HITCHIN GRAFTING REPRESENTATIONS II: DYNAMICS

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ABSTRACT. The Hitchin component of the character variety of representations of a surface group $\pi_1(S)$ into $\mathrm{PSL}_d(\mathbb{R})$ for some $d \geq 3$ can be equipped with a pressure metric whose restriction to the Fuchsian locus equals the Weil-Petersson metric up to a constant factor. We show that if the genus of S is at least 3, then the Fuchsian locus contains quasi-convex subsets of infinite diameter for the Weil-Petersson metric whose diameter for the path metric of the pressure metric is finite. This is established through showing that binfinite paths of bending deformations have controlled bounded length.

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INTRODUCTION

The Teichmüller space $\mathcal{T}(S)$ of a closed oriented surface S of genus $g \geq 2$ is the space of marked hyperbolic structures on S. Equivalently, it can be described as a distinguished component of the space of conjugacy classes of homomorphisms $\pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$, with target the group $\mathrm{PSL}_2(\mathbb{R})$ of orientation preserving isometries of the hyperbolic plane. It was discovered by Hitchin that an analog of the Teichmüller space also exists for conjugacy classes of representations of $\pi_1(S)$ into simple split real Lie groups of higher rank.

The so-called *Hitchin component* $\operatorname{Hit}(S)$ for the target group $\operatorname{PSL}_d(\mathbb{R})$ $(d \geq 3)$ is the component of the *character variety* containing conjugacy classes of discrete representations which factor through an irreducible embedding $\operatorname{PSL}_2(\mathbb{R}) \to \operatorname{PSL}_d(\mathbb{R})$. Hitchin [Hit92] showed that the Hitchin component is homeomorphic to \mathbb{R}^m for some explicit m > 0, and later Labourie [Lab06] and Fock–Goncharov [FG06] independently proved that all representations in the Hitchin component are faithful with discrete image.

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In [BCLS15], a Mod(S)-invariant metric on Hit(S) was introduced, the so-called *pressure metric* (see also [BCLS18; BCS17]). Fix a positive linear functional α_0 on the convex cone of vectors $x = (x_1, \ldots, x_d)$ with $x_1 \ge \cdots \ge x_d$ and $x_1 + \cdots + x_d = 0$, for instance

(1)
$$\alpha_0(x) = (d-1)x_1 + (d-3)x_2 + \dots + (1-d)x_d.$$

The starting point is that α_0 determines a conjugacy invariant translation length function on $\text{PSL}_d(\mathbb{R})$ by applying this functional to the logarithm of the absolute values of the eigenvalues. Hence this yields a length function for the images of $\pi_1(S)$ under a representation in the Hitchin component.

Now let us assume that, after choosing a hyperbolic metric on S, this function can be represented by integration of a Hölder continuous positive function over periodic orbits for the *geodesic flow* Φ^t on the unit tangent bundle T^1S of S. Then one can associate to such a representation the equilibrium state of a positive multiple of the function, chosen to be of vanishing pressure. Using that the pressure is a strictly convex functional, this construction gives rise to a positive semi-definite symmetric bilinear form on the tangent space of the Hitchin component (see (10)). Although this symmetric bilinear form may not be positive definite everywhere, it nevertheless defines a length metric on the component [Sam24].

Using results of [Sam14; BCLS15], we verify in Section 2 that this assumption of representability by Hölder potentials always holds, so that we can talk about the induced pressure metric on Hit(S). Related statements are contained in [BPS19]. Note that while the pressure metric is defined for any Hitchin component, there are several other interesting constructions of Riemannian metrics on the Hitchin component for PSL₃(\mathbb{R}), see [DG96; Li16; KZ17]. It seems unknown whether these metrics coincide with one of the pressure metrics.

As there are many natural choices for the positive linear functional α_0 on the Weyl cone, there are many natural length functions on the Hitchin component, and hence many natural pressure metrics. The linear functional α_0 we shall use for the pressure metric determines a $PSL_d(\mathbb{R})$ -invariant Finsler metric \mathfrak{F} on the symmetric space $\mathbb{X} = PSL_d(\mathbb{R})/PSO_d(\mathbb{R})$, see e.g. Section 5.1 of [KL18], called a *nice* Finsler metric in the sequel. We refer to Section 1 for the precise constraints on the Finsler metrics we shall use, the metric in Equation 1 is an example. The metric \mathfrak{F} depends on α_0 , but its geodesics do not by Lemma 5.10 of [KL18].

The restriction to the Fuchsian locus of each of these metrics is a multiple of the Weil-Petersson metric on the Teichmüller space [Bon88] (relying in an essential way on the article [Wol86], see also Corollary 1.6 of [BCLS15] and [McM08] for an explicit statement). Infinitesimal properties of the pressure metric at the Fuchsian locus were studied in [LW18], and also in [Dai23] in the case d = 3. The large scale geometry of the Weil-Petersson metric on Teichmüller space is quite well understood. In particular, this metric is incomplete [Wol86], and its metric completion $\overline{\mathcal{T}}(S)$, sometimes called augmented Teichmüller space, is a CAT(0) stratified space whose strata are Teichmüller spaces of marked finite area hyperbolic metrics on surfaces obtained from S by replacing some essential simple closed curves by nodes.

For d = 3, Loftin [Lof04] defined an augmented Hitchin space (see also [LZ21]). However, it turns out that unlike in the case of Teichmüller space, this construction is not well

related to geometric properties of the pressure metric. A first instance of this possibility arose in the study of large scale properties of pressure metrics for other kinds of moduli spaces: for instance, for hyperbolic structures with boundary [Xu17], for marked metric graphs [ACR22], and for quasi-Fuchsian representations [FHJZ24]. The following result gives an illustration of this fact in the context of Hitchin representations.

Theorem A. Suppose S has genus at least 3 and consider a pressure metric coming from a nice Finsler metric on the symmetric space $PSL(3, \mathbb{R})/PSO(3)$. Let $Hit_3^{aug}(S)$ be Loftin's augmented Hitchin space and $\overline{Hit}_3(S)$ the pressure metric completion.

Then there exist paths $(x_t)_{t\geq 0}, (y_t)_{t\geq 0} \subset \operatorname{Hit}_3(S)$ converging to two distinct points $x, y \in \operatorname{Hit}_3^{\operatorname{aug}}(S)$, but converging to the same point of $\operatorname{Hit}_3(S)$.

Theorem A is obtained as an application of a large scale geometric study of the path metric for all Hitchin components which we discuss next.

Consider an essential subsurface S_0 of S whose connected boundary ∂S_0 is an essential simple closed curve. Choose a hyperbolic metric X on S and a marked point on the geodesic representing ∂S_0 . Let $\ell > 0$ be the length of ∂S_0 for the metric X, and let $\mathcal{T}(S_0, \ell)$ be the Teichmüller space of marked hyperbolic metrics on S_0 with geodesic boundary of fixed length $\ell > 0$ and one marked point on ∂S_0 . The choice of X determines an embedding $\mathcal{T}(S_0, \ell) \to \mathcal{T}(S)$ by associating to a point $X_0 \in \mathcal{T}(S_0, \ell)$ the metric on Sobtained by gluing X_0 to $X|_{S-S_0}$ identifying marked points.

It is known that this embedding is quasi-isometric for the Weil–Petersson metric on $\mathcal{T}(S_0, \ell)$ and $\mathcal{T}(S)$. Although there does not seem to be an explicit statement available in the literature, this is a fairly easy consequence of the fact that augmented Teichmüller space contains the product of the Teichmüller spaces $\mathcal{T}(S_0)$, $\mathcal{T}(S - S_0)$ of the surfaces $S_0, S - S_0$, with the boundary replaced by cusps, as convex subspaces. Each factor in this product is of infinite diameter, and the image of the embedding of $\mathcal{T}(S_0, \ell)$ contains $\mathcal{T}(S_0) \times \{pt\}$ in a uniformly bounded neighborhood [Yam04]. In particular, the image of the embedding $\mathcal{T}(S_0, \ell) \to \mathcal{T}(S)$ has infinite diameter for the Weil–Petersson metric on $\mathcal{T}(S)$.

The mapping class group Mod(S) acts on the Hitchin component by precomposition of marking. As this action is isometric for the pressure metric, it extends to an action on the metric completion of Hit(S). For any essential subsurface S_0 of S, the mapping class group $Mod(S_0)$ of S_0 is a subgroup of Mod(S). The following theorem is our first main result.

Theorem B. Let $g \ge 3$ and let $S_0 \subset S$ be any essential connected subsurface of genus $g_0 \le g - 2$ with connected boundary.

- (1) The image of an embedding $\mathcal{T}(S_0, \ell) \to \mathcal{T}(S)$ has finite diameter for the pressure metric.
- (2) The action of the subgroup $Mod(S_0)$ of Mod(S) on the metric completion of Hit(S) with respect to the pressure metric has a global fixed point.

As in the case of Teichmüller space, metric completions and partial compactifications of $\operatorname{Hit}(S)$ can be studied through embeddings into the space $\mathcal{PC}(S)$ of projective geodesic currents on S. Namely, the length function f_{ρ} on T^1S defined by a representation $\rho \in \operatorname{Hit}(S)$ determines the projective geodesic current $\Theta(\rho)$ defined by the equilibrium state of a multiple of f_{ρ} . In this way one obtains a continuous mapping Θ of $\operatorname{Hit}(S)$ into the space of projective geodesic currents on S. Its restriction to the Fuchsian locus

coincides with the standard embedding of Teichmüller space [Bon88] which associates to a hyperbolic metric its projective Liouville current.

Since S is compact, the space of projective geodesic currents on S, equipped with the weak*-topology, is a compact space. To establish Theorems A and B, we shall make use the following result, interesting in its own right, on the behavior of Θ along the degenerating sequences of Hitchin representations obtained by bending a fixed Fuchsian representation.

Note that by Corollary 1.4 of [PS17], for each nice Finsler metric on X the entropy of a Hitchin representation is defined and is maximized only on the Fuchsian locus.

Theorem C. Let $S_0 \subset S$ be a proper connected essential subsurface such that no component of $S_1 = S - S_0$ is a pair of pants, and let h be any hyperbolic metric on S_0 so that the boundary of S_0 is geodesic. Then there exists a sequence ρ_i of Hitchin representations with the following properties.

- (1) The projective currents $\Theta(\rho_i)$ converge weakly to the projective current of maximal entropy for the geodesic flow on (S_0, h) .
- (2) The entropies of the representations ρ_i converge to the entropy of the geodesic flow on (S_0, h) .

There exists a natural embedding of the Teichmüller space $\mathcal{T}(S_0, \ell)$ into the space of geodesic currents for S_0 , and the pressure metric on this space of currents is defined. Theorem C implies that the metric completion of the Hitchin component contains a subspace which is naturally isometric to $\mathcal{T}(S_0, \ell)$ equipped with the pressure metric (which does not coincide with the Weil–Petersson metric, see [Xu17]). It then also contains a subspace which is isometric to the space of marked metric graphs equipped with the Weil–Petersson metric [Xu17], as this space is contained in the metric completion of $\mathcal{T}(S_0, \ell)$ equipped with the pressure metric.

By work of Bonahon (Corollary 16 of [Bon88]), the restriction of the map Θ to $\mathcal{T}(S)$ is an embedding into the space of projective geodesic currents $\mathcal{PC}(S)$, and the boundary of the resulting compactification $\overline{\Theta(\mathcal{T}(S))} - \Theta(\mathcal{T}(S))$ is precisely the space $\mathcal{PML}(S)$ of projective measured geodesic laminations, that is, currents with vanishing self-intersection. Theorem C implies that $\overline{\Theta(\text{Hit}(S))} - \Theta(\text{Hit}(S))$ is bigger than $\mathcal{PML}(S)$, since the projective current of maximal entropy for S_0 is not a measured geodesic lamination. Section 1.3 of [BIPP21] contains related results. That the map Θ is an embedding for some choices of length functions, different from ours, is due to Bridgeman, Canary, Labourie and Sambarino (Theorem 1.2 of [BCLS18]).

Question 1. For $n \geq 3$, is $\Theta(\text{Hit}(S))$ dense in the space of projective geodesic currents?

As the map which associates to a Hölder continuous positive length function f on T^1S the entropy of the normalized Gibbs current of f is continuous we obtain the following **Corollary D.** For any number $a \in [0, 1)$ there exists a sequence of degenerating Hitchin representations whose entropy converges to a.

The case a = 0 is due to Zhang [Zha15] and was reworked in [SWZ20], using mainly algebraic methods. Our proof is entirely geometric. For d = 3 and in the context of real projective structures on surfaces, Corollary D is independently due to Nie [Nie15]. In this context, the article [FK16] also contains related results, embarking from the same deformations we use, but with a different geometric interpretation.

Theorems A, B and C rest on a geometric understanding of specific paths in Hit(S) which was established in [BHM25]. These paths are so-called *grafting deformations*, also called *bending deformations* or *bulging deformations*. They are defined as follows.

For $d \geq 2$, the (unique up to conjugation) d-dimensional irreducible representation of $\mathrm{PSL}_2(\mathbb{R})$ defines an embedding $\mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_d(\mathbb{R})$ whose image stabilizes a totally geodesic subspace $\hat{\mathbb{H}}^2 \subset \mathbb{X} = \mathrm{PSL}_d(\mathbb{R})/\mathrm{PSO}(d)$ where \mathbb{X} is equipped with the symmetric metric. Up to scaling, $\hat{\mathbb{H}}^2$ is isometric to the hyperbolic plane. Its tangent bundle $T\hat{\mathbb{H}}^2$ consists of regular tangent vectors in $T\mathbb{X}$. In particular, for $d \geq 3$, every geodesic line $\gamma \subset \hat{\mathbb{H}}^2$ is contained in a unique maximal flat F of dimension d-1. This flat intersects $\hat{\mathbb{H}}^2$ orthogonally along γ .

Let now Γ be the fundamental group of a closed oriented surface S of genus at least 2. Let $\gamma \in \Gamma$ be defined by a separating simple closed curve on S. This curve defines a one edge graph of groups decomposition $\Gamma = \Gamma_1 *_C \Gamma_2$ where C is the infinite cyclic group generated by γ . Let ρ be a discrete and faithful representation of Γ into $\mathrm{PSL}_2(\mathbb{R}) \subset \mathrm{PSL}_d(\mathbb{R})$. Let $\alpha \in \mathrm{PSL}_d(\mathbb{R})$ be an element in the centralizer of $\rho(C)$ but not contained in the one-parameter subgroup containing $\rho(C)$. Partial conjugation of ρ by α then defines a new representation, obtained from the Fuchsian representation ρ by *Hitchin grafting at* γ *with* α . More precisely, this new representation coincides with ρ on Γ_1 , but maps any $\beta \in \Gamma_2$ to $\alpha \rho(\beta) \alpha^{-1}$. More generally, if $t \to \alpha(t)$ is a one-parameter subgroup of the centralizer of $\rho(C)$ not containing $\rho(C)$ then we obtain in this fashion a path in Hit(S) which we call a *Hitchin grafting path*. Such paths are also well defined if γ is non-separating and determines a decomposition of $\pi_1(S)$ as an HNN-extension. We show

Theorem E. Hitchin grafting paths have finite length for the pressure metric.

Theorem 7.1 contains a more precise version of this result. Note that the grafting paths we consider correspond in Teichmüller space to shearing (or twisting) paths along a simple geodesic. These paths are contained in the thick part of Teichmüller space and have infinite Weil-Petersson length, but any two points on the path can be connected by a Weil-Petersson geodesic whose length is bounded from above by a constant only depending on an upper bound for the length of γ .

Hitchin grafting paths are also well defined if the grafting is performed at a simple geodesic multicurve with more than one component and if the starting representation is not contained in the Fuchsian locus. It seems likely that our argument can be extended to show finite pressure length for such paths as well, however we do not carry out such an extension. In view of the work [BD17], it may be possible to extend this analysis to an even larger class of naturally defined paths in Hit(S). This raises the following Question 2. Is the diameter of Hit(S) with respect to the pressure metric finite?

The answer to this question is yes in a different context, namely for the pressure metric on quasi-Fuchsian space [FHJZ24]. Note that any two points in Hit(S) can be connected by finitely many grafting paths [AZ23], however not starting from points in the Fuchsian locus, and it is unclear whether the number of such paths needed has a uniform upper bound.

The proof of Theorem E rests on the main results of the companion article [BHM25] which gives a geometric interpretation of the concept of positivity of Hitchin representations due to Fock and Goncharov [FG06].

Organization of the article and structure of the proof. The first three sections are introductory and mainly used to collect results from the literature, especially from the first part [BHM25] of this work. The results in Section 2 can mostly be found in the literature, although not always in the form we need. We establish that the nice Finsler metrics defined in Section 1 indeed define a pressure metric for the Hitchin component.

Section 4 contains a first instance on the interplay between geometry and dynamics of Hitchin representations. We show that there are sequences of representations in the Hitchin component whose normalized intersection with any Fuchsian representation tend to infinity. Here the normalized intersection number is the entropy normalized intersection number in the space of currents.

The remaining part of the article is devoted to the study of the pressure metric. In Section 5 we use the geometric interpretation of positivity established in [BHM25] to give precise norm bounds for first and second derivatives of the Finsler length of a conjugacy class in $\pi_1(S)$ restricted to two specific classes of paths in Hit(S).

Sections 6 and Sections 7 contain the main dynamical results of this article. We use the geometric information on Hitchin grafting representations obtained in [BHM25] and the results of Section 5 to analyze the geodesic currents defined by such representations. This leads to the proof of Theorem C and Theorem E. The proofs of Theorem A and Theorem B is contained in Section 8.

The appendix collects information on the entropy of the geodesic flow on compact hyperbolic surfaces with boundary which we were unable to find in the literature in the form we need.

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1. LIE GROUPS AND SYMMETRIC SPACES

This section collects some basic facts on Lie groups and symmetric spaces and introduces conventions and notations used lated on.

Consider the unique (up to conjugacy) irreducible representation τ : $\mathrm{PSL}_2(\mathbb{R}) \to G = \mathrm{PSL}_d(\mathbb{R})$, which can be described as follows. A matrix $M = \begin{pmatrix} a & b \\ c & e \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ acts on the algebra $\mathbb{R}[X, Y]$ of polynomials in two variables by $M \cdot X = aX + cY$ and $M \cdot Y = bX + eY$. This action preserves the *d*-dimensional linear subspace $\mathbb{R}^h_{d-1}[X, Y]$ of degree d-1 homogeneous polynomials, which we identify with \mathbb{R}^d .

This representation is regular, in the sense that it maps diagonalisable 2-by-2 matrices with distinct real eigenvalues to diagonalisable d-by-d matrices with distinct real eigenvalues. In fact, using a suitable basis of $\mathbb{R}^h_{d-1}[X,Y]$, the representation τ maps

• the group of diagonal 2-by-2 matrices with positive diagonal entries into the abelian subgroup $A \subset \text{PSL}_d(\mathbb{R})$ of diagonal *d*-by-*d* matrices with positive diagonal entries;

- PSO(2) into $K = PSO(d) \subset PSL_d(\mathbb{R});$
- the subgroup T of triangular 2-by-2 matrices with positive diagonal entries into the subgroup $P \subset \text{PSL}_d(\mathbb{R})$ of triangular d-by-d matrices with positive diagonal entries;
- and 2-by-2 matrices with positive entries to totally positive *d*-by-*d* matrices (namely whose minors are all positive).

As a consequence, τ induces

- an isometric embedding the hyperbolic plane $\mathbb{H}^2 = \mathrm{PSL}_2(\mathbb{R})/\mathrm{PSO}(2)$ into the symmetric space $\mathbb{X} = G/K$, which is endowed with a nonpositively curved *G*-invariant Riemannian metric;
- an embedding of the boundary at infinity $\partial \mathbb{H}^2 = \mathrm{PSL}_2(\mathbb{R})/T$ into the flag variety $\mathcal{F} = G/P$, which can be seen as the space of full flags, i.e. sequences

$$\xi = (\xi_1 \subset \xi_2 \subset \cdots \subset \xi_d = \mathbb{R}^d)$$

where ξ_i is a linear subspace of \mathbb{R}^d of dimension *i* for each $i \leq d$.

We also fix a basepoint $\mathbf{x} = K \in \mathbb{X} = G/K$, whose stabiliser is K. The subspace $A \cdot \mathbf{x}$ is a totally geodesic embedded Euclidean subspace of \mathbb{X} of maximal dimension. This flat identifies with the Cartan subspace \mathfrak{a} , which is the linear space of diagonal (d, d)-matrices with vanishing trace, through the map $v \in \mathfrak{a} \mapsto \exp(v) \cdot \mathbf{x}$. The maximal Euclidean subspaces, called maximal flats, are the translates of $A \cdot \mathbf{x}$ under some $g \in G$.

The stabiliser in K of \mathfrak{a} is finite and acts by permuting the diagonal entries; the quotient by the subgroup acting trivially on \mathfrak{a} is the Weyl group, denoted by Weyl. This action is generated by the swaps of two diagonal entries, which act on \mathfrak{a} by reflections along hypersurfaces called walls. The open Weyl cone $\mathfrak{a}^+ \subset \mathfrak{a}$ is a natural fundamental domain for this action: it is the open cone of diagonal matrices whose entries $(\lambda_1, \ldots, \lambda_d)$ fulfill $\lambda_1 > \lambda_2 > \cdots > \lambda_d$.

Putting $A^+ = \exp(\mathfrak{a}^+)$, the K-orbit of every point $y = g\mathbf{x} \in \mathbb{X}$ intersects the closed Weyl cone $\overline{A^+}\mathbf{x}$ at exactly one point $\exp(u)\mathbf{x}$, and we write $u = \kappa(g)$ and call it the Cartan projection of g. Similarly, the G-orbit of any vector $v \in TX$ intersects $\overline{\mathfrak{a}^+}$ (seen as a subspace of $T_{\mathbf{x}}\mathbb{X}$) in precisely one point $\kappa(v)$ called the *Cartan projection* of v.

Being nonpositively curved, X has a visual boundary $\partial_{\infty} X$ on which acts G. The G-orbit of every point of $\partial_{\infty} X$ intersects exactly once the visual boundary $\partial_{\infty}(\overline{A^+}\mathbf{x})$ of our preferred Weyl cone; in other words the G-translates of $\partial_{\infty}(\overline{A^+}\mathbf{x})$, called the Weyl Chambers (at infinity), cover $\partial_{\infty} X$. The stabiliser of $\partial_{\infty}(\overline{A^+}\mathbf{x})$ is P, so the space of Weyl Chambers identifies with the flag variety $\mathcal{F} = G/P$.

Two flags $\xi = (\xi_1, \ldots, \xi_d)$ and $\eta = (\eta_1, \ldots, \eta_d)$ are transverse if ξ_i and η_{d-i} are in direct sum for every *i*. This is equivalent to the existence of a maximal flat $F(\xi, \eta)$ and two opposite Weyl Cones in it whose boundaries at infinity are ξ and η .

The Jordan projection $\lambda(g) \in \mathfrak{a}^+$ of $g \in G$ is the diagonal matrix whose diagonal entries are the moduli of the eigenvalues of g in descending order. The element $g \in G$ is called *loxodromic* if $\lambda(g)$ is contained in the interior \mathfrak{a}^+ of $\overline{\mathfrak{a}^+}$, which is equivalent to saying that g has an attracting/repelling fixed pair of transverse flags (g^-, g^+) . Then g acts as a translation on the flat $F(g^-, g^+)$ with direction prescribed by its Jordan projection.

A Finsler metric coming from a linear functional on \mathfrak{a}

Notation 1. We fix a linear functional α_0 on \mathfrak{a} which is positive on $\overline{\mathfrak{a}^+}$ and such that $\alpha_0(gv) < \alpha_0(v)$ for all $v \in \mathfrak{a}^+$ and $g \in \text{Weyl}$.

We assume that α_0 is symmetric in the sense that if g is the transformation in the Weyl group that maps \mathfrak{a}^+ to its opposite $-\mathfrak{a}^+$ then $\alpha_0(gv) = -\alpha_0(v)$ for any $v \in \mathfrak{a}$.

An example of a linear functional satisfying the above conditions is given in Equation 1. For any vector $v \in T \mathbb{X}$ we set

(2)
$$\mathfrak{F}(v) = \alpha_0(\kappa(v))$$

where as before, $\kappa(v) \in \overline{\mathfrak{a}^+}$ is the Cartan projection of v.

Proposition 1.1 (Lemmas 5.9-10 of [KL18]). The following hold.

- (1) \mathfrak{F} defines a G-invariant Finsler metric on \mathbb{X} .
- (2) The unparameterized Riemannian geodesics of X are also geodesics for \mathfrak{F} .
- (3) The translation length for \mathfrak{F} of any element $g \in G$ acting on \mathbb{X} is given by $\ell^{\mathfrak{F}}(g) := \alpha_0(\lambda(g))$ where $\lambda(g) \in \overline{\mathfrak{a}^+}$ is the Jordan projection.

In the sequel we always normalize the functional α_0 in such a way that the embedding $\mathbb{H}^2 \to \mathbb{X}$ which is isometric for the symmetric metric also is isometric for the Finsler metric \mathfrak{F} .

Finsler geodesics between two distinct points in X are in general not unique. Indeed, for $x, y \in X$, the *diamond* defined by

$$D(x,y) = \{ z \mid d^{\mathfrak{F}}(x,z) + d^{\mathfrak{F}}(z,y) = d^{\mathfrak{F}}(x,y) \}$$

is the set of all points on a geodesic connecting x to y (see §5.1.3 of [KL18] or §3 of [BHM25]). The diamond is contained in any maximal flat F containing x, y, where it is a compact convex polytope. It is the intersection of a Weyl Cone centered at x and a Weyl Cone centered at y, opposite to each other.

Busemann functions and Gromov product

The Busemann functions, or horofunctions, are generalizations of distance functions on \mathbb{X} : they record relative distances to a point at infinity. The *Busemann function* associated to our choice of Finsler metric is given by

(3)
$$b_{\xi}^{\mathfrak{F}}(x,y) = \lim_{n \to \infty} d^{\mathfrak{F}}(x,z_n) - d^{\mathfrak{F}}(y,z_n) \in \mathbb{R}.$$

where $(z_n)_n \subset \mathbb{X}$ converges to a point of the visual boundary in the interior of ξ .

The *Gromov product* between two transverse flags $\xi, \eta \in \mathcal{F}$ computed at the basepoint $x \in \mathbb{X}$ is defined as

(4)
$$\langle \xi | \eta \rangle_x = \lim_{n \to \infty} \left(d^{\mathfrak{F}}(y_n, x) + d^{\mathfrak{F}}(x, z_n) - d^{\mathfrak{F}}(y_n, z_n) \right) \in \mathbb{R}_{\geq 0}$$

where $(y_n)_n, (z_n)_n \subset \mathbb{X}$ are sequences converging to points of the visual boundary in the interior of ξ and η respectively. Note that we used an unusual convention by not including the factor $\frac{1}{2}$. This will make the computations a bit easier to read.

If x is contained in the flat connecting η to ξ then $\langle \xi | \eta \rangle_p = 0$, which leads to

$$\langle \xi | \eta \rangle_x = b_{\xi}^{\mathfrak{F}}(x,p) + b_{\eta}^{\mathfrak{F}}(x,p)$$

We refer to [KLP18] for more information on this construction, in particular on the existence of the limit in the formula (4).

2. Equilibrium states, Hitchin representations and pressure metrics

In this section we introduce the main structures and tools for this article. It is subdivided into three subsections. In the first subsection we introduce geodesic currents for closed surfaces and the intersection form. The second subsection contains an account of Hitchin representations and length functions defined by Finsler norms. We show, using [BCLS15], that such length functions can be used to construct a pressure metric on the Hitchin component. The third subsection contains a summary of the main properties of Patterson–Sullivan theory we shall use later on.

Throughout, S denotes a closed surface of genus $g \ge 2$, equipped with a fixed choice of a hyperbolic metric. Thus the universal covering \tilde{S} of S can naturally be identified with the hyperbolic plane \mathbb{H}^2 .

2.1. Geodesic currents, length and intersection. A geodesic current for S is a nontrivial $\pi_1(S)$ -invariant Radon measure on the space of oriented geodesics $\partial_{\infty} \mathbb{H}^2 \times \partial_{\infty} \mathbb{H}^2 - \Delta$ of the hyperbolic plane \mathbb{H}^2 (here Δ is the diagonal). Two such currents are projectively equivalent if they are constant multiples of each other. An equivalence class for this equivalence relation is a projective geodesic current. The space $\mathcal{C}(S)$ of geodesic currents for S is equipped with the weak*-topology which descends to a topology on the space $\mathcal{PC}(S)$ of projective geodesic currents. A (projective) measured geodesic lamination is a (projective) geodesic current whose support consists of pairwise disjoint simple geodesics. The space \mathcal{PML} of projective measured geodesic laminations is a closed subset of $\mathcal{PC}(S)$.

Each hyperbolic metric on S determines a geodesic current, the *Liouville current* of the metric. The following is due to Bonahon [Bon88].

Theorem 2.1 (Bonahon). Associating to a hyperbolic metric on S its projective Liouville current defines an embedding of the Teichmüller space into $\mathcal{PC}(S)$, and its complement in its closure is the space of projective measured geodesic laminations.

The idea behind this theorem rests on the existence of an *intersection form*

$$\iota: \mathcal{C}(S) \times \mathcal{C}(S) \to [0,\infty)$$

which extends the geometric intersection number between two closed curves on S. The form ι has the following properties (see Chapter 8 of [Mar16]).

- (1) ι is continuous for the weak*-topology.
- (2) If λ is the Liouville current of a hyperbolic metric ρ on S, and if $\alpha \subset S$ is any closed geodesic, then
- (5) $\iota(\lambda, \cdot)$

$$\iota(\lambda,\alpha) = \ell_{\rho}(\alpha)$$

the ρ -length of α .

The intersection $\iota(\lambda_1, \lambda_2)$ of two Liouville currents λ_1, λ_2 of two hyperbolic metrics on Shas another interpretation which is important for us. Namely, the choice of a hyperbolic metric h_1 on S determines the *geodesic flow* Φ^t on the unit tangent bundle T^1S of S and a Hölder structure on T^1S . Given these data, any geodesic current μ for S extends to a Φ^t -invariant finite Borel measure $\hat{\mu}$ on T^1S . Thus given a Hölder continuous positive function $f: T^1S \to (0, \infty)$, the integral

(6)
$$\int f d\hat{\mu} = \mathbf{I}(\mu, f)$$

is defined. By invariance, this integral only depends on the cohomology class of f. This means that if f' is another Hölder function such that $\int_{\gamma} f' = \int_{\gamma} f$ for every periodic orbit γ for Φ^t then $\int f' d\hat{\mu} = \int f d\hat{\mu}$ and hence $\mathbf{I}(\mu, f') = \mathbf{I}(\mu, f)$.

