A Z-STRUCTURE FOR THE MAPPING CLASS GROUP

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Abstract. We construct a boundary for the mapping class group $\mathrm{Mod}(S)$ of a surface S of finite type. The action of $\mathrm{Mod}(S)$ on this boundary is minimal, strongly proximal and topologically free. The boundary is the boundary of a Z-structure for any torsion free finite index subgroup of $\mathrm{Mod}(S)$.

1. Introduction

The *mapping class group* $\mathrm{Mod}(S)$ of a closed oriented surface S of genus $g \geq 0$ from which $m \geq 0$ points have been removed and so that $3g-3+m \geq 1$ is the group of isotopy classes of diffeomorphisms of S. The mapping class group is well known to be finitely presented, and it admits explicit torsion free finite index subgroups.

A torsion free finite index subgroup Γ of $\mathrm{Mod}(S)$ admits a *finite* classifying space. Such a classifying space can be constructed as follows.

Since the Euler characteristic of S is negative, the *Teichmüller space* $\mathcal{T}(S)$ of S of all *marked* finite area complete hyperbolic structures on S is defined. By elementary hyperbolic geometry, there exists a number $\epsilon_0 > 0$ such that any two closed geodesics on a hyperbolic surface of length at most ϵ_0 are disjoint. The *systole* $\mathrm{systole}(X)$ of a hyperbolic metric X is the length of a shortest closed geodesic. For $\epsilon < \epsilon_0$ define

$$\mathcal{T}_\epsilon(S) = \{X \in \mathcal{T}(S) \mid \mathrm{systole}(X) \geq \epsilon\}.$$

The following is due to Ji and Wolpert [JW10, J14] as reported in Proposition 3.1 and Theorem 3.9 of [J14].

Theorem 1 (Ji-Wolpert)**.** *For sufficiently small $\epsilon < \epsilon_0$, the set $\mathcal{T}_\epsilon(S)$ is a manifold with corners which is a deformation retract of $\mathcal{T}(S)$. The mapping class group $\mathrm{Mod}(S)$ acts on $\mathcal{T}_\epsilon(S)$ properly and cocompactly.*

Since $\mathcal{T}(S)$ is homeomorphic to $\mathbb{R}^{6g-6+2m}$, we obtain that $\mathcal{T}_\epsilon(S)$ is contractible, locally contractible and finite dimensional. As torsion free finite index subgroups Γ of $\mathrm{Mod}(S)$ act freely on $\mathcal{T}_\epsilon(S)$, this implies that $\Gamma\backslash\mathcal{T}_\epsilon(S)$ is a classifying space for Γ.

The goal of this article to construct an explicit compactification $\overline{\mathcal{T}}(S)$ of $\mathcal{T}_\epsilon(S)$ with the property that $\mathcal{X}(S) = \overline{\mathcal{T}}(S) - \mathcal{T}_\epsilon(S)$ is a boundary of $\mathrm{Mod}(S)$ in the following sense.

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Definition 2 (Small boundary)**.** A *boundary* of a finitely generated group Γ is a compact Γ-space Z with the following properties.

- There exists a topology on $\Gamma \cup Z$ which restricts to the discrete topology on Γ, to the given topology on Z and is such that $\Gamma \cup Z$ is compact.
- The left action of Γ on itself extends to the Γ-action on Z.

The boundary is called *small* if the right action of Γ extends to the trivial action of Γ on Z.

The following definition is Lemma 1.3 of [B96].

Definition 3 ($\mathcal{Z}$-structure)**.** A *$\mathcal{Z}$-structure* for a finitely generated torsion free group Γ consists of a pair $(\overline{X}, Z)$ of finite dimensional compact metrizable spaces, with Z nowhere dense in $\overline{X}$, and the following additional properties.

(1) $X = \overline{X} - Z$ is contractible and locally contractible.
(2) For every $z \in Z$ and every neighborhood U of z in $\overline{X}$ there exists a neighborhood $V \subset U$ of z such that the inclusion $V - Z \to U - Z$ is null-homotopic.
(3) X admits a covering space action of Γ with compact quotient.
(4) The collection of all translates of a compact set in X form a null sequence in $\overline{X}$: that is, for every open cover $\mathcal{U}$ of $\overline{X}$, all but finitely many translates are $\mathcal{U}$-small.

An action of a group G on a compact topological space Z is called *minimal* if every G-orbit is dense. It is called *topologically free* if for every $\varphi \in G - \{1\}$ the fixed point set of φ has empty interior. Furthermore, it is called *strongly proximal* if the action of G on the Borel probability measures on Z is such that the closure of any orbit contains some Dirac measure. The following is our main result.

Theorem 4. *There exists a compactification $\overline{\mathcal{T}}(S)$ of $\mathcal{T}_\epsilon(S)$ with the following properties.*

(1) *$\mathcal{X}(S) = \overline{\mathcal{T}}(S) \setminus \mathcal{T}_\epsilon(S)$ is a small boundary for* $\mathrm{Mod}(S)$.
(2) *The action of* $\mathrm{Mod}(S)$ *on $\mathcal{X}(S)$ is minimal, strongly proximal and topologically free.*
(3) *The pair $(\overline{\mathcal{T}}(S), \mathcal{X}(S))$ is a Z-structure for every torsion free finite index subgroup of* $\mathrm{Mod}(S)$.

An alternative approach to this result, based on *hierarchical hyperbolicity*, is due to Durham, Minsky and Sisto [DMS25]. Hierarchical hyperbolicity was also used by Durham, Hagen and Sisto [DHS17] to construct a boundary for $\mathrm{Mod}(S)$. We do not know the relation between these constructions and ours, and hierarchical hyperbolicity for $\mathrm{Mod}(S)$ does not play any role in this article.

We call the space $\mathcal{X}(S)$ the *geometric boundary* of $\mathrm{Mod}(S)$. By work of Kalantar and Kennedy [KK17], it follows from the second part of the above theorem that the mapping class group is C^*-simple, a fact which is however known. For example, it is not hard to see that the mapping class group acts on the compact space of *complete* geodesic laminations minimally, strongly proximally and topologically freely which is sufficient to ensure C^*-simplicity.

The significance of a Z-structure for a torsion free group Γ lies in the fact that the Cech cohomology of the boundary computes the cohomological dimension $\mathrm{cd}(\Gamma)$ of the group, with a dimension shift of one (Theorem 1.7 of [B96]). The virtual cohomological dimension $\mathrm{vcd}(\mathrm{Mod}(S))$ of $\mathrm{Mod}(S)$, that is, the cohomological dimension of a torsion free finite index subgroup was computed by Harer [Har86] to equal $4g-5$ if $m=0$ and $4g-4+m$ if $m>0$. Thus by [B96], the covering dimension of the space $\mathcal{X}(S)$ equals $4g-6$ if $m=0$ and $4g-5+m$ if $m>0$. Theorem 4 can be viewed as giving some evidence that the *asymptotic dimension* of $\mathrm{Mod}(S)$, which is known to be finite and at most quadratic in the virtual cohomological dimension, equals the cohomological dimension of $\mathrm{Mod}(S)$. We refer to [BB19] for a more detailed discussion on this and related questions and results.

We next describe the boundary $\mathcal{X}(S)$ of $\mathrm{Mod}(S)$ as a set.

The *curve complex* $\mathcal{CG}(S_0)$ of a subsurface S_0 of S different from a pair of pants or an annulus is the simplicial complex whose vertices are isotopy classes of simple closed curves and where k such curves span a $k-1$-simplex if they can be realized disjointly. If S_0 is a four-holed sphere or a one holed torus, then this definition has to be modified by connecting two vertices by an edge if they intersect in the minimal number of points.

The curve complex is a hyperbolic geodesic graph of infinite diameter [MM99]. Its *Gromov boundary* $\partial\mathcal{CG}(S_0)$ is the space of *minimal geodesic laminations* on S_0 which *fill* S_0, that is, which intersect every essential simple closed curve on S_0 transversely. The topology on $\partial\mathcal{CG}(S_0)$ is the *coarse Hausdorff topology.* With respect to this topology, a sequence λ_i of minimal filling laminations converges to the lamination λ if and only if the limit of any subsequence which converges in the Hausdorff topology on compact subsets of S_0 contains λ as a sublamination [H06, K99]. The space $\partial\mathcal{CG}(S_0)$ is separable and metrizable. Define the boundary of the curve complex of an essential annulus $A\subset S$ with core curve c to consist of two points c^+,c^-.

If $S_1,\dots,S_k$ is a collection of isotopy classes of pairwise disjoint subsurfaces of S, then we can form the join

$$\mathcal{J}(\cup_{i=1}^k S_i)=\partial\mathcal{CG}(S_1)*\cdots*\partial\mathcal{CG}(S_k).$$

It can be viewed as the set of formal sums $\sum_i a_i\lambda_i$ where $a_i>0$, $\sum_i a_i=1$ and where $\lambda_i\in\partial\mathcal{CG}(S_i)$ for all i. This join is a separable metrizable topological space. Note that if $S_{i_1},\dots,S_{i_s}$ is a subset of the set of surfaces $S_1,\dots,S_k$, then $\mathcal{J}(\cup_{j=1}^s S_{i_j})$ is naturally a non-empty closed subset of $\mathcal{J}(\cup_{i=1}^k S_i)$ corresponding to formal sums $\sum_i a_i\lambda_i$ with $a_i=0$ for $i\not\in\{i_1,\dots,i_s\}$. Define

$$\mathcal{X}(S)=\cup\mathcal{J}(\cup_{i=1}^k S_i)$$

where the union is over all collections of pairwise disjoint essential subsurfaces of S and we use the obvious identification of points which arise in more than one way in this union. Here we view an essential annulus A as an essential subsurface which is disjoint from any subsurface which can be moved off A by an isotopy. Thus $\mathcal{X}(S)$

is just the set of formal sums $\sum_i a_i\lambda_i$ where $a_i > 0, \sum_i a_i = 1$, where $\lambda_1, \dots, \lambda_k$ are pairwise disjoint minimal geodesic laminations on S and where each simple closed curve component λ_i is equipped with an additional label $+, -$. The mapping class group acts naturally on $\mathcal{X}(S)$ as a set.

The following theorem summarizes some more technical properties of the geometric boundary. For its formulation, let us invoke the Nielsen Thurston classification which states that any nontrivial mapping class has a finite power φ with the following property. There exists a decomposition $S = S_1 \cup \dots \cup S_k$ of S into subsurfaces which is preserved by φ and such that for all $i < k$, the restriction of φ to S_i is pseudo-Anosov if S_i is not an annulus, and it is a Dehn twist if S_i is an annulus. The restriction of φ to S_k is trivial. We call a mapping class with this property a *Nielsen Thurston mapping class.*

Let φ be a Nielsen Thurston mapping class. For each $i < k$ such that S_i is not an annulus, the restriction φ_i of φ to S_i preserves precisely two geodesic laminations $\xi_i^\pm$ which are the attracting and repelling laminations of φ_i. Similarly, for any component S_i which is an annulus, the two labeled copies $\xi_i^\pm$ of the core curve of the annulus are preserved as well. Thus φ fixes any point of the form $\sum_i a_i\zeta_i$ where ζ_i is one of the laminations $\xi_i^\pm$ if $i < k$ and where ζ_k is an arbitrary point of the geometric boundary of the (possibly disconnected) surface S_k. We call points of this form the *obvious fixed point set.*

The Gromov boundary $\partial\mathcal{CG}(S_0)$ of the curve graph of S_0 equipped with the Gromov metric is a complete metric space. An *embedding* of a topological space X into a topological space Y is an injective map $f : X \to Y$ which is a homeomorphism onto its image, equipped with the subspace topology.

Proposition 5. *Let $\mathcal{X}(S)$ be the geometric boundary of* $\mathrm{Mod}(S)$.

(1) *For any collection $S_1, \dots, S_k$ of pairwise disjoint subsurfaces of S, the inclusion $\mathcal{J}(\cup_{i=1}^k S_i) \to \mathcal{X}(S)$ is an embedding. In particular, the covering dimension of $\partial\mathcal{CG}(S)$ is at most* $\mathrm{vcd}(\mathrm{Mod}(S)) - 1$.
(2) *The fixed point set for the action of a Nielsen Thurston mapping class φ on $\mathcal{X}(S)$ is precisely the obvious fixed point set of φ.*

That the covering dimension of $\partial\mathcal{CG}(S)$ is bounded from above by $\mathrm{vcd}(\mathrm{Mod}(S))$ is due to Gabai (Proposition 16.3 of [Ga14]).

Our construction is valid for the mapping class group of a once punctured torus or a four punctured sphere. In this case the mapping class group is virtually free and, in particular, it is a hyperbolic group whose Gromov boundary is a Cantor set. It is due to Bestvina and Mess [BM91] that a hyperbolic group admits a Z-structure whose boundary is its Gromov boundary. The boundary we find is the Gromov boundary of the group as well.

The construction of the boundary $\mathcal{X}(S)$ is motivated by the construction of the visual boundary of a CAT(0)-space. Along the way we identify in Section 2 an analog of the familiar Tits boundary of a symmetric space of higher rank.

The advantage of our construction is that the space $\mathcal{X}(S)$ and its topology as well as the action of the group $\mathrm{Mod}(S)$ on $\mathcal{X}(S)$ is completely explicit and can be used to study subgroups of $\mathrm{Mod}(S)$, as for example in Koberda's work [Kb12] who constructed subgroups of $\mathrm{Mod}(S)$ which are isomorphic to right angled Artin groups.

Overview of the article: A significant part of the article is devoted to define a topology on the *set* $\mathcal{X}(S)$ and show that this topology extends to $\mathcal{T}_\epsilon(S) \cup \mathcal{X}(S)$ and defines a compactification of $\mathcal{T}_\epsilon(S)$. This is carried out in Section 3. In Section 2, we introduce the *oriented curve complex* and show that it can be viewed as a Tits type boundary for the mapping class group.

Section 4 is devoted to the proof that the space $\mathcal{X}(S)$ is indeed a small boundary for $\mathrm{Mod}(S)$ and that the action of $\mathrm{Mod}(S)$ on $\mathcal{X}(S)$ is strongly proximal. In Section 5, we show that $\mathcal{X}(S)$ is metrizable. This result depends on the construction of an explicit neighborhood basis of a given point in $\mathcal{X}(S) \subset \overline{\mathcal{T}}(S)$. This neighborhood basis is used in Section 6 to construct another neighborhood basis in $\mathcal{X}(S)$ consisting of sets whose intersections with $\mathcal{T}_\epsilon(S)$ are contractible. Finally in Section 7 the proof of Theorem 4 is completed.

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2. The Tits boundary of $\mathrm{Mod}(S)$

The *join* $X_1 * X_2$ of two topological spaces X_1, X_2 is defined to be the quotient $X_1 \times X_2 \times [0,1]/\sim$ where the equivalence relation $\sim$ collapses $X_1 \times X_2 \times \{0\}$ to X_1 and collapses $X_1 \times X_2 \times \{1\}$ to X_2. For example, the join $S^0_1 * S^0_2$ of two 0-spheres is the circle S^1, thought of as a union of four intervals glued at the endpoints, where each interval has one endpoint in S^0_1 and the second endpoint in S^0_2. The join of two spaces X_1, X_2 contains an embedded copy of X_1, X_2.

Example 2.1. The product of two hyperbolic planes $\mathbb{H}^2 \times \mathbb{H}^2$ is a complete simply connected Riemannian manifold of non-positive curvature. Its *visual boundary* is the join $S^1 * S^1$ of two circles that are the Gromov boundaries of the embedded copies of $\mathbb{H}^2$. This corresponds to the fact that the projection of any geodesic in $\mathbb{H}^2 \times \mathbb{H}^2$ to each of the two factors is a geodesic. Note that the join of two circles is homeomorphic to S^3.

From now on we assume that $3g-3+m \geq 2$ which rules out the once punctured tori and four punctured sphere. Define the *oriented curve complex* $\mathcal{OG}(S)$ of the oriented surface S of finite type to be the complex whose vertices are isotopy classes of oriented simple closed curves in S and where two such vertices are connected by an edge (of length 1) if they can be realized disjointly and are not homotopic up to orientation. Thus any simple closed curve in S defines two distinct vertices in $\mathcal{OG}(S)$, and these vertices are not connected by an edge. Furthermore, we require that any collection of $k \geq 2$ oriented disjoint simple closed curves which are distinct as unoriented curves span a simplex. The union of these simplices defined by a fixed collection of k curves equipped with all combinations of orientations is a sphere of dimension $k-1$. Note that a point in $\mathcal{OG}(S)$ can be viewed as a formal linear combination $\sum_{i=1}^k a_i\lambda_i$ where for some $k\geq 1$, $\lambda_1,\dots,\lambda_k$ are pairwise disjoint oriented simple closed curves, where $a_i>0$ for all i and $\sum_i a_i=1$. In other words, a point in the oriented curve complex can be viewed as a point in the join of a finite collection of oriented pairwise disjoint simple closed curves.

Remark 2.2. If we choose the length of the edges of the oriented curve complex to be $\pi/2$, then this is consistent with the idea that the oriented curve complex can be thought of as being contained in the Tits boundary of $\mathrm{Mod}(S)$, equipped with the angular length metric which identifies each sphere with a sphere of constant curvature one.

A simple closed curve c on S is the core curve of an embedded annulus $A(c)\subset S$. The "curve graph" $\mathcal{CG}(A(c))$ of the annulus $A(c)$ is a graph of isotopy classes of arcs connecting the two boundary components and whose endpoints are allowed to move freely in the complement of a fixed point on each of the two boundary circles. The curve graph of $A(c)$ is a simplicial line. If α is a given vertex of $\mathcal{CG}(A(c))$, then any other isotopy class of arcs can be represented by an arc which is the image of α by a multipe of a Dehn twist about c. The distinction between a positive and a negative Dehn twist about c only depends on the orientation of S but not on the orientation of c. The choice of an orientation of c can be thought of as a spiraling direction about c for oriented arcs connecting the two boundary components of $A(c)$.

In the sequel we denote by c^+ the point in the Gromov boundary of $\mathcal{CG}(A(c))$ (which consists of two points) which corresponds to an iteration of positive Dehn twists about c, and we denote by c^- the point in the Gromov boundary of $\mathcal{CG}(A(c))$ which corresponds to iteration of negative Dehn twists about c. Write $\mathcal{J}(c)=\{c^+,c^-\}$. It will be convenient to think about $\mathcal{J}(c)$ as a set of two distinct points in the oriented curve complex of S, with the same underlying curve.

If S_0 is a subsurface of S different from a pair of pants or an annulus, then we denote its (non-oriented) curve complex by $\mathcal{CG}(S_0)$. The vertices of this complex are isotopy classes of non-peripheral simple closed curves. If S_0 is different from a one-holed torus or a four punctured sphere, then a collection of $k\geq 2$ such disjoint simple closed curves span a simplex of dimension $k-1$. If S_0 is a one-holed torus or a two-holed sphere then two simple closed curves are connected by an edge if they intersect transversely in the minimal number of points. The curve complex of S_0 is hyperbolic and hence it has a *Gromov boundary* $\partial\mathcal{CG}(S_0)$. As a set, the Gromov boundary $\partial\mathcal{CG}(S_0)$ is the set of all minimal filling geodesic laminations on S_0. We refer to [H06] for an account on this result of Klarreich.

There is a natural metrizable topology on the union $\overline{\mathcal{CG}}(S_0)$ of $\mathcal{CG}(S_0)$ with its Gromov boundary, called the *coarse Hausdorff topology.* With respect to this topology, the subspace $\mathcal{CG}(S_0)$, equipped with its simplicial topology, is an open dense subset. To define this topology equip the surface with a hyperbolic metric with geodesic boundary. This choice defines a Hausdorff topology on the space of compact subsets of S_0. A sequence $\lambda_i\subset\mathcal{CG}(S_0)\subset\mathcal{CG}(S_0)\cup\partial\mathcal{CG}(S_0)$ of vertices in $\mathcal{CG}(S_0)$ converges in the coarse Hausdorff topology to $\lambda\in\partial\mathcal{CG}(S_0)$ if and only if the limit of any converging subsequence of λ_i in the Hausdorff topology on compact subsets of S_0 contains λ as a sublamination [H06]. Define

$$\mathcal{J}(S_0) = \partial\mathcal{CG}(S_0),$$

equipped with the topology as a subset of $\overline{\mathcal{CG}}(S_0)$. If S_0 is a pair of pants, then we define $\mathcal{J}(S_0) = \emptyset$.

If $S_1, \dots, S_k$ are *disjoint* connected subsurfaces of S (we allow that they share boundary components, and annuli about such boundary components may be included in the list), then we define

$$\mathcal{J}(\cup_i S_i) = \partial\mathcal{CG}(S_1) * \cdots * \partial\mathcal{CG}(S_k) \tag{1}$$

to be the join of the spaces $\mathcal{J}(S_i) = \partial\mathcal{CG}(S_i)$. For example, if $S_1 \subset S$ is a subsurface which is the complement of a non-separating simple closed curve c, then

$$\mathcal{J}(S_1 \cup A(c)) = \partial\mathcal{CG}(S_1) * \{c^+, c^-\}.$$

A point in $\mathcal{J}(S_1 \cup \cdots \cup S_k)$ can be viewed as a formal linear combination

$$\xi = \sum_i a_i \xi_i$$

where $\xi_i \in \partial\mathcal{CG}(S_i)$, $a_i \geq 0$ for all i and, furthermore, $\sum_i a_i = 1$. The union

$$\operatorname{supp}(\xi) = \cup_{a_i > 0} \xi_i$$

is a geodesic lamination with minimal components ξ_i, and ξ can be viewed as a weighted (and partially labeled if there are simple closed curve components of ξ with positive weight) geodesic lamination. For all $u \leq k$ there is an inclusion $\mathcal{J}(S_1 \cup \cdots \cup S_u) \subset \mathcal{J}(S_1 \cup \cdots \cup S_k)$ which is a topological embedding.

A collection $S_1, \dots, S_k$ of disjoint connected subsurfaces of S is called *maximal* if $S - \cup_i S_i = \emptyset$. By convention, this means that for any boundary component c of one of the surfaces S_i, the annulus $A(c)$ is contained in the collection. Any collection of disjoint connected subsurfaces of S is contained in a maximal collection of such subsurfaces, however this maximal collection is in general not unique. Note that for any collection $S_1, \dots, S_k$ of disjoint connected subsurfaces, there is a canonical maximal collection containing $S_1, \dots, S_k$ which is comprised of the surfaces S_i, the annuli $A(c)$ where c runs through all boundary components of $\cup_i S_i$ which are not already contained in the list, and all connected components of $S - \cup_i S_i$.

