Geometry and dynamics of Hitchin grafting representations

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Abstract

The Hitchin component of the character variety of representations of a surface group $\pi_1(S)$ into $\mathrm{PSL}_d(\mathbb{R})$ for some $d \geq 3$ can be equipped with a pressure metric whose restriction to the Fuchsian locus equals the Weil-Petersson metric up to a constant factor. We show that if the genus of S is at least 3, then the Fuchsian locus contains quasi-convex subsets of infinite diameter for the Weil-Petersson metric whose diameter for the path metric of the pressure metric is finite. This is established through showing that biinfinite paths of bending deformations have controlled bounded length. To this end we give a geometric interpretation of Fock–Goncharov positivity and show that bending deformations of Fuchsian representations stabilize a uniform Finsler quasi-convex disk in the symmetric space $\mathrm{PSL}_d(\mathbb{R})/\mathrm{PSO}(d)$.

Contents

Introduction

The *Teichmüller space* $\mathcal{T}(S)$ of a closed oriented surface S of genus $g \geq 2$ is the space of *marked* hyperbolic structures on S. Equivalently, it can be described as a distinguished component of the space of conjugacy classes of homomorphisms $\pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$, with target the group $\mathrm{PSL}_2(\mathbb{R})$ of orientation preserving isometries of the hyperbolic plane. It was discovered by Hitchin that an analog of the Teichmüller space also exists for conjugacy classes of representations of $\pi_1(S)$ into simple split real Lie groups of higher rank.

The so-called *Hitchin component* $\mathrm{Hit}(S)$ for the target group $\mathrm{PSL}_d(\mathbb{R})$ ($d \geq 3$) is the component of the *character variety* containing conjugacy classes of discrete representations which factor through an irreducible embedding $\mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_d(\mathbb{R})$. Hitchin [Hit92] showed that the Hitchin component is homeomorphic to $\mathbb{R}^m$ for some explicit $m > 0$, and later Labourie [Lab06] and Fock–Goncharov [FG06] independently proved that all representations in the Hitchin component are faithful with discrete image.

In [BCLS15], a $\mathrm{Mod}(S)$-invariant metric on $\mathrm{Hit}(S)$ was introduced, the so-called *pressure metric* (see also [BCLS18; BCS17]). Fix a positive linear functional α_0 on the convex cone of vectors $x = (x_1, \dots, x_d)$ with $x_1 \geq \cdots \geq x_d$ and $x_1 + \cdots + x_d = 0$, for instance $\alpha_0(x) = x_1 - x_d$. The starting point is that α_0 determines a conjugacy invariant translation length function on $\mathrm{PSL}_d(\mathbb{R})$ by applying this functional to the logarithm of the absolute values of the eigenvalues. Hence this yields a length function for the images of $\pi_1(S)$ under a representation in the Hitchin component.

Now let us assume that, after choosing a hyperbolic metric on S, this function can be represented by integration of a Hölder continuous positive function over periodic orbits for the *geodesic flow* Φ^t on the unit tangent bundle T^1S of S. Then one can associate to such a representation the equilibrium state of a positive multiple of the function, chosen to be of vanishing pressure. Using that the pressure is a strictly convex functional, this construction gives rise to a positive semi-definite symmetric bilinear form on the tangent space of the Hitchin component (see (12)). Although this symmetric bilinear form may not be positive definite everywhere, it nevertheless defines a length metric on the component [Sam24].

Using results of [Sam14; BCLS15], we verify in Section 2 that this assumption of representability by Hölder potentials always holds, so that we can talk about the induced pressure metric on $\mathrm{Hit}(S)$. Related statements are contained in [BPS19]. Note that while the pressure metric is defined for any Hitchin component, there are several other interesting constructions of Riemannian metrics on the Hitchin component for $\mathrm{PSL}_3(\mathbb{R})$, see [DG96; Li16; KZ17]. It seems unknown whether these metrics coincide with one of the pressure metrics.

As there are many natural choices for the positive linear functional α_0 on the Weyl cone, there are many natural length functions on the Hitchin component, and hence many natural pressure metrics. The restriction to the *Fuchsian locus* of each of these metrics is a multiple of the *Weil–Petersson metric* on the Teichmüller space [Bon88] (relying in an essential way on the article [Wol86], see also Corollary 1.6 of [BCLS15] and [McM08] for an explicit statement).

The linear functional α_0 we shall use for the pressure metric determines a $\mathrm{PSL}_d(\mathbb{R})$-invariant Finsler metric $\mathfrak{F}$ on the symmetric space $\mathbb{X} = \mathrm{PSL}_d(\mathbb{R})/\mathrm{PSO}_d(\mathbb{R})$, see e.g. Section 5.1 of [KL18], called a *nice* Finsler metric in the sequel. An example of α_0 determining a nice Finsler metric, that the reader may keep in mind thoughout this article, is $\alpha_0(x) = x_1 - x_d$. The metric $\mathfrak{F}$ depends on α_0, but its geodesics do not by Lemma 5.10 of [KL18] (see Proposition 4.11).

Infinitesimal properties of the pressure metric at the Fuchsian locus were studied in [LW15], and also in [Dai23] in the case $d = 3$. The large scale geometry of the Weil–Petersson metric on Teichmüller space is quite well understood. In particular, this metric is incomplete [Wol86], and its metric completion $\bar{\mathcal{T}}(S)$, sometimes called *augmented Teichmüller space*, is a CAT(0) stratified space whose strata are Teichmüller spaces of marked finite area hyperbolic metrics on surfaces obtained from S by replacing some essential simple closed curves by nodes.

For $d = 3$, Loftin [Lof04] defined an augmented Hitchin space (see also [LZ21]). However, it turns out that unlike in the case of Teichmüller space, this construction is not well related to geometric properties of the pressure metric. A first instance of this possibility arose in the study of large scale properties of pressure metrics for other kinds of moduli spaces: for instance, for hyperbolic structures with boundary [Xu17], for marked metric graphs [ACR22], and for quasi-Fuchsian representations [FHJZ24]. The following result gives an illustration of this fact in the context of Hitchin representations.

Theorem A. *Suppose S has genus at least 3 and consider a pressure metric coming from a nice Finsler metric on the symmetric space $\mathrm{PSL}(3,\mathbb{R})/\mathrm{PSO}(3)$. Let $\mathrm{Hit}_3^{\mathrm{aug}}(S)$ be Loftin's augmented Hitchin space and $\overline{\mathrm{Hit}}_3(S)$ the Pressure metric completion.*

Then there exist paths $(x_t)_{t\geq 0}, (y_t)_{t\geq 0} \subset \mathrm{Hit}_3(S)$ converging to two distinct points $x, y \in \mathrm{Hit}_3^{\mathrm{aug}}(S)$, but converging to the same point of $\overline{\mathrm{Hit}}_3(S)$.

Theorem A is obtained as an application of a large scale geometric study of the path metric for all Hitchin components. To formulate our first main result, consider an essential subsurface S_0 of S whose connected boundary ∂S_0 is an essential simple closed curve. Choose a hyperbolic metric X on S and a marked point on the geodesic representing ∂S_0. Let $\ell > 0$ be the length of ∂S_0 for the metric X, and let $\mathcal{T}(S_0, \ell)$ be the Teichmüller space of marked hyperbolic metrics on S_0 with geodesic boundary of fixed length $\ell > 0$ and one marked point on ∂S_0. The choice of X determines an embedding $\mathcal{T}(S_0, \ell) \to \mathcal{T}(S)$ by associating to a point $X_0 \in \mathcal{T}(S_0, \ell)$ the metric on S obtained by gluing X_0 to $X|_{S-S_0}$ identifying marked points.

It is known that this embedding is quasi-isometric for the Weil–Petersson metric on $\mathcal{T}(S_0, \ell)$ and $\mathcal{T}(S)$. Although there does not seem to be an explicit statement available in the literature, this is a fairly easy consequence of the fact that augmented Teichmüller space contains the product of the Teichmüller spaces $\mathcal{T}(S_0)$, $\mathcal{T}(S - S_0)$ of the surfaces $S_0, S - S_0$, with the boundary replaced by cusps, as convex subspaces. Each factor in this product is of infinite diameter, and the image of the embedding of $\mathcal{T}(S_0, \ell)$ contains $\mathcal{T}(S_0) \times \{pt\}$ in a uniformly bounded neighborhood [Yam04]. In particular, the image of the embedding $\mathcal{T}(S_0, \ell) \to \mathcal{T}(S)$ has infinite diameter for the Weil–Petersson metric on $\mathcal{T}(S)$.

The mapping class group $\mathrm{Mod}(S)$ acts on the Hitchin component by precomposition of marking. As this action is isometric for the pressure metric, it extends to an action on the metric completion of $\mathrm{Hit}(S)$. For any essential subsurface S_0 of S, the mapping class group $\mathrm{Mod}(S_0)$ of S_0 is a subgroup of $\mathrm{Mod}(S)$. We show

Theorem B. *Let $g \geq 3$ and let $S_0 \subset S$ be any essential connected subsurface of genus $g_0 \leq g - 2$ with connected boundary.*

1. *The image of an embedding $\mathcal{T}(S_0, \ell) \to \mathcal{T}(S)$ has finite diameter for the pressure metric.*
2. *The action of the subgroup $\mathrm{Mod}(S_0)$ of $\mathrm{Mod}(S)$ on the metric completion of $\mathrm{Hit}(S)$ with respect to the pressure metric has a global fixed point.*

As in the case of Teichmüller space, metric completions and partial compactifications of $\mathrm{Hit}(S)$ can be studied through embeddings into the space $\mathcal{PC}(S)$ of *projective geodesic currents* on S. Namely, the length function f_ρ on T^1S defined by a representation $\rho \in \mathrm{Hit}(S)$ determines the projective geodesic current $\Theta(\rho)$ defined by the equilibrium state of a multiple of f_ρ. In this way one obtains a continuous mapping Θ of $\mathrm{Hit}(S)$ into the space of projective geodesic currents on S. Its restriction to the Fuchsian locus coincides with the standard embedding of Teichmüller space [Bon88] which associates to a hyperbolic metric its projective Liouville current.

Since S is compact, the space of projective geodesic currents on S, equipped with the weak*-topology, is a compact space. To establish Theorems A and B, we shall make use the following result, interesting in its own right, on the behavior of Θ along the degenerating sequences of Hitchin representations obtained by bending a fixed Fuchsian representation. Note that by Corollary 1.4 of [PS17], for each nice Finsler metric on $\mathbb{X}$ the entropy of a Hitchin representation is defined and is maximized only on the Fuchsian locus.

Theorem C. *Let $S_0 \subset S$ be a proper connected essential subsurface such that no component of $S_1 = S - S_0$ is a pair of pants, and let h be any hyperbolic metric on S_0 so that the boundary of S_0 is geodesic. Then there exists a sequence ρ_i of Hitchin representations with the following properties.*

1. *The projective currents $\Theta(\rho_i)$ converge weakly to the projective current of maximal entropy for the geodesic flow on (S_0, h).*
2. *The entropies of the representations ρ_i converge to the entropy of the geodesic flow on (S_0, h).*
3. *For a suitable choice of basepoint, the quotient manifolds $\rho_i(\pi_1(S))\backslash\mathbb{X}$ converge in the pointed Gromov Hausdorff topology to the Fuchsian manifold defined by the bordered hyperbolic surface (S_0, h).*

See Theorem 9.14 for the first two points, and the end of Section 6 for the third property.

There exists a natural embedding of the Teichmüller space $\mathcal{T}(S_0, \ell)$ into the space of geodesic currents for S_0, and the pressure metric on this space of currents is defined. Theorem C implies that the metric completion of the Hitchin component contains a subspace which is naturally isometric to $\mathcal{T}(S_0, \ell)$ equipped with the pressure metric (which does not coincide with the Weil–Petersson metric, see [Xu17]). It then also contains a subspace which is isometric to the space of marked metric graphs equipped with the Weil–Petersson metric [Xu17], as this space is contained in the metric completion of $\mathcal{T}(S_0, \ell)$ equipped with the pressure metric.

By work of Bonahon (Corollary 16 of [Bon88]), the restriction of the map Θ to $\mathcal{T}(S)$ is an embedding into the space of projective geodesic currents $\mathcal{PC}(S)$, and the boundary of the resulting compactification $\overline{\Theta(\mathcal{T}(S))} - \Theta(\mathcal{T}(S))$ is precisely the space $\mathcal{PML}(S)$ of projective measured geodesic laminations, that is, currents with vanishing self-intersection. Theorem C implies that $\overline{\Theta(\mathrm{Hit}(S))} - \Theta(\mathrm{Hit}(S))$ is bigger than $\mathcal{PML}(S)$, since the projective current of maximal entropy for S_0 is not a measured geodesic lamination. Section 1.3 of [BIPP21] contains related results. That the map Θ is an embedding for some choices of length functions, different from ours, is due to Bridgeman, Canary, Labourie and Sambarino (Theorem 1.2 of [BCLS18]).

Question 1. For $n \geq 3$, is $\Theta(\mathrm{Hit}(S))$ dense in the space of projective geodesic currents?

As the map which associates to a Hölder continuous positive length function f on T^1S the entropy of the normalized Gibbs current of f is continuous we obtain the following

Corollary D. *For any number $a \in [0, 1)$ there exists a sequence of degenerating Hitchin representations whose entropy converges to a.*

The case $a = 0$ is due to Zhang [Zha15] and was reworked in [SWZ20], using mainly algebraic methods. Our proof is entirely geometric. For $d = 3$ and in the context of real projective structures on surfaces, Corollary D is independently due to Nie [Nie15]. In this context, the article [FK16] also contains related results, embarking from the same deformations we use, but with a different geometric interpretation.

Theorems A, B and C rest on a geometric understanding of specific paths in $\mathrm{Hit}(S)$ which is of independent interest. These paths are so-called *grafting deformations*, also called *bending deformations* or *bulging deformations*. They are defined as follows.

For $d \geq 2$, the (unique up to conjugation) d-dimensional irreducible representation of $\mathrm{PSL}_2(\mathbb{R})$ defines an embedding $\mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_d(\mathbb{R})$ whose image stabilizes a totally geodesic subspace $\hat{\mathbb{H}}^2 \subset \mathbb{X} = \mathrm{PSL}_d(\mathbb{R})/\mathrm{PSO}(d)$ where $\mathbb{X}$ is equipped with the symmetric metric. Up to scaling, $\hat{\mathbb{H}}^2$ is isometric to the hyperbolic plane. Its tangent bundle $T\hat{\mathbb{H}}^2$ consists of regular tangent vectors in $T\mathbb{X}$. In particular, for $d \geq 3$, every geodesic line $\gamma \subset \hat{\mathbb{H}}^2$ is contained in a unique maximal flat F of dimension $d - 1$. This flat intersects $\hat{\mathbb{H}}^2$ orthogonally along γ.

Let now Γ be the fundamental group of a closed oriented surface S of genus at least 2. Let $\gamma \in \Gamma$ be defined by a simple closed curve on S. This curve defines a one edge graph of groups decomposition $\Gamma = \Gamma_1 *_C \Gamma_2$ where C is the infinite cyclic group generated by γ (we also allow HNN-extensions here). Let ρ be a discrete and faithful representation of Γ into $\mathrm{PSL}_2(\mathbb{R}) \subset \mathrm{PSL}_d(\mathbb{R})$. Let $\alpha \in \mathrm{PSL}_d(\mathbb{R})$ be an element in the centralizer of $\rho(C)$ but not contained in $\rho(C)$. Partial conjugation of ρ by α then defines a new representation, obtained from the Fuchsian representation ρ by *Hitchin grafting at γ with α*. More precisely, this new representation coincides with ρ on Γ_1, but maps any $\beta \in \Gamma_2$ to $\alpha\rho(\beta)\alpha^{-1}$. More generally, if $t \to \alpha(t)$ is a one-parameter subgroup of the centralizer of C then we obtain in this fashion a path in $\mathrm{Hit}(S)$ which we call a *Hitchin grafting path*. We show

Theorem E. *Hitchin grafting paths have finite length for the pressure metric.*

Theorem 10.1 contains a more precise version of this result.

Hitchin grafting paths are also well defined if the grafting is performed at a simple geodesic multicurve with more than one component and if the starting representation is not contained in the Fuchsian locus. It seems likely that our argument can be extended to show finite pressure length for such paths as well, however we do not carry out such an extension. In view of the work [BD17], it may be possible to extend this analysis to an even large class of naturally defined paths in $\mathrm{Hit}(S)$. This raises the following

Question 2. Is the diameter of $\mathrm{Hit}(S)$ *with respect to the pressure metric finite?*

The answer to this question is yes in a different context, namely for the pressure metric on quasi-Fuchsian space [FHJZ24]. Note that any two points in $\mathrm{Hit}(S)$ can be connected by finitely many grafting paths [AZ23], however not starting from points in the Fuchsian locus, and it is unclear whether the number of such paths needed has a uniform upper bound.

The mechanism behind the proof of Theorem E relies on and expands a novel geometric approach towards Anosov representations introduced by Kapovich, Leeb and Porti [KLP17; KL18; KLP18]. By [Lab06; FG06], all Hitchin representations belong to the larger class of Anosov representations introduced by Labourie, which can be thought of as higher rank analogs of the familiar convex cocompact representations in hyperbolic space. The latter are characterized by acting properly and cocompactly on a convex subset of hyperbolic space.

In higher rank, the naive generalisation of this requirement does not lead to interesting examples, in the sense that if a discrete subgroup of $\mathrm{PSL}_d(\mathbb{R})$ acts cocompactly on a subset of the symmetric space $\mathbb{X}$ which is convex for the Riemannian metric, then this group is in fact a uniform lattice, see [KL97; Qui05]. This lead to the development of Anosov representations, which aims at capturing the dynamical properties of convex cocompact representations in rank one. Later several other characterisations of the Anosov property were found, some of them with a more geometric flavor closer to the idea of convex cocompactness, see [Kas24] for a general survey. There are however several inequivalent notions of Anosov representations. Hitchin representations are Anosov representations in the strongest sense.

One of the ideas of Kapovich, Leeb and Porti is to work directly in the symmetric space, but equipped with one of the nice invariant Finsler metrics we mentioned previously (see (3)) instead of with the symmetric Riemannian metric. Another idea is to coarsify the requirement of convexity as follows. For a number $R > 0$, call a subset A of a symmetric space $\mathbb{X}$ *roughly Finsler R-convex* if any two points in A can be connected by *at least one* Finsler geodesic at distance at most R to A. As nice Finsler metrics have the same geodesics, see Proposition 4.11), this notion does not depend on the choice of nice Finsler metrics. Proposition 12.2 of [KL18] shows that for any Anosov representation, there exists R such that the image of the representation acts cocompactly on a roughly Finsler R-convex subset of the symmetric space (they use the terminology "quasiconvex").

The following statement, which is contained in Theorem 6.1, strengthens the previous result in the case of Hitchin *grafting* representations, by making the constant R uniform across large families of representations.

Theorem F. *For every $\sigma > 0$ there exists $R = R(\sigma) > 0$ with the following property. Let $\rho \in \mathrm{Hit}(S)$ be a Hitchin representation, obtained by any Hitchin grafting of a Fuchsian representation at a simple geodesic multicurve all of whose components have length at most σ. Then $\rho(\pi_1(S))$ acts properly and cocompactly on a roughly Finsler R-convex piecewise totally geodesic disk in $\mathbb{X}$.*

The uniform control over the constant R is crucial in the present article, and it is based on a different mechanism. Namely, as the representation varies, the diameters of the quotients of the roughly Finsler convex invariant disks are unbounded in general.

A natural question arises from our result:

Question 3. Does there exists $R > 0$ such that any Hitchin representation acts cocompactly on some roughly Finsler R-convex subset of the symmetric space?

We expect the answer to this question to be no.

The result in [KL18] on rough Finsler convexity of an orbit of an Anosov group relies on a version of a Morse lemma for Anosov representations in higher rank symmetric spaces (Theorem 1.3 of [KLP18]). The proof of Theorem F is independent of the results in [KLP18], but also embarks from a (different) Morse-type lemma for Finsler metrics. In the formulation of this result, $d^{\mathfrak{F}}$ denotes the distance function of a nice Finsler metric on $\mathbb{X}$.

Theorem G. *For every $C > 0$ there exists a number $C' > 0$ with the following property. Let $c : [a, b] \to (\mathbb{X}, d^{\mathfrak{F}})$ be a map such that*

$$d^{\mathfrak{F}}(c(s), c(u)) + d^{\mathfrak{F}}(c(u), c(t)) \leq d^{\mathfrak{F}}(c(s), s(t)) + C$$

for all $s \leq u \leq t$. Then there exists a Finsler geodesic connecting $c(s)$ to $c(t)$ at Hausdorff distance at most C' to c.

This result allows to make a link between the notion of convex cocompactness in the real projective space from [DGK17] and the existence of a roughly Finsler convex subset on which the action is cocompact: Consider a subgroup $\Gamma \subset \mathrm{PSL}_d(\mathbb{R})$ that preserves a properly convex open subset $\Omega \subset \mathbb{RP}^{d-1}$ on which it acts *naively convex cocompactly*, in the sense that it acts cocompactly on a convex subset of Ω (see Definition 1.9 of [DGK17]).

Then Γ acts cocompactly on a roughly Finsler convex subset of the symmetric space $\mathbb{X}$, see Proposition 4.13. In particular, there exists Gromov-hyperbolic groups acting cocompactly on a roughly Finsler convex subset of $\mathbb{X}$ which are not Anosov. This shows that the extra condition of uniform regularity in Kapovich–Leeb's characterisation Proposition 12.2 of [KL18] is necessary.

Organization of the article and structure of the proof The article has three parts. The first part, consisting of Section 1 and Section 2, is introductory, and its results can mostly be found in the literature, although not always in the form we need. Section 1 contains the information on Lie groups and symmetric spaces we are going to use. In particular, we introduce the class of nice invariant Finsler metrics on the symmetric space $\mathbb{X}$. Section 2 turns to the dynamical aspect of this work. We establish that the nice Finsler metrics defined in Section 1 indeed define a pressure metric for the Hitchin component.

In a second part, Section 3 to 6, we focus on the geometry of Hitchin representations. Section 3 introduces abstract grafting and Hitchin grafting representations. To a Hitchin grafting representation ρ we associate an abstract grafted surface S_ρ and a path isometric embedding $Q_\rho : S_\rho \to \rho(\pi_1(S))\backslash\mathbb{X}$. We then introduce a particular class of paths on the grafted surface S_ρ which are piecewise geodesics and are called *admissible*. Their images under the map Q_ρ are piecewise geodesics in the locally symmetric manifold $\rho(\pi_1(S))\backslash\mathbb{X}$.

The preimages of admissible paths in $\mathbb{X}$ play a crucial role for our goal. In Section 6 we show that admissible paths are uniform quasi-geodesics in $\mathbb{X}$, equipped with the Finsler distance function $d^{\mathfrak{F}}$, with precise quantitative control. The proof of this result uses Fock–Goncharov positivity in an essential way. This then leads to the proof of Theorem 6.1, which is a vast and more precise extension of Theorem F. It uses Theorem G as an essential tool, which is proved in Section 4.

A first and fairly easy application of Theorem 6.1 is contained in Section 7. We show that there are sequences of representations in the Hitchin component whose normalized intersection with any Fuchsian representation tend to infinity. Here the normalized intersection number is the entropy normalized intersection number in the space of currents.

The third part of the article turns to the pressure metric. In Section 8 we use the geometric information established in Section 6 and Ehresmann connections to give precise norm bounds for first and second derivative of the Finsler length of a conjugacy class in $\pi_1(S)$ under two specific classes of paths in $\mathrm{Hit}(S)$.

Sections 9 and Sections 10 contain the main dynamical results of this article. We use the geometric information on Hitchin grafting representations obtained in the previous sections to analyze the geodesic currents defined by such representations. This leads to the proof of Theorem C and Theorem E. The proof of Theorem B is contained in Section 11.

The appendix collects information on the entropy of the geodesic flow on compact hyperbolic surfaces with boundary which we were unable to find in the literature in the form we need.

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1 Lie groups and symmetric spaces

This section collects some basic facts on Lie groups and symmetric spaces and introduces conventions and notations used lated on.

Consider the simple Lie group $G = \mathrm{PSL}_d(\mathbb{R})$ and a representation $\tau : \mathrm{PSL}_2(\mathbb{R}) \to G$, that is, a locally injective Lie group homomorphism. Many (but not all) of our results work for other semisimple Lie groups, and their proofs are easier to write using the abstract language of semisimple Lie groups, which we recall below.

Recall that the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of $\mathrm{PSL}_2(\mathbb{R})$ is the Lie algebra of trace free real $(2,2)$-matrices. Denote by $\mathfrak{g} = T_{\mathrm{id}}G$ the Lie algebra of G, by $\mathfrak{a} \subset \mathfrak{g}$ a Cartan subalgebra (maximal abelian subalgebra when G is split) that contains $d\tau\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ and by $\mathfrak{a}^+ \subset \mathfrak{a}$ an open Weyl cone whose closure $\overline{\mathfrak{a}^+}$ contains $d\tau\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ (the definition of Weyl cone is recalled later in this section).

We require the representation τ to be *regular*, that is, $d\tau\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ belongs to the interior $\mathfrak{a}^+$ of the Weyl chamber, or equivalently, there is a unique Cartan subspace $\mathfrak{a}$ containing $d\tau\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$. If G is real split, then such a representation exists and is unique up to conjugation.

Maximal compact subgroup and symmetric space

Let $K \subset G$ be a maximal compact subgroup which contains $\tau(\mathrm{PSO}(2))$ and whose Lie algebra $\mathfrak{k}$ is orthogonal (for the Killing form) to $\mathfrak{a}$.

The symmetric space of G is $\mathbb{X} = G/K$ with basepoint $\mathbf{x} = [\mathrm{id}] = K \in G/K$. Denote by $\pi_{\mathbb{X}} \colon G \to \mathbb{X}$ the projection. The space $\mathbb{X}$ is endowed with a nonpositively curved G-invariant Riemannian metric whose induced norm is denoted by $\|\cdot\|$, and whose distance function is denoted by $d_{\mathbb{X}}$. The metric is normalised so that $\|d\tau\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)\| = 2$.

Maximal Flats

The subspace $\exp(\mathfrak{a}) \cdot \mathbf{x}$ is a totally geodesic embedded Euclidean subspace of $\mathbb{X}$ of maximal dimension. This flat will often be identified with the abelian subgroup $A = \exp(\mathfrak{a})$ which acts simply transitively on it. Each maximal Euclidean subspace of $\mathbb{X}$ can be represented as $g \cdot A$ for some $g \in G$. These maximal Euclidean subspaces are called *maximal flats*.

Root systems

Let $\mathcal{R} \subset \mathfrak{a}^*$ be the set of restricted roots of G, that is, the set of non-zero linear one-forms α on $\mathfrak{a}$ such that

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid [a, X] = \alpha(a)X \ \forall a \in \mathfrak{a}\} \neq 0.$$

Recall that G being split means that

$$\mathfrak{g} = \mathfrak{a} \oplus \bigoplus_{\alpha\in\mathcal{R}} \mathfrak{g}_\alpha.$$

The kernels of the restricted roots are the *walls* of $\mathfrak{a}$, and the *open Weyl cones* are the connected components of the complements of the walls. By our regularity assumption on τ there is a unique open Weyl cone containing $d\tau\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$.

Let $\mathcal{R}^+ := \{\alpha \in \mathcal{R} : \alpha(d\tau\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)) > 0\}$ be the set of positive roots, and $\mathcal{R}^- = -\mathcal{R}^+$ the set of negative roots. Let $\Delta \subset \mathcal{R}^+$ be the set of simple roots (the positive roots that are not sums of several other positive roots).

Minimal parabolic subgroups and flag variety

The normalizer $P := N_G(\bigoplus_{\alpha\in\mathcal{R}^+} \mathfrak{g}_\alpha) \subset G$ for the adjoint representation is a minimal parabolic subgroup. Its Lie algebra is $\mathfrak{p} := \mathfrak{a} \oplus \bigoplus_{\alpha\in\mathcal{R}^+} \mathfrak{g}_\alpha$. The *opposite* parabolic subgroup is $P^- := N_G(\bigoplus_{\alpha\in\mathcal{R}^-} \mathfrak{g}_\alpha)$.

The *flag variety* $\mathcal{F} := G/P$ is compact. In fact, K acts transitively on it, with finite point stabiliser.

A notable subgroup of P is $U = \exp\left(\bigoplus_{\alpha\in\mathcal{R}^+} \mathfrak{g}_\alpha\right)$. The dynamics coming from the geometry of the symmetric space and other homogeneous spaces of G has some contraction properties that are recorded in the following algebraic fact: for any $u = \exp(\sum_{\alpha\in\mathcal{R}^+} X_\alpha)$ in U, for any sequence $(a_n)_n \subset \mathfrak{a}^+$ that diverges from the walls (i.e. $\alpha(a_n) \to +\infty$ for any $\alpha \in \mathcal{R}_+$), we have

$$\exp(-a_n) \cdot u \cdot \exp(a_n) = \exp\left(\sum_{\alpha\in\mathcal{R}_+} e^{-\alpha(a_n)} X_\alpha\right) \to \mathrm{id}\,. \tag{1}$$

If $(a_n)_n$ does not diverge from all the walls but only some of them, then there is still a more complicated weaker contraction property.

Maps induced by τ

Let $T \subset \mathrm{PSL}_2(\mathbb{R})$ be the subgroup of upper triangular matrices. The *ideal boundary* $\partial_\infty\mathbb{H}^2 = \mathrm{PSL}_2(\mathbb{R})/T$ of the hyperbolic plane $\mathbb{H}^2 = \mathrm{PSL}_2(\mathbb{R})/\mathrm{PSO}(2)$ is naturally homeomorphic to the circle $\mathbb{R} \cup \{\infty\}$ under the map $t \mapsto [\left(\begin{smallmatrix}1 & t\\ 0 & 1\end{smallmatrix}\right) \cdot \left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)]$ and $\infty \mapsto [\mathrm{id}] = T \in \mathrm{PSL}_2(\mathbb{R})/T$.

One can check that $\tau(T) \subset P$ and that τ induces an embedding

$$\partial\tau : \partial_\infty\mathbb{H}^2 \hookrightarrow \mathcal{F} = G/P.$$

Transversality in the flag variety

Two flags $\xi, \eta \in \mathcal{F}$ are said to be *transverse* if there exists $g \in G$ such that $g\xi = \partial\tau(0)$ and $g\eta = \partial\tau(\infty)$; in this case we write $\xi \pitchfork \eta$. The set of transverse pairs of flags is an open dense subset of $\mathcal{F}^2$.

The set of flags transverse to $\partial\tau(0)$ is

$$\partial\tau(0)^{\pitchfork} := P^- \cdot \partial\tau(\infty) = \exp\left(\bigoplus_{\alpha\in\mathcal{R}^-} \mathfrak{g}_\alpha\right) \cdot \partial\tau(\infty). \tag{2}$$

Similarly, for any flag ξ denote by $\xi^{\pitchfork}$ the set of flags transverse to ξ. It is an open dense subset of $\mathcal{F}$. Note that by our convention, P^- equals the stabilizer of $\partial\tau(0)$ in $\mathcal{F}$.

Any two transverse flags are contained in the boundary of a unique maximal flat. The maximal flat asymptotic to the transverse flags $\partial\tau(0)$ and $\partial\tau(\infty)$ equals $A \cdot \mathbf{x} \subset \mathbb{X}$.

More generally, the flat between (that is, asymptotic to) any two transverse flags $(\xi, \eta) = g(\partial\tau(0), \partial\tau(\infty))$ is

$$F(\xi, \eta) := gA \cdot \mathbf{x} = gA \subset \mathbb{X}.$$

Jordan and Cartan projections, and loxodromic elements

For any $g \in G$, the *Cartan projection* is the unique element $\kappa(g) \in \overline{\mathfrak{a}^+}$ such that $g \in K\exp(\kappa(g))K$. Putting $A^+ = \exp(\overline{\mathfrak{a}^+})$, it is characterized by the fact that $\exp(\kappa(g))\mathbf{x}$ is the unique intersection point of $A^+\mathbf{x}$ with the K-orbit of $g\mathbf{x}$. Note that $d(\mathbf{x}, g\mathbf{x}) = ||\kappa(g)||$ (here d is the distance of the symmetric metric on $\mathbb{X}$).

Similarly, the G-orbit of any vector $v \in TX$ intersects $\overline{\mathfrak{a}^+} \subset \mathfrak{p}$ in precisely one point $\kappa(v)$ which is called the *Cartan projection* of v.

For the *Jordan projection* $\lambda(g) \in \overline{\mathfrak{a}^+}$ we choose the following unnatural but convenient definition (see Remark 5.31 of [BQ16])

$$\lambda(g) := \lim_{n\to\infty} \frac{1}{n}\kappa(g^n).$$

The element $g \in G$ is called *loxodromic* if $\lambda(g)$ is contained in the interior $\mathfrak{a}^+$ of $\overline{\mathfrak{a}^+}$, which is equivalent to saying that g has an attracting/repelling fixed pair of transverse flags (g^-, g^+). Then g acts as a translation on the flat $F(g^-, g^+)$.

That the representation τ is regular means that the image $\tau(g)$ of any loxodromic $g \in \mathrm{PSL}_2(\mathbb{R})$ is loxodromic in G.

Weyl Chambers and special directions

Since the symmetric space $\mathbb{X}$ is nonpositively curved, it admits a *visual boundary* $\partial_\infty\mathbb{X}$, which is naturally identified with the set of unit speed infinite geodesic rays starting at the basepoint $\mathbf{x}$.

By the normalization of the metric on $\mathbb{X}$, the representation τ induces an isometric embedding $\mathbb{H}^2 \hookrightarrow \mathbb{X}$. The isometric embeddings $A \hookrightarrow \mathbb{X}$, $\mathbb{H}^2 \hookrightarrow \mathbb{X}$ extend to embeddings of the visual boundaries $\partial_\infty A \hookrightarrow \partial_\infty\mathbb{X}$, $\partial_\infty\mathbb{H}^2 \hookrightarrow \partial_\infty\mathbb{X}$.

For any $g \in G$, we identify $\xi = g\partial\tau(\infty) \in \mathcal{F}$ with a compact subset of the visual boundary $\partial_\infty\mathbb{X}$, called a (closed) Weyl Chamber:

$$\xi = g\partial_\infty A^+ = g \cdot \{\lim_{t\to\infty} \exp(tv)\mathbf{x} : v \in \overline{\mathfrak{a}^+}\} \subset g\partial_\infty A \subset \partial_\infty\mathbb{X}.$$

It is the boundary at infinity of the *Weyl Cone* gA^+ based at $g\mathbf{x}$. The facets of $\xi = g\partial\tau(\infty) \in \mathcal{F}$ are the subsets of the form

$$g \cdot \partial_\infty \left(A^+ \cap \bigcap_{\alpha \in S} \ker \alpha \right) \subset \xi,$$

where S is a subset of Δ; they are boundaries at infinity of facets of the Weyl Cone gA^+.

Every G-orbit in $\partial_\infty \mathbb{X}$ intersects exactly *once* every Weyl Chamber. In particular, to every Weyl Chamber $\xi \in \mathcal{F}$ and every point p in the standard Weyl Chamber $\partial\tau(\infty)$ one can associate a point of ξ, which is the intersection point of ξ with $G \cdot p$. The embedding $\partial_\infty \mathbb{H}^2 \to \partial_\infty \mathbb{X}$ determines a special point of $\partial\tau(\infty)$. Its orbit under G determines a special point in every Weyl Chamber.

The Weyl group

Let us recall the definition of the Weyl group, denoted by Weyl. It is the intersection of the maximal compact subgroup K of G with the stabilizer of $\mathfrak{a}$, in restriction to $\mathfrak{a}$ (so quotiented out by the fixator of $\mathfrak{a}$ in G). It can also be described as the group of orthogonal transformations of $\mathfrak{a}$ generated by the reflections along the walls $(\ker \alpha)_{\alpha \in \Delta^+}$ of the closed Weyl Chamber $\overline{\mathfrak{a}^+}$. The Weyl group is finite, and any Weyl-orbit in $\mathfrak{a}$ intersects $\mathfrak{a}^+$ exactly once.

A Finsler metric coming from a linear functional on $\mathfrak{a}$

Notation 1. We fix a linear functional α_0 on $\mathfrak{a}$ which is positive on $\overline{\mathfrak{a}^+}$ and such that $\alpha_0(gv) \leq \alpha_0(v)$ for all $v \in \overline{\mathfrak{a}^+}$ and $g \in$ Weyl*.*

We assume that α_0 is symmetric in the sense that if g is the transformation in the Weyl group that maps $\mathfrak{a}^+$ to its opposite $-\mathfrak{a}^+$ then $\alpha_0(gv) = -\alpha_0(v)$ for any $v \in \mathfrak{a}$.

Let us denote by $\alpha_0^\#$ the vector in $\mathfrak{a}$ such that $\alpha_0(v) = \langle v, \alpha_0^\# \rangle$ for any $v \in \mathfrak{a}$, where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathfrak{a}$ defined by the Riemannian metric on $\mathbb{X}$. Then the assumption above on α_0 is equivalent to asking that $\alpha_0^\# \in \mathfrak{a}^+$. We also denote by $\alpha_0^\# \in \partial_\infty \mathfrak{a}^+$ the point at infinity to which the ray spanned by $\alpha_0^\#$ limits.

Let $\mathfrak{k} \subset \mathfrak{g}$ be the Lie algebra of the maximal compact subgroup K of G. The orthogonal complement $\mathfrak{p}$ of $\mathfrak{k}$ with respect to the Killing form is naturally isomorphic to $T_{\mathbf{x}}\mathbb{X}$, and it contains $\mathfrak{a}$. For any vector $v \in T\mathbb{X}$ we set

$$\mathfrak{F}(v) = \alpha_0(\kappa(v)) \tag{3}$$

where as before, $\kappa(v)$ is the Cartan projection of v.