As a consequence, the pairing $I(\mu, f)$ is a pairing between cohomology classes of (positive) Hölder functions on T^1S and geodesic currents without having to make reference to the geodesic flow Φ^t which depends on the background metric. Furthermore, the pairing is continuous, where as before, C(S) is equipped with the weak*-topology, and the space of Hölder cohomology classes is equipped with the quotient topology obtained from the space of Hölder functions on T^1S for a fixed reference metric. We refer to [Ham99] for more details.

Assume now that f_2 is a Hölder function which integrates over each periodic orbit γ for Φ^t to the length of the free homotopy class of γ for another hyperbolic metric h_2 . If λ_1, λ_2 are the Liouville currents of h_1, h_2 , then we have

$$\iota(\lambda_1,\lambda_2)=\mathbf{I}(\lambda_1,f_2).$$

A Hölder continuous positive function f on T^1S can be used to reparameterize the flow Φ^t . This reparameterization is defined by

$$\Phi_f^t(v) = \Phi^{\sigma(v,t)}(v)$$

where $\int_0^{\sigma(v,t)} f(\Phi^s v) ds = t$. For the reparameterized flow, the function f is cohomologous to the constant function 1. This is equivalent to stating that the period of a periodic orbit γ for the flow Φ_f^t equals the integral of f over the corresponding orbit for Φ^t . The identity $(T^1S, \Phi^t) \to (T^1S, \Phi_f^t)$ is an order preserving orbit equivalence between the flows Φ^t, Φ_f^t .

Denote by h_{μ} the entropy of a Φ^t -invariant Borel probability measure μ on T^1S . For a positive Hölder function f let $\delta(f) > 0$ be such that $\operatorname{pr}(-\delta(f)f) = 0$ where

$$\operatorname{pr}(u) = \sup_{\mu} \left(h_{\mu} + \int u d\mu \right)$$

and μ runs through all Φ^t -invariant Borel probability measures on T^1S . Then

$$h_{\mu} - \delta(f) \int f d\mu \le 0$$

for all μ . A measure μ is called a *Gibbs equilibrium state* for f if the equality in this inequality holds. Using the fact that an order preserving orbit equivalence between two flows induces an isomorphism between the flow invariant probability measures and a formula relating entropies due to Abramov [Abr59], existence and uniqueness of an equilibrium state for the continuous function $\delta(f)f$ is equivalent to existence and uniqueness of a measure of maximal entropy for the geodesic flow Φ_f^t on T^1S , which is well known for Hölder functions (see [KH95] for more details). The constant $\delta(f)$ then equals the topological entropy of Φ_f^t .

Let μ_f be the scalar multiple of the unique Gibbs equilibrium state for f such that $\int f d\mu_f = 1$ (so it is not a necessarily a probability measure). Then μ_f can be obtained as a limit

(7)
$$\mu_f = \lim_{R \to \infty} \frac{1}{\# N_f(R)} \sum_{\ell_f(\gamma) \le R} \frac{\mathcal{D}_{\gamma}}{\ell_f(\gamma)}$$

where $\ell_f(\gamma) = \int_{\gamma} f$ is the period of γ for Φ_f^t , where \mathcal{D}_{γ} is the Φ^t -invariant measure on the periodic orbit γ whose total mass is the Φ^t -period of γ , and where $N_f(R) = \{\gamma \mid \ell_f(\gamma) \leq R\}$. Thus by continuity of the pairing **I**, for any (positive) Hölder function u we have

(8)
$$\mathbf{I}(\mu_f, u) = \int u d\mu_f = \lim_{R \to \infty} \frac{1}{\# N_f(R)} \sum_{\ell_f(\gamma) \le R} \frac{\ell_u(\gamma)}{\ell_f(\gamma)}$$

Following [BCLS15], we also define the normalized intersection number by

$$\mathbf{J}(f, u) = \frac{h(u)}{h(f)} \mathbf{I}(\mu_f, u)$$

where $h(u) = \lim_{R \to \infty} \frac{1}{R} \log \# N_u(R)$.

2.2. Hitchin representations. In this section we introduce Hitchin representations and summarize those of their properties which are important later on. Our main goal is to show that the *G*-invariant Finsler metric \mathfrak{F} defined in (2) induces a pressure metric on the Hitchin component.

The Hitchin component $\operatorname{Hit}(S)$ for conjugacy classes of representations $\pi_1(S) \to \operatorname{PSL}_d(\mathbb{R})$ is the connected component of the set of conjugacy classes of representations which factor through an irreducible representation $\operatorname{PSL}_2(\mathbb{R}) \to \operatorname{PSL}_d(\mathbb{R})$. In the sequel we always work with explicit representations rather than with conjugacy classes.

An important property possessed by Hitchin representations is the Anosov property first introduced by Labourie [Lab06], which plays a central role in [BCLS15] in the definition of the pressure metric. There are many different versions of the Anosov property, and many equivalent characterisations of the Anosov property, see for example [Lab06; GW12; KLP17; GGKW17; BPS19; KP22], and Theorem 4.37 of [Kas24] for more details and history.

Definition 2.2. A representation $\rho : \pi_1(S) \to \mathrm{PSL}_d(\mathbb{R})$ is projective Anosov if there exist ρ -equivariant Hölder continuous maps $\xi : \partial_{\infty} \tilde{S} \to \mathbb{R}P^{d-1}, \ \theta : \partial_{\infty} \tilde{S} \to (\mathbb{R}P^{d-1})^*$ (where $(\mathbb{R}P^{d-1})^*$ is the dual projective space) such that

- (1) if x, y are distinct points in $\partial_{\infty} \tilde{S}$, then $\xi(x) + \ker \theta(y) = \mathbb{R}^d$, and
- (2) if $\gamma_n \in \pi_1(S)$ is a sequence so that for some basepoint $\mathbf{x} \in \tilde{S} = \mathbb{H}^2$, the sequence $\gamma_n \mathbf{x}$ converges to $x \in \partial_\infty \mathbb{H}^2$, and $\gamma_n^{-1} \mathbf{x} \to y \in \partial_\infty \mathbb{H}^2$, then we have $\rho(\gamma_n) p \to \xi(x)$ for any $p \in \mathbb{R}P^{d-1} \ker \theta(y)$ and $\rho(\gamma_n^{-1})q \to \theta(y)$ for any $q \in (\mathbb{R}P^{d-1})^*$ such that $\xi(x) \notin \ker q$.

Remark . In the references given for the characterisations of the Anosov property, the limit map is only required to be continuous, and then the Hölder regularity is derived as a consequence of the other conditions, see for instance Theorem 6.58 of [KLP17].

The following is due to Labourie [Lab06] and Fock–Goncharov [FG06].

Theorem 2.3 (Labourie, Fock–Goncharov). Every representation in the Hitchin component is projective Anosov.

As in [BCLS15], let F be the total space of the bundle over

$$(\mathbb{R}P^{d-1})^{(2)} = \mathbb{R}P^{d-1} \times (\mathbb{R}P^{d-1})^* - \{(U,V) \mid U \subset \ker(V)\}$$

whose fiber at a point (U, V) is the space

$$M(U,V) = \{(u,v) \mid u \in U, v \in V, \langle v \mid u \rangle = 1\} / \sim$$

where $\langle v \mid u \rangle$ is the natural pairing between a vector and a covector and $(u, v) \sim (-u, -v)$. Note that u determines v so that F is an \mathbb{R} -bundle.

The bundle F is equipped with a natural \mathbb{R} -action, given by

$$\Phi_F^t(U, V, (u, v)) = (U, V, (e^t u, e^{-t} v)).$$

Given a projective Anosov representation $\rho : \pi_1(S) \to \mathrm{PSL}_d(\mathbb{R})$ and ξ, θ the associated limit maps, we consider the pullback bundle

$$F_{\rho} = (\xi, \theta)^* F \to \partial_{\infty} \widetilde{S} \times \partial_{\infty} \widetilde{S} - \Delta$$

by the map $\partial_{\infty} \widetilde{S} \times \partial_{\infty} \widetilde{S} - \Delta \xrightarrow{(\xi,\theta)} (\mathbb{R}P^{d-1})^{(2)}$, which inherits an \mathbb{R} -action from the action of Φ_F^t . The actions $\pi_1(S) \curvearrowright_{\rho} \mathbb{R}^d$ and $\pi_1(S) \curvearrowright \partial_{\infty} \widetilde{S} \times \partial_{\infty} \widetilde{S} - \Delta$ extend to an action on F_{ρ} . If we let

$$U_{\rho}S = \pi_1(S) \backslash F_{\rho}$$

then the \mathbb{R} -action on F_{ρ} descends to a flow Φ^t_{ρ} on $U_{\rho}S$ which is called the *spectral radius* flow of the representation (see p.1118 of [BCLS15]).

The following statement combines Propositions 4.1, 4.2 and 6.2 of [BCLS15]. It is valid for any analytic family of projective Anosov representations.

- **Proposition 2.4.** (1) For every representation ρ in the Hitchin component there exists a Hölder continuous order preserving orbit equivalence $\Psi_{\rho} : (T^1S, \Phi^t) \rightarrow (U_{\rho}S, \Phi_{\rho}^t)$. Any primitive element $\gamma \in \pi_1(S)$ has period $\log \Lambda(\rho)(\gamma)$ where $\Lambda(\rho)(\gamma)$ is the spectral radius of $\rho(\gamma) \in \text{PSL}_d(\mathbb{R})$.
 - (2) If D is the unit disk and if ρ_u $(u \in D)$ is a real analytic family of Hitchin representations, then up to decreasing the size of D, there exists a real analytic family $\{f_{\rho_u}: T^1S \to \mathbb{R}\}_{u \in D}$ of positive Hölder functions such that the reparameterization of T^1S by f_{ρ_u} is Hölder conjugate to U_{ρ_u} for all $u \in D$.

As a consequence, the spectral radius length defines a *pressure metric* on Hit(S) as follows. For any smooth deformation ρ_t of a representation $\rho = \rho_0$, put

(9)
$$\|\rho'(0)\|^2 = \frac{d^2}{dt^2}|_{t=0} \mathbf{J}(f_{\rho(0)}, f_{\rho(t)})$$

(Theorem 1.3 of [BCLS15]) where $f_{\rho(s)}$ is the Hölder function constructed from $\rho(s)$ as in Proposition 2.4. That this construction defines indeed a (mildly degenerate) Riemanian metric on the Hitchin component which determines a distance function was established in [BCLS15]. It is based on the fact that projective Anosov representations are dominated in the sense of [BPS19]. We refer to [BCLS15], [BPS19] and [Sam24] for more precise information.

The pressure metric we are interested in is a more geometric version of the metric (9). To define this metric we need to review some additional properties of representations in $\operatorname{Hit}(S)$. Let as before \mathcal{F} be the variety of full flags in \mathbb{R}^d .

Definition 2.5 ([GGKW17; KLP17]). A representation $\rho : \pi_1(S) \to G$ is Borel Anosov if the following holds true.

- (1) There exists a (unique) equivariant Hölder embedding $\partial_{\infty}\rho: \partial \mathbb{H}^2 \to \mathcal{F}$ such that $\partial_{\infty}\rho(\xi) \pitchfork \partial_{\infty}\rho(\eta)$ for all $\xi \neq \eta \in \partial \mathbb{H}^2$.
- (2) For any diverging sequence $(\gamma_n)_n \subset \pi_1(S)$ such that $\gamma_n \to \xi \in \partial \mathbb{H}^2$ and $\gamma_n^{-1} \to \eta$, we have $\rho(\gamma_n)\zeta \to \partial_{\infty}\rho(\eta)$ for any $\zeta \in \mathcal{F}$ transverse to $\partial_{\infty}\rho(\xi)$.

By the groundbreaking work of Labourie and Fock–Goncharov, we have **Theorem 2.6** ([Lab06; FG06]). All Hitchin representations $\pi_1(S) \to \text{PSL}_d(\mathbb{R})$ are Borel Anosov.

Our goal is to construct a pressure metric on $\operatorname{Hit}(S)$ as in (9), but using the fixed length function $\ell^{\mathfrak{F}}(g) = \alpha_0(\lambda(g))$ of the Finsler norm and its associated renormalised intersection form

$$\mathbf{J}(\rho,\rho') = \frac{h(\rho')}{h(\rho)} \lim_{R \to \infty} \frac{1}{\# N_{\rho}(R)} \sum_{\ell^{\mathfrak{F}}(\rho(\gamma)) \leq R} \frac{\ell^{\mathfrak{F}}(\rho'(\gamma))}{\ell^{\mathfrak{F}}(\rho(\gamma))},$$

where

$$N_{\rho}(R) = \{ [\gamma] \in [\pi_1(S)] : \ell^{\mathfrak{F}}(\rho(\gamma)) \le R \} \text{ and } h(\rho) = \lim_{R \to \infty} \frac{1}{R} \log \# N_{\rho}(R).$$

To this end we have to establish an analog of Proposition 2.4 for this new length function. We shall reduce this statement to Proposition 2.4 using the following classical observation.

We shall reduce this statement to Proposition 2.4 using the following classical observation. Let $\xi = (\xi_1 \subset \cdots \subset \xi_d)$ be a full flag in \mathbb{R}^d . Then for each $k \leq d-1$ the k-th exterior power $\Lambda^k(\xi_k)$ is one-dimensional. A non-zero element ω of this vector space defines up to a non-zero multiple a non-zero linear functional $\Psi(\omega) : \Lambda^{d-k}(\mathbb{R}^d) \to \mathbb{R}$ as follows. Choose a non-zero element $\nu \in \Lambda^d(\mathbb{R}^d)$ and put $\Psi(\omega)(\alpha) = c$ if $\omega \wedge \alpha = c\nu$. Note that the kernel of $\Psi(\omega)$ is spanned by all decomposable elements of $\Lambda^{d-k}\mathbb{R}^d$ which are not transverse to ξ_k .

If $\rho : \pi_1(S) \to G$ is Borel Anosov, then by the definition of the transversality relation \pitchfork , for any two distinct points $\xi \neq \eta \in \partial \mathbb{H}^2$, the d - k-th subspace $\partial_{\infty} \rho(\xi)_{d-k}$ of the flag $\partial_{\infty} \rho(\xi)$ defines a line of linear functionals on $\Lambda^k(\mathbb{R}^d)$ which do not evaluate to zero on $\Lambda^k \partial_{\infty} \rho(\eta)_k$, where $\partial_{\infty} \rho(\eta)_k$ is the k-dimensional subspace of the flag $\partial_{\infty} \rho(\eta)$. Thus if $\Lambda^k \rho : \pi_1(S) \to \mathrm{PSL}_{d_k}(\mathbb{R})$ denotes the representation induced by ρ into the full linear group of $\Lambda^k(\mathbb{R}^d)$ where d_k denotes the dimension of $\Lambda^k(\mathbb{R}^d)$, then as the map $\partial_{\infty} \rho : \partial_{\infty} \mathbb{H}^2 \to \mathcal{F}$ is Hölder continuous, the following well-known statement holds true.

Lemma 2.7. If $\rho : \pi_1(S) \to G$ is Borel Anosov, then for any k < d, the induced representation $\Lambda^k \rho$ is projective Anosov.

Remark . It follows from the above discussion that in fact, ρ is Borel Anosov if and only if for each $k \leq d-1$ the induced representation on $\Lambda^k(\mathbb{R}^d)$ is projective Anosov. We refer to Section 4 of [BPS19] for more details on this relation.

Thus we can apply Proposition 2.4 to each representation $\Lambda^k \rho$. Recall from Section 1 the definition of the Jordan projection λ . As implicitly stated in [BPS19], we obtain the regularity statement on Finsler length functions needed to define a pressure metric.

Proposition 2.8. For every Borel Anosov representation $\rho_0 : \pi_1(S) \to \text{PSL}_d(\mathbb{R})$, there exists an open neighborhood U of ρ_0 made of Borel Anosov representations and a real analytic family $\{f_{\rho} : T^1S \to \mathfrak{a}\}_{\rho \in U}$ of Hölder functions, valued in \mathfrak{a}^+ , such that for any $\gamma \in \pi_1(S)$, we have

$$\lambda(\rho(\gamma)) = \int f_{\rho} d\gamma.$$

Proof. Proposition 2.4 implies that there exists an open neighborhood U of ρ_0 and real analytic families $\{g_{\rho}^k : T^1S \to \mathbb{R}\}_{\rho \in U}$ of Hölder functions such that for any $\rho \in U$, each exterior product $\Lambda^k \rho$ is projective Anosov, and for any $\gamma \in \pi_1(S)$, the logarithm

 $\log \Lambda(\Lambda^k \rho(\gamma))$ of the spectral radius of $\Lambda^k(\rho(\gamma))$ equals

$$\log \Lambda(\Lambda^k \rho(\gamma)) = \int g_{\rho}^k d\gamma.$$

Then we can consider the following Hölder function

$$f_{\rho} = (g_{\rho}^1, \ g_{\rho}^2 - g_{\rho}^1, \ g_{\rho}^3 - g_{\rho}^2, \dots, \ g_{\rho}^d - g_{\rho}^{d-1}) \in \mathfrak{a}.$$

By Proposition 2.4, the function f_{ρ} depends analytically on ρ . Moreover, for any $\gamma \in \pi_1(S)$, we have

$$\lambda(\rho(\gamma)) = \int f_{\rho} d\gamma.$$

It is not clear, however, that f_{ρ} is valued in the open Weyl chamber \mathfrak{a}^+ . Let us solve this issue by first replacing f_{ρ_0} by an f'_{ρ_0} valued in \mathfrak{a}^+ , using work of Sambarino, and then extend f'_{ρ_0} to a small neighborhood of representations ρ , using a theorem of Livšic.

We apply Sambarino's reparametrization result to the lengths functions $\alpha_k \circ \lambda \circ \rho_0(\gamma)$ where $\alpha_k(v_1, \ldots, v_d) = v_k - v_{k+1}$, see Theorem 3.2 of [Sam14] (Sambarino proved in pages 481-483 that we can apply this theorem to our setting). This gives us positive Hölder functions $u^k : T^1S \to \mathbb{R}$ such that for any $\gamma \in \pi_1(S)$, we have

$$\alpha_k \circ \lambda \circ \rho_0(\gamma) = \int u^k d\gamma.$$

Let $f'_{\rho_0}: T^1S \to \mathfrak{a}$ be such that $\alpha_k \circ f'_{\rho_0}(v) = u^k(v) > 0$ for all $v \in T^1S$ and $1 \le k \le d-1$. Then f'_{ρ_0} is valued in the interior of \mathfrak{a}^+ by definition, it is Hölder, and $\lambda(\rho_0(\gamma)) = \int f'_{\rho_0} d\gamma$ for any γ .

For any periodic orbit γ in T^1S we have $\int f_{\rho_0}d\gamma = \int f'_{\rho_0}d\gamma$, so by Theorem 1 of [Liv71] f'_{ρ_0} and f_{ρ_0} are cohomologous, in the sense that there exists $F: T^1S \to \mathfrak{a}$ differentiable in the direction of the geodesic flow Φ^t such that $f'_{\rho_0} = f_{\rho_0} + \frac{d}{dt}_{|t=0}F \circ \Phi^t$. Put

$$f'_{\rho} = f_{\rho} + \frac{d}{dt}_{|t=0} F \circ \Phi^t$$

for any $\rho \in U$, so that $\lambda(\rho(\gamma)) = \int f'_{\rho} d\gamma$ for any γ . This yields an analytic family of Hölder functions which take values in \mathfrak{a}^+ for all ρ contained in a sufficiently small neighborhood $U' \subset U$ of ρ_0 . This is what we wanted to show. \Box

Consider now a C^2 -path of representations $(\rho_t)_t$ in the neighborhood U constructed in Proposition 2.8 with initial value ρ_0 . Let μ_0 be the equilibrium state on T^1S associated to f_{ρ_0} introduced in Subsection 2.1, normalized so that $\int f_{\rho_0} d\mu_0 = 1$. Denote by h(t) the entropy associated to f_{ρ_t} . Following [BCLS15] we set

(10)
$$\left\|\frac{d}{dt}_{|t=0}\rho_t\right\|^2 = \frac{1}{h(0)}\int \frac{d^2}{dt^2}_{|t=0}(h(t)\cdot\alpha_0\circ f_{\rho_t})d\mu_0.$$

It follows from Proposition 2.8 and [BCLS15] that this is well defined and is indeed the square norm for a (perhaps degenerate) Riemannian metric on Hit(S) which is a variant of the simple root length metric (9) considered in [BCLS15]. We call this metric the *Finsler pressure metric* on Hit(S).

2.3. Patterson–Sullivan theory.

Patterson–Sullivan theory for hyperbolic metrics. Patterson [Pat76] and Sullivan [Sul79] introduced a construction of measures on $\partial_{\infty}\mathbb{H}^2$ which allows to obtain the entropy maximizing invariant probability measure of the geodesic flow on a compact hyperbolic surface as a product measure. This construction has been generalised in various settings. We recall some important facts about their theory and the generalization to the case of interest for us.

Let as before S be a closed surface of genus $g \ge 2$ and let $\Gamma = \rho(\pi_1(S)) \subset \text{PSL}_2(\mathbb{R})$ be a Fuchsian representation, determined by the choice of a hyperbolic metric on S. For $\xi \in \partial_{\infty} \mathbb{H}^2$ and $x, y \in \mathbb{H}^2$, we denote by $b_{\xi}(x, y)$ the Busemann function of (x, y) based at ξ , defined as in (3). Up to multiplication by a constant, there exists a unique family of finite measures $(\nu^x)_{x \in \mathbb{H}^2}$ which all define the same measure class, and which satisfy the following. For all $x, y \in \mathbb{H}^2$ and $\xi \in \partial_{\infty} \mathbb{H}^2$,

(11)
$$\frac{\partial \nu^x}{\partial \nu^y}(\xi) = e^{b_{\xi}(x,y)}$$

The measures ν^{y} can be obtained as a limit of measure of the form

(12)
$$\frac{1}{c_s} \sum_{g \in \Gamma} e^{sd(y,g \cdot x)} \delta_{g \cdot x}$$

with s converging from above toward 1 (which equals the *critical exponent* of Γ), and the constant c_s is chosen so that for y = x, the measures in (12) are probability measures.

From the measure class ν^x we define a measure on $\partial_\infty \mathbb{H}^2 \times \partial_\infty \mathbb{H}^2 \setminus \Delta$ invariant by the action of Γ . Recall from (4) the definition of the Gromov product $\langle \xi | \eta \rangle_x$ of (ξ, η) based at x. It can be computed by

(13)
$$\langle \xi | \eta \rangle_x = b_{\xi}(x, z) + b_{\eta}(x, z)$$

for any z on the geodesic with endpoints ξ and η . The value does not depend on the choice of z. Then define the measure $\hat{\nu}$ on $\partial_{\infty} \mathbb{H}^2 \times \partial_{\infty} \mathbb{H}^2 \setminus \Delta$ by

(14)
$$d\hat{\nu}(\xi,\eta) = e^{\langle \xi | \eta \rangle_x} \cdot d\nu^x(\xi) \times d\nu^x(\eta)$$

The measure $\hat{\nu}$ is invariant under the action $\Gamma \curvearrowright \partial_{\infty} \mathbb{H}^2 \times \partial_{\infty} \mathbb{H}^2 \setminus \Delta$, is finite on compact sets and does not depend on x [Sul79].

The unit tangent bundle T^1S of the surface $S = \Gamma \setminus \mathbb{H}^2$ is endowed with a geodesic flow Φ^t . It is Anosov, so it admits a unique entropy maximizing invariant probability measure. This measure lifts to a Γ -invariant Φ^t -invariant Radon measure on $T^1\mathbb{H}^2$ which disintegrates to the measure $\hat{\nu}$. Namely, $\partial_{\infty}\mathbb{H}^2 \times \partial_{\infty}\mathbb{H}^2 \setminus \Delta$ is just the set of oriented geodesics in \mathbb{H}^2 , and $d\hat{\nu} \times dt$ defines a Φ^t -invariant Γ -invariant Radon measure on $T^1\mathbb{H}^2$, where dt is the one-dimensional Lebesgue measure on flow lines. This measure projects to a finite Borel measure on T^1S in the Lebesgue measure class, which can be scaled to be a probability measure.

Patterson–Sullivan theory for Hitchin representations. Patterson Sullivan theory was generalized to many different geometric settings. In the setting of Finsler metrics on higher rank symmetric space and Hitchin representations, such a generalization is due to Kapovich and Dey [DK22] (the results are valid for all Anosov representations). Namely,

given a Hitchin representation $\rho: \pi_1(S) \to \mathrm{PSL}_d(\mathbb{R})$, define a Poincaré series

$$P^{\mathfrak{F}}(\rho,s)(x,y) = \sum_{\psi \in \rho(\pi_1(S))} e^{-sd^{\mathfrak{F}}(y,\psi x)}$$

where as before, $d^{\mathfrak{F}}(y,z)$ is the distance between x, y for the Finsler metric \mathfrak{F} . Part (iv) of Theorem A of [DK22] shows that this series diverges at the *critical exponent* δ_{ρ} . Moreover, it defines a family μ^x of Borel measures on the *limit set* $\Lambda \subset \mathcal{F}$ of $\rho(\Gamma)$ in the flag variety \mathcal{F} , that is, the image of $\partial_{\infty} \mathbb{H}^2$ under a ρ -equivariant Hölder continuous map, indexed by the points $x \in \mathbb{X}$. These measures are a *conformal density*, that is, they are equivariant under the action of $\rho(\pi_1(S))$ and transform via

(15)
$$\frac{d\mu^y}{d\mu^x}(\xi) = e^{\delta_\rho b_{\xi}^{\tilde{g}}(x,y)}$$

where $b_{\xi}^{\mathfrak{F}}$ is the Busemann function for the Finsler metric.

Conformal densities had been constructed earlier by Sambarino in [Sam14], using a different method and work of Ledrappier [Led95]. Sambarino's construction is dynamical and does not use the Finsler metric $d^{\mathfrak{F}}$. Here we will need the geometric approach of Dey–Kapovich.

Remark . As the limit curve of a Hitchin representation is a curve in the flag variety rather than the limit set of the representation in the geometric boundary of \mathbb{X} , the above construction can not be carried out for the symmetric metric. Namely, as the limit set of the representation in the geometric boundary $\partial_{\infty} \mathbb{X}$ of \mathbb{X} may have points in asymptotic Weyl chambers which are not opposite in the Weyl chamber and hence can not be connected by a geodesic, it may not be possible to correctly encode translation lengths for the symmetric metric by a global Hölder continuous function on T^1S .

3. HITCHIN GRAFTING REPRESENTATIONS

The Hitchin representations we are interested in are the familiar *bending* or *bulging* deformations of *Fuchsian* representations, that is, representations which factor through the embedding $\tau : \text{PSL}_2(\mathbb{R}) \to \text{PSL}_d(\mathbb{R})$. We refer to [Gol86; AZ23; BHM25] for an account on the bending construction. In this section we introduce these representations and summarize the geometric results from [BHM25] we need.

3.1. **Grafting.** Consider a closed oriented surface S of genus $g \ge 2$ endowed with a hyperbolic metric. A *simple (geodesic) multi-curve* γ^* is the union of pairwise disjoint essential mutually not freely homotopic simple closed curves (geodesics) on S. We fix moreover an orientation on each component of γ^* .

Consider the special direction $u = d\tau \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{a}$ given by τ . For any $z \in \mathfrak{a}$ and $\ell > 0$, let $\operatorname{Cyl}(\ell, z) \subset \mathfrak{a}/\ell u$ be the cylinder obtained by quotienting the strip $\{tu + sz : t \in \mathbb{R}, s \in [0, 1]\} \subset \mathfrak{a}$ under the translation by ℓu . The (Finsler) *height* of such cylinder is defined as

(16)
$$\operatorname{height} = \min\{\mathfrak{F}(tu+z) : t \in \mathbb{R}\}.$$

We fix for every $\gamma \in \gamma^*$ a vector $z_{\gamma} \in \mathfrak{a}$; the collection $z = (z_{\gamma})_{\gamma \in \gamma^*}$ is interpreted as a grafting parameter.

Definition 3.1. The *abstract grafting* of S along the geodesic multi-curve γ^* with grafting parameter z is the surface S_z obtained by cutting S open along each of the components γ of γ^* , inserting flat cylinders $C_{\gamma} = \text{Cyl}(\ell_S(\gamma), z_{\gamma})$ and gluing the surface back with the translation by z_{γ} .

If z_{γ} is not parallel to u for any $\gamma \in \gamma^*$, then this grafting comes with a natural homotopy equivalence $\pi_z : S_z \to S$ projecting the flat cylinders onto γ^* , which allow us to identify $\pi_1(S_z)$ and $\pi_1(S)$.

The abstract grafted surface S_z decomposes into subsurfaces with geodesic boundary which are equipped with a metric of constant curvature. The hyperbolic part S^{hyp} is the union of the subsurfaces with a metric of constant curvature -1 and can be identified with the union of the components of $S \setminus \gamma^*$. The component $S \setminus S^{\text{hyp}}$ is the cylinder part and consists of a union of flat cylinders whose core curves are freely homotopic to the components of γ^* .

As the pressure metric for the Hitchin component we are interested in is defined by a Finsler metric on X using a linear functional α_0 (see (2)) rather than the Riemannian one, we also endow S_z with a Finsler metric by equipping each cylinder C_{γ} with the quotient of the non-Euclidean norm \mathfrak{F} on \mathfrak{a} . Observe that in general, for a given C^1 -structure on S_z as constructed above, this metric is *discontinuous* at the gluing locus between the flat cylinders and the hyperbolic part. Additionally the metric on the flat part is sensitive in the direction of z, and does not depend only on the height of the grafting (contrarily to the Riemannian metric). Nevertheless it induces a well defined path metric on S_z .

Let G_{γ^*} be the oriented graph such that each vertex $v \in V$ corresponds to a component Σ_v of $\Sigma - \gamma^*$, and each edge $e \in E$ corresponds to an oriented component $\vec{\gamma}_e$ of γ^* . Take a discrete and faithful representation $\rho \colon \pi_1(G_{\gamma^*}, T) \to \mathrm{PSL}_2(\mathbb{R}) \xrightarrow{\tau} \mathrm{PSL}_d(\mathbb{R})$ which factors through the embedding $\tau : \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_d(\mathbb{R})$. We use the graphs of groups decomposition of $\pi_1(\Sigma)$ determined by γ^* to perform a bending of the representation in $\mathrm{PSL}_d(\mathbb{R})$ with parameter $z = (z_{\gamma})_{\gamma \in \gamma^*} \in \mathfrak{a}^{\gamma^*}$. This construction can be thought of as bending the surface S along the geodesic multicurve γ^* in the space of representations into G.