Define

$$\mathcal{X}(S) = \cup \mathcal{J}(S_1 \cup \cdots \cup S_k)/\sim \tag{2}$$

where the union is over all collections of disjoint subsurfaces $S_1, \dots, S_k$ of S. The equivalence relation $\sim$ identifies two points $\sum_i a_i \xi_i$ and $\sum_j b_j \zeta_j$ if they coincide as weighted labeled geodesic laminations. Thus a point in $\mathcal{X}(S)$ is nothing else but a formal sum $\sum_{i=1}^k a_i \xi_i$ where $a_i > 0, \sum_i a_i = 1$, where $\xi_1, \dots, \xi_k$ are pairwise disjoint minimal geodesic laminations on S and where every simple closed curve component of this collection is in addition equipped with a label $\pm$. Note that the oriented curve complex $\mathcal{OG}(S)$ of S can naturally be identified with the union of the subsets $\mathcal{J}(A(c_1) \cup \cdots \cup A(c_k))$ of $\mathcal{X}(S)$, and its Gromov boundary (which is just the Gromov boundary $\partial\mathcal{CG}(S)$ of the non-oriented curve complex of S) also is contained in $\mathcal{X}(S)$. The mapping class group $\mathrm{Mod}(S)$ naturally acts on the set $\mathcal{X}(S)$.

Example 2.3. The definition (2) also makes sense if S is a once punctured torus. In this case there are no non-peripheral non-annular subsurfaces of S, and the set $\mathcal{X}(S)$ is just the union of the Gromov boundary of the curve graph $\mathcal{CG}(S)$ with a countable set, consisting of all oriented simple closed curves on S. This set has the following interpretation.

The curve graph of S is the well-known *Farey graph*. It vertices can be represented by the rational points in the boundary $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$ of the hyperbolic plane. If one represents the edges of the Farey graph by geodesics in $\mathbb{H}^2$, then one obtains a tesselation of the hyperbolic plane by ideal triangles which is invariant under the mapping class group $\mathrm{PSL}(2,\mathbb{Z})$ of S. The boundary ∂T of the dual tree T of this tesselation is a Cantor set which admits a surjective continuous map onto the boundary $\partial\mathbb{H}^2$ of the hyperbolic plane. Each irrational point in $\partial\mathbb{H}^2$ has precisely one preimage, and the rational points which correspond to the vertices of the curve graph have two preimages.

The vertices of $\mathcal{CG}(S)$ correspond to the fixed points of the parabolic subgroups of $\mathrm{PSL}(2,\mathbb{Z})$. With this interpretation, the set $\mathcal{X}(S)$ can be identified with the Cantor set ∂T obtained by replacing each rational point in $\mathbb{R} \cup \{\infty\}$ by a compact interval and removing the interior of the interval. This Cantor set in turn has a natural identification with the Gromov boundary ∂T of the virtually free group $\mathrm{PSL}(2,\mathbb{Z})$. In particular, there is a natural invariant topology on $\mathcal{X}(S)$ so that with this topology, $\mathcal{X}(S)$ is a compact $\mathrm{PSL}(2,\mathbb{Z})$-space which contains the Gromov boundary $\partial\mathcal{CG}(S)$ of the curve graph of S as a dense embedded subset. Furthermore, following [BM91], with this topology the set $\mathcal{X}(S)$ is the boundary of a $\mathcal{Z}$-structure for any torsion free finite index subgroup of $\mathrm{PSL}(2,\mathbb{Z})$.

Example 2.4. Let $S_1, \dots, S_k$ be a disjoint union of subsurfaces of S which are different from pairs of pants. Then the join $\mathcal{X}(S_1) * \cdots * \mathcal{X}(S_k)$ is a subset of $\mathcal{X}(S)$.

The oriented curve complex of S is connected, and any non-filling geodesic lamination, that is, a geodesic lamination which is disjoint from some simple closed curve, is disjoint from some vertex of $\mathcal{OG}(S)$. Thus if we equip $\mathcal{X}(S) \setminus \partial\mathcal{CG}(S)$ with the topology of a simplicial complex whose edges are the joins of two disjoint (perhaps labeled) geodesic laminations, then this complex is connected. As a consequence, the *set* $\mathcal{X}(S)$ can be equipped with a topology which coincides with the topology of a (non-locally finite) simplicial complex on $\mathcal{X}(S) \setminus \partial\mathcal{CG}(S)$ and is such that each point in $\partial\mathcal{CG}(S)$ is isolated. We write $\mathcal{X}_T(S)$ for $\mathcal{X}(S)$ equipped with this topology and call $\mathcal{X}_T(S)$ the *Tits boundary* of $\mathrm{Mod}(S)$ (having the Tits boundary of a CAT(0) space as guidance). From this description, we obtain

Lemma 2.5. *The mapping class group* $\mathrm{Mod}(S)$ *of* S *acts on* $\mathcal{X}_T(S)$ *as a group of simplicial automorphisms.*

Proof. The mapping class group acts on the oriented curve complex of S as a group of simplicial automorphisms, and this action extends to an action on the space of formal sums of weighted disjoint minimal geodesic laminations preserving weight and disjointness. Furthermore, it acts on $\partial\mathcal{CG}(S)$ as a group of transformations. Since the topology on $\mathcal{X}_T(S)$ is the topology of a disconnected simplicial complex, constructed from the curve complexes of subsurfaces, the lemma follows. $\square$

Remark 2.6. The Tits boundary of a CAT(0) space X can be viewed as the geometric boundary (that is, the CAT(0) boundary) of X, equipped with a topology which in general is finer than the geometric topology. We shall see in Section 4 that the same holds true for the Tits boundary and the geometric boundary of $\mathrm{Mod}(S)$.

3. A topology for $\mathcal{X}(S)$

The goal of this section is to equip the set $\mathcal{X}(S)$ with a topology which is coarser than the Tits topology so that for this topology, $\mathcal{X}(S)$ becomes a compact $\mathrm{Mod}(S)$-space.

To achieve this goal we use *markings* of (not necessarily proper) essential subsurfaces S_0 of the surface S. Such a marking consists of a pants decomposition P for S_0 together with a collection of *spanning curves*. Each such spanning curve intersects one of the pants curves from P in the minimal number of points (one or two) and is disjoint from all other pants curves. Two spanning curves may not be disjoint, but we require that the number of their intersection points is bounded from above by a universal constant. Since there are only finitely many topological types of pants decompositions, this can clearly be achieved. There is a natural way to equip the set of all markings on S_0 with the structure of a locally finite connected graph on which the mapping class group $\mathrm{Mod}(S_0)$ of S_0 acts properly and cocompactly. We refer to [MM00] for more information on this construction.

Choose a marking μ on S as a basepoint for the proper cocompact action of $\mathrm{Mod}(S)$. For every subsurface S_0 of S which is distinct from a pair of pants or an annulus, this marking can be used to construct a marking $\mu(S_0)$ of S_0 as follows.

There is a coarsely well defined *subsurface projection*

$$\mathrm{pr}_{S_0} : \mathcal{CG}(S) \to \mathcal{CG}(S_0)$$

which associates to a simple closed curve c its intersection $c \cap S_0$ with S_0 in the following sense. If $c \subset S_0$ then put $\mathrm{pr}_{S_0}(c) = c$, and if c is disjoint from S_0 then put $\mathrm{pr}_{S_0}(c) = \emptyset$. In all other cases, $c \cap S_0$ consists of a collection of pairwise disjoint arcs with endpoints on the boundary of S_0. We then put $\mathrm{pr}_{S_0}(c) = u$ for a simple closed curve u in S_0 which is obtained from one of these intersection arcs by choosing a component of the boundary of a tubular neighborhood of the union of the arc with the boundary components of S_0 containing its endpoints.

Given a marking μ for S, the union of the intersections of the marking curves with S_0 consists of a union of arcs and simple closed curves on S_0 with pairwise uniformly bounded intersection numbers which decompose S_0 into simply connected regions. Hence via deleting some of these arcs and modifying some arcs with a surgery to simple closed curves as described in the previous paragraph, the projection $\mathrm{pr}_{S_0}(\mu)$ of μ coarsely defines a marking $\mu(S_0)$ of S_0, called the *subsurface projection* of μ [MM00]. Here a coarse definition means that the construction depends on choices, but any two choices give rise to markings which are uniformly close in the marking graph of S_0, independent of the subsurface S_0.

If S_0 is an annulus, then a similar construction applies. In this case a marking consist of the choice of a marked point on each boundary component of S_0 and an embedded arc in S_0 connecting the two distinct boundary component which is disjoint from the marked points. With a bit of care, a subsurface projection is defined for annuli as well. We refer to [MM00] for more information.

Now let $S = \cup_{i=1}^k S_i$ be a collection of pairwise disjoint subsurfaces of S. Fix as before a marking μ for S. By the discussion in the previous paragraph, each of the surfaces S_i is equipped with a coarsely well defined marking $\mu(S_i)$. Let x_i be one of the marking curves (or arcs if S_i is an annulus) of $\mu(S_i)$. As the intersection number between any two curves (or arcs) of $\mathrm{pr}_{S_i}(\mu)$ is uniformly bounded, the distance in the curve graph of S_i between x_i and any other curve from $\mu(S_i)$ or any other marking of S_i constructed in the same fashion from μ is uniformly bounded. Thus this construction determines a based product space

$$(\mathcal{CG}(\cup_i S_i), x) = (\mathcal{CG}(S_1) \times \cdots \times \mathcal{CG}(S_k), x)$$

where the basepoint $x = (x_1, \ldots, x_k)$ is the product of the coarsely well defined basepoints $x_i \in \mathcal{CG}(S_i)$.

Recall from (3.3) the definition of the sets $\mathcal{J}(\cup_i S_i)$.

Definition 3.1. Define a topology on the union

$$\mathcal{Y}(\cup_i S_i) = \mathcal{CG}(\cup_i S_i) \cup \partial\mathcal{CG}(S_1) * \cdots * \partial\mathcal{CG}(S_k) = \mathcal{CG}(\cup_i S_i) \cup \mathcal{J}(\cup_i S_i)$$

by the following requirements.

- The product space $\mathcal{CG}(\cup_i S_i)$ is equipped with the product topology.
- The subspace $\mathcal{J}(\cup_i S_i)$ is equipped with the topology as a join of the Gromov boundaries of the curve graphs of S_i.
- A sequence of points $(y_1^j, \ldots, y_k^j)_j \subset \mathcal{CG}(\cup_i S_i)$ converges to $\sum_i a_i \xi_i \in \partial\mathcal{CG}(S_1) * \cdots * \partial\mathcal{CG}(S_k)$ if the following two conditions are fulfilled.
  1. For each i with $a_i > 0$, the components $y_i^j \in \mathcal{CG}(S_i)$ converge as $j \to \infty$ to ξ_i in the coarse Hausdorff topology to ξ_i (and hence they converge in $\mathcal{CG}(S_i) \cup \partial\mathcal{CG}(S_i)$ to ξ_i, see [H06]). In particular, we have $d_{\mathcal{CG}(S_i)}(y_i^j, x_i) \to \infty$ $(j \to \infty)$.
  2. Assume without loss of generality that $a_1 > 0$. Then for all $i \geq 2$ we have
  $$\frac{d_{\mathcal{CG}(S_i)}(y_i^j, x_i)}{d_{\mathcal{CG}(S_1)}(y_1^j, x_1)} \to \frac{a_i}{a_1} \quad (j \to \infty).$$

Lemma 3.2. *The notion of convergence in Definition 3.1 defines a topology on $\mathcal{Y}(\cup_i S_i)$ which restricts to the given topology on $\partial\mathcal{CG}(S_1) * \cdots * \partial\mathcal{CG}(S_k)$ and on $\mathcal{CG}(\cup S_i)$. The subspace $\partial\mathcal{CG}(S_1) * \cdots * \partial\mathcal{CG}(S_k)$ is closed in $\mathcal{Y}(\cup_i S_i)$.*

Proof. Define a subset A of $\mathcal{Y}(\cup_i S_i)$ to be *closed* if $A_1 = A \cap \mathcal{CG}(\cup_i S_i)$ is closed, $A_2 = A \cap \partial\mathcal{CG}(S_1) * \cdots * \partial\mathcal{CG}(S_k)$ is closed and if furthermore the following holds true. If $y_i \subset A_1$ is a sequence which converges in the sense described above to a point $y \in \partial\mathcal{CG}(S_1) * \cdots * \partial\mathcal{CG}(S_k)$, then $y \in A_2$. Note that by definition, the empty set is closed, and the same holds true for the total space.

We have to show that complements of closed sets defined in this way fulfill the axioms of a topology, that is, they are stable under arbitrary unions and finite intersections. Equivalently, the family of closed sets is stable under arbitrary intersections and finite unions. As this holds true for the closed subsets of $\mathcal{CG}(\cup_i S_i)$ and for the closed subsets of $\partial\mathcal{CG}(S_1) * \cdots * \partial\mathcal{CG}(S_k)$, all we need to observe is that taking arbitrary intersections and finite unions is consistent with the notion of convergence of points in $\mathcal{CG}(\cup_i S_i)$ to points in the join $\partial\mathcal{CG}(S_1) * \cdots * \partial\mathcal{CG}(S_k)$ in the sense of Definition 3.1.

Consistency with arbitrary intersections is straightforward. To show consistency with taking finite unions let $B_1, \dots, B_\ell \subset \mathcal{Y}(\cup_i S_i)$ be closed in the above sense. Let $y_j \subset \cup_k(B_k \cap \mathcal{CG}(\cup_i S_i))$ be any sequence which converges to a point in $\mathcal{J}(\cup_i S_i)$ according to the definition of convergence. By passing to a subsequence, we may assume that $y_j \in B_m$ for a fixed $m \leq \ell$ and all j. As B_m is closed and the subsequence also fulfills the requirements for convergence, its limit is contained in $B_m \subset \cup_k B_k$. Hence indeed, the notion of a closed set is consistent with taking finite unions. □

So far we have constructed a topology on the spaces $\mathcal{Y}(\cup_i S_i)$ where $S_1, \dots, S_k$ is a collection of disjoint subsurfaces of S. We now use these spaces to define convergent sequences in $\mathcal{X}(S)$ and use this notion of convergent sequence to construct a topology on $\mathcal{X}(S)$ which gives $\mathcal{X}(S)$ the structure of a compact Hausdorff space.

Let $\mathrm{Min}(\mathcal{L})$ be the set of all minimal geodesic laminations on S where as before, a simple closed curve carries in addition a label $\pm$. Let $\xi^j = \sum_m a_m^j \xi_m^j$ be a sequence in $\mathcal{X}(S)$. We shall impose 3 requirements for the sequence to converge to a point $\sum_{i=1}^k b_i\zeta_i \in \mathcal{X}(S)$ (here as before, we require that $a_m^j > 0, b_i > 0, \sum_i b_i = 1 = \sum_m a_m^j$ for all j and that furthermore, $\mathrm{supp}(\sum_i b_i\zeta_i)$ is a disjoint union of minimal components.

Recall that the space of geodesic laminations on S is compact with respect to the Hausdorff topology.

Requirement 1: *Convergence in the coarse Hausdorff topology*
Let ξ^{ℓ_n} be any subsequence of the sequence ξ^j such that the geodesic laminations $\mathrm{supp}(\xi^{\ell_n})$ converge in the Hausdorff topology to a geodesic lamination β. Then β contains $\mathrm{supp}(\zeta)$ as a sublamination.

Example 3.3. Let $S_1, \dots, S_k \subset S$ be disjoint subsurfaces. Example 2.4 shows that $\mathcal{X}(S)$ contains the join $\mathcal{X}(S_1) * \cdots * \mathcal{X}(S_k)$ as a subset. An element $\xi \in \mathcal{X}(S_1) * \cdots * \mathcal{X}(S_k)$ can be represented in the form

$$\xi = \sum_i a_i\xi_i$$

where $\xi_i \in \mathcal{X}(S_i)$, in particular, $\mathrm{supp}(\xi_i) \subset S_i$, and $\sum_i a_i = 1$. Since the subset of geodesic laminations on S which are supported in S_i is closed with respect to the Hausdorff topology, this implies that for any topology on $\mathcal{X}(S)$ which fulfills the first requirement above, the subspace $\mathcal{X}(S_1) * \cdots * \mathcal{X}(S_k)$ of $\mathcal{X}(S)$ is closed.

We next introduce for each collection $S_1,\dots,S_k$ of pairwise disjoint subsurfaces of S a coarsely well defined projection

$$\mathrm{pr}_{\mathcal{Y}(\cup_i S_i)} : \mathrm{Min}(\mathcal{L}) \to \mathcal{Y}(\cup_i S_i) \tag{3}$$

as follows.

Let $\nu \in \mathrm{Min}(\mathcal{L})$. Then for $i \le k$ there are three possibilities.

- The lamination ν is disjoint from S_i up to homotopy.
- $\nu \subset S_i$.
- $\nu \cap S_i$ consists of a collection of simple arcs with endpoints on the boundary of S_i which coarsely define a point in $\mathcal{CG}(S_i)$.

In the third case, the intersection arcs are pairwise disjoint and hence we can choose one of these arcs and replace it by a simple closed curve via surgery as discussed above. Any two choices of such curves are of distance at most two in the curve graph of S_i.

For the definition of $\mathrm{pr}_{\mathcal{Y}(\cup_i S_i)}$, we distinguish three cases.

(1) If $\nu \in \partial\mathcal{CG}(S_i)$ for some i, that is, if $\nu \subset S_i$ and if ν fills S_i, then define
$$\mathrm{pr}_{\mathcal{Y}(\cup_i S_i)}(\nu) = \nu \in \partial\mathcal{CG}(S_1) * \cdots * \partial\mathcal{CG}(S_k).$$
(2) If $\nu \subset S_i$ for some i but if ν is disjoint from an essential simple closed curve $c \subset S_i$ then define $\mathrm{pr}_{\mathcal{Y}(\cup_i S_i)}(\nu) = (x_1,\dots,c,\dots,x_k)$ where x_j is the basepoint in $\mathcal{CG}(S_j)$.
(3) If $\nu \not\subset S_i$ for any i, then the subsurface projections of ν into S_i are either coarsely well defined simple closed curves or empty. Let $\mathrm{pr}_{\mathcal{Y}(\cup_i S_i)}(\nu) = (\nu_1,\dots,\nu_k)$ where for each i, the component ν_i either is a subsurface projection of ν into S_i if ν intersects S_i, or $\nu_i = x_i$ for the basepoint $x_i \in \mathcal{CG}(S_i)$.

Using the projections $\mathrm{pr}_{\mathcal{Y}(\cup_i S_i)}$, we are now ready to define the convergence of a sequence $\xi^j \subset \mathcal{X}$ of minimal geodesic laminations to a limit point $\zeta = \sum_i b_i\zeta_i$.

Requirement 2: Assume that ξ^j is a minimal geodesic lamination for all j. Let $\xi^{j_\ell} \subset \xi^j$ be any subsequence which converges in the Hausdorff topology to a lamination β. By the first requirement, we have $\beta \supset \mathrm{supp}(\zeta)$. Let $\beta_1,\dots,\beta_s$ be the minimal components of β, ordered in such a way that $\beta_i = \zeta_i$ for $i \le k$. We may have $s > k$. For each $i \le s$ let S_i be the subsurface of S filled by β_i, that is, $\beta_i \subset S_i$ and $S_i - \beta_i$ is a union of simply connected components and boundary parallel annuli. We require that

$$\mathrm{pr}_{\mathcal{Y}(\cup_i S_i)}\xi^{j_\ell} \to \zeta \text{ in } \mathcal{Y}(\cup_i S_i).$$

Example 3.4. To illustrate the above construction, let S be a once punctured torus. In Example 2.3, we identified $\mathcal{X}(S)$ with the Gromov boundary of the hyperbolic group $\mathrm{PSL}(2,\mathbb{Z})$ as a set.

Every geodesic lamination on S contains precisely one minimal component. As the Gromov topology on the boundary of the curve graph of S is the coarse Hausdorff topology, an inspection of the construction of the Gromov boundary of the tree dual to the Farey tesselation in Example 2.3, which has a natural identification of with $\mathcal{X}(S)$ as a set, fulfills the requirements (1) and (2). Thus by invariance under the action of $\mathrm{PSL}(2,\mathbb{Z})$, we find that $\mathcal{X}(S)$, equipped with a topology which satisfies these requirements, is indeed equivariantly homeomorphic to the Gromov boundary of $\mathrm{PSL}(2,\mathbb{Z})$.

While requirements (1) and (2) are sufficient to describe convergent sequences consisting of minimal geodesic laminations, the definition of convergence of a general sequence $\xi^j = \sum_i a_i^j \xi_i^j \subset \mathcal{X}(S)$ to a limit point $\sum_{i=1}^k b_i\zeta_i$ is more involved. To obtain a better understanding of what it captures, for a collection $\cup_{i=1}^k S_i$ of disjoint subsurfaces of S define

$$\mathcal{Z}(\cup_i S_i) = \{a\zeta_1 + (1-a)\zeta_2 \mid a\in[0,1], \zeta_1\in\mathcal{J}(\cup_{j\le s}S_{i_j}), \zeta_2\in\mathcal{Y}(\cup_{u\not\in\{i_1,\dots,i_s\}}S_u)\}$$

to be the union of the joins of the spaces $\mathcal{X}(\cup_{j\le s}S_{i_j})$ and $\mathcal{Y}(\cup_{u\not\in\{i_1,\dots,i_s\}}S_u)$ where $\{i_1,\dots,i_s\}$ runs through all (possibly empty) subsets of the set $\{1,\dots,k\}$. Note that $\mathcal{Z}(\cup_i S_i)$ contains $\mathcal{Y}(\cup_i S_i)$ as a subset. However, the union is not meant to be a disjoint union as we identify points if they correspond to the same weighted geodesic laminations.

Construct next a projection $\mathrm{pr}_{\mathcal{Z}(\cup S_i)}: \mathcal{X}(S)\to\mathcal{Z}(\cup_i S_i)$ as follows. Let $\xi = \sum_{j=1}^m a_i\xi_j \in \mathcal{X}(S)$ with $a_j>0$ and $\sum_j a_j = 1$ and write as before $\mathrm{supp}(\xi) = \cup_j\xi_j$. After perhaps a reordering of the components ξ_j, assume that for some $u\le\min\{k,m\}$ the components $\xi_1,\dots,\xi_u$ fill the subsurfaces $S_1,\dots,S_u$, that is, they define points in $\partial\mathcal{CG}(S_i)$, and that for no $j>u$, the component ξ_j fills any of the surfaces S_i. As the components of $\mathrm{supp}(\xi)$ are disjoint, this implies that if $s,t>u$, if $i\in\{u+1,\dots,k\}$ and if the subsurface projections $\mathrm{pr}_{S_i}(\xi_s), \mathrm{pr}_{S_i}(\xi_t)$ of ξ_s,ξ_t into S_j are not empty, then they are of uniformly bounded distance in $\mathcal{CG}(S_i)$ (where we adopt the convention to associate to any non-filling geodesic lamination in S_j a disjoint essential simple closed curve).

