JW10]).

- (1) V(syst) = 1 on  $\{\text{syst} \le 2\epsilon\}$  and
- (2) V vanishes on  $\{\text{syst} \geq 3\epsilon\}$ .

The deformation retraction is then given by the time  $\epsilon$ -map of the flow defined by V. Note however that the image of  $\mathcal{T}(S)$  under this map is the interior of  $\mathcal{T}_{\epsilon}(S)$ . Since the time  $\epsilon$  map of a continuous flow is a homeomorphism, we obtain the following statement as an immediate consequence.

Corollary 4.4. For every  $\epsilon < \epsilon_0/3$  there is a Mod(S)-equivariant homeomorphism  $\Lambda_{\epsilon} : \mathcal{T}(S) \to \mathring{\mathcal{T}}_{\epsilon}(S)$ .

For our purpose, the difficulty arises that we need to construct contractible subsets of  $\mathcal{T}_{\epsilon}(S)$  and not of its interior. But the closure of a contractible open set in a smooth manifold may not be contractible. The following construction will allow us to address this issue.

Define the small closure  $\overline{A}_{\text{small}}$  of a subset A of  $\mathring{\mathcal{T}}_{\epsilon}(S)$  to be the union of A with the set of all points  $z \in \partial \mathcal{T}_{\epsilon}(S)$  so that z has a neighborhood U in  $\mathcal{T}_{\epsilon}(S)$  with  $U \cap (\mathcal{T}_{\epsilon}(S) \setminus \partial \mathcal{T}_{\epsilon}(S)) \subset A$ . Note that  $\overline{A}_{\text{small}} \setminus A$  is an *open* subset of  $\partial \mathcal{T}_{\epsilon}(S)$ . More precisely, we have.

**Lemma 4.5.** (1) The small closure in  $\mathcal{T}_{\epsilon}(S)$  of an open subset of  $\mathring{\mathcal{T}}_{\epsilon}(S)$  is open in  $\mathcal{T}_{\epsilon}(S)$ .

(2) If 
$$U \subset \mathcal{T}_{\epsilon}(S)$$
 is open, then  $U \subset \overline{U \cap \mathring{\mathcal{T}}_{\epsilon}(S)}_{\text{small}}$ .

*Proof.* If  $U \subset \mathring{\mathcal{T}}_{\epsilon}(S)$  is open, then as  $\mathring{\mathcal{T}}_{\epsilon}(S) \subset \mathcal{T}_{\epsilon}(S)$  is open, a point  $x \in U \subset \overline{U}_{\text{small}}$  has a neighborhood in  $\mathcal{T}_{\epsilon}(S)$  which is contained in U.

On the other hand, if  $x \in \overline{U}_{\text{small}} \setminus U$  then it follows from the definition of  $\overline{U}_{\text{small}}$  that x has a neighborhood in  $\mathcal{T}_{\epsilon}(S)$  entirely contained in  $\overline{U}_{\text{small}}$ . This shows the first part of the lemma.

The second part of the lemma is immediate from the definitions.

**Lemma 4.6.** The small closure of a contractible subset of  $\mathcal{T}_{\epsilon}(S)$  is contractible.

*Proof.* It suffices to deformation retract the small closure  $\overline{A}_{\text{small}}$  of a contractible subset A of  $\mathcal{T}_{\epsilon}(S)$  into A. The composition of this deformation retraction with a deformation retraction of A to a point then shows that  $\overline{A}_{\text{small}}$  is contractible.

Since  $\mathcal{T}_{\epsilon}(S) \subset \mathcal{T}(S)$  is a manifold with corners, for  $z \in \overline{A}_{\mathrm{small}} \setminus A$  there is a neighborhood of z in  $\overline{A}_{\mathrm{small}}$  which is homeomorphic to the set  $B_0 = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_i x_i^2 < 1, x_1 \geq 0\}$ , with z corresponding to 0, and such that  $B_0 \cap \{x_1 > 0\} \subset A$ . There is a deformation retraction of  $B_0$  into  $B_0 \setminus V$  where V is a small neighborhood of 0 and such that the following two additional properties are satisfied. The time one map of the deformation retraction is a homeomorphism onto its image, and the support of this deformation retraction is contained in  $\sum_i x_i^2 < \frac{1}{2}$ . Thus for every  $z \in \overline{A}_{\mathrm{small}} \setminus A$  there is a deformation retraction of  $\overline{A}_{\mathrm{small}}$  which moves a neighborhood of z in  $\overline{A}_{\mathrm{small}}$  into A and such that the intersection of the resulting set with  $\partial \mathcal{T}_{\epsilon}(S)$  is contained in the complement of a neighborhood of z in the intersection of  $\overline{A}_{\mathrm{small}}$  with  $\partial \mathcal{T}_{\epsilon}(S)$ .

Each compact subset K of  $\overline{A}_{small} \setminus A \subset \partial \mathcal{T}_{\epsilon}(S)$  can be covered by finitely many open sets in  $\overline{A}_{small}$  which admit a deformation retraction into A. As the composition of finitely many deformation retractions of  $\overline{A}_{small}$  is a deformation retraction, there is a deformation retraction  $\alpha$  of  $\overline{A}_{small}$  with  $\alpha(\overline{A}_{small}) \cap (\overline{A}_{small} \setminus A) \subset \overline{A}_{small} \setminus K$ . By induction and using the fact that  $\overline{A}_{small} \setminus A$  is an open subset of  $\partial \mathcal{T}_{\epsilon}(S)$  and hence has a countable basis, this implies that  $\overline{A}_{small}$  admits a deformation retraction into A. From this the lemma follows.

4.2. A topology on  $\mathcal{T}_{\epsilon}(S) \cup \mathcal{X}(S)$ . The construction of a topology on  $\overline{\mathcal{T}}(S) = \mathcal{T}_{\epsilon}(S) \cup \mathcal{X}(S)$  uses induction on the complexity  $\kappa(S)$  of the surface S. We begin with having a short look at annuli, which are connected surfaces with  $\kappa = 0$ . The only other connected surface S with  $\kappa(S) = 0$  is the sphere with three holes which plays no role for us as its mapping class group has a finite index subgroup which is a direct product of the mapping class groups of its boundary annuli.

**Example 4.7.** In the case S is an annulus, then we have  $\mathcal{T}(S) = \mathbb{R}$ ,  $\mathcal{X}(S) = \{+, -\}$ . If we equip  $\overline{\mathcal{T}}(S)$  with the topology of the compactification of  $\mathbb{R}$  which is homeomorphic to a compact interval and is obtained by attaching two points  $\pm \infty$ , then this construction defines an  $\mathcal{EZ}$ -structure for the infinite cyclic group of Dehn twists along the core curve of the annulus.

Assume now that  $\kappa(S) \geq 1$ . Then the Teichmüller space  $\mathcal{T}(S)$  of S is of dimension at least two. There exists a constant  $\rho = \rho(S) > \epsilon$ , a so-called *Bers constant*, such that any marked hyperbolic surface  $X \in \mathcal{T}(S)$  admits a pants decomposition by simple closed curves of X-length at most  $\rho$  [Bu92]. If  $X \in \mathcal{T}_{\epsilon}(S)$ , then by possibly enlarging  $\rho$ , we may in fact assume that X admits a marking  $\mu(X)$  consisting of simple closed curves of length at most  $\rho$ . We call such a marking *short* for X.

By the collar lemma [Bu92], the geometric intersection number between any two simple closed curves on S of X-length at most  $\rho$  is bounded from above by a

universal constant. In particular, the marking  $\mu(X)$  defines a subset of uniformly bounded diameter in  $\mathcal{CG}(S)$  (see [MM99] for more information).

The marking curves from the marking  $\mu(X)$  decompose S into disks. Thus for every proper essential not necessarily connected subsurface  $S_0$  of S, the subsurface projections of the marking curves from  $\mu(X)$  decompose S into disks and hence they coarsely define a marking  $\operatorname{pr}_{S_0}(\mu(X))$  of  $S_0$ . Here as before, the marking depends on choices, but any two markings obtained in this way are of uniformly bounded distance in the marking graph.

For a proper essential subsurface  $S_0$  of S denote by  $S_0^*$  the finite type surface obtained from  $S_0$  by replacing each boundary component by a puncture.

**Lemma 4.8.** For every proper essential subsurface  $S_0$  of S there exists a natural coarsely well defined projection  $\operatorname{pr}_{S_0^*}: \mathcal{T}_{\epsilon}(S) \to \mathcal{T}_{\epsilon}(S_0^*)$ .

Proof. Choose for  $X \in \mathcal{T}_{\epsilon}(S)$  a short marking  $\mu(X)$  of S. We saw above that  $\mu(X)$  coarsely projects to a marking of  $S_0$  and hence of  $S_0^*$ . On the other hand, by proper discontinuity of the action of  $\operatorname{Mod}(S_0^*)$  on  $\mathcal{T}_{\epsilon}(S_0^*)$  and cocompactness, given a point  $\operatorname{pr}_{S_0^*}(X) \subset \mathcal{T}_{\epsilon}(S_0^*)$  for which this marking is short, any other such point is contained in a uniformly bounded neighborhood of  $\operatorname{pr}_{S_0^*}(X)$ . This construction thus defines a projection as claimed in the lemma.

The above lemma also is valid in the case that  $S_0$  is an annulus, where  $S_0^* = S_0$  and as before,  $\mathcal{T}(S_0) = \mathcal{T}_{\epsilon}(S_0^*) = \mathbb{R}$ . Note that there is a natural identification  $\mathcal{X}(S_0) = \mathcal{X}(S_0^*) * \mathcal{J}(\partial S_0)$  where  $\mathcal{J}(\partial S_0)$  is a join of two element spaces, one for each boundary component of  $S_0$ , so we can view  $\mathcal{X}(S_0^*)$  as a subspace of  $\mathcal{X}(S)$ .

**Definition 4.9.** Let  $S_0 \subset S$  be a proper not necessarily connected subsurface which may have components which are annuli. A topology on  $\overline{\mathcal{T}}(S)$  is called *consistent* with a topology on  $\overline{\mathcal{T}}(S_0^*)$  if the following holds true. If  $\xi \in \mathcal{X}(S_0^*) \subset \mathcal{X}(S)$  then a sequence  $X_i \subset \mathcal{T}_{\epsilon}(S)$  converges to  $\xi$  if  $\operatorname{pr}_{S_0^*}(X_i) \to \xi$  in  $\overline{\mathcal{T}}(S_0^*)$ . The topology is called *consistent* if it is consistent with a family of topologies on  $\overline{\mathcal{T}}(S_0^*)$  for every proper subsurface  $S_0$  of S.

Note that if  $S_2 \subset S_1 \subset S$  are nested essential subsurfaces, then by construction, for every  $X \in \mathcal{T}_{\epsilon}(S)$  the identity  $\operatorname{pr}_{S_2^*}(X) = \operatorname{pr}_{S_2^*}(\operatorname{pr}_{S_1^*}(X))$  coarsely holds. As a consequence, if the topology on  $\overline{\mathcal{T}}(S)$  is consistent, then the same holds true for the topologies of  $\overline{\mathcal{T}}(S_0^*)$  for every essential subsurface  $S_0$  of S.

The following definition is geared at overcoming some purely technical difficulties in the construction of an  $\mathcal{EZ}$ -structure for  $\operatorname{Mod}(S)$ .

**Definition 4.10.** A topology on  $\overline{\mathcal{T}}(S)$  is called *nice* if every point  $\xi \in \mathcal{X}(S)$  has a neighborhood basis consisting of sets  $U_{\xi}$  so that  $U_{\xi} \cap \mathring{\mathcal{T}}_{\epsilon}(S)$  is open and contractible.

**Example 4.11.** If S is a once punctured torus or a four-holed sphere, then we saw in Example 3.3 that  $\mathcal{X}(S)$  has a natural identification with the Gromov boundary  $\partial \text{Mod}(S)$  of the mapping class group of S. Recall that  $\mathcal{X}(S)$  is a disjoint union of the set  $\partial \mathcal{CG}(S)$  of minimal filling geodesic laminations with the set of all labeled simple closed curves.

Since  $\operatorname{Mod}(S)$  is a hyperbolic group which acts properly and cocompactly on  $\mathcal{T}_{\epsilon}(S)$ , the space  $\overline{\mathcal{T}}(S)$  has a natural topology which is inherited from the topology of the union of  $\operatorname{Mod}(S)$  with its Gromov boundary. The restrictions of this topology to the subsets  $\mathcal{T}_{\epsilon}(S)$  and  $\mathcal{X}(S)$  are the given topologies.

Any proper essential subsurface of S is an annulus. Let  $A \subset S$  be such an annulus. We claim that the topology of  $\overline{T}(S)$  is consistent with the topology of the compactification  $\mathbb{R} \cup \{\pm \infty\}$  of the Teichmüller space T(A).

Namely, let c be the core curve of A and let  $c_+ \in \mathcal{X}(S)$  be the curve c equipped with a label. Denote by  $\langle T_c \rangle$  the infinite cyclic group of Dehn twists about c and assume that  $c_+$  corresponds to the limit point of the sequence  $T_c^k$  as  $k \to \infty$ . Note that  $\langle T_c \rangle$  is a quasi-convex subgroup of  $\operatorname{Mod}(S)$ . Let  $X \in \mathcal{T}_{\epsilon}(S)$  be an arbitrary point. With respect to the topology of  $\operatorname{Mod}(S) \cup \partial \operatorname{Mod}(S)$  as the union of a hyperbolic group with its Gromov boundary, a sequence of points  $X_i = \varphi_i(X) \subset \mathcal{T}_{\epsilon}(S)$  for  $\varphi_i \in \operatorname{Mod}(S)$  converges to  $c_+ \in \mathcal{X}(S)$  if the shortest distance projections of the elements  $\varphi_i$  into the quasi-convex infinite cyclic subgroup  $\langle T_c \rangle$  converge to  $c_+$ . Translated into properties of the subsurface projections of points in the Farey graph, this just means that the topology on  $\overline{\mathcal{T}}(S)$  is consistent.

We can also check that the topology is nice. Namely, recall that  $\mathcal{T}_{\epsilon}(S)$  can be identified with the complement in the hyperbolic plane  $\mathbb{H}^2$  of a  $\operatorname{Mod}(S)$  invariant countable collection of horoballs whose closures are pairwise disjoint. The horoballs are based at the rational points of  $\partial \mathbb{H}^2$  and are fixed by an infinite cyclic subgroup of  $\operatorname{Mod}(S)$  of parabolic isometries.