Definition 3.2. We denote by $\operatorname{Gr}_{z}^{\gamma^{*}} \rho \colon \pi_{1}(G_{\gamma^{*}}, T) \to \operatorname{PSL}_{d}(\mathbb{R})$ the representation induced by $\tilde{\rho}_{z}$, and sometimes just ρ_{z} if there is only one hyperbolic structure involved. We call it the *Hitchin grafting representation* with data z along γ^{*} .

Up to conjugation, the representation ρ_z only depends on the grafting parameter z. A *Hichin grafting ray* is a one-parameter family of Hitchin grafting representations $t \to \rho_{tz}$ defined by a ray in $(\mathfrak{a})^k$ where k is the number of components of the multicurve γ^* along which the grafting is performed.

3.2. The characteristic surface for Hitchin grafting representations. Consider a Fuchsian representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_d(\mathbb{R})$ and denote by S the hyperbolic surface defined by this representation. Choose some grafting datum z and let ρ_z be the Hithin grafted representation defined by ρ and z. As this representation is contained in the Hitchin component, it follows from Labourie [Lab06] and Fock–Goncharov [FG06] that ρ_z is faithful, with discrete image. In particular, the quotient manifold $\rho_z \setminus \mathbb{X}$ is homotopy equivalent to S; in fact ρ induces a natural homotopy class of homotopy equivalences between $\rho_z \setminus \mathbb{X}$ and S.

The following statement is Proposition 2.5 of [BHM25].

Proposition 3.3. Consider a Hitchin grafting representation ρ_z obtained from ρ and with grafting datum z. Let S_z be the abstract grafting of S from Definition 3.1, with universal covering \widetilde{S}_z . Then there exists a piecewise totally geodesic immersed surface $\widetilde{S}_z^{\iota} \subset \mathbb{X}$ and a ρ_z -equivariant immersion $\widetilde{Q}_z \colon \widetilde{S}_z \to \widetilde{S}_z^{\iota} \subset \mathbb{X}$.

The map \widetilde{Q}_z is a path isometry for the Riemannian (resp. Finsler) metric on \widetilde{S}_z and the induced path metric on $\widetilde{S}_{z}^{\iota}$ from the Riemannian (resp. Finsler) metric on X.

3.3. Admissible paths. An important tool for the geometric investigation of Hitchin grafting representations are *admissible paths* which were introduced in [BHM25]. They are defined as follows (compare Section 2.5 of [BHM25]).

Definition 3.4. Consider a closed hyperbolic surface S, a multicurve $\gamma^* \subset S$ and a grafting parameter z. Then S_z is the abstract grafted surface with hyperbolic part $S^{\text{hyp}} \subset S_z$ and flat (cylindrical) part $\mathcal{C} \subset S_z$. An admissible path in S_z is a continuous path $c \subset S_z$ such that

- c is geodesic outside $\mu = S^{\text{hyp}} \cap \mathcal{C}$;
- the hyperbolic part $c \cap S^{\text{hyp}}$ intersects γ^* orthogonally;
- a component of the flat part $c \cap C$ connects the two distinct boundary components of the flat cylinder containing it.

Similarly one can define *admissible loops*.

Note that if z is trivial then $S_z = S$ and the above definition still makes sense. The flat part \mathcal{C} is just γ^* , and the path is allowed to contain arcs in γ^* separating two geodesic arcs which emanate to the two distinct sides of γ^* in a tubular neighborhood of γ^* .

An admissible path in the universal cover \tilde{S}_z is the lift of an admissible path in S_z . Any two points of \tilde{S}_z are connected by a unique admissible path; in other words, any path of S_z is homotopic (with fixed endpoints) to a unique admissible path. Similarly, any loop in S_z not homotopic to a component of γ^* is freely homotopic to a unique admissible loop.

We define an *admissible path* in a characteristic surface of a Hitchin grafting representation to be the image of an admissible path in the abstract grafted surface under the natural path isometry. A more conceptual notion of admissible paths which is not needed toward our goal can be found in [BHM25].

For a constant C > 0, a path $\gamma : [0,T] \to X$ in a metric space (X, d_X) is called C-quasi-ruled if for any $0 \le t \le s \le u \le T$ it holds

$$d_X(\gamma(t), \gamma(s)) + d_X(\gamma(s), \gamma(u)) \le d_X(\gamma(t), \gamma(u)) + C.$$

The following is Proposition 4.10 of [BHM25].

Proposition 3.5. For any $\omega > 0$ there exists C_{ω} such that any $(\omega, 0)$ -admissible path c in X is Finsler C_{ω} -quasi-ruled. Moreover, it is at Hausdorff distance at most C'_{ω} from some Finsler geodesic, where C'_{ω} only depends on C_{ω} .

While the above discussion gives a geometric account on admissible paths in the symmetric space \mathbb{X} , we shall also use admissible paths in the group G. the following algebraic definition of admissible paths in X. The description of these paths uses a basepoint for the action of G which is determined by the Fuchsian representation τ as well as the following notation.

Notation 2. We set • $a_t := \tau \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \in G;$

- $r_{\theta} := \tau \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \in G;$ $a'_t := r_{\pi/2} \cdot a_t \cdot r_{\pi/2}^{-1} \in G;$
- for every $t \in \mathbb{R} \cup \{\infty\} = \partial \mathbb{H}^2$ we write $\xi_t = \partial_\infty \tau(t)$.

The group $G = \text{PSL}_d(\mathbb{R})$ identifies with one component of the space of \mathbb{H}^2 -frames Y in $G; g \mapsto g \cdot F_o$, where $F_o = (o, v_o, w_o)$ is a fixed \mathbb{H}^2 -frame, so that o is fixed by r_{θ} , and $v_o = \frac{d}{dt}_{|t=0}a'_t \cdot o$ and $w_o = \frac{d}{dt}_{|t=0}a_t \cdot o$ are tangent to the axes of a'_t and a_t , respectively.

Under this identification, the geodesic flow on the space Y of \mathbb{H}^2 -frames corresponds to the multiplication on the right by a'_t : i.e. $geod_t(gF_o) = (ga'_t)F_o$. On the other hand, the orthogonal sliding flow corresponds to the multiplication on the right by $\exp(z)$: that is, $\operatorname{slide}_z(gF_o) = (g \cdot \exp(z))F_o$ for any $z \in \mathfrak{a}$. This leads us to the following definition of admissible path.

Definition 3.6. A path $c: [0,T] \to G$ or $c: [0,\infty) \to G$ is said to be of

- flat type if $c(t) = g \cdot \exp(tz)$ for some $g \in G$ and $z \in \mathfrak{a}$ of norm 1 for the Finsler metric 3:
- hyperbolic type if $c(t) = ga'_t$ for some $g \in G$.

An *admissible path* of G is a *continuous* (possibly infinite) concatenation of paths of flat and hyperbolic type.

3.4. Geometric control: Uniform quasi-isometry. Recall that S is a hyperbolic closed surface, let $G = \text{PSL}_d(\mathbb{R})$ and $\tau : \text{PSL}_2(\mathbb{R}) \to G$ be the usual irreducible representation. The following is Theorem 5.1 of [BHM25] which was obtained as a consequence of Fock Goncharov positivity.

Theorem 3.7. For every $\sigma > 0$, there exists $C_{\sigma} > 0$ such that the following holds.

Consider a closed hyperbolic surface S, a multicurve $\gamma^* \subset S$ whose components have length at most σ , and a grafting parameter z such that all cylinder heights of the abstract grafting S_z are bounded from below by some number L > 0.

Let us endow X with the G-invariant admissible Finsler metric \mathfrak{F} and S_z with the pullback of this metric under Q_z , denoted by $d_{\tilde{S}_z}^{\mathfrak{F}}$. Then the grafting map $\tilde{Q}_z : \tilde{S}_z \to X$ is an injective quasi-isometric embedding with multiplicative constant $(1 + C_{\sigma}/(L+1))$ and additive constant C_{σ} ; more precisely, for all $x, y \in S_z$ we have

$$\left(1+\frac{C_{\sigma}}{L+1}\right)^{-1}d^{\mathfrak{F}}_{\tilde{S}_{z}}(x,y)-C_{\sigma}\leq d^{\mathfrak{F}}(\tilde{Q}_{z}(x),\tilde{Q}_{z}(y))\leq d^{\mathfrak{F}}_{\tilde{S}_{z}}(x,y).$$

Moreover, the image $\tilde{S}_z^{\iota} = \tilde{Q}_z(\tilde{S}_z)$ is C_{σ} -Finsler-quasiconvex in the sense that for all $x, y \in \tilde{Q}_z(\tilde{S}_z)$, there is a Finsler geodesic from x to y at distance at most C_σ from \tilde{S}_z^{ι} .

There also is the following coarse estimates on length (Theorem 5.2 of [BHM25]). **Theorem 3.8.** In the setting of Theorem 3.7, let $(\rho_z)_z$ be the associated family grafted Hitchin representations. Then there is C'_{σ} only depending on σ such that for any $\gamma \in \pi_1(S)$,

$$\ell^{\mathfrak{F}}(\rho_z(\gamma)) \ge \frac{L+1}{C'_{\sigma}}\iota(\gamma,\gamma^*).$$

Moreover, recalling that z is the datum of a vector $z_e \in \mathfrak{a}$ for each component $e \subset \gamma^*$, then C'_{σ} may be chosen so that if $z_e \in \ker(\alpha_0)$ for any e then

$$\ell^{\mathfrak{F}}(\rho_z(\gamma)) \ge \left(1 + \frac{C'_{\sigma}}{L+1}\right)^{-1} \ell_S(\gamma),$$

where $\ell_S(\gamma)$ is the length of γ in S.

4. INTERSECTION IN THE HITCHIN COMPONENT

This section contains an application of the main results of [BHM25] to dynamical properties of Hitchin grafting representations. Recall from Section 2 the definition of the intersection number $\mathbf{I}(f_1, f_2)$ and the normalized intersection number $\mathbf{J}(f_1, f_2)$ for two Hölder continuous positive functions f_1, f_2 on T^1S . These numbers only depend on the cohomology classes of f_1, f_2 . Thus by the results in Section 2.2, for any two Hitchin representations $\rho_1, \rho_2 : \pi_1(S) \to \mathrm{PSL}_d(\mathbb{R})$, we obtain intersection numbers $\mathbf{I}(\rho_1, \rho_2)$ and normalized intersection numbers $\mathbf{J}(\rho_1, \rho_2)$. We show

Theorem 4.1. There exists a sequence ρ_i of Hitchin representations such that $\mathbf{I}(\nu, \rho_i) \rightarrow \infty$ and $\mathbf{J}(\nu, \rho_i) \rightarrow \infty$ for any Fuchsian representation ν , and this divergence is uniform in ν .

The Hitchin representations which enter Theorem 4.1 are Hitchin grafting representations. More precisely, let as before γ be a simple closed geodesic on the hyperbolic surface S. This datum is used to construct for each L > 0 a Hitchin representation ρ_L obtained by Hitchin grafting along γ of the Fuchsian representation defined by S, with cylinder height L. We do not specify the twisting number of the associated abstract grafting datum as this does not play a role in our discussion, but we assume that $L \to \rho_L$ is a Hitchin grafting ray as introduced in Section 3.1.

The proof of Theorem 4.1 rests on statistical information on length averages, introduced in the next definition. For its formulation, for a Hitchin representation ρ put $R_{\rho}(T) = R_{\ell_{\rho}}(T)$ for all T, where as before, $R_{\ell_{\rho}}(T) = \{\eta \in [\pi_1(S)] \mid \ell_{\rho}(\eta) \leq T\}$ and $\ell_{\rho}(\eta)$ is the Finsler translation length of $\rho(\eta)$. Moreover, $[\pi_1(S)]$ is the set of conjugacy classes of the fundamental group $\pi_1(S)$ of S.

Definition 4.2. Let ρ be a Hitchin representation and A a subset of $[\pi_1(S)]$. We say that A is a *full density* set for ρ if

$$\liminf_{T \to +\infty} \frac{R_{\rho}(T) \cap A}{R_{\rho}(T)} = 1.$$

If \mathcal{P} is an assertion on $[\pi_1(S)]$, we say that a *typical geodesic satisfies* \mathcal{P} if the set $\{\gamma \in [\pi_1(S)] \mid \gamma \text{ satisfies } \mathcal{P}\}$ is a full density set for ρ .

The following statement can be thought of as a statistical version of the duality between length and intersection for hyperbolic metrics on surfaces. Recall from Section 2.1 the definition of the intersection form $\iota : \mathcal{C}(S) \times \mathcal{C}(S) \to [0, \infty)$.

Proposition 4.3. Let ρ be a hyperbolic metric on S, and let $\alpha \subset S$ be a closed geodesic. For any $\epsilon > 0$, for a typical geodesic γ , we have

$$\left|\iota(\gamma,\alpha) - \frac{1}{-4\pi^2 \chi(S)} \ell_{\rho}(\gamma) \ell_{\rho}(\alpha)\right| < \epsilon \ell_{\rho}(\gamma).$$

Proof. The Borel measures

$$\mu_T = \frac{1}{\#R_\rho(T)} \sum_{\ell_\rho(\gamma) \le T} \operatorname{Leb}_{\gamma}$$

converge weakly as $T \to \infty$ to the normalized Lebesgue Liouville measure λ_0 on T^1S (see [Mar04]).

Let $\lambda \in \mathcal{C}(S)$ be the (unnormalized) Liouville *current* of ρ , the current defined by the Lebesgue Liouville measure on T^1S , and for each T let $\hat{\mu}_T$ be the current defined by μ_T . Let α be a closed geodesic on S. As $\iota(\alpha, \lambda) = \ell_{\rho}(\alpha)$ (see Section 2.1), by continuity of the intersection form ι for the weak topology on currents, we know that

$$\iota(\hat{\mu}_T, \alpha) \xrightarrow[T \to \infty]{} \frac{1}{-4\pi^2 \chi(S)} \ell_{\rho}(\alpha).$$

Note to this end that the total volume of T^1S with respect to the Lebesgue Liouville current equals $-4\pi^2 \chi(S)$. Put $\kappa = \frac{1}{-4\pi^2 \chi(S)}$ and let $\epsilon > 0$. To show that the geodesics γ with

$$|\iota(\gamma,\alpha) - \kappa \ell_{\rho}(\gamma)\ell_{\rho}(\alpha)| < \epsilon \kappa \ell_{\rho}(\gamma)$$

are typical we argue as follows. For T > 0 let

$$\mathcal{A}(T) = \{ \gamma \mid \ell_{\rho}(\gamma) \leq T, \iota(\gamma, \alpha) \geq (1 + \epsilon) \kappa \ell_{\rho}(\gamma) \ell_{\rho}(\alpha) \}.$$

We claim that $\frac{\#\mathcal{A}(T)}{\#R_{\rho}(T)} \to 0 \ (T \to \infty)$. To see this assume otherwise. By passing to a subsequence, we may assume that the measures $\nu_T = \frac{1}{\# R_{\rho}(T)} \sum_{\gamma \in \mathcal{A}(T)} \text{Leb}_{\gamma}$ converge weakly to a nontrivial Φ^t -invariant measure ν . By construction, the measure ν is absolutely continuous with respect to the Lebesgue Liouville measure λ . It defines a current $\hat{\nu}$ which satisfies

(17)
$$\iota(\hat{\nu},\alpha)/\nu(T^1S) \ge (1+\epsilon)\kappa\ell_{\rho}(\alpha).$$

But λ is ergodic under the action of Φ^t and hence as ν is absolutely continuous with respect to λ , it is a positive constant multiple of λ . This contradicts the inequality (17) and equation (5).

In the same way we conclude that $\frac{\#\mathcal{B}(T)}{\#\mathcal{B}_{\rho}(T)} \to 0$ as $T \to \infty$ where

$$\mathcal{B}(T) = \{ \gamma \mid \ell_{\rho}(\gamma) \le T, \iota(\gamma, \alpha) \le (1 - \epsilon) \kappa \ell_{\rho}(\gamma) \ell_{\rho}(\alpha) \}$$

Since $\epsilon > 0$ was arbitrary, this shows the proposition.

Let X be a hyperbolic metric on S and let c be a non-separating simple closed geodesic on X of length $\ell > 0$. For $L \ge 0$ denote by ρ_L a representation obtained by Hitchin grafting of X on c of height L. Our goal is to estimate for a hyperbolic metric Y on S the quantities $\mathbf{I}(Y, \rho_L)$ and $\mathbf{J}(Y, \rho_L)$ as $L \to \infty$.

Proof of Theorem 4.1. Let $X \in \mathcal{T}(S)$ be the marked hyperbolic metric which is the basepoint for the Hitchin grafting ray. According to the length control as formulated in

Theorem 3.8, for every $\epsilon > 0$ there exist $C_{\sigma} > 0$ depending on the hyperbolic length σ of the simple closed curve c such that we have

(18)
$$\ell_{\rho_L}(\gamma) \ge \max\left\{C_{\sigma}L\iota(\gamma,c), \frac{L}{L+C_{\sigma}^{-1}}\ell_X(\gamma)\right\}$$

where we use the notations of Theorem 3.8, lengths in X are measured with respect to an admissible Finsler metric, and ℓ_X denotes the length for the hyperbolic metric X.

Let m > 0 be a fixed number. Our goal is to find a number L > 0 so that

$$\mathbf{J}(Y,\rho_L) \ge m$$

for every $Y \in \mathcal{T}(S)$ where as before, $\mathcal{T}(S)$ denotes the Teichmüller space of marked hyperbolic metric on S.

By Theorem 12 of [Bon88], the map which associates to a marked hyperbolic metric on S its Liouville current is a proper topological embedding. More precisely, for the given number m > 0, there exists a compact ball B about X in $\mathcal{T}(S)$ such that $\iota(\lambda_X, \lambda_Y) \ge m$ for all marked hyperbolic metrics $Y \in \mathcal{T}(S) - B$, where λ_X, λ_Y are the currents defined by the normalized Lebesgue Liouville measures. Note that this is symmetric in X, Y. Furthermore, we have $\iota(\lambda_Y, \lambda_X) = \mathbf{J}(Y, X)$. We refer to p.152-153 in [Bon88] for details on these facts.

By the estimate (18), for any $\epsilon > 0$ and all sufficiently large $L \ge 0$ depending on ϵ , say for all $L \ge L(\epsilon)$, we have

$$\ell_{\rho_L}(\gamma) \ge (1 - \epsilon)\ell_X(\gamma).$$

Thus by possibly increasing the ball B we may assume that $\mathbf{J}(Y, \rho_L) \ge m$ for all $L \ge L_0$ and all $Y \notin B$.

We are left with showing that by possibly increasing L_0 , we also have $\mathbf{J}(Y, \rho_L) \geq m$ for all $Y \in B$. However, this follows once more from the estimate (18). Namely, let $Y \in B$. By Proposition 4.3, we know that there exists a constant $\kappa > 0$ such that

$$\iota(\gamma, c) \ge \kappa (1 - \epsilon) \ell_Y(\gamma) \ell_Y(c)$$

for any geodesic γ which is typical for Y.

On the other hand, by compactness of B, there exists a constant $\sigma > 0$ such that $\ell_Y(c) \geq \sigma$ for every $Y \in B$. Then for a geodesic γ which is typical for Y, we have $\ell_Y(\gamma) \leq \frac{1}{\kappa \sigma(1-\epsilon)} \iota(\gamma, c)$. Thus for $L > m/\kappa \sigma(1-\epsilon)C_{\sigma}$ it holds

$$\ell_{\rho_L}(\gamma)/\ell_Y(\gamma) \ge \kappa \sigma (1-\epsilon) C_\sigma L \ge m$$

which is what we wanted to show. Together with the definition, it shows that $\mathbf{I}(\nu, \rho_L) \to \infty$ for every Fuchsian representation ν .

To show that we also have $\mathbf{J}(\nu, \rho_L) \to \infty$ for all Fuchsian representations it suffices to observe that the entropy of ρ_L is bounded from below by a universal positive constant. To see that this is the case, recall that for each L, the restriction of the representation ρ_L to the free subgroup Λ of $\pi_1(S)$ of all based loops which do not cross through c does not depend on L. In particular, the image of Λ under ρ_L stabilizes a totally geodesic hyperbolic plane in \mathbb{X} . As a consequence, for each L the entropy of ρ_L is not smaller than the entropy of the geodesic flow on the bordered surface S - c, which is positive as S - cis a hyperbolic surface with geodesic boundary. Together with the control on $\mathbf{I}(\nu, \rho_L)$ established in the beginning of this proof, this implies that $\mathbf{J}(\nu, \rho_L) \to \infty$ $(L \to \infty)$ for any Fuchsian representation ν .

5. Upper bound on the derivatives of length functions

In this section we show how to control the first and second derivatives of the Finsler length for paths of Hitchin grafting representations. The section is divided into four subsections.

5.1. Derivative bounds for lengths of closed geodesics. Recall that we have fixed a marked hyperbolic surface (S, h), that is, a point in the Teichmüller space $\mathcal{T}(S)$, and a multicurve $\gamma^* = \gamma_1^* \cup \cdots \cup \gamma_N^* \subset S$. In this section we study grafted representations of $\pi_1(S)$ into $G = \mathrm{SL}_d(\mathbb{R})$ via the irreducible representation $\tau : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_d(\mathbb{R})$. Recall that a bending parameter $z \in \mathfrak{a}^N$ above γ^* is the datum of an element $z_i \in \mathfrak{a}$ for each component γ_i^* of γ^* . Given such a bending parameter z, we can construct a conjugacy class of Hitchin representations $[\rho_{h,z}] = [\rho_z] : \pi_1(S) \to \mathrm{SL}_d(\mathbb{R})$. Here $\rho_{h,0}$ is just the image of $h \in \mathcal{T}(S)$ under the representation τ .

The following proposition is the main technical ingredient toward a control of the pressure length of suitably chosen paths in $\operatorname{Hit}(S)$, namely estimates on the derivatives of the map $z \mapsto \lambda \circ \rho_{h,z}(\gamma)$ where $\gamma \in \pi_1(S)$, and $\lambda : G \to \overline{\mathfrak{a}^+}$ is the Jordan projection (see Section 1). To this end recall from Section 2 that for any $z_0 \in \mathfrak{a}^N$ and any $\gamma \in \pi_1(S)$, the differential $d(\lambda \circ \rho_z(\gamma))_{z=z_0}$ of the Jordan projections for the deformations of ρ_{z_0} obtained by grafting along γ^* is defined and can be thought of as a linear map $\mathfrak{a}^N \to \mathfrak{a}$. If we equip \mathfrak{a} with the norm $\| \|$ obtained from the Killing form on $\mathfrak{sl}_d(\mathbb{R})$, then this linear map has an operator norm which we denote by $\| d(\lambda \circ \rho_z(\gamma)) \|_{z=z_0} \|$. Note that the hyperbolic metric h is left fixed in this construction. Similarly, the Hessian $d^2(\lambda \circ \rho_z(\gamma))\|_{z=z_0}$ evaluated on the subspace of $\operatorname{Hit}(S)$ defined by grafting along γ^* can be thought of as an \mathfrak{a} -valued bilinear form on \mathfrak{a}^N which has an operator norm.

Proposition 5.1. For any $\sigma > 0$ there exists $C_{\sigma} > 0$ such that for any $h \in \mathcal{T}(S)$ giving length $\leq \sigma$ to each component of γ^* , for any bending parameter $z_0 \in \mathfrak{a}^N$ along γ^* and for any $\gamma \in \pi_1(S)$, we have

$$\left\| d(\lambda \circ \rho_z(\gamma)) \right\|_{z=z_0} \|, \ \left\| d^2(\lambda \circ \rho_z(\gamma)) \right\|_{z=z_0} \| \le C_{\sigma}\iota(\gamma,\gamma^*) \|_{z=z_0}$$

The idea for the proof of the proposition is the following. Given a family of Hitchin grafting representations $(\rho_z)_{z \in \mathfrak{a}}$ based at a Fuchsian representation ρ_0 , and $\gamma \in \pi_1(S)$, we decompose γ into an admissible path for the hyperbolic structure corresponding to ρ_0 . Say the admissible path travels in $S - \gamma^*$ for the time t_1 , then meets orthogonally some component $\gamma_{i_1}^*$ of γ^* , then travels along it for some time s_1 , then departs from it orthogonally and travels for some time t_2 in $S - \gamma^*$... etc. This gives us the following formula for the conjugacy class of the holonomy $\rho_0(\gamma)$:

$$\rho_0(\gamma) \sim a_{t_1}' a_{s_1} a_{t_2}' \cdots a_{t_k}' a_{s_k},$$

where k is the intersection number of γ with γ^* and where we use the notations interoduced in Definition 3.6.

The grafting operation deforms the above expression in a very explicit way, namely

(19)
$$\rho_z(\gamma) \sim a'_{t_1} a_{s_1} \exp(r_1(z)) a'_{t_2} \cdots a'_{t_k} a_{s_k} \exp(r_k(z)),$$

where $r_n(z)$ is either the *i*-th coordinate z_{i_n} (where $\gamma_{i_n}^*$ is the *n*-th component of γ^* crossed by γ), or $\iota(-z_{i_n})$ where $\iota: \mathfrak{a} \to \mathfrak{a}$ is the Cartan involution, depending on whether we cross $\gamma_{i_n}^*$ from left to right or from right to left. Taking \mathfrak{a} as the vector space of trace free diagonal matrices, this involution permutes the diagonal entries as follows: $\operatorname{diag}(x_1, \ldots, x_d) \mapsto \operatorname{diag}(x_d, \ldots, x_1)$.

To estimate derivatives of the Jordan projection of the above product of 3k matrices we simply differentiate the expression in the equation (19), which is simply a product of matrices, using the product rule for differentiation. This yields the following formula for the first order derivative, where we put $X_n = dr_n(z)|_{z=z_0}$.

 $d(\lambda \circ \rho_{z}(\gamma))|_{z=z}$

$$= \sum_{n=1}^{k} d\lambda (a'_{t_1} a_{s_1} \exp(r_1(z_0)) \cdots a'_{t_n} a_{s_n} \exp(r_n(z_0)) X_n \cdots a'_{t_k} a_{s_k} \exp(r_k(z_0)))$$

$$= \sum_{n=1}^{k} d\lambda (a'_{t_{n+1}} a_{s_{n+1}} \exp(r_{n+1}(z_0)) \cdots a'_{t_1} a_{s_1} \exp(r_1(z_0)) \cdots a'_{t_n} a_{s_n} \exp(r_n(z_0)) X_n)$$

where the second equation in this formula uses the fact that the Jordan projection is invariant under conjugation.

The norm of this differential will be controlled using the fact that the matrix

$$a'_{t_{n+1}}a_{s_{n+1}}\exp(r_{n+1}(z_0))\cdots a'_{t_k}a_{s_k}\exp(r_k(z_0))a'_{t_1}a_{s_1}\exp(r_1(z_0))\cdots a'_{t_n}a_{s_n}\exp(r_n(z_0))$$

is totally positive in a quantitative way. In Section 5.4 we shall establish that it is loxodromic in a quantitative way, and in Section 5.3 we will prove general estimates for derivatives of the form $d\lambda(g \cdot X)$ when g is loxodromic in a quantitative way and $X \in \mathfrak{a}$. These results will be obtained by first estimating $d\lambda_1(g \cdot X)$ with g quantitatively proximal (Section 5.2) and then applying this to exterior powers of loxodromic elements. The computations for the second order derivative are more involved but can also worked out using positivity.

Proposition 5.1 will be useful to bound the pressure length of Hitchin grafting paths in the Hitchin component, namely paths of the form $(\rho_{h,tz}) = (\rho_{tz})_{t\geq 0}$, using the previous notations, that is, $h \in \mathcal{T}(S)$ is a fixed marked hyperbolic metric and $z \in \mathfrak{a}^N$ is a fixed grafting parameter. It will also be used to control the pressure lengths of other kinds of paths, where we fix the grafting parameter z and let the hyperbolic metric vary instead. More precisely, we will deform the hyperbolic metric by *shearing*, which is in fact a particular case of bending, and this allows to give a unified treatment for both types of deformations.

Proposition 5.1 has the following consequence. For its formulation, it is useful to keep in mind that grafting along a multicurve γ^* commutes with modifying the hyperbolic metric in the complement of γ^* . More precisely, fix a basepoint x_i on each component of γ^* . If $S_1 \subset S$ is a component of the complement of γ^* then using the basepoint as a marked point, the Teichmüller space $\mathcal{T}(S_1)$ of S_1 is the space of marked hyperbolic metrics on S_1 with geodesic boundary of fixed length and one marked point in each boundary component. Each choice of a hyperbolic metric h on S determines an embedding of $\mathcal{T}(S_1)$ into $\mathcal{T}(S)$ (where the boundary lengths depend on h) by first marking a point on each geodesic which defines a boundary component of S_1 , cutting S open along these *h*-geodesics and gluing a marked metric $h_1 \in \mathcal{T}(S_1)$ to $S \setminus S_1$ matching marked points. By the definition of grafting, this operation commutes with grafting along γ^* .

Observe that one way to deform the marked hyperbolic metric on S_1 is to shear (or twist) along a closed geodesic entirely contained in S_1 . As twisting is a grafting operation along a grafting parameter contained in the one-dimensional subspace of \mathfrak{a} tangent to the image of the representation τ , we obtain as a corollary of Proposition 5.1 the following result. In its formulation, a constant speed shearing path along a geodesic multicurve η is a path of marked hyperbolic metrics which consists in cutting S open along η and gluing back with a rotation whose rotation speed is constant one along the path, where the speed is the absolute value of the derivative of the signed length of the shearing deformation where length is measured with respect to the length element of the geodesic multicurve. Note that this makes sense as the length element of a shearing multicurve is constant along a shearing path.

Corollary 5.2. Fix a connected component $S_1 \subset S - \gamma^*$. Let $(h_t)_{t \in [0,1]}$ be a smooth path of hyperbolic metrics obtained from h_0 by constant speed shearing along a multicurve $\eta \subset S_1$ (we allow shearing along γ^* and different speed of shearings along the components, including zero speed).

Then there is a constant C > 0 only only depending on an upper bound on the lengths of the multicurve $\gamma^* \cup \eta$ and an upper bound on the shearing speeds along the components of η such that for all z, t, γ we have

$$\left\|\frac{d}{dt}\lambda\circ\rho_{h_t,z}(\gamma)\right\|, \ \left\|\frac{d^2}{dt^2}\lambda\circ\rho_{h_t,z}(\gamma)\right\|\leq C\ell_{h_t}(\gamma\cap S_1),$$

where $\ell_{h_t}(\gamma \cap S_1)$ is the h_t -length of the subarcs of γ contained in S_1 .