Define

$$\mathrm{pr}_{\mathcal{Z}(\cup S_i)} = \sum_{j=1}^u a_i\xi_j + (1-\sum_{j=1}^u a_j)(\mathrm{pr}_{\mathcal{Y}(\cup_{i\ge u+1}S_i)}\cup_{j\ge u+1}\xi_j).$$

Here the term on the right hand side is understood in the following sense. Let us consider a subsurface S_ℓ for some $\ell>u$. If there exists some $s>u$ such that ξ_s intersects S_ℓ, then the component in S_ℓ of the projection $\mathrm{pr}_{\mathcal{Y}(\cup_{i\ge u+1}S_i)}(\cup_{j\ge u+1}\xi_j)$ is a point in $\mathcal{CG}(S_\ell)$. Although this projection depends on choices, it is coarsely well defined, that is, well defined up to a uniformly bounded error. The above remark shows that this projection coarsely does not depend on choices, nor on the component ξ_s of ξ intersecting S_ℓ. If the lamination $\mathrm{supp}(\xi)=\cup_i\xi^i$ is disjoint from the subsurface S_ℓ, then the projection component is defined to be the basepoint of $\mathcal{CG}(S_\ell)$ constructed from the base marking.

Requirement 3: Let ξ^{j_s} be any subsequence of the sequence ξ^j so that the laminations $\mathrm{supp}(\xi^{j_s})$ converge as $s\to\infty$ in the Hausdorff topology to a lamination β with minimal components $\beta_1,\dots,\beta_n$ for some $n\ge k$. By the first requirement, we have $\beta=\cup_i\beta_i\supset\mathrm{supp}(\zeta)$. Assume by reordering that $\beta_i=\zeta_i$ for

$i \leq k$. For each i let S_i be the subsurface filled by β_i; then $\mathrm{pr}_{\mathcal{Z}(\cup_i S_i)}(\xi^{j_s}) \to \zeta$ in $\mathcal{Z}(\cup_i S_i) \supset \mathcal{Z}(\cup_{i\leq k} S_i) \supset \mathcal{J}(\cup_i S_i)$.

Remark 3.5. It follows from the above description that for this notion of convergence, the following holds true. Let ξ^j be a sequence in $\mathcal{X}$ consisting of minimal geodesic laminations which converges to a point $\zeta = \sum_u b_u \zeta_u$.

(a) The lamination $\cup_u \zeta_u$ is a sublamination of the limit in the coarse Hausdorff topology of any convergent subsequence of the sequence $\mathrm{supp}(\xi^j) = \cup_i \xi_i^j$.
(b) For each j let η^j be a minimal geodesic lamination disjoint from ξ^j (we allow $\eta^j = \xi^j$) and let $s_i \in [0,1]$. Then any limit of a convergent subsequence of the sequence $\nu^j = s_i \xi^j + (1 - s_i)\eta^j$ is of the form $s\zeta + (1-s)\eta$ where η is a limit of a subsequence of the sequence η^j and where $s \in [0,1]$.

Definition 3.6. A subset $A \subset \mathcal{X}(S)$ is called *closed for the geometric topology of* $\mathcal{X}(S)$ if the following holds true. Let $\xi_i \subset A$ be any sequence which converges to a point $\xi \in \mathcal{X}(S)$ in the sense described by the requirements (1),(2),(3); then $\xi \in A$.

An *embedding* of a topological space X into a topological space Y is an injective map $f : X \to Y$ which is a homeomorphism onto its image, equipped with the subspace topology. Recall that for any collection $S_1, \dots, S_k$ of pairwise disjoint subsurfaces of S, the space $\mathcal{J}(\cup_{i=1}^k S_i)$ is equipped with a natural topology as a join of the Gromov boundaries of the curve graphs of the subsurfaces S_i. The following statement is the first main step towards the proof of Theorem 4.

Proposition 3.7. (1) *Closed subsets of $\mathcal{X}(S)$ in the sense of Definition 3.6 define a topology $\mathcal{O}$ on $\mathcal{X}(S)$.*
(2) *For any collection $S_1, \dots, S_k$ of pairwise disjoint subsurfaces, the natural inclusion $\mathcal{J}(\cup_{i=1}^k S_i) \to (\mathcal{X}, \mathcal{O})$ is an embedding.*
(3) *The group $\mathrm{Mod}(S)$ acts on $\mathcal{X}(S)$ as a group of transformations.*

Proof. Let $\mathcal{O} \subset \mathcal{X}(S)$ be the family of all subsets of $\mathcal{X}(S)$ whose complement is closed in the above sense. Sets in $\mathcal{O}$ are called *open*. For the first statement in the proposition, we have to show that $\mathcal{O}$ defines a topology on $\mathcal{X}(S)$.

As the empty set and the entire space $\mathcal{X}(S)$ are open, to show that $\mathcal{O}$ is indeed a topology on $\mathcal{X}(S)$ it suffices to show that arbitrary unions of open sets are open, and that finite intersections of open sets are open as well. Or, equivalently, arbitrary intersections of closed sets are closed, and finite unions of closed sets are closed. This can be established using exactly the same reasoning as in the proof of Lemma 3.2.

Namely, that the collection of closed sets is stable under arbitrary intersections is immediate from the definition. So let $B_1, \dots, B_k$ be closed sets and let $B = \cup_i B_i$. Choose a sequence $\xi_i \subset B$ which converges in the sense of requirements (1)-(3) to some point ζ. By passing to a subsequence, we may assume that $\xi_i \in B_\ell$ for some $\ell \leq k$. But then $\zeta \in B_\ell \subset B$ as B_ℓ is closed which completes the proof that $\mathcal{O}$ is indeed a topology on $\mathcal{O}$.

We show next the second property claimed in the proposition. Thus let $S_1, \dots, S_k$ be a collection of pairwise disjoint subsurfaces of S. Our goal is to show that the inclusion $\mathcal{J}(\cup_{i=1}^k S_i) \to (\mathcal{X}(S), \mathcal{O})$ is an embedding. Since the inclusion is injective, and $\mathcal{J}(\cup_{i=1}^k S_i)$ is a separable Hausdorff space, for this it suffices to show that the inclusion is continuous and its image is locally closed. This is equivalent to the following

Claim: A sequence $\xi^j = \sum_i a_i^j \xi_i^j \subset \mathcal{J}(\cup_{i=1}^k S_i)$ converges in $(\mathcal{X}, \mathcal{O})$ to a point $\zeta \in \mathcal{J}(\cup_{i=1}^k S_i)$ if and only if ξ^j converges in $\mathcal{J}(\cup_{i=1}^k S_i)$ to ζ.

To prove the claim we begin with considering a sequence $\xi^j = \sum_i a_i^j \xi_i^j \subset \mathcal{J}(\cup_{i=1}^k S_i)$ which converges in the space $\mathcal{X}(\cup_{i=1}^k S_i)$ to $\zeta = \sum_i b_i \zeta_i$. We aim at showing that ξ^j converges to ζ in $(\mathcal{X}(S), \mathcal{O})$.

By reordering, assume that $1 \leq m \leq k$ is such that $b_i > 0$ if and only if $i \leq m$. Let β be a limit in the coarse Hausdorff topology of a subsequence of the sequence of laminations (supp$(\xi^j) = \cup_{a_i^j > 0} \xi_i^j$. By the definition of convergence in $\mathcal{J}(\cup_{i=1}^k S_i)$, up to reordering, we may assume that for some $m \leq n \leq k$, we have $\beta = \cup_{i \leq n} \beta_i$ where β_i is a (not necessarily minimal and not necessarily filling) geodesic lamination on the surface S_i, and $\beta_i = \zeta_i$ for $1 \leq i \leq m$. The fact that n may be strictly smaller than k arises from the possibility that the formal sum describing ξ^j may not have a positive coefficient corresponding to a surface S_ℓ for $\ell > m$.

Since $\xi^j \in \mathcal{J}(\cup_{i=1}^k S_i)$ for all j, we know that the projection $\mathrm{pr}_{\mathcal{Z}(\cup_{i \leq m} S_i)} \xi^j$ of ξ^j to $\mathcal{Z}(\cup_{i \leq m} S_i)$ is contained in $\mathcal{J}(\cup_{i=1}^m S_i)$. Now by definition of the topology on $\mathcal{J}(\cup_{i=1}^k S_i)$, the subset $\mathcal{J}(\cup_{i=1}^m S_i)$ is an embedded subspace of $\mathcal{J}(\cup_{i=1}^k S_i)$, and the surfaces S_i for $i > m$ are precisely those surfaces with the property that the coefficients a_i^j of the components ξ_i^j of ξ^j in S_i tend to zero as $j \to \infty$. Furthermore, for $i \leq m$ the coefficients a_i^j converge to b_i. Thus an application of the first and third requirement in the definition of convergent sequences for $\mathcal{O}$ shows that indeed, $\xi^j \to \zeta \in \mathcal{X}(S)$.

To complete the proof of the claim, we have to show that a sequence in $\mathcal{J}(\cup_{i=1}^k S_i)$ which converges in $(\mathcal{X}(S), \mathcal{O})$ to a limit point $\zeta = \sum_i b_i \zeta_i \in \mathcal{J}(\cup_{i=1}^k S_i)$ also converges in $\mathcal{X}(\cup_{i=1}^k S_i)$ to the same limit point. However, this can be established with essentially the same argument and will be omitted. The second part of the proposition follows.

That $\mathrm{Mod}(S)$ acts on $\mathcal{X}(S)$ as a group of transformations is immediate from the definition and the fact that $\mathrm{Mod}(S)$ naturally acts on curves graphs and subsurfaces. $\square$

Example 3.8. i) Let $\varphi \in \mathrm{Mod}(S)$ be a pseudo-Anosov element. Then φ acts as a loxodromic isometry on the curve graph of S, with attracting and repelling fixed points $\nu_+, \nu_- \in \partial\mathcal{CG}(S)$. Let $\mu \in \mathcal{X}(S)$ be any minimal geodesic lamination which is distinct from the repelling fixed point ν_- of φ. Then $\varphi^j \mu \to \nu_+$ $(j \to \infty)$ in the coarse Hausdorff topology and therefore $\varphi^j \mu \to \nu_+$ in $\mathcal{X}(S)$.

ii) Now let us assume that $S_0 \subset S$ is a proper connected subsurface different from an annulus and a pair of pants and that $\varphi \in \mathrm{Mod}(S)$ restricts to a pseudo-Anosov mapping class on S_0 and to the trivial mapping class on $S - S_0$. Let $\nu_+ \in \partial\mathcal{CG}(S_0)$

be the attracting geodesic lamination for the action of φ on S_0. Let furthermore $\mu \neq \nu_- \in \mathcal{X}(S)$ be any *minimal* geodesic lamination on S which is different from the repelling fixed point ν_- for the action of φ on $\mathcal{CG}(S_0)$. Then there are two possibilities. In the first case, μ is supported in $S - S_0$. Then we have $\varphi^j(\mu) = \mu$ for all j. However, if μ intersects S_0, then either $\mu = \nu_+$ or μ intersects ν_+ and we have $\varphi^j(\mu) \to \nu_+$ $(j \to \infty)$ in $\mathcal{X}(S)$.

Namely, if μ intersects S_0 then the subsurface projection of μ into any subsurface disjoint from S_0 is a collection of arcs intersecting ∂S_0. In particular, the subsurface projection into any subsurface V of $S - S_0$ is a point of $\mathcal{CG}(V)$. Since φ can be represented by a diffeomorphism which fixes $S - S_0$ pointwise, it acts trivially on $\mathcal{CG}(V)$ which yields the above statement.

Since each of the spaces $\mathcal{J}(\cup_{i=1}^k S_i)$ is a finite join of separable metrizable spaces (namely, the Gromov boundary of a curve graph of a subsurface of S) and hence separable metrizable, the second part of Proposition 3.6 shows that $(\mathcal{X}(S), \mathcal{O})$ is a countable union of (in general not disjoint) separable metrizable spaces and hence is separable.

Corollary 3.9. *$(\mathcal{X}(S), \mathcal{O})$ is a separable Lindelöf space.*

Proof. We have to show that any open cover of $\mathcal{X}(S)$ has a countable subcover. To this end let $\mathcal{U}$ be such an open cover. List the countably many spaces $\mathcal{J}(\cup_i S_i)$ as $\mathcal{X}_1, \mathcal{X}_2, \dots$. Since for each i, the space $\mathcal{X}_i$ is separable and metrizable, the restriction of $\mathcal{U}$ to $\mathcal{X}_i$, which is an open covering of $\mathcal{X}_i$, has a countable subcover, say by sets $U_i^1, U_i^2, \dots$. Now the union $\mathcal{V} = \cup_{i,j} U_i^j$ consists of countably many sets, and for each i, the sets from $\mathcal{V}$ cover $\mathcal{X}_i$. Since $\mathcal{X}(S) = \cup_i \mathcal{X}_i$ (as a set), this shows that $\mathcal{V}$ is a countable subcover of the cover $\mathcal{U}$. In other words, $\mathcal{X}(S)$ is a Lindelöf space as claimed. $\square$

Proposition 3.10. *$(\mathcal{X}(S), \mathcal{O})$ is a compact Hausdorff space.*

Proof. Let us show that the topology $\mathcal{O}$ is Hausdorff. To this end let $\xi = \sum_i a_i \xi_i \neq \zeta = \sum_j b_j \zeta_j \in \mathcal{X}(S)$. We have to show that ξ, ζ have disjoint neighborhoods.

If this is not the case, then any neighborhoods U_ξ of ξ and U_ζ of ζ intersect nontrivially. Since $\mathcal{X}(S)$ is separable, and since points in $\mathcal{X}(S)$ are closed by construction, we conclude that there is a sequence $\xi^j \subset \mathcal{X}$ which converges both to ξ, ζ. But for the notion of convergence used to define the topology $\mathcal{O}$, the limit of a converging sequence is unique. Thus $\mathcal{O}$ is Hausdorff as stated.

As by Corollary 3.9, the space $\mathcal{X}(S)$ is a separable Lindelöf space, moreover it is Hausdorff, to show that $\mathcal{X}(S)$ is compact it suffices to show that $\mathcal{X}(S)$ is sequentially compact.

Thus let $\xi^j = \sum_i a_i^j \xi_i^j \subset \mathcal{X}(S)$ be any sequence. We have to construct a convergent subsequence. Since the space of geodesic laminations equipped with the Hausdorff topology is compact, by passing to a subsequence we may assume that the geodesic laminations $\mathrm{supp}(\xi^j) = \cup_i \xi_i^j$ converge in the Hausdorff topology to a geodesic lamination $\hat{\zeta}$ with minimal components $\zeta_1, \dots, \zeta_k$.

For each $i \leq k$ let $S_i \subset S$ be the subsurface of S filled by ζ_i. Assume by passing to another subsequence that for each component ζ_i of $\hat{\zeta}$, either this component also is a component of $\mathrm{supp}(\xi^j)$ for all j, or it is not a component of $\mathrm{supp}(\xi^j)$ for all j.

By relabeling, assume that for some $u \leq k$ the laminations $\zeta_1, \ldots, \zeta_u$ are those components of $\hat{\zeta}$ which are also components of $\mathrm{supp}(\xi^j)$ for all j. By reordering, we then can write

$$\xi^j = \sum_{i=1}^{u} a_i^j \zeta_i^{\pm} + \sum_{\ell > u} a_\ell^j \xi_\ell^j.$$

By convention, the label $\pm$ is only relevant if ζ_i is a simple closed curve component.

By passing to another subsequence, we may assume that for $i \leq u$, the labels $\pm$ of the components ζ_i are constant along the sequence, and that the weights $a_i^j \in (0,1]$ of the components ζ_i converge to weights $b_i \geq 0$. In particular, the sums $1 - \sum_{i \leq u} a_i^j$ converge to $1 - \sum_{i \leq u} b_i = \kappa$. If $\kappa = 0$, then by the definition of convergent sequences in $\mathcal{X}(S)$, the sequence ξ^j converges to $\sum_i b_i \zeta_i^{\pm}$ and we are done.

Now assume that $\kappa \neq 0$ and hence $\sum_{i > u} a_i^j > \kappa/2 > 0$ for all sufficiently large j. By passing to a subsequence, we may assume that this holds true for all j. For all $i > u$ and for all j, the subsurface of S filled by ξ_i^j is disjoint from the subsurfaces $S_1, \ldots, S_u$ filled by the laminations $\zeta_1, \ldots, \zeta_u$. In other words, if we denote by $\Sigma_{u+1}, \ldots, \Sigma_n$ the components of $S - \cup_{i \leq u} S_i$, with annuli about boundary components of the surfaces S_i $(i \leq u)$ included if they are not one of the surfaces S_i themselves, then for $i \geq u+1$, each of the laminations ξ_i^j, ζ_i is supported in $\cup_{i \geq u+1} \Sigma_i$. Thus by the definition of the topology on $\mathcal{X}(S)$ and writing $\xi^j = (\sum_{i \leq u} a_i^j \zeta_i) + (\sum_{i \geq u+1} a_i^j \xi_i^j)$, viewed as points in the join of two subspaces of $\mathcal{X}(S)$, and similarly for ζ, we conclude that it suffices to construct a convergent subsequence of a sequence ξ^j under the additional assumption that for all j, no component ξ_i^j of $\mathrm{supp}(\xi^j)$ coincides with a component of the limit $\hat{\zeta} = \cup_{i \leq k} \zeta_i$ in the coarse Hausdorff topology.

From now on we assume that the latter assumption holds true. Let as before S_i be the subsurface of S filled by ζ_i. Up to passing to a subsequence, we may assume that there is a number $u \leq k$ such that for each $i \leq u$ and each j, the geodesic lamination $\mathrm{supp}(\xi^j)$ has some component ξ_i^j which is supported in S_i and fills S_i. This means that ξ_i^j defines a point in the Gromov boundary of $\mathcal{CG}(S_i)$ which is different from the point ζ_i. Since $\mathrm{supp}(\xi^j)$ converges as $j \to \infty$ in the Hausdorff topology to a geodesic lamination with minimal components $\zeta_1, \ldots, \zeta_k$, we conclude that for $i \leq u$, the laminations ξ_i^j converge as $j \to \infty$ in the coarse Hausdorff topology to ζ_i. By passing to another subsequence, we may assume that for each $i \leq u$, the coefficients a_i^j of the components contained in S_i converge as $j \to \infty$ to a coefficient b_j. As above, if $\sum_{i \leq u} b_i = 1$, then by the definition of a convergent sequence in $\mathcal{X}(S)$, we know that $\xi^j \to \sum_i b_i \zeta_i$ and hence once again, we are done.

According to what we established so far, it now suffices to assume that for no j there exists a component of $\mathrm{supp}(\xi^j)$ which fills any of the subsurfaces S_i. Then for each i, we can consider the subsurface projection $\mathrm{pr}_{S_i}(\mathrm{supp}(\xi^j))$ of $\mathrm{supp}(\xi^j)$ into the surface S_i. Furthermore, by passing to another subsequence, we may assume that for all j and all $i \le k$, this subsurface projection it is non-empty since the geodesic lamination ζ_i which fills S_i is contained in the limit with respect to the Hausdorff topology of the sequence of laminations $\mathrm{supp}(\xi^j)$. Put differently, we may assume that for each i and all j, the subsurface projection $\mathrm{pr}_{S_i}(\mathrm{supp}(\xi^j))$ of the lamination $\mathrm{supp}(\xi^j)$ into the subsurface S_i is a coarsely well defined point in $\mathcal{CG}(S_i)$. Furthermore, using once more that ζ_i fills S_i and that ζ_i is contained in the Hausdorff limit of the sequence $\mathrm{supp}(\xi^j)$, if we denote by x_i the fixed basepoint in $\mathcal{CG}(S_i)$, then we know that $d_{\mathcal{CG}(S_i)}(\mathrm{pr}_{S_i}(\mathrm{supp}(\xi^j)), x_i) \to \infty$ $(j \to \infty)$.

By passing to another subsequence and reordering indices, we may assume that

$$a_1^j = d_{\mathcal{CG}(S_1)}(\mathrm{pr}_{S_1}(\mathrm{supp}(\xi^j)), x_1) \ge a_i^j = d_{\mathcal{CG}(S_i)}(\mathrm{pr}_{S_i}(\mathrm{supp}(\xi^j)), x_i)$$

for all $i \ge 2$ and all j. Passing to another subsequence, we may assume furthermore that $a_i^j / a_1^j \to a_i \in [0,1]$ for all $i \ge 2$. Put $a_1 = 1$; then we have $\sum_u a_u \ge 1$ and hence defining $b_i = a_i / \sum_u a_u > 0$, we conclude that $\sum_u b_u = 1$. It now follows from the definition of the topology on $\mathcal{X}(S)$ that $\xi^j \to \sum_i b_i \zeta_i$. This completes the proof that $\mathcal{X}(S)$ is sequentially compact.

We are left with showing that the mapping class group $\mathrm{Mod}(S)$ acts on $\mathcal{X}(S)$ as a group of transformations. To this end observe first that by construction, $\mathrm{Mod}(S)$ acts on $\mathcal{X}(S)$ as a group of bijections (equivalently, transformations for the discrete topology). Thus it suffices to show that this action is continuous for the topology $\mathcal{O}$.

By the definition of $\mathcal{O}$, for this it suffices to show the following. Let ξ^j be a sequence converging for the topology $\mathcal{O}$ to a point ξ. Then for every $\varphi \in \mathrm{Mod}(S)$, the sequence $\varphi(\xi^j)$ converges to $\varphi(\xi)$.

That the first defining requirement for convergence is passed on to the image sequence follows from continuity of the action of φ on the space of geodesic laminations, equipped with the Hausdorff topology.

For the second requirement, if $S_1, \dots, S_k$ is a partition of S into disjoint subsurfaces, then the same holds true for $\varphi(S_1), \dots, \varphi(S_k)$, and for any geodesic lamination ν, we have $\mathrm{pr}_{\mathcal{Y}(\cup_i \varphi(S_i))}(\varphi(\nu)) = \varphi(\mathrm{pr}_{\mathcal{Y}(\cup_i S_i)}(\nu))$ up to replacing the basepoints y_i of $\mathcal{CG}(\varphi(S_i))$ by $\varphi(x_i)$. As for all i, we have $d_{\mathcal{CG}(\varphi(S_i))}(\mathrm{pr}_{\varphi(S_i)}(\xi^j), \varphi(x_i)) = d_{\mathcal{CG}(S_i)}(\xi^j, x_i) \to \infty$ $(j \to \infty)$ and the determination of the weights of the limit points are computed using ratios of distances to the basepoint defined by subsurface projections, with the distances tending to infinity along the sequence, we conclude that the second requirement in the definition of convergence is fulfilled for $\varphi(\xi^i)$ if it is fulfilled for ξ^i. The same reasoning also applies to the third requirement. Thus indeed, $\mathrm{Mod}(S)$ acts on $\mathcal{X}(S)$ as a group of transformations. This completes the proof of the proposition. □

Definition 3.11. The space $(\mathcal{X}(S), \mathcal{O})$ is called the *geometric boundary* of $\mathrm{Mod}(S)$.

