HITCHIN GRAFTING REPRESENTATIONS II: CURRENTS AND INTERSECTION

PIERRE-LOUIS BLAYAC, URSULA HAMENSTÄDT, THÉO MARTY, ANDREA EGIDIO MONTI

Abstract. Given any suitable invariant Finsler norm on $\mathrm{SL}_d(\mathbb{R})$, we study the currents defined by Hitchin representations and their limits in the space of projective currents. We find that there are limit points which are projectivizations of currents of maximal entropy for hyperbolic metrics on surfaces with boundary. The entropy of the non-projectivized currents can limit to any prescribed number in the interval $(0, 1)$ while their self-intersection number tends to infinity.

Contents

Introduction

The *Teichmüller space* $\mathcal{T}(S)$ of a closed oriented surface S of genus $g \geq 2$ is the space of *marked* hyperbolic structures on S. Equivalently, it can be described as a distinguished component of the *character variety* of conjugacy classes of homomorphisms $\pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$, with target the group $\mathrm{PSL}_2(\mathbb{R})$ of orientation preserving isometries of the hyperbolic plane. It was discovered by Hitchin that an analog of the Teichmüller space also exists for the character variety of representations of $\pi_1(S)$ into simple split real Lie groups of higher rank.

The *Hitchin component* $\mathrm{Hit}(S)$ for the target group $\mathrm{PSL}_d(\mathbb{R})$ ($d \geq 3$) is the component of the character variety containing the so-called *Fuchsian locus*, consisting of conjugacy classes of discrete faithful representations which factor through an irreducible embedding $\mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_d(\mathbb{R})$. Hitchin [15]

Date: April 7, 2026.

showed that the Hitchin component is homeomorphic to $\mathbb{R}^{(d^2-1)(2g-2)}$, and later Labourie [20] and Fock–Goncharov [9] independently proved that all representations in the Hitchin component are faithful with discrete image.

Choose a hyperbolic metric on S and consider the geodesic flow Φ^t on the unit tangent bundle T^1S for this hyperbolic metric. A *geodesic current* on S can be thought of as a finite Borel measure on T^1S which is invariant under the geodesic flow and the flip $v \to -v$. The space $\mathrm{Curr}(S)$ of geodesic currents is naturally equipped with the weak* topology.

For a suitable choice of a $\mathrm{PSL}_d(\mathbb{R})$-invariant Finsler metric on the symmetric space $\mathbb{X} = \mathrm{PSL}_d(\mathbb{R})/\mathrm{PO}(d)$ of $\mathrm{PSL}_d(\mathbb{R})$, called *nice* in the sequel, one can construct for every Hitchin representation a *length cocycle* on T^1S which is the integration cocycle of a Hölder continuous positive length function f_ρ. Thus a positive normalized multiple of the length cocycle determines an equilibrium state for the geodesic flow on T^1S and hence a geodesic current whose projective class $\Theta(\rho)$ does not depend on the choice of the hyperbolic metric. The thus defined map $\rho \to \Theta(\rho)$ is continuous. Its restriction to the Fuchsian locus coincides with the standard embedding of Teichmüller space [4] into the space of *projective currents* which associates to a hyperbolic metric its projective Liouville current.

Since S is compact, the space of projective geodesic currents on S, equipped with the quotient of the weak*-topology, is a compact space and hence the closure of the image of $\Theta(\mathrm{Hit}(S))$ defines a compactification of $\mathrm{Hit}(S)$ which is invariant under the natural action of the mapping class group $\mathrm{Mod}(S)$ and extends the compactification of Teichmüller space by adding the *Thurston boundary* of projective measured laminations. We obtain concrete information on part of this boundary, which shows that it is surprisingly rich and complicated.

A nice Finsler metric also gives rise to a meaningful numerical invariant on Hitchin representations which associates to a representation $\rho \in \mathrm{Hit}(S)$ its *critical exponent* $h(\rho)$ with respect to the Finsler metric. Corollary 1.4 of [32] shows that this invariant is maximized only on the Fuchsian locus. That the infimum of the critical exponent over $\mathrm{Hit}(S)$ vanishes was established by Zhang [36] and was reworked in [35], using mainly algebraic methods. In the sequel we use the notion *entropy* for the critical exponent as this reflects better our point of view.

By [28], for some particular choices of nice Finsler metrics one can also associate to a Hitchin representation ρ an *intersection current* $\lambda(\rho)$ as follows. Let $\iota : \mathrm{Curr}(S) \times \mathrm{Curr}(S) \to [0, \infty)$ be the *intersection form* on currents, which is a continuous symmetric convex linear functional. The intersection current $\lambda(\rho)$ is characterized by the property that for any free homotopy class c on S we have $\iota(c, \lambda(\rho)) = \ell_\rho(c)$ where $\ell_\rho(c)$ is the translation length of an element of $\rho(\Gamma)$ in the free homotopy class of c and the length is taken with respect to the fixed nice Finsler metric. While for any length cocycle on T^1S defined by a positive flip invariant Hölder function there is a finitely additive signed measure with this property [**H99**], this measure may not be positive. However, for Hitchin representations, one obtains indeed in this way a current [28] which is defined without ambiguity, that is, including the normalization. The self-intersection number $\iota(\lambda(\rho),\lambda(\rho))$ extends the function which associates to a nonpositively curved metric on S its area up to the factor 2π and hence it can be thought of as the volume of the representation.

If ρ is a Fuchsian representation, then the projective class of $\lambda(\rho)$ is just the Liouville current of the hyperbolic metric whose projective class coincides with the class $\Theta(\rho)$. In particular,

The self-intersection number of $\lambda(\rho)$ is constant over the Teichmüller space of Fuchsian representations. However, off the Fuchsian locus, the measure classes of $\lambda(\rho)$ and $\Theta(\rho)$ do not coincide (Theorem C of [**H99**]). The current $\lambda(\rho)$ always determines uniquely a representative $\theta(\rho)$ of the class $\Theta(\rho)$ by requiring that $\iota(\lambda(\rho),\theta(\rho)) = 1$. We call this representative the *mass normalized* representative of $\Theta(\rho)$. The following is our main result.

Theorem 1. *Let $S_0 \subset S$ be a proper connected essential subsurface such that no component of $S_1 = S - S_0$ is a pair of pants, and let h be any hyperbolic metric on S_0 so that the boundary of S_0 is geodesic. Then there exists a sequence ρ_i of Hitchin representations with the following properties.*

(1) The projective currents $\Theta(\rho_i)$ converge weakly to the projective current of maximal entropy for the geodesic flow on (S_0, h).
(2) The entropies of the representations ρ_i converge to the entropy of the geodesic flow on (S_0, h).
(3) The self-intersection numbers $\iota(\lambda(\rho_i),\lambda(\rho_i)) \to \infty$ as $i \to \infty$.
(4) For any Fuchsian representation u, it holds $\iota(\lambda(\rho_i),\lambda(u)) \to \infty$, locally uniformly in u.
(5) For each i let $\theta(\rho_i)$ be the mass normalized representative of $\Theta(\rho_i)$. Then $\iota(\theta(\rho_i),\theta(\rho_i))$ is bounded independent of i, and the same holds true for the intersections $\iota(\theta(\rho_i),\lambda(u))$ for any fixed Fuchsian representation u.
(6) The projectivizations of the currents $\lambda(\rho_i)$ converge weakly to a measured geodesic lamination.

In the appendix, we establish the perhaps well known fact that for any closed surface S of genus $g \geq 2$, any subsurface $S_0 \subset S$ without a complementary component which is a pair of pants and any $a \in (0, 1)$, there is a hyperbolic metric h on S so that the entropy of (S, h) equals δ. Thus part (2) of Theorem 1 leads to the following generalization of the result of Zhang [36] mentioned above, using a different approach.

Corollary. *For any number $a \in [0, 1)$ there exists a sequence of degenerating Hitchin representations whose entropy converges to a.*

For $d = 3$, the Hitchin component can be identified with the space of convex real projective structures on the surface S. Using this viewpoint, Corollary is independently due to Nie [30]. The article [10] also contains related results, embarking from the same deformations we use, but with a different geometric interpretation.

The text below should probably be moved to the other paper To a convex real projective structure on S is associated a so-called *Blaschke metric* which (somewhat indirectly) gives a geometric interpretation of the natural parameterization of $\mathrm{Hit}(S)$ for $d = 3$ as the bundle of cubic differentials over Teichmüller space. This led Loftin [25] to define an augmented Hitchin space (see also [26]) which captures part of this geometric information. It turns out that unlike in the case of Teichmüller space, this construction is not well related to the compactification obtained by the embedding into the space of projective currents.

Theorem 2. *Suppose S has genus at least 3 and consider a nice Finsler metric on the symmetric space $\mathrm{PSL}(3,\mathbb{R})/\mathrm{PSO}(3)$ and the projective geodesic currents it defines for $\mathrm{Hit}(S)$. Let $\mathrm{Hit}_3^{\mathrm{aug}}(S)$ be Loftin's augmented Hitchin space and $\overline{\mathrm{Hit}}_3(S)$ the closure of $\Theta(\mathrm{Hot}(S))$ in the space of projective geodesic currents.*

Then there exist paths $(x_t)_{t\geq 0}, (y_t)_{t\geq 0} \subset \mathrm{Hit}_3(S)$ converging to two distinct points $x, y \in \mathrm{Hit}_3^{\mathrm{aug}}(S)$, but converging to the same projective geodesic current.

Organization of the article and structure of the proof. In Section 1 we introduce the class of nice Finsler Such a metric is determined by a positive linear functional α_0 on the convex cone of vectors $x = (x_1, \dots, x_d)$ with $x_1 \geq \cdots \geq x_d$ and $x_1 + \cdots + x_d = 0$, thought of as the closed positive Weyl chamber in the standard Cartan subalgebra of the Lie algebra $\mathfrak{sl}(d,\mathbb{R})$, which satisfies some extra conditions. An example is given by

$$\alpha_0(x) = (d-1)x_1 + (d-3)x_2 + \cdots + (1-d)x_d. \tag{1}$$

The Finsler metric defined by this datum determines a length function on $\mathrm{PSL}_d(\mathbb{R})$ by applying α_0 to the logarithm of the absolute values of the eigenvalues. For each representation of $\pi_1(S)$ into $\mathrm{PSL}_d(\mathbb{R})$, by precomposition this yields a length function on $\pi_1(S)$. These Finsler metrics are particularly well adapted for a geometric understanding of a Hitchin representations [2].

In Section 2 we verify that after fixing a hyperbolic metric on S with geodesic flow Φ^t on the unit tangent bundle T^1S, each length function from a Hitchin representation ρ corresponds to the integral over periodic Φ^t-orbits of a Hölder continuous positive function f_ρ on T^1S called reparametrisation function. For a slightly different family of length functions, this was established in [5, 3], but for the length functions we use, such a statement does not seem to be available in the literature.

In Section 3 we collect some results from [**BHM24**] which provide a geometric control on the representations we use. This is used in Section 4 to show part (4) of Theorem 1. It seems likely that for $d = 3$, a similar statement can be extracted from the study of Blaschke metrics as for example in [31], but we are not aware of an explicit account in the literature. The proofs of the remaining parts of Theorem 1 are contained in Section 5. The appendix contains information on the entropy of the geodesic flow on compact hyperbolic surfaces with boundary which we were unable to find in the literature in the form we need and which are used in the proof of the corollary.

Acknowledgement: This project started as a working seminar in fall 2021, during the pandemic, held in person at the Max Planck Institute for Mathematics in Bonn. We thank the MPI for the hospitality and financial support, and we thank Gianluca Faraco, Elia Fioravanti, Frieder Jäckel, Yannick Krifka, Laura Monk and Yongquan Zhang for many enjoyable discussions and good company. P.-L.B. is grateful to Dick Canary, Fanny Kassel, Charles Reid, and Ralph Spatzier for helpful discussions, and U.H. and P.-L.B. thank Andrés Sambarino for helpful discussions. All authors thank Beatrice Pozzetti for pointing out an error in an earlier version of this article.

1. Lie groups and symmetric spaces

This section collects some basic facts on Lie groups and symmetric spaces and introduces conventions and notations used lated on.

Consider the unique (up to conjugacy) irreducible representation $\tau : \mathrm{PSL}_2(\mathbb{R}) \to G = \mathrm{PSL}_d(\mathbb{R})$, which can be described as follows. A matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ acts on the algebra $\mathbb{R}[X, Y]$ of polynomials in two variables by $M \cdot X = aX + cY$ and $M \cdot Y = bX + eY$. This action preserves the d-dimensional linear subspace $\mathbb{R}^h_{d-1}[X, Y]$ of degree $d - 1$ homogeneous polynomials, which we identify with $\mathbb{R}^d$.