Proposition 1.1 (Lemmas 5.9-10 of [KL18]). *The following hold.*

1. *$\mathfrak{F}$ defines a G-invariant Finsler metric on $\mathbb{X}$.*

2. *The unparameterized Riemannian geodesics of $\mathbb{X}$ are also geodesics for $\mathfrak{F}$.*

3. *The translation length for $\mathfrak{F}$ of any element $g \in G$ acting on $\mathbb{X}$ is given by $\ell^{\mathfrak{F}}(g) := \alpha_0(\lambda(g))$ where $\lambda(g) \in \mathfrak{a}^+$ is the Jordan projection.*

In the sequel we always normalize the functional α_0 in such a way that the embedding $\mathbb{H}^2 \to \mathbb{X}$ which is isometric for the symmetric metric also is isometric for the Finsler metric $\mathfrak{F}$.

Busemann functions

The Busemann functions, or horofunctions, are generalisations of distance functions on $\mathbb{X}$: they record relative distances to a point at infinity. Geometrically, their level sets, called horospheres, are limits of spheres whose center go to infinity. Since there are several kinds of metrics on $\mathbb{X}$, there are also several kinds of horofunctions.

Because of the contraction property of the subgroup $U \subset G$ explained in (1), horospheres centered at a point p in the interior of the Weyl chamber $\partial\tau(\infty)$ will always be U-invariant, since for $u \in U$, for any $(x_n)_n$ converging to p we have $d(x_n, ux_n) \to 0$ for any G-invariant metric d. In fact every sphere of $\mathbb{X}$ (for any metric) is foliated by orbits of the stabiliser of the center, and U-orbits can be described as limits of these leaves when the center tends to a point in the interior of $\partial\tau(\infty)$.

The leaves foliating the spheres centered at a given point can be parameterised by vectors of $\mathfrak{a}^+$, which one may think of as vector-valued distances. Namely, the G-orbit of any pair $(x, y) \in \mathbb{X}^2$ intersects $\{\mathbf{x}\} \times A^+$ exactly once, and the intersection is denoted $(\mathbf{x}, \exp(\kappa(x, y)))$, where $\kappa(x, y) \in \overline{\mathfrak{a}^+}$ is thought of as a vector-valued distance from x to y. The orbits of the stabiliser of x are the level sets of $\kappa(x, \cdot)$. The Riemannian distance from x to y can be expressed as $||\kappa(x, y)||$, and the Finsler distance as defined in (3) can be expressed as

$$d^{\mathfrak{F}}(x, y) = \alpha_0(\kappa(x, y)). \tag{4}$$

Using the U-orbits one can define a vector-valued Busemann function centered at $\partial\tau(\infty)$: for any $x \in \mathbb{X}$, the U-orbit $U \cdot x$ intersects the standard flat $A \subset \mathbb{X}$ in exactly one point, and taking the logarithm we get a vector $b^{\mathfrak{a}}_{\partial\tau(\infty)}(\mathbf{x}, x) \in \mathfrak{a}$, that records the relative distance from x to $\partial\tau(\infty)$ compared to the basepoint $\mathbf{x}$. One can check using the contraction property of U that if $(y_n)_n \subset A^+$ converge to a point in the interior of the Weyl Chamber $\partial\tau(\infty)$ then the $\mathrm{Stab}(y_n)$-orbits converge to level sets of $b^{\mathfrak{a}}_{\partial\tau(\infty)}(\mathbf{x}, \cdot)$.

Using the action of K, one can extend these vector-valued Busemann functions to other Weyl chamber than $\partial\tau(\infty)$. For any Weyl Chamber $\xi = k\partial\tau(\infty) \in \mathcal{F} = G/P$, the *vector-valued Busemann function* or horofunction centered at ξ between $\mathbf{x}$ and x is $b^{\mathfrak{a}}_{\xi}(\mathbf{x}, x) = b^{\mathfrak{a}}_{\partial\tau(\infty)}(\mathbf{x}, k^{-1}x)$, and more generally for $x, y \in \mathbb{X}$ we have:

$$b^{\mathfrak{a}}_{\xi}(x, y) = -b^{\mathfrak{a}}_{\xi}(\mathbf{x}, x) + b^{\mathfrak{a}}_{\xi}(\mathbf{x}, y) = \lim_{n\to\infty} \kappa(z_n, x) - \kappa(z_n, y) \in \mathfrak{a},$$

where $(z_n)_n \subset \mathbb{X}$ is any sequence converging to a point of the visual boundary in the interior of ξ. One way to check the above formula is to find $k_n \in K$ such that $k_n z_n \in A^+$, use that $\kappa(z_n, x) = \kappa(k_n z_n, k_n x)$, and extract to make k_n converge to some $k \in K$.

A horosphere in $\mathbb{X}$ based at a point in the visual boundary $\partial_\infty A$ of $A = A \cdot \mathbf{x}$ equals the U-obit of a horosphere in A. Given a sequence $(z_n)_n$ in A going to a point p in the visual boundary of A, in the interior of the Weyl chamber $\partial_\infty A^+$, the sequence of Riemannian spheres in A which are centered at z_n and pass through the origin 0 converges to the hyperplane in A containing 0 and perpendicular to the ray from 0 to p. One can check that on the other hand, the sequence of Finsler spheres, that is, the level sets of $d^{\mathfrak{F}}$, converges to the kernel of α_0, which does not depend on p and which, in the notation 1, is the hyperplane perpendicular to $\alpha_0^\# \in \partial_\infty A^+$.

In general the Busemann function $b_p(x, y)$ associated to the Riemannian metric on $\mathbb{X}$ and centered at a point p in the interior of some Weyl chamber $\xi \subset \partial_\infty \mathbb{X}$ is the limit $\lim_{n\to\infty} d(x, z_n) - d(y, z_n)$ where $z_n \to p$. If q is the intersection point of $\partial\tau(\infty)$ with the G-orbit of p then we have $b_p(x, y) = \langle b_\xi^{\mathfrak{a}}(x, y), v\rangle$ where $v \in \mathfrak{a}$ is the unit vector pointing at q and $\langle \cdot, \cdot \rangle$ is the inner product on $\mathfrak{a}$.

The *Busemann function* associated to our choice of Finsler metric is given by

$$b_\xi^{\mathfrak{F}}(x, y) = \alpha_0(b_\xi^{\mathfrak{a}}(x, y)) = \lim_{n\to\infty} d^{\mathfrak{F}}(x, z_n) - d^{\mathfrak{F}}(y, z_n) \in \mathbb{R}. \tag{5}$$

Here the last equality in the identity (5) is valid since $\mathfrak{F}$ is defined by a linear functional on $\overline{\mathfrak{a}^+}$. Moreover, for any loxodromic element $g \in G$ with attracting fixed point $\xi \in \mathcal{F}$, the translation length of g acting on $\mathbb{X}$ endowed with the Finsler metric $d^{\mathfrak{F}}$ equals the quantity $|b_\xi^{\mathfrak{F}}(x, gx)|$.

Note that using Notation 1 we have the following link between the Finsler and Riemannian Busemann functions: if $p \in \partial_\infty \mathbb{X}$ is the intersection point of $\xi \in \mathcal{F}$ with the G-orbit of the point in $\xi \subset \partial_\infty \mathbb{X}$ corresponding to $\alpha_0^\# \in \mathfrak{a}^+$ then

$$b_\xi^{\mathfrak{F}}(x, y) = \langle b_\xi^{\mathfrak{a}}(x, y), \alpha_0^\# \rangle = b_p(x, y), \tag{6}$$

in other words Finsler horospheres are Riemannian horospheres.

When z_n converges to a point of the visual boundary which is not regular, that is, not in the interior of a Weyl Chamber, then the Riemannian Busemann functions are still well defined, the limit $\lim_{n\to\infty} d(x, z_n) - d(y, z_n)$ still exists. For the Finsler metric the situation is more complicated: up to passing to a subsequence of $(z_n)_n$, the limit $\lim_{n\to\infty} d^{\mathfrak{F}}(x, z_n) - d^{\mathfrak{F}}(y, z_n)$ is still well defined for all $x, y \in \mathbb{X}$ (we say that z_n converge in the horoboundary), but it will give a more complicated, less algebraic, function of x and y. This was described by Kapovich–Leeb in Lemma 5.18 of [KL18], and will be used in Section 4.3.

Before we state the result let us analyze geometrically the limits of spheres in the flat A. A Finsler ball is a convex polyhedron whose faces are contained in hyperplanes parallel to $\ker(\alpha_0)$ and their images under the action of the Weyl group by reflections. If the centers of a sequence of such convex polyhedra tend to infinity away from the walls of the Weyl chambers, then the convex polyhedra converge to a halfspace bounded by the image of $\ker(\alpha_0)$ under an element of the Weyl group. If the centers of such a sequence tend to infinity away from all walls *but one*, then the convex polyhedra converge to the intersection of two such halfspaces. For example, they could be $\{x : b_\xi^{\mathfrak{F}}(x, 0) \leq 0\}$ and $\{x : b_\eta^{\mathfrak{F}}(x, 0) \leq 0\}$ determined by two Weyl chambers ξ and η that share a codimension 1 face. So the associated Busemann function associated to this intersection should be $f(x) = \max(b_\xi^{\mathfrak{F}}(x, 0), b_\eta^{\mathfrak{F}}(x, 0))$.

The precise statement is as follows. Let $(z_n)_n \subset \mathbb{X}$ be a sequence converging to a point $p \in \partial_\infty \mathbb{X}$. The point p is contained in finitely many closed Weyl chambers $\xi_1, \ldots, \xi_\ell \in \mathcal{F}$

($\ell \geq 1$). Let $\xi \in \mathcal{F}$ and let C be the Weyl cone based at $\mathbf{x}$ and asymptotic to ξ. If $\xi \notin \mathcal{B} = \{\xi_1, \dots, \xi_\ell\}$ then $d(z_n, C) \to \infty$. Otherwise $d(z_n, C)$ remains bounded; in this case let p_ξ be the intersection point of ξ with $G \cdot \alpha_0^\#$ where we view $\alpha_0^\#$ as a point in $\partial_\infty A^+$. Up to passing to a subsequence, there exists $x_0 \in \mathbb{X}$ such that for any $x \in \mathbb{X}$ we have

$$d^{\mathfrak{F}}(x, z_n) - d^{\mathfrak{F}}(x_0, z_n) \underset{n\to\infty}{\to} \max_{\xi \in \mathcal{B}} b_\xi^{\mathfrak{F}}(x, x_0) = \max_{\xi \in \mathcal{B}} b_{p_\xi}(x, x_0), \tag{7}$$

In other words, the Finsler balls centered at z_n whose boundary contain x_0 converge to the intersections of the Riemannian horoballs centered at p_ξ for $\xi \in \{\xi_1, \dots, \xi_\ell\}$ whose boundary contains x_0.

Gromov product

The *Gromov product* between two transverse flags $\xi, \eta \in \mathcal{F}$ computed at the basepoint $x \in \mathbb{X}$ is defined as

$$\frac{1}{2}\langle \xi | \eta \rangle_x = \lim_{n\to\infty} \frac{1}{2}\Big(d^{\mathfrak{F}}(y_n, x) + d^{\mathfrak{F}}(x, z_n) - d^{\mathfrak{F}}(y_n, z_n) \Big) \in \mathbb{R}_{\geq 0} \tag{8}$$

where $(y_n)_n, (z_n)_n \subset \mathbb{X}$ are sequences converging to points of the visual boundary in the interior of ξ and η respectively. Note that for the notation $\langle \xi | \eta \rangle_x$ we used a different convention than the usual one by not including the factor $\frac{1}{2}$. This will make the computations a bit easier to read.

Note that if x is contained in the flat connecting η to ξ then $\langle \xi | \eta \rangle_p = 0$, which leads to

$$\langle \xi | \eta \rangle_x = b_\xi^{\mathfrak{F}}(x, p) + b_\eta^{\mathfrak{F}}(x, p)$$

We refer to [KLP18] for more information on this construction, in particular on the existence of the limit in the formula (8).

Concrete description of the above objects for $G = \mathrm{PSL}_d(\mathbb{R})$

The Lie algebra $\mathfrak{g}$ is the algebra of trace free (d, d)-matrices. As Cartan subspace $\mathfrak{a}$ we choose the linear subspace of diagonal (d, d)-matrices with vanishing trace, and the open Weyl chamber $\mathfrak{a}^+$ is the open cone of diagonal matrices whose entries $(\lambda_1, \dots, \lambda_d)$ fulfill $\lambda_1 > \lambda_2 > \cdots > \lambda_d$.

The subgroup $K \subset \mathrm{PSL}_d(\mathbb{R})$ is chosen as the group $\mathrm{PSO}_d(\mathbb{R})$, and $P \subset \mathrm{PSL}_d(\mathbb{R})$ is taken as the image in $\mathrm{PSL}_d(\mathbb{R})$ of the set of upper triangular matrices with positive entries on the diagonal and determinant one.

The flag variety $\mathcal{F}$ has the following explicit description. Namely, a *full flag* in $\mathbb{R}^d$ is a sequence

$$\xi = (\xi_1 \subset \xi_2 \subset \cdots \subset \xi_d = \mathbb{R}^d)$$

where ξ_i is a linear subspace of $\mathbb{R}^d$ of dimension i for each $i \leq d$. Clearly $\mathrm{PSL}_d(\mathbb{R})$ acts transitively on the space of all full flags, with point stabilizer a minimal parabolic subgroup. Thus $\mathcal{F}$ is just the space of full flags in $\mathbb{R}^d$.

The Busemann functions also have a concrete description, using the identification between $\mathbb{X}$ and the set of inner products on $\mathbb{R}^d$ that induce the standard volume form. Namely, given $x, y \in \mathbb{X}$, let $||\cdot||_x$ and $||\cdot||_y$ denote the norms of the associated inner products on $\mathbb{R}^d$ and on the exterior products $\Lambda^k\mathbb{R}^d$ ($1 \le k \le d$). Let $\xi = (\xi_1 \subset \xi_2 \subset \cdots \subset \xi_d)$ be a full flag in $\mathbb{R}^d$. Let $v = (v_1, \ldots, v_d) = b^{\mathfrak{a}}_{\xi}(x, y) \in \mathfrak{a}$ with $v_1 + \cdots + v_d = 0$. Then for all $k \le d$, we have

$$v_1 + \cdots + v_k = \log \frac{||X||_x}{||X||_y}$$

where $X \in \Lambda^k\mathbb{R}^d$ is any representative of the k-plane ξ_k.

The homomorphism τ is obtained as follows. For $d \ge 3$ there exists up to conjugation a unique irreducible representation of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R}^d$. This representation determines the following embedding $\tau\colon \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_d(\mathbb{R})$. Let $\mathbb{R}^h_{d-1}[X, Y]$ be the set of degree $d-1$ homogeneous polynomials with real coefficients. A matrix $M = \left(\begin{smallmatrix} a & b \\ c & e \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ acts on the vector space $\mathbb{R}[X, Y]$ of polynomials in two variables by $M \cdot X = aX + cY$ and $M \cdot Y = bX + eY$. This action $\mathrm{SL}_2(\mathbb{R}) \curvearrowright \mathbb{R}[X, Y]$ preserves the d-dimensional linear subspace $\mathbb{R}^h_{d-1}[X, Y]$, with determinant one elements. So it induces an embedding $\mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_d(\mathbb{R})$ which is just the representation τ.

Using a suitable basis we have $\tau(\mathrm{SO}(2)) \subset K$, the representation τ induces a isometric embedding $\mathbb{H}^2 = \mathrm{PSL}_2(\mathbb{R})/\mathrm{PSO}(2) \hookrightarrow \mathbb{X}$. Denote by $\hat{\mathbb{H}}^2 = \pi_{\mathbb{X}} \circ \tau(\mathrm{PSL}_2(\mathbb{R}))$ the image of $\mathbb{H}^2$ inside $\mathbb{X}$.

The following statement is a consequence of the fact that $d\tau(T^1\mathbb{H}^2)$ consists of regular vectors contained in a single G-orbit, and each such vector is tangent to a unique maximal flat. It is well known and immediate from the above discussion.

Fact 1.2. *1. Every geodesic in $\hat{\mathbb{H}}^2$ lies in a unique maximal flat.*

2. For any hyperbolic element $g \in \mathrm{PSL}_2(\mathbb{R})$, the centraliser of $\tau(g)$ in G acts by translations on the unique flat containing the image under $\pi_{\mathbb{X}}$ of the axis of g in $\mathbb{H}^2$.

2 Equilibrium states, Hitchin representations and Pressure metric

In this section we turn to the dynamical aspects of this work. It is subdivided into three subsections. In the first subsection we introduce geodesic currents for closed surfaces and the intersection form. The second subsection contains an account of Hitchin representations and length functions defined by Finsler norms. We show, using [BCLS15], that such length functions can be used to construct a pressure metric on the Hitchin component. The third subsection contains a summary of the main properties of Patterson Sullivan theory we shall use later on.

Throughout, S denotes a closed surface of genus $g \ge 2$, equipped with a fixed choice of a hyperbolic metric. Thus the universal covering $\tilde{S}$ of S can naturally be identified with the hyperbolic plane $\mathbb{H}^2$.

2.1 Geodesic currents, length and intersection

A *geodesic current* for S is a non-trivial $\pi_1(S)$-invariant Radon measure on the space of oriented geodesics $\partial_\infty\mathbb{H}^2 \times \partial_\infty\mathbb{H}^2 - \Delta$ of the hyperbolic plane $\mathbb{H}^2$ (here Δ is the diagonal).

Two such currents are projectively equivalent if they are constant multiples of each other. An equivalence class for this equivalence relation is a *projective geodesic current*. The space $\mathcal{C}(S)$ of geodesic currents for S is equipped with the weak*-topology which descends to a topology on the space $\mathcal{PC}(S)$ of projective geodesic currents. A *(projective) measured geodesic lamination* is a (projective) geodesic current whose support consists of pairwise disjoint simple geodesics. The space $\mathcal{PML}$ of projective measured geodesic laminations is a closed subset of $\mathcal{PC}(S)$.

Each hyperbolic metric on S determines a geodesic current, the *Liouville current* of the metric. The following is due to Bonahon [Bon88].

Theorem 2.1 (Bonahon). *Associating to a hyperbolic metric on S its projective Liouville current defines an embedding of the Teichmüller space into $\mathcal{PC}(S)$, and its complement in its closure is the space of projective measured geodesic laminations.*

The idea behind this theorem rests on the existence of an *intersection form*

$$\iota : \mathcal{C}(S) \times \mathcal{C}(S) \to [0, \infty)$$

which extends the geometric intersection number between two closed curves on S. The form ι has the following properties (see Chapter 8 of [Mar16]).

1. ι is continuous for the weak*-topology.
2. If λ is the Liouville current of a hyperbolic metric ρ, on S and if $\alpha \subset S$ is any closed geodesic, then $\iota(\lambda, \alpha) = \ell_\rho(\alpha)$ the ρ-length of α.

The intersection $\iota(\lambda_1, \lambda_2)$ of two Liouville currents λ_1, λ_2 of two hyperbolic metrics on S has another interpretation which is important for us. Namely, the choice of a hyperbolic metric h_1 on S determines the *geodesic flow* Φ^t on the unit tangent bundle T^1S of S and a Hölder structure on T^1S. Given these data, any geodesic current μ for S extends to a Φ^t-invariant finite Borel measure $\hat{\mu}$ on T^1S. Thus given a Hölder continuous positive function $f : T^1S \to (0, \infty)$, the integral

$$\int f d\hat{\mu} = \mathbf{I}(\mu, f) \tag{9}$$

is defined. By invariance, this integral only depends on the *cohomology class* of f. This means that if f' is another Hölder function such that $\int_\gamma f' = \int_\gamma f$ for every periodic orbit γ for Φ^t then $\int f' d\hat{\mu} = \int f d\hat{\mu}$ and hence $\mathbf{I}(\mu, f') = \mathbf{I}(\mu, f)$.

As a consequence, the pairing $\mathbf{I}(\mu, f)$ is a pairing between cohomology classes of (positive) Hölder functions on T^1S and geodesic currents without having to make reference to the geodesic flow Φ^t which depends on the background metric. Furthermore, the pairing is continuous, where as before, $\mathcal{C}(S)$ is equipped with the weak*-topology, and the space of Hölder cohomology classes is equipped with the quotient topology obtained from the space of Hölder functions on T^1S for a fixed reference metric. We refer to [Ham99] for more details.

Assume now that f_2 is a Hölder function which integrates over each periodic orbit γ for Φ^t to the length of the free homotopy class of γ for another hyperbolic metric h_2. If λ_1, λ_2 are the Liouville currents of h_1, h_2, then we have

$$\iota(\lambda_1, \lambda_2) = \mathbf{I}(\lambda_1, f_2).$$

A Hölder continuous positive function f on T^1S can be used to reparameterize the flow Φ^t. This reparameterization is defined by

$$\Phi_f^t(v) = \Phi^{\sigma(v,t)}(v)$$

where $\int_0^{\sigma(v,t)} f(\Phi^s v)ds = t$. For the reparameterized flow, the function f is cohomologous to the constant function 1. This is equivalent to stating that the period of a periodic orbit γ for the flow Φ_f^t equals the integral of f over the corresponding orbit for Φ^t. The identity $(T^1S, \Phi^t) \to (T^1S, \Phi_f^t)$ is an *order preserving orbit equivalence* between the flows Φ^t, Φ_f^t.

Denote by h_μ the entropy of a Φ^t-invariant Borel probability measure μ on T^1S. For a positive Hölder function f let $\delta(f) > 0$ be such that $\mathrm{pr}(-\delta(f)f) = 0$ where

$$\mathrm{pr}(u) = \sup_\mu \left(h_\mu + \int u d\mu \right)$$

and μ runs through all Φ^t-invariant Borel probability measures on T^1S. Then

$$h_\mu - \delta(f) \int f d\mu \leq 0$$

for all μ. A measure μ is called a *Gibbs equilibrium state* for f if the equality in this inequality holds. Using the fact that an order preserving orbit equivalence between two flows induces an isomorphism between the flow invariant probability measures and a formula relating entropies due to Abramov [Abr59], existence and uniqueness of an equilibrium state for the continuous function $\delta(f)f$ is equivalent to existence and uniqueness of a measure of maximal entropy for the geodesic flow Φ_f^t on T^1S, which is well known for Hölder functions (see [KH95] for more details). The constant $\delta(f)$ then equals the topological entropy of Φ_f^t.

The unique Gibbs equilibrium state μ_f for f, viewed as an invariant measure for the flow Φ_f^t, can be obtained as a limit

$$\mu_f = \lim_{R\to\infty} \frac{1}{\#N_f(R)} \sum_{\ell_f(\gamma)\leq R} \frac{\mathcal{D}_{f,\gamma}}{\ell_f(\gamma)}$$

where $\ell_f(\gamma) = \int_\gamma f$ is the period of γ for Φ_f^t, where $\mathcal{D}_{f,\gamma}$ is the Φ_f^t-invariant measure on the periodic orbit γ of total mass $\ell_f(\gamma)$ and where $N_f(R) = \{\gamma \mid \ell_f(\gamma) \leq R\}$. Thus by continuity of the pairing $\mathbf{I}$, for any (positive) Hölder function u we have

$$\mathbf{I}(\mu_f, u) = \lim_{R\to\infty} \frac{1}{\#N_f(R)} \sum_{\ell_f(\gamma)\leq R} \frac{\ell_u(\gamma)}{\ell_f(\gamma)}. \tag{10}$$

Following [BCLS15], we also define the *normalized intersection number* by

$$\mathbf{J}(f,u) = \frac{h(u)}{h(f)}\mathbf{I}(\mu_f, u)$$

where $h(u) = \lim_{R\to\infty} \frac{1}{R}\log \# N_u(R)$.

2.2 Hitchin representations

In this section we introduce Hitchin representations and summarize those of their properties which are important later on. Our main goal is to show that the G-invariant Finsler metric $\mathfrak{F}$ defined in (3) induces a pressure metric on the Hitchin component.

The *Hitchin component* $\mathrm{Hit}(S)$ for conjugacy classes of representations $\pi_1(S) \to \mathrm{PSL}_d(\mathbb{R})$ is the connected component of the set of conjugacy classes of representations which factor through an irreducible representation $\mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_d(\mathbb{R})$. In the sequel we always work with explicit representations rather than with conjugacy classes.

An important property possessed by Hitchin representations is the Anosov property first introduced by Labourie [Lab06], which plays a central role in [BCLS15] in the definition of the pressure metric. There are many different versions of the Anosov property, and many equivalent characterisations of the Anosov property, see for example [Lab06; GW12; KLP17; GGKW17; BPS19; KP22], and Theorem 4.37 of [Kas24] for more details and history.

Definition 2.2. A representation $\rho : \pi_1(S) \to \mathrm{PSL}_d(\mathbb{R})$ is *projective Anosov* if there exist ρ-equivariant Hölder continuous maps $\xi : \partial_\infty \tilde{S} \to \mathbb{R}P^{d-1}$, $\theta : \partial_\infty \tilde{S} \to (\mathbb{R}P^{d-1})^*$ (where $(\mathbb{R}P^{d-1})^*$ is the dual projective space) such that

1. if x, y are distinct points in $\partial_\infty \tilde{S}$, then $\xi(x) + \ker\theta(y) = \mathbb{R}^d$, and
2. if $\gamma_n \in \pi_1(S)$ is a sequence so that for some basepoint $\mathbf{x} \in \tilde{S} = \mathbb{H}^2$, the sequence $\gamma_n \mathbf{x}$ converges to $x \in \partial_\infty \mathbb{H}^2$, and $\gamma_n^{-1}\mathbf{x} \to y \in \partial_\infty \mathbb{H}^2$, then we have $\rho(\gamma_n)p \to \xi(x)$ for any $p \in \mathbb{R}P^{d-1} - \ker\theta(y)$ and $\rho(\gamma_n^{-1})q \to \theta(y)$ for any $q \in (\mathbb{R}P^{d-1})^*$ such that $\xi(x) \notin \ker q$.

Remark . In the references given for the characterisations of the Anosov property, the limit map is only required to be continuous, and then the Hölder regularity is derived as a consequence of the other conditions, see for instance Theorem 6.58 of [KLP17].

The following is due to Labourie [Lab06] and Fock–Goncharov [FG06].

Theorem 2.3 (Labourie, Fock–Goncharov). *Every representation in the Hitchin component is projective Anosov.*

As in [BCLS15], let F be the total space of the bundle over

$$(\mathbb{R}P^{d-1})^{(2)} = \mathbb{R}P^{d-1} \times (\mathbb{R}P^{d-1})^* - \{(U,V) \mid U \subset \ker(V)\}$$

whose fiber at a point (U,V) is the space

$$M(U,V)=\{(u,v)\mid u\in U, v\in V, \langle v\mid u\rangle=1\}/\sim$$

where $\langle v\mid u\rangle$ is the natural pairing between a vector and a covector and $(u,v)\sim(-u,-v)$. Note that u determines v so that F is an $\mathbb{R}$-bundle.

The bundle F is equipped with a natural $\mathbb{R}$-action, given by

$$\Phi_F^t(U,V,(u,v))=(U,V,(e^t u, e^{-t}v)).$$

Given a projective Anosov representation $\rho:\pi_1(S)\to \mathrm{PSL}_d(\mathbb{R})$ and ξ,θ the associated limit maps, we consider the pullback bundle

$$F_\rho=(\xi,\theta)^*F\to \partial_\infty\widetilde{S}\times\partial_\infty\widetilde{S}-\Delta$$

by the map $\partial_\infty\widetilde{S}\times\partial_\infty\widetilde{S}-\Delta\xrightarrow{(\xi,\theta)}(\mathbb{R}P^{d-1})^{(2)}$, which inherits an $\mathbb{R}$-action from the action of Φ_F^t. The actions $\pi_1(S)\curvearrowright_\rho\mathbb{R}^d$ and $\pi_1(S)\curvearrowright\partial_\infty\widetilde{S}\times\partial_\infty\widetilde{S}-\Delta$ extend to an action on F_ρ. If we let

$$U_\rho S=\pi_1(S)\backslash F_\rho$$

then the $\mathbb{R}$-action on F_ρ descends to a flow Φ_ρ^t on $U_\rho S$ which is called the *spectral radius flow* of the representation (see p.1118 of [BCLS15]).

The following statement combines Propositions 4.1, 4.2 and 6.2 of [BCLS15]. It is valid for any analytic family of projective Anosov representations.

Proposition 2.4. *1. For every representation ρ in the Hitchin component there exists a Hölder continuous order preserving orbit equivalence $\Psi_\rho:(T^1S,\Phi^t)\to(U_\rho S,\Phi_\rho^t)$. Any primitive element $\gamma\in\pi_1(S)$ has period $\log\Lambda(\rho)(\gamma)$ where $\Lambda(\rho)(\gamma)$ is the spectral radius of $\rho(\gamma)\in\mathrm{PSL}_d(\mathbb{R})$.*

2. If D is the unit disk and if ρ_u $(u\in D)$ is a real analytic family of Hitchin representations, then up to decreasing the size of D, there exists a real analytic family $\{f_{\rho_u}:T^1S\to\mathbb{R}\}_{u\in D}$ of positive Hölder functions such that the reparameterization of T^1S by f_{ρ_u} is Hölder conjugate to U_{ρ_u} for all $u\in D$.

As a consequence, the spectral radius length defines a *pressure metric* on $\mathrm{Hit}(S)$ as follows. For any smooth deformation ρ_t of a representation $\rho=\rho_0$, put

$$\|\rho'(0)\|^2=\frac{d^2}{dt^2}|_{t=0}\mathbf{J}(f_{\rho(0)},f_{\rho(t)}) \tag{11}$$

(Theorem 1.3 of [BCLS15]) where $f_{\rho(s)}$ is the Hölder function constructed from $\rho(s)$ as in Proposition 2.4. That this construction defines indeed a (mildly degenerate) Riemanian metric on the Hitchin component which determines a distance function was established in [BCLS15]. It is based on the fact that projective Anosov representations are dominated in the sense of [BPS19]. We refer to [BCLS15], [BPS19] and [Sam24] for more precise information.

The pressure metric we are interested in is a more geometric version of the metric (11). To define this metric we need to review some additional properties of representations in $\mathrm{Hit}(S)$. Let as before $\mathcal{F}$ be the variety of full flags in $\mathbb{R}^d$.

Definition 2.5 ([GGKW17; KLP17]). A representation $\rho : \pi_1(S) \to G$ is *Borel Anosov* if the following holds true.

1. There exists a (unique) equivariant Hölder embedding $\partial_\infty \rho : \partial \mathbb{H}^2 \to \mathcal{F}$ such that $\partial_\infty \rho(\xi) \pitchfork \partial_\infty \rho(\eta)$ for all $\xi \neq \eta \in \partial \mathbb{H}^2$.

2. For any diverging sequence $(\gamma_n)_n \subset \pi_1(S)$ such that $\gamma_n \to \xi \in \partial \mathbb{H}^2$ and $\gamma_n^{-1} \to \eta$, we have $\rho(\gamma_n)\zeta \to \partial_\infty \rho(\eta)$ for any $\zeta \in \mathcal{F}$ transverse to $\partial_\infty \rho(\xi)$.

By the groundbreaking work of Labourie and Fock–Goncharov, we have

Theorem 2.6 ([Lab06; FG06]). *All Hitchin representations $\pi_1(S) \to \mathrm{PSL}_d(\mathbb{R})$ are Borel Anosov.*

Our goal is to construct a pressure metric on $\mathrm{Hit}(S)$ as in (11), but using the fixed length function $\ell^{\mathfrak{F}}(g) = \alpha_0(\lambda(g))$ of the Finsler norm and its associated renormalised intersection form

$$\mathbf{J}(\rho, \rho') = \frac{h(\rho')}{h(\rho)} \lim_{R \to \infty} \frac{1}{\# N_\rho(R)} \sum_{\ell^{\mathfrak{F}}(\rho(\gamma)) \leq R} \frac{\ell^{\mathfrak{F}}(\rho'(\gamma))}{\ell^{\mathfrak{F}}(\rho(\gamma))},$$

where

$$N_\rho(R) = \{[\gamma] \in [\pi_1(S)] : \ell^{\mathfrak{F}}(\rho(\gamma)) \leq R\} \text{ and } h(\rho) = \lim_{R \to \infty} \frac{1}{R} \log \# N_\rho(R).$$

To this end we have to establish an analog of Proposition 2.4 for this new length function. We shall reduce this statement to Proposition 2.4 using the following classical observation.

Let $\xi = (\xi_1 \subset \cdots \subset \xi_d)$ be a full flag in $\mathbb{R}^d$. Then for each $k \leq d-1$ the k-th exterior power $\Lambda^k(\xi_k)$ is one-dimensional. A non-zero element ω of this vector space defines a non-zero linear functional $\Psi(\omega) : \Lambda^{d-k}(\mathbb{R}^d) \to \mathbb{R}$ as follows. Choose a non-zero element $\nu \in \Lambda^d(\mathbb{R}^d)$ and put $\Psi(\omega)(\alpha) = c$ if $\omega \wedge \alpha = c\nu$. Note that the kernel of ω is spanned by all decomposable elements of $\Lambda^{d-k}\mathbb{R}^d$ which are not transverse to ξ_k.

If $\rho : \pi_1(S) \to G$ is Borel Anosov, then by the definition of the transversality relation $\pitchfork$, for any two distinct points $\xi \neq \eta \in \partial \mathbb{H}^2$, the $n-k$-th subspace $\partial_\infty \rho(\xi)_{n-k}$ of the flag $\partial_\infty \rho(\xi)$ defines a line of linear functionals on $\Lambda^k(\mathbb{R}^d)$ which do not evaluate to zero on $\Lambda^k \partial_\infty \rho(\eta)_k$, where $\partial_\infty \rho(\eta)_k$ is the k-dimensional subspace of the flag $\partial_\infty \rho(\eta)$. Thus if $\Lambda^k \rho : \pi_1(S) \to \mathrm{PSL}_{d_k}(\mathbb{R})$ denotes the representation induced by ρ into the full linear group of $\Lambda^k(\mathbb{R}^d)$ where d_k denotes the dimension of $\Lambda^k(\mathbb{R}^d)$, then as the map $\partial_\infty \rho : \partial_\infty \mathbb{H}^2 \to \mathcal{F}$ is Hölder continuous, we have

Lemma 2.7. *If $\rho : \pi_1(S) \to G$ is Borel Anosov, then for any $k < d$, the induced representation $\Lambda^k \rho$ is projective Anosov.*

Remark . *It follows from the above discussion that in fact, ρ is Borel Anosov if and only if for each $k \leq d-1$ the induced representation on $\Lambda^k(\mathbb{R}^d)$ is projective Anosov. We refer to Section 4 of [BPS19] for more details of this relation.*

Thus we can apply Proposition 2.4 to each representation $\Lambda^k\rho$. Recall from Section 1 the definition of the Jordan projection λ. As implicitly stated in [BPS19], we obtain

Proposition 2.8. *For every Borel Anosov representation $\rho_0 : \pi_1(S) \to \mathrm{PSL}_d(\mathbb{R})$, there exists an open neighborhood U of ρ_0 made of Borel Anosov representations and a real analytic family $\{f_\rho : T^1S \to \mathfrak{a}\}_{\rho\in U}$ of Hölder functions, valued in $\mathfrak{a}^+$, such that for any $\gamma \in \pi_1(S)$, we have*

$$\lambda(\rho(\gamma)) = \int f_\rho d\gamma.$$

Proof. Proposition 2.4 implies that there exists an open neighborhood U of ρ_0 and real analytic families $\{g^k_\rho : T^1S \to \mathbb{R}\}_{\rho\in U}$ of Hölder functions such that for any $\rho \in U$, each exterior product $\Lambda^k\rho$ is projective Anosov, and for any $\gamma \in \pi_1(S)$ we have

$$\log \Lambda(\Lambda^k\rho(\gamma)) = \int g^k_\rho d\gamma.$$

Then we can consider the following Hölder function

$$f_\rho = (g^1_\rho,\ g^2_\rho - g^1_\rho,\ g^3_\rho - g^2_\rho, \dots,\ g^d_\rho - g^{d-1}_\rho) \in \mathfrak{a}.$$

By Proposition 2.4, the function f_ρ depends analytically on ρ. Moreover, for any $\gamma \in \pi_1(S)$, we have

$$\lambda(\rho(\gamma)) = \int f_\rho d\gamma.$$

It is not clear, however, that f_ρ is valued in the open Weyl chamber $\mathfrak{a}^+$. We shall use the work [Sam14] to find another Hölder function $f' : \pi_1(S) \to \mathfrak{a}$ taking values in $\mathfrak{a}^+$ and such that $\lambda(\rho_0(\gamma)) = \int f' d\gamma$ for any γ.

For any $1 \le k \le d-1$ consider the cocycle $c_k : \pi_1(S) \times \partial\mathbb{H}^2 \to \mathbb{R}$ given by $c_k(\gamma, p) = \alpha_k(b^{\mathfrak{a}}_{\xi(p)}(\mathbf{x}, \rho_0(\gamma)))$, where $\xi : \partial\mathbb{H}^2 \to \mathfrak{F}$ is the Hölder limit map of ρ_0 and $\alpha_k : \mathfrak{a} \to \mathbb{R}$ is defined by $\alpha_k(v_1, \dots, v_d) = v_k - v_{k+1}$. As ρ_0 is Borel Anosov, these cocycles are Hölder, and their entropy

$$h_k = \lim_{R\to\infty} \frac{1}{R} \log \#\{[\gamma] : \alpha_k \circ \lambda \circ \rho_0(\gamma) \le R\}$$

is well defined and in $(0, \infty)$.

Thus we can apply Sambarino's reparametrization result, see Theorem 3.2 of [Sam14], to obtain positive Hölder functions $u^k : T^1S \to \mathbb{R}$ such that for any $\gamma \in \pi_1(S)$, we have

$$\alpha_k \circ \lambda \circ \rho_0(\gamma) = \int u^k d\gamma.$$

Let $f' : T^1S \to \mathfrak{a}$ be such that $\alpha_k \circ f'(v) = u^k(v) > 0$ for all $v \in T^1S$ and $1 \le k \le d-1$. Then f' is valued in the interior of $\mathfrak{a}^+$ by definition, it is Hölder, and $\lambda(\rho_0(\gamma)) = \int f' d\gamma$ for any γ.