Let us note another naturality property of the geometric boundary of $\mathrm{Mod}(S)$. Namely, if $S_0 \subset S$ is any essential subsuface, then we can construct a geometric boundary $\mathcal{X}(S_0)$ for $\mathrm{Mod}(S_0)$. As a set, this is a subset of the geometric boundary of S which includes the Gromov boundary of the curve graph for peripheral annuli. The above construction immediately yields

Corollary 3.12. *If $S_0 \subset S$ is any subsurface of S, then the geometric boundary of* $\mathrm{Mod}(S_0)$ *is a closed subspace of the geometric boundary of* $\mathrm{Mod}(S)$.

4. A small boundary for $\mathrm{Mod}(S)$

In this section we show that the geometric boundary $\mathcal{X}(S)$ is indeed a small boundary for $\mathrm{Mod}(S)$. For this we have to find a topology on $\mathrm{Mod}(S) \cup \mathcal{X}(S)$ which restricts to the given topology on $\mathcal{X}(S)$ and the discrete topology on $\mathrm{Mod}(S)$ and is such that for this topology, the space $\mathrm{Mod}(S) \cup \mathcal{X}(S)$ is compact.

The construction of this topology is done with the use of Teichmüller geometry. We begin with invoking the properties we need.

By the collar lemma for hyperbolic surfaces, there exists a number $\epsilon_0 > 0$ with the following property. For any closed hyperbolic surface Σ_g of genus $g \geq 2$, any two closed geodesics γ_1, γ_2 on Σ_g of length $\ell(\gamma_1), \ell(\gamma_2) \leq \epsilon_0$ are disjoint.

For $\epsilon \leq \epsilon_0$ define the *ϵ-thick part* $\mathcal{T}_\epsilon(S)$ of Teichmüller space $\mathcal{T}(S)$ by

$$\mathcal{T}_\epsilon(S) = \{X \in \mathcal{T}(S) \mid \text{ systole}(X) \geq \epsilon\}$$

where the *systole* of a hyperbolic surface X is the shortest length of a non-contractible curve on X.

The following statement is well know. We refer to Proposition 1.1 of [JW10] for an explicit statement.

Theorem 4.1. *For $\epsilon < \epsilon_0$, the following holds.*

(1) *The subspace $\mathcal{T}_\epsilon(S) \subset \mathcal{T}(S)$ is non-empty, connected and stable under* $\mathrm{Mod}(S)$*, and its quotient under the action of* $\mathrm{Mod}(S)$ *is compact.*
(2) *$\mathcal{T}_\epsilon(S)$ is a real-analytic manifold with corners and hence admits a* $\mathrm{Mod}(S)$*-invariant triangulation such that* $\mathrm{Mod}(S)\backslash\mathcal{T}_\epsilon(S)$ *is a finite CW-complex.*

There is a coarsely well defined map

$$\Upsilon : \mathcal{T}(S) \to \mathcal{CG}(S)$$

which maps a hyperbolic surface to a systole, that is, a closed non-contractible curve of minimal length. Coarsely well defined means that the map depends on choices, but the images of a surface X for any two choices of such a map are of distance at most two.

Call a map $\Psi : \mathcal{T}(S) \to \mathcal{T}(S)$ *coarsely Υ-invariant* if $d(\Upsilon(\Psi(X)), \Upsilon(X)) \leq 2$ for all X. The following is Theorem 1.2 of [JW10], see also Theorem 3.9 of [J14].

Theorem 4.2 (Ji-Wolpert). *For $\epsilon < \epsilon_0$ there exists a* $\mathrm{Mod}(S)$*-equivariant coarsely* Υ*-invariant deformation retraction* $\mathcal{T}(S) \to \mathcal{T}_\epsilon(S)$.

The following is a consequence of the proof of Theorem 4.2 and will be important later on. For its formulation, choose a torsion free finite index subgroup Γ of $\mathrm{Mod}(S)$. With more care, the result also holds for if we replace Γ by $\mathrm{Mod}(S)$, however we do not need this in the sequel.

Proposition 4.3. *For $\epsilon < \epsilon_0$ there exists a number $\nu > 0$ and there exists a Γ-equivariant diffeomorphism $\Lambda : \mathcal{T}(S) \to \Lambda(\mathcal{T}(S)) \subset \mathcal{T}_\nu(S)$ whose restriction to $\mathcal{T}_\epsilon(S)$ is the identity. The set $\Lambda(\mathcal{T}(S))$ is the interior of a Γ-invariant submanifold with smooth boundary $Q \subset \mathcal{T}_\nu(S)$. The action of Γ on Q is free and cocompact.*

Proof. The proof of Theorem 4.2 relies on the (non-canonical) construction of a smooth vector field on $\mathcal{T}(S) \setminus \mathcal{T}_\epsilon(S)$ which vanishes on $\mathcal{T}_\epsilon(S)$ and which vanishes nowhere on $\mathcal{T}(S) \setminus \mathcal{T}_\epsilon(S)$. Furthermore, this vector field is equivariant under the action of $\mathrm{Mod}(S)$, and it defines a flow Φ^t which retracts $\mathcal{T}(S)$ into $\mathcal{T}_\epsilon(S)$.

Let $\Gamma \subset \mathrm{Mod}(S)$ be a torsion free subgroup of finite index. The flow Φ^t descends to a flow on the smooth manifold $M = \Gamma\backslash\mathcal{T}(S)$. The quotient $\Gamma\backslash\mathcal{T}_\epsilon(S)$ is a compact submanifold of M with corners. The set $\Gamma\backslash\mathcal{T}_{\epsilon/2}(S)$ is a compact neighborhood of $\Gamma\backslash\mathcal{T}_\epsilon(S)$ in M. By construction, the vector field V which generates the flow Φ^t is transverse to the boundary Σ of $\Gamma\backslash\mathcal{T}_{\epsilon/2}(S)$. Note that this makes sense on the smooth part of the boundary of $\Gamma\backslash\mathcal{T}_{\epsilon/2}(S)$, but it also makes sense at the singular locus. We refer to [JW10] for more details on this construction.

Since Σ is a piecewise smooth compact codimension one topological submanifold of M, it can be approximated by a smooth commpact hypersurface $\hat\Sigma$ which is transverse to the vector field V. Then $M\setminus\hat\Sigma = M_0 \cup M_1$ consists of two components. The component M_0 contains $\Gamma\backslash\mathcal{T}_\epsilon(S)$ and its closure $\overline{M_0}$ is compact. The component M_1 is invariant under each of the diffeomorphisms Φ^t for $t \leq 0$. Furthermore, we have

$$\overline{M}_1 = \cup_{t\leq 0}\Phi^t(\hat\Sigma).$$

Let $\sigma : (-\infty, 0] \to (-1, 0]$ be an orientation preserving diffeomorphism which equals the identity in a small neighborhood of 0 and define

$$\Lambda(\Phi^t x) = \Phi^{\sigma(t)}x \quad (x \in \hat\Sigma, t < 0)$$

and $\Lambda(y) = y$ for $y \in M_0$. Then Λ is a diffeomorphism onto its image, which is a relative compact subset of M, and Λ is homotopic to the identity. Since the closure of $\Lambda(M)$ is compact, $\Lambda(M)$ is contained in $\Gamma\backslash\mathcal{T}_\nu(S)$ for some $\nu > 0$.

Since Λ is homotopic to the identity, it lifts to a diffeomorphism $\tilde\Lambda : \mathcal{T}(S) \to \tilde\Lambda(\mathcal{T}(S))$ with the properties stated in the proposition. $\square$

For a fixed torsion free subgroup Γ of $\mathrm{Mod}(S)$ of finite index, the subset $\Lambda(\mathcal{T}(S))$ constructed in Proposition 4.3 is Γ-invariant and contractible, and Γ acts cocompactly on its closure, which is contractible as well. Thus when we talk about a $\mathcal{Z}$-structure for Γ, we shall replace the subset $\mathcal{T}_\epsilon(S)$ on which Γ acts properly and cocompactly by the closure Q of the set $\Lambda(\mathcal{T}(S))$, which is a Γ-invariant smooth manifold with boudnary. As this is a purely technical point and will be not be important in this section, we continue to consider the action of the entire mapping class group on $\mathcal{T}_\epsilon(S)$ for some $\epsilon < \epsilon_0$.

Thus fix a number $\epsilon \leq \epsilon_0$. Since $\mathrm{Mod}(S)$ acts properly and cocompactly on $\mathcal{T}_\epsilon(S)$, to show that $\mathcal{X}(S)$ is a boundary for $\mathrm{Mod}(S)$ it suffices to construct a topology $\mathcal{O}_0$ on $\overline{\mathcal{T}}(S) = \mathcal{T}_\epsilon(S) \cup \mathcal{X}(S)$ with the following property.

(1) $\mathcal{O}_0$ restricts to the standard topology on $\mathcal{T}_\epsilon(S)$ and to the topology $\mathcal{O}$ on $\mathcal{X}(S)$.
(2) $\overline{\mathcal{T}}(S)$ is compact.
(3) The group $\mathrm{Mod}(S)$ acts on $\overline{\mathcal{T}}(S)$ as a group of transformations.

We define a topology $\mathcal{O}_0$ on $\mathcal{T}_\epsilon(S) \cup \mathcal{X}(S)$ by defining what it means for a sequence $(X_i) \subset \mathcal{T}_\epsilon(S)$ to converge to a point $\zeta = \sum_i b_i \zeta_i \in \mathcal{X}(S)$.

There exists a constant $\rho = \rho(S) > \epsilon$, a so-called *Bers constant*, such that any surface $X \in \mathcal{T}(S)$ admits a pants decomposition by simple closed curves of X-length at most ρ [Bu92]. If $X \in \mathcal{T}_\epsilon(S)$, then by possibly enlarging ρ, we may in fact assume that X admits a *marking* $\mu(X)$ consisting of simple closed curves of length at most ρ. We call such a marking *short* for X. By the collar lemma [Bu92], the geometric intersection number between any two simple closed curves on S of X-length at most ρ is bounded from above by a universal constant. In particular, the diameter in $\mathcal{CG}(S)$ of the set $\mathcal{V}(X)$ of simple closed curves from the marking $\mu(X)$ is bounded from above by a universal constant (see [MM99] for more information).

The marking curves from the marking $\mu(X)$ decompose S into disks. Thus for every proper essential subsurface S_0 of S, there exists a simple closed curve $c \in \mathcal{V}(X)$ which has an essential intersection with S_0. Moreover, the diameter of the subsurface projections into $\mathcal{CG}(S_0)$ of the set of all such curves is uniformly bounded. We denote the subsurface projection into S_0 of the set of simple closed curves on S from the set $\mathcal{V}(X)$ by $\mathrm{pr}_{S_0}(X)$. We then view $\mathrm{pr}_{S_0}(X)$ as a non-empty subset of $\mathcal{CG}(S_0)$ whose cardinality is uniformly bounded and whose diameter in the curve graph of S_0 is uniformly bounded as well.

Fix now a hyperbolic metric $X \in \mathcal{T}_\epsilon(S)$. For a subsurface S_0 of S let $\mu(S_0)$ be a marking of S_0 obtained by projecting the short marking $\mu(X)$ to S_0. This projected marking coarsely defines a basepoint in $\mathcal{CG}(S_0)$.

In the sequel, when we say that two geodesic laminations are disjoint then we allow that they contain some common components. All geodesics or geodesic laminations will be realized on a fixed hyperbolic surface $X \in \mathcal{T}_\epsilon(S)$, and the Hausdorff topology for compact subsets of X is the Hausdorff topology defined by the metric X.

Definition 4.4. A sequence $X_j \subset \mathcal{T}_\epsilon(S)$ *converges* to a point $\xi = \sum_i a_i \xi_i \in \mathcal{X}(S)$ if the following holds.

(1) Let $v_j \in \mathcal{V}(X_j)$ be a simple closed curve of X_j-length at most ρ. Then any limit in the Hausdorff topology of a subsequence of the sequence v_j is disjoint from $\mathrm{supp}(\sum_i \xi_i) = \cup_i \xi_i$.
(2) Let S_i be the subsurface of S filled by ξ_i. Then for each i, the projections $\mathrm{pr}_{S_i}(X_j)$ converge in $\mathcal{CG}(S_i) \cup \partial\mathcal{CG}(S_i)$ to ξ_i. Furthermore, if c_i is the basepoint in $\mathcal{CG}(S_i)$ then we have

$$d_{\mathcal{CG}(S_i)}(\mathrm{pr}_{S_i}(X_j), c_i)/d_{\mathcal{CG}(S_1)}(\mathrm{pr}_{S_1}(X_j), c_1) \to a_i/a_1 \text{ for all } i.$$

(3) Let U be any subsurface of S disjoint from the surfaces S_i and let x_U be the basepoint of $\mathcal{CG}(U)$. Then we have

$$d_{\mathcal{CG}(U)}(\mathrm{pr}_U(X_j), x_U)/d_{\mathcal{CG}(S_1)}(\mathrm{pr}_{S_1}(X_j), x_1) \to 0.$$

We first observe that this notion of convergence gives indeed rise to a topology on $\mathcal{T}_\epsilon(S) \cup \mathcal{X}(S)$.

Proposition 4.5. *There exists a topology $\mathcal{O}_0$ on $\overline{\mathcal{T}}(S) = \mathcal{T}_\epsilon(S) \cup \mathcal{X}(S)$ with the property that a set $A \subset \overline{\mathcal{T}}(S)$ is closed for $\mathcal{O}_0$ if and only if the following holds true.*

(1) *$A \cap \mathcal{T}_\epsilon(S)$ is closed in $\mathcal{T}_\epsilon(S)$, and $A \cap \mathcal{X}(S)$ is closed in $\mathcal{X}(S)$.*
(2) *If $X_j \subset A \cap \mathcal{T}_\epsilon(S)$ converges in the above sense to a point $\xi \in \mathcal{X}(S)$, then $\xi \in A$.*

Proof. The proof is completely analogous to the proof of Lemma 3.2 and will be omitted. ☐

The following is the main remaining step towards a proof that $\mathcal{X}(S)$ is a small boundary for $\mathrm{Mod}(S)$.

Proposition 4.6. *The topological space $(\overline{\mathcal{T}}(S), \mathcal{O}_0)$ has the following properties.*

(1) *$\overline{\mathcal{T}}(S)$ is a compact separable Hausdorff space.*
(2) *The mapping class group acts on $\overline{\mathcal{T}}(S)$ as a group of transformations.*

Proof. $\overline{\mathcal{T}}(S)$ is clearly separable since this holds true for $\mathcal{X}(S)$ and $\mathcal{T}_\epsilon(S)$. We show next that $\overline{\mathcal{T}}(S)$ is a Hausdorff space.

Since $\mathcal{X}(S) \subset \overline{\mathcal{T}}(S)$ is closed by construction and hence $\mathcal{T}_\epsilon(S)$ is open in $\overline{\mathcal{T}}(S)$ and is moreover a Hausdorff space, all we need to show is that two points $\xi \neq \eta \in \mathcal{X}(S)$ have disjoint neighborhoods. Now ξ, η have disjoint neighborhoods in $\mathcal{X}(S)$ and hence since $\overline{\mathcal{T}}(S)$ is separable, it suffices to show that the limit of any sequence $X_i \subset \mathcal{T}_\epsilon(S)$ converging to a point in $\mathcal{X}(S)$ is unique. But this is clear from the definitions.

We show next that $\overline{\mathcal{T}}(S)$ is compact. Since $\mathcal{X}(S)$ is compact and $\mathcal{T}_\epsilon(S)$ is a Lindelöf space, the space $\overline{\mathcal{T}}(S)$ is Lindelöf. Since $\overline{\mathcal{T}}(S)$ also is Hausdorff, it suffices to show that $\overline{\mathcal{T}}(S)$ is sequentially compact, and this follows if we can show that any sequence $X_i \subset \mathcal{T}_\epsilon(S)$ has a convergent subsequence in $\overline{\mathcal{T}}(S)$.

If the sequence has a bounded subsequence in $\mathcal{T}_\epsilon(S)$, then as $\mathcal{T}_\epsilon(S)$ is proper, we can extract a converging subsequence. Thus we may assume that the sequence is unbounded.

Since the space of geodesic laminations on S equipped with the Hausdorff topology is compact, by extracting a subsequence we may assume that the sets $\mathcal{V}(X_i)$ converge in the Hausdorff topology to a finite union of geodesic laminations. Note that as some of the curves in $\mathcal{V}(X_i)$ may intersect, these laminations are not necessarily disjoint. However, since the number of components of $\mathcal{V}(X_i)$ is uniformly bounded, the same holds true for the number of limit laminations.

Let $\zeta_1, \dots, \zeta_s$ be the set of all components of these limit laminations which are distinct from simple closed curves. The number of such components is finite. Each of the laminations ζ_j fills a subsurface S_j of S which is different from an annulus or a pair of pants. Thus ζ_j is a point in the Gromov boundary of the curve graph $\mathcal{CG}(S_j)$ of S_j.

Since a sequence c_i^j of simple closed curves on the surface S_j converges to ζ_j in $\mathcal{CG}(S_j) \cup \partial\mathcal{CG}(S_j)$ if and only if their geodesic representatives c_i^j converge to ζ_j in the coarse Hausdorff topology, for each $j \leq s$ the subsurface projections to $\mathcal{CG}(S_j)$ of the sets $\mathcal{V}(X_i)$ converge as $i \to \infty$ in $\mathcal{CG}(S_i) \cup \partial\mathcal{CG}(S_j)$ to the lamination $\zeta_j \in \partial\mathcal{CG}(S_j)$. As the diameter of the subsurface projection of $V(X_i)$ to S_j is bounded independent of i, hyperbolicity of $\mathcal{CG}(S_j)$ implies that this convergence holds true for the subsurface projection to S_j of any of the curves in $\mathcal{V}(X_i)$ which intersects S_j. As a consequence, none of the limits in the Hausdorff topology of any sequence of components of $\mathcal{V}(X_i)$ can intersect ζ_j.

By a similar argument, if ζ_j is a closed curve component, then we can consider the subsurface projections of a component of $V(X_i)$ to an annulus $A(\zeta_j)$ with core curve ζ_j. Up to passing to a further subsequence, we may assume that these projections are either bounded along the sequence, or converge to one of the two boundary components of the curve graph of $A(\zeta_j)$. In the first case call ζ_j *unlabeled.* In the second case, label ζ_j with the corresponding point in the Gromov boundary of the curve graph of $A(\zeta_j)$ and note by the reasoning used in the previous paragraph, no labeled simple closed curve component ζ_j can be intersected by another component ζ_ℓ.

By reordering, let $\zeta_1, \dots, \zeta_k$ be the components of the limit laminations which either are distinct from simple closed curves or which are labeled simple closed curves. We claim that $k \geq 1$, that is, that there is at least one lamination with this property. Namely, by Theorem 1.1 of [R07], as the Teichmüller distance between the basepoint X and X_ℓ tends to infinity with ℓ, there exists at least one subsurface V so that the diameter of the subsurface projection $\mathrm{pr}_V(X) \cup \mathrm{pr}_V(X_\ell)$ tends to infinity which guarantees that there exists at least one component of the limit lamination of this form.

By what we showed so far, $\hat{\zeta} = \cup_{j=1}^k \zeta_k$ is a geodesic lamination. Furthermore, if S_j is the subsurface of S filled by ζ_j, then $d_{\mathcal{CG}}(\mathrm{pr}_{S_j}(\mathcal{V}(X_i)), x_j) \to \infty$ where as before, $x_j \in \mathcal{CG}(S_j)$ is a fixed basepoint for $\mathcal{CG}(S_j)$.

By passing to a subsequence and reordering, we may assume that

$$d_{\mathcal{CG}(S_1)}(\mathrm{pr}_{S_1}(\mathcal{V}(X_i)),x_1) \geq d_{\mathcal{CG}(S_j)}(\mathrm{pr}_{S_j}(\mathcal{V}(X_i)),x_j) - b$$

for all i,j where b is twice the maximal diameter of the subsurface projection of any of the sets $\mathcal{V}(X_i)$. Then by passing to another subsequence, we may assume that

$$d_{\mathcal{CG}(S_j)}(\mathrm{pr}_{S_j}(\mathcal{V}(X_i)),x_j)/d_{\mathcal{CG}(S_i)}(\mathrm{pr}_{S_1}(\mathcal{V}(X_i)),x_1) \to b_i \leq 1.$$

Define $a_i = b_i/\sum_j b_j$ and let $\xi = \sum_{i=1}^k a_i\zeta_i$.

We claim that $X_i \to \xi \in (\overline{\mathcal{T}}(S),\mathcal{O}_0)$. To this end note that the first property in the definition of a convergent sequence is fulfilled by the above discussion, and the second holds true by the observation that if there exists a subsurface U different from the surfaces S_j ($j \leq k$) such that for some subsequence, the subsurface projections of $\mathcal{V}(X_i)$ to U are unbounded, then the subsurface projections $\mathrm{pr}_U(\mathcal{V}(X_i))$ converge up to passing to a subsequence in the Hausdorff topology to a lamination which fills U, violating the choice of the laminations ζ_j. Thus $\overline{\mathcal{T}}(S)$ is sequentially compact Hausdorff Lindelöf space and hence it is compact.

We are left with showing that $\mathrm{Mod}(S)$ acts on $\overline{\mathcal{T}}(S)$ as a group of transformations. However, as $\mathrm{Mod}(S)$ acts on $\mathcal{T}_\epsilon(S)$ and on $\mathcal{X}(S)$ as a group of transformations, and as the definition of convergence which determines the topology $\mathcal{O}_0$ is natural with respect to the action of $\mathrm{Mod}(S)$ on subsurfaces and subsurface projections, this is indeed the case. The proposition is proven. □

Theorem 4.7. *$\mathcal{X}(S)$ is a small boundary for $\mathrm{Mod}(S)$. A pseudo-Anosov mapping class acts on $\mathcal{X}(S)$ with north-south dynamics. In particular, the action of $\mathrm{Mod}(S)$ on $\mathcal{X}(S)$ is strongly proximal.*

Proof. We showed so far that $\mathcal{X}(S)$ defines a boundary of $\mathcal{T}_\epsilon(S)$ and hence of $\mathrm{Mod}(S)$ since $\mathrm{Mod}(S)$ acts properly and cocompactly on $\mathcal{T}_\epsilon(S)$. Furthermore, a pseudo-Anosov element acts on $\mathcal{X}(S)$ with north-south dynamics and hence the action of $\mathrm{Mod}(S)$ on $\mathcal{X}(S)$ is strongly proximal.