This representation is regular, in the sense that it maps diagonalisable 2-by-2 matrices with distinct real eigenvalues to diagonalisable d-by-d matrices with distinct real eigenvalues. As a consequence, τ induces

- an isometric embedding of the hyperbolic plane $\mathbb{H}^2 = \mathrm{PSL}_2(\mathbb{R})/\mathrm{PSO}(2)$ into the symmetric space $\mathbb{X} = G/K$, which is endowed with a non-positively curved G-invariant Riemannian metric;
- an embedding of the boundary at infinity $\partial\mathbb{H}^2 = \mathrm{PSL}_2(\mathbb{R})/T$ into the flag variety $\mathcal{F} = G/P$, which can be seen as the space of full flags, i.e. sequences

$$\xi = (\xi_1 \subset \xi_2 \subset \cdots \subset \xi_d = \mathbb{R}^d)$$

  where ξ_i is a linear subspace of $\mathbb{R}^d$ of dimension i for each $i \leq d$.

We also fix a basepoint $\mathbf{x} = K \in \mathbb{X} = G/K$, whose stabiliser is K. The subspace $A \cdot \mathbf{x}$ is a totally geodesic embedded Euclidean subspace of $\mathbb{X}$ of maximal dimension. This flat identifies with the Cartan subspace $\mathfrak{a}$, which is the linear space of diagonal (d, d)-matrices with vanishing trace, through the map $v \in \mathfrak{a} \mapsto \exp(v) \cdot \mathbf{x}$. The maximal Euclidean subspaces, called maximal flats, are the translates of $A \cdot \mathbf{x}$ under some $g \in G$.

The stabiliser in K of $\mathfrak{a}$ is finite and acts by permuting the diagonal entries; the quotient by the subgroup acting trivially on $\mathfrak{a}$ is the Weyl group, denoted by Weyl. This action is generated by the swaps of two diagonal entries, which act on $\mathfrak{a}$ by reflections along hypersurfaces called walls. The open Weyl cone $\mathfrak{a}^+ \subset \mathfrak{a}$ is a natural fundamental domain for this action: it is the open cone of diagonal matrices whose entries $(\lambda_1, \dots, \lambda_d)$ fulfill $\lambda_1 > \lambda_2 > \cdots > \lambda_d$.

Putting $A^+ = \exp(\mathfrak{a}^+)$, the K-orbit of every point $y = g\mathbf{x} \in \mathbb{X}$ intersects the closed Weyl cone $\overline{A^+}\mathbf{x}$ at exactly one point $\exp(u)\mathbf{x}$, and we write $u = \kappa(g)$ and call it the Cartan projection of g. Similarly, the G-orbit of any vector $v \in TX$ intersects $\overline{\mathfrak{a}^+}$ (seen as a subspace of $T_{\mathbf{x}}\mathbb{X}$) in precisely one point $\kappa(v)$ called the *Cartan projection* of v.

Two flags $\xi = (\xi_1, \dots, \xi_d)$ and $\eta = (\eta_1, \dots, \eta_d)$ are transverse if ξ_i and η_{d-i} are in direct sum for every i. This is equivalent to the existence of a maximal flat $F(\xi, \eta)$ and two opposite Weyl Cones in it whose boundaries at infinity are ξ and η.

The *Jordan projection* $\lambda(g) \in \overline{\mathfrak{a}^+}$ of $g \in G$ is the diagonal matrix whose diagonal entries are the moduli of the eigenvalues of g in descending order. The element $g \in G$ is called *loxodromic* if $\lambda(g)$ is contained in the interior $\mathfrak{a}^+$ of $\overline{\mathfrak{a}^+}$, which is equivalent to saying that g has an attracting/repelling fixed pair of transverse flags (g^-, g^+). Then g acts as a translation on the flat $F(g^-, g^+)$ with direction prescribed by its Jordan projection.

A Finsler metric coming from a linear functional on $\mathfrak{a}$

We fix a linear functional α_0 on $\mathfrak{a}$ which is positive on $\overline{\mathfrak{a}^+}$ and such that $\alpha_0(gv) < \alpha_0(v)$ for all $v \in \mathfrak{a}^+$ and $g \in$ Weyl.

We assume that α_0 is symmetric in the sense that if g is the transformation in the Weyl group that maps $\mathfrak{a}^+$ to its opposite $-\mathfrak{a}^+$ then $\alpha_0(gv) = -\alpha_0(v)$ for any $v \in \mathfrak{a}$.

An example of a linear functional satisfying the above conditions is given in Equation 1.

For any vector $v \in T\mathbb{X}$ we set

$$\mathfrak{F}(v) = \alpha_0(\kappa(v)) \tag{2}$$

where as before, $\kappa(v) \in \overline{\mathfrak{a}^+}$ is the Cartan projection of v.

Proposition 1.1 (Lemmas 5.9-10 of [16]). *The following hold.*

1. *$\mathfrak{F}$ defines a G-invariant Finsler metric on $\mathbb{X}$.*
2. *The unparameterized Riemannian geodesics of $\mathbb{X}$ are also geodesics for $\mathfrak{F}$.*
3. *The translation length for $\mathfrak{F}$ of any element $g \in G$ acting on $\mathbb{X}$ is given by $\ell^{\mathfrak{F}}(g) := \alpha_0(\lambda(g))$ where $\lambda(g) \in \overline{\mathfrak{a}^+}$ is the Jordan projection.*

In the sequel we always normalize the functional α_0 in such a way that the embedding $\mathbb{H}^2 \to \mathbb{X}$ which is isometric for the symmetric metric also is isometric for the Finsler metric $\mathfrak{F}$.

Limit sets and Busemann functions

2. Hitchin representations and equilibrium states

In this section we introduce the main structures and tools for this article. Throughout, S denotes a closed surface of genus $g \geq 2$, equipped with a fixed choice of a hyperbolic metric. Thus the universal covering $\tilde{S}$ of S can naturally be identified with the hyperbolic plane $\mathbb{H}^2$. We denote by T^1S the unit tangent bundle of S, equipped with the geodesic flow Φ^t.

The section is subdivided into two subsections. In the first subsection we briefly discuss geodesic currents for closed surfaces and the intersection form. The second subsection contains an account of Hitchin representations and length functions defined by Finsler norms. Extending some results of [5], we show that the length functions on $\mathrm{Hit}(S)$ defined by such a Finsler metric can be obtained from Hölder continuous functions on the unit tangent bundle T^1S of a fixed hyperbolic surface S depending smoothly on the representation.

2.1. **Currents and intersection.** Throughout, S denotes a closed oriented surface of genus $g \geq 2$ equipped with a fixed choice of a hyperbolic metric, with universal covering the hyperbolic plane $\mathbb{H}^2$ and ideal boundary $\partial\mathbb{H}^2 = S^1$. A *geodesic current* on S can then be thought of as a $\pi_1(S)$-invariant locally finite Borel measure on $S^1 \times S^1 \setminus \Delta$ which is moreover invariant under the flip exchanging the two factors. Particular such currents arise from non-positively curved metrics h on S. There are in fact two distinct such currents. The *Bowen Margulis current* which defines the measure of maximal entropy for the geodesic flow Φ^t on the unit tangent bundle T^1S of S, and the *Liouville current* which is given by the Lebesgue Liouville measure on T^1S. For hyperbolic metrics, these currents coincide. The *intersection form* $\iota : \mathrm{Curr}(S) \times \mathrm{Curr}(S) \to [0, \infty)$ is a convex linear symmetric function, continuous with respect to the weak*-topology.

An arbitrary positive flip invariant Hölder functions on T^1S, thought of as the generalization of the *length cocycle* on T^1S which associates to a periodic orbit for the geodesic flow the length or the closed geodesic defined by this orbit, may also determine two currents. The first current is the *Bowen Margulis current* μ_f, that is, the equilibrium state for Φ^t defined a Hölder continuous representative of the cocycle, only determined up to normalization, and the *intersection current* λ_f (which however may not always exist). Following Lemma 2.6 of [**H99**], the intersection current λ_f is characterized by the property that $\int f d\nu = \iota(\lambda_f, \mu)$ for all currents μ. If $t \to f_t$ is an analytic family of Hölder functions, then the family of currents $t \to \lambda_{f_t}$ is analytic as well, and the same holds true for $t \to \mu_t$ up to normalization.

2.2. **Hitchin representations.** In this section we introduce Hitchin representations and summarize those of their properties which are important later on. Our main goal is to show that the G-invariant Finsler metric $\mathfrak{F}$ defined in (2) induces for each representation in the Hitchin component a positive Hölder continuous function depending in an analytic fashion on the representation whose equilibrium state does not depend on choices and hence is a geodesic current.

The *Hitchin component* $\mathrm{Hit}(S)$ for conjugacy classes of representations $\pi_1(S) \to \mathrm{PSL}_d(\mathbb{R})$ is the connected component of the set of conjugacy classes of representations which factor through an irreducible representation $\mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_d(\mathbb{R})$. In the sequel we always work with explicit representations rather than with conjugacy classes.

An important property possessed by Hitchin representations is the Anosov property first introduced by Labourie [20],. There are many different versions of the Anosov property, see for example [20, 14, 17, 13, 3, 19], and Theorem 4.37 of [18] for more details and history.

Hitchin representations have the strongest possible Anosov property. To define this property let as before $\mathcal{F}$ be the variety of full flags in $\mathbb{R}^d$.

Definition 2.1 ([13, 17])**.** A representation $\rho : \pi_1(S) \to G$ is *Borel Anosov* if the following holds true.

(1) There exists a (unique) equivariant Hölder embedding $\partial_\infty \rho : \partial \mathbb{H}^2 \to \mathcal{F}$ such that $\partial_\infty \rho(\xi) \pitchfork \partial_\infty \rho(\eta)$ for all $\xi \neq \eta \in \partial \mathbb{H}^2$.
(2) For any diverging sequence $(\gamma_n)_n \subset \pi_1(S)$ such that $\gamma_n \to \xi \in \partial \mathbb{H}^2$ and $\gamma_n^{-1} \to \eta$, we have $\rho(\gamma_n)\zeta \to \partial_\infty \rho(\eta)$ for any $\zeta \in \mathcal{F}$ transverse to $\partial_\infty \rho(\xi)$.

By the groundbreaking work of Labourie and Fock–Goncharov [20, 9], we have

Theorem 2.2 ([**Labourie**, **Fock-Goncharov**])**.** *All Hitchin representations $\pi_1(S) \to \mathrm{PSL}_d(\mathbb{R})$ are Borel Anosov.*

Remark 2.3. In the references given for the characterizations of the Anosov property, the limit map is only required to be continuous, and then the Hölder regularity is derived as a consequence of the other conditions, see for instance Theorem 6.58 of [17].

The Borel Anosov property implies the following weaker property which we shall use.

Definition 2.4. A representation $\rho : \pi_1(S) \to \mathrm{PSL}_d(\mathbb{R})$ is *projective Anosov* if there exist ρ-equivariant Hölder continuous maps $\xi : \partial_\infty \tilde{S} \to \mathbb{R}P^{d-1}$, $\theta : \partial_\infty \tilde{S} \to (\mathbb{R}P^{d-1})^*$ (where $(\mathbb{R}P^{d-1})^*$ is the dual projective space) such that

(1) if x, y are distinct points in $\partial_\infty \tilde{S}$, then $\xi(x) + \ker \theta(y) = \mathbb{R}^d$, and

(2) if $\gamma_n \in \pi_1(S)$ is a sequence so that for some basepoint $\mathbf{x} \in \tilde{S} = \mathbb{H}^2$, the sequence $\gamma_n \mathbf{x}$ converges to $x \in \partial_\infty \mathbb{H}^2$, and $\gamma_n^{-1}\mathbf{x} \to y \in \partial_\infty \mathbb{H}^2$, then we have $\rho(\gamma_n)p \to \xi(x)$ for any $p \in \mathbb{R}P^{d-1} - \ker\theta(y)$ and $\rho(\gamma_n^{-1})q \to \theta(y)$ for any $q \in (\mathbb{R}P^{d-1})^*$ such that $\xi(x) \notin \ker q$.

As in [5], let F be the total space of the bundle over

$$(\mathbb{R}P^{d-1})^{(2)} = \mathbb{R}P^{d-1} \times (\mathbb{R}P^{d-1})^* - \{(U,V) \mid U \subset \ker(V)\}$$

whose fiber at a point (U,V) is the space

$$M(U,V) = \{(u,v) \mid u \in U, v \in V, \langle v \mid u \rangle = 1\}/\sim$$

where $\langle v \mid u\rangle$ is the natural pairing between a vector and a covector and $(u,v) \sim (-u,-v)$. Note that u determines v so that F is an $\mathbb{R}$-bundle.