For any periodic orbit γ in T^1S we have $\int f_{\rho_0} d\gamma = \int f' d\gamma$, so by Theorem 1 of [Liv71] f' and f_{ρ_0} are cohomologous, in the sense that there exists $F : T^1S \to \mathfrak{a}$ differentiable in the direction of the geodesic flow Φ^t such that $f' = f_{\rho_0} + \frac{d}{dt}_{|t=0} F \circ \Phi^t$. Put

$$f'_\rho = f_\rho + \frac{d}{dt}_{|t=0} F \circ \Phi^t$$

for any $\rho \in U$, so that $\lambda(\rho(\gamma)) = \int f'_\rho d\gamma$ for any γ. This yields an analytic family of Hölder functions which take values in $\mathfrak{a}^+$ for all ρ contained in a sufficiently small neighborhood $U' \subset U$ of ρ_0. This is what we wanted to show. □

Consider now a C^2–path of representations $(\rho_t)_t$ in the neighborhood U constructed in Proposition 2.8 with initial value ρ_0. Let μ_0 be the equilibrium state on T^1S associated to f_{ρ_0} introduced in Subsection 2.1, normalized so that $\int f_{\rho_0} d\mu_0 = 1$. Denote by $h(t)$ the entropy associated to f_{ρ_t}. Following [BCLS15] we set

$$\left\| \frac{d}{dt}_{|t=0} \rho_t \right\|^2 = \frac{1}{h(0)} \int \frac{d^2}{dt^2}_{|t=0} (h(t) \cdot \alpha_0 \circ f_{\rho_t}) d\mu_0. \tag{12}$$

It follows from Proposition 2.8 and [BCLS15] that this is well defined and is indeed the square norm for a (perhaps degenerate) Riemannian metric on $\mathrm{Hit}(S)$ which is a variant of the simple root length metric (11) considered in [BCLS15]. We call this metric the *Finsler pressure metric* on $\mathrm{Hit}(S)$.

2.3 Patterson–Sullivan theory

Patterson–Sullivan theory for hyperbolic metrics. Patterson [Pat76] and Sullivan [Sul79] introduced a construction of measures on $\partial_\infty \mathbb{H}^2$ which allows to obtain the entropy maximizing invariant probability measure of the geodesic flow on a compact hyperbolic surface as a product measure. This construction has been generalised in various settings. We recall some important facts about their theory and the generalization to the case of interest for us.

Let as before S be a closed surface of genus $g \geq 2$ and let $\Gamma = \rho(\pi_1(S)) \subset \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian representation, determined by the choice of a hyperbolic metric on S. For $\xi \in \partial_\infty \mathbb{H}^2$ and $x, y \in \mathbb{H}^2$, we denote by $b_\xi(x, y)$ the Busemann function of (x, y) based at ξ, defined as in (5). Up to multiplication by a constant, there exists a unique family of finite measures $(\nu^x)_{x\in\mathbb{H}^2}$ which all define the same measure class, and which satisfy the following. For all $x, y \in \mathbb{H}^2$ and $\xi \in \partial_\infty \mathbb{H}^2$,

$$\frac{\partial \nu^x}{\partial \nu^y}(\xi) = e^{b_\xi(x,y)}. \tag{13}$$

The measures ν^y can be obtained as a limit of measure of the form

$$\frac{1}{c_s} \sum_{g\in\Gamma} e^{sd(y,g\cdot x)} \delta_{g\cdot x} \tag{14}$$

with s converging from above toward 1 (which equals the *critical exponent* of Γ), and the constant c_s is chosen so that for $y = x$, the measures in (14) are probability measures.

From the measure class ν^x we define a measure on $\partial_\infty \mathbb{H}^2 \times \partial_\infty \mathbb{H}^2 \setminus \Delta$ invariant by the action of Γ. Recall from (8) the definition of the Gromov product $\langle \xi | \eta \rangle_x$ of (ξ, η) based at x. It can be computed by

$$\langle \xi | \eta \rangle_x = b_\xi(x, z) + b_\eta(x, z) \tag{15}$$

for any z on the geodesic with endpoints ξ and η. The value does not depend on the choice of z. Then define the measure $\hat{\nu}$ on $\partial_\infty \mathbb{H}^2 \times \partial_\infty \mathbb{H}^2 \setminus \Delta$ by

$$d\hat{\nu}(\xi, \eta) = e^{\langle \xi | \eta \rangle_x} \cdot d\nu^x(\xi) \times d\nu^x(\eta) \tag{16}$$

The measure $\hat{\nu}$ is invariant under the action $\Gamma \curvearrowright \partial_\infty \mathbb{H}^2 \times \partial_\infty \mathbb{H}^2 \setminus \Delta$, is finite on compact sets and does not depend on x [Sul79].

The unit tangent bundle $T^1 S$ of the surface $S = \Gamma \backslash \mathbb{H}^2$ is endowed with a geodesic flow Φ^t. It is Anosov, so it admits a unique entropy maximizing invariant probability measure. This measure lifts to a Γ-invariant Φ^t-invariant Radon measure on $T^1 \mathbb{H}^2$ which disintegrates to the measure $\hat{\nu}$. Namely, $\partial_\infty \mathbb{H}^2 \times \partial_\infty \mathbb{H}^2 \setminus \Delta$ is just the set of oriented geodesics in $\mathbb{H}^2$, and $d\hat{\nu} \times dt$ defines a Φ^t-invariant Γ-invariant Radon measure on $T^1 \mathbb{H}^2$, where dt is the one-dimensional Lebesgue measure on flow lines. This measure projects to a finite Borel measure on $T^1 S$ in the Lebesgue measure class, which can be scaled to be a probability measure.

Patterson–Sullivan theory for Hitchin representations Patterson Sullivan theory was generalized to many different geometric settings. In the setting of Finsler metrics on higher rank symmetric space and Hitchin representations, such a generalization is due to Kapovich and Dey [DK22] (the results are valid for all Anosov representations). Namely, given a Hitchin representation $\rho : \pi_1(S) \to \mathrm{PSL}_d(\mathbb{R})$, define a Poincaré series

$$P^{\mathfrak{F}}(\rho, s)(x, y) = \sum_{\psi \in \rho(\pi_1(S))} e^{-s d^{\mathfrak{F}}(y, \psi x)}$$

where as before, $d^{\mathfrak{F}}(y, z)$ is the distance between x, y for the Finsler metric $\mathfrak{F}$. Part (iv) of Theorem A of [DK22] shows that this series diverges at the *critical exponent* δ_ρ. Moreover, it defines a family μ^x of Borel measures on the *limit set* $\Lambda \subset \mathcal{F}$ of $\rho(\Gamma)$ in the flag variety $\mathcal{F}$, that is, the image of $\partial_\infty \mathbb{H}^2$ under a ρ-equivariant Hölder continuous map, indexed by the points $x \in \mathbb{X}$. These measure are a *conformal density*, that is, they are equivariant under the action of $\rho(\pi_1(S))$ and transform via

$$\frac{d\mu^y}{d\mu^x}(\xi) = e^{\delta_\rho b_\xi^{\mathfrak{F}}(x, y)} \tag{17}$$

where $b_\xi^{\mathfrak{F}}$ is the Busemann function for the Finsler metric.

Conformal densities had been constructed earlier by Sambarino in [Sam14], using a different method and work of Ledrappier [Led95]. Sambarino's construction is dynamical and does not use the Finsler metric $d^{\mathfrak{F}}$. Here we will need the geometrical approach of Dey–Kapovich.

Remark . *As the limit curve of a Hitchin representation is a curve in the flag variety rather than the limit set of the representation in the geometric boundary of $\mathbb{X}$, the above construction can not be carried out for the symmetric metric. Namely, as the limit set of the representation in the geometric boundary $\partial_\infty \mathbb{X}$ of $\mathbb{X}$ may have points in asymptotic Weyl chambers which are not opposite in the Weyl chamber and hence can not be connected by a geodesic, it may not be possible to correctly encode translation lengths for the symmetric metric by a global Hölder continuous function on T^1S.*

3 Hitchin grafting representations

The Hitchin representations we are interested in are the familiar *bending* or *bulging* deformations of *Fuchsian* representations, that is, representations which factor through the embedding $\tau : \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_d(\mathbb{R})$. We refer to [Gol86; AZ23] for an account on the bending construction. The goal of this section is to introduce these representations as well as an abstract geometric model for them, and we establish some first geometric properties of the representations and the model. The precise relation between the geometry of bending representations and the geometry of the model will be established in Section 6 and constitutes the main geometric result of this article.

The material in Subsections 3.1 – 3.3 is well known, and the purpose is to summarize the properties and the viewpoint we are going to pursue.

3.1 Abstract grafting

In this subsection we introduce abstract grafting of a hyperbolic surface as initiated by Thurston. We refer to [Tan97] for an early account on this construction. Contrary to the common definition in the literature, our grafting contains a twist which is needed for our purpose.

Consider a closed oriented surface S of genus $g \geq 2$ endowed with a hyperbolic metric. A *simple (geodesic) multi-curve* γ^* is the union of pairwise disjoint essential mutually not freely homotopic simple closed curves (geodesics) on S. We fix moreover an orientation on each component of γ^*.

Consider the special direction $u = d\tau\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathfrak{a}$ given by τ. For any $z \in \mathfrak{a}$ and $\ell > 0$, let $\mathrm{Cyl}(\ell, z) \subset \mathfrak{a}/\ell u$ be the cylinder obtained by quotienting the strip $\{tu + sz : t \in \mathbb{R},\ s \in [0,1]\} \subset \mathfrak{a}$ under the translation by ℓu. The (Finsler) *height* of such cylinder is defined as

$$\mathrm{height} = \min\{\mathfrak{F}(tu + z) : t \in \mathbb{R}\}. \tag{18}$$

We fix for every $\gamma \in \gamma^*$ a vector $z_\gamma \in \mathfrak{a}$; the collection $z = (z_\gamma)_{\gamma \in \gamma^*}$ is interpreted as a grafting parameter.

Definition 3.1. The *abstract grafting* of S along the geodesic multi-curve γ^* is the surface S_z obtained by cutting S open along each of the components γ of γ^*, inserting flat cylinders $C_\gamma = \mathrm{Cyl}(\ell_S(\gamma), z_\gamma)$ and gluing the surface back with the translation by z_γ.

If z_γ is not parallel to u for any $\gamma \in \gamma^*$, then this grafting comes with a natural homotopy equivalence $\pi_z : S_z \to S$ projecting the flat cylinders onto γ^*, which allow us to identify $\pi_1(S_z)$ and $\pi_1(S)$.

More precisely, for each $\gamma \in \gamma^*$, the metric completion of $S - \gamma$ is a surface whose boundary consists of two geodesics γ_1, γ_2 of the same length $\ell_S(\gamma)$. The choice of a parameterisation $\gamma(t)$ defines parameterisations $\gamma_1(t)$, $\gamma_2(t)$. Attach the flat cylinder C_γ to γ_1 and γ_2 by identifying $[tu] \in C_\gamma$ with $\gamma_1(t)$ and $[tu + z_\gamma]$ with $\gamma_2(t)$.

Let $C = \bigcup_{\gamma\in\gamma^*} C_\gamma \subset S_z$ and S' be the metric completion of $S - \gamma^*$, so that $S_z = (S' \sqcup C)/\sim$ where $\sim$ is the gluing procedure explained above. The projection map $\pi_z : S_z \to S$ satisfies the following. Its restriction $S' \to S$ is the continuous extension of the inclusion $S - \gamma^* \hookrightarrow S$. It projects each $[tu + sz_\gamma] \in C_\gamma$ to $\gamma(t) \in S$.

We call this operation *abstract grafting* to distinguish it from the *Hitchin grafting* that we will introduce for Hitchin representations. We shall refer to S_z as a *grafted surface*.

Note that if $z_\gamma = 0$ for every γ in γ^*, then the grafting is trivial and $S_z = S$. If all z_γ are parallel to u, then the grafted surface is hyperbolic and obtained from S by shearing along γ^* with shearing length given by the size of the parameters z_γ.

More generally, S_z is an orientable surface which admits a canonical piecewise smooth structure as well as a natural conformal structure which in turn induces a global C^1-structure on S. Any norm on $\mathfrak{a}$ which coincides with the norm induced by the hyperbolic metric on the distinguished direction u induces a Finsler metric on S_z which coincides with the hyperbolic metric on S' and whose restriction to the cylinders C_γ is flat.

In particular, the norm defined by the Riemannian metric of $\mathbb{X}$ can be used to endow the C^1-surface S_z with a C^0 Riemannian metric which is smooth everywhere except at the gluing locus, has constant curvature -1 in $S_z - \cup_\gamma C_\gamma$ and has constant curvature 0 in the interior of the cylinders C_γ. Since the curvature of this metric is non-positive whereever it is defined and the gluing is performed along geodesics, S_z is non-positively curved in the sense of Alexandrov and hence its universal covering $\tilde{S}_z$ is a CAT(0)-space.

Thus in this case every free homotopy class has a Riemannian geodesic representative whose length is minimal in the free homotopy class. Such a Riemannian geodesic is unique unless it is a core curve of a flat cylinder. If all the z_γ are orthogonal to the special direction u, then the natural homotopy equivalence $\pi_z : S_z \to S$ is 1-Lipschitz and hence in this case, free homotopy classes have longer lengths in S_z than in S. Moreover, the unit tangent bundle $T^1 S_z$ of S_z is well defined, and there is a geodesic flow which is topologically mixing and admits a unique measure of maximal entropy [Kni98].

As the pressure metric for the Hitchin component we are interested in is defined by a Finsler metric on $\mathbb{X}$ using α_0 (see (3)) rather than the Riemannian one, we also endow S_z with a Finsler metric by equipping each cylinder C_γ with the quotient of the non-Euclidean norm $\mathfrak{F}$ on $\mathfrak{a}$. Observe that in general, for a given C^1-structure on S_z as constructed above, this metric is *discontinuous* at the gluing locus between the flat cylinders and the hyperbolic part. Additionally the metric on the flat part is sensitive in the direction of z, and does not depends only on the height of the grafting (contrarily to the Riemannian metric). Nevertheless it induces a well defined path metric on S_z.

The following observation will be useful later on when estimating lengths.

Lemma 3.2. *If all z_e are in $\ker(\alpha_0)$, then the natural projection $\pi_z : S_z \to S$ is 1-Lipschitz for the Finsler metric on S_z. In particular, all free homotopy classes of curves have bigger Finsler lengths in S_z than in S.*

Proof. By definition, the restriction of our projection map $\pi_z : S_z \to S$ to each flat cylinder $C_\gamma = \{tu + sz_\gamma\}/\ell_S(\gamma)u$ comes from the linear projection of $\mathfrak{a}$ onto the line spanned by u, parallel to the direction $z_\gamma \in \ker(\alpha_0)$. To conclude it suffices to note that this projection is 1-Lipschitz for the non-Euclidean norm on $\mathfrak{a}$, which was defined using α_0 (see (3)). □

3.2 Particular case of an amalgamated product

In this section we explain briefly the construction of the following two Sections 3.3 and 3.4 in the special case where γ^* has only one component and is separating.

Let Σ be a closed orientable smooth surface of genus at least 2 and let $\gamma^* \subset \Sigma$ be a separating simple closed curve. Then γ^* splits Σ into two subsurfaces Σ_1 and Σ_2, and $\pi_1(\Sigma)$ can be written as an amalgamated product $\pi_1(\Sigma_1) \underset{\gamma^*}{*} \pi_2(\Sigma_2)$.

Consider a discrete and faithful representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R}) \xrightarrow{\tau} G$ such that $\rho(\gamma^*) = \exp(\ell_\rho(\gamma^*)u)$ where $u = d\tau\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is the special direction of $\mathfrak{a}$. Let $z \in \mathfrak{a}$ be a grafting parameter. The *Hitchin grafting representation* $\rho_z : \pi_1(S) \to G$ is defined by requiring that $\rho_z(\gamma) = \rho(\gamma)$ for any $\gamma \in \pi_1(\Sigma_1)$ and $\rho_z(\gamma) = \exp(z) \cdot \rho(\gamma) \cdot \exp(-z)$ for any $\gamma \in \pi_1(\Sigma_2)$.

One can then define a immersion $Q_z : S_z \to \rho_z \backslash \mathbb{X}$ whose restriction to any of the hyperbolic pieces of S_z and to the flat cylinder is totally geodesic. Indeed, $\rho_z(\pi_1(\Sigma_1)) = \rho(\pi_1(\Sigma_1))$ preserves $\widetilde{\Sigma}_1 \subset \mathbb{H}^2 \subset \mathbb{X}$, inducing a totally geodesic embedding $\Sigma_1 = \rho_z(\pi_1(\Sigma_1))\backslash\widetilde{\Sigma}_1 \hookrightarrow \rho_z\backslash\mathbb{X}$. Similarly, if one identifies Σ_2 with $\rho(\pi_1(\Sigma_2))\backslash\widetilde{\Sigma}_2$ where $\widetilde{\Sigma}_2 \subset \mathbb{H}^2 \subset \mathbb{X}$, then $\rho_z(\pi_1(\Sigma_2)) = \exp(z) \cdot \rho(\pi_1(\Sigma_2)) \cdot \exp(-z)$ preserves $\exp(z)\widetilde{\Sigma}_2$ and hence induces a totally geodesic embedding $\Sigma_2 \hookrightarrow \rho_z\backslash\mathbb{X}$. In general the image of the boundary components $\partial\Sigma_1$ and $\partial\Sigma_2$ in $\rho_z\backslash\mathbb{X}$ are disjoint. However they can be connected by the natural totally geodesic embedding of the cylinder $C = \{tu + sz : t \in \mathbb{R},\ s \in [0,1]\}/\rho_z(\gamma^*)$ into $\rho_z\backslash\mathbb{X}$. Gluing these three embeddings yield an piecewise totally geodesic embedding of $S_z = (\Sigma_1 \cup C \cup \Sigma_2)/\sim$ into $\rho_z\backslash\mathbb{X}$.

3.3 Graphs of groups decomposition and bending

A classical reference for the theory of graph of groups is [Ser77]. We collect some facts we need. Let Σ be a closed orientable smooth surface and let $\gamma^* \subset \Sigma$ be a simple multi-curve. The multi-curve determines the following graph of groups decomposition of $\pi_1(\Sigma)$, which will be used to define a family of Hitchin representations.

Let G_{γ^*} be the oriented graph such that each vertex $v \in V$ corresponds to a component Σ_v of $\Sigma - \gamma^*$, and each edge $e \in E$ corresponds to an oriented component $\vec{\gamma}_e$ of γ^*. Given an edge $e \in E$, we denote by $\bar{e}$ the opposite edge of e, for which $\vec{\gamma}_{\bar{e}}$ corresponds to the curve $\vec{\gamma}_e$ with the reverse orientation. The oriented edge e is adjacent to the two (not necessarily distinct) components $\Sigma_{o(e)}, \Sigma_{t(e)}$ of $\Sigma - \gamma^*$ which contain $\vec{\gamma}_e$ in their boundary. One can embed G_{γ^*} into the surface Σ such that each vertex v lies in the interior of Σ_v and each edge e connects $o(e)$ to $t(e)$, crossing transversally $\vec{\gamma}_e$ once. Since we assume that Σ is oriented, choosing an orientation on γ^* is the same as choosing for each pair of opposite edges $e, \bar{e} \in E$ a preferred one by declaring that the ordered pair (u_1, u_2) consisting of the oriented tangent u_1 of the oriented edge e at x_e and the oriented tangent of the oriented geodesic γ^* defines the orientation of Σ.

The graph of groups decomposition of $\pi_1(S)$ defined by this datum associates to each vertex $v \in V$ the fundamental group $A_v := \pi_1(\Sigma_v, v)$ where v is seen as a point in the interior of Σ_v. To each edge e is associated the fundamental group $A_e := \pi_1(\vec{\gamma}_e, x_e) \simeq \mathbb{Z}$ of $\vec{\gamma}_e$, where x_e is the intersection point of $\vec{\gamma}_e$ with e (which is seen as an arc in Σ transverse to $\vec{\gamma}_e$). The inclusions $\vec{\gamma}_e \hookrightarrow \Sigma_{o(e)}$, $\vec{\gamma}_e \hookrightarrow \Sigma_{t(e)}$ determine the following monomorphisms, by connecting x_e to respectively $o(e)$ and $t(e)$ via e.

$$\alpha_{o(e)} : A_e = \pi_1(\vec{\gamma}_e, x_e) \hookrightarrow A_{o(e)} = \pi_1(\Sigma_{o(e)}, o(e)) \text{ and } \alpha_{t(e)} : A_e \hookrightarrow A_{t(e)}.$$

Note that $\alpha_{o(e)}(\vec{\gamma}_e) = \alpha_{t(\bar{e})}(\vec{\gamma}_{\bar{e}})^{-1}$.

That this construction indeed defines a decomposition of $\pi_1(\Sigma)$ as graph of groups with cyclic edge groups is well known. More precisely, choose a spanning tree $T \subset G_{\gamma*}$ of G_{γ^*}, with edge set $E_T \subset E$ invariant under the orientation reversing map $e \mapsto \bar{e}$. For a vertex $v \in V$ put A_v, and for an edge $e \in E$ put A_e. Denote by $\vec{\gamma}_e$ the oriented geodesic defined by the oriented edge e.

Let $\pi_1(G_{\gamma^*}, T)$ be the quotient group

$$\pi_1(G_{\gamma^*}, T) = {}^{(*_v A_v) * F_E}\!\big/_{R}$$

where $*$ denote the free product, F_E is the free group generated by the edge set E, and R is the normal subgroup of $(*_v A_v) * F_E$ generated by the union of the sets

- $e \cdot \bar{e}$ for all $e \in E$,
- e for all $e \in E_T$,
- $e\alpha_{o(e)}(g)e^{-1}\alpha_{t(e)}(g)^{-1}$ for all $e \in E$ and $g \in A_e$, which we think of as $e\alpha_{o(e)}(g)e^{-1} \equiv \alpha_{t(e)}(g)$.

Thus $\pi_1(\Sigma)$ is obtained from simultaneous HNN-extension of the tree of groups defined by the spanning tree T.

Recall that the isomorphism between $\pi_1(G_{\gamma^*}, T)$ and $\pi_1(S)$ is constructed by choosing a basepoint $v_0 \in V$, embedding each vertex group A_v into $\pi_1(S, v_0)$ by connecting v to v_0 via the spanning tree T, and mapping F_E into $\pi_1(S, v_0)$ by connecting the endpoints $o(e)$ and $t(e)$ of each e to v_0 via the tree T.

Take a discrete and faithful representation $\rho\colon \pi_1(G_{\gamma^*}, T) \to \mathrm{PSL}_2(\mathbb{R}) \xrightarrow{\tau} G$ which factors through the embedding $\tau : \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_d(\mathbb{R})$. We use the graphs of groups decomposition of $\pi_1(\Sigma)$ to perform a bending of the representation in G with parameter $z = (z_\gamma)_{\gamma\in\gamma^*} \in \mathfrak{a}^{\gamma^*}$. This construction can be thought of as bending the surface S along the geodesic multicurve γ^* in the space of representations into G.

Let $\widetilde{\rho} : (*_v A_v) * F_E \to G$ be the composition of ρ with the projection

$$(*_v A_v) * F_E \to \pi_1(G_{\gamma^*}, T). \tag{19}$$

Fix an orientation on γ^*, so that for every $\gamma \in \gamma^*$, we get a preferred edge $e \in E$.

Then there exists $B \in \mathrm{PSL}_d(\mathbb{R})$ such that $\widetilde{\rho}(\alpha_{o(e)}(\vec{\gamma}_e)) = B\exp(\ell_\rho(\gamma)u)B^{-1}$, where $u = d\tau\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ is the special direction of $\mathfrak{a}$ (note that the ℓ_ρ-length of γ does not depend on the orientation of γ since α_0 was taken symmetric).

Set $\zeta_e = B\exp(z_\gamma)B^{-1}$, so that ζ_e commutes with $\widetilde{\rho}(\alpha_{o(e)}(\vec{\gamma}_e))$. Note that by definition of our relations R and $\alpha_{o(\bar{e})}(\vec{\gamma}_{\bar{e}}) = \alpha_{t(e)}(\vec{\gamma}_e)^{-1}$ we have

$$\widetilde{\rho}(\alpha_{o(\bar{e})}(\vec{\gamma}_{\bar{e}})) = \widetilde{\rho}(e)\widetilde{\rho}(\alpha_{o(e)}(\vec{\gamma}_e))^{-1}\widetilde{\rho}(e)^{-1} = \widetilde{\rho}(e)B\exp(-\ell_\rho(\gamma)u)B^{-1}\widetilde{\rho}(e)^{-1}.$$

Set $\zeta_{\bar{e}} = \widetilde{\rho}(e)B\exp(-z_\gamma)B^{-1}\widetilde{\rho}(e)^{-1} = \widetilde{\rho}(e)\zeta_e^{-1}\widetilde{\rho}(e)^{-1}$, that commutes with $\widetilde{\rho}(\alpha_{o(\bar{e})}(\vec{\gamma}_{\bar{e}}))$ and satisfies $\widetilde{\rho}(\bar{e})\zeta_{\bar{e}}\widetilde{\rho}(e)\zeta_e = 1$.

Geometrically, the group A_e acts on $\mathbb{H}^2$ as a translation on a geodesic $\widetilde{\gamma}$. By Fact 1.2, the image of $\widetilde{\gamma} \subset \mathbb{H}^2$ in $\mathbb{X}$ is contained in a unique maximal flat, and ζ_e preserves this flat and acts on it as a translation.

A Hitchin grafting representation is obtained by performing a partial conjugation of $\pi_1(G_{\gamma^*}, T)$ by the elements $\zeta = (\zeta_e)_{e\in E}$. Fix a basepoint $v_0 \in V$. For any $v \in V$, we denote by

$$\omega_v = \zeta_{e_1}\cdots\zeta_{e_n}$$

where $(e_1\cdots e_n)$ is an oriented path in the tree T from v_0 to v. Since $\zeta_{\bar{e}} = \zeta_e^{-1}$ when e is in E_T and T is a tree, the value of ω_v does not depend on the chosen path.

Then define the representation $\widetilde{\rho}_z\colon (*_v A_v) * F_E \to G$ by

i) $\widetilde{\rho}_z(g) = \omega_v\widetilde{\rho}(g)\omega_v^{-1}$ for all $v \in V$ and $g \in A_v = \pi_1(\Sigma_v)$,

ii) $\widetilde{\rho}_z(e) = \omega_{o(e)}\widetilde{\rho}(e)\zeta_e\omega_{t(e)}^{-1}$ for all $e \in E$.

Lemma 3.3. *The representation $\widetilde{\rho}_z$ contains R in its kernel.*

Proof. For all $e \in E$, we have

$$\widetilde{\rho}_z(e\bar{e}) = \left(\omega_{o(e)}\widetilde{\rho}(e)\zeta_e\omega_{t(\bar{e})}^{-1}\right)\left(\omega_{o(e)}\widetilde{\rho}(\bar{e})\zeta_{\bar{e}}\omega_{t(\bar{e})}^{-1}\right) = 1$$

since $\omega_{t(\bar{e})} = \omega_{o(e)}$ and $\widetilde{\rho}(\bar{e})\zeta_{\bar{e}}\rho(e)\zeta_e = 1$.

For all $e \in E_T$, we have $\widetilde{\rho}(e) = 1$ and $\omega_{t(e)} = \omega_{o(e)}\zeta_e$, so

$$\widetilde{\rho}_z(e) = \omega_{o(e)}\widetilde{\rho}(e)\zeta_e\omega_{t(e)}^{-1} = 1$$

Take $e \in E$ and $g \in A_e$. Then

$$\begin{aligned}
\widetilde{\rho}_z(e\alpha_e(g)e^{-1}) &= \left(\omega_{o(e)}\widetilde{\rho}(e)\zeta_e\omega_{t(e)}^{-1}\right)\left(\omega_{t(e)}\widetilde{\rho}(\alpha_e(g))\omega_{t(e)}^{-1}\right)\left(\omega_{t(e)}\zeta_e^{-1}\widetilde{\rho}(e)^{-1}\omega_{o(e)}^{-1}\right)\\
&= \omega_{o(e)}\widetilde{\rho}(e)\zeta_e\widetilde{\rho}(\alpha_e(g))\zeta_e^{-1}\rho(e)^{-1}\omega_{o(e)}^{-1}\\
&= \omega_{o(e)}\widetilde{\rho}(e)\widetilde{\rho}(\alpha_e(g))\widetilde{\rho}(e)^{-1}\omega_{o(e)}^{-1} \qquad \text{since } \zeta_e \text{ and } \widetilde{\rho}(\alpha_e(g)) \text{ commute}\\
&= \omega_{o(e)}\widetilde{\rho}(e\alpha_e(g)\bar{e})\omega_{o(e)}^{-1}\\
&= \omega_{t(\bar{e})}\widetilde{\rho}(\alpha_{\bar{e}}(g))\omega_{t(\bar{e})}^{-1} = \widetilde{\rho}_z(\alpha_{\bar{e}}(g))
\end{aligned}$$

□

Definition 3.4. We denote by $\mathrm{Gr}_z^{\gamma^*}\rho \colon \pi_1(G_{\gamma^*}, T) \to G$ the representation induced by $\widetilde{\rho}_z$, and sometimes just ρ_z if there is only one hyperbolic structure involved. We call it the *Hitchin grafting representation* with data z along γ^*.

Up to conjugation, the representation ρ_z does not depend the choices made for the graph of group decomposition.

3.4 The characteristic surface for Hitchin grafting representations

Consider a Fuchsian representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R}) \to G$ and denote by S the hyperbolic surface defined by this representation. Choose some grafting datum z and let ρ_z be the Hithin grafted representation defined by ρ and z. As this representation is contained in the Hitchin component, it follows from Labourie [Lab06] and Fock–Goncharov [FG06] that ρ_z is faithful, with discrete image. In particular, the quotient manifold $\rho_z\backslash\mathbb{X}$ is homotopy equivalent to S; in fact ρ induces a natural homotopy class of homotopy equivalences between $\rho_z\backslash\mathbb{X}$ and S.

The goal of this subsection is to construct a geometrically controlled homotopy equivalence from an abstract grafted surface into $\rho_z\backslash\mathbb{X}$. The following proposition is the main result of this subsection.

Proposition 3.5. *Consider a Hitchin grafting representation ρ_z obtained from ρ and with grafting datum z. Recall that S_z denotes the abstract grafting of S from Definition 3.1, with universal covering $\widetilde{S}_z$. Then there exists a piecewise totally geodesic immersed surface $\widetilde{S}_z^\iota \subset \mathbb{X}$ and a ρ_z-equivariant immersion $\widetilde{Q}_z \colon \widetilde{S}_z \to \widetilde{S}_z^\iota \subset \mathbb{X}$.*

The map $\widetilde{Q}_z$ is a path isometry for the Riemannian (resp. Finsler) metric on $\widetilde{S}_z$ and the induced path metric on $\widetilde{S}_z^\iota$ from the Riemannian (resp. Finsler) metric on $\mathbb{X}$.

Before we prove the proposition, note that the surface $\widetilde{S}_z^\iota$ is ρ_z-invariant and hence descends to compact piecewise smooth immersed surface $S_z^\iota \subset \rho_z\backslash\mathbb{X}$. We call this surface *characteristic*. We shall show in Proposition 5.6 that the corresponding map $S_z \to S_z^\iota \subset \rho_z\backslash\mathbb{X}$ is actually an embedding, and hence that S_z^ι is not only an immersed surface but an embedded one.

Recall that the Riemannian (resp. Finsler) *cylinder height* of the Hitchin grafting representation ρ_z is the minimum of all $d(z_\gamma, \mathbb{R}u)$ (resp. $d^{\mathfrak{F}}(z_\gamma, \mathbb{R}u)$) for all $\gamma \in \gamma^*$, where $u = d\tau\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ is the special direction in $\mathfrak{a}$.

Proof of Proposition 3.5. Denote by $\widetilde{S}$ and $\widetilde{\gamma}^*$, respectively, the universal cover of S and the pre-image of γ^* in $\widetilde{S}$. Using the hyperbolic metric, we can fix an identification $S = \pi_1(S)\backslash\mathbb{H}^2$ so that $\widetilde{S} = \mathbb{H}^2$.

Let $\widetilde{S}_z$ be the universal cover of the abstract grafted surface S_z. This surface consists of a countable union S_z^{hyp} of simply connected hyperbolic surfaces with geodesic boundary, called hyperbolic pieces in the sequel, and a countable union of flat strips separating these hyperbolic pieces. Let $\mathcal{T} \subset \widetilde{S}$ be an embedded graph with one vertex v in the interior of each of the hyperbolic pieces $\widetilde{\Sigma}_v$ of $\widetilde{S} - \widetilde{\gamma}^*$ and where two such points are connected by an edge e if the pieces containing them are separated by a single component $\tilde{\gamma}_e$ of $\widetilde{\gamma}^*$. To each vertex v of $\mathcal{T}$ is also associated a hyperbolic piece $\widetilde{\Sigma}_v^z \subset \widetilde{S}_z$ which is naturally isometric to $\widetilde{\Sigma}_v$.

By construction, for any vertex v of $\mathcal{T}$ the stabiliser $A_v := \mathrm{Stab}_{\pi_1(S)}(\widetilde{\Sigma}_v)$ is mapped by ρ_z onto a conjugate $g_v\rho(A_v)g_v^{-1}$ of $\rho(A_v)$ in G and hence it stabilises a unique totally geodesic embedded bordered surface $\widehat{\Sigma}_v^z = g_v\widehat{\Sigma}_v \subset \mathbb{X}$ which is naturally isometric to $\widetilde{\Sigma}_v$ and $\widetilde{\Sigma}_v^z$. Define $(\widetilde{Q}_z)_{|\widetilde{\Sigma}_v^s} : \widetilde{\Sigma}_v \to \widehat{\Sigma}_v^z$ to be this natural isometry. By the construction of ρ_z, the thus defined map $\widetilde{Q}_z : \widetilde{S}_z^{hyp} \to \mathbb{X}$ is equivariant with respect to the representation ρ_z.

Consider an edge e of $\mathcal{T}$ between two vertices $v = o(e)$ and $w = t(e)$ that projects onto a component $\gamma \subset \gamma^*$ matching the fixed orientation on γ^*. We also call $e \in \pi_1(S)$ the preferred generator of the stabiliser of $\widetilde{\gamma}_e = \widetilde{\Sigma}_v \cap \widetilde{\Sigma}_w$. Its holonomy $\rho_z(e)$ acts cocompactly by translation on boundary components $\widetilde{c}_1$ and $\widetilde{c}_2$ of $\widehat{\Sigma}_v^z$ and $\widehat{\Sigma}_w^z$, respectively.

Let as before $u = d\tau\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ be the special direction of $\mathfrak{a}$. By construction, there exists a unique $h \in \mathrm{PSL}_d(\mathbb{R})$ such that

- $h\rho_z(A_v)h^{-1} \subset \mathrm{PSL}_2(\mathbb{R}) \subset \mathrm{PSL}_d(\mathbb{R})$;
- $h\widehat{\Sigma}_v^z \subset \mathbb{H}^2 \subset \mathbb{X}$;
- $h\rho_z(e)h^{-1} = \exp(\ell_\rho(e)u)$;
- $h\widetilde{c}_1 \subset \mathbb{H}^2 \cap \mathfrak{a} \subset \mathbb{X}$ is an axis of $\exp(u)$, that is, it is invariant under $\exp(u)$ and $\exp(u)$ acts on it as a translation.

Recall that $h\rho_z(A_v)h^{-1} = hg_v\rho(A_v)g_v^{-1}h^{-1}$ and $hg_v \in \mathrm{PSL}_2(\mathbb{R})$. One can then check the following formula for the holonomy of the adjacent stabiliser A_w:

$$h\rho_z(A_w)h^{-1} = \exp(z_\gamma)hg_v\rho(A_w)g_v^{-1}h^{-1}\exp(-z_\gamma),$$

and hence $h\widehat{\Sigma}_w^z \subset \exp(z_\gamma)\mathbb{H}^2$ and $\widetilde{c}_2 = \exp(z_\gamma)\widetilde{c}_1 \subset \mathfrak{a}$ is another axis of $\exp(u)$. Thus the flat strip $h^{-1}\{tu + sz_\gamma\}$ is $\rho_z(e)$-invariant, connects $\widehat{\Sigma}_v^z$ to $\widehat{\Sigma}_w^z$, is the only such flat strip, and is naturally isometric to the flat strip between $\widetilde{\Sigma}_v^z$ and $\widetilde{\Sigma}_w^z$ in $\widetilde{S}_z$.

Doing this for all flat strips in $\widetilde{S}_z$ yields an extended map $\widetilde{Q}_z \colon \widetilde{S}_z \to \mathbb{X}$, which is an isometry on each hyperbolic and flat piece. Furthermore, by construction, the map $\widetilde{Q}_z$ is continuous and ρ_z-equivariant.

□

ht

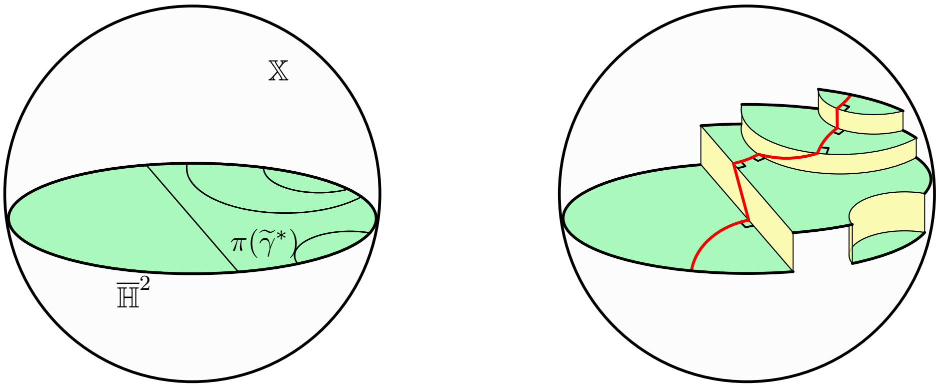

Figure 1: Geometric description of the Hitchin representation: the hyperbolic part in green, the flat part in yellow and an admissible path in red.