We are left with showing that the right action of $\mathrm{Mod}(S)$ induces the identity. However, this action just consists of a change of basepoint. As a sequence of points of uniformly bounded distance from a convergent sequence converges to the same point, this yields the statement of the theorem. □

Example 4.8. Let $\xi = \sum_i a_i\xi_i \in \mathcal{X}(S)$ and let S_i be the subsurface of S filled by ξ_i. Assume that each ξ_i is *uniquely ergodic* and recurrent. By this we mean that a Teichmüller geodesic for S_i with vertical measured geodesic lamination ξ_i is *recurrent*, that is, it returns to a fixed compact subset of moduli space for arbitrarily large times.

Let μ_i be the marking of S_i determined by the fixed marking μ of S. For each i choose a mapping class g_i which preserves the surfaces S_i and such that $g_i\mu_i \to \xi_i$ and that the growth speed condition holds. Such a sequence of mapping classes exists by the recurrence assumptions. Let $U \subset \mathcal{X}(S)$ be an open subset containing ξ which is separated from the repelling fixed points of g_i. Suppose furthermore that

U does not contain any point in $\mathcal{J}(S_1, \dots, S_k, S_{k+1})$ where S_{k+1} is disjoint from S_i. We claim that $\cap_i g_i \overline{U} = \{\xi\}$.

To this end note that the subsurface projections of any lamination $\beta \in U$ to $V \subset S - \cup_i S_i$ remains constant. The subsurface projections of $\beta \in U$ to S_i converges to ξ and the speed condition is fulfilled as well.

5. Metrizability

The goal of this section is to show the following result.

Theorem 5.1. *$(\overline{\mathcal{T}}(S), \mathcal{O}_0)$ is metrizable.*

The strategy for the proof consists in the construction of an explicit neighborhood basis in $\overline{\mathcal{T}}(S)$ for every point $\xi \in \mathcal{X}(S)$. The statement of the theorem then follows with standard tools.

We begin with a general statement about the relation between distances in the marking graph and in the curve graph of S together with all subsurface projections. This statement is closely related to but not an immediate consequence of the distance formula of [MM00, Mi10] for the marking graph. For its formulation, define the *complexity* $\chi(V)$ of a surface V of genus g with s marked points or boundary components as $\chi(V) = 3g - 3 + s$.

Lemma 5.2. *There exists a number $q = q(S) > 0$ with the following property. Let μ, ν be two markings of S. Assume that the distance between μ, ν in the marking graph of S equals $\ell \geq 0$. Then there exists a (not necessarily proper) subsurface V of S so that the diameter in the curve graph of V of the subsurface projection of μ, ν into V is at least $(\ell/q)^{1/(\chi(S)+1)}$.*

Proof. As this statement is a statement about markings, it is valid for surfaces with cusps or boundary. Thus we may argue by induction on the complexity of S.

For the beginning of the induction, consider a surface V of complexity $\chi(V) = 1$. Then V is either a four-holed sphere or a one-holed torus, and two curves in the curve graph of V are connected by an edge if they intersect in the minimal number of points (two in the case of a four-holed sphere and one in the case of a one-holed torus).

For simplicity, we only consider the case of a four-holed sphere V, the case of a one-holed torus is completely analogous. Then a marking μ of V consists of an ordered pair $(b, \hat{b})$ of simple closed curves which intersect in two points. We call these curves components of the marking. Let μ, ν be two markings of V, written as pairs $(b, \hat{b})$ and $(c, \hat{c})$. Let $\eta : [0, n] \to \mathcal{CG}(V)$ be a geodesic connecting b to c. Then we can construct a geodesic in the marking graph as follows (see Section 6 of [MM00] for details, and, in particular, see Theorem 6.12 of [MM00]).

The curves $\hat{b}$ and $\eta(1)$ intersect $\eta(0) = b$ in precisely two points. The curve $\eta(0)$ decomposes V into two pairs of pants P_1, P_2. Since there is a single non-trivial homotopy class of an arc in P_1 with endpoints on $\eta(0)$ (and endpoints are allowed to move freely), $\eta(1)$ can be obtained from $\hat{b}$ by a power of a Dehn twist about $\eta(0)$. Let $q(0) \geq 0$ be the number of twists needed to transform $\hat{b}$ to $\eta(1)$. Up to perhaps an additive constant of ± 1, this number equals the diameter of the subsurface projection of the pair $(\hat{b}, \eta(1))$ into the annulus about $\eta(0)$. If $q(0) = 0$ then $\eta(1) = \hat{b}$ and we replace the pair $(b, \hat{b})$ by $(\hat{b}, b)$, which accounts for an edge in the marking graph. We also replace η by the geodesic $\eta[1, n]$ starting at $\hat{b} = \eta(1)$. Otherwise perform $q(0)$ twists about $\eta(0)$ to transform $\hat{b}$ to $\eta(1)$ and consecutively replace the pair $(\eta(0), \eta(1))$ by the pair $(\eta(1), \eta(0))$. This defines an edge path in the marking graph of length $q(0) + 1$. Proceed inductively.

The length of the resulting path in the marking graph equals $n + \sum_i q(i)$ where up to an error of ± 1, $q(i)$ is the diameter of the subsurface projection of $\hat{b}, \hat{c}$ into the annulus with core curve $\eta(i)$. Furthermore, the length of this path is bounded from above by $p d_V(\mu, \nu)$ where d_V denotes the distance in the marking graph of the surface V and $p \geq 1$ is a universal constant. We refer to Section 6 of [MM00] for more details on these facts.

There are now two cases possible. In the first case, $n = d_{\mathcal{CG}(V)}(b, c) \geq d_V(\mu, \nu)^{1/2}$ and then as $\chi(V) = 1$ we are done. Otherwise we have $d_V(\mu, \nu) \geq (d_{\mathcal{CG}(V)}(b, c))^2$ and hence with the above notation, as the length of the above path is not smaller than $d_V(\mu, \nu)$, there has to be at least one i so that $q(i) + 1 \geq d_V(\mu, \nu)^{1/2}$. If $i \neq 0, n$ then this implies that the diameter of the subsurface projection of b, c into the annulus $A(\eta(i))$ is at least $q(i) - 2 \geq d_V(\mu, \nu)^{1/2} - 1$. If $i = 0$ (or $i = n$) then the same argument applies to the subsurface projection of $\hat{b}, \hat{c}$ into the annulus with core curve $\eta(1)$ and we also obtain the statement we wanted to show.

Assume now that the statement of the lemma is known for all surfaces W of complexity $\chi(W) \leq k - 1$ for some $k \geq 2$, perhaps with punctures and boundary, with control constant $q = q(k-1)$ only depending on $k - 1$. We aim at extending this statement to surfaces V with $\xi(V) = k$.

By the definition of the marking graph (see also the distance formula from Theorem 6.12 of [MM00]), the distance $d_V(\mu, \nu)$ of two markings μ, ν of the surface V is not smaller than the smallest distance in $\mathcal{CG}(V)$ between two pants curves c, d from μ, ν.

Let $\eta : [0, n] \to \mathcal{CG}(V)$ be a geodesic in $\mathcal{CG}(V)$ connecting a pants curve b of μ to a pants curve c of ν. If its length is at least $(d_V(\mu, \nu)/q)^{1/(\chi(V)+1)}$ then we are done. Thus assume that its length does not exceed $(d_V(\mu, \nu)/q)^{1/(\chi(V)+1)}$.

By the distance formula for the marking graph (Theorem 6.12 of [MM00]), there exists at least one number $i \in [0, n]$ such that the sum of the diameters of the subsurface projections into subsurfaces which are disjoint from $\eta(i)$ is not smaller than $(d_V(\mu, \nu)/q)^{\chi(V)/(\chi(V)+1)}$ (up to perhaps replacing q by a bigger constant). Here the endpoints $\eta(0) = b$ and $\eta(n) = c$ are included. If for example $i = 0$, then the sum of the diameters of the subsurface projections into all subsurfaces of $V - b$ between $\eta(1)$ and some marking curve of μ different from b is at least $(d_V(\mu, \nu)/q)^{\chi(V)/(\chi(V)+1)}$.

Now recall that the subsurface projection $\mathrm{pr}_W(c)$ of a simple closed curve $c \subset V$ into a subsurface W of V equals the collection of intersection components of c with W. This implies that for nested subsurfaces $W_1 \subset W_2 \subset V$, the subsurface projection of c into W_1 equals the subsurface projection of $\mathrm{pr}_{W_2}(c)$ into W_1. Thus we can apply the induction hypothesis and conclude that there exists a subsurface W of $V - \eta(i)$ so that the diameter of the subsurface projection of b, c into W is at least $(d_V(\mu,\nu)^{((\chi(V)/(\chi(V)+1))/(\chi(V))}q^{-\chi(V)-1} = d_V(\mu,\nu)^{1/(\chi(V)+1)}q^{-\chi(V)-1}$. But this is what we wanted to show. □

By [MM99], for any surface V of finite type, there is a number $p > 0$ only depending on the complexity of V such that the image under the map Υ of a Teichmüller geodesic $\gamma : \mathbb{R} \to \mathcal{T}(S)$ is an *unparameterized p-quasi-geodesic* in $\mathcal{CG}(V)$. This means the following. There is an increasing homeomorphism $\sigma : (a,b) \subset \mathbb{R} \to \mathbb{R}$ such that the map $\Upsilon \circ \gamma \circ \sigma : (a,b) \to \mathcal{CG}(S)$ is a p-quasi-geodesic. This quasi-geodesic may be bounded, one-sided infinite or two-sided infinite, and it is one-sided infinite if the geodesic recurs to the thick part $\mathcal{T}_\epsilon(S)$ for arbitrarily large times. As a consequence, up to increasing p, any geodesic segment $\alpha : [0,n| \to \mathcal{CG}(S)$ can be extended to a p-quasi-geodesic ray $\alpha : [0,\infty) \to \mathcal{CG}(S)$.

Let $\xi = \sum_{i=1}^k a_i\xi_i \in \mathcal{J}(\cup_i S_i) \subset \mathcal{X}(S)$, that is, we assume that for $i \leq k$ the subsurface S_i of S is *filled* by the geodesic laminations ξ_i. This means that S_i contains ξ_i, and $S_i - \xi_i$ is a union of simply connected regions and annuli about the boundary circles of S_i.

In the construction of the topology on $\mathcal{X}(S)$, we fixed a basepoint marking μ for S and noted that this marking coarsely projects to a marking μ_V of any subsurface V of S. If $V = S - \cup_i S_i$ (where by convention, the annulus about each component of the boundary of one of the surfaces S_i is contained in V if it is not already contained in the set $\{S_j \mid j\}$) then the union of these projected markings $\mu_i = \mu_{S_i}$ and μ_V determine a marking $\hat{\mu}$ of S whose pants decomposition P contains the boundary components of the surfaces S_i. Recall that the marking $\hat{\mu}$ depends on choices, but any two distinct choices are uniformly close in the marking graph of S.

For $i \leq k$ let $c_i \subset \mathcal{CG}(S_i)$ be a component of P if S_i is not an annulus, and otherwise let $c_i \in \mathcal{CG}(S_i)$ be the arc representing a projection of the base marking. For any subsurface V of S let furthermore c_V be a component of the pants decomposition of V defined by the marking μ_V.

For $X \in \mathcal{T}_\epsilon(S)$ and an essential subsurface $V \subset S$, we write $\mathrm{pr}_V(X)$ to denote a marking of V obtained by subsurface projection into V of a short marking μ_X of X. Note as before that this is coarsely well defined. For a vertex c of the curve graph of S we also denote as before by $\mathrm{pr}_V(c)$ the subsurface projection of c into V.

For $\xi = \sum_{i=1}^k a_i\xi_i$ as above and for $j \geq 0$, $\delta \in (0,1)$ define
$$W(\xi, j, \delta) \subset \mathcal{T}_\epsilon(S)$$
to be the set of all hyperbolic metrics $X \in \mathcal{T}_\epsilon(S)$ with the following property.

(1) For all $i \leq k$, we have $d_{\mathcal{CG}(S_i)}(\mathrm{pr}_{S_i}(\Upsilon(X)), c_i) \geq j$.

(2)
$$\frac{d_{\mathcal{CG}(S_i)}(\mathrm{pr}_{S_i}(\Upsilon(X)), c_i)}{d_{\mathcal{CG}(S_1)}(\mathrm{pr}_{S_1}(\Upsilon(X)), c_1)} \in [(1-\delta)a_i/a_1, (1+\delta)a_i/a_1]$$
for all i.

(3) Let $V = S - \cup_i S_i$; if we denote by d_V the distance in the marking graph of V then we have
$$d_V(\mathrm{pr}_V(X), \mu_V) < \delta d_{\mathcal{CG}(S_1)}(\mathrm{pr}_{S_1}(\Upsilon(X)), c_1).$$

(4) For each $i \leq k$, a geodesic in $\mathcal{CG}(S_i)$ connecting c_i to $\mathrm{pr}_{S_i}(\Upsilon(X))$ can be extended to a p-quasi-geodesic in $\mathcal{CG}(S_i)$ whose endpoint is contained in the ball of radius e^{-j} about ξ_i in $\partial\mathcal{CG}(S_i)$, where the metric on $\partial\mathcal{CG}(S_i)$ is the Gromov distance d_{c_i} constructed from the basepoint c_i.

Lemma 5.3. *Let $\xi = \sum_i a_i \xi_i$ and let $j \geq 1, \delta > 0$. Put $a = \min\{a_i \mid i\} > 0$.*

(1) *The closure of $W(\xi, j, \delta)$ in $\overline{\mathcal{T}}(S)$ is a neighborhood of ξ in $\overline{\mathcal{T}}(S)$.*
(2) *Let $\delta' < \delta a^2/4$ and let $\xi' = \sum_i a_i' \xi_i'$ be such that $\xi_i' \in \partial\mathcal{CG}(S_i)$, and $\max\{|a_i - a_i'| \mid i\} \leq \delta'$, and $d_{c_i}(\xi_i, \xi_i') < e^{-4j}$; then $W(\xi', 2j, \delta') \subset W(\xi, j, \delta)$, and the closure of $W(\xi', 2j, \delta')$ is a neighborhood of $\xi \in \overline{\mathcal{T}}(S)$.*

Proof. Let $\xi = \sum_i a_i \xi_i \in \mathcal{X}(S)$ and let $j > 0, \delta > 0$. Since $\mathcal{T}_\epsilon(S)$ is dense in $\overline{\mathcal{T}}(S)$ and by Proposition 4.6, $\overline{\mathcal{T}}(S)$ is a compact separable Hausdorff space, to show that the closure in $\overline{\mathcal{T}}(S)$ of $W(\xi, j, \delta)$ is a neighborhood of ξ in $\overline{\mathcal{T}}(S)$ it suffices to show the following. Let $(X_\ell) \subset \mathcal{T}_\epsilon(S)$ be a sequence converging in $\overline{\mathcal{T}}(S)$ to ξ; then $X_\ell \in W(\xi, j, \delta)$ for all sufficiently large ℓ.

By the second requirement in the Definition 4.4 of convergence, the projections $\mathrm{pr}_{S_i}(\Upsilon(X_\ell))$ converge as $\ell \to \infty$ to ξ_i in $\mathcal{CG}(S_i) \cup \partial\mathcal{CG}(S_i)$. Thus for sufficiently large ℓ, we know that
$$d_{\mathcal{CG}(S_i)}(c_i, \mathrm{pr}_{S_i}(\Upsilon(X_\ell))) \geq 2j \text{ for all } i.$$
Moreover, by hyperbolicity of $\mathcal{CG}(S_i)$ and extendibility of geodesics, since the projections $\mathrm{pr}_{S_i}(\Upsilon(X_\ell))$ converge as $\ell \to \infty$ in $\mathcal{CG}(S_i) \cup \partial\mathcal{CG}(S_i)$ to the point $\xi_i \in \partial\mathcal{CG}(S_i)$, for large enough ℓ the points $\mathrm{pr}_{S_i}(X_\ell)$ are contained in a p-quasi-geodesic connecting c_i to a point in the e^{-j}-ball about ξ_i. Thus for large enough ℓ, the first and fourth requirements in the definition of $W(\xi, j, \delta)$ are fulfilled.

Moreover, it follows from Lemma 5.2 that the requirements (2) and (3) on the relative distances in the curve graphs of S_i and the marking graph of V are also fulfilled provided that ℓ is sufficiently large. As a consequence, for large enough ℓ we have $X_\ell \in W(\xi, j, \delta)$ which shows that the closure of $W(\xi, j, \delta)$ is indeed a neighborhood of ξ in $\overline{\mathcal{T}}(S)$. The first part of the lemma follows.

To show the second part, assume for simplicity that $\delta < \min_i a_i = a$ (this is the only case we need later on). Let $\delta' < \delta a^2$ and let $\xi' = \sum_i a_i' \xi_i'$ with $\max_i |a_i - a_i'| < \delta'/4$ and $d_{c_i}(\xi_i', \xi_i) < e^{-4j}$. Then the ball of radius e^{-2j} about ξ_i' contains the ball of radius e^{-4j} about ξ_i for all i and hence it follows from the definitions that $W(\xi', 2j, \delta') \subset W(\xi, j, \delta)$ and that furthermore, $W(\xi', 2j, \delta')$ is a neighborhood of ξ as it contains a set $W(\xi, 4j, \sigma)$ for some $\sigma > 0$. This completes the proof of the lemma. $\square$

Corollary 5.4. *For each $\xi = \sum_i a_i \xi_i \in \mathcal{X}(S)$, the closures of the sets $W(\xi, j, \delta)$ $(j \geq 1, \delta > 0)$ define a neighborhood basis of ξ in $\overline{\mathcal{T}}(S)$.*

Proof. Observe first that the sets $W(\xi, j, \delta)$ are *nested*: If $m > j$, then $W(\xi, m, \delta) \subset W(\xi, j, \delta)$, and if $\sigma < \delta$ then $W(\xi, j, \sigma) \subset W(\xi, j, \delta)$. Thus by Lemma 5.3, to show that the closures $\overline{W(\xi, j, \delta)}$ in $\overline{\mathcal{T}}(S)$ of the sets $W(\xi, j, \delta)$ define a neighborhood basis of ξ in $\overline{\mathcal{T}}(S)$, it suffices to show that $\cap_{j>0} \cap_{\delta>0} \overline{W(\xi, j, \delta)} = \{\xi\}$.

To see that this is indeed the case note first that $\xi \in \overline{W(\xi, j, \delta)}$ for all j, δ and hence as these sets are compact, the point ξ also is contained in the intersection of these sets. Furthermore, the following holds true. For each ℓ let $X_\ell \in W(\xi, \ell, \delta)$ for some $\delta > 0$; then the distance to the base curve c_i in $\mathcal{CG}(S_i)$ $(i \leq k)$ of the subsurface projection of $\Upsilon(X_\ell)$ tends to infinity with ℓ. This implies that the sequence X_ℓ can not have a convergent subsequence in $\mathcal{T}_\epsilon(S)$.

Thus by compactness of $\overline{\mathcal{T}}(S)$, up to passing to a subsequence, the sequence converges to a point in $\mathcal{X}(S)$. It then follows that this point is contained in a set of the form $\mathcal{J}(S_1, \dots, S_k, V)$ where V is a subsurface of $S - \cup_i S_i$. In other words, for fixed $\delta > 0$ we have $\cap_j \overline{W(\xi, j, \delta)} \subset \mathcal{X}(\cup_i S_i)$, moreover this set is contained in a neighborhood of ξ in $\mathcal{J}(\cup_i S_i)$ whose size is controlled by δ. Letting δ tend to zero yields the corollary. □

We are now ready to show

Proposition 5.5. *$\overline{\mathcal{T}}(S)$ is metrizable.*

Proof. By Uryson's theorem, a second countable Hausdorff space is metrizable. As by Proposition 4.6 the space $\overline{\mathcal{T}}(S)$ is Hausdorff, it suffices to show that $\overline{\mathcal{T}}(S)$ is second countable. Since $\mathcal{T}_\epsilon(S)$ is second countable, this is the case if there exist countably many open sets $U_i \subset \overline{\mathcal{T}}(S)$ which contain a neighborhood basis for any point $x \in \mathcal{X}(S)$.

To see that this is the case let $\mathcal{S}$ be the countable collection of all families of pairwise disjoint subsurfaces $(S_1, \dots, S_k)$ of S. For each $i \leq k$ choose a countable dense subset $\{\xi_{i,\ell} \mid \ell\}$ of minimal filling geodesic laminations on S_i with respect to the coarse Hausdorff topology. Such a set exists since the Hausdorff topology on geodesic laminations is metrizable. Put

$$\mathcal{B}(\cup_i S_i) = \{\sum_i a_i \xi_{i,\ell(i)} \mid a_i \in \mathbb{Q}, \sum_i a_i = 1\}.$$

Then $\mathcal{B}(\cup_i S_i)$ is a dense subset of the joint $\mathcal{J}(\cup_i S_i)$.

Now let $U \subset \overline{\mathcal{T}}(S)$ be open and let $\xi \in U \cap \mathcal{X}(S)$. Then there exists $\mathfrak{S} = (S_1, \dots, S_k) \in \mathcal{S}$, and there are geodesic laminations ξ_i which fill S_i and so that $\xi = \sum_i a_i \xi_i \in \mathcal{J}(S_1, \dots, S_k)$.

By Corollary 5.4, there exists $j \geq 1, \delta > 0$ so that $\overline{W(\xi, j, \delta)} \subset U$. By the choice of the family $\mathcal{B}(\cup_i S_i)$ and the second part of Lemma 5.3, there exists $\xi' \in \mathcal{B}(\cup_i S_i)$ so that $W(\xi', j, 1/q) \subset W(\xi, j', \delta)$ is a neighborhood of ξ contained in U. Together this implies that $\overline{\mathcal{T}}(S)$ is indeed second countable and completes the proof of the proposition. □

6. Neighborhood bases

The goal of this section is to construct for a point in $\mathcal{X}(S) \subset \overline{\mathcal{T}}(S)$ an explicit neighborhood basis in $\overline{\mathcal{T}}(S)$ consisting of sets whose intersections with $\mathcal{T}_\epsilon(S)$ are contractible. Note that the neighborhood basis we constructed in Section 5 does not seem to consist of sets with this property. However, the neighborhoods from that basis will be used in our construction. Here by a contractible subset of $\mathcal{T}_\epsilon(S)$ we mean a subset V which is a contractible space with respect to the subspace topology.

To control contractibility of neighborhoods we now choose a torsion free finite index subgroup Γ of $\mathrm{Mod}(S)$ and use this to construct a Γ-equivariant diffeomorphism $\Lambda : \mathcal{T}(S) \to \Lambda(\mathcal{T}(S))$ as in Proposition 4.3. The group Γ acts properly and cocompactly on the closure $Q = \overline{\Lambda(\mathcal{T}(S))}$, and Q is contractible. Throughout we replace the space $\mathcal{T}_\epsilon(S)$ by the space $Q \supset \mathcal{T}_\epsilon(S)$.