Let  $H \subset \mathbb{H}^2$  be such a horoball, with boundary  $\partial H$ , fixed by the parabolic group  $G \subset \operatorname{Mod}(S)$ . Let  $\eta: \mathbb{R} \to \partial H$  be a parameterization of  $\partial H$  by arc length. The geodesics in  $\mathbb{H}^2$  which are asymptotic to the fixed point of G in  $\partial \mathbb{H}^2$  foliate  $\mathbb{H}^2$  and determine a shortest distance projection  $P: \mathbb{H}^2 \setminus \mathring{H} \to \partial H$ . The set  $U_m = P^{-1}(\eta(m,\infty)) \setminus \eta(m,\infty)$  is contractible and intersects  $\mathring{\mathcal{T}}_{\epsilon}(S)$  in a contractible open set whose small closure is a neighborhood of the labeled point  $\eta(\infty) = c_+$ . These neighborhoods define a countable neighborhood basis of  $c_+$  which are small closures of contractible open subsets of  $\mathring{\mathcal{T}}_{\epsilon}(S)$ .

Alternatively, let  $V_m \supset U_m$  be the union of all leaves of the foliation which pass through  $\eta(m,\infty)$ . Clearly  $V_m$  is contractible. The small closures of the images of the sets  $V_m$  under the homeomorphism  $\mathcal{T}(S) \to \mathring{\mathcal{T}}_{\epsilon}(S)$  then define another neighborhood basis of  $c_+$  in the above topology of  $\overline{\mathcal{T}}(S)$  consisting of small closures of contractible open subsets of  $\mathring{\mathcal{T}}_{\epsilon}(S)$ .

As neighborhood bases of minimal filling laminations will be discussed in detail in a more general context, we omit the discussion here.

If  $S = \bigsqcup_{i=1}^k S_i$  is a disconnected surface of finite type, then a hyperbolic metric on S is a hyperbolic metric on each component  $S_i$  of S. In this vein, the Teichmüller space  $\mathcal{T}(S) = \prod_{i=1}^k \mathcal{T}(S_i)$  is the product of the Teichmüller spaces of the components, and the mapping class group  $\operatorname{Mod}(S) = \prod_{i=1}^k \operatorname{Mod}(S_i)$  is the direct product of the mapping class groups of the surfaces  $S_i$ . Put  $\overline{\mathcal{T}}(S) = \prod \mathcal{T}_{\epsilon}(S_i) \cup \mathcal{X}(\sqcup_i S_i)$ where as before,  $\mathcal{X}(\sqcup_i S_i)$  is the join of the spaces  $\mathcal{X}(S_i)$ .

The following is a more precise version of the third part of Theorem 4. In its formulation, we do not require S to be connected.

**Theorem 4.12.** For a surface S of finite type there exists a topology on  $\overline{\mathcal{T}}(S)$  with the following properties.

- (1) The topology is nice and consistent.
- (2) Let  $\xi = \sum_{i=1}^k a_i \xi_i \in \mathcal{X}(S)$  and let  $S_{k+1} = S \setminus \bigcup_{i=1}^k S_i$ . A sequence  $X^j \subset \mathcal{T}_{\epsilon}(S)$  converges to  $\xi$  if and only if  $(\operatorname{pr}_{S_1}(\mu(X^j)), \dots, \operatorname{pr}_{S_{k+1}}(\mu(X^j))) \to \xi$ in  $\mathcal{Y}(\bigcup_{i=1}^{k+1} S_i)$ . (3) The pair  $(\overline{\mathcal{T}}(S), \mathcal{X}(S))$  is an  $\mathcal{EZ}$ -structure for  $\mathrm{Mod}(S)$ .

**Remark 4.13.** Indirectly, consistency is a consequence of the second property in the theorem. The formulation of the theorem was chosen to keep the technical aspects of this article as simple as possible.

The remainder of this article is devoted to the proof of Theorem 4.12.

We proceed by induction on the complexity  $\kappa(S)$  of S. Example 4.7 and Example 4.11 cover the case of a connected surface of complexity  $\kappa(S) \leq 1$ . In Subsection 7.2 we shall establish Theorem 4.12 for all possibly disconnected surfaces of complexity  $\kappa(S) = 0$ , that is, for disjoint unions of annuli. This allows to use induction on the complexity of the surface. For the remainder of this section, we assume that Theorem 4.12 is known for all surfaces of complexity at most k-1 for some  $k-1 \ge 0$ , and we use this assumption to set up the induction step. To this end consider a connected surface S of finite type and complexity  $\kappa(S) = k$ .

We show next that this notion of convergence gives indeed rise to a topology on  $\mathcal{T}_{\epsilon}(S) \cup \mathcal{X}(S)$ .

**Proposition 4.14.** There exists a Hausdorff topology  $\mathcal{O}_0$  on  $\overline{\mathcal{T}}(S) = \mathcal{T}_{\epsilon}(S) \cup \mathcal{X}(S)$ with the property that a set  $A \subset \overline{\mathcal{T}}(S)$  is closed for  $\mathcal{O}_0$  if and only if the following holds true.

- (1)  $A \cap \mathcal{T}_{\epsilon}(S)$  is closed in  $\mathcal{T}_{\epsilon}(S)$ , and  $A \cap \mathcal{X}(S)$  is closed in  $\mathcal{X}(S)$ .
- (2) If  $X_i \subset A \cap \mathcal{T}_{\epsilon}(S)$  is a sequence which converges to  $\xi \in \mathcal{X}(S)$  in the sense of (2) of Theorem 4.12, then  $\xi \in A$ .

*Proof.* The proof is analogous to the proof of Lemma 3.5. By the Hausdorff property of  $\mathcal{T}_{\epsilon}(S)$  and Lemma 3.5, note first that any limit of a convergent sequence  $X_i \subset$  $\mathcal{T}_{\epsilon}(S)$  is unique.

To show that the notion of convergence defines a topology on  $\overline{\mathcal{T}}(S)$  with the property that  $\mathcal{T}_{\epsilon}(S) \subset \overline{\mathcal{T}}(S)$  is open and a set  $A \subset \overline{\mathcal{T}}(S)$  is closed if  $A \cap \mathcal{T}_{\epsilon}(S)$  and  $A \cap \mathcal{X}(S)$  are closed and if A contains the limit of any sequence  $X_j \subset \mathcal{T}(S)$  which converges to a point in  $\mathcal{X}(S)$ , it suffices to verify that the empty set and the entire space are closed, and the same holds true for finite unions and arbitrary intersections of closed sets. The verification that this is satisfied is identical to the argument used in the proof of Lemma 3.5.

Finally we have to show that the topology thus constructed is Hausdorff. Since  $\mathcal{X}(S) \subset \overline{\mathcal{T}}(S)$  is a closed Hausdorff space and  $\mathcal{T}_{\epsilon}(S) \subset \overline{\mathcal{T}}(S)$  is an open Hausdorff subspace of  $\overline{\mathcal{T}}(S)$ , all we need to show is that two points  $\xi \neq \eta \in \mathcal{X}(S)$  have disjoint neighborhoods. Now  $\xi, \eta$  have disjoint neighborhoods in  $\mathcal{X}(S)$  and hence since  $\overline{\mathcal{T}}(S)$  is separable, it suffices to show that the limit of any sequence  $X_i \subset \mathcal{T}_{\epsilon}(S)$  converging to a point in  $\mathcal{X}(S)$  is unique. But this was established in the beginning of this proof.

Assume from now on that  $\overline{\mathcal{T}}(S)$  is equipped with the topology defined in Proposition 4.14. We have to verify that this topology satisfies the properties stated in Theorem 4.12. The following proposition is the first step.

**Proposition 4.15.** The topological space  $(\overline{\mathcal{T}}(S), \mathcal{O}_0)$  has the following properties.

- (1)  $\overline{\mathcal{T}}(S)$  is compact and separable.
- (2) The mapping class group acts on  $\overline{\mathcal{T}}(S)$  as a group of transformations.

*Proof.*  $\overline{\mathcal{T}}(S)$  is clearly separable since this holds true for  $\mathcal{X}(S)$  and  $\mathcal{T}_{\epsilon}(S)$ . By Proposition 4.14, it is a Hausdorff space.

To show that  $\overline{\mathcal{T}}(S)$  is compact, note that since  $\mathcal{X}(S)$  is compact and  $\mathcal{T}_{\epsilon}(S)$  is a Lindelöf space, the space  $\overline{\mathcal{T}}(S)$  is Lindelöf. Since  $\overline{\mathcal{T}}(S)$  also is Hausdorff, it suffices to show that  $\overline{\mathcal{T}}(S)$  is sequentially compact, and this follows if we can show that any sequence  $X_i \subset \mathcal{T}_{\epsilon}(S)$  has a convergent subsequence in  $\overline{\mathcal{T}}(S)$ .

If the sequence has a bounded subsequence in  $\mathcal{T}_{\epsilon}(S)$  with respect to a fixed basepoint  $X \in \mathcal{T}_{\epsilon}(S)$ , then as  $\mathcal{T}_{\epsilon}(S)$  is proper, we can extract a converging subsequence. Thus it suffices to show the following

Claim: Any unbounded sequence in  $\mathcal{T}_{\epsilon}(S)$  admits a subsequence which converges in  $\overline{\mathcal{T}}(S)$  to a point  $\xi \in \mathcal{X}(S)$ .

*Proof of the claim:* The proof of the claim is essemtially identical with the proof of Proposition 3.12.

Since the space of geodesic laminations on S equipped with the Hausdorff topology is compact, by extracting a subsequence we may assume that the sets of all simple closed curves contained in the marking  $\mu(X_i)$  converge in the Hausdorff topology to a finite union  $\bigcup_{j=1}^{\ell} \beta_j$  of (not necessarily minimal) geodesic laminations. Note that as some of the curves in  $\mu(X_i)$  may intersect, these laminations are not necessarily disjoint, that is,  $\bigcup_{j=1}^{\ell} \beta_j$  may not be a lamination in its own

right. However, since the number of components of  $\mu(X_i)$  is uniformly bounded, the same holds true for the number of limit laminations.

Let  $\zeta_1, \ldots, \zeta_s$  be the collection of all minimal components of the laminations  $\beta_u$  which are distinct from simple closed curves. The number of such components is finite. Each of the laminations  $\zeta_j$  fills a subsurface  $S_j$  of S which is different from an annulus or a pair of pants. Thus  $\zeta_j$  is a point in the Gromov boundary of the curve graph  $\mathcal{CG}(S_j)$  of  $S_j$ .

Now for  $j \leq s$ , a sequence  $c_i^j$  of simple closed curves on the surface  $S_j$  converges to  $\zeta_j$  in  $\mathcal{CG}(S_j) \cup \partial \mathcal{CG}(S_j)$  if and only if their geodesic representatives  $c_i^j$  for some fixed hyperbolic metric on  $S_j$  converge to  $\zeta_j$  in the coarse Hausdorff topology. As the diameter of the subsurface projection of  $\mu(X_i)$  to  $S_j$  is bounded independent of i, hyperbolicity of  $\mathcal{CG}(S_j)$  implies that the subsurface projection to  $S_j$  of any of the curves in  $\mu(X_i)$  which intersects  $S_j$  converges in the coarse Hausdorff topology to  $\zeta_j$ . As a consequence, none of the limits in the Hausdorff topology of any sequence of components of  $\mu(X_i)$  can intersect  $\zeta_j$ .

By a similar argument, if  $\zeta_j$  is a closed curve component, then we can consider the subsurface projections of a component of  $\mu(X_i)$  to an annulus  $A(\zeta_j)$  with core curve  $\zeta_j$ . Up to passing to a further subsequence, we may assume that these projections are either bounded along the sequence, or converge to one of the two boundary components of the curve graph of  $A(\zeta_j)$ . In the first case call  $\zeta_j$  unlabeled. In the second case, label  $\zeta_j$  with the corresponding point in the Gromov boundary of the curve graph of  $A(\zeta_j)$  and note by the reasoning used in the previous paragraph, no labeled simple closed curve component  $\zeta_j$  can be intersected by another component  $\zeta_\ell$ .

By reordering, let  $\zeta_1, \ldots, \zeta_k$  be the components of the limit laminations  $\beta_u$  which either are distinct from simple closed curves or which are labeled simple closed curves. We claim that  $k \geq 1$ , that is, that there is at least one lamination with this property. Namely, if c is an unlabeled simple closed curve, represented by a closed geodesic for the base surface X, and if with respect to the Hausdorff topology on compact subsets of X a limit of the sequence  $\mu(X_i)$  contains c as a unlabeled component, then no component of a limit of the sequence  $\mu(X_i)$  in the Hausdorff topology can spiral about c and hence c is a component of  $\mu(X_i)$  for all but finitely many i. If k = 0 then this holds true for any limit point of the sequence  $\mu(X_i)$  in the Hausdorff topology. But  $\mu(X_i)$  is an arrange of S for all i and hence decomposes S into disks and once punctured disks and consequently the sequence  $\mu(X_i)$  is bounded. But this contradicts the assumption that the sequence  $X_i \subset \mathcal{T}_{\epsilon}(S)$  is an unbounded sequence.

By what we showed so far,  $\hat{\zeta} = \bigcup_{j=1}^k \zeta_k$  is a geodesic lamination. Furthermore, if  $S_j$  is the subsurface of S filled by  $\zeta_j$ , then  $d_{\mathcal{CG}(S_j)}(\operatorname{pr}_{S_j}(\mu(X_i)), x_j) \to \infty$  where as before,  $x_j \in \mathcal{CG}(S_j)$  is a fixed basepoint for  $\mathcal{CG}(S_j)$ .

If  $\hat{\zeta}$  is minimal and fills S then  $X_i \to \hat{\zeta} \in \overline{\mathcal{T}}(S)$  and we are done. Otherwise we use the induction hypothesis which yields the following. Let  $S_{k+1} = S \setminus \bigcup_j S_j$ , then

there is a topology on  $\overline{\mathcal{T}}(\cup_j S_j^*) \cup \mathcal{X}(\cup_j S_j^*)$  with the properties stated in Theorem 4.12, in particular  $\overline{\mathcal{T}}(\cup_j S_j^*)$  is compact and metrizable.