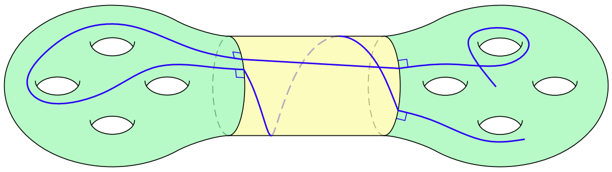

Figure 2: Admissible path in the closed surface obtained as an abstract grafting.

3.5 Admissible paths

In Section 6 we shall show that the characteristic surface not only is embedded, but it also can be used effectively to compare the large scale geometry of the locally symmetric manifold $\rho_z \backslash \mathbb{X}$ to the large scale geometry of the grafted surface. This comparison relies on the analysis of some specific paths which we introduce now.

3.5.1 Admissible paths in abstract grafted surfaces

We begin with introducing a family of paths in grafted surfaces, called *admissible paths*, which are from a technical point of view easier to handle than geodesics.

Let S_z be an abstract grafted surface with hyperbolic part S^{hyp} and cylinder part $\mathcal{C}$. In a nutshell, an admissible path c is a continuous path of S_z which is a geodesic everywhere except possibly at $S^{\mathrm{hyp}} \cap \mathcal{C}$, where it might have a singularity. Moreover, we require that the "hyperbolic part" $c \cap S^{\mathrm{hyp}}$ of the path c is orthogonal to $S^{\mathrm{hyp}} \cap \mathcal{C}$ where it meets it.

It is clear that lifts of admissible paths to the universal cover are quasi-geodesics (although we will not need it). Our goal will be to show that the images of admissible paths under the map constructed in Proposition 3.5 are quasi-geodesics of the symmetric space, with control on the multiplicative constant.

Definition 3.6. Consider a closed hyperbolic surface S, a multicurve $\gamma^* \subset S$ and a grafting parameter z. Then S_z is the abstract grafted surface with hyperbolic part $S^{\mathrm{hyp}} \subset S_z$ and flat (cylindrical) part $\mathcal{C} \subset S_z$. An *admissible path* in S_z is a continuous path $c \subset S_z$ such that

- c is geodesic outside $\mu = S^{\mathrm{hyp}} \cap \mathcal{C}$;
- the hyperbolic part $c \cap S^{\mathrm{hyp}}$ intersects γ^* orthogonally;
- a component of the flat part $c \cap \mathcal{C}$ connects the two distinct boundary components of the flat cylinder containing it.

Similarly one can define *admissible loops.*

Note that if z is trivial then $S_z = S$ and the above definition still makes sense. The flat part $\mathcal{C}$ is just γ^*, and the path is allowed to contain arcs in γ^* separating two geodesic arcs which emanate to the two distinct sides of γ^* in a tubular neighborhood of γ^*.

An *admissible path* in the universal cover $\tilde{S}_z$ is the lift of an admissible path in S_z.

Note that any two points of $\tilde{S}_z$ are connected by a unique admissible path; in other words, any path of S_z is homotopic (with fixed endpoints) to a unique admissible path.

Similarly, any loop in S_z not homotopic to a component of γ^* is freely homotopic to a unique admissible loop.

Observation 3.7. *The image under $\pi_z : S_z \to S$ (or the lift $\widetilde{S}_z \to \widetilde{S}$) of admissible paths in S_z are admissible paths in S.*

In fact, this induces a correspondence in the sense that any admissible path in S is the image under π_z of a unique admissible path in S_z.

3.5.2 Admissible paths in the symmetric space: geometric description

There are complete analogs of admissible paths in grafted surfaces for the symmetric space $\mathbb{X}$ of G, which are also called admissible paths. Such paths include the image of all admissible paths in $\tilde{S}_z$ under a path isometry $Q_\zeta : \tilde{S}_z \to \mathbb{X}$ constructed in Proposition 3.5.

Roughly speaking, admissible paths are piecewise geodesics that alternate between following a geodesic of the same type as the geodesics in the embedded $\mathbb{H}^2 \hookrightarrow \mathbb{X}$, and then following a geodesic in a flat, orthogonal to the previous geodesic, and then following a $\mathbb{H}^2$-type geodesic orthogonal to the previous flat... etc, see Figure 2.

The above description is not quite correct, in particular because it does not encapsulate the positivity assumption which is crucial in our proofs. There are several ways to define rigorously admissible paths. We are going to start with a geometric definition, which is easier to picture, and then in the next section we will give an algebraic definition. In the sequel the geometric definition will never be used, instead all the proofs will use the algebraic one, in particular because the positivity property of admissible paths is more naturally encoded in the algebraic definition.

Recall that $\mathbb{H}^2$ embeds isometrically into $\mathbb{X}$. In fact there are many isometric embeddings, and $\mathrm{PGL}_d(\mathbb{R})$ acts transitively on the set of all isometric embeddings. Let us call an $\mathbb{H}^2$-*frame* the datum of a point $x \in \mathbb{X}$ and a pair of orthogonal unit tangent vectors (v, w) which are tangent to a common embedded $\mathbb{H}^2$. Let Y be the space of $\mathbb{H}^2$-frames, on which $\mathrm{PGL}_d(\mathbb{R})$ also acts transitively. This action is even simply transitive since $\mathrm{PGL}_d(\mathbb{R})$ is split real.

On Y there is a natural geodesic flow $(\mathrm{geod}_t)_{t\in\mathbb{R}}$: given $(x, v, w) \in Y$ one can follow the geodesic ray spanned by v and parallel transport v and w along it. In other words, this action is the action of the one-parameter group of transvections on $\mathbb{X}$ along the geodesic ray spanned by v.

There is also a natural action of $\mathfrak{a}$, which we shall call the "orthogonal sliding action" and denote $(\mathrm{slide}_z)_{z\in\mathfrak{a}}$: given $(x, v, w) \in Y$ there is a unique maximal flat F containing w, and a unique identification of the tangent space of F with $\mathfrak{a}$ such that w is sent into $\mathfrak{a}^+$. Thus given $z \in \mathfrak{a}$ one can follow the geodesic ray spanned by the associated vector in F and parallel transport v and w along it. Note that the image of the vector v under this sliding action remains orthogonal to the flat F.

We define an (ω, L)-admissible path in $\mathbb{X}$ to be a path obtained by choosing an $\mathbb{H}^2$-frame and pushing it via the geodesic flow for some time at least ω, and then sliding orthogonally via $(\mathrm{silde}_{tz})_t$ for some time at least L using some direction $z \in \mathfrak{a}$, and then pushing along the geodesic flow again for time at least ω... etc.

In particular, an admissible path does not backtrack in any obvious way because it remembers directions along which the path can be continued. This is the property which can be thought of as a geometric interpretation of positivity in the sense of [FG06]. In fact we get *quantitative* positivity properties from the lower bound ω on the times we push along the geodesic flow.

3.5.3 Admissible paths in the symmetric space: algebraic definition

Let us now give an algebraic definition of admissible paths in $\mathbb{X}$. For this we will first define admissible paths in G. The description of these paths uses a basepoint for the action of G which is determined by the Fuchsian representation τ.

Notation 2. We set

- $a_t := \tau\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \in G$;
- $r_\theta := \tau\begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \in G$;
- $a'_t := r_{\pi/2} \cdot a_t \cdot r_{\pi/2}^{-1} \in G$;
- *for every* $t \in \mathbb{R} \cup \{\infty\} = \partial\mathbb{H}^2$ *we write* $\xi_t = \partial_\infty \tau(t)$.

The group $G = \mathrm{PSL}_d(\mathbb{R})$ identifies with one component of the space of $\mathbb{H}^2$-frames Y introduced in the previous section via the orbit map $G \to Y$; $g \mapsto g \cdot F_o$, where $F_o = (o, v_o, w_o)$ is a fixed $\mathbb{H}^2$-frame, so that o is fixed by r_θ, and $v_o = \frac{d}{dt}_{|t=0} a'_t \cdot o$ and $w_o = \frac{d}{dt}_{|t=0} a_t \cdot o$ are tangent to the axes of a'_t and a_t, respectively.

Under this identification, the geodesic flow on Y corresponds to the multiplication on the right by a'_t: i.e. $\mathrm{geod}_t(gF_o) = (ga'_t)F_o$. On the other hand, the orthogonal sliding flow corresponds to the multiplication on the right by $\exp(z)$: that is, $\mathrm{slide}_z(gF_o) = (g \cdot \exp(z))F_o$ for any $z \in \mathfrak{a}$. This leads us to the following definition of admissible path.

Definition 3.8. A path $c : [0, T] \to G$ or $c : [0, \infty) \to G$ is said to be of

- *flat type* if $c(t) = g \cdot \exp(tz)$ for some $g \in G$ and $z \in \mathfrak{a}$ of norm 1 for the Finsler metric $\mathfrak{F}$;
- *hyperbolic type* if $c(t) = ga'_t$ for some $g \in G$.

An *admissible path* of G is a *continuous* (possibly infinite) concatenation of paths of flat and hyperbolic type.

It is moreover called (ω, L)-*admissible* for some parameters $\omega, L > 0$ if all hyperbolic (resp. flat) pieces, except maybe the first and last pieces, have length at least ω (resp. L).

A (ω, L)-*admissible path in* $\mathbb{X}$ is a path of the form $t \mapsto c(t) \cdot \mathbf{x}$ where c is a (ω, L)-admissible path of G; note that it is piecewise geodesic.

Remark . Another way to describe admissible paths in G is the following: a path $c : [0, T] \to G$ is admissible, starting with a hyperbolic piece, if there exist $t_0 = 0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = T$ and $z_1, z_3, \ldots, z_k \in \mathfrak{a}$ of norm 1 (where k is the biggest odd integer $< n$) such that for any $0 \leq i < n$, for any $t \in [0, t_{i+1} - t_i]$,

- *if i is even then $c(t_i + t) = c(t_i) \cdot a'_t$,*
- *if i is odd then $c(t_i + t) = c(t_i) \cdot \exp(tz_i)$.*

The following is fairly immediate from the definition of the construction of the characteristic surface of a Hitchin grafting representation ρ_z and the map $\tilde{Q}_z$ from Proposition 3.5. In its formulation, the *collar size* of a simple closed multi-geodesic $\gamma^* \subset S$ is the supremum of all numbers $r > 0$ such that the tubular neighborhood of radius r about γ^* is a union of annuli about the components of γ^*. By hyperbolic geometry, an upper bound on the length of the components of γ^* yields a lower bound on the collar size of γ^*.

Observation 3.9. *Consider a closed hyperbolic surface S, a multicurve $\gamma^* \subset S$ with collar size ω and a grafting parameter z such that all cylinder heights are at least L. Then the image under the grafting map $\tilde{Q}_z$ of any admissible path of $\tilde{S}_z$ is a (ω, L)-admissible path of $\mathbb{X}$.*

Remark 3.10. We define admissible loops *of a quotient of $\mathbb{X}$ as quotients of periodic infinite admissible paths.*

In the next section we recall the notion of positivity in G in the language of Lusztig [Lus94] and some basic results, and see that admissible paths have interesting positive properties, coming from the fact that $(a'_t)_{t>0}$ are totally positive matrices, and $(\exp(z))_{z\in\mathfrak{a}}$ are totally nonnegative matrices.

4 A Morse-type-lemma in the symmetric space

The goal of this section is to establish a Morse-type lemma in the symmetric space. Recall that in δ-hyperbolic geodesic metric spaces, the Morse lemma says that any (λ, C)-quasi-geodesic is at distance at most C' from a geodesic, where C' depends on δ, λ, C. In $\mathbb{R}^2$, equipped with any norm, this lemma does not hold true anymore for all quasi-geodesics, and the same can be said for the higher-rank symmetric spaces endowed with a Finsler or Riemannian metric, since they contain totally geodesic copies of $\mathbb{R}^2$. Recall that a path $c(t)$ in a metric space is a (λ, C)-quasi-geodesic if for all times t, s we have

$$\lambda^{-1}|t-s| - C \leq d(c(t), c(s)) \leq \lambda|t-s| + C.$$

Kapovich–Leeb–Porti [KLP18] (see also Section 12.1 of [KL18]) proved a Morse lemma for certain families of well-behaved quasi-geodesics in symmetric spaces of arbitrary rank, equipped with the standard Riemannian metric, and in Euclidean buildings, see also Section 7 of [BPS19]. There is also a version of the Morse Lemma for quasi-flats instead of quasi-geodesics, see [KL97; EF97]. We propose here a different approach to a generalization of the Morse lemma: we prove that nearby every *Finsler* $(1, C)$-quasi-geodesic, so with *multiplicative* error term of 1, there is at least one Finsler geodesic. Other Finsler geodesic could be far, as in $(\mathbb{X}, d^{\mathfrak{F}})$ there are Finsler geodesics with the same endpoints and arbitrarily large Hausdorff distance.

We first present our result using the notion of quasi-ruled paths as in [BHM11], and then translate it in terms of $(1, C)$-quasi-geodesics. A *C-quasi-ruled* path $c : [0, T] \to X$, where (X, d_X) is a metric space, is a path such that for any $0 \leq t \leq s \leq u \leq T$,

$$d_X(c(t), c(s)) + d_X(c(s), c(u)) \leq d_X(c(t), c(u)) + C.$$

Note that any reparameterization of a quasi-ruled path is quasi-ruled.

Theorem 4.1. *For any $C > 0$ there exists $C' > 0$ such that any Finsler C-quasi-ruled continuous path $c : [0, T] \to \mathbb{X}$ is at Hausdorff distance at most C' from a Finsler geodesic in $(\mathbb{X}, d^{\mathfrak{F}})$ connecting $c(0)$ to $c(T)$.*

One can translate the above theorem in terms of $(1, C)$-quasi-geodesic paths, using the following lemma, which is probably well-known to experts. We provide a proof in Subsection 4.1.

Lemma 4.2. *Let (X, d) be a geodesic metric space and $C \geq 0$. Any $(1, C)$-quasi-geodesic in X is $3C$-quasi-ruled and is at Hausdorff distance at most $1 + C$ from a continuous $6(1 + C)$-quasi-ruled $(1, 2(1 + C))$-quasi-geodesic with the same endpoints.*

Conversely, any continuous C-quasi-ruled path is at Hausdorff distance at most $3C$ from a $(1, 3C)$-quasi-geodesic.

Note that for $C = 0$ this lemma says that any $(1, 0)$-quasi-geodesic is a (continuous, 0-quasi-ruled) geodesic, and that for any continuous 0-quasi-ruled path $c : [0, T] \to X$ from x to y there exists a geodesic $c' : [0, d(x, y)] \to X$ from x to y whose image is exactly the same as that of c.

Remark . *Theorem 4.1 is false for the Euclidean metric on $\mathbb{R}^d$ $(d \geq 2)$, and for the Riemannian metric on $\mathbb{X}$, as can be seen as follows.*

Let $\ell : \mathbb{R} \to \mathbb{R}^d$ be a line through $\ell(0) = 0$ parameterised by arc length for the Euclidean metric. For $n \geq 1$ put $x_n = \ell(-n), y_n = \ell(n)$. Consider the balls B_n^-, B_n^+ of radius n about x_n, y_n. As $n \to \infty$, the boundaries $\partial B_n^\pm$ of the balls $B_n^\pm$ converge in the pointed Gromov Hausdorff topology of $(\mathbb{R}^d, 0)$ to the hyperplane through 0 orthogonal to ℓ. Thus for any $m > 0$ and sufficiently large n, there are points $z_n^\pm$ on $\partial B_n^\pm$ of distance m to ℓ with $d(z_n^-, z_n^+) \leq 1$ (here d is the euclidean distance). Since the subsegment of ℓ connecting x_n to y_n is the unique euclidean geodesic between these points, the piecewise geodesics connecting x_n to y_n with breakpoints at z_n^-, z_n^+ violate the conclusion of Theorem 4.1.

This section is subdivided into four subsections. The first subsection is very short and provides a proof of Lemma 4.2 for the reader's convenience. In the second subsection, which is the longest, we establish Theorem 4.1 for polyhedral norms on $\mathbb{R}^d$, that is, norms whose norm one ball is a finite sided symmetric convex polyhedron. In the third subsection, we prove that any quasi-ruled continuous path in the symmetric space lies near a flat, using a description of Finsler horoballs of Kapovich–Leeb [KL18]. Finally the last section, which is very brief, contains the proof of Theorem 4.1.

4.1 Proof of Lemma 4.2

Let (X, d) be a geodesic metric space. Then for any $(1, C)$-quasi-geodesic $c : [0, T] \to X$ the following holds true.

- c is $3C$-quasi-ruled.
- c is at distance $1 + C$ from a continuous $(1, 2 + 2C)$-quasi-geodesic with the same endpoints.

The first property is an elementary computation which is left to the reader, and the second property is Lemma 1.11 in Chapter III.H of [BH99]. The two properties together yields the first part of the lemma.

Let us prove the second part of the lemma. For an arbitrary $C \geq 0$ consider a continuous C-quasi-ruled path $c : [0, T] \to X$ from x to y.

Let us use an idea we found in the arXiv version of [BHM11], Lemma A.2. Observe that the map

$$f : c[0, T] \to [0, d(x, y) + C]; \quad z \mapsto d(x, z)$$

is continuous and is a $(1, C)$-quasi-isometry. Indeed, if $0 \leq t \leq s \leq T$ then

$$|f(c(t)) - f(c(s))| = |d(x, c(t)) - d(x, c(s))| \leq d(c(t), c(s)),$$

and

$$d(c(t), c(s)) \leq d(x, c(s)) - d(x, c(t)) + C \leq |f(c(t)) - f(c(s))| + C.$$

Moreover, since $c[0, T]$ is path-connected, and f is continuous and attains the values 0 and $d(x, y)$, by the Intermediate Value Theorem f attains all values in $[0, d(x, y)]$ and hence f is C-quasi-surjective. By a classical result from coarse geometry f admits a $(1, 3C)$-quasi-inverse $g : [0, d(x, y)] \to c[0, T] \subset X$.

4.2 A Morse type lemma for euclidean space

This subsection is entirely devoted to the study of the geometry of $\mathbb{R}^d$, equipped with a Finsler metric defined by a translation invariant norm on $T\mathbb{R}^d$. We begin with defining the Finsler metrics we are interested in. To this end call a cone in $\mathbb{R}^d$ *properly convex* if it is convex and its closure does not contain any affine subspace of $\mathbb{R}^d$ of dimension at least 1.

A (symmetric) polyhedral norm $|\cdot|$ on $\mathbb{R}^d$ is a norm of the form

$$|v| = \max\{\alpha(v) : \alpha \in \mathcal{A}\},$$

where $\mathcal{A}$ is a finite set of nonzero linear forms which spans $(\mathbb{R}^d)^*$, and which is symmetric in the sense that $-\mathcal{A} = \mathcal{A}$. This norm induces a metric $d(x, y) = |x - y|$ on $\mathbb{R}^d$ that is invariant under translations. The goal of this section is to show.

Proposition 4.3. *For any polyhedral norm $|\cdot|$ on $\mathbb{R}^d$, there exists $\mu > 0$ such that for any $C \geq 1$, any C-quasi-ruled continuous path $c : [0, T] \to \mathbb{R}^d$ is at Hausdorff distance at most μC from a geodesic in $(\mathbb{R}^d, |\cdot|)$ connecting $c(0)$ to $c(T)$.*

Note that this statement is false for a Euclidean norm on $\mathbb{R}^d$.

Proposition 4.3 has the following reformulation in terms of $(1, C)$-quasi-geodesics, thanks to Lemma 4.2.

Corollary 4.4. *For any polyhedral norm $|\cdot|$ on $\mathbb{R}^d$, there exists $\mu > 0$ such that for any $C \geq 1$, any $(1, C)$-quasi-geodesic $c : [0, T] \to \mathbb{R}^d$ is at Hausdorff distance at most μC from a geodesic in $(\mathbb{R}^d, |\cdot|)$ from $c(0)$ to $c(T)$.*

4.2.1 Diamonds

In this section we introduce several geometric objects relative to our polyhedral norm, including diamonds. We prove that $(1, C)$-quasi-geodesics stay at bounded distance from diamonds, which is the main technical step towards the proof of Proposition 4.3.

For any $\alpha \in \mathcal{A}$, the set $\mathcal{C}_\alpha = \{v \in \mathbb{R}^d : |v| = \alpha(v)\}$ is a polyhedral convex cone based at 0. Note that $\mathcal{C}_{-\alpha} = -\mathcal{C}_\alpha$. Up to removing unnecessary elements of $\mathcal{A}$, we may assume that $\mathcal{C}_\alpha$ has nonempty interior. We call *special cones* (based at $x \in \mathbb{R}^d$) the cones of $\mathbb{R}^d$ that are translates of a cone $\mathcal{C}_\alpha$ (by the translation $y \mapsto y + x$).

The unit closed ball $\bar{B}(0, 1)$, and more generally any closed ball $\bar{B}(x, r)$ for such a norm, is a polyhedral convex set, that is, a finite intersection of (affine) half-spaces of $\mathbb{R}^d$. More precisely,

$$\bar{B}(x, r) = \bigcap_{\alpha \in \mathcal{A}} \{y \in \mathbb{R}^d : \alpha(y - x) \leq r\} = x + r \cdot \bar{B}(0, 1).$$

The codimension-1 faces of $\bar{B}(x, r)$ are the intersections of its boundary $\partial B(x, r)$ with the special cones based at x.

Definition 4.5. Denote by $\mathcal{C}(x \to y)$ the intersection of all special cones based at x that contain y. We define the *diamond* of the pair x, y to be $D(x,y) = \mathcal{C}(x \to y) \cap \mathcal{C}(y \to x)$ (see Figures 3 and 4 for illustrations).

Note that $\mathcal{C}(y \to x) = y - x - \mathcal{C}(x \to y)$. This follows from the fact that for any $\alpha \in \mathcal{A}$, the special cone $x + \mathcal{C}_\alpha$ based at x contains y if and only if the special cone $y + \mathcal{C}_{-\alpha}$ based at y contains x.

Lemma 4.6. *For any $x, y \in \mathbb{R}^d$, we have*

$$\begin{aligned} D(x,y) &= \{z \in \mathbb{R}^d \mid d(x,z) + d(z,y) = d(x,y)\} \\ &= \cup\{\textit{geodesics from } x \textit{ to } y\} \end{aligned}$$

In particular, for any $z \in D(x,y)$, the concatenation of a geodesic from x to z with a geodesic from z to y is a geodesic from x to y.

Given a cone $\mathcal{C} = \mathcal{C}(0 \to y)$, set $\alpha_{\mathcal{C}}$ to be the mean of all $\alpha \in \mathcal{A}$ for which $\mathcal{C}_\alpha$ contains $\mathcal{C}$. It follows from the definition that $|z| \geq \alpha_{\mathcal{C}}(z)$ holds for all $z \in \mathbb{R}^d$, with equality exactly on $\mathcal{C}$.

Proof. For a cone $\mathcal{C} = \mathcal{C}(x \to y)$, consider the form $\alpha_{\mathcal{C}}$ defined above. The point z belongs to $D(x,y)$ if and only if $|z - x| = \alpha_{\mathcal{C}(x\to y)}(z-x)$ and $|z-y| = \alpha_{\mathcal{C}(y \to x)}(z-y) = -\alpha_{\mathcal{C}(x\to y)}(z-y)$. This implies

$$d(x,y) \leq d(x,z) + d(z,y) = \alpha_{\mathcal{C}(x\to y)}(z-x) + \alpha_{\mathcal{C}(x\to y)}(y-z) = \alpha_{\mathcal{C}(x\to y)}(y-x) = d(x,y)$$

and so $d(x,y) = d(z,x) + d(z,y)$.

Conversely if $|z-x| > \alpha_{\mathcal{C}(x\to y)}(z-x)$ or $|z-y| > -\alpha_{\mathcal{C}(x\to y)}(y-z)$, then the above inequality yields $d(z,x) + d(z,y) > d(x,y)$.

It is clear that if z lies on a geodesic from x to y, then $d(x,z) + d(z,y) = d(x,y)$. Reciprocally, if $d(x,z) + d(z,y) = d(x,y)$ then the concatenation of any geodesic from x to z and any geodesic from z to y is a geodesic from x to y. □

From the previous lemma and the triangle inequality we infer that for any point z not too far from a diamond $D(x,y)$ we almost have the triangle equality $d(x,z) + d(z,y) \simeq d(x,y)$. The following lemma is the key technical result towards the proof of Proposiion 4.3; it says that the converse also holds.

Lemma 4.7. *There is $\lambda_1 > 0$ such that for all $x, y, z \in \mathbb{R}^d$ it holds*

$$d(z, D(x,y)) \leq \lambda_1 (d(x,z) + d(z,y) - d(x,y)).$$

The two terms are equal to zero when z belongs to $D(x,y)$. One can think of the lemma in the following way: $z \mapsto f_{x,y}(z) = d(x,z) + d(z,y) - d(x,y)$ is convex, non-negative, and piecewise affine. Take a point $z \in \partial D(x,y)$ and follow a ray $\{z + tv, t \geq 0\}$ for a unit vector v for $|\cdot|$ at z whose euclidean angle (for some fixed euclidean inner product) to $D(x,y)$ is at least $\pi/2$. By this we mean the angle between v and any line segment in $D(x, y)$ starting at z. The restriction of $f_{x,y}$ to the ray is convex, piecewise affine and is equal to zero exactly at z. It follows that it grows at least linearly in t, the slope being given by the derivative at $t = 0$. And so for $z' = z + tv$, one has $f_{x,y}(z') \geq t \cdot f'_{x,y}(0) \geq \mathrm{Cst} f'_{x,y}(0) \cdot d(z', D(x, y))$.

The issue is that the slope does not vary continuously in z, not even lower semi-continuously, so one can not hope to use a compactness argument to obtain a uniform bound on the union of the rays. One might study carefully the combinatorics of the map f to obtain a uniform bound on the slope. We instead take a slightly different approach, which requires one intermediate lemma.

Let us fix an Euclidean inner product $\langle, \rangle$ defining the Euclidean metric d_{eucl} on $\mathbb{R}^d$. By "orthogonal projection" to a closed convex set C we will mean closest-point projection for d_{eucl} to C, which is well defined by convexity of d_{eucl}.

Given a cone $\mathcal{C} \subset \mathbb{R}^d$ based at 0, define the dual cone of $\mathcal{C}$ to be the set

$$\mathcal{C}' = \{x \in \mathbb{R}^d, \langle x, \mathcal{C} \rangle \leq 0\} = \{x \in \mathbb{R}^d \text{ whose orthogonal projection to } \mathcal{C} \text{ is } 0\}.$$

Lemma 4.8. *Let $\mathcal{C}$ be a polyhedral convex cone of $\mathbb{R}^d$ based at 0, that is, the intersection of finitely many closed half-spaces $H_1, \ldots, H_n$ containing 0 in their boundary. Let $\mathcal{C}' \subset \mathbb{R}^d$ be the polyhedral convex dual cone to $\mathcal{C}$. Then there exists $\lambda > 0$ such that for any $x \in \mathcal{C}'$,*

$$d_{\mathrm{eucl}}(x, \mathcal{C}) \leq \lambda \max \left(d_{\mathrm{eucl}}(x, H_1), \ldots, d_{\mathrm{eucl}}(x, H_n)\right).$$

The results holds true for all $x \in \mathbb{R}^d$ (for a bigger constant λ), but the special case $x \in \mathcal{C}'$ is shorter to prove.

Proof. This is an immediate consequence of the fact that the function

$$f(x) = \max \left(d_{\mathrm{eucl}}(x, H_1), \ldots, d_{\mathrm{eucl}}(x, H_n)\right)$$

is homogeneous, continuous, and positive on $\mathcal{C}' - \{0\}$. □

Note that in the previous lemma we allow $\mathcal{C}$ to have empty interior, or be reduced to $\{0\}$, or to be the entire space $\mathbb{R}^d$ (but this last case is not very interesting since then the dual $\mathcal{C}'$ is just $\{0\}$).

Proof of Lemma 4.7. Denote by $\{H_\alpha, \alpha \in \mathcal{A}\}$ the finite family of closed half spaces given by $H_\alpha = \{w \in \mathbb{R}^d, \alpha(w) \leq 0\}$.

For any subset S of $\mathcal{A}$, the intersection

$$\mathcal{C}_S = \cap_{\alpha \in S} H_\alpha$$

is a polyhedral convex cone. Let K_S be the dual cone to $\mathcal{C}_S$. We can apply Lemma 4.8 to K_S, and get a number $\lambda_S > 0$. Let $\lambda = \max\{\lambda_S \mid S \subset \mathcal{A}\}$.

We now prove the inequality for $x, y, z \in \mathbb{R}^d$ fixed. When z belongs to $D(x, y)$, we have $d(x, z) + d(z, y) - d(x, y) = 0 = d(z, D(x, y))$ by Lemma 4.6 and so the inequality holds.

Suppose that $z \notin D(x,y)$. If $d(x,z) \geq d(x,y)$ holds, then one has

$$d(x,z) + d(z,y) - d(x,y) \geq d(z,y) \geq d(z, D(x,y))$$

since $y \in D(x,y)$. So by symmetry in x, y we may assume that $r := d(x,z)$ is smaller than $R := d(x,y)$.

Let $B_x = \bar{B}(x,r)$ and $B_y = \bar{B}(y, R-r)$ be closed balls for the polyhedral norm $|\cdot|$, illustrated in Figure 3. They are polyhedral convex sets, i.e. finite intersection of affine half-spaces. More precisely

$$B_x = H_1 \cap \cdots \cap H_n \quad \text{and} \quad B_y = H'_1 \cap \cdots \cap H'_n$$

where $H_i = \{w \in \mathbb{R}^d, \alpha_i(w-x) \leq r\}$ and $H'_j = \{w \in \mathbb{R}^d, \alpha'_j(w-y) \leq R-r\}$ for some orderings $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\} = \{\alpha'_1, \ldots, \alpha'_n\}$ (it will be convenient later to have two different orderings).

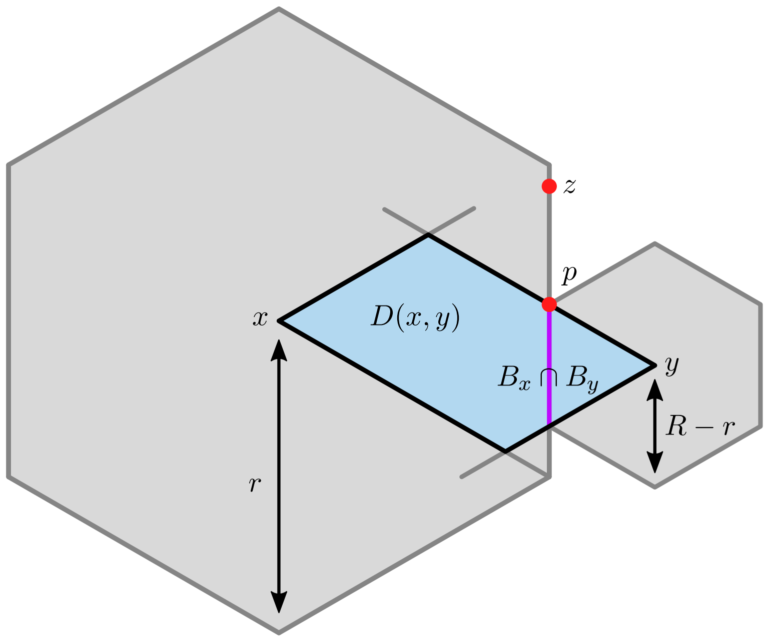

Figure 3: Illustration of the proof of Lemma 4.7. Three points x, y, z and a fourth point $p \in D(x,y)$ with $d(z,p) \leq \lambda(d(x,z) + d(y,z) - d(x,y))$.

The intersection

$$B_x \cap B_y = H_1 \cap \cdots \cap H_n \cap H'_1 \cap \cdots \cap H'_n$$

is a closed polyhedral convex subset of the diamond $D(x,y)$ by Lemma 4.6, with empty interior, and it is not empty (one can verify that $B_x \cap B_y$ contains the point $\frac{r}{R}y + \frac{R-r}{R}x$).

Let p be the Euclidean closest-point projection of z to $B_x \cap B_y$. Up to translation, we may assume that $p = 0$ to be able to use Lemma 4.8. Up to reordering we may also assume that the half-spaces containing p in their boundary are $H_1, \dots, H_k$ and $H'_1, \dots, H'_\ell$; note that k and ℓ are both positive since $p \in \partial B_x \cap \partial B_y$. Since $p = 0$ we have $H_i = \{w : \alpha_i(w) \le 0\}$ for $i \le k$ and $H'_j = \{w : \alpha'_j(w) \le 0\}$ for $j \le l$. Let $S = \{\alpha_i, i \le k\} \cup \{\alpha'_j, j \le \ell\} \subset \mathcal{A}$. Then using the notation introduced at the beginning of the proof we have

$$B_x \cap B_y \subset H_1 \cap \cdots \cap H_k \cap H'_1 \cap \cdots \cap H'_\ell = \mathcal{C}_S.$$

Observe that p is also the Euclidean closest-point projection of z to $\mathcal{C}_S$. Indeed if by contradiction there was $p' \in \mathcal{C}_S$ closer to z, then by convexity of the euclidean distance function, any point of the line segment $(p, p']$ would be closer to z. But any point of $(p, p']$ close enough to p is contained in each $H_{k+1}, \dots, H_n$ and $H'_{\ell+1}, \dots, H'_n$ (since p is in their interior), and hence is in $B_x \cap B_y$, which contradicts that p is the Euclidean closest-point projection of z on $B_x \cap B_y$. In particular, z is contained in the dual cone K_S to $\mathcal{C}_S$.

By Lemma 4.8, the distance $d(z, p)$ is comparable to the distance between z and one of the half spaces $H_1, \dots, H_k, H'_1, \dots, H'_\ell$. But since $z \in B_x$, it must be contained in every H_i, so Lemma 4.8 implies that there exists a half space H'_j, $j \in \{1, \dots, \ell\}$, for which

$$\begin{aligned} d_{\mathrm{eucl}}(z, D(x, y)) &\le d_{\mathrm{eucl}}(z, p) \\ &\le \lambda_S d_{\mathrm{eucl}}(z, H'_j) \\ &\le \lambda d_{\mathrm{eucl}}(z, B_y). \end{aligned}$$

Let us translate what this means for the polyhedral norm, using a constant ν such that $\nu^{-1} d \le d_{\mathrm{eucl}} \le \nu d$. The previous equation yields

$$d(z, D(x, y)) \le \nu^2 \lambda d(z, B_y).$$

Let q be the intersection point of $[y, z]$ with ∂B_y, which satisfies $d(z, q) = d(z, B_y)$. Indeed for any $q' \in \partial B_y$ we have $d(q', y) = d(q, y)$, and so

$$d(z, q') = d(z, q') + d(q', y) - d(q', y) \ge d(z, y) - d(q, y) = d(z, q) = R - r$$

since $[y, z]$ is a geodesic for d. Then

$$\begin{aligned} d(x, z) + d(z, y) - d(x, y) &= r + d(z, y) - R \\ &= d(z, q) + d(q, y) + r - R \\ &= d(z, q) = d(z, B_y) \\ &\ge \nu^{-2} \lambda^{-1} d(z, D(x, y)). \end{aligned}$$

□

4.2.2 Proof of Proposition 4.3

In addition to diamonds we shall consider two other geometric objects: crowns and cores of diamonds. We refer to [KLP18] for closely related constructions and to Figure 4 for an illustration.

- $x, y \in \mathbb{R}^d$ are called *generic* if the diamond between them has nonempty interior, that is, if there is only one $\alpha \in \mathcal{A}$ such that $|x - y| = \alpha(x - y)$.
- The *Crown* $\mathrm{Cr}(x, y)$ is $\partial\mathcal{C}(x \to y) \cap \partial\mathcal{C}(y \to x)$ if x, y are generic, and otherwise it is just $D(x, y)$. Note that if $z \in \mathrm{Cr}(x, y)$ then the pair (x, z) is *not* generic.
- The *Core* $\mathrm{Co}(x, y)$ is the convex hull of the crown, which is just $D(x, y)$ if x and y are not generic.

Note that for generic x, y, the core $\mathrm{Co}(x, y)$ separates x from y in $D(x, y)$, in the sense that x and y are in different connected components of $D(x, y) \smallsetminus \mathrm{Co}(x, y)$.

Note also that for generic x, y the intersection of the core $\mathrm{Co}(x, y)$ with the boundary $\partial D(x, y)$ of the diamond is exactly the crown $\mathrm{Cr}(x, y)$.

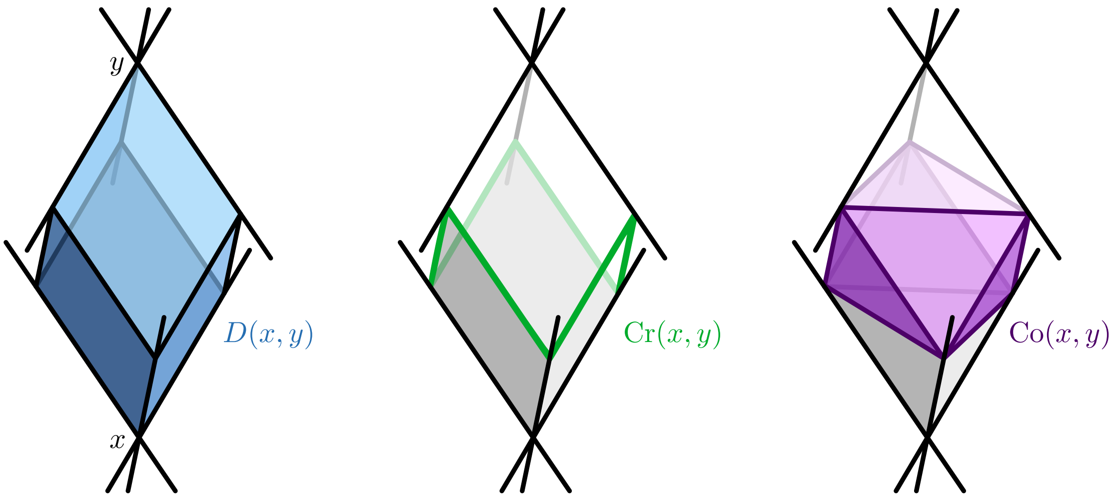

Figure 4: Illustration of the diamond, crown and core of two points $x, y \in \mathbb{R}^d$ in generic position.