Proof. Suppose $(h_t)_{t \in [0,1]}$ is a path of hyperbolic metrics obtained by constant speed shearing h_0 along a multicurve $\eta \subset S_1$. Shearing is a special kind of grafting or bending, with grafting parameter collinear to $d\tau \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Note that $\beta = \gamma^* \cup \eta$ is a multicurve. For a fixed grafting parameter $z \in \mathfrak{a}^N$ and $t \in [0, 1]$ let $\rho_{z,t}$ be the representation

For a fixed grafting parameter $z \in \mathfrak{a}^N$ and $t \in [0, 1]$ let $\rho_{z,t}$ be the representation obtained by grafting h_t along γ^* with grafting parameter z. Then $\rho_{0,t}$ is just the Fuchsian representation defined by the marked hyperbolic metric h_t . Since S_1 is a component of $S \setminus \gamma^*$, it follows from Proposition 5.1 that there exists a number C > 0 only depending on an upper bound for the h_t -lengths of the components of β , which is independent of t, and an upper bound on the shearing speed, such that for each grafting parameter z along γ^* and every $\gamma \in \pi_1(S)$, we have

$$\left\|\frac{d}{dt}\lambda\circ\rho_{h_t,z}(\gamma)\right\|,\ \left\|\frac{d^2}{dt^2}\lambda\circ\rho_{h_t,z}(\gamma)\right\|\leq C\iota(\gamma,\beta).$$

It remains to check that $\iota(\gamma, \beta) \leq C' \ell_{h_t}(\gamma \cap S_1)$ for some constant C' only depending on the upper bound of the lengths of the components of β . For this one just takes C'equal to half the infimum among all t's of the shortest h_t -distance from one component of β to another, which is uniformly bounded from below along the path by the collar lemma.

5.2. Derivatives of lengths of proximal transformations. For a (d, d)-matrix A and any $i \leq d$ we denote by $\lambda_i(A) \in [-\infty, +\infty)$ the logarithm of the absolute value of the *i*-th eigenvalue where the eigenvalues are ordered in nonincreasing absolute values.

We use the convention $\log 0 = -\infty$. Of course the derivatives of λ_i at a point A where $\lambda_i(A) = -\infty$ do not make sense. In the following, every time we need to compute such derivatives, we will always make sure that $\lambda_i(A) > -\infty$. We also write

$$\lambda(A) = (\lambda_1(A), \lambda_2(A), \dots) \in [-\infty, +\infty)^d,$$

which generalizes the Jordan projection when $A \in \mathrm{SL}_d(\mathbb{R})$.

Recall that A is proximal if $\lambda_1(A) > \lambda_2(A)$, which means A has a real eigenvalue of multiplicity one (called the dominant eigenvalue) whose absolute value is strictly greater than that of any other eigenvalue. In particular, we have $\lambda_1(A) > -\infty$. The eigenline associated to the dominant eigenvalue is called the *attracting eigenline*. If A is of maximal rank, then A acts on $\mathbb{R}P^{d-1}$, and the attracting eigenline is an attracting point for the action of A on the projective space. The complementary A-invariant hyperplane (the sum of the remaining generalized eigenspaces) is called the *repelling hyperplane*.

We will need a quantitative version of proximality. For this we endow \mathbb{R}^d with its usual inner product coming from its canonical basis. In the following, all angles come from this fixed inner product.

For any $0 < \theta < \pi/2$ and $\omega \in (0, \infty]$ we denote by $P_{\omega,\theta}$ the set of nonzero proximal (possibly noninvertible) matrices A with the following two properties.

(a) $\lambda_1(A) - \lambda_2(A) \ge \omega$ (allowing $\lambda_2(A) = -\infty$),

(b) the attracting eigenline of A makes an angle $\geq \theta$ with the repelling hyperplane.

We denote by $DP_{\omega,\theta} \subset P_{\omega,\theta} \times P_{\omega,\theta}$ the set of pairs (A_0, A_1) such that A_i 's attracting eigenline forms an angle $\geq \theta$ with A_{1-i} 's repelling hyperplane, and such that the product $A_i A_{1-i}$ is in $P_{\omega,\theta}$. Note that if $A \in P_{\omega,\theta}$ then $(A, A) \in DP_{\omega,\theta}$.

In the next lemma, the norm ||X|| of an element in the Lie algebra $\mathfrak{gl}_d(\mathbb{R}) = \mathfrak{sl}_d(\mathbb{R}) \oplus \mathbb{R}$ is the norm induced by the Killing form of $\mathfrak{sl}_d(\mathbb{R})$ and the choice of a Cartan involution. Furthermore as before, exp: $\mathfrak{gl}_d(\mathbb{R}) \to \mathrm{GL}_d(\mathbb{R})$ denotes the exponential map. By left translation, this exponential map defines for any $A \in \operatorname{GL}_d(\mathbb{R})$ a map $A \cdot \exp : X \in$ $\mathfrak{gl}_d(\mathbb{R}) \to A \cdot \exp(X) \in \mathrm{GL}_d(\mathbb{R})$. In the second statement of the lemma below, the Hessian is taken of a function defined on the direct sum $\mathfrak{gl}_d(\mathbb{R}) \oplus \mathfrak{gl}_d(\mathbb{R})$, and the norm is the operator norm.

Lemma 5.3. For all $0 < \theta < \pi/2$ and $\omega_0 > 0$, there exists $C = C_{\omega_0,\theta} > 0$ such that for any $\omega > \omega_0$ and any $(A, B) \in DP_{\omega,\theta}$, the following is satisfied.

- (1) $||d\lambda_1(A \cdot \exp)|_0|| \le C_{\omega_0,\theta}$ and $||d^2\lambda_1(A \cdot \exp)|_0|| \le C_{\omega_0,\theta};$ (2) For $(X,Y) \in \mathfrak{gl}_d(\mathbb{R}) \oplus \mathfrak{gl}_d(\mathbb{R})$ it holds $||d^2\lambda_1(A \cdot \exp \cdot B \cdot \exp)|_{(0,0)}(X,Y)|| \le$ $C_{\omega_0,\theta}e^{-\omega}$.

Proof. The idea of the proof is to use a compactness argument, by restricting without loss of generality to a compact subset of $P_{\omega,\theta}$. Namely; let $P'_{\omega,\theta} \subset P_{\omega,\theta}$ be the set of proximal

matrices $A \in P_{\omega,\theta}$ with spectral radius $e^{\lambda_1(A)}$ equal to 1, and $DP'_{\omega,\theta} = DP_{\omega,\theta} \cap (P'_{\omega,\theta} \times P'_{\omega,\theta})$. Let us check that restricting to this compact subset does no harm: For all $(A, B) \in DP_{\omega,\theta}$, if $A' = e^{-\lambda_1(A)}A$ and $B' = e^{-\lambda_1(B)}B$ then $\lambda_1(A \cdot \exp(X)) = \lambda_1(A) + \lambda_1(A' \cdot \exp(X))$ and $\lambda_1(A \cdot \exp(X) \cdot B \cdot \exp(Y)) = \lambda_1(A) + \lambda_1(B) + \lambda_1(A' \cdot \exp(X) \cdot B' \cdot \exp(Y))$ for all X, Y. Thus the derivatives we need to estimate are the same for (A, B) and for (A', B').

The set $P'_{\mu\theta}$ is a compact subset of the space $\mathfrak{gl}_d(\mathbb{R}) = \mathbb{R}^{d^2}$ of (d, d)-matrices, and the restriction of the function λ_1 to an open neighborhood of the compact set $P'_{\omega\theta}$ in \mathbb{R}^{d^2} is smooth. and Hence the differential $d\lambda_1$ is smooth section of the restriction of the cotangent bundle of \mathbb{R}^{d^2} to $P'_{\omega,\theta}$, and the Hessian is a smooth section of the bundle of symmetric bilinear forms on \mathbb{R}^{d^2} . Thus there exists a constant K > 0 so that for any $A \in P'_{\omega,\theta}$ we have

(20)
$$\|d\lambda_1(A \cdot \exp)|_0\|, \ \|d^2\lambda_1(A \cdot \exp)|_0\| \le K.$$

This shows the first part of the lemma.

To show the second part of the lemma note that since $DP'_{\omega,\theta}$ is compact, up to increasing K we may assume that $||d^2\lambda_1(A \cdot \exp \cdot B \cdot \exp)|_{(0,0)}|| \leq K$. Furthermore, using smoothness of the map which associates to a pair of points (A, B) in a neighborhood of the compact set $DP_{\omega_0,\theta}$ the Hessian $d^2\lambda_1(A \cdot \exp \cdot B \cdot \exp)|_{(0,0)}$, viewed as a bilinear form on the direct sum $\mathfrak{gl}_d(\mathbb{R}) \oplus \mathfrak{gl}_d(\mathbb{R})$ depending on (A, B), up to increasing once more the control constant K we obtain that

(21)
$$\left\| d^2 \lambda_1 (A \cdot \exp \cdot B \cdot \exp)_{(0,0)} - d^2 \lambda_1 (C \cdot \exp \cdot D \cdot \exp)_{(0,0)} \right\| \le K(||A - C|| + ||B - D||).$$

This estimate allows to proceed by first showing the second part of the lemma in the case $(A, B) \in DP'_{\infty,\theta}$, that is, if all nondominant eigenvalues are zero, equivalently if A and B are rank-one projectors. We will see that in this case $d^2\lambda_1(A \cdot \exp \cdot B \cdot \exp)|_{(0,0)} = 0$, and we will be able to extend to the general case using (21).

By the definition of $DP'_{\omega,\theta}$, if $(A, B) \in DP'_{\omega,\theta}$ and rk(A) = rk(B) = 1 then ker(AB) = ker(B), and for any C, we have $ker(AB) = ker(B) \subset ker(ACB)$. Thus for any C there is a number $\alpha_C \in \mathbb{R}$ such that $ACB = \alpha_C AB$. Then $\lambda_1(ACB) = \log |\alpha_C| + \lambda_1(AB)$ and consequently $\alpha_C = \pm e^{\lambda_1(ACB) - \lambda_1(AB)}$. Similarly, we have $BCA = \beta_C BA$ where $\beta_C = \pm e^{\lambda_1(BCA) - \lambda_1(BA)}$.

Recall that $\lambda_1(CD) = \lambda_1(DC)$ for all matrices. Using that $A^2 = A$ and $B^2 = B$ we note that for all $X, Y \in \mathfrak{gl}_d(\mathbb{R})$ we have

$$\lambda_1(A \cdot \exp(X) \cdot B \cdot \exp(Y)) = \lambda_1(AA \cdot \exp(X) \cdot BB \cdot \exp(Y))$$
$$= \lambda_1(A \cdot \exp(X) \cdot BB \cdot \exp(Y) \cdot A).$$

Now $A \exp(X)B = \alpha_{\exp(X)}AB$ and $B \exp(Y)A = \beta_{\exp(Y)}BA$, so their product is

$$(A\exp(X)B)(B\exp(Y)A) = \alpha_{\exp(X)}\beta_{\exp(Y)}ABBA = \alpha_{\exp(X)}\beta_{\exp(Y)}ABA$$

and hence

$$\lambda_1(A \cdot \exp(X) \cdot B \cdot \exp(Y)) = \log |\alpha_{\exp(X)}| + \log |\beta_{\exp(Y)}| + \lambda_1(ABA).$$

In particular, for the function $\rho_{A,B} : \mathfrak{gl}_d(\mathbb{R}) \oplus \mathfrak{gl}_d(\mathbb{R}) \to \mathbb{R}$ defined by $\rho_{A,B}(X,Y) = \lambda_1(A \cdot \exp(X) \cdot B \cdot \exp(Y))$ we have $d^2\rho_{A,B}(X,Y)_{(0,0)} = 0$. Equivalently, the splitting $\mathfrak{gl}_d(\mathbb{R}) \oplus \mathfrak{gl}_d(\mathbb{R})$ is orthogonal for the Hessian of $\rho_{A,B}$.

Now take arbitrary $(A, B) \in DP'_{\omega,\theta}$. Let $C \in P'_{\omega,\theta}$ (resp. D) be the rank-one projector with kernel the repelling hyperplane of A (resp. B) and with image the attracting line of A (resp. B). By (21), combined with $d^2\rho_{C,D}(X,Y) = 0$ we have

$$\|d^2\lambda_1(A \cdot \exp(X) \cdot B \cdot \exp(Y))\| \le K(||A - C|| + ||B - D||).$$

One can then check that there exists a constant K' that depends on θ such that

$$||A - C|| \le K' e^{-\omega},$$

and similarly for B and D. This proves the second part of the lemma.

5.3. Derivatives of lengths of loxodromic transformations. We are going to deduce from the previous section an estimate for the derivatives of lengths of loxodromic transformations. The argument is classical: it relies on the fact that a matrix $A \in \operatorname{GL}_d \mathbb{R}$ is loxodromic if and only if all its exterior products $\Lambda^k A \in \operatorname{GL}(\Lambda^k \mathbb{R}^d)$ are proximal for $1 \leq k \leq d-1$.

Indeed, suppose $A \in \operatorname{GL}_d(\mathbb{R})$ is loxodromic, i.e. $\lambda_1(A) > \lambda_2(A) > \cdots > \lambda_d(A)$, which means A is diagonalizable such that the eigenvalues have multiplicity 1 and distinct absolute values. Let v_1, \ldots, v_d be an eigenbasis for A ordered by decreasing absolute values of eigenvalues. Then for any k the elements of the form $v_{i_1} \wedge \cdots \wedge v_{i_k}$, where $1 \leq i_1 < \cdots < i_k \leq d$, form an eigenbasis of $\Lambda^k \mathbb{R}^d$ for $\Lambda^k A$, such that the absolute value of the logarithm of the associated eigenvalue is $\lambda_{i_1}(A) + \cdots + \lambda_{i_k}(A)$.

For any $0 \leq k \leq d$ let d_k denote the dimension of the exterior product $\Lambda^k \mathbb{R}^d$. The canonical basis e_1, \ldots, e_d of \mathbb{R}^d yields a natural basis of $\Lambda^k \mathbb{R}^d$ with elements $e_{i_1} \wedge \cdots \wedge e_{i_k}$, where $1 \leq i_1 < \cdots < i_k \leq d$. This induces an identification of $\Lambda^k \mathbb{R}^d$ with \mathbb{R}^{d_k} and an inner product on $\Lambda^k \mathbb{R}^d$.

For any transformation X of $\Lambda^k \mathbb{R}^d$, seen as a matrix of size d_k , we add an upperscript d_k to all quantities previously defined involving X to specify the size of X. For instance $\lambda_1^{d_k}(X), \ldots, \lambda_{d_k}^{d_k}(X)$ are the logarithms of the absolute values of the eigenvalues of X. We also denote by $P_{\omega,\theta}^{d_k}$ the set of proximal transformations of $\Lambda^k \mathbb{R}^d$ that satisfy the quantitative conditions of the previous section.

In particular, coming back to the computations on $A \in \operatorname{GL}_d \mathbb{R}$, we have

(22)
$$\lambda_1^{d_k}(\Lambda^k A) = \lambda_1^d(A) + \dots + \lambda_k^d(A)$$

and

(23)
$$\lambda_2^{d_k}(\Lambda^k A) = \lambda_1^d(A) + \dots + \lambda_{k-1}^d(A) + \lambda_{k+1}^d(A).$$

It is a well known fact that these formulas work for any matrix A of size d, not necessarily invertible. In particular, $\lambda_k^d(A) > -\infty$ if and only if $\lambda_1^{d_k}(\Lambda^k A) > -\infty$, and in this case

(24)
$$\lambda_k^d(A) - \lambda_{k+1}^d(A) = \lambda_1^{d_k}(\Lambda^k A) - \lambda_2^{d_k}(\Lambda^k A).$$

For all $0 < \theta < \pi$ and $\omega > 0$ we denote by $L^d_{\omega,\theta}$ the set of loxodromic invertible matrices A of size d such that $\Lambda^k A \in P^{d_k}_{\omega,\theta}$ for any $1 \le k \le d-1$. We also denote by $DL^d_{\omega,\theta}$ the set of pairs $(A, B) \in L^d_{\omega,\theta} \times L^d_{\omega,\theta}$ such that $(\Lambda^k A, \Lambda^k B) \in DP^{d_k}_{\omega,\theta}$ for any $1 \le k \le d-1$.

For a loxodromic matrix $A \in \operatorname{GL}_d(\mathbb{R})$ put

$$\lambda(A) = (\lambda_1(A), \dots, \lambda_d(A)) \in \mathbb{R}^d.$$

The notations in the following lemma extend the notations in Lemma 5.3. Lemma 5.4. For all $0 < \theta < \pi$ and $\omega_0 > 0$, there exists $C_{\omega_0,\theta} > 0$ such that for all $\omega > \omega_0$ and $(A, B) \in DL_{\omega,\theta}$ we have.

- (1) The differential and the Hessian at X = 0 of the map $X \mapsto \lambda(A \exp(X))$ are bounded above in norm by $C_{\omega_0,\theta}$,
- (2) $\left\| d^2 \lambda (A \cdot \exp(X) \cdot B \cdot \exp(Y)) \right\|_{(0,0)} \le C_{\omega_0,\theta} e^{-\omega}.$

Proof. By definition $\Lambda^k A \in P_{\omega,\theta}^{d_k}$ for any k. By Lemma 5.3 we get a constant C > 0, only depending on ω , such that for any k the first two derivatives at $X = 0_{d_k}$ of $X \mapsto \lambda_1^{d_k}((\Lambda^k A) \cdot \exp(X))$ are bounded above by C.

Since $\Lambda^k(A \cdot \exp(X)) = (\Lambda^k A) \cdot \exp(\Lambda^k X)$ and $X \mapsto \Lambda^k X$ is linear, we deduce that for any k the first two derivatives at $X = 0_d$ of

$$X \mapsto \lambda_1^{d_k}(\Lambda^k(A \cdot \exp(X))) = \lambda_1^d(A \cdot \exp(X)) + \dots + \lambda_k^d(A \cdot \exp(X))$$

are bounded above by some constant C' only depending on C.

This implies that the first two derivatives at $X = 0_d$ of

$$X \mapsto \lambda^d (A \cdot \exp(X)) = (\lambda_1^d (A \cdot \exp(X)), \lambda_2^d (A \cdot \exp(X)), \dots, \lambda_d^d (A \cdot \exp(X)))$$

are bounded above by some constant C'' only depending on C'.

The second part 2 is obtained in exactly the same way, using the corresponding part of Lemma 5.3. $\hfill \Box$

5.4. Totally positive matrices in $a'_{\omega}G_{\geq 0}$ are quantitatively loxodromic. This section simply contains the following result that totally positive matrices in $a'_{\omega}G_{\geq 0}$ are quantitatively loxodromic. It is probably well-known to experts, and was proved in the companion paper [BHM25], Proposition 4.12.

Proposition 5.5. For any $g \in G_{>0}$ there exist $\omega > 0$ and $0 < \theta < \pi/2$ such that

- (1) $gG_{>0} \subset L_{\omega,\theta};$
- (2) for all $h_1, \ldots, h_n \in gG_{\geq 0}$, denoting $h = h_1 \cdots h_n$ we have $\lambda_k(h) \geq \lambda_{k+1}(h) + n\omega$ for any $1 \leq k \leq d-1$, namely $h \in L_{n\omega,\theta}$;
- (3) for all $h, h' \in gG_{\geq 0}$, we have $(h, h') \in DL_{\omega, \theta}$.

5.5. **Proof of Propositions 5.1.** We fix $\gamma \in \pi_1(S)$ and decompose γ into an admissible path for the hyperbolic structure corresponding to ρ_0 . As mentioned before, the admissible path travels in $S - \gamma^*$ for time $t_1 > 0$, then meet orthogonally some component $\gamma_{i_1}^*$ of γ^* , then travels along it for some time s_1 that can be positive or negative, then departs from it orthogonally and travels for some time $t_2 > 0$ in $S - \gamma^*$... etc. This gives us the following formula for the conjugacy class of the holonomy $\rho_0(\gamma)$:

(25)
$$\rho_0(\gamma) \sim a'_{t_1} a_{s_1} a'_{t_2} \cdots a'_{t_k} a_{s_k},$$

where k is the intersection number of γ with γ^* .

After grafting we obtain an admissible path in the characteristic surface, grafted with the parameter z, that first travels in a hyperbolic piece for time t_1 until it meets orthogonally the flat cylinder above $\gamma_{i_1}^*$, then it travels in the flat cylinder along a segment given by a vector of the form $s_1 d\tau \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + r_1(z) \in \mathfrak{a}$ where $r_1(z)$ is either the i_1 -th coordinate z_{i_1} , or the image $\iota(-z_{i_1})$ under the Cartan involution ι , depending whether we cross the component $\gamma_{i_1}^* \subset \gamma^*$ from left to right or from right to left. Then we repeat these steps: we travel in a hyperbolic piece of the characteristic surface for time t_2 , meet orthogonally a flat cylinder, travel along this cylinder... In the end the grafting operation deforms the formula (25) in the following explicit way:

(26)
$$\rho_z(\gamma) \sim a'_{t_1} a_{s_1} \exp(r_1(z)) a'_{t_2} \cdots a'_{t_k} a_{s_k} \exp(r_k(z)).$$

We wish to estimate in norms the derivatives of its Jordan projection in direction of the grafting parameter z. Let us first compute the general formula for the first and second derivatives of a map of the form

$$f(v) = \lambda(A_1 \exp(f_1(v)) \cdots A_k \exp(f_k(v))),$$

where $v \to (f_1(v), \dots f_k(v)) \in \mathfrak{a}^k$ is a smooth path with $f_i(0) = 0$ for any *i*. To make the notations shorter, put $A_i^j = A_i \dots A_j$. We have

(27)
$$\frac{d}{dv}_{|v=0}f(v) = \sum_{i=1}^{k} d\lambda (A_{1}^{i} \exp(\cdot)A_{i+1}^{k})_{0} (\frac{d}{dv}_{|v=0}f_{i}(v)),$$

where the notation $d\lambda(A_1^i \exp(\cdot)A_{i+1}^k)|_0$ means that we take the differential of the map $X \to \lambda(A_1^i \exp(X)A_{i+1}^k)$ at X = 0.

Similarly,

$$\frac{d^2}{dv^2}|_{v=0}f(v) = \sum_{i=1}^k d\lambda (A_1^i \exp(\cdot)A_{i+1}^k)|_0 (\frac{d^2}{dv^2}|_{v=0}f_i(v)) + \sum_{i=1}^k d^2\lambda (A_1^i \exp(\cdot)A_{i+1}^k)|_0 \left(\frac{d}{dv}|_{v=0}f_i(v), \frac{d}{dv}|_{v=0}f_i(v)\right) + 2\sum_{1\le i< j\le k} d^2\lambda (A_1^i \exp(\cdot)A_{i+1}^j \exp(\cdot)A_{j+1}^k)|_{0,0} \left(\frac{d}{dv}|_{v=0}f_i(v), \frac{d}{dv}|_{v=0}f_j(v)\right).$$

Let us now start the computations, starting with the easiest one: According to (26) and (27), an upper bound for $||d\lambda(\rho_{h_0,z}(\gamma))||$ is given by

(29)
$$\sum_{i} \|dr_{i}(z)|_{z=z_{0}}\| \cdot \|d\lambda \left(a_{t_{1}(h_{0})}^{\prime} \cdots a_{t_{i}(h_{0})}^{\prime} a_{s_{i}(h_{0})} \exp(r_{i}(z_{0})) \exp(\cdot)a_{t_{i+1}(h_{0})}^{\prime} \cdots \exp(r_{k}(z_{0}))\right)|_{0}\|.$$

The term $||dr_i(z)|_{z=z_0}||$ is less than or equal to 1 since $r_i: z \in \mathfrak{a} \to r_i(z)$ can be thought of as either the identity map of \mathfrak{a} or the negative of the Cartan involution.

Using that λ is invariant under conjugacy, we can rewrite the term in the second line of equation (29) as

(30)
$$\left\| d\lambda \left(a'_{\omega} \underbrace{a'_{t_{i+1}(h_0)-\omega} \cdots \exp(r_k(z_0)) a'_{t_1(h_0)} \cdots a'_{t_i(h_0)} a_{s_i(h_0)} \exp(r_i(z_0))}_{\in G_{\geq 0}} \right) \exp(\cdot) |_0 \right) \right\|$$

where $\omega > 0$ is the collar size associated to σ .

By Proposition 5.5, there exists $\omega' > 0$ such that $a'_{\omega}G_{\geq 0} \subset L_{\omega'}$. By Lemma 5.4, there exists C > 0 only depending on ω' and hence on σ such that for any $A \in L_{\omega'}$, the first two derivatives at X = 0 of $X \mapsto \lambda(A \cdot \exp(X))$ are bounded above in norm by C.

Then the above quantity in (30) is bounded above by C, and the quantity in (29) is bounded above by $kC = C\iota(\gamma, \gamma^*)$.

Let us now estimate second derivatives. Since the second derivatives of the r_i 's are zero, according to (26) and (28), an upper bound for this quantity is given by

$$\sum_{i} \left\| d^{2}\lambda \left(a_{t_{1}(h_{0})}^{\prime} \cdots \exp(r_{i}(z_{0})) \exp(\cdot) a_{t_{i+1}(h_{0})}^{\prime} \cdots \exp(r_{k}(z_{0})) \right) |_{0} \right\| \\ + 2 \sum_{i < j} \left\| d^{2}\lambda \left(a_{t_{1}(h_{0})}^{\prime} \cdots \exp(r_{i}(z_{0})) \exp(\cdot) \cdots \exp(r_{j}(z_{0})) \exp(\cdot) \cdots \exp(r_{k}(z_{0})) \right) |_{0,0} \right\|.$$

For all $1 \leq i, j \leq k$, let $\mathfrak{d}(i, j) = \min(|j - i|, k - |j - i|)$. Up to taking ω' smaller we can assume $(a'_{\omega}G_{\geq 0}) \times (a'_{\omega}G_{\geq 0}) \subset DL_{\omega'}$, and

$$a'_{t_{i+1}(h_0)} \cdots \exp(r_j(z_0)) \in P_{(j-i)\omega,\theta}$$

and

$$a'_{t_{j+1}(h_0)}\cdots\exp(r_k(z_0))\cdot a'_{t_1(h_0)}\cdots\exp(r_i(z_0))\in P_{(i-j+k)\omega,\theta}$$

Hence we can apply the second estimate of Lemma 5.4 on the derivatives of $(X, Y) \mapsto \lambda(A \cdot \exp(X)B \cdot \exp(Y))$ at 0, which gives a bound on (31) of the form

$$\sum_{i} C + \sum_{i \neq j} C e^{-\mathfrak{d}(i,j)\omega'} = C \sum_{i=1}^{k} \sum_{j=1}^{k} e^{-\mathfrak{d}(i,j)\omega'} \le C \sum_{i=1}^{k} \sum_{m=-\infty}^{+\infty} e^{-|m|\omega'} \le \frac{2Ck}{1 - e^{-\omega'}}$$

for some constant C > 0. This completes the proof.

6. QUANTITATIVE CONVERGENCE OF CURRENTS

In Section 2.2 we introduced the measure of maximal entropy for Hitchin representations with respect to a Finsler metric. In this section we investigate the behavior of these measures along grafting rays in the Hitchin component. Using the geometric control established in Section 3.4, we compare length functions for representations obtained by Hitchin grafting rays to length functions of the corresponding abstract grafted surfaces, viewed as functions on the unit tangent bundle of the hyperbolic surface S which is the starting point for the grafting, and estimate the entropy of the reparameterized flow. This then leads to the proof of Theorem C from the introduction.

The Finsler metric on \mathbb{X} used for the pressure metric is normalized in such a way that its restriction to a hyperbolic plane stabilized by an irreducible representation of $PSL_2(\mathbb{R})$ coincides with the Riemannian metric of constant curvature -1.

We start with a hyperbolic metric on the closed surface S of genus $g \ge 2$ and choose a simple geodesic multicurve γ^* on S (the grafting locus) with $k \ge 1$ components. For each grafting parameter $z = (z_e)_{e \subset \gamma^*} \subset \mathfrak{a}^k$, denote by ρ_z the Hitchin grafting representation with datum z (see Definition 3.2).

By Proposition 2.8, for each z there exists a positive Hölder continuous function f_z on the unit tangent bundle T^1S of S with the property that for every periodic orbit γ for the geodesic flow Φ^t on T^1S , we have that

$$\ell_{f_z}(\gamma) = \int_{\gamma} f_z$$

equals the translation length of the conjugacy class determined by the element $\rho_z(\gamma) \in \text{PSL}_d(\mathbb{R})$ with respect to the Finsler metric.

The Hölder continuous function f_z on T^1S determines a reparameterization $\Phi_{f_z}^t$ of the geodesic flow Φ^t on T^1S , whose measure of maximal entropy corresponds to a Φ^t -invariant Gibbs equilibrium state $\nu(z)$ on T^1S . There are several possible normalizations for this equilibrium state. We assume $\nu(z)$ to be normalized in such a way that

(31)
$$\int f_z d\nu(z) = 1 \text{ for all } z.$$

Note that this normalization only depends on the cohomology class of f_z and hence it does not depend on choices. Our main goal is to determine the possible limits of $\nu(z)$ as the cylinder height of every component z_e of z (that is, at every component of the multi-curve γ^*) tends to infinity, and to show that the intersection numbers with γ^* of the geodesic currents $\hat{\nu}(z)$ determined by the measures $\nu(z)$ decay exponentially fast.

By Section 2.3, the equilibrium measure of the function $-f_z$ can be described in terms of Patterson–Sullivan measures. Denoting as before by \mathcal{F} the flag variety of $\mathrm{PSL}_d(\mathbb{R})$, recall that for $\zeta, \eta \in \mathcal{F}$ and $x, y \in \mathbb{X}$, the function $b_{\zeta}^{\mathfrak{F}}(x, y)$ denotes the Busemann cocycle and $\langle \zeta | \eta \rangle_x$ denotes the Gromov product associated to the Finsler metric \mathfrak{F} (see Equations 3 and 4).

For any non-trivial grafting datum z with nontrivial cylinder height, let $\Xi_z : \partial_\infty \mathbb{H}^2 \to \mathcal{F}$ be the limit map associated to the Hitchin grafting representation ρ_z . Then there exists a family of Patterson Sullivan measures $(\mu_z^x)_{x \in \mathbb{X}}$ on $\partial_\infty \mathbb{H}^2$ such that for all $x, y \in \mathbb{X}$ and $\gamma \in \pi_1(S)$ we have $\mu_z^{\rho_z(\gamma)x} = \gamma_* \mu_z^x$ and

(32)
$$\frac{d\mu_z^y}{d\mu_z^x}(\xi) = e^{\delta(z)b_{\Xi_z(\xi)}^{\mathfrak{F}}(x,y)},$$

where $\delta(z)$ is the *critical exponent* of the group $\rho_z(\pi_1(S))$, or, equivalently, the topological entropy of the reparameterized flow $\Phi_{f_z}^t$ on T^1S . These measure are unique up to a global multiplicative positive constant. Note that the equality 32 is immediate from the fact that the topological entropy of the reparameterized flow equals the expansion rate of the conditional measures on strong unstable manifolds for its unique measure of maximal entropy, which in turn equals the critical exponent by construction.

