Hyperbolic geometry

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1. IV.

Definition 1.0.1. Let (X,d) be a metric space. An (L,C)-*quasi-geodesic* in X is a map $\alpha : J \subset \mathbb{R} \to X$ such that for all s,t it holds

$$|s-t|/L - C \leq d(\alpha(s),\alpha(t)) \leq L|s-t| + C.$$

Proposition 1.0.2. *For all $r>0$ and all $L>1$ there exists a number $\epsilon = \epsilon(r,L) > 0$ with the following property. If $\alpha : J \to \mathbb{H}^3$ is a piecewise geodesic parametrized by arc length which consists of geodesic segments of length $\geq r$ meeting with a breaking angle smaller than ϵ, then α is an (L,C)-quasigeodesic for some unversal constant $C>0$.*

The proof of this proposition is done in three steps.

Lemma 1.0.3. *There exists a number $c>0$ with the following property. Let T be a triangle in $\mathbb{H}^2$ which contains a side a of length at least R and a second side b which meets a with an angle γ of at least $\pi/2$ at a vertex C. Then the angle at the second endpoint of a is at most ce^{-R}.*

Proof. We consider first the case that the angle γ equals $\pi/2$. By moving T with an isometry in the upper half-space we may assume that $C=i$ and that the side a is a subsegment of the unit circle $\Re \geq 0, \Re^2 + \Im^2 = 1$. We furthermore assume that T is contained in $\{z \mid |z|^2 \geq 1\}$. Let $B = b_1 + ib_2$ be the second endpoint of a; we have $b_1 > 0$.

By convexity, T is entirely contained in the vertical strip $\{z \mid |z|^2 \geq 1, 0 \leq \Re \leq b_1\}$. In particular, the angle of T at B is not bigger than the euclidean angle β at B between the unit circle and the vertical line. The hyperbolic length of the geodesic segment a equals

$$\ell(a) = \int_0^{\pi/2-\beta} \frac{1}{\cos t} dt.$$

But for $s = \pi/2 - \beta$ close to zero we have $\frac{1}{\cos s} \sim \frac{1}{s}$ and therefore $\ell(a) \sim \log \beta$ which yields the estimate in the case that the angle γ equals $\pi/2$.

If T is a triangle in $\mathbb{H}^2$ as in the lemma with a side a and such that the angle γ at the vertex C is strictly bigger than $\pi/2$ the there is a geodesic segment ζ issuing from C which makes with a an angle of $\pi/2$ and which is entirely contained in the interior of T. Let A be the first intersection of ζ with the side of T opposite to the vertex C: Then the triangle T' with vertices C, A and the second endpoint of the arc a satisfies the assumption in the first part of this proof. The lemma follows. $\square$

Lemma 1.0.4. *There exists a number $C > 0$ with the following properties. Let $\alpha : [0, u] \to \mathbb{H}^3$ be a piecewise geodesic in $\mathbb{H}^3$ which consists of two geodesic segments of length R_1, R_2 meeting with a breaking angle of at most $\pi/2$. Then*

$$d(\alpha(0), \alpha(u)) \geq \ell(\alpha) - C.$$

Proof. Since α is contained in a totally geodesic hyperbolic plane it suffices to show the estimate for a piecewise geodesic in $\mathbb{H}^2$. Thus let $\alpha_1 : [0, u_1] \to \mathbb{H}^2$ be the first subsegment of α, and let $\alpha_2 : [u_1, u] \to \mathbb{H}^2$ be the second. Consider the variation of geodesics $h(s, t)$ connecting $\alpha(0)$ to $\alpha_2(s)$. Write $h_s(t) = h(s, t)$. By the first variation formula for geodesics, we have

$$\frac{d}{ds}\ell(h_s) = \frac{1}{\|h_s'\|}\langle h_s', \alpha_2' \rangle$$

which is just the cosine of the angle at $\alpha_2(s)$ between h_s and α_2.

Now by Lemma 1.0.3, this inner product tends to one exponentially fast whence the lemma. $\square$

Proof. (of proposition, sketch) Choose $R >> C$ where $C > 0$ is as in Lemma 1.0.4. For this R, show by induction the following. If α is a piecewise geodesic consisting of 2^k segments of length at least R which meet with a breaking angle of at most $\pi/4$. Assume that the breakpoints of α are the points $\alpha(t_i)$. For $\ell < k$ let ζ_ℓ be the piecewise geodesic whose segments connect the points $\alpha(t_{k2^\ell}), \alpha(t_{(k+1)2^\ell})$. Show by induction on ℓ that

- for each $\ell < k$, the breaking angles of the piecewise geodesic ζ_ℓ are not bigger than $\pi/2$
- the lengths of the geodesic segments opf ζ_s are not smaller than $2^\ell(R - C)$.

$\square$

Definition 1.0.5. A continuous map $f : X \to Y$ is called *proper* if for every compact set $K \subset Y$, the preimage $f^{-1}(K) \subset X$ is compact.

Proposition 1.0.6. *Let $\Gamma_\theta \subset PSL(2, \mathbb{C})$ be the group obtained from the fundamental group $\Gamma \subset PSL(2, \mathbb{R})$ by bending with angle θ along a separating simple closed geodesic. There exists a Γ-Γ_θ-equivariant map $F : \mathbf{H}^2 \to H \subset \mathbb{H}^3$ whose image H is a pleated surface. For sufficiently small θ this map is proper.*

Proof. Let γ be the simple closed geodesic along which the bending takes place. The preimage of γ in $\mathbb{H}^2$ is a Γ-invariant collection of disjoint geodesic lines. Since γ has a collar neighborhood of positive radius in $S = \mathbb{H}^2/\Gamma$, the distance between any two distinct of these geodesics is bounded from below by a positive number $r > 0$.

Thus for sufficiently small θ, the image under the equivariant path isometry F of any geodesic in $\mathbb{H}^2$ is an (L, C)-quasi-geodesic in $\mathbb{H}^3$ for numbers (L, C) not depending on the geodesic.

As a consequence, the preimage under F of a ball of radius $R > 0$ about a point in H is contained in a ball of radius $LR + C$ in $\mathbb{H}^2$. But this just means that F is proper. $\square$

Recall that the action of a group Λ on a locally compact metric space X is *proper* if for all compact sets K we have $gK \cap K \neq \emptyset$ for only finitely many $g \in \Lambda$.

Corollary 1.0.7. *The action of Γ_θ on $\mathbb{H}^3$ is proper.*