6.1. **A neighborhood basis for minimal filling laminations.** In this subsection we prove the following result.

Proposition 6.1. *Every point $\xi \in \partial\mathcal{CG}(S) \subset \mathcal{X}(S)$ has a countable neighborhood basis in $\overline{\mathcal{T}}(S)$ consisting of sets whose intersections with Q are contractible.*

To set up the proof, note that any minimal filling geodesic lamination ξ decomposes S into a union of ideal polygons. Each of these polygons which is not an ideal triangle can be subdivided by adding isolated leaves which connect two non-adjacent cusps of the polygon. The various ways to subdivide these polygons determine a finite collection $\xi_0, \dots, \xi_k$ of distinct geodesic laminations which contain ξ as a sublamination. Assume that $\xi_0 = \xi$.

Let d_H be the Hausdorff metric on the space of compact subsets of a fixed hyperbolic surface $X \in \mathcal{T}_\epsilon(S)$. Denote by $\mathrm{Min}_\cup(\mathcal{L})$ the space of geodesic laminations on X which are unions of disjoint minimal components and let

$$\mathrm{supp} : \mathcal{X}(S) \to \mathrm{Min}_\cup(\mathcal{L})$$

be the map which associates to a point $\sum_i a_i\xi_i$ $(a_i > 0)$ the support $\mathrm{supp}(\xi) = \cup_i \xi_i$. We have

Lemma 6.2. *For $i > 0$ let*

$$U_i = \cup_j \{\beta \in \mathrm{Min}_\cup(\mathcal{L}) \mid d_H(\beta, \xi_j) \le 1/i\}$$

and write $V_i = \{\zeta \in \mathcal{X}(S) \mid \mathrm{supp}(\zeta) \in U_i\}$. Then the sets V_i form a neighborhood basis of ξ in $\mathcal{X}(S)$.

Proof. Clearly $\xi \in V_i$ for all i. We first show that for each i the set V_i is a neighborhood of ξ. For this we argue by contradiction and we assume that there exists some i such that this is not the case. Then there exists a sequence $\zeta_j \subset \mathcal{X}(S)$ such that $\zeta_j \to \xi$ and such that $\zeta_j \notin V_i$ for all j.

By the first requirement for convergence in the definition of the topology on $\mathcal{X}(S)$, we know that $\mathrm{supp}(\zeta_j)$ converges in the *coarse* Hausdorff topology to $\xi_0 = \mathrm{supp}(\xi)$. By compactness of the space of compact subsets of S with respect to the Hausdorff topology, by passing to a subsequence we may assume that the sequence $\mathrm{supp}(\zeta_j)$ converges in the Hausdorff topology to a geodesic lamination ζ. Then ζ contains ξ_0 as a sublamination and hence $\zeta = \xi_s$ for some $s \leq k$. By definition, this implies that $\zeta_j \in V_i$ for all sufficiently large j, a contradiction. This shows that indeed, each of the sets V_i is a neighborhood of ξ.

To show that the sets V_i form a neighborhood basis for ξ, note that $V_{i+1} \subset V_i$ and hence it suffices to show that $\cap_i V_i = \{\xi\}$. However, this is immediate from the definitions and the fact that the preimage of $\mathrm{supp}(\xi)$ under the support map supp which associates to $\zeta \in \mathcal{X}(S)$ its support consists of the single point ξ. □

As an immediate consequence of Lemma 6.2, we obtain an alternative proof of a special case of Corollary 5.4.

Corollary 6.3. *Each point $\xi \in \partial\mathcal{CG}(S) \subset \mathcal{X}(S)$ has a countable neighborhood basis.*

A *measured geodesic lamination* on the hyperbolic surface X is a geodesic lamination together with a transverse invariant measure. The space $\mathcal{ML}$ of measured geodesic laminations is equipped with the weak* topology. The quotient of $\mathcal{ML}$ under the natural action of $(0,\infty)$ by scaling is the space $\mathcal{PML}$ of *projective measured geodesic laminations*. This space is homeomorphic to the sphere $S^{6g-7+2m}$. To put Lemma 6.2 into proper context and for later use, we relate the subset $\partial\mathcal{CG}(S) \subset \mathcal{X}(S) \subset \overline{\mathcal{T}}(S)$ to the space $\mathcal{PML}$.

To this end we use a more geometric view on $\mathcal{PML}$. Fix again a point $X \in \mathcal{T}_\epsilon(S)$. To each simple closed curve c on S, we can associate the length $\ell_X(c)$ of the geodesic representative of c for the marked hyperbolic metric defining X. This length is just the *intersection number* $\iota(X,c)$ of the *geodesic current* c with the *Liouville current* λ_X of the metric X (see Section 8.2 of [Mar16] for a comprehensive account on this result of Bonahon).

The *intersection form* ι is a continuous non-negative convex bi-linear function on the space of geodesic currents for S, equipped with the weak*-topology, which is moreover homogeneous in both coordinates (Section 8.2 of [Mar16]). Since the space of measured geodesic laminations on S is a closed subspace of the space of geodesic currents and the Liouville current λ_X has positive intersection with every current, the set

$$\mathcal{ML}(X) = \{\nu \in \mathcal{ML} \mid \iota(\nu, \lambda_X) = 1\}$$

is the image of a section σ of the fibration $\mathcal{ML} \to \mathcal{PML}$.

The *support* $\text{supp}(\nu)$ of a measured geodesic lamination ν is a point in the space $\text{Min}_\cup(\mathcal{L})$. Each of its components is equipped with a transverse invariant measure and hence it is a measured geodesic lamination in its own right. By convex bilinearity of the intersection form, any $\xi \in \mathcal{ML}(X)$ can be represented in the form $\xi = \sum_i \xi_i$ where $\xi_i \in \mathcal{ML}$ are measured geodesic laminations with minimal support, the supports of the laminations ξ_i are pairwise disjoint and $\sum_i \iota(\xi_i, \lambda_X) = \iota(\xi, \lambda_X) = 1$. As a consequence, we can associate to a projective measured geodesic lamination $[\xi] \in \mathcal{PML}$ whose support $\text{supp}([\xi]) = \cup_i \text{supp}(\xi_i)$ does not contain a simple closed curve component a point $F_X(\xi) \in \mathcal{X}(S)$ by writing $\sigma([\xi]) = \sum_i \xi_i$ and putting

$$F_X(\xi) = \sum_i \iota(\xi_i, \lambda_X)\text{supp}(\xi_i).$$

This defines a map $F_X : \mathcal{A} \to \mathcal{X}(S)$ where $\mathcal{A} \subset \mathcal{PML}$ denotes the dense G_δ-set of projective measured geodesic laminations without closed curve component. Note that we have to exclude simple closed curve components in the support of a projective measured geodesic lamination as simple closed curves appearing in the support of a point in $\mathcal{X}(S)$ are oriented.

The cotangent space $T^*_X\mathcal{T}(S)$ of Teichmüller space at X can be identified with the space of measured geodesic laminations on S. Or, equivalently, by the Hubbard Masur theorem, every measured geodesic lamination on S is the vertical measured geodesic lamination for a unique quadratic differential at X. With this identification, we can associate to $\nu \in \mathcal{ML}$ the point $\gamma_\nu(1)$ where $\gamma_\nu : [0, \infty) \to \mathcal{T}(S)$ is the Teichmüller geodesic starting at X whose initial (co)-velocity $\gamma'_\nu(0)$ is the quadratic differential with vertical measured geodesic lamination ν. This construction defines the *Teichmüller exponential map* $\exp_X : \mathcal{ML} \cup \{0\} \to \mathcal{T}(S)$ at X which is a homeomorphism. Via identification of $\mathcal{PML}$ with the sphere of measured geodesic laminations of X-length one, we can view $\mathcal{PML}$ as the space of unit directions at X for the Teichmüller metric. We use these identifications freely in the sequel. Furthermore, for a measured geodesic lamination μ denote by $[\mu] \in \mathcal{PML}$ its projective class.

Let $p > 1$ be a control constant with the following properties.

- The image under the map Υ of any Teichmüller geodesic is an unparameterized p-quasi-geodesic in $\mathcal{CG}(S)$ (see [MM99]).
- Every geodesic segment in $\mathcal{CG}(S)$ can be extended to a p-quasi-geodesic ray.

Let $P(\xi) \subset \mathcal{PML}$ be the set of all projective measured geodesic laminations which are supported in ξ. This is a closed simplex of dimension $\leq 3g - 3 + m$ whose extreme points are the ergodic projective transverse measures supported in ξ. In particular, $P(\xi)$ is compact and contractible. Since $P(\xi)$ is contractible and since $\mathcal{PML}$ is homeomorphic to a sphere of dimension $6g - 7 + 2m$, we can find a descending chain $V_1 \supset V_2 \supset \cdots$ of open contractible neighborhoods of $P(\xi)$ such that $\cap_j V_j = P(\xi)$.

Lemma 6.4. *Let $V_1 \supset V_2 \supset \cdots$ be a descending chain of closed contractible neighborhoods of $P(\xi)$ in $\mathcal{PML}$, with $\cap_i V_i = P(\xi)$, and let $\Lambda : \mathcal{T}(S) \to Q$ be the diffeomorphism from Proposition 4.3. Let moreover* $\exp_X : \mathcal{ML} \to \mathcal{T}(S)$ *be the*

Teichmüller exponential map at X. Then for each $j > 0$, the closure in $\overline{\mathcal{T}}(S)$ of the set

$$Z(i,j) = \Lambda\{\exp_X(\mu) \mid \iota(\lambda_X, \mu) > j, [\mu] \in V_i\}$$

is a neighborhood of ξ, and neighborhoods of this form define a neighborhood basis of ξ.

Proof. We divide the proof of the lemma into two claims.

Claim 1: For all i, j, the closure $\overline{Z(i,j)}$ of $Z(i,j)$ in $\overline{\mathcal{T}}(S)$ is a neighborhood of ξ.

Proof of Claim 1: Recall that $\overline{\mathcal{T}}(S)$ is a compact Hausdorff space. Thus by the definition of the topology on $\overline{\mathcal{T}}(S)$ and the fact that $Q \supset \mathcal{T}_\epsilon(S)$ is dense in $\overline{\mathcal{T}}(S)$, it suffices to show the following. Let $Y_\ell \subset Q$ be a sequence converging to ξ; then for any fixed (i,j), we have $Y_\ell \in \overline{Z(i,j)}$ for all sufficiently large ℓ.

Let $\mathcal{FML} \subset \mathcal{PML}$ be the subset of all projective measured geodesic laminations whose support is a minimal geodesic lamination which fills up S. By Lemma 3.2 of [H09], the support map $F : \mathcal{FML} \to \partial\mathcal{C}(S)$ which associates to a point in $\mathcal{FML}$ its support is continuous and closed. Thus the image $F(\mathcal{FML} \setminus V_i)$ is a closed subset of $\partial\mathcal{CG}(S)$ which does not contain ξ. As a consequence, there exists a number $T = T(i) > 0$ so that the ball of radius $e^{-T(i)}$ about ξ with respect to the Gromov metric on $\partial\mathcal{CG}(S)$ based at $\Upsilon(X)$ is disjoint from $F(\mathcal{FML} \setminus V_i)$.

By the choice of the control constant $p > 1$ and hyperbolicity, there exists a number $\tau(i) > T(i)$ with the following property. Let $\mu \in \mathcal{FML} \setminus V_i$; then the endpoint of a p-quasi-geodesic ray in $\mathcal{CG}(S)$ with starts at the basepoint $\Upsilon(X)$ and which passes through a point on $\Upsilon(\gamma_\mu)$ of distance at least $\tau(i)$ to $\Upsilon(X)$ is not contained in the ball of radius $e^{-T(i)/2}$ about ξ. Since $\exp_X : \mathcal{ML} \cup \{0\} \to \mathcal{T}(S)$ is a homeomorphism and $\mathcal{FML} \subset \mathcal{PML}$ is dense, it follows that for every j there exists ℓ, δ so that the neigbborhood $W(\xi, \ell, \delta)$ of ξ in $\overline{\mathcal{T}}(S)$ is contained in the closure of the set $\{\exp_X(\gamma_\nu(t)) \mid \iota(\lambda_X, \nu) > j, [\mu] \in V_i\}$.

Since the retraction $\Lambda : \mathcal{T}(S) \to \Lambda(\mathcal{T}(S)) \subset Q$ is coarsely Υ-invariant and since the conditions (2),(3) in the definition of the sets $W(\xi, \ell, \delta)$ do not play a role if $\xi \in \partial\mathcal{CG}(S)$, the discussion in the previous paragraph implies that there exists some $\ell > 0$ so that $W(\xi, \ell, \delta) \subset \overline{Z(i,j)}$. Thus by Corollary 5.4, $\overline{Z(i,j)}$ is a neighborhood of ξ in $\overline{\mathcal{T}}(S)$. ■

The proof of the lemma is completed once we established the following.

Claim 2: Let W be a neighborhood of ξ in $\overline{\mathcal{T}}(S)$; then there exists some i, j so that $\overline{Z(i,j)} \subset W$.

Proof of Claim 2: By Claim 1, each of the sets $\overline{Z(i,j)}$ is a neighborhood of ξ and hence contains ξ. Furthermore, these neighborhoods are nested: If $i_1 \leq i_2$ and $j_1 \leq j_2$ then $\overline{Z(i_1,j_1)} \supset \overline{Z(i_2,j_2)}$. Thus since the sets $\overline{Z(i,j)}$ are moreover closed and hence compact, it suffices to show that $\cap_{i,j}\overline{Z(i,j)} = \{\xi\}$.

Since the Teichmüller exponential map $\exp_X$ at X is a homeomorphism, we clearly have $\cap_{i,j}\overline{Z(i,j)} \subset \mathcal{X}(S)$. On the other hand, the map $\Upsilon : \mathcal{T}(S) \to \mathcal{CG}(S)$ is coarsely Lipschitz, and for $\nu \in P(\xi)$, the p-quasigeodesic $t \to \Upsilon(\gamma_\nu(t))$ has infinite diameter. This implies that for any $k > 0$ there are numbers $i(k) > 0, m(k) > 0$ so that for all $\eta \in V_{i(k)}$, the diameter of the image under Υ of the Teichmüller geodesic segment $\exp_X([0, m(k)]\eta)$ is at least k. As a consequence, if $X_i \in Z(i,i)$ for each i, then by compactness of $\overline{\mathcal{T}(S)}$, up to passing to a subsequence the sequence X_i converges to a point $\zeta \in \mathcal{X}(S) \cap \partial\mathcal{CG}(S)$. That this point has to coincide with ξ is an immediate consequence of the discussion in the proof of Claim 1 above. This completes the proof of the claim. ■ □

The image of the diffeomorphism $\Lambda : \mathcal{T}(S) \to \Lambda(\mathcal{T}(S))$ is the interior of the submanifold Q of $\mathcal{T}(S)$ with smooth boundary ∂Q. Define the *small closure* $\overline{A}_{\rm small}$ of a subset A of Q to be the union of Q with the set of all point $z \in \partial Q$ so that z has a neighborhood in ∂Q which is entirely contained in the set theoretic closure of A.

Lemma 6.5. *The small closure of a contractible subset of Q is contractible.*

Proof. It suffices to deformation retract the small closure of a contractible subset A of Q into A. In a second step, one composes this deformation retraction with a deformation retraction of A to a point.

If $z \in \overline{A}_{\rm small}$, then there is a neighborhood of z in $\overline{A}_{\rm small}$ which is diffeomorphic to the set $B_0 = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_i x_i^2 < 1, x_1 \geq 0\}$, with z corresponding to 0. There is a smooth deformation retraction of B_0 into $B_0 \setminus V$ where V is a small neighborhood of 0 and such that the time one map of the deformation retraction is a diffeomorphism onto its image and so that the support of this deformation retraction is contained in $\sum_i x_i^2 < \frac{1}{2}$. Thus for every $z \in \overline{A}_{\rm small}$ there is a deformation retraction of $\overline{A}_{\rm small}$ which moves a neighborhood of z in $\overline{A}$ into $\overline{A} \setminus \partial Q$ and such that the intersection of the resulting set with ∂Q is properly contained in the intersection of $\overline{A}_{\rm small}$ with ∂Q.

Each compact subset K of $\overline{A}_{\rm small} \setminus A$ can be covered by finitely many open sets in $\overline{A}_{\rm small}$ which admit a deformation retraction into A. As the composition of finitely many deformation retractions of $\overline{A}_{\rm small}$ is a deformation retraction, there is a deformation retraction α of $\overline{A}_{\rm small}$ with $\alpha(\overline{A}_{\rm small}) \cap \overline{A}_{\rm small} \setminus A \subset \overline{A}_{\rm small} \setminus K$. The lemma now follows from an inductive iteration of this procedure. □

Lemma 6.6. *The small closures of the subsets $Z(i,j)$ of Q are contractible.*

Proof. By Lemma 6.5, it suffices to show that the sets $Z(i,j)$ are contractible.

To this end recall that since the set V_i is a contractible subset of the set of projectivized measured geodesic laminations, identified with the unit sphere in the cotangent space of $\mathcal{T}(S)$ at X, the set

$$H(i,j) = \cup_{\mu \in V_i}\{\gamma_\mu(t) \mid t \geq j\} \subset \mathcal{T}(S)$$

is contractible since it is homeomorphic to $V_i \times [0,\infty)$. This uses the fact that the Teichmüller exponential map at X is a homeomorphism of $T_X^*\mathcal{T}(S)$ onto $\mathcal{T}(S)$.

But $Z(i,j)$ is the image of $H(i,j)$ under the diffeomorphism $\Lambda : \mathcal{T}(S) \to Q$ and hence $Z(i,j)$ is contractible. Then by Lemma 6.5, the small closure of $Z(i,j)$ in $\mathcal{T}(S)$ is contractible as well. □

Since the cardinality of the family of sets $Z(i,j)$ is countable, we conclude

Corollary 6.7. *Each point $\xi \in \partial\mathcal{CG}(S) \subset \overline{\mathcal{T}}(S)$ has a countable neighborhood basis consisting of sets whose intersections with Q are contractible.*

Proof. The union of $\overline{Z(i,j)}_{\text{small}}$ with $\overline{Z(i,j)} \cap \mathcal{X}(S)$ is a neighborhood of ξ whose intersection with Q is contractible. The countably many such sets define a neighborhood basis of ξ in $\overline{\mathcal{T}(S)}$. □

6.2. **Boundaries of products.** In this section we consider surfaces of finite type $S_1, \dots, S_k$ and the product of the Teichmüller spaces $\mathcal{T}(S_1) \times \dots \times \mathcal{T}(S_k)$. In contrast to the setup in the previous sections, the surfaces S_i may have punctures but no boundary unless S_i is an annulus, which is allowed in the construction. If S_i is an annulus, then by convention, the Teichmüller space of S_i is the real line. If Si is not an annulus, then we require that $\mathcal{T}(S_i)$ is of dimension at least two. Let as before $\mathcal{X}(S_i)$ be the geometric boundary of the compactification $\overline{\mathcal{T}}(S_i)$ of $\mathcal{T}_\epsilon(S_i)$ and consider the join

$$\mathcal{X}(\cup_i S_i) = \mathcal{X}(S_1) * \dots * \mathcal{X}(S_k).$$

Note that in general, this is a bigger space than $\mathcal{J}(\cup_i S_i) = \partial\mathcal{CG}(S_1) * \dots * \partial\mathcal{CG}(S_k)$.

The following observation reflects the fact that the joint of the visual boundaries to two CAT(0)-spaces X, Y is the visual boundary of the product $X \times Y$.

Proposition 6.8. *The space $\mathcal{X}(\cup_i S_i)$ defines a compactification of $\prod \mathcal{T}_\epsilon(S_i)$ which is a small boundary for the direct product $\prod \mathrm{Mod}(S_i)$.*

Proof. We only sketch the proof as it is not properly needed in the sequel. The idea is to mimic the construction from Section 4. Choose a basepoint $(X_1, \dots, X_k) \in \mathcal{T}_\epsilon(S_1) \times \dots \times \mathcal{T}_\epsilon(S_k)$ and corresponding short markings $\mu(X_i)$ for S_i. We define what it means for a sequence $(X_1^j, \dots, X_k^j)$ $(j \ge 1)$ to converge to $\sum_i a_i \xi_i \in \mathcal{X}(\cup_i S_i)$ where $a_i \ge 0, \sum_i a_i = 1$ and where $\xi_i \in \mathcal{X}(S_i)$ for all i. Note that we allow that some of the coefficients a_i vanish.

To this end we use a variation of Definition 4.4 which is given by the following requirements, expressed with the notations from that definition. Assume by reordering that $a_1 \ge a_i$ for $i \ge 2$ and that $a_i > 0$ if and only if $i \le \ell$ for some $\ell \le k$.

- For $i \le \ell$, the points X_i^j converge in $\overline{\mathcal{T}}(S_i)$ to ξ_i.
- For each $i \le \ell$ write $\xi_i = \sum_u b_i^u \xi_i^u$ with $\sum_u b_i^u = 1$, $b_i^1 \ge b_i^u$ for $u \ge 2$. For each $i \le \ell$ and each u let Σ_i^u be the subsurface of S_i filled by ξ_i^u. If c_i is the basepoint in $\mathcal{CG}(S_i)$ then we have

  $d_{\mathcal{CG}(\Sigma_i)}(\mathrm{pr}_{\Sigma_i}(X_i^j), c_i)/d_{\mathcal{CG}(\Sigma_1)}(\mathrm{pr}_{\Sigma_1}(X_1^j), c_1) \to \; a_i b_i^1/a_1 b_1^1$ for all i.

- Let V be any subsurface of S_i disjoint from the surfaces Σ_i^u if $i \leq \ell$ or any subsurface of S_i for $i > \ell$ and let x_V be the basepoint of $\mathcal{CG}(V)$. Then we have

$$d_{\mathcal{CG}(V)}(\mathrm{pr}_V(X_i^j), x_V)/d_{\mathcal{CG}(\Sigma_1^1)}(\mathrm{pr}_{\Sigma_1^1}(X_i^j), x_1) \to 0.$$

It now follows from the proof of Proposition 4.5 that this notion of convergence indeed defines a topology on $\overline{\mathcal{T}}(\cup_i S_i) = \prod \mathcal{T}_\epsilon(S_i) \cup \mathcal{X}(\cup_i S_i)$ which coincides with the given topology of $\mathcal{X}(\cup_i S_i)$ and the product topology on $\prod \mathcal{T}_\epsilon(S_i)$ so that $\overline{\mathcal{T}}(\cup_i S_i)$ is a compact metrizable space. $\square$

To make the notations more uniform we now write a point $\xi = \sum_{i=1}^k a_i \xi_i \in \mathcal{X}(\cup_i S_i)$ of the join $\mathcal{X}(\cup_i S_i)$ in a formal way by summing over all surfaces S_i and allowing that $a_i = 0$ for $\ell < i \leq k$ and some $\ell \geq 1$. Assume by reordering that $a_1 = \max\{a_i \mid i\}$. For technical reason which will be apparent later, in this section we exclusively work in the product of the Teichmüller spaces $\mathcal{T}(S_i)$ without passing to their thick parts. Let $X_i \in \mathcal{T}_\epsilon(S_i)$ be a choice of a basepoint. Put $c_i = \Upsilon(X_i)$. For a measured geodesic lamination ν_i on S_i let $\gamma_{\nu_i} : \mathbb{R} \to \mathcal{T}(S_i)$ be the Teichmüller geodesic starting at X_i which is determined by ν_i.