Finally there is a choice of normalization for the measures μ_z^x such that $\nu(z)$ is the quotient under $\pi_1(S)$ of the measure

(33)
$$e^{\delta(z)\langle \Xi_z(\xi)|\Xi_z(\eta)\rangle_x}d\mu_z^x(\xi)d\mu_z^x(\eta)d\eta_z^x$$

on $\partial_{\infty}\mathbb{H}^2 \times \partial_{\infty}\mathbb{H}^2 \times \mathbb{R}$. Note that the measures μ_z^x are finite but in general they are not probability measures, instead their normalization is determined by the normalization of $\nu(z)$.

Since $\nu(z)$ and hence the geodesic current $\hat{\nu}(z)$ defined by $\nu(z)$ depends continuously (in fact, analytically) on z by Proposition 2.4, we can estimate the intersection $\iota(\hat{\nu}(z), \gamma^*)$ (here γ^* is viewed as a Dirac current) using continuity of the intersection form on the space of currents. However, although the space of projective currents, equipped with the weak*-topology, is compact since this is the case for the space of Φ^t -invariant Borel probability measures on T^1S where Φ^t is the geodesic flow, the family $\hat{\nu}(z)$ may not be precompact as the corresponding Φ^t -invariant measure $\nu(z)$ on T^1S is determined by the normalization (31) and in general is not a probability measure. We shall use the Patterson–Sullivan measures to control the total volume of $\nu(z)$ and overcome this difficulty.

6.1. The entropy of the subsurfaces. The geodesic multicurve γ^* decomposes S into (closed) complementary components S_1, \ldots, S_k . For each $i \leq k$ we denote by $K_i \subset T^1S$ the set of all unit tangent vectors $v \in T^1S_i$ with the property that $\Phi^t v \in T^1S_i$ for all $t \in \mathbb{R}$.

Lemma 6.1. For each *i* the set K_i is compact and Φ^t -invariant.

Proof. The set K_i is clearly Φ^t -invariant and closed by continuity of Φ^t , hence it is compact.

Since S is a closed hyperbolic surface, the geodesic flow Φ^t on T^1S is an Anosov flow and hence for each *i* its restriction to the compact invariant set K_i is an Axiom A flow.

The preimage of the geodesic multicurve γ^* in the universal covering \mathbb{H}^2 of S consists of a countable union of pairwise disjoint geodesic lines. These geodesic lines decompose \mathbb{H}^2 into countably many connected components which are permuted by the action of the fundamental group $\pi_1(S)$ of S. If we denote by $\Gamma \subset \pi_1(S)$ the stabilizer of one of these components $\tilde{\Sigma}$, which is a convex subsurface of \mathbb{H}^2 with geodesic boundary, then Γ acts properly and cocompactly on $\tilde{\Sigma}$, with quotient one of the components S_i of $S - \gamma^*$. Thus Γ is a non-elementary convex cocompact Fuchsian group.

The limit set, that is, the set of accumulation points of a Γ -orbit $\Gamma x \subset \mathbb{H}^2$ $(x \in \tilde{\Sigma})$ in $\mathbb{H}^2 \cup \partial_{\infty} \mathbb{H}^2$, is a Γ -invariant Cantor subset Λ of $\partial_{\infty} \mathbb{H}^2$. The quotient under the action of Γ of the set of all unit tangent vectors of geodesics with both endpoints in Λ has a natural identification with the invariant set $K_i \subset T^1 S$. In particular, the restriction of Φ^t to K_i is topologically transitive. Its topological entropy equals the Hausdorff dimension $\delta_i \in (0,1)$ of Λ [Sul84].

Write $K = \bigcup_i K_i$ and let $\delta > 0$ be the topological entropy of $\Phi_{|K}^t$. We have $\delta = \max\{\delta_i \mid i \leq k\}$. Recall that $\delta(z)$ denotes the topological entropy of the reparameterized flow $\Phi_{f_z}^t$ on T^1S and equals the critical exponent of the group $\rho_z(\pi_1(S)) \subset \text{PSL}_d(\mathbb{R})$.

We have bounds on $\delta(z)$. The upper bound is very general:

Theorem 6.2 (Corollary 1.4 of [PS17]). There is a constant m > 0 that bounds from above the entropy of any Hitchin representation.

The lower bound depends on the choice of the grafting locus γ^* and the hyperbolic metric on S, and its proof is classical.

Lemma 6.3 (e.g. Theorem 4.1 of [CZZ24]). $\delta(z) \in (\delta, m]$ for all z, where $m > \delta$ is the universal constant from the above Theorem 6.2.

Proof. By definition of a Hitchin grafting representation, the image $\rho_z(\Gamma)$ under ρ_z of the fundamental group Γ of any component of $S - \gamma^*$ is conjugate to its image under ρ , and hence has the same critical exponent. Suppose we picked the component with largest critical exponent, namely δ .

Then $\rho_z(\Gamma)$ is also Anosov (Γ is quasi-convex in $\pi_1(S)$) and its limit set is a proper subset of that of $\rho_z(\pi_1(S))$ so by Theorem 4.1 of [CZZ24] it has a strictly smaller critical exponent. Thus the critical exponent of $\rho_z(\pi_1(S))$ is bigger than δ . Let $h_{top}(\Psi^t)$ be the topological entropy of a flow Ψ^t on a compact space; thus $\delta = h_{top}(\Phi^t_{|K})$. A measure of maximal entropy for $\Phi^t_{|K}$ is an invariant probability measure μ with $h_{\mu} = \delta$.

Since $\Phi_{|K_i}^t$ is a topologically transitive Axiom A flow and K_i is compact, it admits a unique measure ν_i of maximal entropy. The measure ν_i is a Gibbs equilibrium state for $\Phi_{|K_i}^t$ with respect to the constant function 1, and it can be obtained from a Patterson Sullivan construction [Sul84]. The following well known fact will be useful later on.

Lemma 6.4. A measure of maximal entropy for $\Phi_{|K}^t$ exists. It is unique if and only if there exists a number $i \leq k$ such that $h_{top}(\Phi_{|K_i}^t) > \max\{h_{top}(\Phi_{|K_j}^t) \mid j \neq i\}$. In this case the measure of maximal entropy is supported in K_i .

Proof. Write again $K = \bigcup_i K_i$. The function which associates to a $\Phi_{|K}^t$ -invariant probability measure μ its entropy h_{μ} is affine: for μ, η and $s \in (0, 1)$ we have $h_{s\mu+(1-s)\eta} = sh_{\mu} + (1-s)h_{\eta}$.

The topologically transitive invariant subsets $K_i \subset K$ intersect at most along a finite number of periodic orbits. As a consequence, any Φ^t -invariant probability measure μ on K can be decomposed as $\mu = \sum_i \mu_i$ where μ_i is supported in K_i . The decomposition is unique if the μ -mass of any periodic orbit for Φ^t which projects to a component of γ^* vanishes.

Since $\Phi_{|K_i}^t$ is a topologically transitive axiom A flow, it admits a unique measure ν_i of maximal entropy. Then we have $h_{\nu_i} = h_{\text{top}}(\Phi_{|K_i}^t)$. Let $\mu = \sum_i \mu_i$ be any Φ^t -invariant Borel probability measure on K. Let $s_i = \mu_i(K_i)$; then $\sum_i s_i = 1$ and

$$h_{\mu} = \sum_{i} s_{i} h_{\mu_{i}} \leq \sum_{i} s_{i} h_{\text{top}}(\Phi_{|K_{i}}^{t}) \leq \delta$$

with equality if and only if $s_j = 0$ for all j such that $h_{top}(\Phi_{|K_j}^t) < \delta$, and $\mu_j = \nu_j$ if $s_j > 0$. In particular, a measure of maximal entropy exists, and if there exists a unique $i \leq k$ such that $h_{top}(\Phi_{|K_j}^t) = \delta$, then such a measure is unique and coincides with ν_i . \Box

6.2. The total mass of the equilibrium state. For the fixed hyperbolic metric on S with unit tangent bundle T^1S and geodesic flow Φ^t denote by $\nu^1(z)$ the Φ^t -invariant probability measure on T^1S which is a multiple of $\nu(z)$. It turns out that the two normalisations $\nu(z)$ and $\nu^1(z)$ for the equilibrium states are comparable independently of z, as soon as the grafting datum z is taken in ker α_0 where α_0 is the linear functional which determines the Finsler norm of the tangent of a Riemannian geodesic in X which is invariant under $\rho(\gamma^*)$ (or a component of $\rho(\gamma^*)$).

Lemma 6.5. For any $\sigma > 0$ there exists a constant C > 0 such that if the length of each component of $\gamma^* \subset S$ is at most σ , then for any grafting parameter $z \subset \ker \alpha_0^{\perp}$,

$$C^{-1} \le \|\nu(z)\| = \nu(z)(T^1S) \le C.$$

Proof. Put $\nu^1(z) = \frac{\nu(z)}{\|\nu(z)\|}$ so that $\nu^1(z)$ is a probability measure on T^1S . Then $\|\nu(z)\| = (\int f_z d\nu^1(z))^{-1}$ since by equation (31), $\nu(z)$ was normalized so that $\int f_z d\nu(z) = 1$.

By definition of the equilibrium state of $-f_z$ and the fact that the entropy of the reparameterized flow $\Phi_{f_z}^t$ equals $\delta(z)$, we have

(34)
$$\int f_z d\nu^1(z) = \frac{h_{\nu^1(z)}}{\delta(z)}.$$

Since $h_{\nu^1(z)} \leq 1$ (the topological entropy of Φ^t is 1, and is greater than or equal to the entropy of any invariant measure) and $\delta(z) > \delta$ by Lemma 6.3, it holds $\int f_z d\nu^1(z) \leq \frac{1}{\delta}$. It remains to get a lower bound.

By Theorem 3.8, we have

$$\int f_z \frac{d\zeta}{\ell(\zeta)} \ge \left(1 + \frac{C}{L+1}\right)^{-1},$$

for any $\zeta \in \pi_1(S)$, represented by a periodic orbit for Φ^t of length $\ell(\gamma)$, and where $L \ge 0$ is any lower bound on the heights of the cylinders added along the components of γ^* to construct S_z (see Definition 3.1).

Then by density of the convex hull of currents supported on closed geodesics in the space of all currents, we get

$$\int f_z d\nu^1(z) \ge (1+C)^{-1} \,.$$

6.3. The total mass of the Patterson–Sullivan measure. In this section we establish estimates on the total mass of some of the Patterson–Sullivan measures (see Equations (32) and (33)). To this end we use the equivariant path isometry $\tilde{Q}_z : \tilde{S}_z \to \mathbb{X}$ to view the family (μ_z^x) of Patterson Sullivan measures on $\partial \mathbb{H}$ as a family of measures parameterized by points in the universal covering \tilde{S}_z of the abstract grafted surface S_z . In the sequel we use this convention without further mention.

Proposition 6.6. For any $\sigma > 0$ there is a constant C > 0 such that if the length of each component of γ^* is at most σ , then for any grafting parameter $z \subset \ker \alpha_0^{\perp}$, in any hyperbolic piece of \widetilde{S}_z there exists a point x such that

$$\mu_z^x(\partial_\infty \mathbb{H}^2) \le C.$$

The strategy of the proof is as follows (see Figure 1). Assume that each component of $S - \gamma^*$ is a pair of pants. We fix one of them, say the pair of pants Σ , and its fundamental group Γ . Let $\tilde{\Sigma}$ be the universal covering of Σ . Then $\tilde{\Sigma} \subset \mathbb{H}^2$ is a convex hyperbolic surface with geodesic boundary. We find two disjoint intervals $I, J \subset \partial_{\infty} \mathbb{H}^2$, numbers $C_1, C_2 > 0$ and a fundamental domain for the action $\Gamma \curvearrowright \tilde{\Sigma}$, made of two right-angle hexagons $H \cup H'$ whose diameters are bounded from above by a constant only depending on σ and which depend in a suitable sense continuously on the data, so that the following holds. Let x be the center of H. First, the masses $\mu_z^x(I)$ and $\mu_z^x(J)$ are bounded from below by $C_1\mu_z^x(\partial_\infty \mathbb{H}^2)$. Second, each geodesic connecting a point in Ito a point in J intersects H in an arc of length at least C_2 and hence passes uniformly near x. We then can estimate the Gromov product and bound the product measure $\mu_z^x(I) \times \mu_z^x(J) = \mu_z^x \times \mu_z^x(I \times J)$ from above by a constant multiple of $\nu(z)(T^1S)$, which is uniformly bounded from above by Lemma 6.5.

We begin with establishing a few estimates in a more general setting involving representations of the fundamental group of a pair of pants (the free group F_2 with two



FIGURE 1. Control of $\nu(T^1S)$ using the measure for μ of two intervals $I = \bar{a}_u \cdot A_u$ and $J = \bar{a}_u^{-1} \cdot A_u$ in $\partial \mathbb{H}^2$. The hexagon H_u is half of a fundamental domain of a pair of pant P_u . The geodesic from J to Iintersects H_u in an arc whose length is bounded from below.

generators) into $PSL_2(\mathbb{R})$. Let us introduce some notations. Let P be a topological pair of pants, equipped with a fixed orientation. We fix a basepoint p_0 in P and three generators a, b, c of the fundamental group $\pi_1(P, p_0) = F_2$ such that $c \cdot b \cdot a = 1$ and each generator corresponds to one of the boundary components of P.

For a set of lengths $u = (u_a, u_b, u_c) \in [0, \infty)^3$ there is a unique hyperbolic structure on P whose boundary components have these lengths on a, b, c, and up to conjugation, there is a unique representation $j_u : \pi_1(P) \to \mathrm{PSL}_2(\mathbb{R})$ associated to this hyperbolic structure which is normalized so that the following ordering assumption holds.

Put a_u, b_u, c_u instead of $j_u(a), j_u(b), j_u(c)$. Then u_i are loxodromic elements of $PSL_2(\mathbb{R})$ with axes $\bar{a}_u, \bar{b}_u, \bar{c}_u \subset \mathbb{H}^2$, oriented to define the boundary orientation for the oriented pair of pants P, with endpoints $\bar{a}_u^{\pm}, \bar{b}_u^{\pm}, \bar{c}_u^{\pm} \subset \partial_{\infty} \mathbb{H}^2 = S^1$. We use the abuse of notation that if say $u_a = 0$, then $j_u(a)$ has only one fixed point on $\partial \mathbb{H}^2$, and $\bar{a}_u = \bar{a}_u^+ = \bar{a}_u^-$. We require that the cycle $(\bar{a}_u^-, \bar{a}_u^+, \bar{b}_u^-, \bar{b}_u^+, \bar{c}_u^-, \bar{c}_u^+)$ is oriented clockwise for the circular order on $\partial_{\infty}\mathbb{H}^2$. We may also assume that the center 0 of the unit disk $D = \mathbb{H}^2$ is contained in the convex hull of the limit set of j_u and that j_u varies continuously in u.

Consider the three intervals of $\partial_{\infty} \mathbb{H}^2$ (that is, the segment in $\partial_{\infty} \mathbb{H}^2$ determined by the clockwise orientation of S^1 and its endpoints)

- $A_u = [\bar{b}_u^-, \bar{c}_u^+],$ $B_u = [\bar{c}_u^-, \bar{a}_u^+],$ $C_u = [\bar{a}_u^-, \bar{b}_u^+].$

By construction, we have $A_u \cup B_u \cup C_u = \partial_\infty \mathbb{H}^2$, so for any finite measure μ on $\partial_\infty \mathbb{H}^2$, one of the intervals has mass at least $\frac{1}{3}\mu(\partial_\infty \mathbb{H}^2)$. Put $A_u^+ = a_u \cdot A_u$, $A_u^- = a_u^{-1} \cdot A_u$, and similarly for $B_u^+, B_u^-, C_u^+, C_u^-$.

Lemma 6.7. The intervals A_u^+ and A_u^- are disjoint. Similarly $B_u^+ \cap B_u^- = \emptyset$ and $C_u^+ \cap C_u^- = \emptyset$.

Proof. First notice that $a_u \cdot \overline{c}_u^+$ belongs to $[\overline{a}_u^+, \overline{c}_u^+]$ since a_u is an hyperbolic element with attractive fixed point \overline{a}_u^+ . By construction, we have $c_u \cdot b_u \cdot a_u = 1$, so that

$$a_u \cdot \bar{c}_u^+ = (b_u^{-1} \cdot c_u^{-1}) \cdot \bar{c}_u^+ = b_u^{-1} \cdot \bar{c}_u^+$$

So $a_u \cdot \bar{c}_u^+$ is included in both $[\bar{a}_u^+, \bar{c}_u^+]$ and $[\bar{c}_u^+, \bar{b}_u^-]$, whose intersection is equal to the interval $[\bar{a}_u^+, \bar{b}_u^-]$. Similarly, $a_u \cdot \bar{b}_u^-$ lies inside $[\bar{a}_u^+, a_u \cdot \bar{c}_u^+] \subset [\bar{a}_u^+, \bar{b}_u^-)$. And since $A_u = [\bar{b}_u^-, \bar{c}_u^+]$, $a_u \cdot A_u \subset [\bar{a}_u^+, \bar{b}_u^-)$, and similarly $a_u^{-1} \cdot A_u \subset (\bar{c}_u^+, \bar{a}_u^-]$, it follows that they are disjoint.

Write $\Gamma_0 = \{a, a^{-1}, b, b^{-1}, c, c^{-1}\} \subset \pi_1(P)$

Corollary 6.8. If μ is a $j_u(\pi_1(P))$ -quasi-invariant finite measure on $\partial_{\infty}\mathbb{H}^2$, then the measure for $\mu \times \mu$ of one of the three products $A_u^- \times A_u^+$, $B_u^- \times B_u^+$, $C_u^- \times C_u^+$ has mass at least $\frac{C^2}{9}\mu(\partial_{\infty}\mathbb{H}^2)^2$, where

$$C = C_{\mu,u} = \inf \left\{ \frac{dj_u(\gamma)_*\mu}{d\mu}(\xi) : \xi \in \partial_\infty \mathbb{H}^2, \ \gamma \in \Gamma_0 \right\}.$$

We will also need an estimate on the lengths of the intersection of geodesics from $A_u^$ to A_u^+ , with $H_u \subset \mathbb{H}^2$ the (possibly degenerate) right-angled hexagon adjacent to the axes of $j_u(a), j_u(b), j_u(c)$.

Lemma 6.9. For any $\sigma > 0$ there exists L_{σ} such that for any $u \in [0, \sigma]^3$, for all (x, y) in $A_u^- \times A_u^+$, $B_u^- \times B_u^+$ or $C_u^- \times C_u^+$, the length of the intersection of the hexagon H_u with the geodesic from x to y is at least L_{σ} .

Proof. This is a direct consequence of the following three facts. A_u^{\pm} varies continuously with u. The length $length(\gamma \cap H_u)$ for a geodesic γ with ends in $A_u^- \times A_u^+$ is positive and continuous in the pair (u, γ) . And $[0, \sigma]^3$ is compact.

Proof of Proposition 6.6. According to Theorem A.2, we can choose a pair of pants in $S \setminus \gamma^*$, whose boundary curves have length bounded from above by a constant only depending on the genus of S and σ . Let us identify it topologically with P, and identify a convex subsurface \tilde{P} of a hyperbolic piece inside the universal covering \tilde{S}_z of the abstract grafted surface with the universal cover of P. We denote by $u = (u_a, u_b, u_c)$ the lengths of the boundary components of this pair of pants. The surface \tilde{P} contains a right angled hexagon H_u whose double is a fundamental domain for the deck group $\pi_1(P)$.

Identify $\operatorname{PSL}_2(\mathbb{R})$ with a subgroup of $\operatorname{PSL}_d(\mathbb{R})$ and \mathbb{H}^2 with a totally geodesic subspace of X. Up to conjugation of the Hitchin grafting representation ρ_z , we may assume that its restriction to $\pi_1(P)$ coincides with $j_u : \pi_1(P) \to \operatorname{PSL}_2(\mathbb{R})$, so that the fixed hyperbolic piece \widetilde{P} of the characteristic surface is contained in $\mathbb{H}^2 \subset X$. More precisely, it is the convex hull of the limit set of $j_u(\pi_1(P))$. Note that this makes sense since the boundary of \mathbb{H}^2 embeds naturally into the flag variety \mathcal{F} as well as the visual boundary $\partial_{\infty} X$ of X. We choose a point x in the interior of the hexagon $H_u \subset \widetilde{P}$ as a basepoint in \mathbb{H}^2 .

By Lemma 6.5, Corollary 6.8 and Lemma 6.9, and since the Gromov product is nonnegative, there is a constant C such that

$$C \ge \left(e^{\delta \langle \cdot | \cdot \rangle_x} \mu_z^x \times \mu_z^x \times \text{Leb}\right) (T^1 H_u) \ge \frac{L_\sigma C_\delta^2}{9} \mu_z^x (\partial_\infty \mathbb{H}^2)^2$$

where T^1H_u denotes the set of unit tangent vectors in $T^1\mathbb{H}^2$ with footpoint in the hexagon H_u , and C_δ is the infimum of the constants $C_{\mu,u}$ appearing in the corollary, that is

$$C_{\delta} = \inf \left\{ e^{\delta b_{\xi}^{\delta}(x, j_u(\gamma)x)} : \xi \in \mathcal{F}, \ u \in [0, \sigma]^3, \ \gamma \in \Gamma_0 \right\}.$$

To conclude the proof one can use Theorem 6.2, which implies $C_{\delta} \geq C_m$ where $C_m > 0$ is a constant that only depends on the genus of S and the choice of length function on $\mathrm{PSL}_d(\mathbb{R})$ (i.e. the choice of a linear functional α_0 on \mathfrak{a}). \Box

Remark 6.10. In the proposition 6.6, we may actually take x to be any point in ϵ -thick part of S, for $\epsilon > 0$ first fixed. To see that, modify the proof as follow. Fix $\epsilon > 0$, and instead of taking a unique representation j_u for a fixed data of $u \in [0, \sigma]^3$, take a larger compact set of representations J so that each representation in J is conjugated to one j_u as above. Furthermore, for each j_u and each point p in the ϵ -thick part of the hyperbolic pair of pants obtained as the quotient by j_u of the convex core of j_u , there exists a representation $j_{u,p}$ in J and an isometry f of \mathbb{H}^2 which conjugates $j_{u,p}$ to j_u and which sends the origin of \mathbb{H}^2 to a preimage of p in the convex core of $j_{u,p}$. The same compactness argument holds except that the constant C my be larger. Additionally, for $\epsilon > 0$ small enough, the ϵ -thick part of a surface S is equal to the union of the ϵ -thick parts of the pairs of pants which are contained in some pants decomposition of S.

6.4. Estimating the intersection number of the equilibrium state with the grafting locus. Recall that γ^* is the grafting locus and $\hat{\nu}(z)$ is the geodesic current defined by the equilibrium measure $\nu(z)$ for a grafting parameter z. The goal of this section is to show that if the heights of the flat cylinders in the grafted surface go to infinity, then $\nu(z)$ concentrates more and more on the components of $S - \gamma^*$ and avoids crossing the grafting locus, and this with exponential speed. A more precise consequence will be that the intersection number $\iota(\hat{\nu}(z), \gamma^*)$ goes to zero with speed $Ce^{-\delta L/2}$ where L is the minimal height of the flat cylinders, δ is the entropy of $S - \gamma^*$, and C is a constant that depends on the hyperbolic length of γ^* .

In practice, we will prove a stronger result, which in vague terms states that if we see the equilibrium state $\nu(z)$ as a measure on the unit tangent bundle of the grafted surface (instead of the hyperbolic surface), then as $L \to \infty$ the mass given by $\nu(z)$ to the flat cylinders (which have length at least L) goes to zero exponentially fast. Let us now explain more rigorously what this means.

Let γ_0^* be a component of γ^* . Let $\tilde{\gamma}^* \subset \tilde{S} \simeq \mathbb{H}^2$ be the preimage of γ^* and choose a component $\hat{\gamma}_0^* \subset \tilde{\gamma}^*$ of the preimage of γ_0^* . Denote by I^- and I^+ the two connected components of $\partial_{\infty}\mathbb{H}^2 - \hat{\gamma}_0^*(\pm\infty)$. Recall that an oriented geodesic in \mathbb{H}^2 can be thought of as an ordered pair of distinct points $(\xi^-, \xi^+) \in \partial_{\infty}\mathbb{H}^2 \times \partial_{\infty}\mathbb{H}^2 - \Delta$. Then a geodesic (ξ^-, ξ^+) intersects $\hat{\gamma}_0^*$ transversely if and only if $(\xi^-, \xi^+) \in I^- \times I^+ \cup I^+ \times I^-$. Let $g_u \in \pi_1(S)$ be a generator of the infinite cyclic subgroup $\langle g_u \rangle$ of $\pi_1(S)$ which preserves $\hat{\gamma}_0^*$ and acts on it as a group of translations. Choose a fundamental domain Ω^{\pm} for the action of $\langle g_u \rangle$ on I^{\pm} of the form $\Omega^{\pm} = [\xi_0^{\pm}, g_u \xi_0^{\pm}) \subset I^{\pm}$, where $\xi_0^{\pm} \in I^{\pm}$ are taken so that the geodesic (ξ_0^-, ξ_0^+) crosses $\hat{\gamma}_0^*$ orthogonally at a point x.

To every pair $(\xi^-, \xi^+) \in I^- \times I^+$ is associated an infinite admissible path a in the grafted surface \tilde{S}_z that projects onto the admissible path in $\tilde{S} \simeq \mathbb{H}^2$ from ξ^- to ξ^+ , and hence that crosses the flat band $B \subset \tilde{S}_z$ sitting above $\gamma_0^* \subset \tilde{S}$. We define $l_{\gamma_0^*, z}^{\text{flat}}(\xi^-, \xi^+)$ to be the Finsler length of $a \cap B$, and use it to define the length in the flat part of any current $\hat{\lambda}$:

(35)
$$\ell_{\gamma_0^*,z}^{\text{flat}}(\hat{\lambda}) := \int_{\Omega_- \times I_+ \cup \Omega_+ \times I_-} l_{\gamma_0^*,z}^{\text{flat}} d\hat{\lambda} \quad \text{and} \quad \ell_z^{\text{flat}}(\hat{\lambda}) = \sum_{\gamma_0^* \subset \gamma^*} \ell_{\gamma_0^*,z}^{\text{flat}}(\hat{\lambda}).$$

Remark . (1) If $\hat{\lambda}$ is the current associated to a closed curve $\eta \subset S$ then we retrieve $\ell_z^{\text{flat}}(\hat{\lambda}) = \ell_z^{\text{flat}}(\eta)$ from Theorem 5.1.

- (2) One has $\iota(\hat{\lambda}, \gamma^*) = \hat{\lambda}(\Omega_- \times I_+ \cup \Omega_+ \times I_-)$, whence $\iota(\hat{\lambda}, \gamma^*) \leq L\ell_z^{\text{flat}}(\hat{\lambda})$ (with L the minimal height of the flat cylinders). For a reminder on currents that implies this, see the appendix of [Bon88] and Chapter 8.2.11 of [Mar16]).
- (3) Let $\hat{\gamma}_1, \hat{\gamma}_2 \subset \mathbb{H}^2$ be two other components of $\tilde{\gamma}^*$ such that $\hat{\gamma}_1, \hat{\gamma}_0^*, \hat{\gamma}_2$ lie in this order in \mathbb{H}^2 with no other components of $\tilde{\gamma}^*$ in between, and $\hat{\gamma}_1, \hat{\gamma}_2$ bound intervals $J_1, J_2 \subset \partial \mathbb{H}^2$ such that $J_1 \subset I_-$ and $J_2 \subset I_+$. Then $l_{\gamma_0^*,z}^{\text{flat}}$ is constant on $J_1 \times J_2$. The reason is that the admissibles paths associated to two different geodesics from J_1 to J_2 coincide on their three pieces from $\hat{\gamma}_1$ to $\hat{\gamma}_2$: the first piece follow the orthogeodesic from $\hat{\gamma}_1$ to $\hat{\gamma}_0^*$, the last piece is the orthogeodesic from $\hat{\gamma}_0^*$ to $\hat{\gamma}_2$, and the middle piece is the geodesic in the flat band B that connects the first and third pieces, and whose length is computed by $l_{\gamma_0^*,z}^{\text{flat}}$.
- (4) As a function on the space of currents, ℓ_z^{flat} is linear but not continuous. However, it is continuous at currents that give zero measure to the set of geodesics that are asymptotic to a components of $\tilde{\gamma}^*$ in \tilde{S} , because of the previous point of this remark. Since $\hat{\nu}(z)$ satisfies this condition (it is of the form $f\mu \otimes \mu$ where μ is a nonatomic measure on $\partial \mathbb{H}^2$, see (33)), by (7) we have

(36)
$$\ell_z^{\text{flat}}(\hat{\nu}(z)) = \lim_{R \to \infty} \frac{1}{\# N_{\rho_z}(R)} \sum_{\ell^{\mathfrak{F}}(\rho_z(\alpha)) \le R} \frac{\ell_z^{\text{flat}}(\alpha)}{\ell^{\mathfrak{F}}(\rho_z(\alpha))}$$

The main result of this section is the following.

Proposition 6.11. For any $\sigma > 0$ there are $C, C', \delta_{\sigma} > 0$ such that if every component of γ^* has hyperbolic length at most σ , then for any grafting parameter z we have

$$\ell_z^{\text{flat}}(\hat{\nu}(z)) \le C(L+1)^2 e^{-\delta(z)L} \le C' e^{-\delta_\sigma L},$$

where L is the minimum of the heights of the flat cylinders in the abstract grafting (see Equation (16)).

Remark . (1) Recall that $\delta(z)$ is bounded from below by the topological entropy δ of the geodesic flow on $S - \gamma^*$, which, by Proposition A.3, is itself bounded from below by a positive constant that only depends on σ . So in the above proposition, one can take δ_{σ} to be half of the smallest possible entropy of hyperbolic surfaces homeomorphic to $S - \gamma^*$, with boundary lengths at most σ . Then C' could be C times the maximum of the function $x \in [0, \infty) \mapsto (x + 1)^2 e^{-\delta_{\sigma} x}$.

(2) One can check that the proof of Proposition 6.11 gives the following estimate on the intersection number of $\hat{\nu}(z)$ with γ^* :

$$\iota(\hat{\nu}(z),\gamma^*) \le C(L+1)e^{-\delta(z)L},$$

where C is a constant that depends on σ .