As a consequence, by passing to a subsequence we may assume that  $\operatorname{pr}_{\cup_j S_j^*}(X_i)$  converges to a point in  $\overline{\mathcal{T}}(\cup_i S_i^*)$ . It follows from the above discussion that this point  $\xi$  is contained in  $\mathcal{X}(\cup_i S_i^*)$ . But then by consistency, we have  $X_i \to \xi \in \overline{\mathcal{T}}(S)$  which completes the proof of the claim.

To summarize, we showed that  $\overline{\mathcal{T}}(S)$  is sequentially compact Hausdorff Lindelöf space and hence it is compact.

We are left with showing that  $\operatorname{Mod}(S)$  acts on  $\overline{T}(S)$  as a group of transformations. However, as  $\operatorname{Mod}(S)$  acts on  $\mathcal{T}_{\epsilon}(S)$  and on  $\mathcal{X}(S)$  as a group of transformations, and it maps subsurfaces of S to subsurfaces of the same topological type, moreover the definition of convergence which determines the topology  $\mathcal{O}_0$  is natural with respect to the action of  $\operatorname{Mod}(S)$  on subsurfaces and subsurface projections, this is indeed the case. The proposition is proven.

**Theorem 4.16.**  $\mathcal{X}(S)$  is a small boundary for  $\operatorname{Mod}(S)$ . A pseudo-Anosov mapping class acts on  $\mathcal{X}(S)$  with north-south dynamics. In particular, the action of  $\operatorname{Mod}(S)$  on  $\mathcal{X}(S)$  is strongly proximal.

*Proof.* We showed so far that  $\mathcal{X}(S)$  defines a boundary of  $\mathcal{T}_{\epsilon}(S)$  and hence of  $\operatorname{Mod}(S)$  since  $\operatorname{Mod}(S)$  acts properly and cocompactly on  $\mathcal{T}_{\epsilon}(S)$ . Furthermore, a pseudo-Anosov element acts on  $\mathcal{X}(S)$  with north-south dynamics and hence the action of  $\operatorname{Mod}(S)$  on  $\mathcal{X}(S)$  is strongly proximal.

We are left with showing that the right action of Mod(S) induces the identity. However, this action just consists of a change of basepoint. As a sequence of points of uniformly bounded distance from a convergent sequence converges to the same point, this yields the statement of the theorem.

## 5. Metrizability

The goal of this section is to show the following result.

**Theorem 5.1.**  $(\overline{\mathcal{T}}(S), \mathcal{O}_0)$  is metrizable.

The strategy for the proof consists in the construction of an explicit neighborhood basis in  $\overline{\mathcal{T}}(S)$  for every point  $\xi \in \mathcal{X}(S)$ . The statement of the theorem then follows with standard tools.

By [MM99], for any surface V of finite type there is a number p>0 only depending on the complexity of V such that the image under the map  $\Upsilon$  of a Teichmüller geodesic  $\gamma: \mathbb{R} \to \mathcal{T}(S)$  is an unparameterized p-quasi-geodesic in  $\mathcal{CG}(V)$ . This means the following. There is an increasing homeomorphism  $\sigma: (a,b) \subset \mathbb{R} \to \mathbb{R}$  such that the map  $\Upsilon \circ \gamma \circ \sigma: (a,b) \to \mathcal{CG}(S)$  is a p-quasi-geodesic. This quasi-geodesic may be bounded, one-sided infinite or two-sided infinite. A sufficient but not necessary condition for being one-sided infinite in the positive direction is that

the geodesic recurs in the positive direction to the thick part  $\mathcal{T}_{\epsilon}(S)$  for arbitrarily large times. Since the directions of Teichmüller geodesic rays with this property are dense in the cotangent bundle of Teichmüller space, up to increasing p, any geodesic segment  $\alpha:[0,n]\to\mathcal{CG}(S)$  can be extended to a p-quasi-geodesic ray  $\alpha:[0,\infty)\to\mathcal{CG}(S)$ .

Consider first a point  $\xi \in \partial \mathcal{CG}(S)$ . Choose a basepoint  $X_0 \in \mathcal{T}_{\epsilon}(S)$  and let c be a pants curve from  $\mu(X_0)$ . For  $j \geq 0$  define

$$W(\xi,j) \subset \mathcal{T}_{\epsilon}(S)$$

to be the set of all hyperbolic metrics  $X \in \mathcal{T}_{\epsilon}(S)$  with the following properties.

- (1)  $d_{\mathcal{CG}(S)}(\mu(X), c) \geq j$ .
- (2) A geodesic in  $\mathcal{CG}(S)$  connecting c to  $\mu(X)$  can be extended to a p-quasi-geodesic in  $\mathcal{CG}(S)$  whose endpoint is contained in the ball of radius  $e^{-j}$  about  $\xi$  in  $\partial \mathcal{CG}(S)$ , where the metric on  $\partial \mathcal{CG}(S_i)$  is the Gromov distance  $d_c$  constructed from the basepoint c.

If  $\xi = \sum_{i=1}^k a_i \xi_i$  where the lamination  $\xi_i$  fills a proper subsurface  $S_i$  of S then put  $S_{k+1} = S \setminus \bigcup_i S_i$ . Using the induction hypothesis and consistency, choose a countable neighborhood basis  $V_i$  of  $\xi$  in  $\overline{\mathcal{T}}(\bigcup_i S_i^*)$  and define

$$W(\xi, j) = \{ X \in \mathcal{T}_{\epsilon}(S) \mid \operatorname{pr}_{\cup_{i} S^{*}}(X) \in V_{j} \}.$$

Then we have

**Proposition 5.2.** For each  $\xi$ , j the closures of  $W(\xi, j)$  in  $\overline{\mathcal{T}}(S)$  define a neighborhood basis of  $\xi$  in  $\overline{\mathcal{T}}(S)$ .

*Proof.* We show first that for each  $\xi, j$  the closure of  $W(\xi, j)$  in  $\overline{\mathcal{T}}(S)$  is a neighborhood of  $\xi$ . Since  $\mathcal{T}_{\epsilon}(S)$  is dense in  $\overline{\mathcal{T}}(S)$  and by Proposition 4.15,  $\overline{\mathcal{T}}(S)$  is a compact separable Hausdorff space, it suffices to show the following. Let  $(X_{\ell}) \subset \mathcal{T}_{\epsilon}(S)$  be a sequence converging in  $\overline{\mathcal{T}}(S)$  to  $\xi$ ; then  $X_{\ell} \in W(\xi, j)$  for all sufficiently large  $\ell$ .

Now if  $\xi \in \partial \mathcal{CG}(S)$  then by the definition of the topology of  $\overline{\mathcal{T}}(S)$ , we know that  $\mu(X_{\ell}) \to \xi$  in  $\mathcal{CG}(S) \cup \partial \mathcal{CG}(S)$ . But this immediately implies that  $X_{\ell} \in W(\xi, j)$  for all sufficiently large  $\ell$ .

Similarly, if  $\xi = \sum_i a_i \xi_i$  then  $\operatorname{pr}_{\bigcup_i S_i}(X_\ell) \to \xi$  in  $\overline{\mathcal{T}}(\bigcup_i S_i^*)$  and hence by the definition of the sets  $W(\xi,j)$  we also have  $X_\ell \in W(\xi,j)$  for all sufficiently large  $\ell$ .

A similar argument also shows that the sets  $W(\xi,j)$  define a neighborhood basis of  $\xi$ . Namely, we may assume that the sets  $W(\xi,j)$  are *nested*: If m>j, then  $W(\xi,m)\subset W(\xi,j)$ . Thus since  $\mathcal{X}(S)$  is a compact Hausdorff space, to show that the closures  $\overline{W}(\xi,j)$  in  $\overline{\mathcal{T}}(S)$  of the sets  $W(\xi,j)$  define a neighborhood basis of  $\xi$  in  $\overline{\mathcal{T}}(S)$ , it suffices to show that  $\cap_{j>0} \overline{W}(\xi,j) = \{\xi\}$ .

To see that this is indeed the case note first that  $\xi \in \overline{W(\xi,j)}$  for all j and hence as these sets are compact, the point  $\xi$  also is contained in the intersection of these sets. Furthermore, the following holds true.

For each  $\ell$  let  $X_{\ell} \in W(\xi, \ell)$ ; if  $\xi \in \partial \mathcal{CG}(S)$  then the distance of  $\mu(X_{\ell})$  to the base curve c in  $\mathcal{CG}(S)$  tends to infinity with  $\ell$ . This implies that the sequence  $X_{\ell}$  can not have a convergent subsequence in  $\mathcal{T}_{\epsilon}(S)$ . Thus by compactness of  $\overline{\mathcal{T}}(S)$ , up to passing to a subsequence, the sequence converges to a point in  $\mathcal{X}(S)$ . It then follows from the definitions that this point equals  $\xi$ .

The case  $\xi = \sum_i a_i \xi_i$  follows in exactly the same way from the consistency and will be omitted.

We are now ready to show

**Proposition 5.3.**  $\overline{\mathcal{T}}(S)$  is metrizable.

*Proof.* By Uryson's theorem, a second countable Hausdorff space is metrizable. As by Proposition 4.15 the space  $\overline{\mathcal{T}}(S)$  is Hausdorff, it suffices to show that  $\overline{\mathcal{T}}(S)$  is second countable. Since  $\mathcal{T}_{\epsilon}(S)$  is second countable, this is the case if there exist countably many open sets  $U_i \subset \overline{\mathcal{T}}(S)$  which contain a neighborhood basis for any point  $x \in \mathcal{X}(S)$ .

Since the Hausdorff topology on geodesic laminations is metrizable, there exists a countable dense subset  $\{\xi_i \mid i\} \subset \partial \mathcal{CG}(S)$ . For each i, j let  $\mathring{W}(i, j)$  be the interior of the set  $\overline{W(\xi_i, j)} \subset \overline{\mathcal{T}}(S)$  and put

$$\mathcal{W}_S = \{ \mathring{W}(i,j) \mid i,j \}$$

which is a countable collection of open sets.

We claim that the sets from the collection  $\mathcal{W}_S$  contain a neighborhood basis of every point in  $\partial \mathcal{CG}(S) \subset \mathcal{X}(S)$ . To this end let  $U \subset \overline{\mathcal{T}}(S)$  be open and let  $\xi \in \partial \mathcal{CG}(S) \cap U$ . By Proposition 5.2 there exists some j so that  $W(\xi,j) \subset U$ . By the fact that the set  $\{\xi_i \mid i\}$  is dense in  $\partial \mathcal{CG}(S)$  and standard consequences of hyperbolicity of  $\mathcal{CG}(S)$ , there exists some i and some  $\ell > 10j$  so that the set  $W(\xi_i,\ell)$  is a neighborhood of  $\xi$  contained in  $W(\xi,j)$  which is what we wanted to show.

Second, for a proper essential subsurface  $S_0 \in \mathcal{S}$  choose a countable basis  $\{V_j \mid j\}$  of the topology of  $\overline{\mathcal{T}}(S_0^*)$  which exists by the induction hypothesis. Then the sets  $\{X \mid \operatorname{pr}_{S_0^*}(X) \in V_i\}$  defines a countable neighborhood basis of  $\mathcal{X}(S_0^*)$  in  $\overline{\mathcal{T}}(S)$ . Since there are only countably many subsurfaces in S, and we have  $\mathcal{X}(S) = \partial \mathcal{CG}(S) \cup \bigcup_V \mathcal{X}(V^*)$ , this shows that  $\overline{\mathcal{T}}(S)$  is indeed second countable and completes the proof of the proposition.

### 6. Dimensions and smallness

We showed so far that the pair  $(\overline{\mathcal{T}}(S), \mathcal{X}(S))$  is a pair of compact metrizable spaces, with  $\mathcal{X}(S)$  nowhere dense in  $\overline{\mathcal{T}}(S)$ . In this section we show that these spaces are finite dimensional and that the collection of all translates of a compact set in X form a null sequence in  $\overline{\mathcal{T}}(S)$ . Throughout we assume that S is connected, of genus  $g \geq 0$  with  $m \geq 0$  punctures. The extension of the results in this section to disconnected surfaces is straightforward.

Recall that the covering dimension of a topological space X is the minimum of the numbers  $n \geq 0$  so that the following holds true. Any open cover  $\mathcal{U}$  of X has a refinement  $\mathcal{V}$  so that a point in X is contained in at most n+1 of the sets  $V \in \mathcal{V}$ . With this terminology, the covering dimension of  $\mathbb{R}^n$  is n, and hence the covering dimension of any subset of  $\mathbb{R}^n$  equipped with the subspace topology is at most n. In particular, the covering dimension of  $\mathcal{T}(S)$  equals 6g-6+2m.

The following result relies on work of Gabai [Ga14], see also [BB19].

**Proposition 6.1.** The covering dimension of  $\mathcal{X}(S)$  is finite

*Proof.* We proceed by induction on the complexity of the surface S. If S is an annulus, then its geometric boundary consists of two points and there is nothing to show.

Consider next a four-holed sphere or a one-holed torus S. By Example 2.3 and Example 3.3, the geometric boundary as a topological space is homeomorphic to the Gromov boundary of the hyperbolic group  $\mathrm{PSL}(2,\mathbb{Z})$ . Since the group  $\mathrm{PSL}(2,\mathbb{Z})$  is virtually free, the boundary  $\mathcal{X}(S)$  of S is a Cantor set, which has covering dimension zero.

Let X and Y be compact spaces with covering dimensions m,n. We claim that the covering dimension of the join X\*Y is at most m+n+1. To see that this is the case recall that X\*Y is the quotient of  $X\times Y\times [0,1]$  under an equivalence relation  $\sim$  which is only nontrivial on  $X\times Y\times \{0\}$  and  $X\times Y\times \{1\}$ . The projection  $X\times Y\times [0,1]\to X\times Y\times [0,1]/\sim \max X\times Y\times \{0\}$  to  $X\times \{0\}$  and maps  $X\times Y\times \{1\}$  to  $Y\times \{1\}$ . Thus we have  $X*Y=X\cup Y\cup C$  where  $X\subset X*Y$  is the closed set which is the quotient of  $X\times Y\times \{0\}$ ,  $Y\subset X*Y$  is the closed set which is the quotient of  $X\times Y\times \{1\}$ , and the set C is homeomorphic to  $X\times Y\times \{0,1\}$ .