We will need the following elementary result on properly convex cones, which implies that if a tip of a diamond is not too close to the crown then it is not too close to the core.

Given a set $K \subset \mathbb{R}^d$, denote by $\mathrm{Conv}(K)$ the convex hull of K.

Lemma 4.9. *Let $\mathcal{C} \subset \mathbb{R}^r$ be a closed properly convex cone with vertex 0. Then there exists $\lambda_2 > 0$ such that for any compact set $K \subset \mathcal{C}$, we have*

$$d(0, K) \le \lambda_2 d(0, \mathrm{Conv}(K)).$$

Proof. Since $\mathcal{C}$ is properly convex, 0 is an extremal point of it and does not belong to the convex hull $\mathrm{Conv}(\mathcal{C} - B(0, 1))$ where $B(0, 1)$ is the open ball of radius one around 0 for the metric d. Let $\lambda^{-1} = d(0, \mathrm{Conv}(\mathcal{C} - B(0, 1)))$ be the distance from 0 to $\mathrm{Conv}(\mathcal{C} - B(0, 1))$.

Let $K \subset \mathcal{C}$ be a compact subset. If $0 \in K$ then $d(0, K) = 0 \le \lambda d(0, \mathrm{Conv}(K))$.

Suppose $0 \notin K$ and and put $a = d(0,K)^{-1}$. Then the compact $a \cdot K$ is included in $\mathcal{C} - B(0,1)$ and so $\mathrm{Conv}(a \cdot K) \subset \mathrm{Conv}(\mathcal{C} - B(0,1))$, which yields $d(0, \mathrm{Conv}(a \cdot K)) \geq \lambda^{-1}$. As $\mathrm{Conv}(a \cdot K) = a \cdot \mathrm{Conv}(K)$, one has

$$d(0, \mathrm{Conv}(K)) = d(0,K) \cdot d(0, a \cdot \mathrm{Conv}(K)) \geq \lambda^{-1} d(0,K). \qquad \square$$

Finally we will need the following observation about quasi-ruled paths, whose proof is a simple calculation.

Observation 4.10. *Let $a, b : [0,T] \to X$ and $c : [T, T'] \to X$ be paths in a metric space.*

1. *If a is C-quasi-ruled and $d(a(t), b(t)) \leq C'$ for all t then b is $C + 6C'$-quasi-ruled.*
2. *If a is C-quasi-ruled and if $d(c(t), a(T)) \leq C'$ for all t then the concatenation of the path a with c is $C + 2C'$-quasi-ruled.*

Proof of Proposition 4.3. We proceed by induction on the dimension d. In the case $d = 1$ there is nothing to show, so assume that for some $d \geq 2$, the claim holds true for all dimensions $< d$, with a constant depending on the dimension and on the polyhedral norm.

Let $|\cdot|$ be a polyhedral norm on $\mathbb{R}^d$. Note that the restriction of $|\cdot|$ to any linear subspace is polyhedral. Thus by the induction assumption, there exists a constant $\mu > 1$ so that the proposition is valid with this constant for paths contained in the linear subspaces $\{\alpha_{i_1} = \cdots = \alpha_{i_k} \mid \alpha_{i_j} \in \mathcal{A}\}$ which are the linear spans of the faces of the special cones $\mathcal{C}_\alpha$ for $\alpha \in \mathcal{A}$.

Let $\lambda_1 > 0$ be as in Lemma 4.7, $\nu > 1$ be such that $\nu^{-1} d \leq d_{\mathrm{eucl}} \leq \nu d$ where $d(x,y) = |x - y|$, and $\lambda_1' = \nu^2 \lambda_1$. Then for all $x, y \in \mathbb{R}^d$, if

$$\Pi_{xy} : \mathbb{R}^d \to D(x,y)$$

is the Euclidean closest-point projection onto $D(x,y)$ (which is well defined continuous since $D(x,y)$ is compact and convex, contrarily to the closest-point projection for d), then by Lemma 4.7 we get

$$d(z, \Pi_{xy}(z)) \leq \nu d_{\mathrm{eucl}}(z, \Pi_{xy}(z)) = \nu d_{\mathrm{eucl}}(z, D(x,y)) \leq \lambda_1'(d(x,z) + d(z,y) - d(x,y)). \tag{20}$$

Let $\lambda_2 > 0$ be the maximum of the constants from Lemma 4.9, applied to the special cones $\mathcal{C}_\alpha$, $\alpha \in \mathcal{A}$. Then for all $x, y \in \mathbb{R}^d$ we have

$$d(x, \mathrm{Cr}(x,y)) \leq \lambda_2 d(x, \mathrm{Co}(x,y)). \tag{21}$$

We claim that the statement of the proposition holds true for $(\mathbb{R}^d, |\cdot|)$ with the constant $\mu' = (1 + 2\lambda)(\mu(1 + 6\lambda_1') + \lambda_1')$, where $\lambda = \max(\lambda_1', \lambda_2)$.

To this end we proceed by induction on k where $d(x,y) \in (k-1, k]$. Note first that by Lemma 4.2, by adjusting the constant μ' we may assume that all considered quasi-ruled paths are continuous.

Let $C \geq 1, k = 1$ and let c be a C-quasi-ruled continuous path from x to y such that $d(x,y) \leq 1$. Then the segment $[x,y]$ is at distance at most $C + 1$ from c and hence the claim holds true in this case.

Let $k \geq 2$ and assume that the claim holds true for all continuous C-quasi-ruled paths from x to y such that $d(x, y) \leq k - 1$. Let c be a C-quasi-ruled continuous path from x to y such that $d(x, y) \in (k - 1, k]$.

There are two possible cases. In the first case, c stays $C\lambda$-far from the crown $\mathrm{Cr}(x, y)$. The projection $\Pi_{xy} \circ c$ is a continuous path from x to y in $D(x, y)$, so it must cross the core $\mathrm{Co}(x, y)$ at some time t.

Note that we have $\Pi_{xy}(c(t)) = c(t)$. Namely, otherwise $\Pi_{xy}(c(t))$ is contained in the boundary $\partial D(x, y)$, and hence contained in the crown. But since $\Pi_{xy}(c(t))$ is $C\lambda_1'$-close to $c(t)$ by Inequality (20), then $c(t)$ is $C\lambda_1'$-close and hence $C\lambda$-close to the crown, which contradicts our assumption.

Moreover, we must also have the inequalities

$$d(x, c(t)) \geq d(x, \mathrm{Co}(x, y)) \geq 1 \text{ and } d(y, c(t)) \geq 1.$$

Otherwise by Equation (21), it holds $d(x, \mathrm{Cr}(x, y)) \leq \lambda_2 \leq \lambda \leq C\lambda$, which contradicts our assumption that c stays $C\lambda$-away from the crown. Lemma 4.6 yields that

$$d(x, c(t)) = d(x, y) - d(y, c(t)) \leq k - 1$$

and similarly $d(y, c(t)) \leq k - 1$. We can apply the induction hypothesis (on k) to the path $c[0, t]$ and the path $c[t, T]$. As the concatenation of a geodesic connecting x to $c(t) \in D(x, y)$ and $c(t)$ to y is a geodesic, this suffices for the induction step.

In the second case, c passes at some time t at distance less than $C\lambda$ from the crown. Let $p \in \mathrm{Cr}(x, y)$ be such that $d(p, c(t)) \leq C\lambda$. Recall that this means the pairs (x, p) and (y, p) are not generic.

Concatenate $c_1 = c[0, t]$ with a geodesic from $c(t)$ to p, to get a continuous $(1 + 2\lambda)C$-quasi-ruled path c_1' from x to p by Observation 4.10. By Inequality (20), the projection $\Pi_{xp} \circ c_1'$ is at distance at most $\lambda_1'(1 + 2\lambda)C$ from c_1. Using again Observation 4.10, it is a continuous $(1 + 2\lambda)C(1 + 6\lambda_1')$-quasi-ruled path in $D(x, p)$ from x to p.

As the pair (x, p) is not generic, we can apply our induction on the dimension and deduce that there is a geodesic $c_1'' \subset D(x, p)$ at distance at most $\mu C(1 + 2\lambda)(1 + 6\lambda_1')$ from $\Pi_{xp} \circ c_1'$, which is then at distance at most $C(1 + 2\lambda)(\mu(1 + 6\lambda_1') + \lambda_1')$ from the original path c.

With a similar construction for $c_2 = c[t, T]$, one obtains a geodesic c_2'' from p to y which is at distance at most $C(1 + 2\lambda)(\mu(1 + 6\lambda_1') + \lambda_1')$ from c.

The concatenation of c_1'' and c_2'' is by Lemma 4.6 a geodesic from x to y at distance at most $C(1 + 2\lambda)(\mu(1 + 6\lambda_1') + \lambda_1')$ from c, which concludes the proof. □

4.3 Projecting to a flat

In this subsection we extend Proposition 4.3 to the symmetric space $\mathbb{X} = \mathrm{PSL}_d(\mathbb{R})/\mathrm{PSO}(d)$ equipped with the Finsler metric $d^{\mathfrak{F}}$. We begin with extending the geometric notions from Section 4.2 to $\mathbb{X}$.

Recall, for instance from [KL18], that the *diamond* between two points $x, y \in \mathbb{X}$ is defined as follows. A *Weyl cone* of a flat of $\mathbb{X}$ is the translation under an element of

$\mathrm{PSL}_d(\mathbb{R})$ of the standard Weyl cone $\exp \mathfrak{a}^+ \subset \exp \mathfrak{a}$ based at the basepoint $\mathbf{x}$ of the standard flat $\exp \mathfrak{a}$. Consider a flat F containing x and y, a Weyl cone $W \subset F$ based at x and containing y, and the opposite Weyl cone W' based at y (which automatically contains x). Then the *diamond* $D(x,y)$ is defined as

$$D(x,y) = W \cap W'.$$

It does not depend on choices, and as an intersection of convex subsets of $\mathbb{X}$, it is convex. The analog of Lemma 4.6 holds true.

Proposition 4.11 (Lemma 5.10 of [KL18]). *For all $x, y \in \mathbb{X}$, the diamond $D(x,y)$ is the set of points $z \in \mathbb{X}$ such that $d^{\mathfrak{F}}(x,z) + d^{\mathfrak{F}}(z,y) = d^{\mathfrak{F}}(x,y)$.*

The following non-uniform version of Lemma 4.7 for $(\mathbb{X}, d^{\mathfrak{F}})$ is used to reduce Theorem 4.1 to Proposition 4.3.

Proposition 4.12. *For any $C \geq 1$ there exists $C' > 0$ such that for all $x, y \in \mathbb{X}$, any $z \in \mathbb{X}$ such that $d^{\mathfrak{F}}(x,z) + d^{\mathfrak{F}}(z,y) \leq d^{\mathfrak{F}}(x,y) + C$ is at distance at most C' from $D(x,y)$.*

Proof. Our goal is to reduce the proof of the proposition to Lemma 4.7. To this end denote for $x \neq y \in \mathbb{X}$ by $F(x,y)$ the intersection of all maximal flats containing the diamond $D(x,y)$. Then $F(x,y)$ is a (not necessarily maximal) flat in $\mathbb{X}$ containing $D(x,y)$. We claim that for any $C > 0$ there exists a $C' > 0$ with the following property. If $x, y, z \in \mathbb{X}$ are such that $d^{\mathfrak{F}}(x,z) + d^{\mathfrak{F}}(z,y) \leq d^{\mathfrak{F}}(x,y) + C$, then $d^{\mathfrak{F}}(z, F(x,y)) \leq C'$.

Assuming this claim, let $x, y, z \in \mathbb{X}$ be such that $d^{\mathfrak{F}}(x,z) + d^{\mathfrak{F}}(z,y) \leq d^{\mathfrak{F}}(x,y) + C$. Choose a point $z' \in F(x,y)$ of shortest distance to z. Then we have

$$d^{\mathfrak{F}}(x,z') + d^{\mathfrak{F}}(z',y) \leq d^{\mathfrak{F}}(x,y) + 2C' + C$$

and hence the proposition is a consequence of Lemma 4.7, applied to $x, y, z' \subset F(x,y)$.

To show the claim we argue by contradiction and we assume that the claim does not hold true. Then there exists a number $C > 0$ and a sequence of counter examples, consisting of pairs of points (x_n, y_n) and points z_n such that

$$d^{\mathfrak{F}}(x_n, z_n) + d^{\mathfrak{F}}(z_n, y_n) \leq d^{\mathfrak{F}}(x_n, y_n) + C,$$

but $d^{\mathfrak{F}}(z_n, F(x_n, y_n)) \geq n$. In particular, we have $d^{\mathfrak{F}}(x_n, z_n) \geq n$ and $d^{\mathfrak{F}}(x_n, y_n) \geq 2n - C$. By equivariance under the action of the isometry group of $\mathbb{X}$ we may assume that $z_n = z$ is a fixed point. Furthermore, by passing to a subsequence, we may assume that the sequence x_n converges to a point $\xi \in \partial_\infty \mathbb{X}$, and the sequence y_n converges to a point $\eta \in \partial_\infty \mathbb{X}$.

Recall that the Finsler metric $d^{\mathfrak{F}}$ is equivalent to the Riemannian metric and hence complete. Thus after passing to a subsequence, we may assume that the functions $\alpha_n : u \to d^{\mathfrak{F}}(x_n, u) - d^{\mathfrak{F}}(x_n, z)$ and $\beta_n : u \to d^{\mathfrak{F}}(y_n, u) - d^{\mathfrak{F}}(y_n, z)$ converge locally uniformly to Finsler horofunctions b_ξ, b_η on $\mathbb{X}$. These functions are 1-Lipschitz for the metric $d^{\mathfrak{F}}$.

For each n consider the Riemannian geodesic γ_n connecting z to x_n, parameterized to be a geodesic for $d^{\mathfrak{F}}$. As $x_n \to \xi$, the geodesics γ_n converge locally uniformly to a geodesic ray γ_ξ, parameterized by arc length with respect to $d^{\mathfrak{F}}$, with endpoint ξ. The convergence $\gamma_n \to \gamma$ is uniform on compact sets, and the same holds true for the convergence $\alpha_n \to b_\xi$. Since furthermore we have $\alpha_n(\gamma_n(t)) = -t$ for all t, we also have $b_\xi(\gamma_\xi(t)) = -t$ for all t. Similarly, if γ_η is a limit of Riemannian geodesics η_n connecting z to y_n, parameterized by arc length with respect to $d^{\mathfrak{F}}$, then we have $b_\eta(\gamma_\eta(t)) = -t$ for all t.

On the other hand, by assumption on x_n, y_n, z_n and passing to a limit, we also have $b_\xi(\gamma_\eta(t)) \geq t - C$ and $b_\eta(\gamma_\xi(t)) \geq t - C$ for all t. Namely, for each n the function α_n is one-Lipschitz, and hence its restriction to the geodesic η_n connecting z to y_n satisfies $\alpha_n(\eta_n(t)) \geq \alpha_n(y_n) - d^{\mathfrak{F}}(z_n, y_n) + t \geq d^{\mathfrak{F}}(z_n, y_n) - C - d^{\mathfrak{F}}(z_n, y_n) + t = t - C$ and similarly $\alpha_n(\eta_n(t)) \leq t$. Consequently we have $\alpha_n(\eta_n(t)) \in [t - C, t]$. This estimate then passes on to the limit.

By the discussion in Section 1 following Section 5 of [KL18], horofunctions for the Finsler metric can be characterized as follows.

To each maximal simplex $\sigma \subset \partial_\infty \mathbb{X}$ there corresponds an interior point $\xi_\sigma \in \sigma$ so that the following holds true. If $q_n \subset \mathbb{X}$ is a sequence converging in $\mathbb{X} \cup \partial_\infty \mathbb{X}$ to an interior point of σ, then the functions $u \to d^{\mathfrak{F}}(q_n, u) - d^{\mathfrak{F}}(q_n, z)$ converge to the Riemannian Busemann function b_σ defined by ξ_σ. If $q_n \subset \mathbb{X}$ converges to a point in $\partial_\infty \mathbb{X}$ which is an interior point of a simplex τ which is not maximal, then up to passing to a subsequence and slightly modifying the sequence so that the geodesic connecting the basepoint z to q_n is regular, that is, its direction is contained in the interior of a Weyl chamber, we may assume that the functions $u \to d^{\mathfrak{F}}(q_n, u) - d^{\mathfrak{F}}(q_n, z)$ converge to the function b_σ where σ is a maximal simplex containing τ. However, any such function can arise in the limit, and the Busemann function for the Finsler metric corresponding to the simplex τ equals

$$b_\tau = \max\{b_\sigma \mid \sigma \supset \tau\}.$$

We refer to Lemma 5.16 and Lemma 5.18 of [KL18] and their proofs for this statement.

Thus let σ be a maximal simplex containing ξ so that the function b_σ arises as a limit of the functions $u \to d^{\mathfrak{F}}(x_n, u) - d^{\mathfrak{F}}(x_n, z)$. We showed that the restriction of b_σ to the geodesic ray γ_η satisfies $b_\sigma(\gamma_\eta(t)) \geq t - C$ for all t.

Let $F \subset \mathbb{X}$ be a maximal flat containing both the simplex σ as well as the endpoint η of γ_η in its boundary. Let $\nu \subset \partial_\infty F$ be the maximal simplex in the boundary $\partial_\infty F$ of F which is antipodal to σ. We claim that ν contains η in its closure.

To this end consider the restriction of b_σ to F. It is given by a linear functional on $F = \mathbb{R}^d$. For any basepoint $o \in F$, with $b_\sigma(o) = 0$, the restriction of b_σ to the closed Weyl cone based at o whose boundary $\nu \subset \partial_\infty F$ is antipodal to the cone containing σ in its boundary coincides with the linear functional defining the Finsler distance from o for points in this cone. On the other hand, for any geodesic ray $\zeta : [0, \infty) \to F$ with $\zeta(0) = o$ whose endpoint is contained in the complement of ν and parameterized by arc length for $d^{\mathfrak{F}}$, there is a number $\delta(\zeta) > 0$ such that $\limsup_{t \to \infty} \frac{1}{t} b_\sigma(\zeta(t)) \leq 1 - \delta(\zeta)$.

Now the geodesic $\gamma_\eta \subset F$ is strongly asymptotic to a geodesic ray $\zeta \subset F$ starting at o.

Thus we have

$$\limsup_{t\to\infty} \frac{1}{t} b_\sigma(\gamma_\eta(t)) = \limsup_{t\to\infty} \frac{1}{t} b_\sigma(\zeta(t)).$$

As the left hand side of this expression equals one by the above discussion, we conclude that the endpoint η of γ_η is contained in the closure of ν.

To summarize, we showed that for any maximal simplex σ which contains ξ in its closure, the point η is contained in the closure of a simplex which is antipodal to σ. But this just means that η is contained in a simplex θ which is antipodal to the simplex τ. Furthermore, the geodesic rays γ_ξ, γ_η are asymptotic to the (perhaps not maximal but unique) flat F which contains τ, θ as maximal antipodal simplices and hence they are contained in a uniformly bounded neighborhood of F. As $x_n \to \xi, y_n \to \eta$ we know that up to passing to a further subsequence, the flats $F(x_n, y_n)$ converge to a flat containing F. But then the distance between z and $F(x_n, y_n)$ is uniformly bounded which is a contradiction. This completes the proof of the proposition. □

4.4 Proof of Theorem 4.1

We are now ready for the proof of Theorem 4.1.

Consider a continuous path $c : [0, T] \to \mathbb{X}$ such that for any $0 \le t \le s \le u \le T$, we have $d^{\mathfrak{F}}(c(t), c(s)) + d^{\mathfrak{F}}(c(s), c(u)) \le d^{\mathfrak{F}}(c(t), c(u)) + C$.

Let F be a flat containing $c(0)$ and $c(T)$, so that it also contains the diamond $D(c(0), c(T))$. By Proposition 4.12 there exists $C' > 0$ only depending on C such that for any $0 \le t \le T$, there exists $a(t) \in D(c(0), c(T))$ at distance at most C' from $c(t)$, with $a(0) = c(0)$ and $a(T) = c(T)$. By the triangle inequality, the path $t \to a(t)$ is $C + 6C'$ quasi-ruled. By Lemma 4.2, up to enlarging $C + 6C'$ to a constant which also only depends on C, we may assume that $t \to a(t)$ is continuous.

Thus we can apply Proposition 4.3 to a, to find a geodesic $b : [0, T] \to F$ such that $d^{\mathfrak{F}}(a(t), b(t)) \le C''$ for some $C'' > 0$ which depends only on C. Recall that geodesics for the restricted metric on F are also geodesics for the metric on $\mathbb{X}$.

We conclude that $d^{\mathfrak{F}}(c(t), b(t)) \le C'' + C'$ for any t where $C'' + C'$ depends on C but not on the path c.

4.5 Rough Finsler convexity and naive convex cocompactness

In this section we apply Theorem 4.1 to prove that naively convex cocompact groups, in the sense of [DGK17], act cocompactly on a roughly Finsler convex subset of the symmetric space. Recall that a domain of the projective space $\Omega \subset \mathbb{RP}^{d-1}$ is called properly convex if it is bounded and convex in some affine chart.

Proposition 4.13. *Let $\Gamma \subset \mathrm{PSL}_d(\mathbb{R})$ preserve a properly convex domain $\Omega\mathbb{RP}^{d-1}$ and act naively convex cocompactly on it, in the sense that there exist a convex subset $C \subset \Omega$ on which Γ acts cocompactly. Then Γ acts cocompactly on a roughly Finsler convex subset of the symmetric space $\mathbb{X}$.*

Proof. The properly convex domain Ω can be endowed with a natural Finsler metric d_Ω called the *Hilbert metric*, which is invariant under projective transformations, and such that projective segments in Ω are geodesics for this metric (see [PT14] for a broad introduction to the Hilbert metric). Since Γ acts cocompactly on C there is $R > 0$ such that any point of C is at Hilbert distance at most R from the orbit Γp.

Recall that by Proposition 4.11 all choices of the linear functional α_0 as in Notation 1 determine Finsler metrics on the symmetric space that have the same geodesics. Hence we can choose the linear functional $\alpha_0(x) = \frac{1}{2}(x_1 - x_d)$, which is well adapted to the Hilbert metric d_Ω for the following reason. By Proposition 10.1 of [DGK17], fixing some $p \in C \subset \Omega$, there exists a constant K such that for any $\gamma \in \Gamma$ we have

$$|d_\Omega(p, \gamma p) - d_{\mathbb{X}}^{\mathfrak{F}}(\mathbf{x}, \gamma \mathbf{x})| \leq K. \tag{22}$$

Let us combine this with Theorem 4.1 to prove that the orbit $\Gamma \mathbf{x} \subset \mathbb{X}$ (on which Γ obviously acts cocompactly) is roughly Finsler convex. Fix $\gamma \in \Gamma$ and let us find a Finsler geodesic from $\mathbf{x}$ to $\gamma \mathbf{x}$ at bounded distance from $\Gamma \mathbf{x}$. The segment $[p, \gamma p] \subset \Omega$ is a geodesic for the Hilbert metric.

Let $p_0 = p, p_1, \dots, p_n = p$ be points in this order on this geodesic, with at successive distance at most 1. For any i there is $\gamma_i \in \Gamma$ such that $d_\Omega(p_i, \gamma_i p) \leq R$, and $\gamma_0 = \mathrm{id}$ and $\gamma_n = \gamma$. For all $i < j < k$, since $p_0, \dots, p_n$ are on a geodesic, we have

$$d_\Omega(\gamma_i p, \gamma_j p) + d_\Omega(\gamma_j p, \gamma_k p) \leq d_\Omega(\gamma_i p, \gamma_k p) + 6R. \tag{23}$$

Now by (22) we get

$$d_{\mathbb{X}}^{\mathfrak{F}}(\gamma_i \mathbf{x}, \gamma_j \mathbf{x}) + d_{\mathbb{X}}^{\mathfrak{F}}(\gamma_j \mathbf{x}, \gamma_k \mathbf{x}) \leq d_{\mathbb{X}}^{\mathfrak{F}}(\gamma_i \mathbf{x}, \gamma_k \mathbf{x}) + 6R + 3K. \tag{24}$$

Moreover, for any i we have

$$d_{\mathbb{X}}^{\mathfrak{F}}(\gamma_i \mathbf{x}, \gamma_{i+1} \mathbf{x}) \leq 2R + 1 + K. \tag{25}$$

Linking the $n+1$ points $\mathbf{x} = \gamma_0 \mathbf{x}, \gamma_1 \mathbf{x}, \dots, \gamma_n \mathbf{x} = \gamma \mathbf{x} \in \mathbb{X}$ by geodesics we clearly get a continuous path which is C-quasiruled, where $C = 6R + 3K + 6(2R + 1 + K)$.

By Theorem 4.1 there is a Finsler geodesic from $\mathbf{x}$ to $\gamma \mathbf{x}$ at distance at most C' from this quasiruled path, where C' only depends on C. Then by (25) this geodesic is at distance at most $C' + 2R + 1 + K$ from $\Gamma \mathbf{x}$. ☐

An example of a Gromov-hyperbolic naively convex cocompact group which is not Anosov can be produced using a ping-pong argument as in Proposition 12.4 of [DGK17].

Consider a Schottky subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ generated by two elements coming from two matrices $g, h \in \mathrm{SL}_2(\mathbb{R})$, so that Γ preserves the projective model $D \subset \mathbb{RP}^2$ of the Poincaré disk, and acts cocompactly on a convex subset $C \subset D$.

Let $\lambda > \lambda^{-1}$ be the eigenvalues of g. Consider the 3-by-3 matrices

$$g' = \begin{pmatrix} \lambda^{\frac{1}{3}} \cdot g & 0 \\ 0 & \lambda^{-\frac{2}{3}} \end{pmatrix} \quad \text{and} \quad h' = \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix},$$

so that g' is *not* loxodromic. They generate a discrete subgroup $\Gamma' \subset \mathrm{PSL}_3(\mathbb{R})$ which preserves a projective plane $\mathbb{RP}^2 \subset \mathbb{RP}^3$, where its action is exactly the action of Γ, so it preserves the disk D and acts cocompactly on the convex subset $C \subset D$.

Note that Γ' also preserves in $\mathbb{RP}^3$ an open convex cylinder $D \times \mathbb{R}$ containing D, which is not properly convex: it is contained in an affine chart, in which it is convex but not bounded. Fix a small compact neighborhood B of a point of D in the cylinder $D \times \mathbb{R}$. Up to taking g and h with larger hyperbolic translation length, another ping-pong argument shows that the Γ'-orbit of B in $D \times \mathbb{R}$ remains bounded, hence the convex hull $\Omega \subset D \times \mathbb{R}$ of this orbit and of D is a Γ'-invariant properly convex domain. Moreover Ω contains D, and hence C, which is a convex subset on which Γ' acts cocompactly. Thus Γ' acts naively convex cocompactly in the sense of Definition 1.9 of [DGK17].

5 Fock–Goncharov positivity

This section is devoted to a geometric interpretation of positivity as introduced by Lusztig [Lus94] and imported into the context of Hitchin representations by Fock and Goncharov [FG06]. We collect the relevant algebraic results and relate them to admissible paths on the characteristic surface of a Hitchin grafting representation. Throughout this section, we put $G = \mathrm{SL}_d(\mathbb{R})$ although most of the discussion is valid for all split real simple Lie groups and although ultimately we are interested in $\mathrm{PSL}_d(\mathbb{R})$. For completeness, note that for $G = \mathrm{SL}_d(\mathbb{R})$, most of Lusztig's results and concepts were already known (see for instance [And87]), but we still use Lusztig's notation and formalism. In particular, we will use Lusztig's work to introduce the subsets

$$G_{>0} \subset G_{\geq 0} \subset G \text{ and } \mathcal{F}_{>0} \subset \mathcal{F}_{\geq 0} \subset \mathcal{F}$$

and some of their basic properties. Most of the time we shall work with the matrix group $\mathrm{SL}_d(\mathbb{R})$ without further mentioning as this allows for more concrete computations.

As $G = \mathrm{SL}_d(\mathbb{R})$, the subset $G_{>0} \subset G$ is the set of *totally positive matrices*, which are the matrices $A \in \mathrm{SL}_d(\mathbb{R})$ such that for any $1 \leq k \leq d-1$ the exterior product $\Lambda^k A \in \mathrm{SL}(\Lambda^k \mathbb{R}) = \mathrm{SL}_{d_k}(\mathbb{R})$ is positive, i.e. all its entries are positive. For general split Lie groups the definition of $G_{>0}$ is more complicated, see Sections 2.2, 2.12, 5.10 and 8.8 of [Lus94].

One can check that $\tau : \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_d(\mathbb{R})$ is positive in the sense that it maps projectivizations of positive matrices to projectivizations of totally positive matrices. Note also that Lusztig did not introduce the concept of positive representation from $\mathrm{PSL}_2(\mathbb{R})$ into G: this is due to Fock–Goncharov [FG06], who used it to prove among other things that all Hitchin representations are discrete. Many ideas in this section are inspired by the work of Fock–Goncharov.

Finally, the concept of positivity also exists in certain *nonsplit* real Lie group: see [BH12; GW18]. However in these settings the positive cone lies in a partial flag manifold instead of the full flag manifold, which is not enough for our purposes.

5.1 Reminders on positivity

The following summarizes the results from [Lus94] we are going to use. As before, $\mathcal{F}$ denotes the variety of full flags in $\mathbb{R}^d$.

Theorem 5.1 ([Lus94]). *There exist semigroups $G_{>0} \subset G_{\geq 0} \subset G$ and a subset $\mathcal{F}_{>0} \subset \mathcal{F}$ with the following properties.*

1. *[By definition, see §2.2]* $\exp(\mathfrak{a}) \subset G_{\geq 0}$*, in particular* $1 \in G_{\geq 0}$*.*
2. *[Th. 4.8] For the standard embedding $\tau : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_d(\mathbb{R})$, the set $G_{>0}$ is an (open) connected component of*
$$\{g \in G : g\partial\tau(\infty) \pitchfork \partial\tau(\infty) \text{ and } g\partial\tau(0) \pitchfork \partial\tau(0)\}.$$
3. *[Th. 5.6] Every element of $G_{>0}$ is loxodromic, and in particular does not fix any point of $\mathbb{X}$.*
4. *[Th. 4.3 & Rem. 4.4] $G_{\geq 0}$ is the closure of $G_{>0}$.*
5. *[before Prop. 2.13] $G_{>0}G_{\geq 0} \subset G_{>0}$ and $G_{\geq 0}G_{>0} \subset G_{>0}$.*
6. *[Prop. 8.14] $\mathcal{F}_{>0}$ is an (open) connected component of $\partial\tau(\infty)^{\pitchfork} \cap \partial\tau(0)^{\pitchfork}$.*
7. *[Prop. 8.12] $G_{\geq 0}\mathcal{F}_{>0} \subset \mathcal{F}_{>0}$.*
8. *[By (7) above and by definition, see Th. 8.7] $\mathcal{F}_{>0} = G_{>0} \cdot \partial\tau(\infty)$.*

Example . In the special case $G = \mathrm{SL}_3(\mathbb{R})$, one can visualise $\mathcal{F}_{>0}$: namely, recall that in this case, the flag variety $\mathcal{F}$ is identified with the set of pairs (p, ℓ) where p is a point of $\mathbb{RP}^2$ and ℓ is a line containing p.

Let x, y and $z \in \mathbb{RP}^2$ be the image of the canonical basis of $\mathbb{R}^3$, let $[x, y]$, $[y, z]$ and $[z, x] \subset \mathbb{RP}^2$ be the image of the segments between the vectors of the canonical basis, and let $T \subset \mathbb{RP}^2$ be the triangle enclosed by these segments. Then $\mathcal{F}_{>0}$ is the set of pairs (p, ℓ) such that p is contained in the interior of T and ℓ intersects the (relative) interior of the segments $[x, y]$ and $[y, z]$. Then
$$G_{>0} = \{g \in \mathrm{SL}_3(\mathbb{R}) \mid g\overline{\mathcal{F}_{>0}} \subset \mathcal{F}_{>0}\}.$$
One can see that $g \in G_{>0}$ maps $\overline{T}$ into its interior (this corresponds to the fact that the entries of g, i.e. the minors of size 1, are positive), and hence g has an attracting fixed point in T. In general it is true that any totally positive matrix is diagonalisable with distinct positive eigenvalues.

We will also use the following, which should be well-known to the experts, but for which we did not find a reference.

Lemma 5.2. *9. $\mathcal{F}_{>0}$ is the interior of its closure, denoted by $\mathcal{F}_{\geq 0}$.*

10. $G_{>0}\mathcal{F}_{\geq 0} \subset \mathcal{F}_{>0}$.

11. *Let us denote* $G_{<0} := (G_{>0})^{-1}$ *and* $\mathcal{F}_{<0} = G_{<0} \cdot \partial\tau(\infty)$*, and* $G_{\leq 0}$ *and* $\mathcal{F}_{\leq 0}$ *their respective closures. Then any pair in* $\mathcal{F}_{<0} \times \mathcal{F}_{\geq 0}$ *is transverse.*

Remark . *It is clear that* $\mathcal{F}_{\geq 0}$ *contains* $G_{\geq 0} \cdot \partial\tau(\infty)$ *but they are not equal in general. For instance consider the case* $\mathrm{SL}_3(\mathbb{R})$*. Denote the usual basis of* $\mathbb{R}^3$ *by* (e_1, e_2, e_3)*. Consider the flags* $\partial\tau(\infty) = (span(e_3), span(e_2, e_3))$ *and* $F = (span(e_2), span(e_2, e_3 - e_1)$*.*

Then F *lies in* $\mathcal{F}_{\geq 0}$ *but not in* $G_{\geq 0} \cdot \partial\tau(\infty)$*. Indeed one can check* $F \in \mathcal{F}_{\geq 0}$ *by computing that* $F = \lim_{\lambda \to 0^+} A_\lambda \partial\tau(\infty)$ *where* $A_\lambda = \begin{pmatrix} 1 & 3\lambda & \lambda \\ \lambda & 2 & 1 \\ \lambda^3 & \lambda & \lambda \end{pmatrix}$ *is totally positive for small positive values of* λ*. (One can renormalize* A_λ *to make it determinant* 1*.)*

But if a matrix $B \in G_{\geq 0}$ *sends* $\partial\tau(\infty)$ *into the flag* F*, then it sends* e_2 *into a point* $B \cdot e_2$ *in* $span(e_2, e_3 - e_1) \setminus span(e_2)$ *In particular one can write* $B \cdot e_3 = ae_2 + b(e_3 - e_1)$ *with* $b \neq 0$*. Then one of the coefficients of* B*, in position* $(3,1)$ *or* $(3,3)$*, has a negative entry. It contradicts the fact that* $B \in G_{\geq 0}$*, and that matrices in* $G_{\geq 0}$ *have non-negative entries. The flag* F *may be thought to correspond to a point at infinity of* $G_{\geq 0}$ *(for the compactification* $\mathrm{SL}_3(\mathbb{R}) \hookrightarrow \mathrm{PGL}_3(\mathbb{R})$*).*

Proof of Lemma 5.2. Proof of (9). Put $\xi_\infty = \partial\tau(\infty)$ and $\xi_0 = \partial\tau(0)$. By definition, the sets $\xi_\infty^\pitchfork$ and $\xi_0^\pitchfork$ are open Bruhat cells in $\mathcal{F}$ and hence $Z = \xi_\infty^\pitchfork \cap \xi_0^\pitchfork$ is open. The complements of the Bruhat cells $\xi_\infty^\pitchfork$ and $\xi_0^\pitchfork$ are (real) connected projective varieties all of whose irreducible components are of codimension one. Then every point of the boundary of $\xi_\infty^\pitchfork \cap \xi_0^\pitchfork$ is contained in an irreducible subvariety of codimension one. Thus by property (6), the statement(9) is equivalent to the following. Let V be a component of Z. Denoting by $\overline{U}$ the closure of a set U, it holds

$$V = \mathcal{F} - \overline{\mathcal{F} - \overline{V}}. \tag{26}$$

As for any two open sets U_1, U_2 we have $\overline{U_1 \cup U_2} = \overline{U_1} \cup \overline{U_2}$, it suffices to write V as a finite intersection $V = \cap_{j=1}^{d-1} U_j$ where each of the sets U_j has property (26). To construct such sets note that if $\xi_\iota = (\xi_\iota^1 \subset \cdots \subset \xi_\iota^{d-1})$ $(\iota = \infty, 0)$ then the linear hyperplanes $\xi_\infty^{d-1}, \xi_0^{d-1}$ are transverse and hence they decomposes $\mathbb{R}^d$ into four connected components which are paired be the reflection $x \to -x$, say the components $A, -A, B, -B$. The closures of the components $A, -A$ and $B, -B$ intersect in a linear subspace of codimension 2. A component Z of $\xi_\infty^\pitchfork \cap \xi_0^\pitchfork$ consists of flags $\zeta = \zeta_1 \subset \cdots \subset \zeta_{d-1}$ with the property that ζ_1 is transverse to both $\xi_\infty^{d-1}, \xi_0^{d-1}$. But this means that for either A or B, say for A, any nonzero point on the line ζ_1 is contained in $A \cup -A$. As a consequence, if we define U_1 to be the set of all flags $\zeta = \zeta_1 \subset \cdots \subset \zeta_{d-1}$ such that $\zeta_1 - \{0\} \subset A \cup -A$ then $U_1 \subset Z$ is an open set of the required form.

Now note that U_1 can be described as follows. Choose a generator ω_ι of $\wedge^{d-1}\xi_\iota^{d-1}$ $(\iota = 0, \infty)$ and define U_1 to be the set of all flags ζ with the property that for some basis element e of ζ_1, the wedge products $e \wedge \omega_0, e \wedge \omega_\infty$ define the same (or the opposite) orientation of $\mathbb{R}^d$. Then U_1 is one of the sets determined in the previous paragraph.

For $j \leq d-1$ define the set U_j as the set of all flags ζ so that for some generator e of $\Lambda^j \zeta_j$ and some generators $\omega_0^{d-j}, \omega_\infty^{d-j}$ of $\Lambda^{d-j}\xi_0^{d-j}, \Lambda^{d-j}\xi_\infty^j$ the orientations defined by

$e \wedge \omega_0^{d-j}, e \wedge \omega_\infty^{d-j}$ coincide (or are opposite). For suitably choices of the sets U_j, we then have $\xi_0^{\pitchfork} \cap \xi_\infty^{\pitchfork} = \cap_j U_j$. Together with the first paragraph of this proof, (9) follows.