Recall from Section 6.1 that for each i and every measured geodesic lamination ν_i on S_i the function

$$t \to d_{\mathcal{CG}(S_i)}(\Upsilon(\gamma_{\nu_i}(t)), c_i)$$

is *coarsely non-decreasing*. By this we mean that there exists a number $q > 0$ such that $f(y) \geq f(x) - p$ for all $x \leq y$. Namely, there exists a number $p > 0$ so that the map $t \to \Upsilon(\gamma_{\nu_i}(t))$ is an unparameterized p-quasi-geodesic in $\mathcal{CG}(S_i)$, and quasi-geodesics in a hyperbolic geodesic metric space do not coarsely backtrack. The following was shown in [H09].

Lemma 6.9. *There exists a continuous* $\mathrm{Mod}(S_i)$*-equivariant function*

$$\delta_{c_i} : \mathcal{T}(S_i) \to [0, \infty)$$

which is at uniformly bounded distance from the function $X_i \to d(c_i, \Upsilon(X_i))$.

To construct contractible subsets of $\prod \mathcal{T}(S_i)$ whose closures define neighborhoods of $\sum_i a_i \xi_i$ in $\mathcal{X}(\cup_i S_i)$, viewed as the join of the boundaries of the factors $\mathcal{T}(S_i)$, we have to control the speed of progress in the curve graph of each of the surfaces S_i. To this end note that for every Teichmüller geodesic $\gamma : \mathbb{R} \to \mathcal{T}(S)$ the function $t \to \delta_{c_i}(\gamma(t))$ is coarsely non-decreasing and continuous. We use this to construct a new parameterization of a Teichmüller geodesic starting from X_i which encapsulates the progress in the curve graph based on the following elementary observation. Here the distance between two functions $f, g : \mathbb{R} \to \mathbb{R}$ is defined as $\sup\{|f(t) - g(t)| \mid t\}$.

Lemma 6.10. *Let* $f : [0, \infty) \to [0, \infty)$ *be a continuous coarsely non-decreasing function. Then* $F = \min\{g \mid g \geq f, g$ *non-decreasing*$\}$ *is non-decreasing, continuous and at distance at most* $q + 1$ *from* f.

Proof. Let $\mathcal{F} = \{g \mid g$ non-decreasing continuous, $g \geq f\}$. We claim that if $g_1, g_2 \in \mathcal{F}$ then the same holds true for $\min\{g_1, g_2\}$.

Note first that the minimum $\min\{g_1, g_2\}$ of two continuous functions is well known to be continuous, furthermore $\min\{g_1, g_1\} \geq f$ by assumption on g_1, g_2.

To see that $u = \min\{g_1, g_2\}$ is non-decreasing if this holds true for g_1, g_2, let $x < y$ and assume without loss of generality that $u(y) = g_1(y)$. Then we have $u(x) \leq g_1(x) \leq g_1(y) = u(y)$ which shows the claim.

Define $u(x) = \min\{g(x) \mid g \in \mathcal{F}\}$; then $u \geq f$, furthermore the argument in the previous paragraph shows that u is non-decreasing. To see that u also is continuous, note that since u is non-decreasing, it has at most countably many discontinuities. Assume to the contrary that x is a discontinuity of u. Since f is continuous, and $u \geq f$, we have
$$\lim_{s \nearrow x} u(s) \geq f(x)$$
and hence by the definition of u, we may assume that $u(x) = \lim_{s \nearrow x} u(s)$. But if there exists a number $\delta > 0$ such that $\lim_{s \searrow x} u(s) = u(x) + \delta$, then by continuity of f and the fact that f is non-decreasing, there exists a number $\epsilon > 0$ so that $f(s) \leq u(x) + \delta/2 \leq u(s) - \delta/2$ for all $s \in [x, x+\epsilon]$. But then there exists a function $v \in \mathcal{F}$ with $v(s) = u(x) + \delta/2 < u(s) - \delta/2$ for $s \in (-\infty, x] \cup \{x + \frac{\epsilon}{4} \leq s \leq x + \frac{\epsilon}{2}\}$ and coincides with u on $[x+\epsilon, \infty)$. Since $v \in \mathcal{F}$, we have $v \geq u$ which is a contradiction. We conclude that u is indeed continuous.

We are left with showing that for all x we have $u(x) - f(x) \leq q + 1$ where $q > 0$ is as in the definition of a coarsely increasing function. Namely, fix a point x and let $T = \sup\{t \mid f(t) \leq f(x) + 1\}$. If $T = \infty$ then the constant function $s \to g(s) = f(x) + p + 1$ is non-decreasing and $\geq f$ and hence $g \in \mathcal{F}$ and $g \geq u$, in particular $u(x) \leq f(x) + q + 1$.

Otherwise $T \in (x, \infty)$. Since $f(s) \leq f(x) + q + 1$ for all $s \in [x, T]$, we can connect the constant function $s \to f(x) + q + 1$ on $[-\infty, x]$ to the restriction of the function $s \to u(s) + c$ on $[T, \infty)$ for some $c \in [0, p]$ by a continuous non-decreasing function defined on $[x, T]$ such that the resulting function g is continuous and contained in $\mathcal{F}$. But then $\min\{g, u\} \in \mathcal{F}$ and hence $u(x) \leq g(x) = f(x) + q + 1$ as claimed. This completes the proof of the lemma. $\square$

Let $\gamma_i : [0, \infty) \to \mathcal{T}(S_i)$ be a Teichmüller geodesic starting at X_i. As the projection $\Upsilon(\gamma_i)$ of γ_i to $\mathcal{CG}(S_i)$ is a uniform unparameterized quasi-geodesic, the function $t \to \delta_{c_i}(\gamma_i(t))$ is coarsely non-decreasing. If the unparameterized quasi-geodesic $\Upsilon(\gamma_i[0, \infty))$ is of infinite diameter, then it is unbounded.

Let f_{γ_i} be the function constructed in Lemma 6.10 from the function $\delta_{c_i}|\gamma_i$. Since the function δ_{c_i} on $\mathcal{T}(S_i)$ is continuous, and Teichmüller geodesics depend continuously on their initial direction, this function depends continuously on the initial velocity of γ_i.

The function f_{γ_i} is non-decreasing, but it may be constant on arbitrarily large intervals. However, we can modify the function f_{γ_i} to a function f'_{γ_i} in such a way that f'_{γ_i} has the following properties.

(1) The function f'_{γ_i} is continuous and strictly increasing on $[0,T(\gamma_i)]$ where
$T(\gamma_i)=\inf\{t\mid \delta_{c_i}(\Upsilon(\gamma(0)),\Upsilon(\gamma(t)))=\sup\{\delta_{c_i}(\Upsilon(\gamma(0)),\Upsilon(\gamma(t)))\mid t\}-p.$
(2) f'_{γ_i} is constant for $t\geq T(\gamma_i)$.
(3) $\sup\{f_{\gamma_i}(t)-f'_{\gamma_i}(t)|t\}\leq 1$.

In particular, if f_{γ_i} is unbounded then $T(\gamma_i)=\infty$ and f'_{γ_i} is a homeomorphism. Let $\tau(\gamma_i)=f'_{\gamma_i}(T(\gamma_i))$. Note that $\tau(\gamma_i)=\infty$ if the support of the geodesic lamination on S_i which determines γ_i fills S_i.

Since f'_{γ_i} is continuous and strictly increasing on $[0,T(\gamma_i)]$, with image $[0,\tau(\gamma_i)]$, it can be inverted on this interval. We then can define a function g_{γ_i} by

$$g_{\gamma_i}(t)=\begin{cases}(f'_{\gamma_i})^{-1}(t) \text{ for } t\leq\tau(\gamma_i)\\ T(\gamma_i)+t-\tau(\gamma_i) \text{ for } t\geq \tau(\gamma_i).\end{cases} \tag{4}$$

For a point $\sum_i a_i\xi_i\in\mathcal{X}(\cup_i S_i)$ with the property that ξ_i fills S_i if $a_i>0$ we can now use this construction to define a neighborhood basis in $\overline{\mathcal{T}}(\cup_i S_i)$ that consists of sets whose intersection with $\prod Q_i$ is contractible following the route laid out in Section 6.1. To this end we define a subset $W(\xi,j,\delta)\subset\prod\mathcal{T}(S_i)$ as in Section 5.

For $1\leq i\leq k$ let $V_1^i\supset V_2^i\supset\cdots$ be an open descending chain of contractible neighborhoods of the set $P(\xi_i)$ of projective measured geodesic laminations supported in ξ_i in the sphere $\mathcal{PML}(S_i)$ of projective measured geodesic laminations on S_i.

Choose once and for all a marking μ_i for S_i which is short for a hyperbolic metric $X_i\in\mathcal{T}_\epsilon(S_i)$. Let $Z_i(j,\ell)$ be the open contractible subsets of $\overline{\mathcal{T}}(S_i)$ whose closures $\overline{Z_i(j,\ell)}$ in $\overline{\mathcal{T}}(S_i)$ are a neighborhood basis of ξ_i in $\overline{\mathcal{T}}(S_i)$ that were constructed in Subsection 6.1 from the sets V_j^i and the points X_i. Choose an essential simple closed curve $c_i\in S_i$ which is a pants curve for this marking. Each point $(\nu_1,\dots,\nu_k)\in V_j^1\times\cdots\times V_j^k$ determines a k-tuple $(\gamma_{\nu_1},\dots,\gamma_{\nu_k})$ of geodesics γ_{ν_i} in the Teichmüller space $\mathcal{T}(S_i)$. We refer to Section 6.1 for more details. In the formulation of the following proposition, we view $\mathcal{X}(\cup_i S_i)$ as the boundary of the product $\prod\mathcal{T}(S_i)$.

Proposition 6.11. *Assume that $\sum_{i=1}^\ell a_i\xi_i$ is such that $1\leq\ell\leq k$ and that $a_i>0$ for all $i\leq\ell$. For each j,δ there is a neighborhood $V(\delta,j)$ of ξ in $\overline{\mathcal{T}}(\cup_j S_j)$ with the following property.*

(1) *$V(\delta,j)\cap\prod\mathcal{T}(S_i)$ is contractible.*
(2) *The sets $V(\delta,j)$ define a neighborhood basis of $\sum_i a_i\xi_i$ in $\overline{\prod\mathcal{T}(S_i)}\cap\mathcal{X}(\cup_i S_i)$.*

Proof. For $j\leq\ell$ let V_j^i be as above. For each $\nu_j\in V_j^i$ choose a parameterization of γ_{ν_j} as given in formula (4) and note that this depends coarsely continuously on ν_j. Denote by $\hat\gamma_{\nu_1}$ this parameterization. Let also $\tau(\nu_j)=\tau(\gamma_{\nu_j})>0$ be the parameter defined above. Note that by the assumption on the laminations ξ_i, for all i and any $\nu_i\in P(\xi_i)$ we have $\tau(\nu_i)=\infty$.

For $(\nu_1,\dots,\nu_\ell)\in V_1^i\times\cdots\times V_\ell^i$ and $\delta>0$ define

$$F(\nu_1,\dots,\nu_\ell)=\{(\hat\gamma_{\nu_1}(t_1),\hat\gamma_{\nu_2}(t_2),\dots,\hat\gamma_{\nu_\ell}(t_\ell),z_{\ell+1},\dots,z_k)\in\prod\mathcal{T}(S_i)\mid$$
$$|t_i/a_i-t_1/a_1|\le\delta, d_{\mathcal{T}(S_i)}(z_i,X_i)<\delta t_1\}.$$

Claim: $\Phi(i,\delta)=\cup_{\nu_j\in V_j^i}F(\nu_1,\dots,\nu_\ell)$ is contractible.

Proof of the claim. Note first that if $(z_1,\dots,z_\ell,z_{\ell+1},\dots,z_k)\in\Phi(i,\delta)$ then the same holds true for $(z_1,\dots,z_\ell,z'_{\ell+1},\dots,z'_k)$ for any z'_i which is contained in the Teichmüller geodesic connecting X_i to z_i and all $i\ge\ell+1$. Thus retracting component wise the last $k-\ell$ components z_i to the basepoint X_i $(i\ge\ell+1)$ along the unique Teichmüller geodesic connecting X_i to z_i and keeping the remaining components fixed defines a retraction of $\Phi(i,\delta)$ to $\Phi(i,\delta)\cap\{(z_1,\dots,z_k)\mid z_i=X_i \text{ for } \ell+1\le i\le k\}$. In particular, in the remainder of the construction, it suffices to assume that $\ell=k$.

Next consider the set $F(\nu_1,\dots,\nu_k)$. It is defined using a parameterization of Teichmüller geodesics and a contractible domain $\{|t_i/a_i-t_1/a_1|\le\delta\}$ in the domain space of the parameterization. Since the Teichmüller exponential map is a homeomorphism, the set is contractible.

For $j\le k$ choose some $\nu_j\in P(\xi_j)$. We claim that the set $\Phi(i,\delta)$ admits a deformation retraction onto

$$\hat\Phi(i,\delta)=\Phi(i,\delta)\cap\mathcal{T}(S_i)\times\cdots\times\mathcal{T}(S_{k-1})\times\gamma_{\nu_k}[0,\infty).$$

To show that this indeed holds true let $R_k:[0,1]\times V_k^i\to V_k^i$ be a deformation retraction of V_k^i onto $\{\nu_k\}=R_k(1,V_k^i)$. Such a deformation retraction exists since the sets V_i^k are contractible neighborhoods of $P(\xi_k)$ and $P(\xi_k)$ is homeomorphic to a finite dimensional simplex and hence is contractible.

Let $(\mu_1,\dots,\mu_k)\in V_1^i\times\cdots\times V_k^i$ and let $(t_1,\dots,t_k)$ be such that

$$(\hat\gamma_{\mu_1}(t_1),\dots,\hat\gamma_{\mu_k}(t_k))\in F(\mu_1,\dots,\mu_k).$$

By construction, for each $s\in[0,1]$ the point $(\hat\gamma_{\mu_1}(t_1),\dots,\hat\gamma_{\mu_{k-1}}(t_{k-1}),\hat\gamma_{R(s,\mu_k)}(t_k))$ also is contained in $\Phi(j,\delta)$. Thus the assignment $s\to(\hat\gamma_{\nu_1}(t_1),\dots,\hat\gamma_{R(s,\mu_k)}(t_k))$ defines a path in $\Phi(j,\delta)$, and the union of these paths define a deformation retraction of $\Phi(j,\delta)$ onto its intersection with $\mathcal{T}(S_1)\times\cdots\times\mathcal{T}(S_{k-1})\times\gamma_{\nu_k}[0,\infty)$.

This construction can successively be repeated. Namely, in a second step, we construct in this way a retraction of $\Phi(i,j)\cap\mathcal{T}(S_1)\times\cdots\times\mathcal{T}(S_{k-1})\times\gamma_{\nu_k}[0,\infty)$ onto

$$\Phi(i,j)\cap\mathcal{T}(S_1)\times\cdots\times\mathcal{T}(S_{k-2})\times\hat\gamma_{\nu_{k-1}}[0,\infty)\times\hat\gamma_{\nu_k}[0,\infty).$$

In k steps we obtain a retraction of $\Phi(i,\delta)$ onto the set

$$F(\nu_1,\dots,\nu_k)=\Phi(i,\delta)\cap\gamma_{\nu_1}[0,\infty)\times\cdots\times\gamma_{\nu_k}[0,\infty)$$

Since $F(\nu_1,\dots,\nu_k)$ is contractible, this completes the proof of the claim. ∎

For $j>0$ define similarly a set $\Psi(i,\delta,j)\subset\Phi(i,\delta)$ by requiring that it only contains points $(\hat\gamma_1(t),\dots,\hat\gamma_k(t))$ with $t\ge j$, that is, whose projection to $\Upsilon(\mathcal{T}(S_i))$

in $\mathcal{CG}(S_i)$ to the basepoint c_i is roughly of distance at least j. The same argument as before shows that this set is contractible as well.

So far we constructed from a tuple of contractible neighborhoods V_i^j ($j = 1, \dots, k$) and numbers $j > 0, \delta > 0$ a contractible subset $\Psi(i, \delta, j)$ of $\prod \mathcal{T}(S_i)$.

We claim that if $X_\ell \subset \prod \mathcal{T}_\epsilon(\cup_i S_i)$ is a sequence converging to ξ; then $X_\ell \in \Xi(i, \delta, j)$ for large enough ℓ. This in turn follows if we can show that any point in $\overline{\mathcal{T}}(\cup_i S_i)$ sufficiently close to ξ is contained on a geodesic $(\gamma_{\nu_1}, \dots, \gamma_{\nu_k})$ with $\nu_i \in V_i^j$. However, this can be seen in exactly the same way as in the proof of Lemma 6.6.

Finally we have to show that the closures $\overline{\Psi(i, \delta, j)}$ of the sets $\Psi(i, \delta, j) \cap \prod \mathcal{T}_\epsilon(S_i)$ form a neighborhood basis of ξ in $\overline{\mathcal{T}}(\cup_i S_i)$. Since by Proposition 6.8 the space $\overline{\mathcal{T}}(\cup_i S_i)$ is a compact metrizable space, to this end it suffices to show that the intersection $\cap_{j,i,\delta} \overline{\Psi(j, i, \delta)} \cap \prod \mathcal{T}_\epsilon(S_i) = \{\xi\}$. As ξ clearly is contained in this intersection, it suffices to show that it is unique with this property.

Following the reasoning in the proof of Lemma 6.4, note that

$$\cap \overline{\Psi(i, \delta, j)} \cap \prod \mathcal{T}_\epsilon(S_i) = \emptyset.$$

Namely, since the map Υ is coarsely Lipschitz, for all j this set only contains points which project to tuples of points of Teichmüller distance at least aj to the basepoint $(X_1, \dots, X_k)$ where $a > 0$ is a universal constant. But this immediately implies that the closures of the intersections of the sets $\Psi(i, \delta, j)$ do not contain points in $\prod \mathcal{T}_\epsilon(S_i)$.

In the same way we see that ξ is the unique boundary point by letting i tend to infinity and letting $\delta \to 0$. □

6.3. **A neighborhood basis of an arbitrary point $\xi \in \mathcal{X}(S)$.** In this subsection we consider an arbitrary point $\xi = \sum_{i=1}^m a_i \xi_i \in \mathcal{X}(S)$ with $a_i > 0, \sum_i a_i = 1$. We assume that for some $k \leq m$ and all $i \leq k$ the components ξ_i are minimal and fill a subsurface $S_i \subset S$ of negative Euler characteristic, and for $i \geq k+1$ the component ξ_i is an oriented simple closed curve. Our goal is to use the results from Section 6.2 to construct for the point $\xi \in \mathcal{X}(S)$ a neighborhood basis in $\overline{\mathcal{T}}(S)$ consisting of sets whose intersections with $\Lambda(\mathcal{T}(S)) = Q$ are contractible where Q is constructed as in Section 6.1.

The union of the boundary curves of the surfaces S_i and the simple closed curves ξ_j for $k < j \leq m$ is a multi-curve α in S which decomposes S into $\ell \geq k$ subsurfaces. Up to reordering, we may assume that the first k of these subsurfaces are just the surfaces S_i. We denote the remaining subsurfaces by $S_{k+1}, \dots, S_\ell$. For each component c of the multi-curve α we add to the list of surfaces in the collection S_i an annulus with core curve c. Then each of the laminations ξ_i ($i \leq m$) fills precisely one of the surfaces S_i. We retain the notation that for $i \leq \ell$ the surface S_i is not an annulus.

Let S_i ($i \leq \ell$) be one of the above non-annular surfaces with boundary and let $\mathrm{Mod}(S_i)$ be the pure mapping class group of isotopy classes of diffeomorphisms of S_i fixing the boundary pointwise. The center of $\mathrm{Mod}(S_i)$ equals the free abelian subgroup generated by the Dehn twists about the boundary components of S_i. If $s > 0$ is the number of boundary components of S_i then the quotient $\mathrm{Mod}(S_i^*)$ of $\mathrm{Mod}(S_i)$ by its center fits into an exact sequence

$$1 \to \mathbb{Z}^s \to \mathrm{Mod}(S_i) \to \mathrm{Mod}(S_i^*) \to 1. \tag{5}$$

We can identify $\mathrm{Mod}(S_i^*)$ with the pure mapping class group of the surface S_i^* obtained by replacing each boundary component of S_i by a marked point (puncture).

The subgroup $\mathrm{Stab}(\cup_{i\leq \ell} S_i)$ of the mapping class group $\mathrm{Mod}(S)$ of S which fixes the multicurve α componentwise is a quotient of the direct product $\prod \mathrm{Mod}(S_i)$ by the free abelian group generated by the Dehn twists about the boundary components of the surfaces S_i, embedded diagonally into the mapping class groups $\mathrm{Mod}(S_i)$. The mapping class group $\prod \mathrm{Mod}(S_i^*)$ of the product $\prod \mathcal{T}(S_i^*)$ is the quotient of $\mathrm{Stab}(\cup_i S_i)$ by the free abelian group $\mathbb{Z}^p$ of Dehn twists about the boundary components of the surfaces S_i. Here p is the number of components of the multicurve α which decomposes S into the surfaces S_i, that is, we have $S \setminus \alpha = \cup_i S_i$.

Our strategy is to reduce the construction of neighborhoods of $\sum_i a_i \xi_i$ to the results of Section 6.2. To this end consider the *augmented Teichmüller space* $\mathcal{T}^{\mathrm{aug}}(S)$ of S [Wo03, Ya04]. This is a stratified space whose open stratum of maximal dimension equals the Teichmüller space $\mathcal{T}(S)$. For each multi-curve β on S there exists a stratum $\mathcal{S}(\beta)$ which equals the Teichmüller space of the surface $(S \setminus \beta)^*$ obtained from $S \setminus \beta$ by replacing each boundary component by a puncture. This Teichmüller space is a direct product of Teichmüller spaces, one for each component of $S \setminus \beta$, and where the Teichmüller space associated to each of these components is just the Teichmüller space of the surface obtained by replacing each boundary circle by a cusp. The strata in the boundary of $\mathcal{S}(\beta)$ correspond to multi-curves containing β as a subset.