By Alexandrov's definition of dimension (see Theorem 3.4 of [Dr18]), we have  $\dim(A \times B) \leq \dim(A) + \dim(B)$  and hence  $\dim(C) \leq \dim(X) + \dim(Y) + 1$ . The compact space X \* Y is the union of the closed subset  $X \cup Y$  with C and hence the theorem of Menger and Uryson (see Theorem 3.1 of [Dr18]) shows that  $\dim(X * Y) = \dim(C) \leq \dim(X) + \dim(Y) + 1 = m + n + 1$  as claimed.

Assume now that the proposition was established for all surfaces of complexity at most k-1. Let S be a surface of complexity k. We have  $\mathcal{X}(S) = \partial \mathcal{CG}(S) \cup \mathcal{Y}$  (disjoint union) where  $\mathcal{Y} = \cup \mathcal{X}(S_1) * \cdots * \mathcal{X}(S_p)$  and the union in the definition of  $\mathcal{Y}$  is over all disjoint collections of proper subsurfaces  $S_1, \ldots, S_p$  of S. The union  $\mathcal{Y}$  is not disjoint.

The number of disjoint surfaces in one of the joins appearing in the definition of  $\mathcal{Y}$  is uniformly bounded in terms of k. Thus by the induction hypothesis and the above dimension estimate for joins, applied inductively, there exists a number n > 0 which bounds from above the covering dimension of each of the sets  $\mathcal{X}(S_1) * \cdots * \mathcal{X}(S_p)$ . Example 3.2 shows that as subsets of  $\mathcal{X}(S)$ , the sets  $\mathcal{X}(S_1) * \cdots * \mathcal{X}(S_p)$  are closed and hence compact. As a consequence, the subspace  $\mathcal{Y}$  of  $\mathcal{X}(S)$ , equipped with the induced topology, is a  $\sigma$ -compact Hausdorff space as it is a countable union of compact spaces.

If  $K \subset \mathcal{Y}$  is compact, then K can be represented as a countable union of the compact spaces  $K \cap \mathcal{X}(S_1) * \cdots * \mathcal{X}(S_p)$ . Then the countable union theorem Theorem 3.2 of [Dr18] shows that  $\dim(K) = \sup\{\dim(K \cap \mathcal{X}(S_1) * \cdots * \mathcal{X}(S_p))\}$  where the supremum is over all disjoint unions of proper subsurfaces of S. By the above estimate for the dimension of the spaces  $\mathcal{X}(S_1) * \cdots * \mathcal{X}(S_p)$ , we have

$$\dim(K \cap \mathcal{X}(S_1) * \cdots * \mathcal{X}(S_p)) \le n$$

for any such intersection. But then the dimension of  $\mathcal{Y}$  is at most n (see p.316 of [Mu14] for a sketch of a proof).

Following [Ga14], the covering dimension of  $\partial \mathcal{CG}(S)$  is at most 4g-5+2m. Then by the Uryson-Menger formula (see Theorem 3.3 of [Dr18]), the dimension of the compactum  $\mathcal{X}(S)$  is at most

$$\dim(\mathcal{X}(S)) = \dim(\partial \mathcal{CG}(S)) + \dim(\mathcal{Y}) + 1$$

and hence it is finite.

As a consequence, we obtain

Corollary 6.2. The pair  $(\overline{\mathcal{T}}(S), \mathcal{X}(S))$  is a pair of spaces of finite dimension.

*Proof.* By Proposition 6.1, the dimension of  $\mathcal{X}(S)$  is finite. As the compactum  $\overline{\mathcal{T}}(S) = \mathcal{T}_{\epsilon}(S) \cup \mathcal{X}(S)$  is a union of two subspaces of finite dimension, with  $\mathcal{X}(S) \subset \overline{\mathcal{T}}(S)$  closed, we have

$$\dim(\overline{\mathcal{T}}(S)) = \max\{\dim(\mathcal{T}_{\epsilon}(S),\dim(\mathcal{X}(S))\} < \infty.$$

We now verify the requirement (4) in the definition of a  $\mathbb{Z}$ -set.

**Proposition 6.3.** The action of Mod(S) on  $\mathcal{T}(S)$  is  $\mathcal{U}$ -small for every open covering  $\mathcal{U}$  of  $\overline{\mathcal{T}}(S)$ .

*Proof.* Let  $\mathcal{U}$  be an open covering of  $\overline{\mathcal{T}}(S)$ . By compactness, we may extract a finite subcovering, so we may assume that  $\mathcal{U}$  is in fact finite, that is, we have  $\mathcal{U} = \bigcup_{0 \leq i \leq m} U_i$  for some open sets  $U_i \subset \overline{\mathcal{T}}(S)$ . Assume without loss of generality that  $U_i \cap \mathcal{X}(S) \neq \emptyset$  for all  $i \geq 1$ .

We argue now by contradiction and we assume that there exists a compact set  $K \subset \overline{\mathcal{T}}_{\epsilon}(S)$  and infinitely many elements  $\varphi_i \in \Gamma$  such that  $g_i K \not\subset U_j$  for all  $j \leq m$ . Since the action of  $\Gamma$  on Q is proper and cocompact, we may assume that  $K = \cup_{j=1}^{\ell} \psi_j K_0$  where  $K_0$  is a compact fundamental domain for the action of  $\Gamma$  and  $\psi_j \in \Gamma$ .

Let  $X \in K_0$ . Since the action of  $\Gamma$  on Q is proper and  $\overline{\mathcal{T}}(S)$  is compact, we conclude that up to passing to a subsequence, the sequence  $\varphi_i X$  converges in  $\overline{\mathcal{T}}(S)$  to a point  $\xi \in \mathcal{X}(S)$ . Since the right action of  $\Gamma$  on itself extends to the trivial action on  $\mathcal{X}(S)$ , we then have  $\varphi_i(\psi_j X) \to \xi$  for all  $j \leq \ell$ . In particular, for sufficiently large i, we have  $\varphi_i(\psi_j X) \in U_p$  for some fixed p > 0. But then it follows from the definition of the topology on  $\overline{\mathcal{T}}(S)$  that in fact  $\varphi_i K \to \xi$  and hence  $\varphi_i K \subset U_p$  for

all sufficiently large p. This is a contradiction which completes the proof of the proposition.

### 7. Neighborhood bases

The main goal of this section is to construct for a point in  $\mathcal{X}(S) \subset \overline{\mathcal{T}}(S)$  an explicit neighborhood basis in  $\overline{\mathcal{T}}(S)$  consisting of small closures of open contractible subsets of  $\mathring{\mathcal{T}}_{\epsilon}(S)$ . Note that the neighborhood basis we constructed in Section 5 does not seem to consist of sets with this property. However, the neighborhoods from that basis will be used in our construction. Here by a contractible subset of  $\mathcal{T}_{\epsilon}(S)$  we mean a subset V which is a contractible space with respect to the subspace topology. This result then completes the proof of Theorem 4.12 and hence of Theorem 4. The construction is the most involved part of the proof. It is carried out in three steps, each of which is contained in a separate subsection. The first two steps do not rely on an induction hypothesis and are used in the third subsection to complete the inductive construction of a topology on  $\overline{\mathcal{T}}(S)$  with the properties stated in Theorem 4.12.

7.1. A neighborhood basis for minimal filling laminations. In this subsection we prove the following result.

**Proposition 7.1.** Every point  $\xi \in \partial \mathcal{CG}(S) \subset \mathcal{X}(S)$  has a countable neighborhood basis in  $\overline{\mathcal{T}}(S)$  consisting of sets whose intersections with  $\mathcal{T}_{\epsilon}(S)$  are small closures of contractible open subsets of  $\mathring{\mathcal{T}}_{\epsilon}(S)$ .

The proof of Proposition 7.1 introduces the ideas used in the general case, but it is technically easier. To set it up, note that any minimal filling geodesic lamination  $\xi$  decomposes S into a union of ideal polygons. Each of these polygons which is not an ideal triangle can be subdivided by adding isolated leaves which connect two non-adjacent cusps of the polygon. The various ways to subdivide these polygons determine a finite collection  $\xi_0, \ldots, \xi_k$  of distinct geodesic laminations which contain  $\xi$  as a sublamination. Assume that  $\xi_0 = \xi$ .

Let  $d_H$  be the Hausdorff metric on the space of compact subsets of a fixed hyperbolic surface  $X \in \mathcal{T}_{\epsilon}(S)$ . Denote as before by  $\mathrm{Min}_{\cup}(\mathcal{L})$  the space of geodesic laminations on X which are unions of disjoint minimal components. Equivalently, the only isolated leaves of a geodesic lamination in  $\mathrm{Min}_{\cup}(\mathcal{L})$  are simple closed curves. As before, let supp :  $\mathcal{X}(S) \to \mathrm{Min}_{\cup}(\mathcal{L})$  be the map which associates to a point  $\sum_i a_i \xi_i$   $(a_i > 0)$  the support  $\mathrm{supp}(\xi) = \bigcup_i \xi_i$ . We have

Lemma 7.2. For i > 0 let

$$U_i = \bigcup_i \{ \beta \in \operatorname{Min}_{\cup}(\mathcal{L}) \mid d_H(\beta, \xi_i) \le 1/i \}$$

and write  $V_i = \{\zeta \in \mathcal{X}(S) \mid \text{supp}(\zeta) \in U_i\}$ . Then the sets  $V_i$  form a neighborhood basis of  $\xi \in \partial \mathcal{CG}(S) \subset \mathcal{X}(S)$  in  $\mathcal{X}(S)$ .

*Proof.* Clearly  $\xi \in V_i$  for all i. We first show that for each i the set  $V_i$  is a neighborhood of  $\xi$ . For this we argue by contradiction and we assume that there exists some i such that this is not the case. Then there exists a sequence  $\zeta_j \subset \mathcal{X}(S)$  such that  $\zeta_j \to \xi$  and such that  $\zeta_j \notin V_i$  for all j.

By the first requirement for convergence in the definition of the topology on  $\mathcal{X}(S)$ , we know that  $\operatorname{supp}(\zeta_j)$  converges in the coarse Hausdorff topology to  $\xi_0 = \operatorname{supp}(\xi)$ . By compactness of the space of compact subsets of S with respect to the Hausdorff topology, by passing to a subsequence we may assume that the sequence  $\operatorname{supp}(\zeta_j)$  converges in the Hausdorff topology to a geodesic lamination  $\zeta$ . Then  $\zeta$  contains  $\xi_0$  as a sublamination and hence  $\zeta = \xi_s$  for some  $s \leq k$ . By definition, this implies that  $\operatorname{supp}(\zeta_j) \in U_i$  and hence  $\zeta_j \in V_i$  for all sufficiently large j, a contradiction. This shows that indeed, each of the sets  $V_i$  is a neighborhood of  $\xi$ .

To show that the sets  $V_i$  form a neighborhood basis for  $\xi$ , note that  $V_{i+1} \subset V_i$  and hence it suffices to show that  $\cap_i V_i = \{\xi\}$ . However, this is immediate from the definitions and the fact that the preimage of  $\text{supp}(\xi)$  under the support map supp which associates to  $\zeta \in \mathcal{X}(S)$  its support consists of the single point  $\xi$ .

A measured geodesic lamination on the surface S is a geodesic lamination together with a transverse invariant measure. The space  $\mathcal{ML}$  of measured geodesic laminations is equipped with the weak\* topology. The quotient of  $\mathcal{ML}$  under the natural action of  $(0,\infty)$  by scaling is the space  $\mathcal{PML}$  of projective measured geodesic laminations. This space is homeomorphic to the sphere  $S^{6g-7+2m}$ . To put Lemma 7.2 into proper context and for later use, we relate the subset  $\partial \mathcal{CG}(S) \subset \mathcal{X}(S) \subset \overline{\mathcal{T}}(S)$  to the space  $\mathcal{PML}$ .

To this end we use a more geometric view on  $\mathcal{PML}$ . Fix again a point  $X \in \mathcal{T}_{\epsilon}(S)$ . The cotangent space  $T_X^*\mathcal{T}(S)$  of Teichmüller space at X can be identified with the space of measured geodesic laminations on S. Or, equivalently, by the Hubbard Masur theorem, every measured geodesic lamination  $\nu$  on S is the vertical measured geodesic lamination of a unique marked quadratic differential  $q(\nu)$  for the complex structure on S defined by X. With this identification, we can associate to  $\nu \in \mathcal{ML}$  the point  $\gamma_{\nu}(1)$  where  $\gamma_{\nu}:[0,\infty)\to\mathcal{T}(S)$  is the Teichmüller geodesic starting at X whose initial (co)-velocity  $\gamma'_{\nu}(0)$  is the quadratic differential with vertical measured geodesic lamination  $\nu$ . This construction defines the Teichmüller exponential map  $\exp_X:\mathcal{ML}\cup\{0\}\to\mathcal{T}(S)$  at X which is a homeomorphism.

The area  $\operatorname{area}(q(\nu))$  of the flat metric defined by  $q(\nu)$  defines a norm on  $\mathcal{ML}$  depending on X. Associating to  $[\nu] \in \mathcal{PML}$  the unique measured lamination  $\rho([\nu])$  with the property that the area of  $q(\rho[\nu])$  equals one then defines a section  $\rho: \mathcal{PML} \to \mathcal{ML}$ . In this way we can identify  $\mathcal{PML}$  with the sphere of unit directions for the Teichmüller metric.

The  $support \operatorname{supp}(\nu)$  of a measured geodesic lamination  $\nu$  is a point in the space  $\operatorname{Min}_{\cup}(\mathcal{L})$ . Each of its components is equipped with a transverse invariant measure and hence it is a measured geodesic lamination in its own right.

Let p > 1 be a control constant with the following properties.

- The image under the map  $\Upsilon$  of any Teichmüller geodesic is an unparameterized p-quasi-geodesic in  $\mathcal{CG}(S)$  (see [MM99]).
- Every geodesic segment in  $\mathcal{CG}(S)$  can be extended to a p-quasi-geodesic ray.