We will need the following result about Hitchin representations. In its formulation, $\partial \pi_1(S)$ denotes the Gromov boundary of the surface group $\pi_1(S)$.

Lemma 6.12. Let $\rho' : \pi_1(S) \to G$ be a Hitchin representation with limit map $\Xi' : \partial \pi_1(S) \to \mathcal{F}$, and let $(\gamma_n)_n \subset \pi_1(S)$ be a sequence converging to $\xi \in \partial \pi_1(S)$. Then for any compact set $K \subset \mathbb{X}$, the accumulation points of $\rho'(\gamma_n)K$ in the visual boundary $\partial_{\infty}\mathbb{X}$ of \mathbb{X} are contained in the interior of the Weyl Chamber $\Xi'(\xi)$.

Proof. This is a consequence of the Anosov property discussed in Section 2.2, which is satisfied by Hitchin representations, and a characterisation of this property in terms of Cartan decompositions of the images $\rho'(\gamma)$ with $\gamma \in \pi_1(S)$.

Let $\rho'(\gamma_n) = k_n \exp(a_n)\ell_n$ be a Cartan decomposition, so that $k_n, \ell_n \in K$ (the maximal compact subgroup) and $a_n \in \mathfrak{a}^+$. By a characterisation of the Anosov property (see Theorem 4.37 of [Kas24] for more details and a history of this result), the angle formed by a_n with each wall of the Weyl Cone \mathfrak{a}^+ is bounded from below independently of n. In other words, denoting by $\|\cdot\|$ the Euclidean norm on \mathfrak{a} , we have $\alpha(a_n) \geq \operatorname{Cst} ||a_n||$ for any positive root α , which means precisely that $(\exp(a_n))_n$ accumulates in the interior of the Weyl Chamber $\partial_{\infty} \exp(\mathfrak{a}^+) \subset \partial_{\infty} \mathbb{X}$ in the ideal boundary of the flat cone $\exp(\mathfrak{a}^+) \subset \mathbb{X}$.

Up to passing to a subsequence we may assume that $k_n \to k$ and $\ell_n \to \ell$. Let \mathfrak{a}^- be the Weyl chamber opposite to \mathfrak{a}^+ , with boundary $\partial_{\infty} \exp(\mathfrak{a}^-)$, viewed as a point in \mathcal{F} . Then for any $\eta \in \mathcal{F}$ transverse to $\ell^{-1}\partial_{\infty} \exp(\mathfrak{a}^-)$ we have $\rho'(\gamma_n)\eta \to k\partial_{\infty} \exp(\mathfrak{a}^+)$. By the definition of the limit map Ξ' , this implies that $\Xi'(\xi) = k\partial_{\infty} \exp(\mathfrak{a}^+)$ (see Definition 2.5).

Then $\rho'(\gamma_n)\mathbf{x} = k_n e^{a_n} \mathbf{x}$ only accumulates in the interior of the Weyl Chamber $\Xi'(\xi)$, and the same holds for $\rho'(\gamma_n)K$ which lies at bounded distance from $\rho'(\gamma_n)\mathbf{x}$.

Proof of Proposition 6.11. Let γ_0^* be a component of γ^* . Let $\tilde{\gamma}^* \subset \mathbb{H}^2$ be the preimage of γ^* and choose a component $\hat{\gamma}_0^* \subset \tilde{\gamma}^*$ of the preimage of γ_0^* . Denote by I^- and I^+ the two connected components of $\partial_{\infty}\mathbb{H}^2 - \hat{\gamma}_0^*(\pm\infty)$. Recall that an oriented geodesic in \mathbb{H}^2 can be thought of as an ordered sets of distinct points $(\xi^-, \xi^+) \in \partial_{\infty}\mathbb{H}^2 \times \partial_{\infty}\mathbb{H}^2 - \Delta$. Then a geodesic (ξ^-, ξ^+) intersects $\hat{\gamma}_0^*$ transversely if and only if $(\xi^-, \xi^+) \in I^- \times I^+ \cup I^+ \times I^-$.

Let $g_u \in \pi_1(S)$ be a generator of the infinite cyclic subgroup $\langle g_u \rangle$ of $\pi_1(S)$ which preserves $\hat{\gamma}_0^*$ and acts on it as a group of translations. Choose a fundamental domain Ω^{\pm} for the action of $\langle g_u \rangle$ on I^{\pm} of the form $\Omega^{\pm} = [\xi_0^{\pm}, g_u \xi_0^{\pm}) \subset I^{\pm}$, where $\xi_0^{\pm} \in I^{\pm}$ are taken so that the geodesic (ξ_0^-, ξ_0^+) crosses $\hat{\gamma}_0^*$ orthogonally at a point x.

Using these notations, it follows from the definitions of the intersection number between two geodesic currents of S (see the appendix of [Bon88] and Chapter 8.2.11 of [Mar16]) that

(37)
$$\iota(\hat{\nu}(z), \gamma_0^*) = \hat{\nu}(z)(\Omega^- \times I^+) + \hat{\nu}(z)(I^+ \times \Omega^-).$$

Namely, the intersection number of $\hat{\nu}(z)$ with γ_0 is just the total $\hat{\eta}(z)$ -mass of all geodesics crossing transversely through a fundamental domain for the action of g_u on $\hat{\gamma}_0^*$. This set in turn is a fundamental domain for the action of $\langle g_u \rangle$ on the space of all geodesics crossing through $\hat{\gamma}_0^*$. As $(\Omega^- \cup I^+) \cup (I^+ \cup \Omega^-)$ is another such fundamental domain, and $\hat{\eta}(z)$ is $\langle g_u \rangle$ - invariant, this yields the formula (37).

By symmetry, it suffices to bound $\hat{\nu}(z)(\Omega^- \times I^+)$ from above. Recall that the map $\Xi_z : \partial_\infty \mathbb{H}^2 \to \mathcal{F}$ is the limit map induced by the Hitchin grafting representation ρ_z . Our computations rely on the characterisation of $\hat{\nu}(z)$ as the product

$$\hat{\nu}(z) = e^{\delta(z) \langle \Xi_z(\xi) | \Xi_z(\eta) \rangle_p} d\mu_z^p(\xi) d\mu_z^p(\eta)$$

where $\langle \cdot | \cdot \rangle_p$ is the Gromov product based at p and p is any point in X. The measure $\hat{\nu}(z)(\Omega^- \times I^+)$ can then be bounded as follows, using that $I^+ = \bigcup_n g_u^n \Omega^+$:

$$\begin{split} \hat{\nu}(z)(\Omega^{-} \times I^{+}) &= \sum_{n} \int_{\Omega^{-} \times g_{u}^{n} \Omega^{+}} e^{\delta(z) \langle \Xi_{z}(\xi) | \Xi_{z}(\eta) \rangle_{p}} d\mu_{z}^{p}(\xi) d\mu_{z}^{p}(\eta) \\ &\leq \mu_{z}^{p}(\Omega^{-}) \cdot \max_{(\xi,\eta) \in \Omega^{-} \times I^{+}} (e^{\delta(z) \langle \Xi_{z}(\xi) | \Xi_{z}(\eta) \rangle_{p}}) \cdot \sum_{n} \mu_{z}^{p}(g_{u}^{n} \Omega^{+}). \end{split}$$

The strategy for estimating these quantities and completing the proof is the following.

- (i) Make a suitable choice of basepoint p.
- (ii) Use Proposition 6.6 to find a constant C_1 only depending on σ such that $\mu_z^p(\Omega^-) \leq C_1$.
- (iii) Use admissible paths and Proposition 3.5 to find a constant C_2 only depending on σ such that $\langle \Xi_z(\xi), \Xi_z(\eta) \rangle_p \leq C_2$ for all $\xi \in \Omega^-$ and $\eta \in I^+$.
- (iv) Use admissible paths and Propositions 6.6 and 3.5 to bound $\mu_z^p(g_u^n\Omega^+)$ and conclude the proof.

The most involved part will be the last step (iv) of the above list.

First step (i). Put $\ell := \ell_S(\gamma_0^*) = \ell^{\mathfrak{F}}(\rho_z(g_u))$, which is bounded from above by σ by assumption, and let $\omega = \sinh^{-1}\left(\frac{1}{\sinh(\ell/2)}\right)$ be the size of the collar in S around γ_0^* , so that the two boundaries of the collar are in the ϵ_0 -thick part of S for some universal constant ϵ_0 .

The geodesic line $\hat{\gamma}_0^*$ is adjacent to two connected components \tilde{S}^-, \tilde{S}^+ of $\mathbb{H}^2 - \tilde{\gamma}^*$. Denote by H^+, H^- the two closed half-planes of \mathbb{H}^2 with boundary $\hat{\gamma}_0^*$ and assume that $\tilde{S}^{\pm} \subset H^{\pm}$ and that $I^{\pm} \subset \partial_{\infty} H^{\pm}$. Let x^-, x^+ be the points lying in this order on the geodesic (ξ_0^-, ξ_0^+) , both at distance exactly ω from the intersection point x of (ξ_0^-, ξ_0^+) with $\hat{\gamma}_0^*$. In particular, x^{\pm} projects into the ϵ_0 -thick part of S.

Recall that for the abstract grafting surface S_z there exists a natural projection map $\pi_z : S_z \to S$ which is injective outside of the flat cylinders (see Definition 3.1). Lift π_z to a $\pi_1(S)$ -equivariant map $\tilde{\pi}_z : \tilde{S}_z \to \tilde{S} = \mathbb{H}^2$, which is injective on the preimages $\tilde{S}_z^{\pm} \subset \tilde{S}_z$ of the components \tilde{S}^{\pm} of $\mathbb{H}^2 - \tilde{\gamma}^*$. Then $x^{\pm} \in \mathbb{H}^2$ (but not x) have unique preimages $\tilde{x}^{\pm} \in \tilde{S}_z^{\pm}$. The basepoint p we were looking for is $p = \tilde{Q}_z(\tilde{x}^-)$.

Second step (ii). Pulling the Patterson Sullivan measure μ_p based at p for the action of $\rho_z(\pi_1(S))$ back to a measure $\mu_z^{\tilde{x}^-}$ on $\partial_\infty \mathbb{H}^2$, this is an immediate application of Proposition 6.6 (and 6.10), which says that $\mu_z^{\tilde{x}^-}(\partial_\infty \mathbb{H}^2)$ is bounded from above by a constant depending only on σ .

Third step (iii). Let $\xi \in \Omega^-$ and $\eta \in I^+$. There is a unique bi-infinite admissible path $a: \mathbb{R} \to \mathbb{H}^2$ from ξ to η , which is a lift of an admissible path in the hyperbolic surface S, defined with respect to the multicurve γ^* . Recall that a is a concatenation of geodesic

pieces, alternating between arcs contained in $\tilde{\gamma}^*$, called flat-type, and geodesic arcs with endpoints on $\tilde{\gamma}^*$ and orthogonal to $\tilde{\gamma}^*$, called hyperbolic-type.

Up to parameterization of the flat pieces, a is the image under π_z of a unique admissible path $\tilde{a}: \mathbb{R} \to \tilde{S}_z \subset \mathbb{X}$ (see [BHM25]) for details). By Lemma 6.12, $\tilde{Q}_z(\tilde{a}(t))$ accumulates as $t \to \infty$ (resp. $t \to -\infty$) in the interior of the Weyl Chamber $\Xi_z(\eta)$ (resp. $\Xi_z(\xi)$).

Using the map \tilde{Q}_z , which is a $\pi_1(S)$ -equivariant embedding of \tilde{S}_z onto the universal covering of the characteristic surface of $\rho_z(\pi_1(S))$, pull the Finsler distance $d^{\mathfrak{F}}$ back to \tilde{S}_z and denote this distance by the same symbol. With this notation and by definition of the Gromov product (see (4)), we have

$$\langle \Xi_z(\eta), \Xi_z(\xi) \rangle_{x^-} = \lim_{T \to \infty} \frac{1}{2} \left(d^{\mathfrak{F}}(\widetilde{a}(-T), \widetilde{x}^-) + d^{\mathfrak{F}}(\widetilde{x}^-, \widetilde{a}(T)) - d^{\mathfrak{F}}(\widetilde{a}(-T), \widetilde{a}(T)) \right).$$

By Proposition 3.5, the path $\tilde{Q}_z(\tilde{a})$ is C_2 -quasi-ruled for some constant C_2 depending only on σ , so $d_{\mathbb{X}}^{\mathfrak{F}}(a(-T), \tilde{a}(t)) + d_{\mathbb{X}}^{\mathfrak{F}}(\tilde{a}(t), \tilde{a}(T)) \leq d_{\mathbb{X}}^{\mathfrak{F}}(\tilde{a}(-T), \tilde{a}(T)) + C_2$ for all $-T \leq t \leq T$. Thus, to find an upper bound on $\langle \Xi_z(\eta), \Xi_z(\xi) \rangle_{\tilde{x}^-}$, it suffices to prove that there exists a number R > 0 only depending on σ such that \tilde{a} intersects the ball of radius R about \tilde{x}^- .

As a intersects $\hat{\gamma}_0^*$, it contains a (possibly degenerate) piece of flat type which is a subarc of $\hat{\gamma}_0^*$. Choose a parameterization of a so that $a(0) \in \hat{\gamma}_0^*$ is the starting point of this segment. Then the piecewise geodesic ray $a|_{(-\infty,0]} : (-\infty,0] \to H^-$ ends on $\hat{\gamma}_0^*$ with a hyperbolic-type geodesic piece of length at least ω (the collar size). By the definition of Ω^- , the shortest distance projection of ξ to $\hat{\gamma}_0^*$ is at distance at most ℓ from the shortest distance projection x of x^- . The constant R we are looking for is provided by the following lemma, whose proof (which relies on hyperbolic trigonometry) is postponed until after this proof.

Lemma 6.13. For any $\sigma > 0$ there is R > 0 such that the following holds. Let $0 < \ell \leq \sigma$ and let $\omega = \sinh^{-1}\left(\frac{1}{\sinh(\ell/2)}\right) > 0$ be the collar size associated to ℓ by the hyperbolic collar lemma.

Let $\mathcal{L} \subset \mathbb{H}^2$ be a line and $a : [0, \infty) \to \mathbb{H}^2$ an admissible path starting on \mathcal{L} orthogonally to it, and with a hyperbolic-type piece of length at least ω . Suppose a(t) tends as $t \to \infty$ to $\xi \in \partial \mathbb{H}^2$ whose orthogonal projection is ℓ -close to the starting point of a ray $r : [0, \infty) \to \mathbb{H}^2$ orthogonal to \mathcal{L} and in the same half-plane as a. Then $d_{\mathbb{H}^2}(a(\omega), r(\omega)) \leq R$.

Lemma 6.13 exactly tells us that the point $a(-\omega)$ is contained in the ball of radius R about x^- . It remains to check that the distance between $\tilde{a}(-\omega)$ and \tilde{x}^- is at most R as well. This holds true because $a(-\omega)$ and x^- are contained in \tilde{S}^- , so their preimages $\tilde{a}(-\omega)$ and \tilde{x}^- are contained in the same hyperbolic piece $\tilde{S}_z^- \subset \tilde{S}_z$. As this piece is isometrically embedded in \tilde{S}_z and the Finsler distance $d^{\mathfrak{F}}$ is not larger than the path distance on the grafted surface, this completes the distance estimate.

Fourth step (iv). This part of the proof is the longest and most involved. By equivariance, we have

$$\mu_z^{\widetilde{x}^-}(\xi) = e^{\delta(z)b_{\Xi_z(\xi)}^{\widetilde{\mathfrak{F}}}(\widetilde{x}^+, \widetilde{x}^-)} d\mu_z^{\widetilde{x}^+}(\xi)$$

where $\delta(z)$ is the critical exponent of ρ_z and $b_{\eta}^{\mathfrak{F}}(q,q')$ is the Busemann function of (q,q') based at $\eta \in \mathcal{F}$ (for the Finsler metric), see Section 1.

Since $\mu_z^{\widetilde{x}^+}(\partial_\infty \mathbb{H}^2) \leq C_1$ for a constant $C_1 > 0$ only depending on σ by Proposition 6.6, to get the desired upper bound on $\mu_z^{\widetilde{x}^-}(g_u^n \Omega^+)$ it suffices for $\xi^+ \in g_u^n \Omega^+$ to bound from

above the Busemann function $b_{\Xi_z}^{\mathfrak{F}}(g_u^n \widetilde{x}^+, \widetilde{x}^-)$. For this we use the admissible path from \widetilde{x}^- to $\Xi_z(\xi^+)$, which is quasi-ruled and passes near $g_u^n \widetilde{x}^+$, and we use our knowledge of the lengths of the pieces of admissible paths. We will see that the Busemann function is roughly $-\max(L, |n-n_0|\ell) - 2\omega$ for some n_0 independent of n. We will then be able to conclude our estimate of $\mu_z^{\widetilde{x}^-}(I^+)$ by computing

$$\begin{split} \hat{\nu}(z)(\Omega^{-} \times I^{+}) &\leq \mu_{z}^{p}(\Omega^{-}) \cdot \max_{(\xi,\eta) \in \Omega^{-} \times I^{+}} (e^{\delta(z)\langle \Xi_{z}(\xi) | \Xi_{z}(\eta) \rangle_{p}}) \cdot \sum_{n} \mu_{z}^{p}(g_{u}^{n}\Omega^{+}) \\ &\leq \operatorname{Cst} e^{-\delta(z)\max(L,|n-n_{0}|\ell) - 2\delta(z)\omega} \end{split}$$

Fix $\xi \in g_u^n \Omega^+ \subset I^+$. There exists a unique admissible path $a_{\xi} : [0, +\infty) \to \widetilde{S} = \mathbb{H}^2$ from x^- to ξ (lifting an admissible path of S), and it is the image under $\widetilde{\pi}_z$ of a unique admissible path $\widetilde{a}_{\xi} : [0, \infty) \to \widetilde{S}_z \subset \mathbb{X}$ that starts at \widetilde{x}^- and accumulates in the interior of the simplex $\Xi_z(\xi)$ by Lemma 6.12. By the definition of Finsler Busemann cocycles (see Section 1), this means that we have

(38)
$$b_{\Xi_{z}(\xi)}^{\mathfrak{F}}(\rho_{z}(g_{u}^{n})\widetilde{x}^{+},\widetilde{x}^{-}) = \lim_{T \to \infty} d^{\mathfrak{F}}(\rho_{z}(g_{u}^{n})\widetilde{x}^{+},\widetilde{a}_{\xi}(T)) - d^{\mathfrak{F}}(\widetilde{x}^{-},\widetilde{a}_{\xi}(T)).$$

Notice that the third geodesic piece of \tilde{a}_{ξ} (the one that leaves the flat strip $\hat{\gamma}_0^*$) is the isometric image by g_u^n of an admissible path going from $\hat{\gamma}_0^*$ to $g_u^{-n}\xi \in \Omega^+$. And therefore \tilde{a}_{ξ} passes within distance R of $\rho_z(g_u^n)\tilde{x}^+$ at some time t.

By Proposition 3.5, \tilde{a}_{ξ} is C_2 -quasi-ruled (and starts at \tilde{x}^-) so

$$d^{\mathfrak{F}}(\widetilde{a}_{\xi}(t),\widetilde{a}_{\xi}(T)) - d^{\mathfrak{F}}(\widetilde{x}^{-},\widetilde{a}_{\xi}(T)) \leq -d^{\mathfrak{F}}(\widetilde{x}^{-},\widetilde{a}_{\xi}(t)) + C_{2} \text{ for any } T \geq t.$$

This, combined with $d^{\mathfrak{F}}(\rho_z(g_u^n)\widetilde{x}^+, \widetilde{a}_{\xi}(t)) \leq R$ and (38) yields:

(39)
$$b^{\mathfrak{F}}_{\Xi_{z}(\xi)}(\rho_{z}(g_{u}^{n})\widetilde{x}^{+},\widetilde{x}^{-}) \leq -d^{\mathfrak{F}}(\rho_{z}(g_{u}^{n})\widetilde{x}^{+},\widetilde{x}^{-}) + C_{2} + 2R.$$

We now need to estimate $d^{\mathfrak{F}}(\rho_z(g_u^n)\widetilde{x}^+,\widetilde{x}^-)$, and we also do this using that the admissible path from \widetilde{x}^- to $\rho_z(g_u^n)\widetilde{x}^+$ is quasi-ruled, except that this time this path is completely explicit. The unique admissible path c in \mathbb{H}^2 from x^- to $g_u^n x^+$ has three geodesic pieces: first the geodesic from x^- to x, of length ω , then the geodesic from x to $g_u^n x$, of length $|n|\ell$, and finally the geodesic from $g_u^n x$ to $g_u^n x^+$, of length ω . It's image under $\widetilde{\pi}_z$ is the unique admissible path \widetilde{c} from \widetilde{x}^- to $\rho_z(g_u^n)\widetilde{x}^+$, which is also made of three explicit geodesic pieces. The first and last pieces are just translates of the corresponding pieces of c, and hence have length ω .

The middle piece, however, is more complicated because instead of sliding along $\hat{\gamma}_0^*$ like c, we are navigating in a flat strip that lifts the flat cylinder above $\gamma_0^* \subset \gamma^*$, and we must move diagonally in this flat strip to realise at the same time the horizontal translation prescribed by the middle piece of c and the vertical translation prescribed by the grafting parameter z. Let $z_0 \in \mathfrak{a}$ be the coordinate of z associated to the component $\gamma_0^* \subset \gamma^*$. Then the above mentioned flat strip is conjugate to the strip $\{tv_0 \pm sz_0 : t \in \mathbb{R}, s \in [0, 1]\}$ where $v_0 = d\tau \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the special direction of \mathfrak{a}^+ and the sign \pm depends on the choice of orientation on γ_0^* (see Section 3.1). Since we are moving horizontally a distance $|n|\ell$, the middle piece of \tilde{c} is conjugate in this strip to the geodesic segment from 0 to $n\ell v_0 \pm z_0$. As a consequence, the length of this middle piece is exactly $\mathfrak{F}(n\ell v_0 \pm z_0)$ where \mathfrak{F} is

the norm on \mathfrak{a} defined in Equation (2). Finally, using again that \tilde{c} is C_2 -quasi-ruled (Proposition 3.5), we get

(40)
$$d^{\mathfrak{F}}(\widetilde{x}^{-},\rho_{z}(g_{u}^{n})\widetilde{x}^{+}) \geq 2\omega + \mathfrak{F}(z_{0} \pm n\ell v_{0}) - 2C_{2}.$$

Now we must estimate $\mathfrak{F}(z_0 \pm n\ell v_0)$. By the assumption, the height of the cylinder at $\widetilde{\gamma}_0^*$ is $\min_{t \in \mathbb{R}} \mathfrak{F}(z_0 + tv_0) \geq L$ (see (16)). Let t_0 be the unique point of \mathbb{R} such that $z_0 + t_0 v_0 \in \ker \alpha_0$ (α_0 is the linear form which is equal to the Finsler norm in the Weyl cone that contains w_0). Then

$$\mathfrak{F}(z_0 + tv_0) \ge |\alpha_0(z_0 + tv_0)| = |\alpha_0((t - t_0)v_0)| = |t - t_0|$$

for any $t \in \mathbb{R}$. Let n_0 be the integer closest to t_0/ℓ , so that $\mathfrak{F}(z_0 + n\ell v_0) \ge |n - n_0|\ell - \ell$, which is bounded below by $|n - n_0|\ell - \sigma$. Combining this with (39) and (40) we get

$$b_{\Xi_z(\xi)}^{\mathfrak{F}}(\rho_z(g_u^n)\widetilde{x}^+,\widetilde{x}^-) \le -2\omega - \max(|n_0 \pm n|\ell,L) + C_2 + 2R + \sigma + 2C_2.$$

(Recall that \pm is just some fixed sign depending on the choice of orientation of γ_0^* .)

Recall the quasi-invariance of Patterson–Sullivan measures:

$$\mu_z^{\widetilde{x}}(g_u^n\Omega^+) = \int_{\xi \in g_u^n\Omega^+} e^{\delta(z)b_{\Xi_z(\xi)}^{\widetilde{\mathfrak{Z}}}(\rho_z(g_u^n)\widetilde{y},\widetilde{x})} d\mu_z^{\rho_z(g_u^n)\widetilde{y}}(\xi)$$

Since $\mu_z^{\rho_z(g_u^n)\widetilde{y}}(\partial \mathbb{H}^2) = \mu_z^{\widetilde{y}}(\partial \mathbb{H}^2)$ is bounded from above by some constant C_1 that only depends on σ by Proposition 6.6, and $\delta(z) \leq m$ for some constant m depending only on α_0 by Lemma 6.3, we get

$$\mu_{z}^{\widetilde{x}}(\rho_{z}(g_{u}^{n})\Omega^{+}) \leq e^{m(2R+3C_{2}+\sigma)}C_{1}e^{-\delta(z)\max(|n_{0}\pm n|\ell,L)}e^{-2\delta(z)\omega}$$
$$= C_{3}e^{-\delta(z)\max(|n_{0}\pm n|\ell,L)}e^{-2\delta(z)\omega}.$$

where C_3 only depends on σ . After some computations, and using that (for $\alpha = \delta(z)\ell$ and $\beta = L/\ell$)

$$e^{-\omega} \le \ell \cosh(\sigma)$$
 and $\sum_{n \ge \beta} e^{-\alpha n} \le \frac{e^{-\alpha \beta}}{\alpha}$,

we get

$$\begin{aligned} \mu_z^{\widetilde{x}}(I^+) &\leq C_3 e^{-2\delta(z)\omega} \sum_n e^{-\delta(z)\max(|n_0\pm n|\ell,L)} \\ &\leq 2C_3 e^{-2\delta(z)\omega} \left(\sum_{0 \leq n < L/\ell} e^{-\delta(z)L} e^{-2\delta(z)\omega} + \sum_{n \geq L/\ell} e^{-\delta(z)n\ell} \right) \\ &\leq 2C_3 \left(\frac{L}{\ell} + \frac{1}{\delta(z)\ell} \right) e^{-\delta(z)L} (e^{-\omega})^{2\delta(z)} \\ &\leq 2C_3 \left(\frac{1}{\ell} + \frac{1}{\delta(z)\ell} \right) (L+1) e^{-\delta(z)L} \ell^{2\delta(z)} \cosh(\sigma)^{2m} \\ &\leq C_4 \max(\ell^{2\delta(z)-1}, 1) (L+1) e^{-\delta(z)L}. \end{aligned}$$

To obtain C_4 only depending on σ , we use that $\delta(z) \geq \delta$ and that δ is bounded from below by a constant that only depends on σ , by Proposition A.3.

By Proposition A.3, there exists $\epsilon_{\sigma} \leq 1$ such that if $\ell \leq \epsilon_{\sigma}$ then $\delta > \frac{1}{2}$. Thus, if on one hand $2\delta(z) - 1 \geq 0$ then $\ell^{2\delta(z)-1} \leq \sigma^{2\delta(z)-1} \leq \sigma^{2m-1}$. On the other hand, if $2\delta(z) - 1 < 0$ then we must have $\ell > \epsilon_{\sigma}$ so $\ell^{2\delta(z)-1} \leq \epsilon_{\sigma}^{2\delta(z)-1} \leq \epsilon_{\sigma}^{2m-1}$. In any case $\ell^{2\delta(z)-1}$ is bounded above by a constant that only depends on σ , which concludes the proof.

We now prove the technical estimate we used in the proof.

Proof of Lemma 6.13. Parallel transport \mathcal{L} along the first geodesic piece of a until time ω , to obtain \mathcal{L}' at distance ω from \mathcal{L} . Let H be the half-plane delimited by \mathcal{L}' that does not contain \mathcal{L} . Then by definition of admissible path one can check that $a(t) \in H$ for any $t \geq \omega$.

By a classical formula of hyperbolic trigonometry, see Theorem 7.17.1 of [Bea83], the orthogonal projection of any $x \in H$ is at distance at most $\sinh^{-1}(\frac{1}{\sinh(\omega)})$ from a(0).

In particular the orthogonal projection of ξ is at distance at most $\ell' := \sinh^{-1}\left(\frac{1}{\sinh(\omega)}\right)$ from a(0), and by triangle inequality a and r start at distance at most $\ell + \ell'$. Using for instance again Theorem 7.17.1 of [Bea83], one can check that $a(\omega)$ and $r(\omega)$ are at distance at most twice the following:

$$\sinh^{-1}(\sinh(\ell/2)\cosh(\omega)) + \sinh^{-1}(\sinh(\ell'/2)\cosh(\omega)) \le 2\sinh^{-1}\left(\frac{\cosh\omega}{\sinh\omega}\right),$$

which can be bounded above in terms of σ because ω can be bounded below in terms of σ (since $\ell \leq \sigma$).

6.5. Convergence of currents. Recall $\delta > 0$ is the topological entropy of $\Phi_{|K}^t$. The following is the main result of this section.

Proposition 6.14. Let $L_i \to \infty$ and let $\rho_i = \rho_{z_i}$ be a sequence of Hitchin representations obtained by Hitchin grafting of a Fuchsian representation at the simple geodesic multicurve γ^* with cylinder heights bounded from below by L_i . Then $\delta(z_i) \to \delta$, and up to passing to a subsequence, the equilibrium measures $\nu_i = \nu(z_i)$ converge weakly to a measure of maximal entropy for $\Phi^t | K$.

Proof. Recall that $f_i = f_{z_i}$ denotes a positive Hölder continuous potential on T^1S whose periods are the Finsler translation lengths of the elements of $\rho_i(\pi_1(S))$.

Up to passing to a subsequence, we may assume that the Φ^t -invariant probability measures $\nu_i^1 = \nu_i/||\nu_i||$ converges weakly to a Φ^t -invariant probability measure ν on T^1S . By Lemma 6.5, we may also assume that the geodesic currents $\hat{\nu}(z)$ converge weakly to a current $\hat{\nu}$ which is a positive multiple of the current defined by ν .

By Proposition 6.11, we have $\iota(\hat{\nu}, \gamma^*) = 0$ and hence the limit measure ν must be supported on K. By Lemma 6.4, we are thus left with showing that $h_{\nu} \geq \delta$.