Proof of (10). Since $G_{>0}$ is open (by (2)), every $G_{>0}$-orbit in $\mathcal{F}$ is open. Being a union of such orbits, $G_{>0}\mathcal{F}_{\geq 0}$ is also open. Moreover it is contained in $\mathcal{F}_{\geq 0}$ (by (7)) and hence it is contained in its interior, which is precisely $\mathcal{F}_{>0}$ by (9).

Proof of (11). Consider $(\xi, \eta) \in \mathcal{F}_{<0} \times \mathcal{F}_{\geq 0}$. Since $\mathcal{F}_{>0}$ is open, $G_{>0}$ contains 1 in its closure, and $G_{>0}\mathcal{F}_{\geq 0} \subset \mathcal{F}_{>0}$, there exists $g \in G_{>0}$ such that $g\xi \in \mathcal{F}_{<0}$ and $g\eta \in \mathcal{F}_{>0}$.

By definition, there exists $h \in G_{>0}$ such that $g\xi = h^{-1}\xi_\infty$. Then $hg\xi = \xi_\infty$ and $hg\eta \in h\mathcal{F}_{>0} \subset \mathcal{F}_{>0} \subset \xi_\infty^{\pitchfork}$ by (6). Therefore $hg\xi$ and $hg\eta$ are transverse, and so are ξ and η. □

5.2 Positivity and injectivity of admissible paths

We now explain the assumption that the representation $\tau : \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_d(\mathbb{R})$ is positive: it means that τ maps every 2×2 matrix with positive entries into $G_{>0}$. It has the following consequences, which should be well-known to experts.

Lemma 5.3. *We have the following.*

1. $a'_t \in G_{>0}$ *for any* $t > 0$.

2. $G_{<0} = r_\pi G_{>0} r_\pi$ *(see Notation 2).*

Proof. Proof of (1). This is an immediate consequence of the fact that

$$\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

has positive entries.

Proof of (2). Let us prove that $r_\pi G_{>0}^{-1} r_\pi = G_{>0}$. The maps $g \mapsto g^{-1}$ and $g \mapsto r_\pi g r_\pi$ both preserve

$$\{g \in G : g\partial\tau(\infty) \pitchfork \partial\tau(\infty) \text{ and } g\partial\tau(0) \pitchfork \partial\tau(0)\},$$

and hence permute the connected components, and so does $g \mapsto r_\pi g^{-1} r_\pi$. To prove that this map preserves the connected component $G_{>0}$ (see Theorem 5.1 (2)), it suffices to show that it fixes a point of $G_{>0}$. It is clear that it fixes for instance $a'_1 \in G_{>0}$. □

A first important consequence of all the facts about positivity that we have listed is the following.

Corollary 5.4. *For any admissible path* $c : [0, T] \to G$*, for all* $0 \leq s < t \leq T$ *we have* $c(s)^{-1}c(t) \in G_{\geq 0}$.

Proof. By definition, $c(s)^{-1}c(t)$ is a product of elements of G of the form a'_r for some $r > 0$ or $\exp(v)$ for some $v \in \mathfrak{a}$. All these elements belong to $G_{\geq 0}$ by Theorem 5.1 and Lemma 5.3, and so does their product since $G_{\geq 0}$ is a semigroup. □

The previous result, combined with the fact that totally positive matrices are not the identity, tells us that admissible paths are injective, as explained below.

Proposition 5.5. *Any admissible path in $\mathbb{X}$ is injective*

Proof. Let $c : [0, T] \to G$ be an admissible path and let $\mathbf{x} \in \mathbb{X}$ be an arbitrarily fixed point. Consider $0 \le s < t \le T$, and let us prove that $c(s)\mathbf{x} \neq c(t)\mathbf{x}$, i.e. that $c(s)^{-1}c(t)\mathbf{x} \neq \mathbf{x}$. By definition, $c(s)^{-1}c(t)$ is a product of elements of G of the form a'_r for some $r > 0$ or $\exp(v)$ for some $v \in \mathfrak{a}$. There are two cases.

Case 1: $c(s)^{-1}c(t) = \exp(v)$ for some nonzero $v \in \mathfrak{a}$, then it is clear that $\exp(v)\mathbf{x} \neq \mathbf{x}$.

Case 2: $c(s)^{-1}c(t)$ is a product of elements of G of the form a'_r for some $r > 0$ or $\exp(v)$ for some $v \in \mathfrak{a}$, with at least one element of the form a'_r. All the $\exp(v)$'s belong to $G_{\ge 0}$ by Theorem 5.1.1 and all the a'_r's belong to $G_{>0}$ by Lemma 5.3.1, and so $c(s)^{-1}c(t)$ belongs to $G_{>0}$ by Theorem 5.1.5. Therefore $c(s)^{-1}c(t)$ does not fix any point of $\mathbb{X}$ by Theorem 5.1.3, and $c(s)^{-1}c(t)\mathbf{x} \neq \mathbf{x}$. □

This implies that the characteristic surface we have constructed in Section 3.4 is embedded.

Corollary 5.6. *The map $Q_z : S_z \to \rho_z\backslash\mathbb{X}$ constructed in Proposition 3.5 is injective.*

Proof. Consider $x, y \in S_z$ such that $Q_z(x) = Q_z(y)$, and let us prove that $x = y$.

Consider two lifts $\tilde{x}$ and $\tilde{y} \in \tilde{S}_z$ of respectively x and y. Then $\tilde{Q}_z(\tilde{x})$ and $\tilde{Q}_z(\tilde{y})$ have the same projection in $\rho_z\backslash\mathbb{X}$, which means that there exists $\gamma \in \pi_1(S)$ such that

$$\tilde{Q}_z(\tilde{x}) = \rho_z(\gamma)\tilde{Q}_z(\tilde{y}) = \tilde{Q}_z(\gamma\tilde{y}).$$

Consider an admissible path $c : [0, T] \to \tilde{S}_z$ from $\tilde{x}$ to $\gamma\tilde{y}$. Then by Observation 3.9 $\tilde{Q}_z \circ c : [0, T] \to \mathbb{X}$ is an admissible path from $\tilde{Q}_z(\tilde{x})$ to itself.

Since admissible paths of $\mathbb{X}$ are injective by Proposition 5.5, this means that $T = 0$ and $\tilde{x} = \gamma\tilde{y}$, and hence that $x = y$. □

5.3 Positivity gives a control on default of the triangle inequality

The goal of this section is to prove the following result about totally positive transformations. We then use it to prove that admissible paths are (Finsler) quasi-ruled, quasi-convex and quasi-geodesics, which implies that the characteristic surface in the symmetric space associated to a Hitchin grafting representation is Finsler quasi-convex.

As before, we write $G = \mathrm{SL}_d(\mathbb{R})$, and we choose a basepoint $\mathbf{x} \in \mathbb{X} = G/K$, thought of as the projection of the identity in G.

Lemma 5.7. *For any $\omega > 0$, there exists $C_\omega > 0$ such that for all $g_+ \in G_{\ge 0}$ and $g_- \in a'_{-\omega}G_{\le 0}$, we have*

$$d^{\mathfrak{F}}(g_-\mathbf{x}, \mathbf{x}) + d^{\mathfrak{F}}(\mathbf{x}, g_+\mathbf{x}) \le d^{\mathfrak{F}}(g_-\mathbf{x}, g_+\mathbf{x}) + C_\omega.$$

To prove it we will need the following technical result. In its formulation, we use a K-invariant metric $d_{\mathcal{F}}$ on the flag variety $\mathcal{F}$. Furthermore, we denote by $\xi_0 \in \mathcal{F}$ the simplex $\partial\tau(\infty) = \exp(\mathfrak{a}^+) \in \mathcal{F}$. Distances are taken with respect to the distance function d defined by the symmetric Riemannian metric.

Lemma 5.8. *For every $\epsilon > 0$ there exists a number $C_\epsilon > 0$ only depending on ϵ with the following property. Consider $g \in G$ decomposed as $g = k\exp(\kappa(g))\ell$ with $k, \ell \in K$ (the maximal compact subgroup) and $\kappa(g) \in \mathfrak{a}^+$. Let $\xi \in \mathcal{F}$ at $d_{\mathcal{F}}$-distance at least $\epsilon > 0$ from $k\xi_0^{\not\pitchfork}$. Then $\mathbf{x}$ is at distance at most C_ϵ from the Weyl cone with vertex at $g\mathbf{x}$ and boundary at infinity the simplex in $\partial_\infty\mathbb{X}$ corresponding to ξ.*

Proof. Since $d_{\mathcal{F}}(\xi, k\xi_0^{\not\pitchfork}) \geq \epsilon$, the simplices $\xi, k\xi_0$ are transverse and hence they are contained in a unique maximal flat F. We claim that there is $x \in F$ at distance at most $C_\epsilon > 0$ from $\mathbf{x}$, where C_ϵ only depends on ϵ. Indeed, this follows from the compactness of the set $\{(\xi, \eta) \in \mathcal{F}^2, d_{\mathcal{F}}(\xi, \eta) \geq \epsilon\}$ and continuity of the map which associates to two transverse flags the unique maximal flat whose visual boundary contains the Weyl chambers that corresponds to the two flags.

Note that $g\mathbf{x}$ is contained in the Weyl cone connecting $\mathbf{x}$ to $k\xi_0$ (because $k\mathbf{x} = \mathbf{x}, \ell\mathbf{x} = \mathbf{x}$ and $\exp(\kappa(g))\mathbf{x}$ is contained in the Weyl Cone connecting $\mathbf{x}$ to ξ_0). Thus as Weyl cones are convex cones, the endpoint $p \in \partial_\infty\mathbb{X}$ of the ray starting at $\mathbf{x}$ and passing through $g\mathbf{x}$ is contained in the Weyl Chamber $k\xi_0$. The ray $[x, p)$ is contained in the Weyl Cone from x to $k\xi_0$.

We now apply the CAT(0)-property for the Riemannian symmetric metric to the asymptotic rays $[x, p)$ and $[\mathbf{x}, p)$. It yields that the point $y \in [x, p)$ at distance exactly $d(\mathbf{x}, g\mathbf{x})$ from x is of distance at most $d(\mathbf{x}, x) \leq C_\epsilon$ from $g\mathbf{x}$.

By construction, the geodesic ray $[x, p)$ is contained in the flat F, and its endpoint is contained in the Weyl chamber $k\xi_0$. The unique geodesic line η extending $[x, p)$ is contained in F and is backward asymptotic to a point q in the unique Weyl chamber ξ in the visual boundary of F which is transverse to $k\xi_0$. Recall that η passes through the C_ϵ-neighborhood of $\mathbf{x}$.

Using once more the CAT(0)-property, this time applied to the subray of η which connects y to q and the geodesic ray ζ connecting $g\mathbf{x}$ to q, we conclude that ζ passes through the C_ϵ-neighborhood of x and hence through the $2C_\epsilon$-neighborhood of $\mathbf{x}$. On the other hand, by construction, this ray is contained in the Weyl cone connecting $g\mathbf{x}$ to ξ. Together this is what we wanted to show. □

Proof of Lemma 5.7. Decompose $g_\pm = k_\pm e^{\kappa(g_\pm)}\ell_\pm$ with $k_\pm, \ell_\pm \in K$ (the maximal compact subgroup) and $\kappa(g_\pm) \in \mathfrak{a}^+$. The plan is to use positivity and Lemma 5.8 to find Weyl Chambers $\xi_\pm$ such that their images $g_\pm\xi_\pm$ are transverse, the flat F through them passes near $\mathbf{x}$, and the Weyl Cone from x to $g_\pm\xi_\pm$ passes near $g_\pm\mathbf{x}$.

Recall the definition of the set $\mathcal{F}_{\geq 0}$, and for $\omega > 0$ the element a'_ω. Since $a'_\omega\mathcal{F}_{\geq 0}$ has nonempty interior and $\xi_0^{\not\pitchfork}$ is a closed set with empty interior, there exists $\epsilon = \epsilon_\omega > 0$ such that for every $k \in K$ there is $\xi \in a'_\omega\mathcal{F}_{\geq 0}$ at $d_{\mathcal{F}}$-distance at least ϵ from $k\xi_0^{\not\pitchfork}$. Similarly, for every $k \in K$ there is $\xi \in a'_{-\omega}\mathcal{F}_{\leq 0}$ at $d_{\mathcal{F}}$-distance at least ϵ from $k\xi_0^{\not\pitchfork}$.

Let $\xi_+ \in a'_\omega \mathcal{F}_{\geq 0}$ be at $d_\mathcal{F}$-distance at least ϵ from $\ell_+^{-1}\xi_0^{\not{\sigma}}$ and $\xi_- \in a'_{-\omega}\mathcal{F}_{\leq 0}$ be at $d_\mathcal{F}$-distance at least ϵ from $\ell_-^{-1}\xi_0^{\not{\sigma}}$.

We know $\xi_+ \in \mathcal{F}_{\geq 0}$ and $g_+ \in G_{\geq 0}$, so by Theorem 5.1.7 we have $g_+\xi_+ \in \mathcal{F}_{\geq 0}$, and similarly $g_-\xi_- \in a'_{-\omega}\mathcal{F}_{\leq 0}$. In particular, by Lemma 5.2.11 $g_+\xi_+$ and $g_-\xi_-$ are transverse. More precisely, if we denote as before by d the distance function of the symmetric metric, then the flat $F = F(g_-\xi_-, g_+\xi_+)$ through them contains a point x with $d(x, \mathbf{x}) \leq q_\omega$ for some $q_\omega > 0$ only depending on ω, because every pair in the set $a'_{-\omega}\mathcal{F}_{\leq 0} \times \mathcal{F}_{\geq 0}$ is transverse and this set is compact.

Since $d_\mathcal{F}(\xi_\pm, \ell_\pm^{-1}\xi_0^{\not{\sigma}}) \geq \epsilon$, Lemma 5.8 implies that $g_\pm\mathbf{x}$ is at Riemannian distance at most C_ϵ to the Weyl Cone connecting $\mathbf{x}$ to $g_\pm\xi_\pm$. By the CAT(0) property of $(\mathbb{X}, d)$, applied to the geodesics connecting $\mathbf{x}$ and x to all points in $g_\pm\xi_\pm$, the Hausdorff distance (for d) between this Weyl cone and the Weyl cone $W_\pm \subset F$ connecting x to $g_\pm\xi_\pm$ is at most $d(\mathbf{x}, x) \leq q_\omega$. As a consequence, there is $x_\pm \in W_\pm$ with

$$d(x_\pm, g_\pm\mathbf{x}) \leq q_\omega + C_\epsilon.$$

Since W_+ and W_- are two opposite Weyl Cones in F based at x, and since $x_\pm \in W_\pm$, we deduce from Proposition 4.11 that

$$d^{\mathfrak{F}}(x_-, x) + d^{\mathfrak{F}}(x, x_+) = d^{\mathfrak{F}}(x_-, x_+).$$

To conclude, recall that the Riemannian and Finsler metrics are comparable; that is, there exists $\lambda > 0$ such that $\lambda^{-1}d \leq d^{\mathfrak{F}} \leq \lambda d$. Then by the triangle inequality

$$\begin{aligned} d^{\mathfrak{F}}(g_-\mathbf{x}, \mathbf{x}) &+ d^{\mathfrak{F}}(\mathbf{x}, g_+\mathbf{x}) - d^{\mathfrak{F}}(g_-\mathbf{x}, g_+\mathbf{x}) \\ &\leq d^{\mathfrak{F}}(x_-, x) + d^{\mathfrak{F}}(x, x_+) - d^{\mathfrak{F}}(x_-, x_+) + 6\lambda(q_\omega + C_\epsilon) \\ &\leq 6\lambda(q_\omega + C_\epsilon). \end{aligned}$$ □

We now use the previous result to prove that admissible paths are quasi-ruled. For this we will need the following general fact about quasi-ruled paths.

Lemma 5.9. *Let (X, d) be a metric space and $x_1, \dots, x_n \in X$ such that for some constant $C > 0$ we have $d(x_i, x_j) + d(x_j, x_k) \leq d(x_i, x_k) + C$ for all $i < j < k$. Then any concatenation of geodesics $[x_1, x_2], [x_2, x_3], \dots, [x_{n-1}, x_n]$ is $4C$-quasi-ruled.*

Proof. Let x, y, z be three points on such a concatenation, in this order, and let us check that $d(x, y) + d(y, z) \leq d(x, z) + 4C$. Let $[x_i, x_{i+1}]$, $[x_j, x_{j+1}]$ and $[x_k, x_{k+1}]$ be three geodesic pieces of the concatenation containing respectively x, y, z, such that $i \leq j \leq k$. Let us assume for the rest of the proof that $i < j < k$; if instead we had $i = j < k$ or $i < j = k$ then the proof would be similar (in fact easier), and the case $i = j = k$ is obvious.

Using the triangle inequality and our assumption, and denoting $(a, b) = d(a, b)$ to lighten our estimates on distances, we have the following, which concludes the proof.

$$\begin{aligned}
(x,y)+(y,z) &\leq (x,x_{i+1})+(x_{i+1},x_j)+(x_j,y)+(y,x_{j+1})+(x_{j+1},x_k)+(x_k,z)\\
&\leq -(x_i,x)+(x_i,x_{i+1})+(x_{i+1},x_j)+(x_j,x_{j+1})+(x_{j+1},x_k)+(x_k,x_{k+1})-(z,x_{k+1})\\
&\leq -(x_i,x)+(x_i,x_j)+C+(x_j,x_k)+C+(x_k,x_{k+1})-(z,x_{k+1})\\
&\leq 2C-(x_i,x)+(x_i,x_k)+C+(x_k,x_{k+1})-(z,x_{k+1})\\
&\leq 3C-(x_i,x)+(x_i,x_{k+1})+C-(z,x_{k+1})\\
&\leq 4C+(x,z).
\end{aligned}$$
□

We now prove that admissible paths are quasi-ruled.

Proposition 5.10. *For any $\omega > 0$ there exists C_ω such that any $(\omega, 0)$-admissible path c in $\mathbb{X}$ is Finsler C_ω-quasi-ruled, and hence is at Hausdorff distance at most C'_ω from some Finsler geodesic by Theorem 4.1, where C'_ω only depends on C_ω.*

Proof. By definition $c(r) = a(r)\mathbf{x}$ for any r, where $a(r)$ is an admissible path in G.

Take $0 \leq t < s < u \leq T$, and let us show $d(c(t), c(s))+d(c(s), c(u)) \leq d(c(t), c(u))+C_\omega$ for a well-chosen C_ω. By Lemma 5.9 above we may assume that each of the points $a(t), a(s), a(u)$ is at the junctions of two pieces of the admissible path a, one of hyperbolic type, and the other of flat type (see Definition 3.8).

By Corollary 5.4, $a(s)^{-1}a(u) \in G_{\geq 0}$ and $a(s)^{-1}a(t) \in G_{\leq 0}$.

Note that $a(s)$ is adjacent to a hyperbolic-type piece of a, which has length at least ω, unless this piece is the first or last piece of a. If this hyperbolic-type piece is first or last and has length less than ω, then $c(s)$ is ω-close to either $c(t)$ or $c(u)$, and one conclude easily with the triangle inequality (taking $C_\omega \geq \omega$). Let us assume this hyperbolic-type piece has length at least ω.

If this piece is after $a(s)$ then $a(s)^{-1}a(u) \in a'_\omega G_{\geq 0}$. If on the contrary this piece is before $a(s)$ then $a(s)^{-1}a(t) \in a'_{-\omega} G_{\leq 0}$.

In any case, by Lemma 5.7, we can conclude:

$$d^{\mathfrak{F}}(c(t), c(s)) + d^{\mathfrak{F}}(c(s), c(u)) \leq d^{\mathfrak{F}}(c(t), c(u)) + C_\omega.$$
□

Finally, we deduce that the characteristic surface associated to a Hitchin grafting representation is quasi-convex.

Corollary 5.11. *Consider the map $\widetilde{Q}_z : \widetilde{S}_z \to \widetilde{S}^\iota_z \subset \mathbb{X}$ constructed in Proposition 3.5, such that the grafting locus $\gamma^* \subset S$ has collar size at least $\omega > 0$. Then $\widetilde{S}^\iota_z$ is Finsler C_ω-quasi-convex for some C_ω depending on ω, in the sense that any two points of $\widetilde{S}^\iota_z$ can be connected by a Finsler geodesic that stays at distance at most C_ω from $\widetilde{S}^\iota_z$.*

Proof. This is an immediate consequence of Proposition 5.10 and Observation 3.9. □

5.4 Estimates on eigenvalues of products of totally positive matrices

We will also need a quantitative version of the classical result that all elements of $G_{>0}$ are loxodromic. This quantitative result is probably well known to experts; since we did not find a precise reference for what we need, we give a proof (in the case $G = \mathrm{SL}_d(\mathbb{R})$).

Recall that for any matrix g and $1 \leq k \leq d$, we denote by $\lambda_k(g)$ the logarithm of the norm of the k-th eigenvalue of g, such that $\lambda_1(g) \geq \lambda_2(g) \geq \cdots \geq \lambda_d(g)$.

Proposition 5.12. *For any $g \in G_{>0}$ there exists $C > 0$ such that for all $h_1, \ldots, h_n \in G_{\geq 0}$, denoting $h = gh_1gh_2 \cdots gh_n$ we have $\lambda_k(h) \geq \lambda_{k+1}(h) + \frac{n}{C}$ for any $1 \leq k \leq d-1$.*

In particular, $d^{\mathfrak{F}}(\mathbf{x}, h\mathbf{x}) \geq \frac{n}{C'}$ for some constant C' that only depends on C and hence only depends on g.

To prove Proposition 5.12, we first establish an intermediate result about positive matrices and use the fact that a matrix of size d is totally positive if and only if all its exterior products, seen as matrices of size $d_k = \dim(\Lambda^k \mathbb{R}^d)$ where $1 \leq k \leq d-1$, are *positive*, i.e. with positive entries.

Lemma 5.13. *For any positive matrix g of size d, there exists $\omega > 0$ such that for all $A_1, \ldots, A_n$ nonnegative, if $A = gA_1 \cdots gA_n$ then $\lambda_1(A) \geq \lambda_2(A) + n\omega$.*

Proof. By density of positive matrices it suffices to prove the lemma for $A_1, \ldots, A_n$ positive.

Let $\mathcal{C} \subset \mathbb{R}^d$ be the open convex cone of positive vectors, and $\Omega \subset \mathbb{RP}^{d-1}$ its projectivisation, which is a properly convex domain in the sense that there is an affine chart of $\mathbb{RP}^{d-1}$ containing Ω and in which Ω is bounded and convex.

Note that $A\overline{\mathcal{C}} \subset \mathcal{C} \cup \{0\}$ for any positive matrix A, so $A\overline{\Omega} \subset \Omega$.

Any properly convex domain $\Omega' \subset \mathbb{RP}^{d-1}$ can be endowed with a classical Finsler metric $d_{\Omega'}$ called the Hilbert metric, locally equivalent to the usual Riemannian metric of $\mathbb{RP}^{d-1}$, such that (see [Bir57], or see [PT14] for a broad introduction to Hilbert geometry)

1. it is projectively equivariant: $d_{h\Omega'} \circ h = d_{\Omega'}$ for any projective transformation h;
2. it is monotone with respect to inclusion: $d_{\Omega'} \leq d_{\Omega''}$ (on Ω'') for any $\Omega'' \subset \Omega'$;
3. if $\overline{\Omega}'' \subset \Omega'$ then there is $r < 1$ such that $d_{\Omega'} \leq r d_{\Omega''}$ (on Ω'').

Let g be a positive matrix: As $g\overline{\Omega} \subset \Omega$ there is $r < 1$ such that $d_{\Omega} \leq r d_{g\Omega}$. Then $g : \Omega \to \Omega$ is r-Lipschitz for d_{Ω} — hence it is a contraction — by equivariance of the Hilbert metric. In fact, $gA : \Omega \to \Omega$ also is an r-Lipschitz map for any positive matrix A since $gA\Omega \subset g\Omega$.

By the Banach fixed-point theorem, gA has a fixed point $p \in \Omega$ such that $(gA)^n x \to p$ for any $x \in \Omega$. In particular, gA is proximal with attracting line $p \in g\Omega$, and the complementary invariant hyperplane does not intersect Ω.

This implies the existence of a positive lower bound θ on the angle (for the standard Euclidean metric) between the attracting line of gA and the complementary invariant hyperplane of gA' for all positive matrices A, A'.

Since the restriction of the Hilbert metric d_Ω to the compact subset $\overline{g\Omega}$ is uniformly comparable to the standard Riemannian metric on $\mathbb{RP}^{d-1}$, this yields that for any proximal matrix h of the form $h = gA$ for a positive matrix A the number $e^{\lambda_2(h)-\lambda_1(h)}$ is comparable to the contraction rate of h at its attracting fixed point in $\mathbb{RP}^{d-1}$ for the usual Riemannian metric, where the comparison error does not depend on A. Thus there exists a number $C > 1$ such that for any proximal transformation $h : \Omega \to g\Omega$, if h is R-d_Ω-Lipschitz for some $R < 1$ then

$$e^{\lambda_2(h)-\lambda_1(h)} \leq CR.$$

In particular, for all $A_1, \ldots, A_n$ positive, $A = gA_1 \cdots gA_n$ is r^n-d_Ω-Lipschitz so

$$\lambda_1(A) - \lambda_2(A) \geq -\log(Cr^n).$$

In fact, for any k, the transformation A^k is r^{nk}-d_Ω-Lipschitz so

$$\lambda_1(A) - \lambda_2(A) = \frac{1}{k}(\lambda_1(A^k) - \lambda_2(A^k)) \geq \frac{-\log C}{k} + n\log\left(\frac{1}{r}\right),$$

and letting $k \to \infty$ we get

$$\lambda_1(A) - \lambda_2(A) \geq n\log\left(\frac{1}{r}\right).$$

Hence $\omega = \log\frac{1}{r}$ is the positive number we were looking for. □

Proof of Proposition 5.12. This is an immediate consequence of the previous lemma, the fact that a matrix $g \in \mathrm{GL}_d(\mathbb{R})$ is totally positive if and only if $\Lambda^k g \in \mathrm{GL}_{d_k}(\mathbb{R})$ is positive for any $1 \leq k \leq d$, and the fact that

$$\lambda_k(g) - \lambda_{k+1}(g) = \lambda_1(\Lambda^k g) - \lambda_2(\Lambda^k g).$$

The fact that $d^{\mathfrak{F}}(\mathbf{x}, h\mathbf{x}) \geq \frac{n}{C'}$ comes from that $d^{\mathfrak{F}}(\mathbf{x}, h\mathbf{x})$ is bounded from below by the Finsler translation length of h acting on $\mathbb{X}$, which is given by $\alpha_0(\lambda_1(h), \ldots, \lambda_d(h))$ and α_0 is positive on the set of vectors $(v_1, \ldots, v_d)$ such that $v_k \geq v_{k+1}$ for each k, see Notation (1). □

Proposition 5.12 has the following consequence in terms of admissible paths.

Proposition 5.14. *For any $\omega > 0$ there exists $C_\omega > 0$ such that for any $(\omega, 0)$-admissible path $c : [0, T] \to G$, we have*

$$d^{\mathfrak{F}}(c(0) \cdot \mathbf{x}, c(T) \cdot \mathbf{x}) \geq \frac{k-2}{C_\omega},$$

where k is the number of singularities (i.e. $k + 1$ is the number of geodesic pieces of c).

Observe that we need the -2 term in $(k-2)/C_\omega$ because we allow the first and last pieces of c to have length less than ω.

Proof. Without loss of generality we can assume $k \geq 3$, so that c contains at least one piece of hyperbolic type of length at least ω, and that $c(0) = 1$. Let r be the number of hyperbolic pieces of c of length at least ω; note that

$$\frac{k-2}{2} \leq r \leq \frac{k+2}{2}.$$

Then by definition of admissible (Definition 3.8) and Theorem 5.1, we can write $c(T) \in G_{>0}$ as the following product:

$$c(T) = g_0 a'_\omega g_1 a'_\omega g_2 \cdots g_{r-1} a'_\omega g_r,$$

where $g_0, \ldots, g_r \in G_{\geq 0}$.

By Proposition 5.12, there is $C > 0$ only depending on ω such that

$$d^{\mathfrak{F}}(c(0)\mathbf{x}, c(T)\mathbf{x}) \geq \frac{r}{C} \geq \frac{k-2}{2C}. \qquad \square$$

6 Geometric control: Uniform quasi-isometry

This section contains the main geometric results of this article. Recall that S is a hyperbolic closed surface, let $G = \mathrm{PSL}_d(\mathbb{R})$ and $\tau : \mathrm{PSL}_2(\mathbb{R}) \to G$ be the usual irreducible representation. In Section 3.3, given a collection of disjoint closed curves on S and an element of the Cartan subspace of G for each of these curves, we have recalled the definition of bending $\tau(\pi_1(S))$ inside G along these closed curves via the elements of the Cartan subspace. Moreover, in Section 3.4, we associated to such a bending an abstract grafting S_z of S (where z is the grafting parameter) and an equivariant, 1-Lipschitz, and piecewise totally geodesic map

$$\tilde{Q}_z : \tilde{S}_z \to \mathbb{X}$$

from its universal covering $\tilde{S}_z$ to the symmetric space $\mathbb{X}$ of G which projects to $Q_z : S_z \to \rho_z \backslash \mathbb{X}$.

Note that G is real split and τ is a regular and positive representation in the sense that it maps (projectivizations of) positive matrices in $\mathrm{PSL}_2(\mathbb{R})$ to (projectivizations of) totally positive matrices in G (see Section 5). Then the bent representation of $\pi_1(S)$ is Hitchin, which implies by independent (and different) work of Labourie [Lab06] and Fock–Goncharov [FG06] that our equivariant map $\tilde{S}_z \to \mathbb{X}$ is a quasi-isometric embedding.

In this section we give an upper bound for the multiplicative error of this quasi-isometric embedding. As mentioned in the introduction, our proof does not rely directly on the work of Labourie and Fock–Goncharov, but follows from the results on totally positive matrices proved in Section 5, which were greatly inspired by Fock–Goncharov's use of positivity.

Theorem 6.1. *For every $\sigma > 0$, there exist $C_\sigma > 0$ such that the following holds.*

Consider a closed hyperbolic surface S, a multicurve $\gamma^* \subset S$ whose components have length at most σ, and a grafting parameter z such that all cylinder heights of the abstract grafting S_z are bounded from below by some number $L > 0$.

Let us endow $\mathbb{X}$ with the G-invariant admissible Finsler metric $\mathfrak{F}$ and S_z with the pullback of this metric under Q_z, denoted by $d^{\mathfrak{F}}_{\tilde{S}_z}$. Then the grafting map $\tilde{Q}_z : \widetilde{S}_z \to \mathbb{X}$ is an injective quasi-isometric embedding with multiplicative constant $(1 + C_\sigma/(L+1))$ and additive constant C_σ; more precisely, for all $x, y \in \tilde{S}_z$ we have

$$\left(1 + \frac{C_\sigma}{L+1}\right)^{-1} d^{\mathfrak{F}}_{\tilde{S}_z}(x,y) - C_\sigma \leq d^{\mathfrak{F}}(\tilde{Q}_z(x), \tilde{Q}_z(y)) \leq d^{\mathfrak{F}}_{\tilde{S}_z}(x,y).$$

Moreover, the image $\tilde{S}^\iota_z = \tilde{Q}_z(\tilde{S}_z)$ is C_σ-Finsler-quasiconvex in the sense that for all $x, y \in \tilde{Q}_z(\tilde{S}_z)$, there is a Finsler geodesic from x to y at distance at most C_σ from $\tilde{S}^\iota_z$.

The facts that $\tilde{Q}_z$ is injective and $\tilde{S}^\iota_z$ is quasi-convex have already been established in Corollaries 5.6 and 5.11.

Note that the upper bound for $d^{\mathfrak{F}}(\tilde{Q}_z(x), \tilde{Q}_z(y))$ is an immediate consequence of the definition of $d^{\mathfrak{F}}_{\tilde{S}_z}$ as the pullback of $d^{\mathfrak{F}}$.

The remaining estimate will be obtained as a consequence of Observation 3.9 and an intermediate proposition stating that the images of admissible paths in $\mathbb{X}$ are quasi-geodesics, with control on the multiplicative error term. This intermediate result will be proved using Proposition 5.10 (that admissible paths are quasi-ruled) and Proposition 5.14 (a lower bound on the displacement of admissible paths).

The collar lemma for hyperbolic surfaces states that for any $\sigma > 0$, if

$$\omega = \sinh^{-1}\left(\frac{1}{\sinh(\sigma/2)}\right)$$

then any simple closed geodesic $\gamma^* \subset S$ of length at most σ will have a *collar size* bounded from below by ω, in the sense that $N_\omega(\gamma^*) = \{z \mid d(z, \gamma^*) \leq \omega\}$ is an annulus. Then Observation 3.9 says that for any multicurve $\gamma^* \subset S$ with components of length at most σ, for any grafting parameter z, the image of an admissible path of $\widetilde{S}_z$ under $\widetilde{Q}_z : \widetilde{S}_z \to \mathbb{X}$ will be a $(\omega, 0)$-admissible path of $\mathbb{X}$.

We will also prove the following coarse estimates on lengths.

Theorem 6.2. *In the setting of Theorem 6.1, let $(\rho_z)_z$ be the associated family grafted Hitchin representations. Then there is C'_σ only depending on σ such that for any $\gamma \in \pi_1(S)$,*

$$\ell^{\mathfrak{F}}(\rho_z(\gamma)) \geq \frac{L+1}{C'_\sigma} \iota(\gamma, \gamma^*).$$

Moreover, recalling that z is the datum of a vector $z_e \in \mathfrak{a}$ for each component $e \subset \gamma^*$, then C'_σ may be chosen so that if $z_e \in \ker(\alpha_0)$ for any e then

$$\ell^{\mathfrak{F}}(\rho_z(\gamma)) \geq \left(1 + \frac{C'_\sigma}{L+1}\right)^{-1} \ell_S(\gamma),$$

where $\ell_S(\gamma)$ is the length of γ in S.

6.1 Elementary observations

The triangle inequality easily yields the following.

Observation 6.3. *For any piecewise geodesic curve* $c : [0, T] \to (\mathbb{X}, d^{\mathfrak{F}})$ *with* $m \geq 1$ *geodesic pieces, that is additionally* C*-quasi-ruled, we have*

$$d^{\mathfrak{F}}(c(0), c(T)) \geq \mathrm{Len}^{\mathfrak{F}}(c) - (m-1)C.$$

We will also need the following technical estimates.

Lemma 6.4. *Consider* $a, b, k, L \geq 0$ *and* $C \geq 1$, *such that*

$$a \geq b - kC \quad \text{and} \quad b \geq (k-2)L \quad \text{and} \quad a \geq \frac{k-2}{C}.$$

Then

$$a \geq \left(1 + \frac{4(C+1)^3}{L+1}\right)^{-1} b - 2C.$$

Proof. We can assume $L > 0$. Consider first the case that $L \geq 2C \geq 2$. We use the second equation and get

$$k \leq \tfrac{b}{L} + 2$$

By the first equation, we have

$$a \geq b(1 - \tfrac{C}{L}) - 2C$$

with the additional inequality

$$1 - \tfrac{C}{L} \geq \left(1 + 2\tfrac{C}{L}\right)^{-1} \geq \left(1 + \tfrac{4(C+1)^3}{L+1}\right)^{-1}$$

(indeed one can check that $0 \leq x \leq \frac{1}{2}$ implies $(1-x)(1+2x) \geq 1$).

If $L < 2C$ then use the third equation and get $k \leq \frac{b}{L} + 2$ and so

$$a \geq \frac{b - 2C}{1 + C^2} \geq \frac{b}{1 + C^2} - 2C$$

followed by

$$\left(1 + C^2\right)^{-1} \geq \left(1 + \tfrac{4(C+1)^3}{L+1}\right)^{-1}.$$

□

6.2 Admissible paths are uniform quasi-geodesics

To prove that the embedding $\widetilde{Q}_z : \widetilde{S}_z \to X$ is quasi-isometric, by Observation 3.9 it suffices to show that admissible paths in $\mathbb{X}$ are quasi-geodesics. We prove this now with uniform constants using the consequences of positivity established in Section 5 and the elementary observations of the previous section.

Proposition 6.5. *For all $\omega > 0$, there exist $C_\omega > 0$ such that for every $L \geq 0$, all (ω, L)-admissible paths are $\left(1 + \frac{C_\omega}{(L+1)}, C_\omega\right)$-quasi-Finsler-geodesics (where the first constant is the multiplicative constant).*

Proof. Let $c : [0, T] \to \mathbb{X}$ be an (ω, L)-admissible path. It is clear that $d^{\mathfrak{F}}(c(0), c(T)) \leq \mathrm{Len}^{\mathfrak{F}}(c)$, so we only need to obtain a converse inequality.

By Proposition 5.10, c is C_ω-quasi-ruled, for some constant C_ω depending on ω. By Observation 6.3 we get

$$d^{\mathfrak{F}}(c(0), c(T)) \geq \mathrm{Len}^{\mathfrak{F}}(c) - kC_\omega,$$

where k is the number of singularities of c (i.e. $k+1$ is the number of geodesic pieces).

Since c contains at least $(k-2)/2$ geodesic pieces of flat type and length at least L, we also know that

$$\mathrm{Len}^{\mathfrak{F}}(c) \geq (k-2)\tfrac{L}{2}.$$

Finally by Proposition 5.14 we also have

$$d^{\mathfrak{F}}(c(0), c(T)) \geq \frac{k-2}{C'_\omega}$$

for some constant C'_ω depending on ω.

We conclude thanks to Lemma 6.4. □

6.3 Proof of Theorem 6.1

It is an immediate consequence of Observation 3.9, Corollary 5.6, Corollary 5.11 and Proposition 6.5. More precisely, by compactness of S, Corollary 5.6 ensures that the natural map $S_z \xrightarrow{Q_z} \rho_z \backslash \mathbb{X}$ is an embedding.