Consider as before a minimal filling geodesic lamination  $\xi \in \partial \mathcal{CG}(S)$ . Let  $P(\xi) \subset \mathcal{PML}$  be the set of all projective measured geodesic laminations which are supported in  $\xi$ . This is a non-empty compact polytope of dimension  $\leq 3g - 3 + m$  whose extreme points are the ergodic projective transverse measures supported in  $\xi$ . In particular,  $P(\xi)$  is compact and contractible. Since  $P(\xi)$  is contractible and since  $\mathcal{PML}$  is homeomorphic to a sphere of dimension 6g - 7 + 2m, we can find a descending chain  $V_1 \supset V_2 \supset \cdots$  of closed contractible neighborhoods of  $P(\xi)$ , each of which is homeomorphic to a closed ball, such that  $V_{i+1} \subset \mathring{V}_i$  and  $\cap_i V_i = P(\xi)$ .

In the sequel we use the terminology small closure  $\bar{A}_{small}$  in  $\overline{\mathcal{T}}(S)$  of a set  $A \subset \mathcal{T}_{\epsilon}(S)$  to denote the union of the small closure of A in  $\mathcal{T}_{\epsilon}(S)$  with the intersection with  $\mathcal{X}(S)$  of the closure  $\bar{A}$  of A in  $\overline{\mathcal{T}}(S)$ . Thus for any set  $A \subset \mathcal{T}_{\epsilon}(S)$ ,  $\bar{A}_{small} \cap \mathcal{X}(S)$  is closed, but  $\bar{A}_{small} \cap \mathcal{T}_{\epsilon}(S)$  may be open.

**Lemma 7.3.** Let  $V_1 \supset V_2 \supset \cdots$  be a descending chain of closed contractible neighborhoods of  $P(\xi)$  in  $\mathcal{PML}$ , each of which is homeomorphic to a closed ball, with  $\cap_i V_i = P(\xi)$ . Let  $\Lambda_{\epsilon} : \mathcal{T}(S) \to \mathring{\mathcal{T}}_{\epsilon}(S)$  be the homeomorphism from Proposition 4.4 and let  $\exp_X : \mathcal{ML} \cup \{0\} \to \mathcal{T}(S)$  be the Teichmüller exponential map at X. Then for each j > 0, the small closure  $\overline{Z(i,j)}_{small}$  in  $\overline{\mathcal{T}}(S)$  of the open set

$$Z(i,j) = \Lambda_{\epsilon} \{ \exp_X(\nu) \mid \operatorname{area}(\rho[\nu]) > j, [\nu] \in \mathring{V}_i \}$$

is a neighborhood of  $\xi$ , and neighborhoods of this form define a neighborhood basis of  $\xi$ .

*Proof.* We divide the proof of the lemma into two claims.

Claim 1: For all i, j, the small closure  $\overline{Z(i, j)}_{\text{small}}$  of Z(i, j) in  $\overline{\mathcal{T}}(S)$  is a neighborhood of  $\xi$ .

Proof of Claim 1: By the definition of the topology on  $\overline{\mathcal{T}}(S)$  and the fact that  $\mathring{\mathcal{T}}_{\epsilon}(S)$  is dense in  $\overline{\mathcal{T}}(S)$ , it suffices to show the following. Let  $Y_{\ell} \subset \mathring{\mathcal{T}}_{\epsilon}(S)$  be a sequence converging to  $\xi$ ; then for any fixed (i,j), we have  $Y_{\ell} \in Z(i,j)$  for all sufficiently large  $\ell$ .

Let  $\mathcal{FML} \subset \mathcal{PML}$  be the subset of all projective measured geodesic laminations whose support is a minimal geodesic lamination which fills up S. By Lemma 3.2 of [H09], the support map  $F: \mathcal{FML} \to \partial \mathcal{C}(S)$  which associates to a point in  $\mathcal{FML}$  its support is continuous and closed. Thus the image  $F(\mathcal{FML} \setminus \mathring{V}_i)$  is a closed subset of  $\partial \mathcal{CG}(S)$  which does not contain  $\xi$ . As a consequence, there exists a number T(i) > 0 so that the ball of radius  $e^{-T(i)}$  about  $\xi$  with respect to the Gromov metric on  $\partial \mathcal{CG}(S)$  based at  $\Upsilon(X) \in \mathcal{CG}(S)$  is disjoint from  $F(\mathcal{FML} \setminus \mathring{V}_i)$ .

By the choice of the control constant p > 1 and hyperbolicity, there exists a number  $\tau(i) > T(i)$  with the following property. Let  $[\nu] \in \mathcal{FML} \setminus \mathring{V_i}$ ; then the endpoint of a p-quasi-geodesic ray in  $\mathcal{CG}(S)$  with starts at the basepoint  $\Upsilon(X)$  and

which passes through a point on the *p*-quasi-geodesic  $\Upsilon\{\exp_X(t\rho[\nu]) \mid t \geq 0\}$  of distance at least  $\tau(i)$  to  $\Upsilon(X)$  (provided that such a point exists) is not contained in the ball of radius  $e^{-T(i)/2}$  about  $\xi$ .

Since the homeomorphism  $\Lambda_{\epsilon}: \mathcal{T}(S) \to \mathring{\mathcal{T}}_{\epsilon}(S)$  is coarsely  $\Upsilon$ -invariant, the map  $\exp_X: \mathcal{ML} \cup \{0\} \to \mathcal{T}(S)$  is a homeomorphism and  $\mathcal{FML} \subset \mathcal{PML}$  is dense, it follows that for all i, j there exists  $\ell$  so that the neighborhood  $W(\xi, \ell)$  of  $\xi$  in  $\overline{\mathcal{T}}(S)$  constructed in Proposition 5.2 is contained in the closure of the set  $\Lambda_{\epsilon}(\{\exp_X(t\rho[\nu]) \mid t \geq j, [\nu] \in \mathring{V}_i\})$ . Thus by Proposition 5.2,  $\overline{Z}(i,j)_{\text{small}}$  is a neighborhood of  $\xi$  in  $\overline{\mathcal{T}}(S)$ .

The proof of the lemma is completed once we established the following. In its formulation,  $\overline{Z(i,j)}$  is the closure of Z(i,j) in  $\overline{\mathcal{T}}(S)$ .

Claim 2: Let W be a neighborhood of  $\xi$  in  $\overline{\mathcal{T}}(S)$ ; then there exists some i, j so that  $\overline{Z(i,j)} \subset W$ .

Proof of Claim 2: By Claim 1, each of the sets  $\overline{Z(i,j)}$  is a neighborhood of  $\xi$  and hence contains  $\xi$ . Furthermore, these neighborhoods are nested: If  $i_1 \leq i_2$  and  $j_1 \leq j_2$  then  $\overline{Z(i_1,j_1)} \supset \overline{Z(i_2,j_2)}$ . Thus since the sets  $\overline{Z(i,j)}$  are moreover closed and hence compact, it suffices to show that  $\bigcap_{i,j} \overline{Z(i,j)} = \{\xi\}$ .

Since the Teichmüller exponential map  $\exp_X$  at X is a homeomorphism, we clearly have  $\bigcap_{i,j} \overline{Z(i,j)} \subset \mathcal{X}(S)$ . On the other hand, the map  $\Upsilon: \mathcal{T}(S) \to \mathcal{CG}(S)$  is coarsely Lipschitz, and for  $\nu \in P(\xi)$ , the p-quasigeodesic  $t \to \Upsilon(\exp(t\rho[\nu]))$  has infinite diameter. This implies that for any k>0 there are numbers i(k)>0, m(k)>0 so that for all  $[\eta] \in V_{i(k)}$ , the diameter of the image under  $\Upsilon$  of the Teichmüller geodesic segment  $\exp_X([0,m(k)]\rho[\eta])$  is at least k. As a consequence, if  $X_i \in \overline{Z(i,i)}$  for each i, then by compactness of  $\overline{\mathcal{T}(S)}$ , up to passing to a subsequence the sequence  $X_i$  converges to a point  $\zeta \in \mathcal{X}(S) \cap \partial \mathcal{CG}(S)$ . That this point has to coincide with  $\xi$  is an immediate consequence of the discussion in the proof of Claim 1 above. This completes the proof of the claim.

**Lemma 7.4.** The sets  $\overline{Z(i,j)}_{\text{small}} \cap \mathcal{T}_{\epsilon}(S)$  are contractible.

*Proof.* Since for each i the set  $\mathring{V}_i$  is a contractible subset of the set of projectivized measured geodesic laminations, identified with the unit sphere in the cotangent space of  $\mathcal{T}(S)$  at X, the set

$$H(i,j) = \bigcup_{[\nu] \in V_i} \{ \exp_X(t\rho[\nu]) \mid t > j \} \subset \mathcal{T}(S)$$

is open and contractible since it is homeomorphic to  $\mathring{V}_i \times (j, \infty)$ . This uses the fact that the Teichmüller exponential map at X is a homeomorphism of  $T_X^*\mathcal{T}(S)$  onto  $\mathcal{T}(S)$ .

But Z(i,j) is the image of H(i,j) under the homeomorphism  $\Lambda_{\epsilon}: \mathcal{T}(S) \to \mathring{\mathcal{T}}_{\epsilon}(S)$  and hence Z(i,j) is contractible. Then by Lemma 4.6, the small closure of Z(i,j) in  $\mathcal{T}(S)$  is contractible as well.

Proof of Proposition 7.1. The small closure of  $\overline{Z(i,j)}_{small}$  of Z(i,j) in  $\overline{T}(S)$  is a neighborhood of  $\xi$  which is the small closure of a contractible open subset  $\mathring{\mathcal{T}}_{\epsilon}(S)$  and hence contractible. The countably many such sets define a countable neighborhood basis of  $\xi$  in  $\overline{T}(S)$  whence the proposition.

7.2. Neighborhoods of minimal filling laminations for disconnected surfaces. In this section we consider a disjoint union  $S = \bigsqcup_{i=1}^k S_i$  of finitely many connected surfaces of finite type. Our goal is to construct for any point in

$$\mathcal{E} = \partial \mathcal{CG}(S_1) * \cdots * \partial \mathcal{CG}(S_k) \subset \mathcal{X}(S)$$

a neighborhood basis in  $\overline{\mathcal{T}}(S)$  consisting of sets whose intersections with  $\mathcal{T}_{\epsilon}(S)$  are small closures of open contractible subsets of  $\mathring{\mathcal{T}}_{\epsilon}(S)$ .

Remark 7.5. In [Ti11], it was shown that if two groups  $\Gamma_1$ ,  $\Gamma_2$  admit  $\mathcal{E}\mathcal{Z}$ -structures  $(X_1, Z_1)$  and  $(X_2, Z_2)$ , then the direct product  $\Gamma_1 \times \Gamma_2$  admits an  $\mathcal{E}\mathcal{Z}$ -structure consisting of a compactification of the product  $(X_1 \setminus Z_1) \times (X_2 \setminus Z_2)$  by adding the join  $Z_1 * Z_2$ . Unfortunately, we can not use this result directly as we need more precise information for the proof of Theorem 4.12.

The set  $\mathcal{E}$  is the set of sums  $\sum_i a_i \xi_i$  where  $\xi_i \in \partial \mathcal{CG}(S_i)$  and  $a_i \geq 0$ ,  $\sum_i a_i = 1$ , Recall from Section 7.1 that for each i the choice of the basepoint  $X_i$  determines a section  $\rho_i : \mathcal{PML}(S_i) \to \mathcal{ML}(S_i)$ . Let  $x_i \in \mathcal{CG}(S_i)$  be a component of the pants decomposition of  $\mu(X_i)$ . For simplicity of notation, call a function  $f : \mathbb{R} \to \mathbb{R}$  coarsely non-decreasing, with control constant q > 0, if we have  $f(t) \geq f(s) - q$  for all  $s \leq t$ . Then for every projective measured geodesic lamination  $[\nu_i]$  on  $S_i$  the function

$$t \to d_{\mathcal{CG}(S_i)}(\Upsilon(\exp_{X_i}(t\rho_i[\nu_i])), x_i)$$

is coarsely non-decreasing, with control constant only depending on the complexity of  $S_i$  [MM99]. The following was shown in [H09].

Lemma 7.6. There exists a continuous function

$$\delta_{x_i}: \mathcal{T}(S_i) \to [0, \infty)$$

which is at uniformly bounded distance from the function  $Y_i \to d_{\mathcal{CG}(S_i)}(\Upsilon(Y_i), x_i)$ .

To construct open contractible subsets of  $\prod \tilde{\mathcal{T}}_{\epsilon}(S_i)$  whose closures define neighborhoods of  $\sum_i a_i \xi_i$  in  $\overline{\mathcal{T}}(\cup_i S_i)$ , we shall control the speed of progress in the curve graph of each of the surfaces  $S_i$ . To this end note that by Lemma 7.6, for every Teichmüller geodesic  $\gamma: \mathbb{R} \to \mathcal{T}(S_i)$  starting at the fixed basepoint  $X_i$ , the function  $t \to \delta_{x_i}(\gamma(t))$  is coarsely non-decreasing and continuous. We use this to construct a new parameterization of a Teichmüller geodesic starting from  $X_i$  which encapsulates its progress in the curve graph. The construction is based on the following elementary observation. Here the distance between two functions  $f, g: J \subset \mathbb{R} \to \mathbb{R}$  is defined as  $||f - g|| = \sup\{|f(t) - g(t)| \mid t\}$ .

**Lemma 7.7.** Let  $f: \mathbb{R}^n \to [0, \infty)$  be a continuous function whose restriction to each ray  $t \to tx$   $(x \in S^{n-1} \subset \mathbb{R}^n)$  is coarsely non-decreasing, with fixed control constant q > 0. Then

$$u = \inf\{g \mid g \ge f, g \text{ continuous, non-decreasing on rays}\}$$

is non-decreasing on rays, continuous and at distance at most q from f.

*Proof.* For  $x \in S^{n-1}$  and  $t \ge 0$  put

$$u(tx) = \max\{f(sx) \mid s \le t\}.$$

This makes sense since f is continuous. By definition, u is non-decreasing on rays,  $u \ge f$  and  $u - f \le q$  as f is coarsely non-decreasing.

Since f is continuous, it is also immediate that u is continuous. This shows the lemma.

Let  $f_i$  be the function on  $T_{X_i}\mathcal{T}(S_i) \sim \mathbb{R}^{m_i}$  constructed in Lemma 7.7 from the function  $\delta_{x_i} \circ \exp_{X_i}$ . For each  $[\nu] \in \mathcal{PML}(S_i)$  the restriction of the function  $f_i$  to the ray  $t\rho_i[\nu]$   $(t \in (0, \infty))$  is non-decreasing, but it may be constant on arbitrarily large intervals. However, by replacing  $f_i$  by  $f_i + \alpha_i$  where  $\alpha_i(t\rho_i[\nu]) = a(t)$  for a smooth strictly increasing function  $[0, \infty) \to [0, 1)$ , we may assume that the function  $f_i$  has the following properties.