From Lemma 6.5 we have $\delta(z_i) \in (\delta, m]$ for any *i*. Recall from (34) that

(41)
$$h_{\nu_i^1} = \delta(z_i) \int f_i d\nu_i^1.$$

By Theorem 3.8, it holds

$$\int f_i \frac{d\eta}{\ell_S(\eta)} \ge \left(1 + \frac{C}{L_i + 1}\right)^{-1}$$

for any $\eta \in \pi_1(S)$, and hence since the Φ^t -invariant Borel probability measures supported on closed geodesics are weak^{*}-dense in the space of all Φ^t -invariant Borel probability measures, we get

$$\int f_i d\nu_i^1 \ge \left(1 + \frac{C}{L_i + 1}\right)^{-1},$$

and hence

$$\lim \inf_{i \to \infty} h_{\nu_i} \ge \lim \inf_{i \to \infty} \delta(z_i) \ge \delta.$$

Since the entropy function is lower semi-continuous, we conclude that $h_{\nu} \geq \delta$. As ν is supported in K, this implies that indeed, ν is a measure of maximal entropy for the restriction of Φ^t to K by Lemma 6.4.

Using the above results we are now ready to complete the proof of Theorem C from the introduction.

Proof of Theorem C. Part (3) of Theorem C was shown in Section 3.4, so we are left with showing part (1) and (2). Let $\gamma^* \subset S$ be a pair of pants decomposition of $S_1 = S - S_0$ that contains $\partial S_0 = \partial S_1$. The metric h on S_0 prescribes lengths for the components of γ^* in ∂S_0 .

Since no component of S_1 is a pair of pants, every pair of pants in $S_1 - \gamma^*$ has a boundary component in $\gamma^* - \partial S_1$. By Proposition A.4, one can choose lengths large enough for each component of $\gamma^* - \partial S_1$ such that each pair of pants of $S_1 - \gamma^*$ has entropy very close to zero, and in particular strictly smaller than the entropy of S_0 .

Then by Hitchin grafting along γ^* flat cylinders with bigger and bigger heights, we get a sequence $\rho_i = \rho_{z_i}$ of Hitchin representations satisfying the first two statement of Theorem C, according to Proposition 6.14.

7. Pressure length control

Define the entropy gap of the pair consisting of a hyperbolic surface and a separating simple closed geodesic to be the absolute value of the difference between the entropies of the two components of $S - \gamma^*$. If γ^* is non-separating then we define the entropy gap to be one.

Consider a path $t \to \rho_{tz}$ of Hitchin representations obtained by Hitchin grafting along a single geodesic γ and grafting parameters a ray in \mathfrak{a} with direction in the kernel of the linear functional defining the Finsler length of γ^* . The first goal of this section is to show **Theorem 7.1**. The pressure metric length of the path $t \to \rho_{tz}$ is finite and bounded from above by a constant only depending on z, and an upper bound for the length of the grafting geodesic γ^* .

We also show

Theorem 7.2. Consider a subsurface $S_1 \subset S$ with $\partial S_1 \subset \gamma^*$. Let $(\rho^t)_{t \in [0,T]}$ be a path of hyperbolic structures on S obtained by concatenating shearing paths along multicurves contained in S_1 , such that for any $t \in [0,T]$ the entropy of the geodesic flow on T^1S_1 and the restriction of the metric ρ_t is strictly smaller than the entropy of the geodesic flow on T^1S_0 for $S_0 = S - S_1$ (which does not depend on t). Denote by ρ_z^t the Hitchin grafting of ρ^t alogn γ^* with parameter z. Then the pressure length of the smooth path $(\rho_z^t)_{t \in [0,T]}$ tends to zero as the cylinder height associated to z tends to infinity.

This section is subdivided into three subsections. In the first subsection we recall the definition of the pressure length of a path in the Hitchin component, and we give an upper bound purely in terms of the nonnormalized intersection form $\mathbf{I}(\rho_1, \rho_2)$ (see Section 2.1). In the second subsection, we give upper bounds for the derivatives of this intersection form along the two paths in the above theorems, and use this to conclude.

7.1. A general upper bound for pressure lengths. Let $[\rho_t]_{a \le t \le b}$ be a smooth path in the Hitchin component. By definition, its length for the pressure metric is

$$\int_{a}^{b} \left(\frac{d^2}{ds^2} \mathbf{J}(\rho_t, \rho_{t+s})|_{s=0} \right)^{\frac{1}{2}} dt,$$

where, denoting by $\delta(t)$ the entropy of ρ_t , we have $\mathbf{J}(\rho_t, \rho_{t+s}) = \frac{\delta(t+s)}{\delta(t)} \mathbf{I}(\rho_t, \rho_{t+s})$ and $\mathbf{I}(\rho_t, \rho_{t+s})$ is the nonnormalized intersection form. We want an upper bound for this length in terms of $\mathbf{I}(\rho_t, \rho_{t+s})$ and its derivatives. This is possible thanks to the following classical lemma.

Lemma 7.3. $\delta'(t) = -\delta(t) \frac{d}{ds} \mathbf{I}(\rho_t, \rho_{t+s})|_{s=0}.$

Proof. We fix a hyperbolic structure on S and use its unit tangent bundle T^1S , equipped with the geodesic flow, as the underlying phase space for all computations. Let f_t be a reparametrisation function associated with ρ_t . Recall that $P(-\delta(t)f_t) = 0$, where P is the pressure function (see Section 2.1). We are going to differentiate this equality, using the fact due to Parry–Pollicott, see Propositions 4.10-11 of [PP90], and Ruelle [Rue78], that for any C^1 one-parameter family of Hölder functions $(g_t)_t$ we have $\frac{d}{dt}P(g_t) = \int (\frac{d}{dt}g_t)d\mu_t$ where μ_t is the equilibrium state associated to g_t .

Let ν_t be the equilibrium state associated with $-\delta(t)f_t$. Then

$$0 = \int \left(\delta'(t) f_t + \delta(t) \left(\frac{d}{dt} f_t \right) \right) d\nu_t$$

Now recall that

$$\mathbf{I}(\rho_t, \rho_{t+s}) = \frac{\int f_{t+s} d\nu_t}{\int f_t d\nu_t} \quad \text{and} \quad \frac{d}{ds} \mathbf{I}(\rho_t, \rho_{t+s})|_{s=0} = \frac{\int \left(\frac{d}{dt} f_t\right) d\nu_t}{\int f_t d\nu_t},$$

which concludes the proof.

We can now prove the following upper bound for the pressure length of $[\rho_t]_{a \le t \le b}$. **Proposition 7.4.** An upper bound for $\int_a^b \left(\frac{d^2}{ds^2} \mathbf{J}(\rho_t, \rho_{t+s})|_{s=0}\right)^{\frac{1}{2}} dt$ is

$$\sqrt{b-a} \left(-\frac{d}{ds} \mathbf{I}(\rho_b, \rho_{b+s})|_{s=0} + \frac{d}{ds} \mathbf{I}(\rho_a, \rho_{a+s})|_{s=0} + \int_a^b \frac{d^2}{ds^2} \mathbf{I}(\rho_t, \rho_{t+s})|_{s=0} dt \right)^{\frac{1}{2}}.$$

Proof. Let us differentiate twice the formula $\mathbf{J}(\rho_t, \rho_{t+s}) = \frac{\delta(t+s)}{\delta(t)} \mathbf{I}(\rho_t, \rho_{t+s})$. We get

$$\frac{d^2}{ds^2} \mathbf{J}(\rho_t, \rho_{t+s})|_{s=0} = \frac{\delta''(t)}{\delta(t)} + 2\frac{\delta'(t)}{\delta(t)}\frac{d}{ds}\mathbf{I}(\rho_t, \rho_{t+s})|_{s=0} + \frac{d^2}{ds^2}\mathbf{I}(\rho_t, \rho_{t+s})|_{s=0}$$

Using
$$\frac{\delta''}{\delta} = (\log \delta)'' + (\frac{\delta'}{\delta})^2$$
 and $\frac{\delta'(t)}{\delta(t)} = -\frac{d}{ds} \mathbf{I}(\rho_t, \rho_{t+s})|_{s=0}$ from Lemma 7.3, we get

$$\frac{d^2}{ds^2} \mathbf{J}(\rho_t, \rho_{t+s})|_{s=0} = (\log \delta)''(t) - \left(\frac{\delta'(t)}{\delta(t)}\right)^2 + \frac{d^2}{ds^2} \mathbf{I}(\rho_t, \rho_{t+s})|_{s=0}$$

$$\leq (\log \delta)''(t) + \frac{d^2}{ds^2} \mathbf{I}(\rho_t, \rho_{t+s})|_{s=0}$$

We now conclude:

$$\int_{a}^{b} \left(\frac{d^{2}}{ds^{2}}\mathbf{J}(\rho_{t},\rho_{t+s})|_{s=0}\right)^{\frac{1}{2}} dt \leq \sqrt{b-a} \left(\int_{a}^{b} \frac{d^{2}}{ds^{2}}\mathbf{J}(\rho_{t},\rho_{t+s})|_{s=0} dt\right)^{\frac{1}{2}}$$

$$\leq \sqrt{b-a} \left(\int_{a}^{b} (\log \delta)''(t) dt + \int_{a}^{b} \frac{d^{2}}{ds^{2}}\mathbf{I}(\rho_{t},\rho_{t+s})|_{s=0} dt\right)^{\frac{1}{2}}$$

$$\leq \sqrt{b-a} \left((\log \delta)'(b) - (\log \delta)'(a) + \int_{a}^{b} \frac{d^{2}}{ds^{2}}\mathbf{I}(\rho_{t},\rho_{t+s})|_{s=0} dt\right)^{\frac{1}{2}}$$

$$\leq \sqrt{b-a} \left(-\frac{d}{ds}\mathbf{I}(\rho_{b},\rho_{b+s})|_{s=0} + \frac{d}{ds}\mathbf{I}(\rho_{a},\rho_{a+s})|_{s=0} + \int_{a}^{b} \frac{d^{2}}{ds^{2}}\mathbf{I}(\rho_{t},\rho_{t+s})|_{s=0} dt\right)^{\frac{1}{2}}.$$

7.2. Proofs of Theorems 7.1 and 7.2. First we give an upper bound of the derivatives of the intersection form in the case of the grafting path from Theorem 7.1.

Proposition 7.5. Let $(\rho_t = \rho_{tz})_{t\geq 0}$ be a grafting path as in Theorem 7.1. Then there exist numbers $\kappa > 0$ and C > 0 only depending on z and an upper bound σ for the length of γ such that

$$\left|\frac{d}{ds}\mathbf{I}(\rho_t, \rho_{t+s})|_{s=0}\right| \le Ce^{-\kappa t} \text{ and } \frac{d^2}{ds^2}\mathbf{I}(\rho_t, \rho_{t+s})|_{s=0} \le Ce^{-\kappa t}$$

Proof. Resuming the notations from Section 2.2, Proposition 2.8 shows that a Hitchin grafting path $t \to \rho_{tz}$ in the Hitchin component gives rise to a real analytic family $f_t: T^1S \to (0, \infty)$ of Hölder functions defining a reparameterization of the geodesic flow on T^1S corresponding to the Finsler length of $\rho_{tz}(\pi_1(S))$.

For each t let $\nu(t)$ be the Gibbs equilibrium state of $-\delta(t)f_t$ where $\delta(t) > 0$ is such that the pressure of $-\delta(t)f_t$ vanishes. By our convention, $\nu(t)$ is a probability measure on T^1S which is invariant under the geodesic flow Φ^t on S. For $\eta \in \pi_1(S)$ we also put $f_t(\eta) = \int_{\eta} f_t$, the Finsler translation length of the element $\rho_{tz}(\eta)$.

To bound the derivatives of the intersection form \mathbf{I} we will use the following formula (see Section 2.1)

$$\mathbf{I}(\rho_t, \rho_{t+s}) = \frac{\int f_{t+s} d\nu_t}{\int f_t d\nu_t} = \int f_{t+s} d\nu_t,$$

which is easier to differentiate with respect to s.

We know that

$$\nu(t) = \lim_{T \to \infty} \frac{1}{\# N_{f_t}(T)} \sum_{\eta \in N_{f_t}(T)} \frac{\mathcal{D}_{\eta}}{f_t(\eta)},$$

where \mathcal{D}_{η} is the flow-invariant measure supported on the periodic orbit η (in our fixed hyperbolic structure). Exchanging derivatives and integrals, we get

$$\begin{aligned} \frac{d}{ds} \int f_{t+s} d\nu(t)|_{s=0} &= \int \frac{d}{ds} f_{t+s}|_{s=0} d\nu(t) \\ &= \lim_{T \to \infty} \frac{1}{\# N_{f_t}(T)} \sum_{\eta \in N_{f_t}(T)} \frac{\int \frac{d}{ds} f_{t+s}|_{s=0} d\mathcal{D}_{\eta}}{f_t(\eta)} \\ &= \lim_{T \to \infty} \frac{1}{\# N_{f_t}(T)} \sum_{\eta \in N_{f_t}(T)} \frac{\frac{d}{ds} f_{t+s}(\eta)|_{s=0}}{f_t(\eta)} \end{aligned}$$

By Theorem 5.1 we know

$$\left|\frac{d}{ds}f_{t+s}(\eta)|_{s=0}\right| \le A_{\epsilon,\sigma}\iota(\eta,\gamma^*)$$

and hence by the continuity of ι and Hölder continuity of the function $\frac{d}{ds}f_{t+s},$ we conclude that

$$\left|\frac{d}{ds}\int f_{t+s}d\nu(t)|_{s=0}\right| \le A_{\epsilon,\sigma}\iota(\nu(t),\gamma^*).$$

By Proposition 6.11 we have

$$\left|\frac{d}{ds}\int f_{t+s}d\nu(t)|_{s=0}\right| \le Ce^{-\kappa t}.$$

With a similar argument, which is omitted here, one proves

$$\frac{d^2}{ds^2} \int f_{t+s} d\nu(t)|_{s=0} \le C e^{-\kappa t}.$$

We can now conclude the proof of Theorem 7.1.

Proof of Theorem 7.1. Using Propositions 7.4 and 7.5 we have the following estimates on the pressure length of $(\rho_t = \rho_{tz})_{t \ge 0}$.

$$\begin{split} &\int_{0}^{\infty} \left(\frac{d^{2}}{ds^{2}} \mathbf{J}(\rho_{t}, \rho_{t+s})|_{s=0} \right)^{\frac{1}{2}} dt = \sum_{m \ge 0} \int_{m}^{m+1} \left(\frac{d^{2}}{ds^{2}} \mathbf{J}(\rho_{t}, \rho_{t+s})|_{s=0} \right)^{\frac{1}{2}} dt \\ &\leq \sum_{m \ge 0} \left(\frac{d}{ds} \mathbf{I}(\rho_{m+1}, \rho_{m+1+s})|_{s=0} - \frac{d}{ds} \mathbf{I}(\rho_{m}, \rho_{m+s})|_{s=0} + \int_{m}^{m+1} \frac{d^{2}}{ds^{2}} \mathbf{I}(\rho_{t}, \rho_{t+s})|_{s=0} dt \right)^{\frac{1}{2}} \\ &\leq \sum_{m \ge 0} \left(3Ce^{-\kappa m} \right)^{\frac{1}{2}} \\ &\leq \frac{\sqrt{3C}}{1 - e^{-\kappa/2}}. \end{split}$$

We now turn to the proof of Theorem 7.2, which is very similar, except that it uses Corollary 5.2 instead of Theorem 5.1.

Proof of Theorem 7.2. Recall that in the present setting, the geodesic γ divides S into subsurfaces S_0, S_1 , the smooth path $t \to \rho^t$ ($t \in [0, T]$) in the Teichmüller space of marked Riemann surfaces is such that:

- (1) The restriction of the marked hyperbolic metric ρ^t to the subsurface S_0 does not depend on t.
- (2) The entropy of the geodesic flow of ρ^t restricted to the subspace of all geodesics entirely contained in S_0 is strictly larger than the entropy of the restriction of the flow to the subspace of all geodesics entirely contained in S_1 .

We denote by ρ_z^t the representation obtained by grafting ρ^t at γ with parameter z. Let f_z^t be a corresponding positive Hölder function and let ν_z^t be the corresponding Gibbs state, such that $\int f_z^t d\nu_z^t = 1$ (on the hyperbolic surface associated to ρ^t).

As in the proof of Proposition 7.5, denoting $f_z^t(\eta) = \int_{\eta} f_z^t$ the Finsler length of $\rho_z^t(\eta)$, by Corollary 5.2 we have

$$\begin{split} \left| \frac{d}{ds} \mathbf{I}(\rho_z^t, \rho_z^{t+s}) |_{s=0} \right| &= \left| \frac{d}{ds} \int f_z^{t+s} d\nu_z^t |_{s=0} \right| \\ &\leq \lim_{T \to \infty} \frac{1}{\# N_{f_z^t}(T)} \sum_{\eta \in N_{f_z^t}(T)} \frac{\left| \frac{d}{ds} f_z^{t+s}(\eta) \right|_{s=0} \right|}{f_z^t(\eta)} \\ &\leq \lim_{T \to \infty} \frac{1}{\# N_{f_z^t}(T)} \sum_{\eta \in N_{f_z^t}(T)} C \frac{\ell_{\rho^t}^{S_0}(\eta)}{f_z^t(\eta)} \\ &= C \lim_{T \to \infty} \ell_{\rho^t}^{S_0} \left(\frac{1}{\# N_{f_z^t}(T)} \sum_{\eta \in N_{f_z^t}(T)} \frac{\mathcal{D}_{\eta}}{f_z^t(\eta)} \right) \\ &= C \ell_{\rho^t}^{S_0}(\nu_z^t), \end{split}$$

where C is a constant that depends on $(\rho^t)_{0 \le t \le T}$, and $\ell_{\rho^t}^{S_0}(\nu)$ is simply the mass given by ν to the set of unit tangent vectors of S that are footed on S_0 . This is a linear function of ν that is continuous at ν when it gives zero measure to the set of geodesics asymptotic to γ ; in particular it is continuous at ν_z^t .

Similarly one can check that

$$\frac{d^2}{ds^2} \mathbf{I}(\rho_z^t, \rho_z^{t+s})|_{s=0} \le C \ell_{\rho^t}^{S_0}(\nu_z^t)$$

We deduce from this, Proposition 6.11 and Proposition 6.14 that for fixed t, the derivatives $\left|\frac{d}{ds}\mathbf{I}(\rho_z^t,\rho_z^{t+s})\right|_{s=0}$ and $\frac{d^2}{ds^2}\mathbf{I}(\rho_z^t,\rho_z^{t+s})|_{s=0}$ converge to zero when the cylinder height associated to z goes to infinity. Moreover these quantities are bounded above by a constant which depends on the whole path $(\rho^s)_{0\leq s\leq T}$ but not on t (notice that $\ell_{\rho^t}^{S_0}(\nu_z^t)$ is bounded above by the total mass of ν_z^t , which is bounded above by Lemma 6.5). By the dominated convergence theorem, we conclude that

$$\int_0^T \frac{d^2}{ds^2} \mathbf{I}(\rho_z^t, \rho_z^{t+s})|_{s=0} dt \xrightarrow[L \to \infty]{} 0,$$

where L denotes the cylinder height associated to z.

We now conclude using Proposition 7.4, as in the proof of Theorem 7.1.

8. DISTORTION

The restriction of the pressure metric to the Fuchsian locus is a multiple of the Weil Petersson metric on Teichmüller space [BCLS15] and hence its intrinsic large scale geometric properties are well understood. Moreover, by [PS17], the Fuchsian locus can be characterized as the set of Hitchin representations whose critical exponent for the symmetric metric as well as for the Hilbert metric (and other sufficiently well behaved Finsler metrics) assumes a maximum. This intrinsic geometric characterization of the Fuchsian locus does however not reveal information on its significance for the large scale geometry of the Hitchin component.

In fact, the pressure metric for the *Hilbert length*, which by definition is induced from the Hilbert metric for convex domains in projective space, is degenerate and hence *not* a Finsler metric for the Hitchin component. Namely, the contragredient involution of $PSL_d(\mathbb{R})$ acts isometrically on the character variety equipped with the pressure metric. If d = 2m is even, then this involution is just conjugation with the standard symplectic form, with fixed point set the symplectic group $PSp_{2m}(\mathbb{R})$. It turns out that the pressure metric for the Hilbert length is degenerate on the normal bundle of the space of representations with image in $PSp_{2m}(\mathbb{R})$. Note that since the involution is an isometry for the pressure metric, the locus of representations into $PSp_{2m}(\mathbb{R})$ is totally geodesic.

In the case d = 3, the fixed point set of the involution equals the image PSO(2, 1) of $PSL_2(\mathbb{R})$ under the irreducible representation and hence the Fuchsian locus is totally geodesic for the pressure metric (see e.g. [Dai23]). However, in spite of recent refined information on the restriction of the pressure metric to the Fuchsian locus [LW18], the following seems to be an open question.

Question 3. For $d \ge 4$, is the Fuchsian locus totally geodesic for the pressure metric for representations into $\text{PSL}_d(\mathbb{R})$?

On purpose, we leave the specification of the length function defining the pressure metric open.

The main goal of this section is to show that from a global geometric perspective, the Fuchsian locus is distorted for the pressure distance on the Hitchin component for $n \ge 3$ and genus $g \ge 3$, where the pressure distance is taken with respect to the Finsler length considered in the previous sections. We believe that similar arguments should lead to corresponding results for all variants of the pressure metric.

8.1. Regions of finite diameter. Let S be a closed surface of genus $g \ge 3$ and a simple closed curve $\gamma^* \subset S$ that splits S into two subsurfaces S_0, S_1 of genus $g_0 = 2, g_1 = g - 2 \ge 1$. Let $\sigma > 0$ and $0 < \ell \le \sigma$.

Let $\mathcal{T}(S_i, \ell)$ (with i = 0, 1) and $\mathcal{T}(S, \ell)$ be the Teichmüller spaces of marked hyperbolic metrics on S_i and S such that γ^* has length ℓ . The restriction map

$$(r_0, r_1) : \mathcal{T}(S, \ell) \to \mathcal{T}(S_0, \ell) \times \mathcal{T}(S_1, \ell)$$

is a fiber bundle with fiber \mathbb{R} , on which the twist flow along γ^* acts by translation. Using for instance Fenchel–Nielsen coordinates, one can find a section and have a parametrisation

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of $\mathcal{T}(S, \ell)$ of the form

$$\mathcal{T}(S_0,\ell) \times \mathcal{T}(S_1,\ell) \times \mathbb{R} \simeq \mathcal{T}(S,\ell) \subset \mathcal{T}(S).$$

Given a different section we will obtain a different parametrisation but with the same image, and the resulting change of coordinate will be of the form

$$\mathcal{T}(S_0, \ell) \times \mathcal{T}(S_1, \ell) \times \mathbb{R} \to \mathcal{T}(S_0, \ell) \times \mathcal{T}(S_1, \ell) \times \mathbb{R}$$
$$(X_0, X_1, t) \mapsto (X_0, X_1, t + b)$$

where b > 0 is a fixed constant.

The following is the main result of this section.

Theorem 8.1. Let $X_0 \in \mathcal{T}(S_0, \ell)$. Consider the subset $r_0^{-1}\{X_0\} \subset \mathcal{T}(S)$: the points whose restriction to S_0 is isometric to X_0 . Equivalently, it is the image of the embedding

$$\{X_0\} \times \mathcal{T}(S_1, \ell) \times \mathbb{R} \hookrightarrow \mathcal{T}(S) \hookrightarrow \operatorname{Hit}(S).$$

Then for the pressure metric on the Hitchin component, this set has diameter bounded by a number C that only depends on an upper bound σ for ℓ .

Question 4. Is the diameter for the pressure metric of the Fuchsian locus finite? Is the diameter of the Hitchin component for the pressure metric finite?

The proof of Theorem 8.1 has three main ingredients: Theorem 7.1 (upper bounds for pressure lengths of Hitchin grafting paths), Theorem 7.2 (under an entropy gap assumption, certain paths $\mathcal{T}(S)$ pushed in Hit(S) via grafting see their pressure length decrease to zero), and a celebrated result of Wolpert that the Weil–Petersson length of a path in $\mathcal{T}(S)$ obtained by pinching a simple closed curve is bounded above by a constant only depending on the length of the curve. The idea will be, starting from a well chosen path between two arbitrary points of $\{X_0\} \times \mathcal{T}(S_1, \ell) \times \mathbb{R}$, to deform this path by first pinching curves in S_0 (to create an entropy gap) and then grafting along γ^* to shrink the pressure length of this path to zero.

We now recall Wolpert's result more precisely. Given essential disjoint simple closed curves $\alpha_1, \ldots, \alpha_4 \subset S_0$ and $\beta_1, \ldots, \beta_{3g-8} \subset S_1$ that, together with γ^* , decompose Sinto pairs of pants, the Fenchel–Nielsen coordinates give a diffeomorphism from $\mathcal{T}(S)$ to $\mathbb{R}^{3g-3}_{>0} \times \mathbb{R}^{3g-3}$. To a hyperbolic metric is associated the lengths of $\alpha_i, \beta_j, \gamma^*$ and twist parameters along these curves. Pinching $\alpha_1, \ldots, \alpha_4$ by multiplying their length by $\lambda < 1$ in Fenchel–Nielsen coordinates (while keeping all other coordinates constant) is a transformation of $\mathcal{T}(S)$ that does not depend on the choice of $\beta_1, \ldots, \beta_{3g-8}$.

Fact 8.2 ([Wol86]). For any $\sigma' > 0$ there is a constant C > 0 such that the following holds. Let $\gamma_1, \ldots, \gamma_4 \subset S_0$ be a multicurve splitting S_0 into pairs of pants. Then for any $S \in \mathcal{T}(S)$ giving length at most σ' to each $\gamma_1, \ldots, \gamma_4, \gamma^*$, pinching the length of $\gamma_1, \ldots, \gamma_4$ to zero, while keeping all other Fenchel-Nielsen coordinates constant, yields a path in $\mathcal{T}(S)$ with Weil-Petersson length at most C.

Proof of Theorem 8.1. Let $X_0 \in \mathcal{T}(S_0, \ell)$ be a marked hyperbolic metric. Let $X, Y \in \{X_0\} \times \mathcal{T}(S_1, \ell) \times \mathbb{R}$. We want to show that X can be connected to Y by a path of uniformly bounded pressure metric length. This path will be constructed by concatenating five paths of Hitchin representations, illustrated in Figure 2.

Let $X_1, Y_1 \in \mathcal{T}(S_1, \ell)$ be the restrictions of X, Y to S_1 . A result of Wolpert (Corollary 3.5 of [Wol82]) says that the tangent space of $\mathcal{T}(S_1, \ell)$ at each point is spanned by the

vector fields of twist flows along 6g - 16 well chosen simple closed curves (Wolpert's result is stated for closed surfaces, but it also applies to compact surfaces with boundary). By a classical argument from differential geometry, one can hence connect X_1 and Y_1 via a path $(X_1(t))_{0 \le t \le 1}$ which is a finite concatenation of twisting paths along these well chosen simple closed curves.

As a consequence, one can connect X and Y via a path

$$(X(t))_{0 \le t \le 1} \subset \{X_0\} \times \mathcal{T}(S_1, \ell) \times \mathbb{R}$$

which is a finite concatenation of twisting paths along simple closed curves of S_1 (adding to $({X_0} \times {X_1(t)} \times {0})_t$ a final twist along γ^* , if necessary).

By Proposition A.1, for every t the entropy of $X_1(t)$ is strictly less than 1, and it varies continuously with t. By compactness, this entropy is bounded from above by h < 1 independent of t.

Now we want to pinch along curves in S_0 to create an entropy gap. By Theorem A.2 there is a pair of pants decomponsition $\alpha_1, \ldots, \alpha_4 \subset X_0$ with length at most $\max(\ell, 4\pi)$. For any $Z \in \{X_0\} \times \mathcal{T}(S_1, \ell) \times \mathbb{R}$ and $\lambda < 1$, denote by $p_{\lambda}(Z) \in \mathcal{T}(S_0, \ell) \times \mathcal{T}(S_1, \ell) \times \mathbb{R}$ the metric obtained by pinching $\alpha_1, \ldots, \alpha_4$ with factor λ (multipliving lengths by λ in Fenchel–Nielsen coordinates). By Fact 8.2, the Weil–Petersson length of $(p_{\lambda}(Z))_{0 < \lambda < 1}$ is bounded from above by some constant $C_3 > 0$ that depends on σ .

By Proposition A.1, and since $\alpha_1, \ldots, \alpha_4$ split S_0 into two pairs of pants one of which is not adjacent to γ^* , for λ_0 small enough the entropy of $p_{\lambda_0}(X_0)$ is strictly greater than h. In other words, for each t there is an entropy gap in $p_{\lambda_0}(X(t))$ between the S_0 , that has entropy greater than h, and the S_1 , whose entropy is bounded by h.

Fix a grafting parameter z transverse to the twist direction, and for $s \geq 0$ and $Z \in \mathcal{T}(S_0, \ell) \times \mathcal{T}(S_1, \ell) \times \mathbb{R}$ denote by $g_s(Z) \in \text{Hit}(S)$ the Hitchin grafting representation obtained by grafting Z along γ^* with parameter sz. By Theorem 7.1 the pressure length of $(g_s(Z))_{s>0}$ is bounded above by a constant $C_4 > 0$ that only depends on σ and z.

Notice that $(p_{\lambda_0}(X(t)))_{0 \le t \le 1}$ is, like $(X(t))_{0 \le t \le 1}$, a concatenation of paths obtained by twisting along closed curves in S_1 . Hence we can apply Theorem 7.2, which says that the pressure length of $(g_s \circ p_{\lambda_0}(X(t)))_{0 \le t \le 1}$ goes to zero as s goes to infinity. Let s_0 be such that this length is less than 1.

Then we consider the concatenation of five paths where we first pinch $(p_{\lambda}(X(0)))_{1 \geq \lambda \geq \lambda_0}$, then graft $(g_s \circ p_{\lambda_0}(X(0)))_{0 \leq s \leq s_0}$, then let t vary $(g_{s_0} \circ p_{\lambda_0}(X(t)))_{0 \leq t \leq 1}$, then ungraft $(g_s \circ p_{\lambda_0}(X(1)))_{s_0 \geq s \geq 0}$, and finally we unpinch $(p_{\lambda}(X(1)))_{\lambda_0 \leq \lambda \leq 1}$. This connects X(0) to X(1) in Hit (Σ) and has pressure length at most 2C + 2C' + 1, which only depends on σ .

8.2. Length comparison with a separating curve graph when $g \ge 5$. In this section we assume $g \ge 5$. Let SCG(S) be the graph whose vertices are separating simple closed curves which decompose S into a surface of genus 2 and a surface of genus g - 2 and where two such curves are connected by an edge if they can be realized disjointly. We have

Lemma 8.3. The graph SCG(S) is connected.

Proof. The mapping class group Mod(S) of S clearly acts transitively on the vertices of SCG(S). Thus to check connectedness, we can apply a trick due to Putman [Put08]:

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FIGURE 2. Bounded path of Hitchin representations for the pressure metric. Each path is bounded by a constant that depends only on the length of γ^* and on the systole of Σ_0 .