The bundle F is equipped with a natural $\mathbb{R}$-action, given by

$$\Phi_F^t(U,V,(u,v)) = (U,V,(e^t u, e^{-t}v)).$$

Given a projective Anosov representation $\rho : \pi_1(S) \to \mathrm{PSL}_d(\mathbb{R})$ and ξ, θ the associated limit maps, we consider the pullback bundle

$$F_\rho = (\xi,\theta)^* F \to \partial_\infty \widetilde{S} \times \partial_\infty \widetilde{S} - \Delta$$

by the map $\partial_\infty \widetilde{S} \times \partial_\infty \widetilde{S} - \Delta \xrightarrow{(\xi,\theta)} (\mathbb{R}P^{d-1})^{(2)}$, which inherits an $\mathbb{R}$-action from the action of Φ_F^t. The actions $\pi_1(S) \curvearrowright_\rho \mathbb{R}^d$ and $\pi_1(S) \curvearrowright \partial_\infty \widetilde{S} \times \partial_\infty \widetilde{S} - \Delta$ extend to an action on F_ρ. If we let

$$U_\rho S = \pi_1(S) \backslash F_\rho$$

then the $\mathbb{R}$-action on F_ρ descends to a flow Φ_ρ^t on $U_\rho S$ which is called the *spectral radius flow* of the representation (see p.1118 of [5]). It obtains its name from the fact that the length of periodic orbits is measured by the spectral radius of the conjugacy class in the image of the representation defining the orbit.

The following statement combines Propositions 4.1, 4.2 and 6.2 of [5]. It is valid for any analytic family of projective Anosov representations.

Proposition 2.5. *(1) For every projective Anosov representation $\rho : \pi_1(S) \to \mathrm{SL}_d(\mathbb{R})$ there exists a Hölder continuous order preserving orbit equivalence $\Psi_\rho : (T^1S, \Phi^t) \to (U_\rho S, \Phi_\rho^t)$. Any primitive element $\gamma \in \pi_1(S)$ has period $\log \Lambda(\rho)(\gamma)$ where $\Lambda(\rho)(\gamma)$ is the spectral radius of $\rho(\gamma) \in \mathrm{PSL}_d(\mathbb{R})$.*

(2) If D is the unit disk and if ρ_u $(u \in D)$ is a real analytic family of Hitchin representations, then up to decreasing the size of D, there exists a real analytic family $\{f_{\rho_u} : T^1S \to \mathbb{R}\}_{u\in D}$ of positive Hölder functions such that the reparameterization of T^1S by f_{ρ_u} is Hölder conjugate to U_{ρ_u} for all $u \in D$.

The goal of this subsection is to extend Proposition 2.5 to length functions $\ell^{\mathfrak{F}}(g) = \alpha_0(\lambda(g))$ for Hitchin representations defined by one of the Finsler norms introduced in the introduction. We shall reduce this statement to Proposition 2.5 using the following classical observation.

Let $\xi = (\xi_1 \subset \cdots \subset \xi_d)$ be a full flag in $\mathbb{R}^d$. Then for each $k \le d-1$ the k-th exterior power $\Lambda^k(\xi_k)$ is one-dimensional. A non-zero element ω of this vector space defines up to a non-zero multiple a non-zero linear functional $\Psi(\omega) : \Lambda^{d-k}(\mathbb{R}^d) \to \mathbb{R}$ as follows. Choose a non-zero element $\nu \in \Lambda^d(\mathbb{R}^d)$ and put $\Psi(\omega)(\alpha) = c$ if $\omega \wedge \alpha = c\nu$. Note that the kernel of $\Psi(\omega)$ is spanned by all decomposable elements of $\Lambda^{d-k}\mathbb{R}^d$ which are not transverse to ξ_k.

If $\rho : \pi_1(S) \to G$ is Borel Anosov, then by the definition of the transversality relation $\pitchfork$, for any two distinct points $\xi \neq \eta \in \partial\mathbb{H}^2$, the $d-k$-th subspace $\partial_\infty\rho(\xi)_{d-k}$ of the flag $\partial_\infty\rho(\xi)$ defines a line of linear functionals on $\Lambda^k(\mathbb{R}^d)$ which do not evaluate to zero on $\Lambda^k\partial_\infty\rho(\eta)_k$, where $\partial_\infty\rho(\eta)_k$ is the k-dimensional subspace of the flag $\partial_\infty\rho(\eta)$. Thus if $\Lambda^k\rho : \pi_1(S) \to \mathrm{PSL}_{d_k}(\mathbb{R})$ denotes the representation induced by ρ into the full linear group of $\Lambda^k(\mathbb{R}^d)$ where d_k denotes the dimension of $\Lambda^k(\mathbb{R}^d)$, then as the map $\partial_\infty\rho : \partial_\infty\mathbb{H}^2 \to \mathcal{F}$ is Hölder continuous, the following well-known statement holds true.

Lemma 2.6. *If $\rho : \pi_1(S) \to G$ is Borel Anosov, then for any $k < d$, the induced representation $\Lambda^k\rho$ is projective Anosov.*

Remark 2.7. It follows from the above discussion that in fact, ρ is Borel Anosov if and only if for each $k \le d-1$ the induced representation on $\Lambda^k(\mathbb{R}^d)$ is projective Anosov. We refer to Section 4 of [3] for more details on this relation.

Thus we can apply Proposition 2.5 to each representation $\Lambda^k\rho$. Recall from Section 1 the definition of the Jordan projection λ. As implicitly stated in [3], we obtain the following regularity statement on Finsler length functions.

Proposition 2.8. *For every Borel Anosov representation $\rho_0 : \pi_1(S) \to \mathrm{PSL}_d(\mathbb{R})$, there exists an open neighborhood U of ρ_0 made of Borel Anosov representations and a real analytic family $\{f_\rho : T^1S \to \mathfrak{a}\}_{\rho\in U}$ of Hölder functions, valued in $\mathfrak{a}^+$, such that for any $\gamma \in \pi_1(S)$, we have*

$$\lambda(\rho(\gamma)) = \int f_\rho d\gamma.$$

Proof. Proposition 2.5 implies that there exists an open neighborhood U of ρ_0 and real analytic families $\{g^k_\rho : T^1S \to \mathbb{R}\}_{\rho\in U}$ of Hölder functions such that for any $\rho \in U$, each exterior product $\Lambda^k\rho$ is projective Anosov, and for any $\gamma \in \pi_1(S)$, the logarithm $\log\Lambda(\Lambda^k\rho(\gamma))$ of the spectral radius of $\Lambda^k(\rho(\gamma))$ equals

$$\log\Lambda(\Lambda^k\rho(\gamma)) = \int g^k_\rho d\gamma.$$

Then we can consider the following Hölder function

$$f_\rho = (g^1_\rho,\ g^2_\rho - g^1_\rho,\ g^3_\rho - g^2_\rho, \ldots,\ g^d_\rho - g^{d-1}_\rho) \in \mathfrak{a}.$$

By Proposition 2.5, the function f_ρ depends analytically on ρ. Moreover, for any $\gamma \in \pi_1(S)$, we have

$$\lambda(\rho(\gamma)) = \int f_\rho d\gamma.$$

It is not clear, however, that f_ρ is valued in the open Weyl chamber $\mathfrak{a}^+$. Let us solve this issue by first replacing f_{ρ_0} by an f'_{ρ_0} valued in $\mathfrak{a}^+$, using work of Sambarino, and then extend f'_{ρ_0} to a small neighborhood of representations ρ, using a theorem of Livšic.

We apply Sambarino's reparametrization result to the lengths functions $\alpha_k \circ \lambda \circ \rho_0(\gamma)$ where $\alpha_k(v_1, \ldots, v_d) = v_k - v_{k+1}$, see Theorem 3.2 of [33] (Sambarino proved in pages 481-483 that we can apply this theorem to our setting). This gives us positive Hölder functions $u^k : T^1S \to \mathbb{R}$ such that for any $\gamma \in \pi_1(S)$, we have

$$\alpha_k \circ \lambda \circ \rho_0(\gamma) = \int u^k d\gamma.$$

Let $f'_{\rho_0} : T^1S \to \mathfrak{a}$ be such that $\alpha_k \circ f'_{\rho_0}(v) = u^k(v) > 0$ for all $v \in T^1S$ and $1 \leq k \leq d-1$. Then f'_{ρ_0} is valued in the interior of $\mathfrak{a}^+$ by definition, it is Hölder, and $\lambda(\rho_0(\gamma)) = \int f'_{\rho_0} d\gamma$ for any γ.

For any periodic orbit γ in T^1S we have $\int f_{\rho_0} d\gamma = \int f'_{\rho_0} d\gamma$, so by Theorem 1 of [22] f'_{ρ_0} and f_{ρ_0} are cohomologous, in the sense that there exists $F : T^1S \to \mathfrak{a}$ differentiable in the direction of the geodesic flow Φ^t such that $f'_{\rho_0} = f_{\rho_0} + \frac{d}{dt}_{|t=0} F \circ \Phi^t$. Put

$$f'_\rho = f_\rho + \frac{d}{dt}_{|t=0} F \circ \Phi^t$$

for any $\rho \in U$, so that $\lambda(\rho(\gamma)) = \int f'_\rho d\gamma$ for any γ. This yields an analytic family of Hölder functions which take values in $\mathfrak{a}^+$ for all ρ contained in a sufficiently small neighborhood $U' \subset U$ of ρ_0. This is what we wanted to show. $\square$

3. Hitchin grafting representations

The Hitchin representations we are interested in are the familiar *bending* or *bulging* deformations of *Fuchsian* representations, that is, representations which factor through the embedding $\tau : \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_d(\mathbb{R})$. We refer to [11, 1, 2] for an account on the bending construction. In this section we introduce these representations and summarize the geometric results from [2] we need.

3.1. **Grafting.** Consider a closed oriented surface S of genus $g \geq 2$ endowed with a hyperbolic metric. A *simple (geodesic) multi-curve* γ^* is the union of pairwise disjoint essential mutually not freely homotopic simple closed curves (geodesics) on S. We fix moreover an orientation on each component of γ^*.

Consider the special direction $u = d\tau \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{a}$ given by τ. For any $z \in \mathfrak{a}$ and $\ell > 0$, let $\mathrm{Cyl}(\ell, z) \subset \mathfrak{a}/\ell u$ be the cylinder obtained by quotienting the strip $\{tu + sz : t \in \mathbb{R},\ s \in [0,1]\} \subset \mathfrak{a}$ under the translation by ℓu. The (Finsler) *height* of such cylinder is defined as

$$\text{height} = \min\{\mathfrak{F}(tu + z) : t \in \mathbb{R}\}. \tag{3}$$

We fix for every $\gamma \in \gamma^*$ a vector $z_\gamma \in \mathfrak{a}$; the collection $z = (z_\gamma)_{\gamma \in \gamma^*}$ is interpreted as a grafting parameter.

Definition 3.1. The *abstract grafting* of S along the geodesic multi-curve γ^* with grafting parameter z is the surface S_z obtained by cutting S open along each of the components γ of γ^*, inserting flat cylinders $C_\gamma = \mathrm{Cyl}(\ell_S(\gamma), z_\gamma)$ and gluing the surface back with the translation by z_γ.

If z_γ is not parallel to u for any $\gamma \in \gamma^*$, then this grafting comes with a natural homotopy equivalence $\pi_z : S_z \to S$ projecting the flat cylinders onto γ^*, which allow us to identify $\pi_1(S_z)$ and $\pi_1(S)$.

The abstract grafted surface S_z decomposes into subsurfaces with geodesic boundary which are equipped with a metric of constant curvature. The *hyperbolic part* S^{hyp} is the union of the subsurfaces with a metric of constant curvature -1 and can be identified with the union of the components of $S \setminus \gamma^*$. The component $S \setminus S^{\mathrm{hyp}}$ is the *cylinder part* and consists of a union of flat cylinders whose core curves are freely homotopic to the components of γ^*.

We endow S_z with a Finsler metric by equipping each cylinder C_γ with the quotient of the non-Euclidean norm $\mathfrak{F}$ on $\mathfrak{a}$. Observe that in general, for a given C^1-structure on S_z as constructed above, this metric is *discontinuous* at the gluing locus between the flat cylinders and the hyperbolic part. Additionally the metric on the flat part is sensitive in the direction of z, and does not depend only on the height of the grafting (contrarily to the Riemannian metric). Nevertheless it induces a well defined path metric on S_z.

Let G_{γ^*} be the oriented graph such that each vertex $v \in V$ corresponds to a component Σ_v of $\Sigma - \gamma^*$, and each edge $e \in E$ corresponds to an oriented component $\vec{\gamma}_e$ of γ^*. Take a discrete and faithful representation $\rho\colon \pi_1(G_{\gamma^*}, T) \to \mathrm{PSL}_2(\mathbb{R}) \xrightarrow{\tau} \mathrm{PSL}_d(\mathbb{R})$ which factors through the embedding $\tau : \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_d(\mathbb{R})$. We use the graphs of groups decomposition of $\pi_1(\Sigma)$ determined by γ^* to perform a bending of the representation in $\mathrm{PSL}_d(\mathbb{R})$ with parameter $z = (z_\gamma)_{\gamma \in \gamma^*} \in \mathfrak{a}^{\gamma^*}$. This construction can be thought of as bending the surface S along the geodesic multicurve γ^* in the space of representations into G.