- (1) The function  $f_i: T_{X_i}\mathcal{T}(S_i) \to [0, \infty)$  is continuous and strictly increasing on rays starting at 0.
- $(2) \sup |f_i \delta_{x_i} \circ \exp_{X_i}| \le q + 1.$

In particular, if  $f_i|\{t\rho_i[\nu]\mid t\in (0,\infty)\}$  is unbounded, then  $f_i|\{t\rho_i[\nu]\mid t\}$  is a homeomorphism onto  $[0,\infty)$ .

Put  $\tau[\nu] = \sup\{f_i(t\rho_i[\nu]) \mid t\}$ . Note that  $\tau[\nu] = \infty$  if the support of the geodesic lamination  $[\nu]$  on  $S_i$  fills  $S_i$ .

Since  $f_i$  is continuous and its restriction to each ray  $\{t\rho[\nu] \mid t \geq 0\}$  is a homeomorphism onto  $[0, \tau[\nu])$ , it can be inverted. We then can define a function  $g_{[\nu]}$  on  $[0, \tau[\nu])$  by

$$g_{[\nu]}(t) = (f_i | \{t \rho[\nu] \mid t\})^{-1}.$$

Using this function, we obtain a parameterization  $t \to \hat{\gamma}_{[\nu]}(t)$  of the Teichmüller geodesic  $t \to \exp_{X_i}(t\rho[\nu])$  on the interval  $[0,\tau[\nu])$  by defining

(3) 
$$\hat{\gamma}_{[\nu]}(t) = \exp_{X_{\cdot}}(g_{[\nu]}(t)).$$

With this definition, we know that  $|d_{\mathcal{CG}(S_i)}(x_i, \hat{\gamma}_{[\nu]}(t)) - t| \leq b$  where b > 0 is a universal constant not depending on t or i.

Let  $\xi = \sum_i a_i \xi_i \in \mathcal{X}(\cup_i S_i)$  be such that  $\xi_i \in \partial \mathcal{CG}(S_i)$  for all i. Assume by reordering that there exists a number  $\ell \leq k$  such that  $a_i > 0$  if and only if  $i \leq \ell$ . For  $1 \leq i \leq \ell$  let  $V_1^i \supset V_2^i \supset \cdots$  be a closed descending chain of contractible neighborhoods of the polytope  $P(\xi_i)$  of projective measured geodesic laminations supported in  $\xi_i$  in the sphere  $\mathcal{PML}(S_i)$  of projective measured geodesic laminations on  $S_i$ . If  $S_i$  is an annulus, then by convention,  $\mathcal{PML}(S_i)$  consists of two points. We assume that each of the sets  $V_j^i$  is homeomorphic to a closed ball and that for each  $j \geq 1$  there exists a deformation retraction  $R_j^i : V_j^i \to V_{j+1}^i$  which maps  $V_j^i \setminus V_{j+1}^i$  into  $V_{j+1}^i \setminus V_{j+2}^i$ . We also may assume that there exists an increasing sequence

 $m(j) \to \infty$  so that for every  $i \le \ell$  and every  $[\nu] \in V_j^i$  the following properties are satisfied.

- (1)  $\tau[\nu] \ge 2m(j)$ .
- (2) If the support of  $\zeta \in V_j^i$  is minimal and fills, and if c is a shortest distance projection of supp $(\zeta)$  into a p-quasi-geodesic connecting the basepoint  $x_i$  to  $\xi_i$ , then  $d_{\mathcal{CG}(S_i)}(c, x_i) \geq 2m(j)$ .

Recall to this end that  $\tau[\nu] = \infty$  for every  $\nu \in P(\xi_i)$  since  $\xi_i$  is minimal and filling by assumption, and that a shortest distance projection of  $\mathcal{CG}(S_i)$  into any p-quasigeodesic connecting  $x_i$  to  $\xi_i$  extends to  $\partial \mathcal{CG}(S_i) \setminus \xi_i$ .

For a pair of points  $X, Y \in \mathcal{T}(S_i)$  define

$$\hat{d}_{\mathcal{T}}(X,Y) = \max\{d_{\mathcal{CG}(V)}(\operatorname{pr}_{V}(\mu(X)), \operatorname{pr}_{V}(\mu(Y))) \mid V\}$$

where the maximum is over all subsurfaces V of  $S_i$  and  $\mu(X), \mu(Y)$  are short markings.

**Theorem 7.8** (Theorem B of [R14]). For any Teichmüller geodesic  $\gamma : [0, \infty) \to \mathcal{T}(S_i)$ , the function  $t \to \hat{d}_{\mathcal{T}}(\gamma(0), \gamma(t))$  is coarsely non-decreasing with control constant not depending on  $\gamma$ .

Proof. By Theorem B of [R14], there is a number p>0 only depending on the complexity of S such that for every subsurface V of S, the image under the map  $\operatorname{pr}_V \circ \mu$  of a Teichmüller geodesic  $\gamma: \mathbb{R} \to \mathcal{T}(S)$  is an unparameterized p-quasigeodesic in  $\mathcal{CG}(V)$ . This means the following. There is an increasing homeomorphism  $\sigma: (a,b) \subset \mathbb{R} \to \mathbb{R}$  such that the map  $\operatorname{pr}_V \circ \mu \circ \gamma \circ \sigma: (a,b) \to \mathcal{CG}(V)$  is a p-quasi-geodesic. This quasi-geodesic may be bounded, one-sided infinite or two-sided infinite. Since  $\mathcal{CG}(V)$  is a hyperbolic geodesic metric space, this implies that the path  $t \to \operatorname{pr}_V \circ \mu \circ \gamma(t)$  coarsely does not backtrack: There exists a universal constant q>0 not depending on the subsurface V such that for  $0 \le s \le t$ , it holds

$$d_{\mathcal{CG}(V)}(\operatorname{pr}_{V}(\mu(\gamma(0))), \operatorname{pr}_{V}(\mu(\gamma(t)))) \geq d_{\mathcal{CG}(V)}(\operatorname{pr}_{V}(\mu(\gamma(0))), \operatorname{pr}_{V}(\mu(\gamma(s)))) - b.$$

As the projections  $\operatorname{pr}_V(\mu(X))$  only coarsely determine a point in the curve graph of V, the distances in this formula are only coarsely well defined, but this does not affect the validity of the estimate.

As a consequence, for every subsurface V of  $S_i$  and every Teichmüller geodesic  $\gamma:[0,\infty)\to \mathcal{T}(S_i)$  the function

$$t \to d_{\mathcal{CG}(V)}(\operatorname{pr}_V(\mu(\gamma(0))), \operatorname{pr}_V(\mu(\gamma(t))))$$

is coarsely non-decreasing, with control constant q not depending on V. Then the same holds true for  $\hat{d}_{\mathcal{T}}$ .

The following proposition is the technically most involved part of the proof of our main theorem. In its formulation, we denote by  $\Lambda_i = \Lambda_{\epsilon,i} : \mathcal{T}(S_i) \to \mathring{\mathcal{T}}_{\epsilon}(S_i)$  a homeomorphism as constructed in Proposition 4.4. Then  $\Lambda_{\epsilon} = \prod \Lambda_{i,\epsilon}$  is a homeomorphism of  $\prod \mathcal{T}(S_i)$  onto  $\prod \mathring{\mathcal{T}}_{\epsilon}(S_i)$ .

**Proposition 7.9.** Assume that  $\xi = \sum_{i=1}^{k} a_i \xi_i$  is such that  $\xi_i \in \partial \mathcal{CG}(S_i)$  for all i. For integers  $j, n \geq 1$  and for  $\delta > 0$  there is an open subset  $E(j, n, \delta)$  of  $\prod \mathcal{T}(S_i)$  with the following property.

- (1)  $E(j, n, \delta)$  is contractible.
- (2)  $E(j, n, \delta) \subset E(j', n', \delta')$  for  $j \ge j', n \ge n', \delta \le \delta'$ .
- (3) The small closures of the sets  $\Lambda_{\epsilon}E(j,n,\delta)$   $(j \geq 1, n \geq 1, \delta > 0)$  in  $\overline{\mathcal{T}}(\cup_{i}S_{i})$  define a neighborhood basis of  $\xi$ .

*Proof.* Assume by reordering that for some  $1 \le \ell \le k$  we have  $a_i > 0$  if and only if  $i \le \ell$ . For  $i \le \ell$  let  $V_j^i$  be as above. For each  $[\nu_i] \in V_j^i$  choose a parameterization of the geodesic  $t \to \exp_{X_i}(t\sigma[\nu_i])$  on  $[0,\tau[\nu_i])$  as constructed in equation (3) above. Note that by the choice of the constants m(j), the domain of definition of this parameterization contains the interval [0,m(j)], and the restriction of this parameterization to [0,m(j)] depends continuously on  $[\nu_i]$ . Denote by  $\hat{\gamma}_{[\nu_i]}:[0,\tau[\nu_i])\to \mathcal{T}(S_i)$  this parameterization.

Theorem 7.8 shows that for any Teichmüller geodesic  $\gamma:[0,\infty)\to \mathcal{T}(S_i)$ , the function  $t\to \hat{d}_{\mathcal{T}}(\gamma(0),\gamma(t))$  is coarsely non-decreasing, with fixed control constant q>0. Put  $\tilde{d}_{\mathcal{T}}(\gamma(0),\gamma(t))=\sup_{s\leq t}\hat{d}_{\mathcal{T}}(\gamma(0),\gamma(s))$ . By uniqueness of Teichmüller geodesics between any pair of points, this defines a function  $\mathcal{T}(S_i)\times\mathcal{T}(S_i)\to[0,\infty)$  which however may not be symmetric. For any Teichmüller geodesic  $\gamma$ , the function  $t\to\tilde{d}_{\mathcal{T}}(\gamma(0),\gamma(t))$  is non-decreasing.

For an  $\ell$ -tuple  $(j_1, ..., j_{\ell}) \in \mathbb{N}^{\ell}$  put  $m(j_1, ..., j_{\ell}) = \min\{m(j_i) \mid i\}$ . For  $i \leq \ell$  and  $j \geq 1$  put  $\hat{V}^i_j = V^i_j \setminus V^i_{j+1}$ . For  $([\nu_1], ..., [\nu_{\ell}]) \in \hat{V}^1_{j_1} \times \cdots \times \hat{V}^{\ell}_{j_{\ell}}$  and  $\delta > 0$ ,  $n < m(j_1 - 1, ..., j_{\ell} - 1)/2$  define

$$F(n, \delta, [\nu_1], \dots, [\nu_{\ell}]) = \{ (\hat{\gamma}_{[\nu_1]}(t_1), \hat{\gamma}_{[\nu_2]}(t_2), \dots, \hat{\gamma}_{[\nu_{\ell}]}(t_{\ell}), z_{\ell+1}, \dots, z_k) \in \prod \mathcal{T}(S_i) \mid t_i \ge n, |t_i/t_1 - a_i/a_1| \le \delta \text{ if } t_i < m(j_1 - 1, \dots, j_{\ell} - 1) \text{ for } i \le \ell,$$

$$\tilde{d}_{\mathcal{T}(S_i)}(z_i, X_i) < \delta t_1 \text{ for } i \ge \ell + 1 \}.$$

Claim: The set  $\Phi(j, n, \delta) = \bigcup_{[\nu_i] \in V_j^i} F(n, \delta, [\nu_1], \dots, [\nu_\ell])$  is contractible for every  $n \leq m(j_1 - 1, \dots, j_\ell - 1)$ .

Proof of the claim. Note first that if  $(z_1,\ldots,z_\ell,z_{\ell+1},\ldots,z_k)\in\Phi(j,n,\delta)$  then the same holds true for  $(z_1,\ldots,z_\ell,z'_{\ell+1},\ldots,z'_k)$  for any  $z'_i$  which is contained in the Teichmüller geodesic connecting  $X_i$  to  $z_i$  and all  $i\geq \ell+1$ . Thus retracting component wise the last  $k-\ell$  components  $z_i$  to the basepoint  $X_i$   $(i\geq \ell+1)$  along the unique Teichmüller geodesic connecting  $X_i$  to  $z_i$  and keeping the remaining components fixed defines a retraction of  $\Phi(j,n,\delta)$  to  $\Phi(j,n,\delta)\cap\{(z_1,\ldots,z_k)\mid z_i=X_i \text{ for }\ell+1\leq i\leq k\}$ . In particular, in the remainder of the construction, it suffices to assume that  $\ell=k$ .

Next observe that  $\Phi(j+1,n,\delta) \subset \Phi(j,n,\delta)$  for all  $j,n,\delta$ . We construct a homotopy of  $\Phi(j,n,\delta)$  into its subset  $\Phi(j+1,n,\delta)$  as follows.

The set

$$S(m(j-1)) = \{(t_1, \dots, t_k) \in [n, \infty)^k \mid |t_i/t_1 - a_i/a_1| \le \delta \text{ if } t_i < m(j-1) \,\forall i\}$$

admits a deformation retraction onto its subset

$$S(m(j)) = \{(t_1, \dots, t_k) \in [n, \infty)^k \mid |t_i/t_1 - a_i/a_1| \le \delta \text{ if } t_i < m(j) \,\forall i\}.$$

Namely, define a homotopy  $h:[0,1]\times[n,\infty)\to[n,\infty)$  by

$$h(u,t) = \begin{cases} \min\{t(1 - u + u(m(j)/m(j-1))), m(j)\} & \text{if } t < m(j) \\ t & \text{if } t \ge m(j). \end{cases}$$

Then for any  $t_1, t_2 \in [n, m(j-1))$  and u with  $h(u, t_1) < m(j), h(u, t_2) < m(j)$  we have  $h(u, t_1)/h(u, t_2) = t_1/t_2$ . As a consequence, the map  $(u, (t_1, \ldots, t_k)) \rightarrow (h(u, t_1), \ldots, h(u, t_k))$  preserves S(m(j-1)), and it defines a homotopy of S(m(j-1)) into S(m(j)).