Consider an equivariant lift $\tilde{Q}_z : \tilde{S}_z \to \mathbb{X}$ of Q_z. By Corollary 5.11, any two points in $\tilde{Q}_z(\tilde{S}_z)$ can be connected by a Finsler geodesic in $\mathbb{X}$ which remains at distance at most C_ω from $Q_z(S_z)$. This establishes the last statement in Theorem 6.1. Note that these Finsler geodesics then project to Finsler geodesics in the quotient $\rho_z \backslash \mathbb{X}$ in the C_ω-neighborhood of $Q_z(S_z)$.

To show the distance-length control, note that any admissible path in S_z is sent by Q_z to an (ω, L)-admissible path inside $\rho_z \backslash \mathbb{X}$ (Observation 3.9), which is therefore a quasi-geodesic (Proposition 6.5).

6.4 Proof of Theorem 6.2

Fix $[\gamma] \in [\pi_1(S)]$ transverse to γ^* and recall that it corresponds to a free homotopy class in the characteristic surface $S_z^\iota \subset \rho_z \backslash \mathbb{X}$. Let $c \subset S_z^\iota \subset \rho_z \backslash \mathbb{X}$ be the unique admissible loop in this free homotopy class. It has a $\rho_z(\gamma)$-invariant lift $\tilde{c} : \mathbb{R} \to \mathbb{X}$ such that 0 is a singularity and the geodesic piece of $\tilde{c}$ starting at time 0 is of hyperbolic type.

Denote by T the period of c, so that $\widetilde{c}(t+T) = \rho_z(\gamma)\widetilde{c}(t)$ for any t. By Proposition 6.5 for any $n \geq 1$ we have

$$d^{\mathfrak{F}}(\widetilde{c}(0), \rho_z(\gamma)^n \widetilde{c}(T)) = d^{\mathfrak{F}}(\widetilde{c}(0), \widetilde{c}(n\ell)) \geq \left(1 + \frac{C_\sigma}{L+1}\right)^{-1} n \operatorname{Len}^{\mathfrak{F}}(c) - C_\sigma.$$

Dividing by n and letting $n \to \infty$ we get

$$\ell^{\mathfrak{F}}(\rho_z(\gamma)) \geq \left(1 + \frac{C_\sigma}{L+1}\right)^{-1} \operatorname{Len}^{\mathfrak{F}}(c). \tag{27}$$

We may assume $\iota(\gamma, \gamma^*) \geq 1$ (the case $\iota(\gamma, \gamma^*) = 0$ is trivial). The number $2\iota(\gamma, \gamma^*)$ of singularities of c is even and bounded from below by 2, and also is the number of geodesic pieces, half of which are of flat type and have length at least L, and the other half are of hyperbolic type and have length at least $\omega = \sinh^{-1}(\sinh(\sigma/2)^{-1})$, so

$$\ell^{\mathfrak{F}}(\rho_z(\gamma)) \geq \left(1 + \frac{C_\sigma}{L+1}\right)^{-1} \frac{L+\omega}{2} \iota(\gamma, \gamma^*) \geq \frac{L+\omega}{2+2C'_\sigma} \iota(\gamma, \gamma^*).$$

Finally, if for each component $e \subset \gamma^*$ the vector z_e is taken in $\ker(\alpha_0)$ then we can apply Lemma 3.2, which says that $\operatorname{Len}^{\mathfrak{F}}(c)$ is bounded from below by the length of the image of c under the projection maps $S_z \to S$, which is itself greater than or equal to $\ell_S(\gamma)$, and this means Equation 27 implies the desired inequality.

6.5 Proof of part (3) of Theorem C

Let S be a closed hyperbolic surface. Choose a simple closed geodesic $\gamma \subset S$ and if γ is separating, choose a component S_1 of S. Put $S_1 = S - \gamma$ otherwise. Consider a one-parameter family ρ_t of Hitchin grafting representations with grating datum tz for a point $z \in \mathfrak{a}$ which is linearly independent from the direction of a tangent vector of $\mathbb{H} \subset \mathbb{X}$. Then for each t, the bordered hyperbolic surface S_1 is totally geodesic embedded in $\rho_t \backslash \mathbb{X}$.

Choose once and for all a basepoint $x \in S_1$ and view this a basepoint in $\rho_t \backslash \mathbb{X}$ for all t. By Theorem 6.2 and equivalence of the Riemannian and the Finsler metric, for each $R > 0$ there exists a number $t = t(R) > 0$ so that the shortest closed geodesic in $\rho_t \backslash \mathbb{X}$ which is not contained in S_1 intersects the complement of the R-ball about S_1. Thus for $t > t(R)$, the normal injectivity radius for the Riemannian metric is at least R and the ball $B(S_1, R)$ of radius R about S_1 is homotopy equivalent to S_1.

By passing to a subsequence, we may assume that the pointed manifolds $(\rho_t \backslash \mathbb{X}, x)$ converge in the pointed Gromov Hausdorff topology to a locally symmetric pointed manifold (N, x). This manifold contains S_1 as a totally geodesic embedded surface of infinite normal injectivity radius. But this just means that N equals the manifold defined by the Fuchsian representation $\rho|S_1$. This is precisely the statement of the third part of Theorem C.

7 Intersection in the Hitchin component

This section contains a first application of the results from Section 6. Namely, recall from Section 2 the definition of the intersection number $\mathbf{I}(f_1, f_2)$ and the normalized intersection number $\mathbf{J}(f_1, f_2)$ for two Hölder continuous positive functions f_1, f_2 on T^1S. These numbers only depend on the cohomology classes of f_1, f_2. Thus by the results in Section 2.2, for any two Hitchin representations $\rho_1, \rho_2 : \pi_1(S) \to \mathrm{PSL}_d(\mathbb{R})$, we obtain intersection numbers $\mathbf{I}(\rho_1, \rho_2)$ and normalized intersection numbers $\mathbf{J}(\rho_1, \rho_2)$. We show

Theorem 7.1. *There exists a sequence ρ_i of Hitchin representations such that $\mathbf{I}(\nu, \rho_i) \to \infty$ and $\mathbf{J}(\nu, \rho_i) \to \infty$ for any Fuchsian representation ν, and this divergence is uniform in ν.*

The Hitchin representations which enter Theorem 7.1 are Hitchin grafting representations. More precisely, let as before γ be a simple closed geodesic on the hyperbolic surface S. This datum is used to construct for each $L > 0$ a Hitchin representation ρ_L obtained by Hitchin grafting along γ of the Fuchsian representation defined by S, with cylinder height L. We do not specify the twisting number of the associated abstract grafting datum as this does not play a role in our discussion, but we assume that $L \to \rho_L$ is a Hitchin grafting ray as introduced in Section 3.3.

The proof of Theorem 7.1 rests on statistical information on length averages, introduced in the next definition. For its formulation, for a Hitchin representation ρ put $R_\rho(T) = R_{\ell_\rho}(T)$ for all T, where as before, $R_{\ell_\rho}(T) = \{\eta \in [\pi_1(S)] \mid \ell_\rho(\eta) \le T\}$ and $\ell_\rho(\eta)$ is the Finsler translation length of $\rho(\eta)$. Moreover, $[\pi_1(S)]$ is the set of conjugacy classes of the fundamental group $\pi_1(S)$ of S.

Definition 7.2. Let ρ be a Hitchin representation and A a subset of $[\pi_1(S)]$. We say that A is a *full density* set for ρ if

$$\liminf_{T \to +\infty} \frac{R_\rho(T) \cap A}{R_\rho(T)} = 1.$$

If $\mathcal{P}$ is an assertion on $[\pi_1(S)]$, we say that a *typical geodesic satisfies* $\mathcal{P}$ if the set $\{\gamma \in [\pi_1(S)] \mid \gamma \text{ satisfies } \mathcal{P}\}$ is a full density set for ρ.

The following statement can be thought of as a statistical version of the duality between length and intersection for hyperbolic metrics on surfaces. Recall from Section 2.1 the definition of the intersection form $\iota : \mathcal{C}(S) \times \mathcal{C}(S) \to [0, \infty)$.

Proposition 7.3. *Let ρ be a hyperbolic metric on S, and let $\alpha \subset S$ be a closed geodesic. For any $\epsilon > 0$, for a typical geodesic γ, we have*

$$\left| \iota(\gamma, \alpha) - \frac{1}{-4\pi^2 \chi(S)} \ell_\rho(\gamma)\ell_\rho(\alpha) \right| < \epsilon \ell_\rho(\gamma).$$

Proof. The Borel measures

$$\mu_T = \frac{1}{\# R_\rho(T)} \sum_{\ell_\rho(\gamma) \le T} \mathrm{Leb}_\gamma$$

converge weakly as $T \to \infty$ to the normalized Lebesgue Liouville measure λ_0 on T^1S (see [Mar04]).

Let $\lambda \in \mathcal{C}(S)$ be the (unnormalized) Liouville *current* of ρ, the current defined by the Lebesgue Liouville measure on T^1S, and for each T let $\hat{\mu}_T$ be the current defined by μ_T. Let α be a closed geodesic on S. As $\iota(\alpha, \lambda) = \ell_\rho(\alpha)$ (see Section 2.1), by continuity of the intersection form ι for the weak topology on currents, we know that

$$\iota(\hat{\mu}_T, \alpha) \underset{T\to\infty}{\longrightarrow} \frac{1}{-4\pi^2\chi(S)}\ell_\rho(\alpha).$$

Note to this end that the total volume of T^1S with respect to the Lebesgue Liouville current equals $-4\pi^2\chi(S)$.

Put $\kappa = \frac{1}{-4\pi^2\chi(S)}$ and let $\epsilon > 0$. To show that the geodesics γ with

$$|\iota(\gamma, \alpha) - \kappa\ell_\rho(\gamma)\ell_\rho(\alpha)| < \epsilon\kappa\ell_\rho(\gamma)$$

are typical we argue as follows. For $T > 0$ let

$$\mathcal{A}(T) = \{\gamma \mid \ell_\rho(\gamma) \leq T, \iota(\gamma, \alpha) \geq (1+\epsilon)\kappa\ell_\rho(\gamma)\ell_\rho(\alpha)\}.$$

We claim that $\frac{\#\mathcal{A}(T)}{\#R_\rho(T)} \to 0$ $(T \to \infty)$.

To see this assume otherwise. By passing to a subsequence, we may assume that the measures $\nu_T = \frac{1}{\#R_\rho(T)}\sum_{\gamma\in\mathcal{A}(T)} \mathrm{Leb}_\gamma$ converge weakly to a nontrivial Φ^t-invariant measure ν. By construction, the measure ν is absolutely continuous with respect to the Lebesgue Liouville measure λ_0. It defines a current $\hat{\nu}$ which satisfies

$$\iota(\hat{\nu}, \alpha)/\nu(T^1S) \geq (1+\epsilon)\kappa\ell_\rho(\alpha). \tag{28}$$

But λ is ergodic under the action of Φ^t and hence as ν is absolutely continous with respect to λ, it is a positive constant multiple of λ. This contradicts the inequality (28).

In the same way we conclude that $\frac{\#\mathcal{B}(T)}{\#R_\rho(T)} \to 0$ as $T \to \infty$ where

$$\mathcal{B}(T) = \{\gamma \mid \ell_\rho(\gamma) \leq T, \iota(\gamma, \alpha) \leq (1-\epsilon)\kappa\ell_\rho(\gamma)\ell_\rho(\alpha)\}.$$

Since $\epsilon > 0$ was arbitrary, this shows the proposition. □

Let X be a hyperbolic metric on S and let c be a non-separating simple closed geodesic on X of length $\ell > 0$. For $L \geq 0$ denote by ρ_L a representation obtained by Hitchin grafting of X on c of height L. Our goal is to estimate for a hyperbolic metric Y on S the quantities $\mathbf{I}(Y, \rho_L)$ and $\mathbf{J}(Y, \rho_L)$ as $L \to \infty$.

Proof of Theorem 7.1. Let $X \in \mathcal{T}(S)$ be the marked hyperbolic metric which is the basepoint for the Hitchin grafting ray. According to the length control as formulated in Theorem 6.2, for every $\epsilon > 0$ there exist $C_\sigma > 0$ depending on the hyperbolic length σ of the simple closed curve c such that we have

$$\ell_{\rho_L}(\gamma) \geq \max\left\{C_\sigma L\iota(\gamma, c), \frac{L}{L + C_\sigma^{-1}}\ell_X(\gamma)\right\} \tag{29}$$

where we use the notations of Theorem 6.2, lengths in $\mathbb{X}$ are measured with respect to an admissible Finsler metric, and ℓ_X denotes the length for the hyperbolic metric X.

Let $m > 0$ be a fixed number. Our goal is to find a number $L > 0$ so that

$$\mathbf{J}(Y, \rho_L) \geq m$$

for every $Y \in \mathcal{T}(S)$ where as before, $\mathcal{T}(S)$ denotes the Teichmüller space of marked hyperbolic metric on S.

By Theorem 12 of [Bon88], the map which associates to a marked hyperbolic metric on S its Liouville current is a proper topological embedding. More precisely, for the given number $m > 0$, there exists a compact ball B about X in $\mathcal{T}(S)$ such that $\iota(\lambda_X, \lambda_Y) \geq m$ for all marked hyperbolic metrics $Y \in \mathcal{T}(S) - B$, where λ_X, λ_Y are the currents defined by the normalized Lebesgue Liouville measures. Note that this is symmetric in X, Y. Furthermore, we have $\iota(\lambda_Y, \lambda_X) = \mathbf{J}(Y, X)$. We refer to p.152-153 in [Bon88] for details on these facts.

By the estimate (29), for any $\epsilon > 0$ and all sufficiently large $L \geq 0$ depending on ϵ, say for all $L \geq L(\epsilon)$, we have

$$\ell_{\rho_L}(\gamma) \geq (1 - \epsilon)\ell_X(\gamma).$$

Thus by possibly increasing the ball B we may assume that $\mathbf{J}(Y, \rho_L) \geq m$ for all $L \geq L_0$ and all $Y \notin B$.

We are left with showing that by possibly increasing L_0, we also have $\mathbf{J}(Y, \rho_L) \geq m$ for all $Y \in B$. However, this follows once more from the estimate (29). Namely, let $Y \in B$. By Proposition 7.3, we know that there exists a constant $\kappa > 0$ such that

$$\iota(\gamma, c) \geq \kappa(1 - \epsilon)\ell_Y(\gamma)\ell_Y(c)$$

for any geodesic γ which is typical for Y.

On the other hand, by compactness of B, there exists a constant $\sigma > 0$ such that $\ell_Y(c) \geq \sigma$ for every $Y \in B$. Then for a geodesic γ which is typical for Y, we have $\ell_Y(\gamma) \leq \frac{1}{\kappa\sigma(1-\epsilon)}\iota(\gamma, c)$. Thus for $L > m/\kappa\sigma(1 - \epsilon)C_\sigma$ it holds

$$\ell_{\rho_L}(\gamma)/\ell_Y(\gamma) \geq \kappa\sigma(1 - \epsilon)C_\sigma L \geq m$$

which is what we wanted to show. Together with the definition, this shows that $\mathbf{I}(\nu, \rho_L) \to \infty$ for every Fuchsian representation ν.

To show that we also have $\mathbf{J}(\nu, \rho)_L) \to \infty$ for all Fuchsian representations it suffices to observe that the entropy of ρ_L is bounded from below by a universal positive constant. To see that this is the case, recall that for each L, the restriction of the representation ρ_L to the free subgroup Λ of $\pi_1(S)$ of all based loops which do not cross through c does not depend on L. In particular, the image of Λ under ρ_L stabilizes a totally geodesic hyperbolic plane in $\mathbb{X}$. As a consequence, for each L the entropy of ρ_L is not smaller than the entropy of the geodesic flow on the bordered surface $S - c$, which is positive as $S - c$ is a hyperbolic surface with geodesic boundary. Together with the control on $\mathbf{I}(\nu, \rho_L)$ established in the beginning of this proof, this implies that $\mathbf{J}(\nu, \rho_L) \to \infty$ $(L \to \infty)$ for any Fuchsian representation ν. □

8 Upper bound on the derivatives of length functions via Ehresmann connections

In this section we show how to control the first and second derivatives of the Finsler length for paths of Hitchin grafting representations. The section is divided into three subsection.

8.1 Derivative bounds for lengths of closed geodesics

For simplicity, all Hitchin grafting representations will be obtained by grafting a Fuchsian representation ρ of the fundamental group of a surface S of genus $g \geq 2$ along a single simple closed geodesic $\gamma \subset S$. If γ is separating, then we let S_0, S_1 be the metric completions of the two components of $S - \gamma$. Recall that a Fuchsian representation is determined by a marked hyperbolic structure on S.

Resuming the notations from Section 3, consider a discrete and faithful representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R}) \to G = \mathrm{PSL}_d(\mathbb{R})$ and let $u = d\tau \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{a}$ be the special direction, normalized to be tangent to a preimage of $\rho(\gamma)$. Choose a vector $z \in \mathfrak{a}$ transverse to u and for $t > 0$ denote by

$$\rho_t : \pi_1(S) \to G$$

the Hitchin grafting representation defined by tz. We assume that the normalization of the vector z is such that the Finsler width of the flat strip in $\mathfrak{a}$ bounded by the line $\mathbb{R}u$ and its translate by z equals one. This amounts to saying that in the abstract grafting model for ρ_t, the height of the cylinder inserted at γ equals precisely t.

Recall that according to Theorem 6.1, the geometry of ρ_t is coarsely governed by the characteristic surface $S_z^\iota \subset \rho_t \backslash \mathbb{X}$, which is a piecewise totally geodesic embedded surface. Its lift to $\mathbb{X}$ looks like terraces with horizontal pieces conjugate into $\mathbb{H}^2$ and vertical pieces conjugate into the standard flat, see Figure 1.

The first main result of this section provides a quantitative control on the change of lengths of closed geodesics along the grafting path $t \to \rho_t$. For its formulation, recall that for any abstract grafted surface S_z, any $\eta \in \pi_1(S)$ can be represented by a unique admissible path (up to free homotopy). This admissible path is a piecewise geodesic which is a concatenation of hyperbolic and flat pieces. We denote by $\ell_z^{flat}(\eta)$ the sum of the lengths of the flat pieces, and by $\ell_z^{hyp}(\eta)$ the sum of the length of the hyperbolic pieces.

Proposition 8.1. *For any $\epsilon > 0, \sigma > 0$ there exists $L = L_{\sigma,\epsilon} > 0$ and $A = A_{\sigma,\epsilon} > 0$ with the following properties. Suppose that the injectivity radius of the Fuchsian representation ρ is at least ϵ and that $\ell_\rho(\gamma) \leq \sigma$ and that $t_0 \geq L$. Then for any $\eta \in \pi_1(S)$, we have*

$$\left| \frac{d}{dt} \ell_{\rho_t}(\eta)|_{t=t_0} \right| \leq A \ell_{t_0 z}^{flat}(\eta) \quad \text{and} \quad \frac{d^2}{dt^2} \ell_{\rho_t}(\eta)|_{t=t_0} \leq A \ell_{t_0 z}^{flat}(\eta).$$

Another way to deform the grafted representation ρ_t is to deform ρ, in other words alter the hyperbolic metric on S. We shall analyze this situation in the case that γ is separating and that one only changes ρ on the component S_1 of $S-\gamma$. Thus we consider a path $s \to \rho^s$ of Fuchsian representations whose restrictions to the fundamental group $\pi_1(S_0) < \pi_1(S)$ is constant. For a fixed grafting parameter z and the representation ρ_z obtained by grafting ρ along γ with parameter z we can consider the path $s \to \rho_z^s$. The second main result of this section states that the length of a curve along this path only changes proportionally to the time spent in S_1.

Proposition 8.2. *For any $\epsilon > 0, \sigma > 0$ there exists $L = L_{\sigma,\epsilon} > 0$ and $a = a_{\sigma,\epsilon} > 0$ with the following properties. Suppose that the injectivity radius of the Fuchsian representation ρ is at least ϵ, that $\ell_\rho(\gamma) \leq \sigma$ and that the cylinder height of the grafting by $u \in \mathfrak{a}$ is at least L. Let $(\rho^s)_{-\epsilon<s<\epsilon}$ be a smooth path of Fuchsian representations, parameterized by arc length with respect to the Teichmüller metric and such that $\rho^0 = \rho$ and $\rho^s(\eta) = \rho(\eta)$ for any $\eta \in \pi_1(S_0)$. Then for any $\eta \in \pi_1(S)$, we have*

$$\left|\frac{d}{ds}\ell_{\rho_u^s}(\eta)|_{s=0}\right| \leq a\ell_\rho^{S_1}(\eta) \quad \textit{and} \quad \frac{d^2}{ds^2}\ell_{\rho_u^s}(\eta)|_{s=0} \leq a\ell_\rho^{S_1}(\eta),$$

where $\ell_\rho^{S_1}(\eta)$ is the length of $\hat{\eta} \cap S_1$ for the admissible path $\hat{\eta}$ freely homotopic to η with respect to the hyperbolic metric on S_1 induced by ρ (which does not depend on u),

Remark . *The dependence of the constants in Proposition 8.1 Proposition 8.2 on an upper bound of the length of the grafting geodesic seems necessary, but this is less clear for the dependence on the injectivity radius of ρ.*

Let us stress that the above results are not elementary or quick consequences of the estimates of Theorems 6.1 and 6.2 which, by their *coarse* nature, do not directly give estimates on the derivatives of lengths. In fact, the proof only uses Finsler quasi-convexity of the characteristic surface $\widetilde{S}_z^\iota \subset \mathbb{X}$ of a Hitchin grafting representation as established in Theorem 6.1. This implies that every $\eta \in \pi_1(S)$ has a (Finsler) geodesic representative in $\rho_z \backslash N_C(S_z^\iota)$, where $N_C(S_z^\iota)$ is the C-neighbourhood of S_z^ι in $\rho_z \backslash \mathbb{X}$ and where $C > 0$ only depends only on σ.

The strategy is as follows. Consider a path $s \to \rho_u^s$ of deformations of the grafted representation ρ_u considered in Propositions 8.1 or 8.2. Thus we have $u = t_0 z$ for some sufficiently large t_0 and either $\rho_s = \rho_{(s+t_0)z}$ as in Proposition 8.1, or $\rho_u^s = \rho_{t_0 z}^s$ as in Proposition 8.2. Let also $(S^\iota)_u^s \subset \rho_u^s \backslash \mathbb{X}$ be the associated characteristic surfaces, and N_u^s their uniform C-neighbourhoods. We are going to construct geometrically controlled diffeomorphisms $N_u = N_u^0 \to N_u^s$ such that the pullbacks $\mathfrak{F}_s$ to N_u of the Finsler metrics on the sets N_u^s depend smoothly on s, and their first and second time-derivatives $\partial_s \mathfrak{F}_s$ and $\partial_s^2 \mathfrak{F}_t$ are pointwise uniformly bounded at regular tangent vectors. Furthermore, these derivatives vanish on a neighborhood of controlled size of the hyperbolic parts of the surface which are left fixed by the deformation.

For any $\eta \in \pi_1(S)$, fixing a geodesic representative $\hat{\eta} \subset N_u$ for $\mathfrak{F} = \mathfrak{F}_0$ which we can assume to be piecewise smooth, we will have

$$\ell_{\rho_u^s}(\eta) \leq \int \mathfrak{F}_s(\hat{\eta}'(t))dt$$

with equality at $s = 0$. As $\hat{\eta}$ is a geodesics and hence of minimal Finsler length, this will imply that the first derivatives at $s = 0$ of the two sides are equal. We then apply a trick due to Pollicott [Pol94] and deduce that the second derivative of the left hand side is bounded from above by the second derivative of the right hand side. Then we shall conclude using the estimates on $\partial_s \mathfrak{F}_t$ and $\partial_s^2 \mathfrak{F}$.

8.2 Controlled Ehresmann connections

To set up the proof, recall from Proposition 2.8 that each of the representations ρ_u^s defines a length cocycle ℓ_u^s which can be represented by a Hölder continuous positive function f_u^s on the unit tangent bundle T^1S of S. The function f_u^s is only well defined up to a coboundary, that is, by a Hölder function whose integral over each periodic orbit vanishes, but locally near $\rho_0 = \rho_u^0$, it can be chosen to depend in an analytic fashion on the representation ρ_u^s.

Recall that for each s, the image $\rho_u^s(\pi_1(S))$ is a discrete subgroup of $\mathrm{PSL}_d(\mathbb{R})$ acting freely on the symmetric space $\mathbb{X}$. The quotient under this action is a locally symmetric manifold M_s with fundamental group isomorphic to $\pi_1(S)$. Put $M = M_0$ and $\rho_u = \rho_u^0$ for simplicity of notation.

The group $\rho_u^s(\pi_1(S))$ acts via the adjoint representation of $\mathrm{PSL}_d(\mathbb{R})$ on the Lie algebra $\mathfrak{sl}(d, \mathbb{R})$, and it acts from the left on $\mathbb{X}$. Then $V = \rho_u^s \backslash (\mathbb{X} \times \mathfrak{sl}(d, \mathbb{R}))$ is a flat vector bundle $V \to M_0$ with fiber $\mathfrak{sl}(d, \mathbb{R})$ over $M_s := \rho_u^s \backslash \mathbb{X}$. The tangent space at ρ_u^s of the deformation space of ρ_u^s can be identified with the first cohomology group $H^1(M_s, V)$ of M_s with values in V.

Consider the trivial bundle

$$\Pi : \mathcal{E} = \mathbb{X} \times (-1, 1) \to (-1, 1).$$

Each of the fibers of the bundle will be equipped with the symmetric metric. In particular, there is a natural smooth Riemannian metric on the *vertical bundle*, that is, the kernel of $d\Pi$. The group $\pi_1(S)$ acts on $\mathcal{E}$ as a group of smooth fiberwise isometric bundle maps by

$$\psi(x, s) = (\rho_u^s(\psi)(x), s) \quad (\psi \in \pi_1(S)).$$

Namely, by the construction of the path $s \to \rho_u^s$, for each $\psi \in \pi_1(S)$ the path $s \to \rho_u^s(\psi)$ is a smooth path in the group $\mathrm{PSL}_d(\mathbb{R})$ and hence $(x, s) \to \rho_u^s(\psi(x))$ is smooth. This action is also fiber preserving, and the differentials preserve the Riemannian metric on the vertical bundle. The quotient $\pi_1(S) \backslash \mathcal{E}$ has again a natural structure of a fiber bundle over $(-1, 1)$, however this bundle is not trivial.

An *Ehresmann connection* in $\mathcal{E}$ is the choice of a smooth *horizontal bundle* $H \subset T\mathcal{E}$ which is complementary to the vertical bundle $\ker d\Pi$. Given such an Ehresmann connection H, the basic vector field $\frac{\partial}{\partial t}$ on $(-1, 1)$ lifts to a smooth section of H which induces a fiber preserving flow on $\mathcal{E}$ (at least locally). Furthermore, an Ehresmann connection determines a smooth Riemannian metric on $\mathcal{E}$ which restricts to the given metric on the fibers, for which the decomposition $T\mathcal{E} = \ker d\Pi \oplus H$ is orthogonal and such that the projection Π is a Riemannian submersion. Note that as the bundle $\mathcal{E}$ is *naturally* a product, its product structure defines an Ehresmann connection on $\mathcal{E}$, however this connection does not descend to $\pi_1(S)\backslash\mathcal{E}$.

Our first goal is to construct a geometrically controlled $\pi_1(S)$-invariant Ehresmann connection on $\mathcal{E}$. Such an Ehresmann connection then descends to an Ehresmann connection on $\pi_1(S)\backslash\mathcal{E}$.

For $s \in (-1,1)$ let S_s be the surface obtained from S by abstract grafting at γ with grafting datum determined by ρ_u^s. By Proposition 3.5 and Proposition 5.6, there exists a natural path isometric piecewise totally geodesic embedding $Q_s : S_s \to M_s$ whose image is called the characteristic surface of M_s. The characteristic surface of M_s depends continuously on s. Hence the union of these surfaces as s ranges through the open interval $(-1,1)$ is the image in $\pi_1(S)\backslash\mathcal{E}$ of $S \times (-1,1)$ under a continuous (and in fact piecewise smooth) map. We call this image $\mathcal{S}$ the *characteristic family* for $\pi_1(S)\backslash\mathcal{E}$.

Let $S_1 \subset S - \gamma$ be a component of $S - \gamma$. If γ is non-separating then $S_1 = S - \gamma$. Choose a basepoint $p \in S_1 \subset S - \gamma$. We assume that the path $t \to \rho_u^s$ is such that up to adjusting with a conjugation, the restriction of ρ_u^s to the subgroup $\pi_1(S_1,p)$ of $\pi_1(S,p)$ does not depend on s. Then the set

$$V = \bigcup_{-1<t<1} (Q_s(S_1), s) \subset \mathcal{S} \subset \pi_1(S)\backslash\mathcal{E}$$

is a smooth embedded submanifold with boundary. Furthermore, the product metric on $\mathcal{E}$ descends to a smooth metric on V. In particular, the horizontal vector field $\frac{\partial}{\partial s}$ on the bundle $\mathcal{E} \to (-1,1)$ descends to a vector field on the bundle $V \subset \mathcal{S} \subset \pi_1(S)\backslash\mathcal{E} \to (-1,1)$ and hence it defines an Ehresmann connection on V.

We aim at extending this Ehresmann connection to a geometrically controlled Ehresmann connection on a neighborhood of $\mathcal{S} \subset \pi_1(S)\backslash\mathcal{E}$ of prescribed radius. To construct such a connection we work directly in $\mathcal{E}$. Note that the preimage $\tilde{\mathcal{S}}$ of the characteristic family $\mathcal{S}$ in $\mathcal{E}$ is homeomorphic to a disk bundle over $(-1,1)$. We call this preimage again a characteristic family. A component of the preimage in $\mathcal{E}$ of the trivial bundle $V \subset \mathcal{S}$ is a trivial subbundle $\tilde{V}$ of $\mathcal{E}$.

Throughout this section, distance, norms $\| \; \|$ in tensor bundles and covariant derivatives ∇ are taken with respect to the product Riemannian metric on $\mathcal{E}$. Proposition 8.3 below will be used to control the pressure metric along the paths $t \to \rho_u^s$ introduced above. Recall that the starting time parameter t_0 of the deformation encodes the cylinder height of the grafting representation.

Proposition 8.3. *For every $R > 0, \epsilon > 0, \sigma > \varepsilon$ there exist numbers $\tau(R,\epsilon,\sigma) > 0$ and $C = C(R,\sigma) > 0$ only depending on R, ϵ and σ with the following property.*

Assume that the systole of the Fuchsian representation ρ is at least ϵ and that the length of the grafting geodesic is at most σ. Let $t_0 \geq \tau(R,\epsilon,\sigma)$; then there exists a $\pi_1(S)$-invariant Ehresmann connection on the R-neighborhood $N_R(\tilde{\mathcal{S}})$ of the characteristic family $\tilde{\mathcal{S}} \subset \mathcal{E}$ which is spanned by a smooth $\pi_1(S)$-invariant vector field ξ on $N_R(\tilde{\mathcal{S}})$ with the following properties.

1. *$d\Pi(\xi) = \frac{\partial}{\partial t}$ on $N_R(\tilde{\mathcal{S}})$.*

2. $\xi = \frac{\partial}{\partial t}$ *on the R-neighborhood of* $\tilde{V}$.

3. $\sup\{\|\xi\|_x, \|\nabla\xi\|_x \mid x \in N_R(\tilde{\mathcal{S}})\} \le C$.

Before we present the proof of Proposition 8.3 we begin with an observation about hyperbolic surfaces which is probably well known but hard to find in the literature. This observation will be the starting point for the proof of Proposition 8.3.

For its formulation, note that any hyperbolic surface S can be cut open along disjoint simple geodesic arcs based at a fixed point $p \in S$ such that the resulting polygon D with $4g$ geodesic sides is a (not necessarily convex) fundamental domain for the action of $\pi_1(S)$ on $\mathbb{H}^2$ and that S is the quotient of D by standard isometric side pairing transformations. For a counterclockwise cyclic order $a_1, b_1, a_1^{-1}, b_1^{-1}, \dots, a_g, b_g, a_g^{-1}, b_g^{-1}$ of the oriented edges of D, the side pairings identify a_i with a_i^{-1} and b_i with b_i^{-1}. Similarly, if S is a compact hyperbolic surface of genus $g \ge 1$ with connected geodesic boundary, then there is a fundamental polygon D for the action of $\pi_1(S)$ on the universal covering $\tilde{S} \subset \mathbb{H}^2$ of S with $4g+1$ sides and such that the last side is contained in a boundary geodesic of $\tilde{S}$ and projects to the boundary of S. We call such fundamental domains *standard*. For clarity, we call the side of D contained in a boundary geodesic of $\tilde{S}$ the *open side* of D.

Lemma 8.4. *For all $\sigma > \epsilon > 0$ there exists a number $k = k(\epsilon, \sigma) > 0$ with the following property. Let S be a hyperbolic surface of genus g and injectivity radius at least ϵ.*

1. *Let $\gamma \subset S$ be a non-separating simple closed geodesic on S of length at most σ. Then there is a standard fundamental domain D for S of diameter at most k and such that the sides a_1, a_1^{-1} project onto γ.*

2. *Let $\gamma \subset S$ be a separating simple closed geodesic on S of length at most σ and let S_1, S_2 be the two component of $S - \gamma$. Then there exists a standard fundamental domain D for the action of $\pi_1(S)$ of diameter at most k such that $D = D_1 \cup D_2$ where D_i is a standard fundamental domain for the action of $\pi_1(S_i)$ with open side $D_1 \cap D_2$ projecting onto γ $(i = 1, 2)$.*

Furthermore, for every $R > 0$ there exists a number $m = m(\epsilon, \sigma, R) > 0$ with the property that there are at most $m(\epsilon, \sigma, R)$ elements $\psi \in \pi_1(S)$ with $\psi(D) \cap N_R(D) \neq \emptyset$ where $N_R(D)$ denotes the R-neighborhood of D.

Proof. We only show the first part of the lemma, the second part follows in the same way and its proof will be omitted.

Thus let S be a hyperbolic surface of injectivity radius at least $\epsilon > 0$. Then the diameter δ of S is bounded from above by a constant $\kappa > 0$ only depending on the genus of S and ϵ. Let $\gamma \subset S$ be a simple closed geodesic of length at most σ. Cut S open along γ and let $\hat{S}$ be a corresponding bordered surface with two boundary components $\partial_-\hat{S}, \partial_+\hat{S}$ corresponding to two copies of γ.

Since the diameter of S is uniformly bounded from above, the length of a shortest geodesic arc in $\hat{S}$ which connects $\partial_-\hat{S}$ to $\partial_+\hat{S}$ is uniformly bounded. The concatentation of such an arc with one of the two components of σ cut out by its endpoints is a simple closed curve which intersects γ in a single point. Since the length of γ is at most σ, the length of this curve is uniformly bounded and hence the same holds true for a shortest simple closed geodesic ν which intersects γ in a single point p. The curves γ, ν fill a one-holed torus $T_0 \subset S$ with geodesic boundary. By this we mean that a tubular neighborhood of $\gamma \cup \nu$ in S is a one holed torus, and T_0 is obtained by replacing the boundary of this torus by its geodesic representative in S.

Let $p_\pm$ be the two copies of p on $\partial_\pm \hat{S}$. Cut $\hat{S}$ open along ν. The resulting surface S' has connected piecewise geodesic boundary with edges of uniformly bounded length. The endpoints of these edges project to the point p in S. Choose one of these preimages of p , say the point q, and let β' be the oriented boundary of S' as a based loop at q. This loop is piecewise geodesic, with four breakpoints. By construction, the oriented angles at any two consecutive breakpoints add up to π.

Let $\tilde{\beta}$ be a lift of β' to the hyperbolic plane $\mathbb{H}^2$. By the Gauss-Bonnet theorem, $\tilde{\beta}$ is simple, and the geodesic segment connecting its endpoints projects to a simple geodesic loop β in S' based at q which is homotopic with fixed endpoints to β' and meets β' only at q. The length of β is bounded from above by a universal constant only depending on the diameter of S and the upper bound σ for the length of γ. Then β can be viewed as a geodesic loop in S.

As β is homotopic to β' as loops in S' based at q, the geodesic loop β is freely homotopic to the boundary of T_0 and hence it is contained in T_0 by convexity.

Choose $2g-2$ non-separating simple closed geodesics $a_2, b_2, \dots, a_g, b_g$ in $S_0 = S - T_0$ of uniformly bounded length so that a_i is disjoint from a_j, b_j for $j \neq i$ and that a_i, b_i intersect in a single point. The existences of such simple closed geodesic is well known and, for example, an easy consequence of Theorem A.2. These closed geodesics can be replaced by geodesic loops based at q of uniformly bounded length so that cutting S_0 along these loops yields a polygon with $4g-3$ geodesic sides of uniformly bounded length, one of which is β. This polygon can be glued along β to the polygon obtained by cutting T_0 open along γ, ν to yield a fundamental domain D with the required properties.

To show the last statement of the lemma, let $R > 0$ and let $k > 0$ be an upper bound for the diameter of D. Then if $x \in D$ and if $\psi \in \pi_1(S)$ is such that $\psi(D) \cap N_R(D) \neq \emptyset$, then we have $\psi(x) \subset B(x, R+2k)$ where $B(x, R+2k)$ is denotes the ball of radius $R+2k$ about x. On the other hand, as the injectivity radius of S is at least ϵ, the $\pi_1(S)$-images of the ball $B(x, \epsilon)$ are pairwise disjoint. Thus there are at most $\mathrm{vol}(B(x, R+2k+\epsilon))/\mathrm{vol}(B(x, \epsilon))$ elements of $\pi_1(S)$ with $\psi(D) \cap N_R(D) \neq \emptyset$. ☐

Lemma 8.5. *For all $\sigma > \epsilon > 0$ and all $R > 0$ there exist numbers $k = k(\epsilon, \sigma, R) > 0$ and $h = h(\epsilon, \sigma, R) > 0$ with the following property. Let S be a hyperbolic surface of genus g and injectivity radius at least ϵ and let $\gamma \subset S$ be a simple closed geodesic on S of length at most σ. Let S_u be the surface obtained by S by abstract grafting at γ with grafting datum u and cylinder height at least h.*

1. *If $\gamma \subset S$ is non-separating then there is a standard fundamental domain $D \subset \tilde{S}_u$ for S_u with side pairings $\phi_1, \dots, \phi_{2g}$ so that $\phi_2, \dots, \phi_{2g} \in \pi_1(S - \gamma)$ and that moreover the following holds true. If $y \in D$ and if $\psi \in \pi_1(S)$ satisfies $d(D, \psi(y)) \leq R$, then*

ψ can be represented as a word of length at most k in the side pairings $\phi_1, \ldots, \phi_{2g}$, with at most one occurrence of $\phi_1^{\pm}$, either at the beginning or the end of the word.