Start with a countable family $\mathcal{V} = \{V_i \mid i\}$ of contractible subsets of $\prod \mathcal{T}(S_i^*)$ as in Section 6.2 whose retractions to $\prod \mathcal{T}_\epsilon(S_i^*)$ determine a neighborhood basis of ξ in the compactification of $\prod \mathcal{T}_\epsilon(S_i^*)$ with boundary $\mathcal{X}(\cup_i S_i)$ the join of the spaces $\mathcal{X}(S_i)$. Here the annular surfaces are contained in the list of factors of the product. Choose a boundary curve c of one of the surfaces S_i $(i \leq \ell)$, that is, a curve c in S which is contracted to a puncture in $\prod \mathcal{T}(S_i^*)$. Assume that this curve is the core curve of the annulus $S_{\ell+1}$. The curve c determines a stratum $\mathcal{S}'$ in $\mathcal{T}^{\mathrm{aug}}(S)$ whose closure contains $\mathcal{S}$, and the compactification $\mathcal{X}(\mathcal{S}')$ of this stratum is a compact subspace of $\mathcal{X}(S)$ containing $\mathcal{X}(\cup_i S_i)$ and hence ξ (in a formal sense as the simple closed curve components ξ_i different from c are only formally contained in $\mathcal{X}(\mathcal{S}')$). We use the sets from the family $\mathcal{V}$ to construct a countable collection of contractible sets in $\mathcal{S}'$ whose retractions to the thick part of $\mathcal{S}'$ determine a neighborhood basis of ξ in $\mathcal{X}(\mathcal{S}')$. In finitely many such steps we successively replace the punctures of the surfaces S_i^* by simple closed curves and recover the surface S, together with a neighborhood basis of ξ in $\overline{\mathcal{T}}(S)$ consisting sets whose intersections with $\mathcal{T}(S)$ are contractible.

Let as before $\epsilon > 0$ be a sufficiently small number. The set $N(c)$ of points $Y \in \mathcal{S}'$ so that the length of c is at most ϵ is a tubular neighborhood of $\mathcal{S}$ in the augmentation of the Teichmüller space $\mathcal{S}'$, which is a closed subset of $\mathcal{T}^{\mathrm{aug}}(S)$. Its boundary $\partial N(c)$ is invariant under the action of the infinite cyclic group of Dehn twists about c. The quotient of $\partial N(c)$ under this action is a circle bundle over the stratum $\mathcal{S}$.

Recall that the choice of a base marking for S determines for each surface $X \in \partial N(c)$ and for the simple closed curve c a *twist parameter* $\tau(X, c) \in \mathbb{Z}$, uniquely up to an additive error of ± 1. If T_c denotes the left Dehn twist about c, then for all k and up to an additive error in $[-2, 2]$, we have $\tau(T_c^k X, c) = \tau(X, c) + k$.

Using for example a shortest distance projection $\Pi : \partial N(c) \to \mathcal{S}$ for the metric completion of the *Weil-Petersson metric*, which is a CAT(0)-metric on $\mathcal{T}^{\rm aug}(S)$ with the property that each stratum is convex, we observe that the hypersurface $\partial N(c) \to \mathcal{S}$ can be viewed as a fiber bundle over the contractible space $\mathcal{S}$, with fiber $\mathbb{R}$. The fiber is invariant under the action of the infinite cyclic group of Dehn twists about c.

The function which associates to $Y \in \partial N(c)$ the twist parameter $\tau(Y, c)$ can be modified to a continuous function of uniformly bounded distance and so that the modified function, denoted again by $\tau(Y, c)$, satisfies $\tau(T_c(Y), c) = \tau(x, c) + 1$. Then the restriction of the function $Y \to \tau(Y, c)$ to a fiber of the bundle $\partial N(c) \to \mathcal{S}$ is proper and hence using local sections of $\partial N(c) \to \mathcal{S}$ so that the value of $\tau(\cdot, c)$ on the image is uniformly bounded in norm and gluing of the local sections with a partition of unity, we obtain the following.

Lemma 6.12. *There exists a continuous map $\sigma : \mathcal{S} \to \partial N(c)$ with the following properties.*

1. $\Pi(\sigma(x)) = x$ *for all* x.
2. *There exists a constant $b > 0$ so that $\tau(\sigma(x), c) \in [-b, b]$ for all $x \in \mathcal{S}$.*

By invariance of the fibers of $\Pi : \partial N(c) \to \mathcal{S}$ is invariant under the action of the group $\mathbb{Z}$ of Dehn twists about c, using the section σ and another partition of unity we can construct a continuous fiber preserving flow Ψ^t on $\partial N(c)$ so that $\Psi^1(x) = T_c(x)$ for all x. Using this flow and the section σ to define a basepoint, this yields an identification of $\partial N_\chi(\mathcal{S})$ with the product space $\mathbb{R} \times \mathcal{S}$ in such a way that the group generated by T_c acts as a group of integral translations on the factor $\mathbb{R}$.

A section σ as in Lemma 6.12 is an embedding of $\mathcal{S}$ into $\partial N(c)$, This embedding and the transverse flow Ψ^T determine a homeomorphism $\mathcal{S} \times \mathbb{R} \to \partial N(c)$ which maps (x, t) to $\Psi^t(x)$. This map is equivariant with respect to the action of $\mathbb{Z}$ on $\mathbb{R}$ by translation and the action of the infinite cyclic group of Dehn twists about c on $\partial N(c)$. As $\mathcal{S} \times \mathbb{R}$ is naturally identified with the product of the Teichmüller space $\mathcal{S}$ and the Teichmüller space of the annulus with core curve c, the same holds true for $\partial N(c)$. Using this identification, we obtain from the construction in Section 6.2 a countable collection $\mathcal{V} = \{V_i \mid i\}$ of ξ of contractible subsets of $\partial N(c)$.

Let S_c be the component of $(S \setminus \alpha) \cup c$ which contains c and write as before S_c^* to denote the surface obtained from S_c by replacing each boundary component by a puncture. The stratum $\mathcal{S}'$ is a product of Teichmüller spaces containing the Teichmüller space $\mathcal{T}(S_c)$ of S_c as a factor. Thus if we denote by $[\mu]$ the projective measured geodesic lamination on the surface S_c whose support equals the simple closed curve c, then by the Hubbard Masur theorem, for each $Y \in \partial N(c)$ the projective measured geodesic lamination $[\mu]$ determines a Teichmüller geodesic γ_Y through Y. These Teichmüller geodesics foliate the stratum $\mathcal{S}'$, and they project to points by a factor projection onto a factor of $\mathcal{S}'$ different from $\mathcal{T}(S_c)$. By choosing $\epsilon > 0$ sufficiently small we may assume that these Teichmüller geodesics intersect $\partial N(c)$ transversely, and two Teichmüller geodesic passing through different points of $\partial N(c)$ do not intersect. We parameterize these geodesics γ_Y in such a way that $\gamma_Y(0) = Y$ and $\gamma_Y(-\infty, 0) \subset N(c)$. Then the geodesic lines $\gamma_Y(-\infty, \infty)$ foliate the stratum $\mathcal{S}'$).

Proposition 6.13. *Put $U_i = \{\gamma_Y(-\infty,\infty) \mid Y \in V_i \subset \partial N(c)\}$; then the sets U_i are contractible, and the closures of their intersections with the ϵ-thick part $\mathcal{S}'_\epsilon$ of the stratum $\mathcal{S}'$ define a neighborhood basis of ξ in the compactification of $\mathcal{S}'_\epsilon$.*

Proof. Since the geodesics with horizontal projective measured lamination $[\mu]$ foliate $\mathcal{S}'$ and $\partial N(c)$ is transverse to these geodesics, the set U_i admits a deformation retraction onto V_i. Thus since the sets V_i are contractible, the same holds true for the sets U_i.

Denote by S' the surface obtained from $(S \setminus \alpha) \cup c$ by replacing each boundary component by a puncture and adding for each boundary component a factor (R). Write $\overline{\mathcal{T}(S')}$ to denote the compactification of the Teichmüller space of S' by the joint $/calX(\cup_i S_i)$. We have to show that the closures of the sets U_i in $\overline{\mathcal{T}}(S')$ define a neighborhood basis of ξ in $\overline{\mathcal{T}}(S')$. To this end choose as basepoint the image of the basepoint for $\prod \mathcal{T}(S_i^*)$ in $\partial N(c)$. We have to show that for any sequence $X_u \subset \mathcal{S}'$, all but finitely many X_u are contained in U_i.

Let $\Omega \subset \partial N(c)$ be a fundamental domain for the action of the stabilizer $\mathrm{Stab}(c)$ of c in the mapping class group of $S' = (S \setminus \alpha) \cup c$ which contains the basepoint X. We may assume that the closure $\overline{\Omega}$ of Ω in $\mathcal{T}^{\rm aug}(S)$ is compact. Recall that the length function for the hyperbolic metric X determines a section of the bundle $\mathcal{ML} \to \mathcal{PML}$. Let $A \subset \mathcal{ML}$ be the closure of the image of this section of the set of vertical projective laminations for the geodesics $\gamma_Y, Y \in \Omega$, and denote by ν the unique measured geodesic lamination of X-length one with support c. We claim that there exists a number $\rho > 0$ so that $\iota(\mu,\nu) \geq \rho$ for each $\mu \in A$, where as before, ι is the intersection form.

To this end note that for each $Y \in \Omega$, the quadratic differential defining γ_Y in $\mathcal{T}(S_c)$ is a one cylinder area one Strebel differential, and the extremal length of the core curve of the cylinder is contained in an interval $[a, a^{-1}]$ for a number $a > 0$ not depending on Y. The singular flat metric on S_c defined by such a Strebel differential is obtained by gluing an area one flat cylinder of uniformly bounded height and circumference along the boundary to the surface S_c by identifying subarcs of the boundary isometrically in pairs. Degeneration of simple closed curves disjoint from c to nodes corresponds to degeneration of boundary arcs to points, resulting in the convergence of the Strebel differential to a Strebel differential on a component of a stratum in the closure of $\mathcal{S}'$ in $\mathcal{T}^{\rm aug}(S)$ which is obtained by shrinking some curves disjoint from c to nodes. In other words, allowing to pass to boundary strata of

$\mathcal{S}'$, the Strebel differentials defined by the points in Ω define a precompact family of differentials, and the intersection at the basepoint X of the defining measured laminations is uniformly bounded from above and below.

As a consequence, there exists a number $R > 0$ with the following property. For $Y \in \Omega$, the diameter of the subsurface projection of $\Upsilon(\gamma_Y(t)), \Upsilon(\gamma_Y(0))$ into a component S_i of S' is bounded from above by R. Note that this subsurface projection only depends on the component of Y in S_c. Thus the closure in $\mathcal{X}(\mathcal{S}')$ of the set $\{\gamma_Y(t) \mid Y \in \Omega, t \in \mathbb{R}\}$ does not contain ξ. Furthermore, by equivariance of the geodesics γ_Y under the action of $\mathrm{Stab}(c) \subset \mathrm{Mod}(S')$ and invariance of the components S_i under this action, the diameter of the subsurface projection of any of the geodesics γ_Y into S_i is bounded from above by R.

Now the group $\mathrm{Stab}(c)$ acts properly on $\mathcal{X}(\mathcal{S}') \setminus \mathcal{X}(\mathcal{S})$, and it acts properly on $\partial N(c)$. As taking the accumulation points in $\mathcal{X}(\mathcal{S}')$ of the geodesics γ_Y is equivariant with respect to this action, it follows from the previous paragraph that the starting points $x_u \in \partial N(c)$ of the geodesics γ_{x_u} which contain u converge in $\partial N(c) \sim \prod \mathcal{T}(S_i)^*$ to ξ. This implies that indeed, for large enough u we have $x_u \in V_i$ and hence U_i is a neighborhood of ξ in $\mathcal{T}(\mathcal{S}')$.

Similarly, the same discussion also shows that $\cap_i \overline{U_i} = \{\xi\}$. Together this shows the proposition. □

Using this construction inductively, and using a torsion free finite index subgroup and a diffeomorphism $\Lambda : \mathcal{T}(S) \to \Lambda(\mathcal{T}(S)) \subset \mathcal{T}_\epsilon(S)$ completes the proof of the existence of a neighborhood basis for an arbitrary point whose intersection with $\sqcup(S)$ is contractible.

Example 6.14. In the case that S is a once puncture torus, whose Teichmüller space $\mathcal{T}(S)$ is the hyperbolic plane $\mathbb{H}^2$, a rational point $x \in \partial\mathbb{H}^2$ corresponds to an essential simple closed curve c on S. Remove from $\mathbb{H}^2$ a horosphere H which is small enough that its images under the group $\mathrm{PSL}(2,\mathbb{Z})$ are pairwise disjoint. The boundary ∂H of this horosphere is invariant under the stabilizer of x. As $H \subset \mathbb{H}^2$ is convex, there is a shortest distance projection $P : \mathbb{H}^2 \to \mathbb{H}^2 \setminus H$. Given a parameterization $\alpha : \mathbb{R} \to \partial H$, a neighborhood basis of the point in $\mathcal{X}(S)$ defined by the curve c equipped with an orientation, denoted by c^+, is then given by the sets $P^{-1}(\alpha(u,\infty)) \setminus \mathrm{PSL}(2,\mathbb{Z})(H)$ where $u \in \mathbb{R}$ (up to perhaps changing the orientation of c). This amounts precisely to a neighborhood basis of the point defined by c^+ in the Cantor set which is the Gromov boundary of the hyperbolic group $\mathrm{PSL}(2,\mathbb{Z})$.

7. A Z-set for $\mathrm{Mod}(S)$

In this section we complete the proof of Theorem 4. We begin with establishing one more property of the geometric boundary $\mathcal{X}(S)$ of the surface S of finite type. Recall that the *covering dimension* of a topological space X is the minimum of the numbers $n \geq 0$ so that the following holds true. Any open cover $\mathcal{U}$ of X has a refinement $\mathcal{V}$ so that a point in X is contained in at most $n+1$ of the sets $V \in \mathcal{V}$. With this terminology, the covering dimension of $\mathbb{R}^n$ is n, and hence the covering dimension of any subset of $\mathbb{R}^n$ equipped with the subspace topology is at most n. In particular, the covering dimension of $\mathcal{T}(S)$ equals $6g-6+2m$.

The following result relies on work of Gabai [Ga14], see also [BB19].

Proposition 7.1. *The covering dimension of $\mathcal{X}(S)$ is finite*

Proof. We proceed by induction on the complexity of the surface S. If S is an annulus, then its geometric boundary consists of two points and there is nothing to show.

Consider next a four-holed sphere or a one-holed torus S. Then the boundary $\mathcal{X}(S)$ of S is a Cantor set, which has covering dimension zero (corresponding to the fact that the mapping class group of S is virtually free).

Let X and Y be compact spaces with covering dimensions m, n. We claim that the covering dimension of the join $X * Y$ is at most $m+n+1$. To see that this is the case recall that $X * Y$ is the quotient of $X \times Y \times [0,1]$ under the equivalence relation which is only nontrivial on $X \times Y \times \{0\}$ and $X \times Y \times \{1\}$ and projects $X \times Y \times \{0\}$ to $X \times \{0\}$ and projects $Y \times Y \times \{1\}$ to $Y \times \{1\}$. Thus we have $X * Y = X \cup Y \cup C$ where $X \subset X * Y$ is the closed subset which is identified with the quotient of $X \times Y \times \{0\}$, $Y \subset X * Y$ is the closed subset which is identified with the quotient of $X \times Y \times \{1\}$, and the set C is homeomorphic to $X \times Y \times (0,1)$.

By Alexandrov's definition of dimension (see Theorem 3.4 of [Dr18]), we have $\dim(A \times B) \leq \dim(A) \cup \dim(B)$ and hence $\dim(C) \leq \dim(X) + \dim(Y) + 1$. Now the compact space $X * Y$ is the union of the closed subset $X \cup Y$ with C and hence the theorem of Menger and Uryson (see Theorem 3.1 of [Dr18]) shows that $\dim(X * Y) = \dim(C) \leq \dim(X) + \dim(Y) + 1$ as claimed.

Assume now that the proposition was established for all surfaces of complexity at most $k-1$. Let S be a surface of complexity k. We have $\mathcal{X}(S) = \partial\mathcal{CG}(S) \cup \mathcal{Y}$ (disjoint union) where $\mathcal{Y} = \cup \mathcal{X}(S_1) * \cdots * \mathcal{X}(S_p)$ and the union in the definition of $\mathcal{Y}$ is over all disjoint collections of subsurfaces $S_1, \ldots, S_p$ of S. The union $\mathcal{Y}$ is not disjoint. Note that the number of disjoint surfaces in this join is uniformly bounded in terms of k.

By induction hypothesis and the discussion of dimension for joins, there exists a number $n > 0$ which bounds from above the covering dimension of each of the sets $\mathcal{X}(S_1) * \cdots * \mathcal{X}(S_p)$. Following Example 3.3, $\mathcal{X}(S_1) * \cdots * \mathcal{X}(S_p)$ is a closed subset of $\mathcal{X}(S)$ and hence it is compact. As a consequence, the subspace $\mathcal{Y}$ of $\mathcal{X}(S)$ is a σ-compact Hausdorff space as it is a countable union of compact spaces. If $K \subset \mathcal{Y}$ is compact, then K is a compact space which can be written as a countable union of the compact spaces $K \cap \mathcal{X}(S_1) * \cdots * \mathcal{X}(S_p)$. Then the countable union theorem Theorem 3.2 of [Dr18] shows that $\dim(K) = \sup\{\dim(K \cap \mathcal{X}(S_1) * \cdots * \mathcal{X}(S_p)\}$ where the supremum is over all disjoint unions of subsurfaces of S. By the induction hypothesis, this dimension is at most n.

To summarize, the space $\mathcal{Y}$ is a σ-compact Hausdorff space with the property that the dimension of each compact subset K of $\mathcal{Y}$ is bounded from above by a fixed number $n \geq 1$. Then the dimension of $\mathcal{Y}$ is at most n (see p.316 of [Mu14] for a sketch of a proof). Following [Ga14], the covering dimension of $\partial\mathcal{CG}(S)$ is at most $4g-5+2s$. Then by the Uryson-Menger formula (see Theorem 3.3 of [Dr18]), the dimension of the compactum $\mathcal{X}(S)$ is at most

$$\dim(\mathcal{X}(S)) = \dim(\partial\mathcal{CG}(S)) + \dim(\mathcal{Y}) + 1$$

and hence it is finite. □

As a consequence, we obtain

Corollary 7.2. *The pair $(\overline{\mathcal{T}}(S), \mathcal{X}(S))$ is a pair of spaces of finite dimension.*

Proof. By Proposition 7.1, the dimension of $\mathcal{X}(S)$ is finite. As the compactum $\overline{\mathcal{T}}(S) = \mathcal{T}_\epsilon(S) \cup \mathcal{X}(S)$ is a union of two subspaces of finite dimension, with $\mathcal{X}(S) \subset \overline{\mathcal{T}}(S)$ closed, we have

$$\dim(\overline{\mathcal{T}}(S)) = \max\{\dim(\mathcal{T}_\epsilon(S), \dim(\mathcal{X}(S))\} < \infty.$$

□

Let now Γ be a torsion free finite index subgroup of $\mathrm{Mod}(S)$. It acts freely on $\mathcal{T}_\epsilon(S)$, with compact quotient (or on the Γ-invariant subspace Q of $\mathcal{T}(S)$ as introduced in Proposition 4.3.

Theorem 7.3. *The pair $(\overline{\mathcal{T}}(S), \mathcal{X}(S))$ defines a $\mathcal{Z}$-structure for Γ.*

Proof. We have to verify the axioms for a $\mathcal{Z}$-structure.

We showed so far that the Γ-invariant subspace $\mathcal{Q}$ is contractible, locally contractible and of dimension $6g-6$. The group Γ admits a covering space action on Q, We also know that a point $z \in \mathcal{X}(S)$ has a neighborhood basis in $\overline{\mathcal{T}}(S)$ consisting of sets whose intersections with Q are contractible. Thus we are left with showing that the action of Γ is $\mathcal{U}$-small for every open covering $\mathcal{U}$ of $\overline{\mathcal{T}}(S)$.

To this end let $\mathcal{U}$ be an open covering of $\overline{\mathcal{T}}(S)$. By compactness, we may extract a finite subcovering, so we may assume that $\mathcal{U}$ is in fact finite, that is, we have $\mathcal{U} = \cup_{0\leq i\leq m} U_i$ for some open sets $U_i \subset \overline{\mathcal{T}}(S)$. Assume without loss of generality that $U_i \cap \mathcal{X}(S) \neq \emptyset$ for all $i \geq 1$.

We argue now by contradiction and we assume that there exists a compact set $K \subset \overline{\mathcal{T}}_\epsilon(S)$ and infinitely many elements $\varphi_i \in \Gamma$ such that $g_i K \not\subset U_j$ for all $j \leq m$. Since the action of Γ on Q is proper and cocompact, we may assume that $K = \cup_{j=1}^{\ell} \psi_j K_0$ where K_0 is a compact fundamental domain for the action of Γ.

Let $X \in K_0$. Since the action of Γ on $\mathcal{T}_\epsilon(S)$ is proper and $\overline{\mathcal{T}}(S)$ is compact, we conclude that up to passing to a subsequence, the sequence $\psi_i X$ converges in $\overline{\mathcal{T}}(S)$ to a point $\xi \in \mathcal{X}(S)$. Since the right action of Γ on itself extends to the trivial action on $\mathcal{X}(S)$, we then have $\psi_i(\varphi_j X) \to \xi$ for all $j \leq \ell$. In particular, for sufficiently large i, we have $\psi_i(\varphi_j X) \in U_p$ for some fixed $p > 0$. But then it follows from the definition of the topology on $\overline{\mathcal{T}}(S)$ that in fact $\psi_i K \to \xi$ and hence $g_i K \subset U_p$ for all sufficiently large p. This is a contradiction which completes the proof of the theorem. □

As an application, we obtain.

Corollary 7.4. $\dim(\partial\mathcal{CG}(S)) \le 4g-6$.

Proof. Since $\mathcal{X}(S)$ is a $\mathcal{Z}$-set for Γ, the cohomological dimension of $\mathcal{X}(S)$ equals $\mathrm{vd}(\mathrm{Mod})(S)-1$ [B96]. Furthermore, this dimension also equals the covering dimension of $\mathcal{X}$.

Now as $\partial\mathcal{CG}(S)$ is embedded in $\mathcal{X}(S)$, it is equipped with the subspace topology. This means that any open covering of $\partial\mathcal{CG}(S)$ is the restriction of an open covering of $\mathcal{X}(S)$. Such a covering then has a $4g-6$-finite refinement and hence the same holds true for the refinement of the original cover of $\partial\mathcal{CG}(S)$. □

The following conjecture is taken from [BB19]. We believe that the results in this work support this conjecture.

Conjecture. $\mathrm{asdim}(\mathrm{Mod}(S)) = \mathrm{vd}(\mathrm{Mod}(S)) = 4g-5$.

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