Composing this deformation of the domain S(m(j-1)) into S(m(j)) with the map

$$(t_1, \ldots, t_k) \to (\hat{\gamma}_{[\nu_1]}(t_1), \ldots, \hat{\gamma}_{[\nu_k]}(t_k))$$

defines a homotopy of  $\Phi(j, n, \delta)$  into its subset

$$\Xi = \Phi(j, n, \delta) \cap \{ (\hat{\gamma}_{[\nu_1]}(t_1), \dots \hat{\gamma}_{[\nu_k]}(t_k)) \mid |t_i/t_1 - a_i/a_1| \le \delta \text{ if } t_i < m(j) \, \forall i \}.$$

The deformation retractions  $R^i_j:[0,1[\times V^i_j\to V^i_j \text{ of } V^i_j \text{ onto } R^i_j(V^i_j\times\{1\})=V^i_{j+1}$  induce a deformation retraction

$$R_j: [0,1] \times V_i^1 \times \cdots \times V_i^k \to V_i^1 \times \cdots \times V_i^k$$

onto  $V^1_{j+1} \times \cdots \times V^k_{j+1}$  by applying the deformation retractions  $R^i_j$  component wise. Since for each i, the image of  $V^i_j \setminus V^i_{j+1}$  is contained in  $V^i_{j+1} \setminus V^i_{j+2}$ , we obtain a deformation retraction of  $\Xi$  onto its subset  $\Phi(j+1,n,\delta)$  by defining

$$(s, (\gamma_{[\nu_1]}(t_1), \dots, \gamma_{[\nu_k]}(t_k))) \to (\gamma_{R_j^1(s, [\nu_1])}(t_1), \dots, \gamma_{R_j^k(s, [\nu_k])}(t_k)).$$

The composition of these two homotopies yields a homotopy of  $\Phi(j, n, \delta)$  into  $\Phi(j + 1, n, \delta)$ .

Now  $\cap_j \Phi(j, n, \delta) = \bigcup_{[\nu_i] \in P(\xi_i)} F(n, \delta, [\nu_1], \dots, [\nu_k])$ , and since  $P(\xi_i)$  is contractible for all i, this set is contractible as well. This completes the proof of the claim.

So far we constructed from a tuple of contractible neighborhoods  $V_i^j$  (j = 1, ..., k) and numbers  $j > 0, \delta > 0$  a contractible subset  $\Phi(j, n, \delta)$  of  $\mathcal{T}(S) = \prod \mathcal{T}(S_i)$ . We aim at using these sets to construct contractible neighborhoods of  $\xi$  in  $\overline{\mathcal{T}}(\cup_i S_i)$ .

Claim: For fixed  $(j, n, \delta)$ , if  $X^{\ell} \subset \prod \mathcal{T}_{\epsilon}(S_i)$  is a sequence converging to  $\xi$ , then  $X^{\ell} \in \overline{\Lambda_{\epsilon}\Phi(j, n, \delta)}$  for large enough  $\ell$ .

Proof of the claim: Let  $X^u = (X_1^u, \dots, X_k^u) \subset \prod \mathcal{T}_{\epsilon}(S_i)$  be a sequence converging to  $\xi$ . We show first that  $X^u \in \Phi(j, n, \delta)$  for large enough u.

For  $i \leq \ell$  let  $[\nu_i] \in \mathcal{PML}(S_i)$  be such that  $\operatorname{supp}([\nu_i]) = \xi_i$ . Let  $\hat{\gamma}_i : [0, \infty) \to \mathcal{CG}(S_i)$  be the p-quasi-geodesic constructed as a reparameterization of the Teichmüller geodesic  $\gamma_i(t) = \Upsilon(\exp_{X_i} t[\nu_i])$  as before and let  $\Pi_i : \mathcal{CG}(S_i) \to \hat{\gamma}_i$  be a shortest distance projection. We then have

$$(4) \quad d_{\mathcal{CG}(S_i)}(\Pi_i(\Upsilon(X_i)), \Pi_i(\Upsilon(X_i^u))) / d_{\mathcal{CG}(S_1)}(\Pi_1(\Upsilon(X_1)), \Pi_1(\Upsilon(X_1^u))) \to a_i/a_1.$$

Furthermore, for  $i \ge \ell + 1$  it holds

(5) 
$$\hat{d}_{\mathcal{T}}(X_i, X_i^u) / \min_{i \leq \ell} d_{\mathcal{CG}(S_i)}(\Pi_i(\Upsilon(X_i)), \Pi_i(\Upsilon(X_1^u))) \to 0.$$

Let  $[\eta_i^u] \in \mathcal{PML}(S_i)$  and  $t_i^u \geq 0$  be such that  $X_i^u = \hat{\gamma}_{[\eta_i^u]}(t_i^u)$ . Then for all  $i \leq \ell$ , we have  $t_i^u \to \infty$   $(u \to \infty)$ , moreover by Lemma 7.3 and its proof, it holds  $[\eta_i^u] \to P(\xi_i)$   $(u \to \infty)$ . Thus for large enough u and all  $i \leq \ell$ , there is some  $j(i,u) \geq 1$  so that  $[\eta_i^u] \in \hat{V}^i_{j(i,u)}$ . As  $t_i^u \to \infty$   $(u \to \infty)$  for all i, equation (5) shows that for sufficiently large u and all  $i \geq \ell+1$  we have  $\hat{d}_{\mathcal{T}}(X_i, X_i^u) < \min_{i \leq \ell} \delta t_i^u/2$ . By the definition of the set  $\Phi(j, n, \delta)$ , this implies that  $X^u \in \Phi(j, n, \delta)$  for large enough u if and only if this holds true for  $(X_1^u, \ldots, X_\ell^u, X_{i+1}, \ldots, X_k)$  (here as before,  $X_i$  is the basepoint). Consequently we obtain as in the beginning of this proof that it suffices to assume that  $a_i > 0$  for all  $i \leq k$ . Thus assume from now on that  $\ell = k$ .

For large enough u put  $n(u) = m(j(1,u) - 1, \ldots, j(k,u) - 1)$ . If  $t_i^u \leq n(u)$  for all u, then by the choice of the constants m(j), the shortest distance projections of  $\Upsilon(X_i^u)$  into a p-quasi-geodesic  $\gamma_i$  connecting the basepoint  $x_i$  to  $\xi$  is uniformly close to  $\Upsilon(X_i^u)$ . By perhaps increasing u, it then follows from (4) that the point  $X^u = (X_1^u, \ldots, X_k^u)$  is contained in the set  $\Phi(j, n, \delta)$ . Otherwise  $X^u$  is contained in  $\Phi(j, n, \delta)$  because there is no constraint on the distance ratios in the definition of the sets  $F(n, \delta, [\eta_1^u], \ldots, [\eta_k^u])$  provided that  $t_i^u \geq m(j)$  for at least one i. From this we deduce that indeed, we have  $X^u \in \Phi(j, n, \delta)$  for large u.

We are left with deducing that in fact  $X^u \in \overline{\Lambda_{\epsilon}(\Phi(j, n, \delta))}$ . However, the map  $\Lambda_{\epsilon} = \Lambda_{1,\epsilon} \times \cdots \times \Lambda_{k,\epsilon}$  is coarsely  $\Upsilon$ -invariant for each i. As the defining properties of the sets  $\Phi(j, n, \delta)$  only depend on distances in the curve graph of the surfaces  $S_i$ , we conclude that  $X^u \in \overline{\Lambda_{\epsilon}(\Phi(j, n, \delta))}$  for large u.

We are left with showing that the closures  $\Lambda_{\epsilon}\Phi(j,n,\delta)$  of the sets  $\Lambda_{\epsilon}\Phi(j,n,\delta)$  form a neighborhood basis of  $\xi$  in  $\overline{\mathcal{T}}(\cup_i S_i)$ . Since by Proposition 4.15 and Theorem 5.1 the space  $\overline{\mathcal{T}}(\cup_i S_i)$  is compact and metrizable, to this end it suffices to show that the intersection  $\cap_{j,n,\delta}\overline{\Lambda_{\epsilon}\Phi(j,n,\delta)}=\{\xi\}$ . As  $\xi$  clearly is contained in this intersection, it suffices to show that it is unique with this property.

Following the reasoning in the proof of Lemma 7.3, note that

$$\cap \overline{\Lambda_{\epsilon}\Phi(j,n,\delta)} \cap \prod \mathcal{T}(S_i) = \emptyset.$$

Namely, since the map  $\Upsilon$  is coarsely Lipschitz, for all j this set only contains points which project to tuples of points of Teichmüller distance at least an to the basepoint  $(X_1, \ldots, X_k)$  where a > 0 is a universal constant. But this immediately implies that the closures of the intersections of the sets  $\Phi(j, n, \delta)$  do not contain points in  $\prod \mathcal{T}(S_i)$ .

In the same way we see that  $\xi$  is the unique boundary point by letting j tend to infinity and letting  $\delta \to 0$ .

Corollary 7.10. Each  $\xi = \sum_i a_i \xi_i \in \mathcal{E}$  admits a neighborhood basis in  $\overline{\mathcal{T}}(\cup_i S_i)$  consisting of small closures of open contractible subsets of  $\prod \mathring{\mathcal{T}}_{\epsilon}(S_i)$ .

Proof. Let  $E_1 \supset E_2 \supset \ldots$  be a countable neighborhood basis of  $\xi \in \mathcal{X}(\cup_i S_u)$  consisting of open sets whose intersections with  $\prod \mathring{\mathcal{T}}_{\epsilon}(S_i)$  are contractible. Then the small closure of  $E_j$  in  $\mathcal{T}(S)$  is contractible, and it is the intersection with  $\prod \mathcal{T}_{\epsilon}(S_i)$  of a neighborhood of  $\xi$  in  $\overline{\mathcal{T}}(\cup_i S_i)$ . These sets form a neighborhood basis of  $\xi$  with the properties stated in the corollary.

Since a surface of complexity zero is a disjoint union of annuli, whose geometric boundary consists of two points, we obtain as an immediate consequence of Corollary 7.10

Corollary 7.11. Theorem 4.12 holds for surfaces of complexity zero.

7.3. The induction step: Neighborhoods of arbitrary points. In this section we complete the proof of Theorem 4.12. We proceed by induction on the complexity  $\kappa(S)$  of the possibly disconnected surface S. Corollary 7.11 contains the case  $\kappa(S) = 0$ , so assume that Theorem 4.12 has been established for all surfaces of complexity at most k-1 for some  $k-1 \geq 0$ . Let S be a possibly disconnected surface of complexity  $\kappa(S) = k$ . By Section 7.1 and Section 7.2, we are left with constructing neighborhood bases for points  $\xi = \sum_{i=1}^k a_i \xi_i \in \mathcal{X}(S)$  where each  $\xi_i$  fills a proper subsurface  $S_i$  of S (which may be a connected component of S) and that furthermore there exists at least one i such that  $\xi_i$  does not fill a connected component of S. In particular,  $\text{supp}(\xi)$  fills a subsurface of S with the property that there exists at least one non-peripheral simple closed curve  $c \subset S$  contained in the boundary of  $\text{supp}(\xi)$ .

Let c be such a simple closed curve. Then  $S_c = S \setminus c$  is a (possibly disconnected) surface of complexity k-1 and (with a small abuse of notation) we can write  $S = S_c \sqcup A_c$  where  $A_c$  is the annulus with core curve c. We then can view  $\xi$  as an element in the geometric boundary of the disconnected surface  $S_c^* \sqcup A_c$ . Since the complexity of  $S_c$  is at most k-1, by the induction hypothesis, Theorem 4.12 holds true for  $S_c \sqcup A_c$ .

The infinite cyclic group generated by the left Dehn twist  $T_c$  about c equals the mapping class group of  $A_c$ . The stabilizer Stab(c) of c in the mapping class group Mod(S) fits into the exact sequence

(6) 
$$1 \to \langle T_c \rangle \to \operatorname{Stab}(c) \to \operatorname{Mod}(S_c^*) \to 1,$$

however this sequence does not split in general. We will nevertheless use the  $\mathcal{EZ}$ -structure for  $\mathrm{Mod}(S_c^*) \times \langle T_c \rangle$ , viewed as the mapping class group of  $\mathcal{T}(S_c^*) \times A_c$ , to construct a neighborhood basis of  $\xi$  in  $\overline{\mathcal{T}}(S)$  as follows.

Consider the augmented Teichmüller space  $\mathcal{T}^{\operatorname{aug}}(S)$  of S [Wo03, Ya04]. This is a stratified space whose open stratum of maximal dimension equals the Teichmüller space  $\mathcal{T}(S)$ . For each multi-curve  $\beta$  on S there exists a stratum  $\mathcal{S}(\beta)$  which equals

the Teichmüller space of the surface  $(S \setminus \beta)^*$  obtained from  $S \setminus \beta$  by replacing each boundary component by a puncture. This Teichmüller space is a direct product of Teichmüller spaces, one for each component of  $S \setminus \beta$ . The strata in the boundary of  $S(\beta)$  correspond to multi-curves containing  $\beta$  as a subset.

The set N(c) of points  $Y \in \mathcal{T}(S)$  so that the Y-length of c is at most  $\epsilon$  is a tubular neighborhood of S(c) in the augmented Teichmüller space  $\mathcal{T}^{\operatorname{aug}}(S)$ . Its boundary  $\partial N(c)$  is invariant under the action of the infinite cyclic group of Dehn twists  $\langle T_c \rangle$  about c.

The augmented Teichmüller space  $\mathcal{T}^{\operatorname{aug}}(S)$ , equipped with the completion of the Weil-Petersson metric, is a (non-locally compact) CAT(0) space. The strata are convex subspaces. In particular, there exists a shortest distance projection  $\Pi: \partial N(c) \to \mathcal{S}(c)$ . Since Dehn twists are isometries, this projection is invariant under the action of the group  $\langle T_c \rangle$ . More precisely, the fibers of  $\Pi$  are homeomorphic to  $\mathbb{R}$  and invariant under  $\langle T_c \rangle$  and hence via the map  $\Pi$ , the quotient  $\langle T_c \rangle \backslash \partial N(c)$  obtains the structure of a circle bundle over  $\mathcal{S}(c)$ . As  $\mathcal{S}(c)$  is contractible, this circle bundle is trivial (however non-canonically).

Recall that the choice of a base marking of S coarsely determines for each  $Y \in \mathcal{T}(S)$  a twist parameter  $\tau(Y,c) \in \mathbb{Z}$  about c, unique up to an error of  $\pm 1$ .