Definition 3.2. We denote by $\mathrm{Gr}_z^{\gamma^*}\rho\colon \pi_1(G_{\gamma^*}, T) \to \mathrm{PSL}_d(\mathbb{R})$ the representation induced by $\widetilde{\rho}_z$, and sometimes just ρ_z if there is only one hyperbolic structure involved. We call it the *Hitchin grafting representation* with data z along γ^*.

Up to conjugation, the representation ρ_z only depends on the grafting parameter z. A *Hichin grafting ray* is a one-parameter family of Hitchin grafting representations $t \to \rho_{tz}$ defined by a ray in $(\mathfrak{a})^k$ where k is the number of components of the multicurve γ^* along which the grafting is performed.

3.2. **The characteristic surface for Hitchin grafting representations.** Consider a Fuchsian representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_d(\mathbb{R})$ and denote by S the hyperbolic surface defined by this representation. Choose some grafting datum z and let ρ_z be the Hithin grafted representation defined by ρ and z. As this representation is contained in the Hitchin component, it follows from Labourie [20] and Fock–Goncharov [9] that ρ_z is faithful, with discrete image. In particular, the quotient manifold $\rho_z\backslash\mathbb{X}$ is homotopy equivalent to S; in fact ρ induces a natural homotopy class of homotopy equivalences between $\rho_z\backslash\mathbb{X}$ and S.

The following statement is Proposition 2.5 of [2].

Proposition 3.3. *Consider a Hitchin grafting representation ρ_z obtained from ρ and with grafting datum z. Let S_z be the abstract grafting of S from Definition 3.1, with universal covering $\widetilde{S}_z$. Then there exists a piecewise totally geodesic immersed surface $\widetilde{S}^\iota_z \subset \mathbb{X}$ and a ρ_z-equivariant immersion $\widetilde{Q}_z \colon \widetilde{S}_z \to \widetilde{S}^\iota_z \subset \mathbb{X}$.*

The map $\widetilde{Q}_z$ is a path isometry for the Riemannian (resp. Finsler) metric on $\widetilde{S}_z$ and the induced path metric on $\widetilde{S}^\iota_z$ from the Riemannian (resp. Finsler) metric on $\mathbb{X}$.

3.3. **Geometric control: Uniform quasi-isometry.** Recall that S is a hyperbolic closed surface, let $G = \mathrm{PSL}_d(\mathbb{R})$ and $\tau : \mathrm{PSL}_2(\mathbb{R}) \to G$ be the usual irreducible representation. The following is Theorem 5.1 of [2] which was obtained as a consequence of Fock Goncharov positivity.

Theorem 3.4. *For every $\sigma > 0$, there exists $C_\sigma > 0$ such that the following holds.*

Consider a closed hyperbolic surface S, a multicurve $\gamma^* \subset S$ whose components have length at most σ, and a grafting parameter z such that all cylinder heights of the abstract grafting S_z are bounded from below by some number $L > 0$.

Let us endow $\mathbb{X}$ with the G-invariant admissible Finsler metric $\mathfrak{F}$ and S_z with the pullback of this metric under Q_z, denoted by $d^{\mathfrak{F}}_{\widetilde{S}_z}$. Then the grafting map $\tilde{Q}_z : \widetilde{S}_z \to \mathbb{X}$ is an injective quasi-isometric embedding with multiplicative constant $(1 + C_\sigma/(L + 1))$ and additive constant C_σ; more *precisely, for all $x, y \in \tilde{S}_z$ we have*

$$\left(1+\frac{C_\sigma}{L+1}\right)^{-1} d^{\mathfrak{F}}_{\tilde{S}_z}(x,y) - C_\sigma \leq d^{\mathfrak{F}}(\tilde{Q}_z(x), \tilde{Q}_z(y)) \leq d^{\mathfrak{F}}_{\tilde{S}_z}(x,y).$$

Moreover, the image $\tilde{S}^\iota_z = \tilde{Q}_z(\tilde{S}_z)$ is C_σ-Finsler-quasiconvex in the sense that for all $x, y \in \tilde{Q}_z(\tilde{S}_z)$, there is a Finsler geodesic from x to y at distance at most C_σ from $\tilde{S}^\iota_z$.

There also is the following coarse estimates on length (Theorem 5.2 of [2]).

Theorem 3.5. *In the setting of Theorem 3.4, let $(\rho_z)_z$ be the associated family grafted Hitchin representations. Then there is C'_σ only depending on σ such that for any $\gamma \in \pi_1(S)$,*

$$\ell^{\mathfrak{F}}(\rho_z(\gamma)) \geq \frac{L+1}{C'_\sigma} \iota(\gamma, \gamma^*).$$

Moreover, recalling that z is the datum of a vector $z_e \in \mathfrak{a}$ for each component $e \subset \gamma^*$, then C'_σ may be chosen so that if $z_e \in \ker(\alpha_0)$ for any e then

$$\ell^{\mathfrak{F}}(\rho_z(\gamma)) \geq \left(1+\frac{C'_\sigma}{L+1}\right)^{-1} \ell_S(\gamma),$$

where $\ell_S(\gamma)$ is the length of γ in S.

4. Intersection in the Hitchin component

This section contains an application of the main results of [2] to dynamical properties of Hitchin grafting representations. Recall from the introduction the definition of the intersection form $\iota : \mathrm{Curr}(S) \times \mathrm{Curr}(S) \to [0, \infty)$. In the sequel we always normalize the current $\mu(\rho)$ of maximal entropy defined by a fixed choice of a Finsler metric $\mathfrak{F}$ and a Hitchin representation ρ in such a way that $\int f_\rho d\mu(\rho) = 1$. Note that this normalization does not depend on choices. If ρ is Fuchsian, then $\mu(\rho)$ coincides with the mass normalized Liouville current, that is, $\frac{1}{4\pi^2|\chi(S)|}\lambda(\rho)$ where $\lambda(\rho)$ is the current defined by the Lebesgue Liouville measure on T^1S of the hyperbolic metric ρ.

Martone and Zhang [28] showed that Hitchin representations are *positive ratioed.* This means that the length function f_ρ defined by the Finsler metric arising from a fundamental weight, which is among the Finsler metrics we introduced in Section 3, defines without ambiguity a so-called *intersection current* $\lambda(\rho)$. It then holds $\iota(\mu(\rho), \lambda(\rho)) = 1$ for all ρ.

The following is part 3 of Theorem 1.

Proposition 4.1. *There exists a sequence ρ_i of Hitchin representations such that $\iota(\lambda(u), \lambda(\rho_i)) \to \infty$ for any Fuchsian representation u, and this divergence is uniform in u.*

The Hitchin representations which enter Theorem 4.1 are Hitchin grafting representations. More precisely, let as before γ be a simple closed geodesic on the hyperbolic surface S. This datum is used to construct for each $L > 0$ a Hitchin representation ρ_L obtained by Hitchin grafting along γ of the Fuchsian representation defined by S, with cylinder height L. We do not specify the twisting number of the associated abstract grafting datum as this does not play a role in our discussion, but we assume that $L \to \rho_L$ is a Hitchin grafting ray as introduced in Section 3.1.

The proof of Theorem 4.1 rests on statistical information on length averages, introduced in the next definition. For its formulation, for a Hitchin representation ρ put $R_\rho(T) = R_{\ell_\rho}(T)$ for all T, where as before, $R_{\ell_\rho}(T) = \{\eta \in [\pi_1(S)] \mid \ell_\rho(\eta) \leq T\}$ and $\ell_\rho(\eta)$ is the Finsler translation length of $\rho(\eta)$. Moreover, $[\pi_1(S)]$ is the set of conjugacy classes of the fundamental group $\pi_1(S)$ of S.

Definition 4.2. Let ρ be a Hitchin representation and A a subset of $[\pi_1(S)]$. We say that A is a *full density* set for ρ if

$$\liminf_{T\to+\infty} \frac{R_\rho(T) \cap A}{R_\rho(T)} = 1.$$

If $\mathcal{P}$ is an assertion on $[\pi_1(S)]$, we say that a *typical geodesic satisfies* $\mathcal{P}$ if the set $\{\gamma \in [\pi_1(S)] \mid \gamma \text{ satisfies } \mathcal{P}\}$ is a full density set for ρ.

The following statement can be thought of as a statistical version of the duality between length and intersection for hyperbolic metrics on surfaces. Recall from Section **??** the definition of the intersection form $\iota : \mathcal{C}(S) \times \mathcal{C}(S) \to [0, \infty)$.

Proposition 4.3. *Let ρ be a hyperbolic metric on S, and let $\alpha \subset S$ be a closed geodesic. For any $\epsilon > 0$, for a typical geodesic γ, we have*

$$\left| \iota(\gamma, \alpha) - \frac{1}{-4\pi^2\chi(S)} \ell_\rho(\gamma)\ell_\rho(\alpha) \right| < \epsilon \ell_\rho(\gamma).$$

Proof. The Borel measures

$$\mu_T = \frac{1}{\#R_\rho(T)} \sum_{\ell_\rho(\gamma)\leq T} \mathrm{Leb}_\gamma$$

converge weakly as $T \to \infty$ to the normalized Lebesgue Liouville measure λ_0 on T^1S (see [27]).

Let $\lambda \in \mathcal{C}(S)$ be the (unnormalized) Liouville *current* of ρ, the current defined by the Lebesgue Liouville measure on T^1S, and for each T let $\hat{\mu}_T$ be the current defined by μ_T. Let α be a closed geodesic on S. As $\iota(\alpha, \lambda) = \ell_\rho(\alpha)$ (see Section **??**), by continuity of the intersection form ι for the weak topology on currents, we know that

$$\iota(\hat{\mu}_T, \alpha) \underset{T\to\infty}{\longrightarrow} \frac{1}{-4\pi^2\chi(S)} \ell_\rho(\alpha).$$

Note to this end that the total volume of T^1S with respect to the Lebesgue Liouville current equals $-4\pi^2\chi(S)$.

Put $\kappa = \frac{1}{-4\pi^2\chi(S)}$ and let $\epsilon > 0$. To show that the geodesics γ with

$$|\iota(\gamma,\alpha) - \kappa\ell_\rho(\gamma)\ell_\rho(\alpha)| < \epsilon\kappa\ell_\rho(\gamma)$$

are typical we argue as follows. For $T > 0$ let

$$\mathcal{A}(T) = \{\gamma \mid \ell_\rho(\gamma) \le T, \iota(\gamma,\alpha) \ge (1+\epsilon)\kappa\ell_\rho(\gamma)\ell_\rho(\alpha)\}.$$

We claim that $\frac{\#\mathcal{A}(T)}{\#R_\rho(T)} \to 0$ $(T \to \infty)$.

To see this assume otherwise. By passing to a subsequence, we may assume that the measures $\nu_T = \frac{1}{\#R_\rho(T)}\sum_{\gamma\in\mathcal{A}(T)} \mathrm{Leb}_\gamma$ converge weakly to a nontrivial Φ^t-invariant measure ν. By construction, the measure ν is absolutely continuous with respect to the Lebesgue Liouville measure λ. It defines a current $\hat\nu$ which satisfies

$$\iota(\hat\nu,\alpha)/\nu(T^1S) \ge (1+\epsilon)\kappa\ell_\rho(\alpha). \tag{4}$$

But λ is ergodic under the action of Φ^t and hence as ν is absolutely continous with respect to λ, it is a positive constant multiple of λ. This contradicts the inequality (4) and equation (**??**).

In the same way we conclude that $\frac{\#\mathcal{B}(T)}{\#R_\rho(T)} \to 0$ as $T \to \infty$ where

$$\mathcal{B}(T) = \{\gamma \mid \ell_\rho(\gamma) \le T, \iota(\gamma,\alpha) \le (1-\epsilon)\kappa\ell_\rho(\gamma)\ell_\rho(\alpha)\}.$$

Since $\epsilon > 0$ was arbitrary, this shows the proposition. □

Let X be a hyperbolic metric on S and let c be a non-separating simple closed geodesic on X of length $\ell > 0$. For $L \ge 0$ denote by ρ_L a representation obtained by Hitchin grafting of X on c of height L. Our goal is to estimate for an arbitrary hyperbolic metric u on S the quantities $\mu(\lambda(u),\mu(\rho_L))$ as $L \to \infty$ where $\mu(\rho_L)$ is the mass normalized current of maximal entropy for ρ_L and where as before, $\lambda(u)$ is the Liouville current of a hyperbolic metric.