Choose a vertex c of $\mathcal{SCG}(S)$ and a generating set ψ_1, \ldots, ψ_k of Mod(S). If for each j the vertex c can be connected to $\psi_j(c)$ by an edge path in $\mathcal{SCG}(S)$, then the graph is connected.

To see that this condition is satisfied we choose the Humphries generating set $\psi_1, \ldots, \psi_{2g+1}$ of Mod(S) consisting of Dehn twists about the non-separating simple closed curves $a_1, \ldots, a_g, c_1, \ldots, c_{g-1}, m_1, m_2$ in S as shown in Figure 4.5 of [FM11]. That these elements generate Mod(S) is explained in Theorem 4.14 of [FM11]. Let furthermore c be the separating simple closed curve which intersects the simple closed curve c_2 in precisely two points and is disjoint from any of the curves a_i, c_j, m_u for $j \neq 2$. Then $\psi_s(c) = c$ for $s \neq g + 2$, moreover both $c, \psi_{g+2}(c)$ are disjoint from the vertex b of SCG(S) which intersects c_{g-2} in precisely two points (this is where we need $g \geq 5$) and is disjoint from the remaining curves. Thus $c, b, \psi_{g+2}(c)$ is an edge path connecting c to $\psi_{g+2}(c)$, which suffices for the proof of the lemma.

Let $\Upsilon : \mathcal{T}(S) \to \mathcal{SCG}(S)$ be a map which associates to $X \in \mathcal{T}(S)$ a point in $\mathcal{SCG}(S)$ whose length is minimal among the lengths of all separating geodesics which cut S into a surface of genus 2 and a surface of genus S_2 . The length of $\Upsilon(X)$ in X is bounded above by a constant σ that only depends on g by Theorem A.2. We use Lemma 8.3 to show **Theorem 8.4**. For any $d \geq 3$ there exists a number C(d, g) > 0 with the following property. Let $X, Y \in \mathcal{T}(S)$ be any two points in the Fuchsian locus of the Hitchin component of representations $\pi_1(S) \to \mathrm{PSL}_d(\mathbb{R})$. Then the pressure metric distance between X, Y is at most $C(d, g)d(\Upsilon(X), \Upsilon(Y)) + C(d, g)$.

Proof. In this proof, distances between Hitchin representations are always taken with respect to the path metric defined by the pressure metric.

Let $X, Y \in \mathcal{T}(S)$. Suppose first that $d(\Upsilon(X), \Upsilon(Y)) = 1$ (the case $d(\Upsilon(X), \Upsilon(Y)) = 0$ is similar) The curves $\Upsilon(X)$ and $\Upsilon(Y)$ split S into three subsurfaces S_1, S_2, S_3 such that S_1 have genus 2 and one boundary component $\partial S_1 = \Upsilon(X)$, S_2 has genus g - 4 and two boundary components, and S_3 have genus 2 and one boundary component $\partial S_3 = \Upsilon(Y)$.

Let $Z \in \mathcal{T}(S)$ that coincides with X on S_1 and coincides with Y on S_3 . By Theorem 8.1 the distance from X to Z is bounded by some constant that depends on σ (upper bound on the length X and Z give to $\Upsilon(X) = \partial S_1$). Similarly the distance from Z to Y is bounded by some constant that depends on σ .

Now if $m = d(\Upsilon(X), \Upsilon(Y)) \geq 2$ then let $\alpha_0 = \Upsilon(X), \alpha_1, \ldots, \alpha_m = \Upsilon(Y)$ be a minimizing path in $\mathcal{SCG}(S)$. For each $1 \leq i \leq m-1$ let $X_i \in \mathcal{T}(S)$ such that $\Upsilon(X_i) = \alpha_i$, and let $X_0 = X$ and $X_m = Y$. For every *i* we have $d(\Upsilon(X_i), \Upsilon(X_{i+1})) = 1$ so we can apply the above: the distance from X_i to X_{i+1} is bounded above by some constant *C* that only depends on σ , and hence on *g*. Thus the distance from *X* to *Y* is at most mC. \Box

8.3. Fixed point for a subgroup of the mapping class group. The mapping class group Mod(S) of S acts by precomposition of markings on the Hitchin component Hit(S) preserving the Fuchsian locus $\mathcal{T}(S)$ and the pressure metric (whose restriction to $\mathcal{T}(S)$ is the Weil–Petersson metric). Thus Mod(S) also acts on the Weil–Petersson metric completion $\overline{\mathcal{T}(S)}$ of $\mathcal{T}(S)$ and on the pressure metric completion $\overline{Hit}(S)$ of Hit(S).

Note that the embedding of $\mathcal{T}(S) \hookrightarrow \operatorname{Hit}(S)$ extends to a continuous but noninjective map $\overline{\mathcal{T}(S)} \to \overline{\operatorname{Hit}(S)}$ which is equivariant under the actions of the mapping class group.

Recall that $\overline{\mathcal{T}(S)}$ is stratified. A stratum is defined by a simple geodesic multicurve $c \subset S$, and it consists of the Teichmüller space of all marked complete finite volume hyperbolic metrics on S - c. By this we mean that each component of S - c is an essential subsurface of S of negative Euler characteristic, and hence it determines a Teichmüller space of marked complete finite volume hyperbolic metrics on the component. The stratum of S - c is then the product of these Teichmüller spaces.

The action of the mapping class group Mod(S) of S on boundary points for the metric completion of $\mathcal{T}(S)$ projects to the action of the mapping class group on the curve complex, thought of as remembering the nodes (or cusps) of the completion points. Dehn multitwists have global fixed points acting on this boundary: if T_c is a Dehn twist about c, then any surface with node at c is fixed by T_c . However, there is no subgroup of the mapping class group containing a free group with two generators which acts with a global fixed point.

In contrast, the action of the outer automorphism group of the free group F_k with $k \ge 3$ generators on the metric completion of Outer space of marked graphs with fundamental group F_k , equipped with an analog of the pressure metric, has a global fixed point (see [ACR22]).

Our final result shows that a weaker but related statement holds true for the action of the mapping class group Mod(S) on the metric completion of the Hitchin component, equipped with the pressure metric, provided that the genus of S is at least 3. For the formulation of our result, recall that for every essential subsurface S_1 of the surface S with connected boundary, the mapping class group $Mod(S_1)$ of S_1 embeds into the mapping class group Mod(S) of S as a group of isotopy classes of homeomorphisms of S which fix $S - S_1$ pointwise.

We will prove that if S_1 has genus g - 2 and one boundary component, then $Mod(S_1)$ fixes a point of the metric completion of Hit(S) for the Pressure metric, and this point is explicit: let us describe it now.

Let ϕ_1, \ldots, ϕ_k be a generating set of $\operatorname{Mod}(S_1)$ consisting of Dehn twists. It suffices to find a point fixed by these generators. Denote by $\phi_i^t : \mathcal{T}(S_1) \to \mathcal{T}(S_1)$ the twist flow whose time 1 map is ϕ_i . Fix a hyperbolic metric X_1 on S_1 . Let a < 1 be the maximum of the entropies of all points $\phi_i^t(X_1) \in \mathcal{T}(S_1)$, for $1 \le i \le k$ and $0 \le t \le 1$. Let X_0 be a hyperbolic metric on $S - S_1$ with entropy greater than a and with same boundary length as X_1 (using that $S - S_1$ has genus 2). Let $X \in \mathcal{T}(S)$ be a gluing of X_0 and X_1 , and for L > 0 let $X(L) \in \operatorname{Hit}(S)$ be a grafting of X along ∂S_1 with cylinder height L (and fixed grafting direction). By Theorem 7.1, the path $(X(L))_{L\geq 0}$ has finite length and hence converges to a point of the completion $X(\infty) \in \operatorname{Hit}(S)$

Theorem 8.5. The subgroup $Mod(S_1) \subset Mod(S)$ fixes $X(\infty)$.

Proof. As mentioned it suffices to fix $1 \le i \le k$ and prove $\phi_i(X(\infty)) = X(\infty)$.

The point $\phi_i(X(\infty))$ is the limit as $L \to \infty$ of $\phi_i(X(L))$, which is obtained by gluing X_0 to $\phi_i(X_1)$ and grafting along ∂S_1 with cylinder height L.

By the discussion in the proof of Theorem 8.1, as $L \to \infty$, the length for the pressure metric of the path $t \to \phi_i^t(X(L))$ tends to zero. Hence the Pressure distance between X(L) and $\phi_i(X(L))$ tends to zero as $L \to \infty$, which proves that $\phi_i(X(\infty)) = X(\infty)$. \Box

8.4. **Proof of Theorem A.** As mentioned in the introduction, in [Lof04; Lof19], Loftin constructed a natural bordification of the space of Hitchin representations $\text{Hit}_3(S)$ of a closed surface S, called the augmented Hitchin space $\text{Hit}_3^{\text{aug}}(S)$, which extends the augmented Teichmüller space. This construction applies more generally to noncompact finite type surfaces and their moduli spaces of convex projective structures. Our goal in this section is to relate Loftin's bordification with our grafting procedure. More precisely we want to show that, starting with a Fuchsian representation and grafting it with grafting parameter going to infinity in a specific direction, the resulting family of Hitchin representations will converge to a point in Loftin's bordification.

Let S be a connected surface of finite type, seen as a closed surface with punctures. Recall that a projective structure is an atlas of charts on S into the projective plane such that the change of charts are projective transformations. To such a structure can be associated a holonomy representation of the fundamental group into the group of projective transformations $PSL_3(\mathbb{R})$, and a holonomy-equivariant developing map from the universal cover \tilde{S} into the projective plane. A projective structure is called convex if the developing map is injective and its image is properly convex (convex and bounded in some affine chart), which implies the holonomy representation is faithful with discrete image. In this case, the projective structure is completely determined by the data of the holonomy representation and the image of the developing map by Proposition 2.5 of [LZ21].

The moduli space of convex projective structures $\mathcal{C}(S)$ can be described as the quotient under the action of $\mathrm{PSL}_3(\mathbb{R})$ of the set of pairs (Ω, ρ) , where $\Omega \subset \mathbb{RP}^2$ is open and properly convex and ρ is a discrete and faithful representation of $\pi_1(S)$ into $\mathrm{PSL}_3(\mathbb{R})$ that preserves Ω . It is topologized so that $(\Omega_n, \rho_n) \to (\Omega, \rho)$ if $\Omega_n \to \Omega$ for the Hausdorff topology and $\rho_n \to \rho$ on a set of generators (up to the action of $\mathrm{PSL}_3(\mathbb{R})$). The projective structures around punctures can be classified, and in particular the conjugacy class of the holonomy of a curve enclosing a puncture can be of three types: parabolic $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, quasi-hyperbolic $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 1 \\ 0 & 0 & \mu \end{pmatrix}$ or hyperbolic $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}$ (where λ, μ, ν are distinct). As explained in the Appendix A of [LZ21], the projective structure around the puncture is determined by this holonomy in the parabolic and quasi-hyperbolic cases. However in the hyperbolic case there are many structures with the same holonomy. In particular any such structure can be deformed locally with out changing the holonomy by a bulging procedure (one can "inflate" or "deflate" the structure near the puncture). The two special degenerate structures obtained by inflating or deflating to infinity any other structure are called respectively bulge $+\infty$ and bulge $-\infty$. See e.g. Figure 4 of [LZ21]. To conclude, for any pair (Ω, ρ) , the convex set Ω is determined by ρ and the projective structure around punctures of hyperbolic type.

In particular, if S is closed then every point of $\mathcal{C}(S)$ is determined by the holonomy representation. By work of Choi and Goldman [Gol90; CG93], $\mathcal{C}(S)$ is connected, open and closed as a subset of the set of representations of $\pi_1(S)$, and it contains the representations coming from hyperbolic structures, so $\mathcal{C}(S) = \text{Hit}_3(S)$.

To define the augmented Hitchin space, Loftin first defines admissible convex projective structures by allowing only bulge $\pm \infty$ structures near the punctures of hyperbolic type. Then $\operatorname{Hit}_{3}^{\operatorname{aug}}(S)$ is defined as the set, over all multicurves $\mathcal{D} \subset S$, of admissible convex projective structures $(\Omega_1, \rho_1), \ldots, (\Omega_k, \rho_k)$ on the connected components S_1, \ldots, S_k of $S - \mathcal{D}$ that satisfy some compatibility conditions between the pairs of ends corresponding to the same curve $\gamma \subset \mathcal{D}$: they have the same holonomy and a bulge $+\infty$ end must face a bulge $-\infty$ end. It is further topologised so that $(\Omega^{(n)}, \rho^{(n)}) \in \operatorname{Hit}(S)$ converge to $((\Omega_1, \rho_1), \ldots, (\Omega_k, \rho_k))$ in the boundary if $(\Omega^{(n)}, \rho^{(n)}_{|\pi_1 S_i}) \to (\Omega_i, \rho_i)$ for every i (up to the action of $\operatorname{PSL}_3(\mathbb{R})$).

Let us now relate the above construction with the algebraic bending deformation of a Fuchsian representation ρ along a multicurve $\mathcal{D} \subset S$, as recalled in Section 3.1: it was defined by partially conjugating the image by ρ of the fundamental groups of the connected components S_1, \ldots, S_k of $S - \mathcal{D}$. We gave in [BHM25] and 3.2 a geometric interpretation of this deformation, inside the symmetric space of $PSL_3(\mathbb{R})$, in terms of grafting a flat cylinder along the multicurve \mathcal{D} . Suppose now that all the grafting parameters (which are vectors of the Cartan subspace) are parallel to the special direction $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then there is another geometric interpretation of bending due to Goldman [Gol90, §5.5] using convex projective geometry: bending induces a deformation of the underlying convex projective structure called *bulging*, which is the same procedure as the local surgery around punctures mentioned previously. The idea is the same as before (when k = 2 and \mathcal{D} has only one curve): suppose $\rho_z(\pi_1(S_1)) = \rho(\pi_1(S_1))$ is unchanged and $\rho_z(\pi_1(S_2)) = e^z \rho(\pi_1(S_2)) e^{-z}$. The ρ -invariant convex domain $\Omega \subset \mathbb{RP}^2$ is made of a tree of infinitely many copies of universal covers of S_1 and S_2 , each copy being invariant under a conjugate of $\rho(\pi_1(S_1))$ or $\rho(\pi_1(S_2))$. The ρ_z -invariant convex domain Ω_z is then produced by deforming each of these copies using e^z and e^{-z} and conjugates of them: e.g. if Ω_2 is a $\rho(\pi_1(S_2))$ -invariant copy of \widetilde{S}_2 then $e^z \Omega_2$ is $\rho_z(\pi_1(S_2))$ -invariant. One can see that e^z acts by inflating Ω_2 , without disconnecting it from the adjacent copies of \widetilde{S}_1

(so there is no need to graft a flat cylinder as in the symmetric space). The following fact is an immediate consequence of Goldman's work and the above definition of Loftin's bordification. Fix a grafting parameter z parallel to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Fact 8.6. For any t > 0 let $[\rho_t] \in \operatorname{Hit}_3(S)$ be obtained by grafting ρ along \mathcal{D} with

Fact 8.6. For any t > 0 let $[\rho_t] \in \text{Hit}_3(S)$ be obtained by grafting ρ along \mathcal{D} with parameter tz. Then as t goes to infinity, $[\rho_t]$ converges to $[(\Omega_1, \eta_1), \ldots, (\Omega_k, \eta_k)] \in \text{Hit}_3^{\text{aug}}(S)$ (projective structures on S_1, \ldots, S_k) such that the projective structures near the two ends associated to a $\gamma \subset \mathcal{D}$ are of hyperbolic type with bulge $+\infty$ and $-\infty$ respectively, and the holonomies η_i are the restrictions of ρ to $\pi_1(S_i)$.

To prove Theorem A, we consider the case where S is cut into two subsurfaces S_1, S_2 such that the entropy of η_1 is strictly greater than that of η_2 . We slightly perturb ρ into $(\rho^s)_{-\epsilon \leq s \leq \epsilon}$ so that $\eta_1^s = \eta_1$ for any s with entropy still greater than that of η_2^s , and η_2^s and η_2^σ are not conjugate for $s \neq \sigma$. Now we graft, and by Theorem 7.2 the pressure length of $(\rho_t^s)_{-\epsilon \leq s \leq \epsilon}$ goes to zero as t diverges, which implies all $(\rho_t^s)_{t\to\infty}$ converge to the same point of the pressure metric completion of $\operatorname{Hit}_3(S)$, independent of s. However by the above fact they converge to different points of Loftin's augmented Hitchin space. Heuristically, the pressure metric is not fine enough to distinguish points in $\operatorname{Hit}_3^{\operatorname{aug}}(S)$, because it focuses too much on the component with bigger entropy and can only see changes there.

Another interesting remark can be made about another description of the augmented Hitchin space (which is in fact Loftin's original definition), in terms of cubic differentials. Recall that by independent work of Labourie [Lab07] and Loftin [Lof01], there is a vector bundle structure π : Hit₃(S) $\rightarrow \mathcal{T}(S)$ such that the fiber above a point of $\mathcal{T}(S)$, seen as a (marked) complex structure on S, is the vector space of holomorphic cubic differentials on S. It turns out this vector bundle structure extends to π : Hit^{aug}₃(S) $\rightarrow \mathcal{T}^{aug}(S)$. Moreover, using the notations from the above fact and denoting the limit of [ρ_t] as $t \rightarrow \infty$ by [ρ_{∞}] = [(Ω_1, η_1), ..., (Ω_k, η_k)], it follows from Theorem 12 of [Lof19] that the projection $\pi[\rho_{\infty}] \in \mathcal{T}^{aug}(S)$ is the noded hyperbolic surface obtained by pinching to zero the multicurve $\mathcal{D} \subset S$.

Hence for t large the Hitchin grafting representation ρ_t , which we think of in this paper as the hyperbolic structure ρ where we grafted long flat cylinder along \mathcal{D} , naturally stands above another hyperbolic structure $\pi(\rho_t)$ on S with long and narrow hyperbolic collars around \mathcal{D} . Since pinching a curve in $\mathcal{T}(S)$ is a finite length surgery for the Weil–Petersson metric, it seems likely that $(\pi[\rho_t])_{t>0}$ has finite length. As $([\rho_t])_{t>0}$ also has finite length, for any t the pressure distance from $\pi[\rho_t]$ to $[\rho_t]$ is bounded independently of t. Moreover, there is a natural straight-line path between these two points, since $[\rho_t]$ lies in the fiber above $\pi[\rho_t]$, which is a vector space. A natural question is then: is the pressure length of this path bounded above independently of t?

APPENDIX A. ENTROPY OF HYPERBOLIC SURFACES WITH BOUNDARY

The goal of this appendix is to collect some basic results on the entropy of hyperbolic surfaces with boundary. We give proofs for the ones we did not find in the literature, although they should be well known by the experts. Some of the following statements are consequences of more general theorems.

Consider a compact surface Σ , of genus g, with at least one boundary component. Let S be a hyperbolic surface obtained by equipping Σ with a hyperbolic metric, so that its boundary is geodesic, that is, S belongs to the Teichmüller space $\mathcal{T}(\Sigma)$ for Σ . Denote by h(S) the topological entropy of the geodesic flow on T^1S . We also denote by $\delta(S)$ the critical exponent of any representation $\pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{R})$ representing the metric S. **Proposition A.1.** The following holds true:

- (1) $h(S) = \delta(S)$ (see [Sul79]).
- (2) The function $\delta(S)$ is real analytic in S and invariant under the action of $Mod(\Sigma)$ (see [Rue78]).
- (3) h(S) < 1.
- (4) Take a pants decomposition of Σ . When sending to zero the lengths of all boundary curves of a fixed pair of pants, the entropy goes to one.

Proof. Statement 3. It follows from Proposition 5 of [PS98] that the Poincaré series $P(\delta(S))$ is diverging at the critical exponent $\delta(S)$. Consider a closed hyperbolic surface Σ_d obtained by doubling Σ along its boundary, equipped with the double S_d of the given hyperbolic metric S. It follows from Proposition 2 of [DOP00] that we have $\delta(S) < \delta(S_d)$ (it uses as hypothesis that $P(\delta(S))$ is diverging). The latter is known to be equal to one. Namely, for a hyperbolic metric S_d without boundary and with finite volume, the limit set of $\pi_1(\Sigma)$ in $\partial_{\infty} \mathbb{H}^2$, that is, the accumulation points of the orbit $\pi_1(\Sigma) \cdot x$ for any $x \in \mathbb{H}^2$, is equal to all of $\partial_{\infty} \mathbb{H}^2$. It follows from Theorem 1.1 of [BJ97] that the critical exponent of S_d is one.

Statement 4. Take a compact hyperbolic surface with boundary and pinch all boundary components. The critical exponent of Kleinian groups is lower semi-continuous for the so called algebraic convergence, see Theorem 2.4 of [BJ97]. It implies that when decreasing the lengths of the boundary curves to zero, the limit inferior of the critical exponents is at least the critical exponent of the surface obtained by pinching the boundary curves. That is one according to the proof of statement (3).

We mention a result of Hugo Parlier, which is a neat improvement of results already known previously.

Theorem A.2 ([Par24]). Let S be a hyperbolic surface, possibly with boundary, and with finite volume. Then S admits a pant decomposition for which the length of each curve is at most $\max(length(\partial S), area(S))$.

Proposition A.3. There exists a function f_1 depending on Σ ($\partial \Sigma \neq \emptyset$) such that the following holds. If every boundary component has length at most σ and at least one of them has length at most $\epsilon \leq \sigma$ then $\delta(S) \geq f_1(\sigma, \epsilon) > 0$ with $\liminf_{\epsilon \to 0} f_1(\sigma, \epsilon) > \frac{1}{2}$ for fixed σ .

Proof. Denote by S_n a sequence of metrics as in the proposition, so that all boundary components of Σ have length at most σ . Using Theorem A.2, S_n admits a decomposition into hyperbolic pairs of pants $P_1^{(n)}, \dots, P_r^{(n)}$ so that the decomposing curves have a length bounded from above by some constant $C(\Sigma)$, and so that the shortest boundary component of S_n is in $P_1^{(n)}$. Our goal is to show that $\delta(S_n)$ is bounded from below by a universal constant.

Suppose by contradiction that $\delta(S_n) \to 0$. Then $\delta(P_1^{(n)}) \to 0$ since it is bounded from above by $\delta(S_n)$. Up to extraction we may assume that the boundary lengths of $P_1^{(n)}$ converge, which implies that $P_1^{(n)}$ converge to some hyperbolic pair of pants P, possibly with cusps. By lower semicontinuity of δ (see Theorem 2.4 of [BJ97]) we get that $0 = \lim_{n} \delta(P_1^{(n)})$, which is absurd. Thus the critical exponents are bounded away from zero.

Let us now prove the second part of the statement. Suppose by contradiction that the shortest boundary curve of S_n has length tending to zero, but $\liminf_n \delta(S_n) \leq 1/2$. Then $\liminf_n \delta(P_1^{(n)}) \leq 1/2$. Once again, up to extracting we may assume $P_1^{(n)} \to P$, with P having a cusp (since the shortest boundary of $P_1^{(n)}$, which is that of S_n , is pinched to zero). By lower semicontinuity of δ (see Theorem 2.4 of [BJ97]) we get that $\liminf_n \delta(P_1^{(n)}) \geq \delta(P)$. This is absurd as $\delta(P) > 1/2$ by Proposition 2 of [DOP00], since the critical exponent of a neighbourhood of a cusp is 1/2, with a diverging Poincaré series at the critical exponent.

Hyperbolic pairs of pants. Here suppose that Σ is a sphere with three boundary components, and $S_{a,b,c}$ is the metric of a hyperbolic pair of pants with boundary length a, b and c.

Proposition A.4. There exists a function f_2 depending on Σ ($\partial \Sigma \neq \emptyset$) with the following property. If Σ is a pair of pants, two boundary components of S have length at least $\sigma > 0$ and the third at least $\ell \ge \sigma$, then $\delta(S) \le f_2(\sigma, \ell)$ with $f_2(\sigma, \ell) \to 0$ for fixed $\sigma > 0$ as $\ell \to \infty$.

We use the notations from [MZ19], where the authors give some control on the entropy of a hyperbolic surface using the lengths of the small curves on the surface. Denote by L(S) the systole of S, that is, the length of the shortest closed geodesic in S. Denote by K(S) the length of the shortest closed geodesic in $S \setminus \partial S$ (K(S) is more complicated to define when S is not a pair of pants). Also let $\delta(S)$ be the critical exponent of S.

Theorem A.5 (Particular case of Theorem 1.4 of [MZ19]). There exists a constant C > 0 for which we have

$$\frac{1}{4}\log(2) \le \delta(S)K(S) \le C\left(\log(4) + 1 + \log\left(1 + \frac{1}{x_0}\right)\right)$$

where x_0 is the unique positive solution of the equation $(1+x)^{\left\lceil \frac{K(S)}{L(S)}-1 \right\rceil} x = 1$. Lemma A.6. Let S be a pair of pants with boundary lengths a, b, c. Then $K(S) \ge \max(a, b, c)$.

Proof. Up to reordering we may assume $\max(a, b, c) = c$. The surface S is obtained by gluing two isometric right-angled hyperbolic hexagons $H_1 = H, H_2$ along three nonadjacent sides, such that the three other sides have lengths $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$. In particular, there is a natural projection $\pi : S \to H$. Let $\overline{A}, \overline{B}, \overline{C}$ be the sides of H which are glued, so that the hyperbolic distance from \overline{B} to \overline{C} is a/2, the distance from \overline{C} to \overline{A} is b/2, and the distance from \overline{A} to \overline{B} is c/2.

Let γ be a closed geodesic in $S \setminus \partial S$, and let us check it has length at least c. Note that $\pi(\gamma) \subset H$ is a concatenation of geodesics between the sides $\overline{A}, \overline{B}, \overline{C}$. This path has to intersect all three sides, for if it was alternating between only two sides, then γ is freely homotopic to a multiple of the boundary curve of S between these two sides.

Say γ starts on the side \bar{A} at some point x, then travels until it hits \bar{B} at some point y (maybe bouncing off \bar{C} and \bar{A} in between), and then comes back to x. The first part of

the path from x to y must have length at least the distance from \overline{A} to \overline{B} , which is c/2, and similarly the second part has length at least c/2 too, so in total γ has length at least c.

Proof of Proposition A.4. Let $(a_n)_n, (b_n)_n, (c_n)_n$ be three sequences in \mathbb{R}^+ so that a_n and b_n are bounded away from zero, and c_n tends to to infinity with n. Let $S_n = S_{a_n,b_n,c_n}$ be the pair of pants with boundary lengths a_n, b_n, c_n . By Lemma A.6, $K(S_n)$ tends to infinity with n.

By assumption, $L(S_n)$ is bounded away from zero. So up to passing to a subsequence, we can assume that $\frac{K(S_n)}{L(S_n)}$ converges to $y \in (0, +\infty]$. If $y < +\infty$, then the solutions x_n of $(1+x)^{\left\lceil \frac{K(S_n)}{L(S_n)} - 1 \right\rceil} x = 1$ remain bounded away from zero. So $C\left(\log(4) + 1 + \log\left(1 + \frac{1}{x_n}\right)\right)$ is bounded, and $\delta(S_n) \leq \frac{cste}{K(S_n)}$ goes to zero.

If $y = +\infty$, then x_n goes to zero, and a simple analysis yields that $-\frac{\log(x_n)}{x_n}$ is equivalent to $\frac{K(S_n)}{L(S_n)}$. It follows that

(42)
$$\delta(S_n)K(S_n) \le C\left(\log(4) + 1 + \log\left(1 + \frac{1}{x_n}\right)\right)$$

(43)
$$\leq \operatorname{Cst} \cdot x_n \frac{K(S_n)}{L(S_n)}$$

(44) and hence
$$\delta(S_n) \leq \operatorname{Cst} \cdot \frac{x_n}{L(S_n)} \xrightarrow[n \to 0]{} 0$$

Surfaces with one boundary component. Assume now that the surface Σ is of genus genus $g = g(\Sigma)$, with exactly one boundary component. Let $\mathcal{T}(\Sigma, \ell)$ be the Teichmüller space of marked hyperbolic structures on Σ with geodesic connected boundary of length ℓ . Denote also by $\mathcal{T}_{\epsilon}(\Sigma, \ell) \subset \mathcal{T}(\Sigma, \ell)$ the subset of structures whose systole is at least ϵ . Lemma A.7. The following holds true:

(1) If g = 1 and $S \in \mathcal{T}(\Sigma, \ell)$, then $\delta(S)$ is bounded from above by some $b(\ell) < 1$.

(2) If $g \ge 2$ then for all $\nu, \ell > 0$ there exists a surface $S \in \mathcal{T}(\Sigma, \ell)$ with $\delta(S) > 1 - \nu$.

(3) If $S \in \mathcal{T}_{\epsilon}(\Sigma, \ell)$, then $\delta(S)$ is bounded from above by some $b(\epsilon, \ell) < 1$

Proof. Statement 1. Note that the critical exponent is invariant under the action of the mapping class group. Let $S_i \subset \mathcal{T}(\Sigma, \ell)$ be a sequence so that

$$\delta(S_i) \xrightarrow[i \to +\infty]{} \sup\{\delta(Z) \mid Z \in \mathcal{T}(\Sigma, \ell)\}$$

Up to passing to a subsequence, we may assume that the projections of the marked surfaces S_i to the moduli space $Mod(S) \setminus \mathcal{T}(\Sigma, \ell)$ converge in the Deligne–Mumford compactification of the moduli space to a surface Z with connected geodesic boundary of length ℓ , of genus $g' \leq 1$, possibly with one node. Either Z is smooth and $\delta(Z) < 1$ (see point 3 of Proposition A.1). Or the surface obtained by removing the node is a sphere with 3 punctures. In this case the entropies of the surfaces S_i converge to the *metric* entropy $\delta(Z)$ of the geodesic flow on the surface Z, equipped with the normalized Liouville measure, which is also less than 1.

Statement 2. It follows from the statement 4 of Proposition A.1. Find a pair of pants decomposition of S, take one pair of pants disjoint from ∂S and shrink all its boundary components. The critical exponent of the resulting metrics goes to one.

Statement 3. This part of the lemma follows from invariance under the mapping class group and compactness. Namely, let us assume that $S_i \subset \mathcal{T}_{\epsilon}(\Sigma, \ell)$ is a sequence of marked metrics so that the entropy

$$\delta(S_i) \to \sup\{\delta(S) \mid S_i \in \mathcal{T}_{\epsilon}(\Sigma, \ell)\}.$$

By adjusting with elements of the mapping class group, we may assume that $S_i \to S$ in $\mathcal{T}_{\epsilon}(\Sigma, \ell)$. Then $\delta(S_i) \to \delta(S)$, on the other hand we have $\delta(S) < 1$. This completes the proof of the lemma.

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