**Lemma 7.12.** There exists a continuous map  $\sigma : \mathcal{S}(c) \to \partial N(c)$  with the following properties.

- (1)  $\Pi(\sigma(x)) = x$  for all x.
- (2) There exists a constant b > 0 so that  $\tau(\sigma(Y), c) \in [-b, b]$  for all  $Y \in \mathcal{S}(c)$ .

Proof. Since  $\partial N(c)$  is a fiber bundle over the smooth contractible manifold  $\mathcal{S}(c)$  with contractible fiber, for each point  $z \in \mathcal{S}(c)$  we can find a neighborhood  $U_z$  of z and a local section  $\sigma_z : \mathcal{S}(c) \to \partial N(c)$  for the projection  $\Pi$  so that  $\tau(\sigma_z(y), c) \in [-4, 4]$  for all  $y \in U_z$ . Note that this makes sense in spite of the fact that  $\tau(\cdot, c)$  is only coarsely well defined. Using a partition of unity and the fact that the fiber of  $\Pi$  is contractible, these local sections can be patched together to a global section with the properties in the lemma.

A section  $\sigma$  as in Lemma 7.12 is an embedding of  $\mathcal{S}(c)$  into  $\partial N(c)$ . This embedding can be used to construct a homeomorphism  $\Sigma: \mathcal{S}(c) \times \mathbb{R} \to \partial N(c)$  which is equivariant with respect to the action of  $\mathbb{Z}$  on  $\mathbb{R}$  by translation and the action of the infinite cyclic group of Dehn twists about c on  $\partial N(c)$ . As  $\mathcal{S}(c) \times \mathbb{R}$  is just the product of the Teichmüller space  $\mathcal{T}(S_c^*)$  and the Teichmüller space of the annulus with core curve c, the same holds true for  $\partial N(c)$ .

Let  $[\mu]$  be the projective measured geodesic lamination on the surface S whose support equals the simple closed curve c. By the Hubbard Masur theorem, the Teichmüller space  $\mathcal{T}(S)$  of S is foliated by Teichmüller geodesics with horizontal measured geodesic lamination in the class of  $[\mu]$ . As Dehn twists about c act as isometries for the Teichmüller metric and  $[\mu]$  is  $T_c$ -invariant, the foliation is invariant under the action of the infinite cyclic group generated by  $T_c$  and the

action of the elements of Mod(S) which can be represented by homeomorphisms fixing  $S_c$  pointwise.

If  $\gamma: \mathbb{R} \to \mathcal{T}(S)$  is a Teichmüller geodesic with horizontal measured geodesic lamination in the class of  $[\mu]$ , then as  $t \to \infty$ , the simple closed curve c is collapsed to a node. As a consequence, by perhaps replacing  $\epsilon$  by a smaller constant, we may assume that these Teichmüller geodesics intersect  $\partial N(c)$  transversely, and two Teichmüller geodesic passing through different points of  $\partial N(c)$  do not intersect. In other words, each of these Teichmüller geodesics passes through a unique point  $Y \in \partial N(c)$  and hence they can be parameterized by a map  $\gamma_Y: \mathbb{R} \to \mathcal{T}(S)$  in such a way that  $\gamma_Y(0) = Y$  and  $\gamma_Y(-\infty, 0) \subset N(c)$ . Then the geodesic lines  $\gamma_Y(-\infty, \infty)$  foliate  $\mathcal{T}(S)$ .

The following lemma is a consequence of the article [R14].

**Lemma 7.13.** There exists a number D > 0 with the following property. Let  $Y \in \partial N(c)$ ; then for any not necessarily proper subsurface V of  $S_c$  we have  $\operatorname{diam}(\operatorname{pr}_{V^*}(\gamma_Y(-\infty,\infty)) \leq D$ .

Proof. Let  $t \to q(t)$  be the cotangent line of the geodesic  $\gamma_Y$ . For each t, q(t) is an area one quadratic differential over the surface  $\gamma_Y(t)$ . As  $[\mu]$  is defined by a single simple closed curve, each q(t) is a one cylinder Strebel differential with core curve c. This means that the connected component S' containing c of the surface S, equipped with the singular flat metric defined by q(t), consists of a single flat cylinder glued along its sides to the surface S' by identifying subarcs of the boundary of the same lengths in pairs. The image in S' of the sides of this cylinder is the critical graph of the flat metric, which is a finite graph G composed of horizontal saddle connections.

In this flat metric, the surface  $S'_c = S' \setminus c$  is degenerate, that is, the critical graph is a deformation retraction of  $S'_c$ . The mass deposited on the edges of the critical graph by the vertical measured geodesic lamination of q(t) equips G with the structure of a metric graph. Any simple closed curve  $\alpha$  in  $S'_c = S' \setminus c$  is then homotopic to a closed edge path in G, and a closed edge path of minimal length is unique up to parameterization and is the geodesic representing the free homotopy class of  $\alpha$  for the locally CAT(0)-metric q(t).

The singular flat metric for q(t) is obtained from the singular flat metric on q(0) by multiplying the horizontal length, that is, the circumference of the cylinder, with  $e^{t/2}$ , and the vertical length, that is, the height, with  $e^{-t/2}$ . As a consequence, if for a subsurface V of  $S'_c$  we define the size  $\operatorname{size}_{q(t)}(V)$  of V with respect to the metric q(t) as the shortest q(t)-length of a closed q(t)-geodesic in V and if for a non-peripheral simple closed curve  $\alpha \subset S'_c$  we denote by  $\ell_{q(t)}(\alpha)$  the length of its q(t)-geodesic representative, then

$$\log \frac{\operatorname{size}_{q(t)}(S'_c)}{\ell_{q(t)}(\alpha)}$$

does not depend on t. From Theorem 3.1 of [R14], one deduces that the extremal length of any non-peripheral simple closed curve in  $S'_c$  along the Teichmüller geodesic  $t \to \gamma_Y(t)$  is bounded from below by a universal positive constant. The same

holds true for the extremal length of c along the ray  $\gamma_Y[0,\infty)$ , which is exponentially increasing along the ray.

As a consequence of Section 5 of [R14], for any (not necessarily proper) subsurface V of  $S'_c$  the active interval for V is empty along  $\gamma_Y[0,\infty)$ . Theorem A of [R14] then shows that for each such V, the diameter of the projection  $\{\operatorname{pr}_V(\mu(\gamma_Y(t))) \mid t \in [0,\infty)\}$  is bounded from above by a universal constant. This is what we wanted to show.

Recall that the Teichmüller space of the annulus  $A_c$  is naturally identified with the real line  $\mathbb{R}$ . Start with a countable family  $\mathcal{V} = \{V_i \mid i\}$  of open contractible subsets of  $\mathring{\mathcal{T}}_{\epsilon}(S_c^*) \times \mathbb{R}$  whose small closures define a neighborhood basis of  $\xi$  in  $\overline{\mathcal{T}}(S_c^* \sqcup A_c) = \mathcal{T}_{\epsilon}(S_c^*) \times \mathbb{R} \cup \mathcal{X}(S_c^*) * \mathcal{X}(A_c)$ . Such a neighborhood basis exists since by the induction hypothesis, Theorem 4.12 holds true for  $S_c \sqcup A_c$ . Let  $\Lambda_{\epsilon} : \mathcal{T}(S_c^*) \to \mathring{\mathcal{T}}_{\epsilon}(S_c^*)$  be the  $\mathrm{Mod}(S_c)$ -equivariant homeomorphism from Corollary 4.4 and for each i define  $W_i = (\Lambda_{\epsilon}^{-1} \times \mathrm{Id})(V_i)$ . Then  $W_i$  is an open and contractible subset of  $\mathcal{T}(S_c^*) \times \mathbb{R}$ . Denote by  $E_i \subset \partial N(c)$  its image under the identification of  $\partial N(c)$  with  $\mathcal{T}(S_c^*) \times \mathbb{R}$  using the section  $\sigma$ .

**Proposition 7.14.** Put  $A_i = \{ \gamma_Y(-\infty, \infty) \mid Y \in E_i \subset \partial N(c) \}$ ; then the sets  $\Lambda_{\epsilon}(A_i) \subset \mathring{\mathcal{T}}_{\epsilon}(S)$  are open and contractible, and their small closures in  $\overline{\mathcal{T}}(S)$  define a neighborhood basis of  $\xi$  in  $\overline{\mathcal{T}}(S)$ .

*Proof.* Since the geodesics with horizontal projective measured lamination  $[\mu]$  foliate  $\mathcal{T}(S)$  and  $\partial N(c)$  is transverse to these geodesics, the set  $A_i$  admits a deformation retraction onto  $E_i$ . Thus since the sets  $E_i$  are contractible, the same holds true for the sets  $A_i$  and for the sets  $U_i = \Lambda_{\epsilon}(A_i)$ .

We have to show that the small closures of the sets  $U_i \subset \mathring{\mathcal{T}}_{\epsilon}(S)$  in  $\overline{\mathcal{T}}(S)$  define a neighborhood basis of  $\xi$  in  $\overline{\mathcal{T}}(S)$ . This is the case if for any sequence  $X_u \subset \mathring{\mathcal{T}}_{\epsilon}(S)$  converging to  $\xi$ , all but finitely many  $X_u$  are contained in  $U_i$ .

By Lemma 7.13, for each  $Y \in \partial N(c)$  and any subsurface V of  $S_c$ , the diameter of the subsurface projection  $\operatorname{pr}_V(\mu(\gamma_Y(-\infty,\infty)))$  is uniformly bounded, independent of Y. By the definition of the topology on  $\overline{\mathcal{T}}(S)$  and coarse  $\Upsilon$ -invariance of the projection  $\Lambda_\epsilon$ , this implies that for each i the small closure  $\overline{U}_{i,\text{small}}$  in  $\overline{\mathcal{T}}(S)$  of the set  $U_i$  is indeed a neighborhood of  $\xi$  in  $\mathcal{T}(S)$ . The same discussion also shows that  $\cap_i \overline{U}_{i,\text{small}} = \{\xi\}$  and hence the sets  $\overline{U}_{i,\text{small}}$  define a neighborhood basis of  $\xi$  in  $\overline{\mathcal{T}}(S)$  as claimed in the proposition.

Proof of Theorem 4.12. We proceed by induction on the complexity  $\kappa(S)$  of S. The case  $\kappa(S) = 0$  is contained in Corollary 7.11. Thus assume that the statement holds true for all surfaces of complexity at most k-1 for some  $k-1 \geq 0$ .

Let S be a surface of complexity k. By Proposition 4.14, there exists a consistent topology on  $\overline{\mathcal{T}}(S)$  with properties (2) and (3) in the statement of Theorem 4.12. By Theorem 5.1, this topology is metrizable, and Corollary 6.2 shows that the pair  $(\overline{\mathcal{T}}(S), \mathcal{X}(S))$  is a pair of spaces of finite dimension. Furthermore, by Proposition 6.3, the action of  $\operatorname{Mod}(S)$  on  $\overline{\mathcal{T}}(S)$  is  $\mathcal{U}$ -small for every open covering  $\mathcal{U}$  of  $\overline{\mathcal{T}}(S)$ .

As a consequence, to complete the proof of Theorem 4.12, it suffices to verify that the topology on  $\overline{\mathcal{T}}(S)$  is nice. If S is connected, then this follows from Proposition 7.1 and Proposition 7.14. If S is disconnected, then it is a consequence of Proposition 7.9 and Proposition 7.14.

As an application, we obtain.

Corollary 7.15.  $\dim(\partial \mathcal{CG}(S)) \leq \operatorname{vcd}(\operatorname{Mod}(S)) - 1$ .

*Proof.* Since  $\mathcal{X}(S)$  is a  $\mathcal{Z}$ -set for a torsion free finite index subgroup  $\Gamma$  of Mod(S), the cohomological dimension of  $\mathcal{X}(S)$  equals vcd(Mod)(S) - 1 [B96]. Furthermore, this dimension also equals the covering dimension of  $\mathcal{X}(S)$  [B96].

Now as  $\partial \mathcal{CG}(S)$  is embedded in  $\mathcal{X}(S)$ , it is equipped with the subspace topology. This means that any open covering of  $\partial \mathcal{CG}(S)$  is the restriction of an open covering of  $\mathcal{X}(S)$ . Such a covering then has a  $\operatorname{vcd}(\operatorname{Mod}(S)) - 1$ -finite refinement and hence the same holds true for the refinement of the original cover of  $\partial \mathcal{CG}(S)$ .

The following conjecture is taken from [BB19]. We believe that the results in this work support this conjecture.

Conjecture. For any surface S of finite type,  $\operatorname{asdim}(\operatorname{Mod}(S)) = \operatorname{vcd}(\operatorname{Mod}(S))$ .

We are left with showing Corollary 6 and Corollary 7 from the introduction.

Proof of Corollary 6. By Theorem 4,  $\operatorname{Mod}(S)$  admits an  $\mathcal{EZ}$ -structure  $(\overline{X}, Z)$  where  $X = \overline{X} \setminus Z$  is a manifold with boundary of dimension 6g - 6 + 2m. Assume that  $6g - 6 + 2m \geq 5$ . By Lemma 2.3 of [FL05], there exists a new  $\mathcal{EZ}$ -structure for  $\operatorname{Mod}(S)$  obtained by doubling X along the boundary. By Proposition 2.1 of [FL05], this structure is of the form  $(\overline{Y}, Z)$  where  $\overline{Y}$  is a manifold with boundary of dimension 6g - 5 + 2m.

Proposition 2.2 of [FL05] then shows that another application of this construction to the pair  $(\overline{Y}, Z)$  results in an  $\mathcal{E}\mathcal{Z}$ -structure given by unit ball in  $\mathbb{R}^{6g-4+2m}$  and a subset Z of its boundary, the sphere of dimension 6g-5+2m. This is what we wanted to show.

Proof of Corollary 7. The corollary follows from the fact that  $\partial \mathcal{CG}(S)$  is embedded in  $\mathcal{X}(S)$ , and by Corollary 6 and its proof,  $\mathcal{X}(S)$  embeds into a manifold of dimension 6g - 6 + 2m and into the sphere  $S^{6g-5+2m}$ .

**Question.** What is the smallest dimension n so that Mod(S) admits an  $\mathcal{EZ}$  structure on a pair  $(\mathbb{D}^n, \Delta)$  where  $\Delta$  is a subset of  $S^{n-1}$ ?

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