Denote by $\ell_u(\gamma), \ell_\rho(\gamma)$ the translation length of γ for the hyperbolic metric u and the Finsler translation length for the representation ρ, respectively, and for a number $R > 0$ let $N_{\rho_L}(R)$ be the number of periodic geodesic for the geodesic flow on T^1S. Since for a hyperbolic metric Y, the measure of maximal entropy for the geodesic flow coincides with the Lebesgue Liouville measure, for any Hitchin representation ρ with associated Hölder function f_ρ, Lemma 2.6 of [**H99**] shows

$$\iota(\mu(u),\mu(\rho_L)) = \mathbf{J}(u,\rho_L) = \int f_u d\mu(\rho_L) = \lim_{R\to\infty}\frac{1}{N_{\rho_L}(R)}\sum_{\ell_{\rho_L}(\gamma)\le R}\frac{\ell_u(\gamma)}{\ell_{\rho_L}(\gamma)}. \tag{5}$$

We also consider the quantity

(6)
$$\iota(\lambda(\rho_L),\lambda(u)) = \mathbf{J}(\rho_L,u) = \int f_{\rho_L} d\lambda(u) = \lim_{R\to\infty} \frac{1}{\#N_f(R)} \sum_{\ell_u(\gamma)\leq R} \frac{\ell_{\rho_L}(\gamma)}{\ell_u(\gamma)}.$$

Proof of Theorem 4.1. Let $X \in \mathcal{T}(S)$ be the marked hyperbolic metric which is the basepoint for the Hitchin grafting ray. According to the length control as formulated in Theorem 3.5, for every $\epsilon > 0$ there exist $C_\sigma > 0$ depending on the hyperbolic length σ of the simple closed curve c such that we have

$$\ell_{\rho_L}(\gamma) \geq \max\left\{C_\sigma L\iota(\gamma,c), \frac{L}{L+C_\sigma^{-1}}\ell_X(\gamma)\right\} \tag{7}$$

where we use the notations of Theorem 3.5, lengths in $\mathbb{X}$ are measured with respect to an admissible Finsler metric, and ℓ_X denotes the length for the hyperbolic metric X.

Let $m > 0$ be a fixed number. Our goal is to find a number $L > 0$ so that

$$\mathbf{J}(Y,\rho_L) \geq m$$

for every $Y \in \mathcal{T}(S)$ where as before, $\mathcal{T}(S)$ denotes the Teichmüller space of marked hyperbolic metric on S.

By Theorem 12 of [4], the map which associates to a marked hyperbolic metric on S its Liouville current is a proper topological embedding. More precisely, for the given number $m > 0$, there exists a compact ball B about X in $\mathcal{T}(S)$ such that $\iota(\lambda_X,\lambda_Y) \geq m$ for all marked hyperbolic metrics $Y \in \mathcal{T}(S) - B$, where λ_X, λ_Y are the currents defined by the normalized Lebesgue Liouville measures. Note that this is symmetric in X, Y. Furthermore, we have $\iota(\lambda_Y,\lambda_X) = \mathbf{J}(Y,X)$. We refer to p.152-153 in [4] for details on these facts.

By the estimate (7), for any $\epsilon > 0$ and all sufficiently large $L \geq 0$ depending on ϵ, say for all $L \geq L(\epsilon)$, we have

$$\ell_{\rho_L}(\gamma) \geq (1-\epsilon)\ell_X(\gamma).$$

Thus by possibly increasing the ball B we may assume that $\mathbf{J}(Y,\rho_L) \geq m$ for all $L \geq L_0$ and all $Y \notin B$.

We are left with showing that by possibly increasing L_0, we also have $\mathbf{J}(Y,\rho_L) \geq m$ for all $Y \in B$. However, this follows once more from the estimate (7). Namely, let $Y \in B$. By Proposition 4.3, we know that there exists a constant $\kappa > 0$ such that

$$\iota(\gamma,c) \geq \kappa(1-\epsilon)\ell_Y(\gamma)\ell_Y(c)$$

for any geodesic γ which is typical for Y.

On the other hand, by compactness of B, there exists a constant $\sigma > 0$ such that $\ell_Y(c) \geq \sigma$ for every $Y \in B$. Then for a geodesic γ which is typical for Y, we have $\ell_Y(\gamma) \leq \frac{1}{\kappa\sigma(1-\epsilon)}\iota(\gamma,c)$. Thus for $L > m/\kappa\sigma(1-\epsilon)C_\sigma$ it holds

$$\ell_{\rho_L}(\gamma)/\ell_Y(\gamma) \geq \kappa\sigma(1-\epsilon)C_\sigma L \geq m$$

which is what we wanted to show. Together with the definition, it shows that $\mathbf{I}(\nu, \rho_L) \to \infty$ for every Fuchsian representation ν.

To show that we also have $\mathbf{J}(\nu, \rho_L) \to \infty$ for all Fuchsian representations it suffices to observe that the entropy of ρ_L is bounded from below by a universal positive constant. To see that this is the case, recall that for each L, the restriction of the representation ρ_L to the free subgroup Λ of $\pi_1(S)$ of all based loops which do not cross through c does not depend on L. In particular, the image of Λ under ρ_L stabilizes a totally geodesic hyperbolic plane in $\mathbb{X}$. As a consequence, for each L the entropy of ρ_L is not smaller than the entropy of the geodesic flow on the bordered surface $S - c$, which is positive as $S - c$ is a hyperbolic surface with geodesic boundary. Together with the control on $\mathbf{I}(\nu, \rho_L)$ established in the beginning of this proof, this implies that $\mathbf{J}(\nu, \rho_L) \to \infty$ $(L \to \infty)$ for any Fuchsian representation ν. $\square$

5. Quantitative convergence of currents

In Section 2.2 we introduced the measure of maximal entropy for Hitchin representations with respect to a Finsler metric. In this section we investigate the behavior of these measures along grafting rays in the Hitchin component. Using the geometric control established in Section 3.3, we compare length functions for representations obtained by Hitchin grafting rays to length functions of the corresponding abstract grafted surfaces, viewed as functions on the unit tangent bundle of the hyperbolic surface S which is the starting point for the grafting, and estimate the entropy of the reparameterized flow. This then leads to the proof of Theorem 1 from the introduction.

The Finsler metric on $\mathbb{X}$ used for the pressure metric is normalized in such a way that its restriction to a hyperbolic plane stabilized by an irreducible representation of $\mathrm{PSL}_2(\mathbb{R})$ coincides with the Riemannian metric of constant curvature -1.

We start with a hyperbolic metric on the closed surface S of genus $g \geq 2$ and choose a simple geodesic multicurve γ^* on S (the grafting locus) with $k \geq 1$ components. For each grafting parameter $z = (z_e)_{e \subset \gamma^*} \subset \mathfrak{a}^k$, denote by ρ_z the Hitchin grafting representation with datum z (see Definition 3.2).

By Proposition 2.8, for each z there exists a positive Hölder continuous function f_z on the unit tangent bundle T^1S of S with the property that for every periodic orbit γ for the geodesic flow Φ^t on T^1S, we have that

$$\ell_{f_z}(\gamma) = \int_\gamma f_z$$

equals the translation length of the conjugacy class determined by the element $\rho_z(\gamma) \in \mathrm{PSL}_d(\mathbb{R})$ with respect to the Finsler metric.

The Hölder continuous function f_z on T^1S determines a reparameterization $\Phi^t_{f_z}$ of the geodesic flow Φ^t on T^1S, whose measure of maximal entropy corresponds to a Φ^t-invariant Gibbs equilibrium state $\nu(z)$ on T^1S. There are several possible normalizations for this equilibrium state. We assume

$\nu(z)$ to be normalized in such a way that

$$\int f_z d\nu(z) = 1 \text{ for all } z. \tag{8}$$

Note that this normalization only depends on the cohomology class of f_z and hence it does not depend on choices. Our main goal is to determine the possible limits of $\nu(z)$ as the cylinder height of every component z_e of z (that is, at every component of the multi-curve γ^*) tends to infinity, and to show that the intersection numbers with γ^* of the geodesic currents $\hat{\nu}(z)$ determined by the measures $\nu(z)$ decay exponentially fast.

By Section **??**, the equilibrium measure of the function $-f_z$ can be described in terms of Patterson–Sullivan measures. Denoting as before by $\mathcal{F}$ the flag variety of $\mathrm{PSL}_d(\mathbb{R})$, recall that for $\zeta, \eta \in \mathcal{F}$ and $x, y \in \mathbb{X}$, the function $b_\zeta^{\mathfrak{F}}(x,y)$ denotes the Busemann cocycle and $\langle \zeta | \eta \rangle_x$ denotes the Gromov product associated to the Finsler metric $\mathfrak{F}$ (see Equations **??** and **??**).

For any non-trivial grafting datum z with nontrivial cylinder height, let $\Xi_z : \partial_\infty \mathbb{H}^2 \to \mathcal{F}$ be the limit map associated to the Hitchin grafting representation ρ_z. Then there exists a family of Patterson Sullivan measures $(\mu_z^x)_{x\in\mathbb{X}}$ on $\partial_\infty \mathbb{H}^2$ such that for all $x, y \in \mathbb{X}$ and $\gamma \in \pi_1(S)$ we have $\mu_z^{\rho_z(\gamma)x} = \gamma_* \mu_z^x$ and

$$\frac{d\mu_z^y}{d\mu_z^x}(\xi) = e^{\delta(z) b^{\mathfrak{F}}_{\Xi_z(\xi)}(x,y)}, \tag{9}$$

where $\delta(z)$ is the *critical exponent* of the group $\rho_z(\pi_1(S))$, or, equivalently, the topological entropy of the reparameterized flow $\Phi^t_{f_z}$ on T^1S. These measure are unique up to a global multiplicative positive constant. Note that the equality 9 is immediate from the fact that the topological entropy of the reparameterized flow equals the expansion rate of the conditional measures on strong unstable manifolds for its unique measure of maximal entropy, which in turn equals the critical exponent by construction.

Finally there is a choice of normalization for the measures μ_z^x such that $\nu(z)$ is the quotient under $\pi_1(S)$ of the measure

$$e^{\delta(z)\langle \Xi_z(\xi) | \Xi_z(\eta)\rangle_x} d\mu_z^x(\xi) d\mu_z^x(\eta) dt \tag{10}$$

on $\partial_\infty \mathbb{H}^2 \times \partial_\infty \mathbb{H}^2 \times \mathbb{R}$. Note that the measures μ_z^x are finite but in general they are not probability measures, instead their normalization is determined by the normalization of $\nu(z)$.

Since $\nu(z)$ and hence the geodesic current $\hat{\nu}(z)$ defined by $\nu(z)$ depends continuously (in fact, analytically) on z by Proposition 2.5, we can estimate the intersection $\iota(\hat{\nu}(z), \gamma^*)$ (here γ^* is viewed as a Dirac current) using continuity of the intersection form on the space of currents. However, although the space of projective currents, equipped with the weak*-topology, is compact since this is the case for the space of Φ^t-invariant Borel probability measures on T^1S where Φ^t is the geodesic flow, the family $\hat{\nu}(z)$ may not be precompact as the corresponding Φ^t-invariant measure $\nu(z)$ on T^1S is determined by the normalization (8) and in general is not a probability measure. We shall use the Patterson–Sullivan measures to control the total volume of $\nu(z)$ and overcome this difficulty.

5.1. **The entropy of the subsurfaces.** The geodesic multicurve γ^* decomposes S into (closed) complementary components $S_1, \ldots, S_k$. For each $i \le k$ we denote by $K_i \subset T^1S$ the set of all unit tangent vectors $v \in T^1S_i$ with the property that $\Phi^t v \in T^1S_i$ for all $t \in \mathbb{R}$.

Lemma 5.1. *For each i the set K_i is compact and Φ^t-invariant.*

Proof. The set K_i is clearly Φ^t-invariant and closed by continuity of Φ^t, hence it is compact. □

Since S is a closed hyperbolic surface, the geodesic flow Φ^t on T^1S is an Anosov flow and hence for each i its restriction to the compact invariant set K_i is an Axiom A flow.

The preimage of the geodesic multicurve γ^* in the universal covering $\mathbb{H}^2$ of S consists of a countable union of pairwise disjoint geodesic lines. These geodesic lines decompose $\mathbb{H}^2$ into countably many connected components which are permuted by the action of the fundamental group $\pi_1(S)$ of S. If we denote by $\Gamma \subset \pi_1(S)$ the stabilizer of one of these components $\tilde{\Sigma}$, which is a convex subsurface of $\mathbb{H}^2$ with geodesic boundary, then Γ acts properly and cocompactly on $\tilde{\Sigma}$, with quotient one of the components S_i of $S - \gamma^*$. Thus Γ is a non-elementary convex cocompact Fuchsian group.

The *limit set*, that is, the set of accumulation points of a Γ-orbit $\Gamma x \subset \mathbb{H}^2$ ($x \in \tilde{\Sigma}$) in $\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$, is a Γ-invariant Cantor subset Λ of $\partial_\infty \mathbb{H}^2$. The quotient under the action of Γ of the set of all unit tangent vectors of geodesics with both endpoints in Λ has a natural identification with the invariant set $K_i \subset T^1S$. In particular, the restriction of Φ^t to K_i is topologically transitive. Its topological entropy equals the Hausdorff dimension $\delta_i \in (0,1)$ of Λ [34].

Write $K = \cup_i K_i$ and let $\delta > 0$ be the topological entropy of $\Phi^t_{|K}$. We have $\delta = \max\{\delta_i \mid i \le k\}$. Recall that $\delta(z)$ denotes the topological entropy of the reparameterized flow $\Phi^t_{f_z}$ on T^1S and equals the critical exponent of the group $\rho_z(\pi_1(S)) \subset \mathrm{PSL}_d(\mathbb{R})$.

We have bounds on $\delta(z)$. The upper bound is very general:

Theorem 5.2 (Corollary 1.4 of [32])**.** *There is a constant $m > 0$ that bounds from above the entropy of any Hitchin representation.*

The lower bound depends on the choice of the grafting locus γ^* and the hyperbolic metric on S, and its proof is classical.

Lemma 5.3 (e.g. Theorem 4.1 of [7])**.** *$\delta(z) \in (\delta, m]$ for all z, where $m > \delta$ is the universal constant from the above Theorem 5.2.*

Proof. By definition of a Hitchin grafting representation, the image $\rho_z(\Gamma)$ under ρ_z of the fundamental group Γ of any component of $S-\gamma^*$ is conjugate to its image under ρ, and hence has the same critical exponent. Suppose we picked the component with largest critical exponent, namely δ.

Then $\rho_z(\Gamma)$ is also Anosov (Γ is quasi-convex in $\pi_1(S)$) and its limit set is a proper subset of that of $\rho_z(\pi_1(S))$ so by Theorem 4.1 of [7] it has a strictly smaller critical exponent. Thus the critical exponent of $\rho_z(\pi_1(S))$ is bigger than δ. ☐

Let $h_{\rm top}(\Psi^t)$ be the topological entropy of a flow Ψ^t on a compact space; thus $\delta = h_{\rm top}(\Phi^t_{|K})$. A *measure of maximal entropy* for $\Phi^t_{|K}$ is an invariant probability measure μ with $h_\mu = \delta$.

Since $\Phi^t_{|K_i}$ is a topologically transitive Axiom A flow and K_i is compact, it admits a unique measure ν_i of maximal entropy. The measure ν_i is a Gibbs equilibrium state for $\Phi^t_{|K_i}$ with respect to the constant function 1, and it can be obtained from a Patterson Sullivan construction [34]. The following well known fact will be useful later on.

Lemma 5.4. *A measure of maximal entropy for $\Phi^t_{|K}$ exists. It is unique if and only if there exists a number $i \leq k$ such that $h_{\rm top}(\Phi^t_{|K_i}) > \max\{h_{\rm top}(\Phi^t_{|K_j}) \mid j \neq i\}$. In this case the measure of maximal entropy is supported in K_i.*

Proof. Write again $K = \cup_i K_i$. The function which associates to a $\Phi^t_{|K}$-invariant probability measure μ its entropy h_μ is affine: for μ, η and $s \in (0,1)$ we have $h_{s\mu+(1-s)\eta} = sh_\mu + (1-s)h_\eta$.

The topologically transitive invariant subsets $K_i \subset K$ intersect at most along a finite number of periodic orbits. As a consequence, any Φ^t-invariant probability measure μ on K can be decomposed as $\mu = \sum_i \mu_i$ where μ_i is supported in K_i. The decomposition is unique if the μ-mass of any periodic orbit for Φ^t which projects to a component of γ^* vanishes.

Since $\Phi^t_{|K_i}$ is a topologically transitive axiom A flow, it admits a unique measure ν_i of maximal entropy. Then we have $h_{\nu_i} = h_{\rm top}(\Phi^t_{|K_i})$. Let $\mu = \sum_i \mu_i$ be any Φ^t-invariant Borel probability measure on K. Let $s_i = \mu_i(K_i)$; then $\sum_i s_i = 1$ and

$$h_\mu = \sum_i s_i h_{\mu_i} \leq \sum_i s_i h_{\rm top}(\Phi^t_{|K_i}) \leq \delta$$

with equality if and only if $s_j = 0$ for all j such that $h_{\rm top}(\Phi^t_{|K_j}) < \delta$, and $\mu_j = \nu_j$ if $s_j > 0$. In particular, a measure of maximal entropy exists, and if there exists a unique $i \leq k$ such that $h_{\rm top}(\Phi^t_{|K_i}) = \delta$, then such a measure is unique and coincides with ν_i. ☐

5.2. **The total mass of the equilibrium state.** For the fixed hyperbolic metric on S with unit tangent bundle T^1S and geodesic flow Φ^t denote by $\nu^1(z)$ the Φ^t-invariant probability measure on T^1S which is a multiple of $\nu(z)$. It turns out that the two normalisations $\nu(z)$ and $\nu^1(z)$ for the equilibrium states are comparable independently of z, as soon as the grafting datum z is taken in $\ker \alpha_0$ where α_0 is the linear functional which determines the Finsler norm of the tangent of a Riemannian geodesic in X which is invariant under $\rho(\gamma^*)$ (or a component of $\rho(\gamma^*)$).

Lemma 5.5. *For any $\sigma > 0$ there exists a constant $C > 0$ such that if the length of each component of $\gamma^* \subset S$ is at most σ, then for any grafting parameter $z \subset \ker \alpha_0^\perp$,*

$$C^{-1} \leq \|\nu(z)\| = \nu(z)(T^1S) \leq C.$$

Proof. Put $\nu^1(z) = \frac{\nu(z)}{\|\nu(z)\|}$ so that $\nu^1(z)$ is a probability measure on T^1S. Then $\|\nu(z)\| = (\int f_z d\nu^1(z))^{-1}$ since by equation (8), $\nu(z)$ was normalized so that $\int f_z d\nu(z) = 1$.

By definition of the equilibrium state of $-f_z$ and the fact that the entropy of the reparameterized flow $\Phi^t_{f_z}$ equals $\delta(z)$, we have

$$\int f_z d\nu^1(z) = \frac{h_{\nu^1(z)}}{\delta(z)}. \tag{11}$$

Since $h_{\nu^1(z)} \leq 1$ (the topological entropy of Φ^t is 1, and is greater than or equal to the entropy of any invariant measure) and $\delta(z) > \delta$ by Lemma 5.3, it holds $\int f_z d\nu^1(z) \leq \frac{1}{\delta}$. It remains to get a lower bound.

By Theorem 3.5, we have

$$\int f_z \frac{d\zeta}{\ell(\zeta)} \geq \left(1 + \frac{C}{L+1}\right)^{-1},$$

for any $\zeta \in \pi_1(S)$, represented by a periodic orbit for Φ^t of length $\ell(\gamma)$, and where $L \geq 0$ is any lower bound on the heights of the cylinders added along the components of γ^* to construct S_z (see Definition 3.1).

Then by density of the convex hull of currents supported on closed geodesics in the space of all currents, we get

$$\int f_z d\nu^1(z) \geq (1 + C)^{-1}. \qquad \square$$

Corollary. *$\mathbf{J}(\rho_i, \rho)$ is uniformly bounded, independent of i.*

As a consequence of the above discussion, we can take a weak limit of the Φ^t-invariant measures $\nu(z)$ as $z \to \infty$.

Lemma 5.6. *A weak limit of the measures $\nu(z)$ as $z \to \infty$ is a measure of maximal entropy for the restriction of Φ^t to $S \setminus c$.*

Proof. We first claim that any weak limit is supported on the unit tangent bundle of $S \setminus c$. Namely, otherwise by ergodicity, there is a subsequence and a number $\epsilon > 0$ and a number δ such that the following holds true. Let V_δ be the set of all unit tangent vectors v with footpoint on c and so that the angle between the tangent of v and the tangent of c is bounded from below by δ. Then $\nu(z)(\{\Phi^t w \mid -\epsilon < t < \epsilon, w \in V_\delta\} \geq \tau$ where $\tau > 0$ and all other constants are independent of z and where σ is sufficiently small that the footpoints of unit vectors in this set are contained in a collar neighborhood of c.

But up to changing the functions f_z by Hölder equivalent ones, this implies that $\int f_z d\nu(z) \to \infty$ which is a contradiction. □

Now consider intersection numbers. For a hyperbolic metric ρ we have $\mathbf{J}(\mu, \nu) = \iota(\mu, \nu(z)) \to \infty$ as $z \to \infty$ by the results in Section . On the other hand, by [28], we know that a Hitchin representation is positive ratioed, so the intersection current $\sigma(z)$ is well defined.

Proposition 5.7. *$\iota(\nu(z), \nu(z)) \to \infty$ as $z \to \infty$.*

Proof. By the above and the results of Section 4, we have $\iota(\mu, \nu(z)) \to \infty$ as $z \to \infty$. But $\iota(\mu, \nu) = \int f d\sigma(z)$ where f is the length function of the hyperbolic metric. Since $f_z \geq f$, the same holds true for the $\int f_z d\sigma(z) = \iota(\nu(z), \nu(z))$. □

5.3. **Convergence of currents.** Recall $\delta > 0$ is the topological entropy of $\Phi^t_{|K}$. The following is the main result of this section.

Proposition 5.8. *Let $L_i \to \infty$ and let $\rho_i = \rho_{z_i}$ be a sequence of Hitchin representations obtained by Hitchin grafting of a Fuchsian representation at the simple geodesic multicurve γ^* with cylinder heights bounded from below by L_i. Then $\delta(z_i) \to \delta$, and up to passing to a subsequence, the equilibrium measures $\nu_i = \nu(z_i)$ converge weakly to a measure of maximal entropy for $\Phi^t|K$.*

Proof. Recall that $f_i = f_{z_i}$ denotes a positive Hölder continuous potential on T^1S whose periods are the Finsler translation lengths of the elements of $\rho_i(\pi_1(S))$.

Up to passing to a subsequence, we may assume that the Φ^t-invariant probability measures $\nu_i^1 = \nu_i/||\nu_i||$ converges weakly to a Φ^t-invariant probability measure ν on T^1S. By Lemma 5.5, we may also assume that the geodesic currents $\hat{\nu}(z)$ converge weakly to a current $\hat{\nu}$ which is a positive multiple of the current defined by ν.

By Proposition **??**, we have $\iota(\hat{\nu}, \gamma^*) = 0$ and hence the limit measure ν must be supported on K. By Lemma 5.4, we are thus left with showing that $h_\nu \geq \delta$.

From Lemma 5.5 we have $\delta(z_i) \in (\delta, m]$ for any i. Recall from (11) that

$$h_{\nu_i^1} = \delta(z_i) \int f_i d\nu_i^1. \tag{12}$$

By Theorem 3.5, it holds

$$\int f_i \frac{d\eta}{\ell_S(\eta)} \geq \left(1 + \frac{C}{L_i + 1}\right)^{-1}$$

for any $\eta \in \pi_1(S)$, and hence since the Φ^t-invariant Borel probability measures supported on closed geodesics are weak*-dense in the space of all Φ^t-invariant Borel probability measures, we get

$$\int f_i d\nu_i^1 \geq \left(1 + \frac{C}{L_i + 1}\right)^{-1},$$

and hence

$$\liminf_{i \to \infty} h_{\nu_i} \geq \liminf_{i \to \infty} \delta(z_i) \geq \delta.$$

Since the entropy function is lower semi-continuous, we conclude that $h_\nu \geq \delta$. As ν is supported in K, this implies that indeed, ν is a measure of maximal entropy for the restriction of Φ^t to K by Lemma 5.4. □

Using the above results we are now ready to complete the proof of Theorem 1 from the introduction.

Proof of Theorem 1. Part (3) of Theorem 1 was shown in Section 3.3, so we are left with showing part (1) and (2). Let $\gamma^* \subset S$ be a pair of pants decomposition of $S_1 = S - S_0$ that contains $\partial S_0 = \partial S_1$. The metric h on S_0 prescribes lengths for the components of γ^* in ∂S_0.

Since no component of S_1 is a pair of pants, every pair of pants in $S_1 - \gamma^*$ has a boundary component in $\gamma^* - \partial S_1$. By Proposition A.1, one can choose lengths large enough for each component of $\gamma^* - \partial S_1$ such that each pair of pants of $S_1 - \gamma^*$ has entropy very close to zero, and in particular strictly smaller than the entropy of S_0.

Then by Hitchin grafting along γ^* flat cylinders with bigger and bigger heights, we get a sequence $\rho_i = \rho_{z_i}$ of Hitchin representations satisfying the first two statement of Theorem 1, according to Proposition 5.8. □

5.4. **Proof of Theorem 2.** As mentioned in the introduction, in [25, 24], Loftin constructed a natural bordification of the space of Hitchin representations $\mathrm{Hit}_3(S)$ of a closed surface S, called the augmented Hitchin space $\mathrm{Hit}_3^{\mathrm{aug}}(S)$, which extends the augmented Teichmüller space. This construction applies more generally to noncompact finite type surfaces and their moduli spaces of convex projective structures. Our goal in this section is to relate Loftin's bordification with our grafting procedure. More precisely we want to show that, starting with a Fuchsian representation and grafting it with grafting parameter going to infinity in a specific direction, the resulting family of Hitchin representations will converge to a point in Loftin's bordification.

Let S be a connected surface of finite type, seen as a closed surface with punctures. Recall that a projective structure is an atlas of charts on S into the projective plane such that the change of charts are projective transformations. To such a structure can be associated a holonomy representation of the fundamental group into the group of projective transformations $\mathrm{PSL}_3(\mathbb{R})$, and a holonomy-equivariant developing map from the universal cover $\tilde{S}$ into the projective plane. A projective structure is called convex if the developing map is injective and its image is properly convex (convex and bounded in some affine chart), which implies the holonomy representation is faithful with discrete image. In this case, the projective structure is completely determined by the data of the holonomy representation and the image of the developing map by Proposition 2.5 of [26].

The moduli space of convex projective structures $\mathcal{C}(S)$ can be described as the quotient under the action of $\mathrm{PSL}_3(\mathbb{R})$ of the set of pairs (Ω, ρ), where $\Omega \subset \mathbb{RP}^2$ is open and properly convex and ρ is a discrete and faithful representation of $\pi_1(S)$ into $\mathrm{PSL}_3(\mathbb{R})$ that preserves Ω. It is topologized so that $(\Omega_n, \rho_n) \to (\Omega, \rho)$ if $\Omega_n \to \Omega$ for the Hausdorff topology and $\rho_n \to \rho$ on a set of generators (up to the action of $\mathrm{PSL}_3(\mathbb{R})$).

The projective structures around punctures can be classified, and in particular the conjugacy class of the holonomy of a curve enclosing a puncture can be of three types: parabolic $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, quasi-hyperbolic $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 1 \\ 0 & 0 & \mu \end{pmatrix}$ or hyperbolic $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}$ (where λ, μ, ν are distinct). As explained in the Appendix A of [26], the projective structure around the puncture is determined by this holonomy in the parabolic and quasi-hyperbolic cases. However in the hyperbolic case there are many structures with the same holonomy. In particular any such structure can be deformed locally with out changing the holonomy by a bulging procedure (one can "inflate" or "deflate" the structure near the puncture). The two special degenerate structures obtained by inflating or deflating to infinity any other structure are called respectively bulge $+\infty$ and bulge $-\infty$. See e.g. Figure 4 of [26]. To conclude, for any pair (Ω, ρ), the convex set Ω is determined by ρ and the projective structure around punctures of hyperbolic type.

In particular, if S is closed then every point of $\mathcal{C}(S)$ is determined by the holonomy representation. By work of Choi and Goldman [12, 8], $\mathcal{C}(S)$ is connected, open and closed as a subset of the set of representations of $\pi_1(S)$, and it contains the representations coming from hyperbolic structures, so $\mathcal{C}(S) = \mathrm{Hit}_3(S)$.

To define the augmented Hitchin space, Loftin first defines admissible convex projective structures by allowing only bulge $\pm\infty$ structures near the punctures of hyperbolic type. Then $\mathrm{Hit}_3^{\mathrm{aug}}(S)$ is defined as the set, over all multicurves $\mathcal{D} \subset S$, of admissible convex projective structures $(\Omega_1, \rho_1), \ldots, (\Omega_k, \rho_k)$ on the connected components $S_1, \ldots, S_k$ of $S - \mathcal{D}$ that satisfy some compatibility conditions between the pairs of ends corresponding to the same curve $\gamma \subset \mathcal{D}$: they have the same holonomy and a bulge $+\infty$ end must face a bulge $-\infty$ end. It is further topologised so that $(\Omega^{(n)}, \rho^{(n)}) \in \mathrm{Hit}(S)$ converge to $((\Omega_1, \rho_1), \ldots, (\Omega_k, \rho_k))$ in the boundary if $(\Omega^{(n)}, \rho^{(n)}_{|\pi_1 S_i}) \to (\Omega_i, \rho_i)$ for every i (up to the action of $\mathrm{PSL}_3(\mathbb{R})$).

Let us now relate the above construction with the algebraic bending deformation of a Fuchsian representation ρ along a multicurve $\mathcal{D} \subset S$, as recalled in Section 3.1: it was defined by partially conjugating the image by ρ of the fundamental groups of the connected components $S_1, \ldots, S_k$ of $S - \mathcal{D}$. We gave in [2] and 3.2 a geometric interpretation of this deformation, inside the symmetric space of $\mathrm{PSL}_3(\mathbb{R})$, in terms of grafting a flat cylinder along the multicurve $\mathcal{D}$. Suppose now that all the grafting parameters (which are vectors of the Cartan subspace) are parallel to the special direction $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then there is another geometric interpretation of bending due to Goldman [12, §5.5] using convex projective geometry: bending induces a deformation of the underlying convex projective structure called *bulging*, which is the same procedure as the local surgery around punctures mentioned previously. The idea is the same as before (when $k = 2$ and $\mathcal{D}$ has only one curve): suppose $\rho_z(\pi_1(S_1)) = \rho(\pi_1(S_1))$ is unchanged and $\rho_z(\pi_1(S_2)) = e^z \rho(\pi_1(S_2)) e^{-z}$. The ρ-invariant convex domain $\Omega \subset \mathbb{RP}^2$ is made of a tree of infinitely many copies of universal covers of S_1 and S_2, each copy being invariant under a conjugate of $\rho(\pi_1(S_1))$ or $\rho(\pi_1(S_2))$. The ρ_z-invariant convex domain Ω_z is then produced by deforming each of these copies using e^z and e^{-z} and conjugates of them: e.g. if Ω_2 is a $\rho(\pi_1(S_2))$-invariant copy of $\widetilde{S}_2$ then $e^z \Omega_2$ is $\rho_z(\pi_1(S_2))$-invariant. One can see that e^z acts by inflating Ω_2, without disconnecting it from the adjacent copies of $\widetilde{S}_1$ (so there is no need to graft a flat cylinder as in the symmetric space). The following fact is an immediate consequence of Goldman's work and the above definition of Loftin's bordification. Fix a grafting parameter z parallel to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Fact: For any $t > 0$ let $[\rho_t] \in \mathrm{Hit}_3(S)$ be obtained by grafting ρ along $\mathcal{D}$ with parameter tz. Then as t goes to infinity, $[\rho_t]$ converges to $[(\Omega_1, \eta_1), \ldots, (\Omega_k, \eta_k)] \in \mathrm{Hit}_3^{\mathrm{aug}}(S)$ (projective structures on $S_1, \ldots, S_k$) such that the projective structures near the two ends associated to a $\gamma \subset \mathcal{D}$ are of hyperbolic type with bulge $+\infty$ and $-\infty$ respectively, and the holonomies η_i are the restrictions of ρ to $\pi_1(S_i)$.

To prove Theorem 2, we consider the case where S is cut into two subsurfaces S_1, S_2 such that the entropy of η_1 is strictly greater than that of η_2. We slightly perturb ρ into $(\rho^s)_{-\epsilon \le s \le \epsilon}$ so that $\eta_1^s = \eta_1$ for any s with entropy still greater than that of η_2^s, and η_2^s and η_2^σ are not conjugate for $s \neq \sigma$. Now we graft, and by Theorem **??** the pressure length of $(\rho_t^s)_{-\epsilon \le s \le \epsilon}$ goes to zero as t diverges, which implies all $(\rho_t^s)_{t \to \infty}$ converge to the same point of the pressure metric completion of $\mathrm{Hit}_3(S)$, independent of s. However by the above fact they converge to different points of Loftin's augmented Hitchin space. Heuristically, the pressure metric is not fine enough to distinguish points in $\mathrm{Hit}_3^{\mathrm{aug}}(S)$, because it focuses too much on the component with bigger entropy and can only see changes there.

Another interesting remark can be made about another description of the augmented Hitchin space (which is in fact Loftin's original definition), in terms of cubic differentials. Recall that by independent work of Labourie [21] and Loftin [23], there is a vector bundle structure $\pi : \mathrm{Hit}_3(S) \to \mathcal{T}(S)$ such that the fiber above a point of $\mathcal{T}(S)$, seen as a (marked) complex structure on S, is the vector space of holomorphic cubic differentials on S. It turns out this vector bundle structure extends to $\pi : \mathrm{Hit}_3^{\mathrm{aug}}(S) \to \mathcal{T}^{\mathrm{aug}}(S)$. Moreover, using the notations from the above fact and denoting the limit of $[\rho_t]$ as $t \to \infty$ by $[\rho_\infty] = [(\Omega_1, \eta_1), \dots, (\Omega_k, \eta_k)]$, it follows from Theorem 12 of [24] that the projection $\pi[\rho_\infty] \in \mathcal{T}^{\mathrm{aug}}(S)$ is the noded hyperbolic surface obtained by pinching to zero the multicurve $\mathcal{D} \subset S$.

Hence for t large the Hitchin grafting representation ρ_t, which we think of in this paper as the hyperbolic structure ρ where we grafted long flat cylinder along $\mathcal{D}$, naturally stands above another hyperbolic structure $\pi(\rho_t)$ on S with long and narrow hyperbolic collars around $\mathcal{D}$. Since pinching a curve in $\mathcal{T}(S)$ is a finite length surgery for the Weil–Petersson metric, it seems likely that $(\pi[\rho_t])_{t>0}$ has finite length. As $([\rho_t])_{t>0}$ also has finite length, for any t the pressure distance from $\pi[\rho_t]$ to $[\rho_t]$ is bounded independently of t. Moreover, there is a natural straight-line path between these two points, since $[\rho_t]$ lies in the fiber above $\pi[\rho_t]$, which is a vector space. A natural question is then: is the pressure length of this path bounded above independently of t?

Appendix A. Entropy of hyperbolic surfaces with boundary

The goal of this appendix is to establish some basic results on the entropy of hyperbolic surfaces with boundary which is used in Section **??** but for which we did not find a reference.

To begin with, consider a sphere Σ with three boundary components, and let $S_{a,b,c}$ be a the metric on Σ as a hyperbolic pair of pants with geodesic boundary of length a, b and c. Our goal is to show.

Proposition A.1. *There exists a function f depending on Σ ($\partial\Sigma \neq \emptyset$) with the following property. If Σ is a pair of pants, two boundary components of S have length at least $\sigma > 0$ and the third at least $\ell \geq \sigma$, then $\delta(S) \leq f(\sigma, \ell)$ with $f(\sigma, \ell) \to 0$ for fixed $\sigma > 0$ as $\ell \to \infty$.*

We use the notations from [29], where the authors give some control on the entropy of a hyperbolic surface using the lengths of the small curves on the surface. Denote by $L(S)$ the systole of S, that is, the length of the shortest closed geodesic in S. Denote by $K(S)$ the length of the shortest closed geodesic in $S \setminus \partial S$ ($K(S)$ is more complicated to define when S is not a pair of pants). Also let $\delta(S)$ be the critical exponent of S.

Theorem A.2 (Particular case of Theorem 1.4 of [29])**.** *There exists a constant $C > 0$ for which we have*

$$\frac{1}{4}\log(2) \leq \delta(S)K(S) \leq C\left(\log(4) + 1 + \log\left(1 + \frac{1}{x_0}\right)\right)$$

where x_0 is the unique positive solution of the equation $(1+x)^{\left\lceil \frac{K(S)}{L(S)} - 1 \right\rceil} x = 1$.

Lemma A.3. *Let S be a pair of pants with boundary lengths a, b, c. Then $K(S) \geq \max(a, b, c)$.*

Proof. Up to reordering we may assume $\max(a,b,c) = c$. The surface S is obtained by gluing two isometric right-angled hyperbolic hexagons $H_1 = H, H_2$ along three nonadjacent sides, such that the three other sides have lengths $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$. In particular, there is a natural projection $\pi : S \to H$. Let $\bar{A}, \bar{B}, \bar{C}$ be the sides of H which are glued, so that the hyperbolic distance from $\bar{B}$ to $\bar{C}$ is $a/2$, the distance from $\bar{C}$ to $\bar{A}$ is $b/2$, and the distance from $\bar{A}$ to $\bar{B}$ is $c/2$.

Let γ be a closed geodesic in $S \setminus \partial S$, and let us check it has length at least c. Note that $\pi(\gamma) \subset H$ is a concatenation of geodesics between the sides $\bar{A}, \bar{B}, \bar{C}$. This path has to intersect all three sides, for if it was alternating between only two sides, then γ is freely homotopic to a multiple of the boundary curve of S between these two sides.

Say γ starts on the side $\bar{A}$ at some point x, then travels until it hits $\bar{B}$ at some point y (maybe bouncing off $\bar{C}$ and $\bar{A}$ in between), and then comes back to x. The first part of the path from x to y must have length at least the distance from $\bar{A}$ to $\bar{B}$, which is $c/2$, and similarly the second part has length at least $c/2$ too, so in total γ has length at least c. □

Proof of A.1. Let $(a_n)_n, (b_n)_n, (c_n)_n$ be three sequences in $\mathbb{R}^+$ so that a_n and b_n are bounded away from zero, and c_n tends to to infinity with n. Let $S_n = S_{a_n,b_n,c_n}$ be the pair of pants with boundary lengths a_n, b_n, c_n. By Lemma A.3, $K(S_n)$ tends to infinity with n.

By assumption, $L(S_n)$ is bounded away from zero. So up to passing to a subsequence, we can assume that $\frac{K(S_n)}{L(S_n)}$ converges to $y \in (0, +\infty]$. If $y < +\infty$, then the solutions x_n of $(1+x)^{\left\lceil \frac{K(S_n)}{L(S_n)} - 1 \right\rceil} x = 1$ remain bounded away from zero. So $C\left(\log(4) + 1 + \log\left(1 + \frac{1}{x_n}\right)\right)$ is bounded, and $\delta(S_n) \leq \frac{cste}{K(S_n)}$ goes to zero.

If $y = +\infty$, then x_n goes to zero, and a simple analysis yields that $-\frac{\log(x_n)}{x_n}$ is equivalent to $\frac{K(S_n)}{L(S_n)}$. It follows that

$$\delta(S_n)K(S_n) \leq C\left(\log(4) + 1 + \log\left(1 + \frac{1}{x_n}\right)\right) \tag{13}$$

$$\leq \mathrm{Cst} \cdot x_n \frac{K(S_n)}{L(S_n)} \tag{14}$$

$$\text{and hence} \quad \delta(S_n) \leq \mathrm{Cst} \cdot \frac{x_n}{L(S_n)} \xrightarrow[n \to 0]{} 0 \tag{15}$$

□

References

[1] J. Audibert and M. Zshornack. *Rational approximation for Hitchin representations.* 2023. DOI: 10.48550/ARXIV.2310.15121. arXiv: 2310.15121. URL: https://arxiv.org/abs/2310.15121.

[2] P.L. Blayac, U. Hamenstädt, and T. Marty. *Hitchin grafting representations I: Geometry.* 2025.

[3] J. Bochi, R. Potrie, and A. Sambarino. "Anosov representations and dominated splittings". In: *J. Eur. Math. Soc. (JEMS)* 21.11 (2019), pp. 3343–3414. ISSN: 1435-9855,1435-9863. DOI: 10.4171/JEMS/905. URL: https://doi.org/10.4171/JEMS/905.

[4] F. Bonahon. "The geometry of Teichmüller space via geodesic currents". In: *Inventiones Mathematicae* 92.1 (Feb. 1988), pp. 139–162. DOI: 10.1007/bf01393996. URL: https://doi.org/10.1007/bf01393996.

[5] M. Bridgeman et al. "The pressure metric for Anosov representations". In: *Geometric and Functional Analysis* 25.4 (June 2015), pp. 1089–1179. DOI: 10.1007/s00039-015-0333-8. URL: https://doi.org/10.1007/s00039-015-0333-8.

[6] M. Burger et al. "Currents, systoles, and compactifications of character varieties". In: *Proceedings of the London Mathematical Society* 123.6 (Aug. 2021), pp. 565–596. ISSN: 1460-244X. DOI: 10.1112/plms.12419. URL: http://dx.doi.org/10.1112/plms.12419.

[7] R. Canary, T. Zhang, and A. Zimmer. "Patterson-Sullivan measures for transverse subgroups". In: *Journal of Modern Dynamics* 20 (2024), pp. 319–377. DOI: 10.2307/120995.

[8] S. Choi and W. M. Goldman. "Convex real projective structures on closed surfaces are closed". In: *Proc. Amer. Math. Soc.* 118.2 (1993), pp. 657–661.

[9] V. Fock and A. Goncharov. "Moduli spaces of local systems and higher Teichmüller theory". In: *Publications mathématiques de l'IHÉS* 103.103 (June 2006), pp. 1–211. ISSN: 0073-8301. DOI: 10.1007/s10240-006-0039-4. URL: https://doi.org/10.1007/s10240-006-0039-4.

[10] P. Foulon and I. Kim. *Topological Entropy and bulging deformation of real projective structures on surface.* 2016. arXiv: 1608.06799.

[11] W. Goldman. "Invariant functions on Lie groups and Hamiltonian flows of surface group representations". In: *Inventiones Mathematicae* 85.2 (1986), pp. 263–302. ISSN: 1432-1297. DOI: 10.1007/bf01389091. URL: http://dx.doi.org/10.1007/BF01389091.

[12] W. M. Goldman. "Convex real projective structures on compact surfaces". In: *J. Diff. Geom* 31.3 (1990), pp. 791–845.

[13] F. Guéritaud et al. "Anosov representations and proper actions". In: *Geom. Topol.* 21.1 (2017), pp. 485–584. ISSN: 1465-3060,1364-0380. DOI: 10.2140/gt.2017.21.485. URL: https://doi.org/10.2140/gt.2017.21.485.

[14] O. Guichard and A. Wienhard. "Anosov representations: domains of discontinuity and applications". In: *Invent. Math.* 190.2 (2012), pp. 357–438. ISSN: 0020-9910,1432-1297. DOI: 10.1007/s00222-012-0382-7. URL: https://doi.org/10.1007/s00222-012-0382-7.

[15] N. J. Hitchin. "Lie groups and Teichmüller space". English. In: *Topology* 31.3 (1992), pp. 449–473. ISSN: 0040-9383. DOI: 10.1016/0040-9383(92)90044-I.

[16] M. Kapovich and B. Leeb. "Finsler bordifications of symmetric and certain locally symmetric spaces". In: *Geom. Topol.* 22.5 (2018), pp. 2533–2646. ISSN: 1465-3060,1364-0380. DOI: 10.2140/gt.2018.22.2533. URL: https://doi.org/10.2140/gt.2018.22.2533.

[17] M. Kapovich, B. Leeb, and J. Porti. "Anosov subgroups: dynamical and geometric characterizations". In: *Eur. J. Math.* 3.4 (2017), pp. 808–898. ISSN: 2199-675X,2199-6768. DOI: 10.1007/s40879-017-0192-y. URL: https://doi.org/10.1007/s40879-017-0192-y.

[18] F. Kassel. *Discrete subgroups of semisimple Lie groups, beyond lattices.* 2024. arXiv: 2402.16833.

[19] F. Kassel and R. Potrie. "Eigenvalue gaps for hyperbolic groups and semigroups". In: *J. Mod. Dyn.* 18 (2022), pp. 161–208. ISSN: 1930-5311,1930-532X. DOI: 10.3934/jmd.2022008. URL: https://doi.org/10.3934/jmd.2022008.

[20] F. Labourie. "Anosov flows, surface groups and curves in projective space". In: *Inventiones mathematicae* 165.1 (Mar. 2006), pp. 51–114. ISSN: 0020-9910. DOI: 10.1007/s00222-005-0487-3. URL: https://doi.org/10.1007/s00222-005-0487-3.

[21] F. Labourie. "Flat projective structures on surfaces and cubic holomorphic differentials". English. In: *Pure Appl. Math. Q.* 3.4 (2007), pp. 1057–1099. ISSN: 1558-8599. DOI: 10.4310/PAMQ.2007.v3.n4.a10.

[22] A. Livšic. "Certain properties of the homology of Y-systems". In: *Mat. Zametki* 10 (1971), pp. 555–564. ISSN: 0025-567X.

[23] J. Loftin. "Affine spheres and convex $\mathbb{RP}^n$-manifolds". English. In: *Am. J. Math.* 123.2 (2001), pp. 255–274. ISSN: 0002-9327. DOI: 10.1353/ajm.2001.0011. URL: muse.jhu.edu/journals/american_journal_of_mathematics/toc/ajm123.2.html.

[24] J. Loftin. "Convex $\mathbb{RP}^2$ structures and cubic differentials under neck separation". English. In: *J. Differ. Geom.* 113.2 (2019), pp. 315–383. ISSN: 0022-040X. DOI: 10.4310/jdg/1571882429.

[25] J. Loftin. "The compactification of the moduli space of convex $\mathbb{RP}^2$ surfaces. I". In: *J. Differ. Geom.* 68.2 (2004), pp. 223–276. ISSN: 0022-040X. DOI: 10.4310/jdg/1115669512.

[26] J. Loftin and T. Zhang. "Coordinates on the augmented moduli space of convex $\mathbb{RP}^2$ structures". In: *J. Lond. Math. Soc., II. Ser.* 104.4 (2021), pp. 1930–1972. ISSN: 0024-6107. DOI: 10.1112/jlms.12488.

[27] G. Margulis. *On Some Aspects of the Theory of Anosov Systems.* Springer Berlin Heidelberg, 2004. ISBN: 9783662090701. DOI: 10.1007/

978-3-662-09070-1. URL: http://dx.doi.org/10.1007/978-3-662-09070-1.

[28] G. Martone and T. Zhang. "Positively ratioed representations". In: *Commentarii Mathematici Helvetici* 94.2 (Apr. 2019), pp. 273–345. ISSN: 1420-8946. DOI: 10.4171/cmh/461. URL: http://dx.doi.org/10.4171/CMH/461.

[29] G. Martone and T. Zhang. "Positively ratioed representations". In: *Commentarii Mathematici Helvetici* 94.2 (Apr. 2019), pp. 273–345. DOI: 10.4171/cmh/461. URL: https://doi.org/10.4171/cmh/461.

[30] X. Nie. "Entropy degeneration of convex projective surfaces". In: *Conform. Geom. Dyn.* 19 (2015), pp. 318–322. ISSN: 1088-4173. DOI: 10.1090/ecgd/286. URL: https://doi.org/10.1090/ecgd/286.

[31] Charles Ouyang and Andrea Tamburelli. "Limits of Blaschke metrics". In: *Duke Math. J.* 170.8 (2021), pp. 1683–1722. ISSN: 0012-7094,1547-7398. DOI: 10.1215/00127094-2021-0027. URL: https://doi.org/10.1215/00127094-2021-0027.

[32] R. Potrie and A. Sambarino. "Eigenvalues and entropy of a Hitchin representation". In: *Invent. Math.* 209.3 (2017), pp. 885–925. ISSN: 0020-9910,1432-1297. DOI: 10.1007/s00222-017-0721-9. URL: https://doi.org/10.1007/s00222-017-0721-9.

[33] A. Sambarino. "Quantitative properties of convex representations". In: *Comment. Math. Helv.* 89.2 (2014), pp. 443–488. ISSN: 0010-2571,1420-8946. DOI: 10.4171/CMH/324. URL: https://doi.org/10.4171/CMH/324.

[34] D. Sullivan. "Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups". In: *Acta Mathematica* 153 (1984), pp. 259–277. DOI: 10.1007/bf02392379. URL: https://doi.org/10.1007/bf02392379.

[35] Z. Sun, A. Wienhard, and T. Zhang. "Flows on the $\mathbf{PGL}(\mathbf{V})$-Hitchin Component". In: *Geometric and Functional Analysis* 30.2 (Apr. 2020), pp. 588–692. ISSN: 1420-8970. DOI: 10.1007/s00039-020-00534-4. URL: http://dx.doi.org/10.1007/s00039-020-00534-4.

[36] T. Zhang. "Degeneration of Hitchin representations along internal sequences". In: *Geometric and Functional Analysis* 25.5 (Sept. 2015), pp. 1588–1645. ISSN: 1420-8970. DOI: 10.1007/s00039-015-0342-7. URL: http://dx.doi.org/10.1007/s00039-015-0342-7.

graftinghit

Pierre-Louis Blayac
Université de Strasbourg, IRMA, 7 rue René-Descartes 67084 Strasbourg Cedex, France
e-mail: blayac@unistra.fr

Ursula Hamenstädt
Math. Institut der Univ. Bonn, Endenicher Allee 60, 53115 Bonn, Germany
e-mail: ursula@math.uni-bonn.de

Théo Marty
Institut de mathématiques de Bourgogne, 9 avenue Alain Savary, 21078 Dijon, France
e-mail: theo.marty@u-bourgogne.fr

Andrea Egidio Monti
Max Planck Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
e-mail: amonti@math.uni-bonn.de