2. *If $\gamma \subset S$ is separating let S_1, S_2 be the two component of $S - \gamma$. There exist standard fundamental domains D_i for S_i with side pairings $\phi_1^i, \ldots, \phi_{2g_i}^i$ $(i = 1, 2)$ and that moreover the following holds true. The domains D_1, D_2 are contained in a standard fundamental domain D for S_u, and if $y \in D$ and $\psi \in \pi_1(S)$ satisfies $d(y, \psi(y)) \leq R$ then ψ can be represented as a word of length at most k in either the elements ϕ_j^1 or the elements ϕ_j^2.*

Proof. We only show the first part of the lemma, the second part is completely analogous and its proof will be omitted.

Thus let S be a hyperbolic surface of genus g and injectivity radius at least ϵ, and let $\gamma \subset S$ be a non-separating simple closed geodesic of length at most σ. The hyperbolic structure on S defines a free isometric action of $\pi_1(S)$ on the hyperbolic plane $\mathbb{H}^2$. Fix once and for all a standard fundamental domain D for this action of $\pi_1(S)$ on $\mathbb{H}^2$ with piecewise geodesic boundary as in the first part Lemma 8.4 using the geodesic γ. Then there exists a lift $\tilde{\gamma} \subset \mathbb{H}^2$ of the grafting geodesic γ which intersects D in the side $\alpha = a_1$ which is paired by the side pairing $\phi_1 \in \pi_1(S)$ with another side $a_1^{-1} = \phi_1(\alpha)$. The subarc α of $\tilde{\gamma}$ is a fundamental domain for the action of the stabilizer of $\tilde{\gamma}$ in $\pi_1(S)$. In particular, the length of α equals the length of the geodesic γ and hence it is contained in the interval $[\epsilon, \sigma]$. Assume that the stabilizer of $\tilde{\gamma}$ in $\pi_1(S)$ is generated by the side pairing ϕ_2. Then ϕ_2 identifies the endpoints of α.

Recall the notational convention from the beginning of this section. Choose a grafting parameter z of cylinder height 1. For $t > 0$ let S_t be the surface obtained from the hyperbolic surface S by abstract grafting at γ with grafting datum tz. The fundamental domain D determines a fundamental domain D_t for the action of $\pi_1(S_t)$ on $\tilde{S}_t$ as follows. Attach to both α and $\phi_1(\alpha)$ a flat parallelogram $R(t, \alpha)$, $R(t, \phi_1(\alpha))$ of cylinder height $t/2$ whose slope is determined by the grafting datum z. Here by slope we mean the oriented angle between the oriented side α and a side adjacent to α, oriented to move away from α.

The polygonal surface D_t with piecewise geodesic boundary is equipped with a natural metric which restricts to the original hyperbolic metric on D, and it equals the flat metric on the attached parallelograms. The side pairing transformations $\phi_2, \ldots, \phi_{2g}$ for the standard fundamental domain D for the hyperbolic metric S naturally determine side pairing transformations of D_t, where we require that the extension of the side pairing ϕ_2 which generates the stabilizer of $\tilde{\gamma}$ in $\pi_1(S)$ identifies the sides of $R(t, \alpha)$ adjacent to α, that is, glue $R(t, \alpha)$ to a flat cylinder of cylinder height $t/2$. Replace ϕ_1 by the side pairing of D_t which identifies the two boundary components of $R(t, \alpha), R(s, \phi_1(\alpha))$ disjoint from D and parallel to $\alpha, \phi_1(\alpha)$. The quotient of D_t under these side pairing transformations is isometric to the abstract grafted surface S_t.

Namely, the above side pairings are gluing operations which glue D_t along its piecewise geodesic sides to a compact polygonal surface homeomorphic to S. The metric on D_t descends to a natural piecewise smooth metric on this surface. By construction, this metric does not have any cone points and is isometric to the metric of S_t.

The polygonal surface D_t naturally and isometrically embeds into the universal covering $\tilde{S}_t$ of S_t, and the above abstract side pairings of D_t are restrictions of a generating subset $\{\phi_1, \ldots, \phi_{2g}\}$ of $\pi_1(S_t)$ acting as the deck group on $\tilde{S}_t$. Note that for all t, the hyperbolic polygon D is a subsurface of D_t. It is contained in a component $\tilde{\Sigma}$ of the preimage of the bordered hyperbolic surface $S - \gamma \subset S_t$ in the universal covering $\tilde{S}_t$ of S_t. The full preimage of $S - \gamma$ in $\tilde{S}_t$ equals the $\pi_1(S)$-orbit of $\tilde{\Sigma}$.

Let $R > 0$ and let $t > 4R$. Let $\Gamma \subset \pi_1(S)$ be the stabilizer of $\tilde{\Sigma}$. This is the subgroup of $\pi_1(S)$ generated by the side pairing transformations $\phi_2, \ldots, \phi_{2g}$. The Γ-orbit ΓD_t of D_t equals the $t/2$-neighborhood of $\tilde{\Sigma}$ in $\tilde{S}_t$. Furthermore, the distance between ΓD_t and any hyperbolic piece disjoint from $\tilde{\Sigma}$ equals $t/2 > 2R$.

Let $y \in D_t$ be such that $d(D_t, \psi(y)) \leq R$ for some $\psi \in \pi_1(S)$. Assume first that y is contained in the R-neighborhood of $\tilde{\Sigma}$. Since the infinite cyclic group generated by ϕ_1 acts freely on hyperbolic pieces in $\tilde{S}_t$, and the distance between D_t and any hyperbolic piece different from $\tilde{\Sigma}$ is bigger than $2R$, we conclude that ψ can be represented as a word in the generators $\phi_2, \ldots, \phi_{2g}$.

If y is contained in the complement of the R-neighborhood of $\tilde{\Sigma}$ then an admissible path connecting y to $\psi(y)$ projects to a path in S which starts in the flat cylinder C_t of S_t, crosses a boundary component of C_t and returns to C_t by crossing through the second boundary component. Then the homotopy class of such a path, that is, the element ψ, can be written in the form $\psi = \phi_1^{\pm} \cdot w$ or $\psi = w \cdot \phi_1^{\pm}$ for some $w \in \pi_1(S - \gamma)$. This is what we wanted to show. □

Proof of Proposition 8.3. We loosely follow Ehresmann as in [Sle21], beginning with the case that the curve γ is non-separating and hence the path $s \to \rho_u^s$ $(u = t_0 z)$ is a Hitchin grafting path. At the end of this proof we point out the changes necessary to cover the case that γ is separating.

The idea is as follows. The group $\pi_1(S)$ acts properly discontinuously on $\mathcal{E}$, and for any $R > 0$, it acts compactly on the closed R-neighborhood N of the characteristic surface $\tilde{\mathcal{S}}$. Thus we can find a compact set $K \subset \mathcal{E}$ whose $\pi_1(S)$-translates cover N. For such a set K there are only finitely many elements $\psi \in \pi_1(S)$ so that $\psi(K) \cap K \neq \emptyset$, say the elements $\psi_1, \ldots, \psi_m$, and these elements generate the group $\pi_1(S)$.

By abuse of notation, let $\frac{\partial}{\partial s}$ be the vector field on $\mathcal{E}$ which is tangent to the factor $(-1, 1)$. Multiply the vector field $\frac{\partial}{\partial s}$ with a suitably chosen cutoff function θ with compact support in a compact set containing K. The push-forwards of the compactly supported vector field $\theta \frac{\partial}{\partial s}$ by the maps ψ_j can be averaged and equivariantly be extended to yield a $\pi_1(S)$-invariant Ehresmann connection on $N \subset \mathcal{E}$. The main task is to do this construction in such a way that the norm of this Ehresmann connection and its covariant derivative can be controlled globally, only depending on the injectivity radius of the Fuchsian representation ρ, the length of the grafting geodesic γ and its topological type (non-separating, or cutting from S a surface of genus g_1 for some $g_1 \in \{1, \ldots, \lfloor g/2 \rfloor\}$) as well as the given control constant $R > 0$ provided that $t_0 > 0$ is sufficiently large.

The implementation of this idea is divided into 4 steps.

Step 1: Construction of an adapted fundamental domain for the action of

$\pi_1(S)$.

Fix once and for all a hyperbolic plane $\hat{\mathbb{H}}^2 \subset \mathbb{X}$ which is stabilized by the Fuchsian representation ρ. In Lemma 8.5, we constructed for each s a fundamental domain D_s for the action of $\pi_1(S)$ on the universal covering of the abstract grafted surface $\tilde{S}_s$ corresponding to the representation ρ_u^s. This fundamental domain contains the fundamental domain D for the Fuchsian group ρ used as starting point for the grafting construction. Furthermore, $D_s - D$ is the union of two flat parallelograms contained in a flat piece of $\tilde{S}_s$. These parallelograms are determined by the grafting data and hence for the grafting path we use, they all have the same slope, and the height of a parallelogram in $D_s - D$ equals precisely $(t_0 + s)/2$ (by the choice of normalization).

The representation ρ_u^s determines a piecewise isometric embedding of the fundamental domains D_s into $\mathbb{X}$, with fixed image of D. The images of the parallelograms $D_s - D$ are contained in a fixed maximal flat in $\mathbb{X}$, in such a way that for $s < t$ we have $D_s \subset D_t$. The two sides of D_s which are disjoint from $\hat{\mathbb{H}}^2$ are parallel to a fixed side of $D \subset \hat{\mathbb{H}}^2$ and vary in a real analytic fashion on s. More precisely, for each s there is a natural diffeomorphism $\eta_s : D_0 \to D_s$ which equals the identity on D and whose restriction to each component of $D_0 - D$ is the restriction of an affine map with respect to the Euclidean structure of the flat. For each point $z \in D_0$, the map $s \to \eta_s(z)$ is smooth, with uniformly bounded derivative and covariant derivative. If for each s we view the set D_s as a subset of $\mathbb{X} \times \{s\}$, then $\mathcal{D} = \cup_s D_s \subset \tilde{\mathcal{S}}$ and furthermore, $\mathcal{D}$ is a fundamental domain for the action of $\pi_1(S)$ on $\tilde{\mathcal{S}}$.

By Theorem 6.1, there exists a number $b = b(\sigma) \in (0, 1]$ and numbers $\tau(\sigma) > 1, C > 0$ depending on the *upper* bound σ for the length of γ such that for $t_0 \geq \tau(\sigma)$ and $s \in (-1, 1)$ as before, the following properties hold true. For $x, y \in \tilde{Q}_s(\tilde{S}_s) \subset \mathbb{X}$ we have

$$d^{\mathfrak{F}}(x, y) \geq b d_s(x, y) - C \tag{30}$$

where d_s equals the intrinsic path distance in $\tilde{S}_s \sim \tilde{Q}_s(\tilde{S}_s) \subset \mathbb{X}$ and $d^{\mathfrak{F}}$ is the Finsler distance function on $\mathbb{X}$ as before. Here we use the notations from Theorem 6.1, in particular $\tilde{Q}_s$ denotes the grafting map

Fix a number $R > C$ and let

$$t_0 > \max\{\tau(\sigma), 4R/b + C\}.$$

For $q > 0$ let $U_{s,q}$ be the q-neighborhood of $\tilde{Q}_s(D_s)$ in $(\mathbb{X}, d^{\mathfrak{F}})$ and write $U_q = \cup_s U_{s,q}$. Let $N_q(\tilde{\mathcal{S}}^{hyp})$ be the union of the q-neighborhoods in $(\mathbb{X}, d^{\mathfrak{F}})$ of the subspace $\tilde{Q}_s(\tilde{S}_s^{hyp})$ $(s \in (-1, 1))$. Note that $N_q(\tilde{\mathcal{S}}^{hyp})$ is invariant under the action of $\pi_1(S)$ on $\mathcal{E}$.

We claim that there exists a number $m = m(\epsilon, \sigma, R) > 0$ with the following properties.

- There are at most m elements $\psi \in \pi_1(S)$ such that $\psi(U_R) \cap U_R \neq \emptyset$.
- Each such element can be written as a word of length at most m in the generators $\{\phi_1, \ldots, \phi_{2g}\}$ of $\pi_1(S)$ and their inverses, with at most one occurrence of ϕ_1 or its inverse, and such an occurrence happens either at the beginning or the end of the word.

- The restriction of the vector field $d\psi(\frac{\partial}{\partial s})$ to $\psi(U_R)\cap U_R$ is of the form $\xi_\psi+\frac{\partial}{\partial s}$ where ξ_ψ is a section of the vertical tangent bundle of $\mathcal{E}$ whose norm and norm of its covariant derivative is bounded from above by a constant only depending on ϵ,σ,R.

To see that this holds indeed true, note that since the map $\tilde{Q}_s:\tilde{S}_s\to\mathbb{X}$ is a $\pi_1(S)$-equivariant path isometry (for the Finsler metric on $\tilde{S}_s$ and the Finsler metric on $\mathbb{X}$), we can argue as in the proof of Lemma 8.5.

Consider first an element $\psi\in\pi_1(S_s)$ with the property that

$$\psi(U_R)\cap U_R\cap(\mathbb{X}-N_{4R/b}(\tilde{\mathcal{S}}^{hyp}))\neq\emptyset. \tag{31}$$

As the action of $\pi_1(S)$ preserves the fibers of the fibration $\mathcal{E}\to(-1,1)$, we can choose some s so that $\psi(U_{s,R})\cap U_{s,R}\cap(\mathbb{X}-N_{4R/b}(\tilde{\mathcal{S}}^{hyp}))\neq\emptyset$.

As ψ is an isometry for $d^{\mathfrak{F}}$ and $U_{s,R}$ equals the R-neighborhood of $\tilde{Q}_s(D_s)$ in $\mathbb{X}$, we have

$$\psi(U_{s,R})\cap U_{s,R}\neq\emptyset \text{ if and only if } \psi(\tilde{Q}_s(D_s))\cap U_{s,2R}\neq\emptyset.$$

More precisely, if $u\in\psi(U_{s,R})\cap U_{s,R}\cap(\mathbb{X}-N_{4R/b}(\tilde{S}_s^{hyp}))$ then there are $y\in U_{s,2R}\cap\psi(\tilde{Q}_s(D_s))$ and $x\in\tilde{Q}_s(D_s)$ with $d^{\mathfrak{F}}(x,u)<R$, $d^{\mathfrak{F}}(u,y)<R$ and hence $d^{\mathfrak{F}}(x,y)<2R$.

By the estimate (30) and the choice of R and b, the intrinsic path distance $d_s(x,y)$ between x,y in $\tilde{Q}(\tilde{S}_s)$ is at most $2R/b+C/b$. Furthermore, as $d^{\mathfrak{F}}(x,u)<R$, $d^{\mathfrak{F}}(y,u)<R$ and $u\notin N_{4R/b}(\tilde{S}_s^{hyp})$, we have

$$d_s(\{x\cup y\},\tilde{Q}_s(\tilde{S}_s^{\rm hyp}))\geq 3R/b>2R/b+C/b.$$

Since $\tilde{S}_s$ with the intrinsic path metric is a geodesic metric space, and the map $\tilde{Q}_s$ is a path isometric embedding, we conclude that the points x,y are contained in the image under $\tilde{Q}_s$ of a fixed flat strip in $\tilde{S}_s$, and the same holds true for the geodesic arc connecting them.

In particular, we have $x\in\tilde{Q}_s(D_s-D)$ and hence the point x either is contained in the image of the parallelogram $R(s,\alpha)$ or of the parallelogram $R(s,\phi_1(\alpha))$. Then the flat strip in $\tilde{S}_s$ whose image under $\tilde{Q}_s$ contains x,y either is obtained by gluing $R(s,\phi_1(\alpha))$ to the side of $R(s,\alpha)$ parallel to α with the side pairing ϕ_1^{-1} and translating the resulting flat parallelgram with powers of the side pairing ϕ_2, or gluing $R(s,\alpha)$ to $R(s,\phi_1(\alpha))$ with the map ϕ_1 and translating the resulting flat rectangle with powers of the conjugate $\phi_1\phi_2\phi_1^{-1}$ (here composition of maps are written from right to left, that is, in a product ab, the map b is applied first). In both cases, the element ψ has the form claimed in the second item above.

To study the image of the vector field $\frac{\partial}{\partial s}$ under the map ψ on the R-neighborhood of x, note that by the discussion in the previous paragraph, the restriction ψ_s of ψ to neighborhoods of $\tilde{Q}_s(D_s)$ in $\mathbb{X}\times\{s\}\subset\mathcal{E}$ consists in postcomposition of the map ψ_0 with a one-parameter group of transvections $s\to\alpha(s)$ whose derivatives and covariant derivative near x are uniformly bounded. As the derivative of this one-parameter group of transvections is a Killing field, its derivative and covariant derivative also is uniformly bounded on the R-neighborhood of x and hence at u. This shows the third item above.

Since the translation length of ϕ_2 is bounded from below by $\epsilon > 0$, we conclude as in the proof of Lemma 8.4 that the cardinality of the set $\mathcal{A}(\sigma, R) \subset \pi_1(S)$ of elements ψ with $\psi(U_R) \cap U_R \neq \emptyset$ is bounded from above by a universal constant only depending on ϵ, σ and R which was claimed in the first item.

The same argument as used above and in the proof of Lemma 8.5 also shows that if $x \in \tilde{Q}_s(D_s)$ is such that $d^{\mathfrak{F}}(x, y) \leq 2R$ for some $y \in \psi(\tilde{Q}_s(D_s))$ and some $s \in (-1, 1)$ and if furthermore we have $x \in N_{4R/b}(\tilde{S}_s^{hyp})$ then ψ can be written as a word of uniformly bounded length in the generators $\phi_2, \ldots, \phi_{2g}$ alone.

Step 2: Construction of the Ehresmann connection

Recall from Step 1 above the definition of the set $U_R = \cup_s U_{s,R}$ $(R \geq 1)$. It is an open relatively compact subset of $\mathcal{E}$. As $\cup_s \tilde{Q}_s(D_s)$ is a fundamental domain for the action of $\pi_1(S)$ on the $\pi_1(S)$-invariant preimage $\tilde{\mathcal{S}} \subset \mathcal{E}$ of the characteristic surface $\mathcal{S} \subset \pi_1(S) \backslash \mathcal{E}$, the set $\pi_1(S)(U_R)$ is $\pi_1(S)$-invariant and contains the R-neighborhood of $\tilde{\mathcal{S}}$ (for the path metric induced by the product of the standard metric on $(-1, 1)$ and the metric $d^{\mathfrak{F}}$ on $\mathbb{X}$).

Let $\alpha : \mathcal{E} \to [0, 1]$ be a smooth function which is constant 1 on U_{R+1} and vanishes on $\mathcal{E} - U_{R+2}$. Since the symmetric distance function on $\mathbb{X}$ is smooth away from the diagonal, and since $R > 1$, we may assume that the pointwise norms of the differential of α and its covariant derivative, taken with respect to the product metric on $\mathcal{E}$, are uniformly bounded, independent of t_0 provided that $t_0 > 1$ is large enough.

Define

$$\eta(x, s) = \alpha(x, s) \frac{\partial}{\partial s}$$

and let

$$\xi' = \sum_{\psi \in \pi_1(S)} \psi_* \eta \tag{32}$$

where the action of $\pi_1(S)$ is the action by bundle automorphisms of $\mathcal{E} \to (-1, 1)$.

We showed in Step 1 above that there exists a number $m > 0$ only depending on ϵ, σ but not depending on t_0 provided that t_0 is sufficiently large, such that there are at most m elements $\psi \in \pi_1(S)$ with $\psi(U_{R+2}) \cap U_{R+2} \neq \emptyset$. Furthermore, if $\psi(U_{R+2}) \cap U_{R+2} \cap (\mathbb{X} - \cup_s N_{3(R+2)/b}(\tilde{S}_s^{hyp})) \neq \emptyset$ then $\psi \in \mathcal{A}(\sigma, R)$.

Thus by equivariance under the action of $\pi_1(S)$, for all sufficiently large t_0 and for each $y \in \mathcal{E}$, all but at most m of the functions $\psi_* \alpha$ $(\psi \in \pi_1(S))$ vanish at y. Recall that the action of $\pi_1(S)$ on $\mathcal{E}$ is via the representations ρ_u^s.

As a consequence, the vector field ξ' is smooth and invariant under the action of $\pi_1(S)$. Furthermore, it can be represented in the form

$$\xi' = \xi_0' + \theta \frac{\partial}{\partial s}$$

where ξ_0' is a section of the vertical bundle $\ker d\Pi$ and where $\theta(x, s)$ is a smooth non-negative function on $\mathcal{E}$. On the $\pi_1(S)$-orbit of U_R, which for each $s \in (-1, 1)$ contains the R-neighborhood of $\tilde{\mathcal{S}} \cap \mathbb{X} \times \{s\}$ with respect to the Finsler metric on the fibers, the value of the weight function $\theta(x, s)$ is bounded from below by 1, moreover ξ' is invariant under the action of $\pi_1(S)$. Thus ξ' spans a smooth one-dimensional subbundle H of $T\mathcal{E}|\pi_1(S)(U_R)$ which defines a $\pi_1(S)$-invariant Ehresmann connection

$$\xi = \theta^{-1}\xi'$$

on $\pi_1(S)(U_R) \subset \mathcal{E}$.

Step 3: Verifying the properties of the Ehresmann connection

We have to show that the Ehresmann connection constructed in Step 2 has properties (1)-(3) stated in Proposition 8.3. Note that these properties only involve the restriction of the Ehresmann connection to the set $\pi_1(S)(U_R)$.

The first property is immediate from the construction.

To verify the second property, recall from Step 1 of this proof that if $x \in U_{s,R+2} \cap N_{4(R+2)/b}(\tilde{S}_s^{hyp})$ and if $\psi \in \pi_1(S)$ is such that $\psi(x) \in U_{s,R+2}$, then ψ can be written as word in the generators $\phi_2, \ldots, \phi_{2g} \in \pi_1(S-\gamma)$ alone. But the restrictions of the representations ρ_u^s to $\pi_1(S-\gamma)$ do not depend on s and hence the vector field $\frac{\partial}{\partial s}$ is invariant under ψ. On the other hand, in the formula (32) for the vector field ξ' at x, the sum is taken over elements $\psi \in \pi_1(S)$ so that x is contained in the image under ψ of the set U_{R+2} (by the assumption on the support of the function α). Thus this sum is over positive multiples of the vector field $\frac{\partial}{\partial s}$. This implies the second property stated in Proposition 8.3.

To show the third property, we noted in Step 1 above that if $u \in U_{R+2} \cap \psi(U_{R+2})$ for some $\psi \in \pi_1(S)$, then the restriction of the vector field $d\psi(\frac{\partial}{\partial s})$ to $U_{R+2} \cap \psi(U_{R+2})$ can be represented in the form $\xi_\psi + \frac{\partial}{\partial s}$ where the vector field ξ_ψ and its covariant derivatives are pointwise uniformly bounded in norm. On the other hand, by the first item in Step 1 and the definition, the number of nontrivial terms in sum which defines the Ehresmann connection at u is bounded from above by a universal constant m. But this implies that indeed, the Ehresmann connection has the third property stated in Proposition 8.3, completing the proof of the lemma in the case that γ is non-separating.

Step 4: The case that γ is separating

In the case that γ is separating let S_1, S_2 be the two components of $S-\gamma$ and fix a basepoint $p \in S_1$. Let g_i be the genus of S_i. By the second part of Lemma 8.4, for $i = 1, 2$ there exists a standard fundamental domain for $\pi_1(S_i)$ of diameter at most k and with open side projecting onto γ. Gluing the open sides of D_1, D_2 then yields a fundamental domain D for $\pi_1(S)$.

As in the case when γ is non-separating, the fundamental domains D_1, D_2 can be used to construct for any s a fundamental domain D_s for the abstract grafted surface S_s along γ defined by the grafting path through the Fuchsian representation ρ, determined by the grafting datum $(t_0+s)z$ $(t>0)$, obtained by cutting D open along the image $\alpha \subset D$ of the arcs α_i and inserting a flat parallelogram of height t_0+s and slope determined by the grafting element.

The side pairing transformations of D_i generate the subgroup $\pi_1(S_i)$ of $\pi_1(S)$, and their union generates $\pi_1(S)$. Choose a basepoint $p \in \gamma$. Similar to the beginning of this proof, let $\mathcal{C}(k)$ be the subset of $\pi_1(S)$ which can be represented as a word of length k in either the generators of $\pi_1(S_1)$ or the generators of $\pi_1(S_2)$. Let $R > 0$ and let $t_0 > 2R$ and $u = t_0 z$. Let $N_R(D_s)$ be the R-neighborhood of D_s in $\tilde{S}_s$. It follows from the arguments used in the case that γ is non-separating that there exists a number $k > 0$ such that an element $\psi \in \pi_1(S)$ for which $\psi N_R(D_s) \cap N_R(D_s) \neq \emptyset$ is contained in $\mathcal{C}(k)$.

With this information, the argument used for the case that γ is non-separating carries over identically to yield the statement of the proposition. □

Remark . The estimates established in the proof of Proposition 8.3 are far from optimal, but they are sufficient for our purpose.

8.3 Differentiating Finsler metrics

The vector field ξ constructed in Proposition 8.3 defines an Ehresmann connection on the R-neighborhood of the characteristic surface in the manifold $M = \rho_u\backslash\mathbb{X}$. As $d\Pi(\xi) = \frac{\partial}{\partial s}$, it generates a fiber preserving local flow Ψ^s, that is, Ψ^s maps the fiber over t to the fiber over $t + s$. In particular, for some fixed t and sufficiently small $|s|$, we can consider the family of pull-back Finsler metrics $\mathfrak{F}(s) = (\Psi^s)^{-1}\mathfrak{F}$ on $\Pi^{-1}(t)$.

Denote by $\mathcal{L}_\xi$ the Lie derivative in direction of the vector field ξ. Recall that a tangent vector of a quotient of $\mathbb{X}$ is regular if it is an interior point of a Weyl chamber.

The Finsler metrics we are interested in are determined by the choice of a closed positive Weyl chamber $\overline{\mathfrak{a}^+}$ and a positive linear functional α_0 on $\overline{\mathfrak{a}^+}$. The linear functional defines a norm on $\overline{\mathfrak{a}^+}$ which is then extended to a norm on $\mathfrak{a}$ by invariance under the Weyl group. As the flow Ψ^t does not need to preserve regular vectors, we shall take derivatives on the set of regular vectors and then argue by continuity.

The following computation is well known in the Riemannian setting. As we are working with Finsler metrics, a bit more care is required.

Lemma 8.6. *If $X \in T(\rho_u\backslash\mathbb{X})$ is a regular vector then for $t \in (-1, 1)$ sufficiently close to 0 we have*

$$\frac{d}{ds}\mathfrak{F}(s)(X)|_{s=t} = (\Psi^t)^*\mathcal{L}_\xi\mathfrak{F}(X).$$

Proof. Since the set of regular vectors is an open dense subset of $T(\rho_u\backslash\mathbb{X})$, if $X \in T(\rho_u\backslash\mathbb{X})$ is regular then the same holds true for $d\Psi^s(X)$ provided that $|s|$ is sufficiently small.

By the definition of the Finsler metric $\mathfrak{F}$, this means that there is a fixed linear functional α_0 on $\mathfrak{a}$ such that the Finsler norm of $d\Psi^s(X)$ equals the value of α_0 on $d\Psi^t X$, viewed as an element in $\mathfrak{a}$. Then

$$\frac{d}{ds}\mathfrak{F}(s)(X)|_{s=t} = \frac{d}{ds}(\Psi^s)^*\alpha_0(X)|_{s=t} = (\Psi^t)^*\mathcal{L}_\xi(\alpha_0)(X) = (\Psi^t)^*\mathcal{L}_\xi\mathfrak{F}(X)$$

as claimed. □

Corollary 8.7. *If $X \in T(\rho_u\backslash\mathbb{X})$ is a regular vector then for all sufficiently small $|t|$, the derivative $\frac{d}{ds}\mathfrak{F}(s)(X)|_{s=t}$ exists, and its absolute value is bounded from above by $C\mathfrak{F}(X)$, where $C > 0$ is a constant only depending on $\mathfrak{F}$ and an upper bound for the norm of the vector field ξ.*

Write $q(t) = \frac{d}{ds}\mathfrak{F}(s)|_{s=t}$, thought of in the following way. For each x in the domain of the flow Ψ^s and each open Weyl chamber $A \subset T_x\rho_u\backslash\mathbb{X}$, there exists a number $\epsilon > 0$ and a smooth path $s \to \alpha(A,s)$ of uniformly bounded linear functionals on A so that for every vector $X \in A$ and all sufficiently small $|s| < \epsilon$ depending on X, we have $\frac{d}{ds}\mathfrak{F}(s)(X)|_{s=t} = \alpha(A,t)(X) = q(t)(X)$.

Using this terminology, we compute second derivatives. Namely, since $s \to q(s)$ is smooth in the sense described above, near $s = 0$ and evaluated on a fixed Weyl chamber, the assignment $s \to q(s)$ is a smooth time dependent local section of the vector space of linear functionals on A. This implies that the Lie derivative $\mathcal{L}_\xi(q(s))$ of $q(s)$ along the flow Ψ^s, evaluated on regular vectors, is defined. The following lemma computes the second derivative of the family of Finsler metrics $s \to \mathfrak{F}(s)$ using these notations in a formal way.

Lemma 8.8. *For any regular vector $X \in T(\rho_u\backslash\mathbb{X})$, we have*

$$\frac{d^2}{dt^2}\mathfrak{F}(t)(X)|_{t=0} = \frac{d}{dt}q(t)|_{t=0}(X) + \mathcal{L}_\xi(q(0))(X).$$

Proof. Write $(\Psi^t)^*(q(t)) = (\Psi^t)^*(q(t) - q(s)) + (\Psi^t)^*(q(s))$ and differentiate at $t = s = 0$ in direction of t. □

We are now ready to prove Proposition 8.1 using the notations introduced before its formulation.

Proof of Proposition 8.1. For $\sigma > 0$ let $R = C_\sigma > 0$ be as in Theorem 6.1. For this number R and for $\epsilon \in (0,\sigma)$ let $\tau_0 = \tau(R+2,\epsilon,\sigma) > 4R$ be as in Proposition 8.3.

Using the conventions in the beginning of this proof, let $u = t_0 z$ for some $t_0 \geq \tau_0$ and let $t \to \rho_{(t_0+s)z} = \rho_u^s$ be a Hitchin grafting path where the grafting is performed at a simple closed geodesic of length at most σ. Let $\eta \subset S$ be any closed geodesic and let α_η be the admissible path in the abstract grafted surface S_u defined by the grafting datum u which represents the free homotopy class of η. The path α_η intersects the cylinder part of S_u in precisely $k = \iota(\eta,\gamma)$ components.

Equivalently, if $\tilde{\alpha}_\eta$ denotes a lift of α_η to the universal covering $\tilde{S}_u$ of S_u, and if $\psi \in \pi_1(S)$ is an element which preserves $\tilde{\alpha}_\eta$ and such that the quotient of $\tilde{\alpha}_\eta$ under ψ defines the free homotopy class of η, then there exists a fundamental domain in the admissible path $\tilde{\alpha}_\eta$ for the action of ψ of the form

$$\alpha_\eta^0 = \alpha_1 \circ \cdots \circ \alpha_{2k+1},$$

where for each i, the arc α_{2i-1} is a geodesic contained in a hyperbolic piece of S_z, and the arc α_{2i} is a geodesic contained in a flat cylinder. We have

$$\ell_u^{flat}(\eta) = \sum_{i=1}^{k} \ell(\alpha_{2i}).$$

By Theorem 6.1, there is a Finsler geodesic $\zeta \subset \rho_u\backslash\mathbb{X}$ representing the free homotopy class $\rho_u(\eta)$ which is contained in the R-neighborhood of $Q_u(\alpha_\eta)$ where $Q_u : S_u \to \rho_u\backslash\mathbb{X}$ is the path isometry defining the characteristic surface of ρ_u.

As a consequence, by the triangle inequality, the Finsler geodesic ζ, which is supposed to be parameterized by arc length, can be decomposed as

$$\zeta = \zeta_1 \circ \cdots \circ \zeta_{2k+1}$$

where for all j, the subarc ζ_j is contained in the R-neighborhood of $Q_u(\alpha_j)$ and the length of each of the arcs ζ_j does not exceed $\ell(\alpha_j) + 2R$. Furthermore, the arcs ζ_j are contained in the R-neighborhood of $Q_u(S_u)$. We may replace each of the arcs by a piecewise geodesic for the symmetric metric of the same Finsler length and with the same endpoints which is contained in the $R + 1$-neighborhood of S_u.

Consider now the Hitchin grafting path $s \to \rho_u^s$. By Proposition 8.3, there is an Ehresmann connection ξ on the corresponding bundle $\pi_1(S)\backslash\mathbb{X} \times (-1, 1)$ which can be thought of as a geometric realization of the deformation. Let Ψ^s be the corresponding flow. By Proposition 8.3, the restriction of Ψ^s to the $R + 2$-neighborhood of $Q_u(S_u^{hyp})$ is an isometry. Furthermore, using Lemma 8.6 and the control on the Ehresmann connection established in Proposition 8.3, for a tangent vector X of ζ_{2i} we have

$$\left|\frac{d}{ds}\mathfrak{F}(\Psi^s X)|_{s=0}\right| \leq C_0$$

where $C_0 > 0$ only depends on σ but not on η or $u = t_0 z$.

Summing over the subarcs ζ_{2i} and using the fact that as $\tau_0 > 4R$, the flat length of each component α_{2i} of the intersection of η is at least $4R$ to subsume the error in the length estimate for the arcs ζ_{2i}, we conclude that there exists a number $C_1 > C_0$ such that

$$\left|\frac{d}{ds}\ell_{\rho_u^s}(\eta)|_{s=0}\right| \leq C_1 \ell_u^{flat}(\eta) \tag{33}$$

which shows the first part of the proposition.

Namely, the path ζ is a Finsler geodesic and length minimizing in its free homotopy class. This implies that

$$\int_\zeta \frac{d}{ds}\ell(\Psi^s(\zeta))|_{s=0} dt$$

indeed computes the first derivative of the Finsler lengths of η by the first variation formula for the energy which also holds true in the Finsler setting on the locally symmetric manifolds $\rho_u\backslash\mathbb{X}$, and by construction, this derivative equals the sum

$$\sum_{i=1}^k \frac{d}{ds}\ell(\Psi^s(\zeta_{2i})|_{s=0}.$$

The second part of the proposition is shown in a similar way. Namely, using once more the control of the Ehresmann connection established in Proposition 8.3 and Lemma 8.8, we know that there exists a constant $C_2 > C_1$ such that

$$\frac{d^2}{ds^2}\ell(\Psi^s\zeta)|_{s=0} \le C_2\ell_u^{flat}(\eta).$$

However, for $s \neq 0$ the path $\Psi^s\zeta$ on $\rho_u^s\backslash\mathbb{X}$ may not be a Finsler geodesic, that is, its length may be larger than the shortest length of a path in the free homogopy class of $\rho_u^s(\eta)$.

In other words, for each $s \in (-1,1)$, the push-forward $\Psi^s(\zeta)$ is a closed curve in $\rho_u^s\backslash\mathbb{X}$ which is freely homotopic to a closed Finsler geodesic in the homotopy class of $\rho_u^s(\eta)$. In particular, the ρ^s-length $b(s)$ of this geodesic does not exceed the length $a(s)$ of γ for the pull-back metric $\mathfrak{F}(s)$, and equality holds for $s = 0$.

We now use the following simple comparison lemma. Let $a, b : (-1,1) \to \mathbb{R}$ be two C^2-functions. Assume that $a(\lambda) \ge b(\lambda)$ for all λ and $a(0) = b(0)$; then $da(0) = db(0)$ and $d^2a(0) \ge d^2b(0)$ (compare Section 2 of [Pol94]).

From this comparison, we deduce that

$$\frac{d^2}{ds^2}\ell_{\rho_u^s}(\eta) \le \frac{d^2}{ds^2}\ell(\Psi^s\zeta)|_{s=0} \le C_2\ell_u^{flat}(\eta)$$

which is what we wanted to show. □

We are left with showing Proposition 8.2. Recall the setup. For a fixed number $\sigma > 0$ let γ be a separating geodesic on the hyperbolic surface S of length $\ell \le \sigma$. It decomposes the surface S into two hyperbolic surfaces S_1, S_2 with connected boundary. Let $s \to \beta(s)$ a smooth path in the Teichmüller space $\mathcal{T}(S_2, \ell)$ of hyperbolic metrics on S_2 with fixed boundary length. This path determines a deformation $s \to S(s)$ of the hyperbolic surface S preserving the hyperbolic metric on S_1. We require that for each s, there exists a diffeomorphism $\Lambda^s : S \to S(s)$ with the property that the pull-back $g(s)$ of the hyperbolic metric on $S(s)$ by the map Λ^s satisfies

$$\left|\frac{d}{ds}g(s)\right| \le 1, \left|\frac{d^2}{ds^2}g(s)\right| \le 1$$

where the norms are taken with respect to the hyperbolic metric on S. We call such a deformation a *geometrically controlled* deformation.

For each s and $u = t_0 z$ as before let ρ_u^s be obtained from $S(s)$ by grafting along γ with datum u. Then $s \to \rho_u^s$ is a smooth path of Hitchin rerpresentations which we call a *half-surface deformation of* S_2.

Proposition 8.2 is the analog of Proposition 8.1 for half-surface deformations of S_2. For its formulation, defines the *half-surface length* $\ell_u^{S_2}(\eta)$ of a free homotopy class of η in S as follows. Represent η by an admissible path in S_2 and let $\ell_u^{S_2}(\eta)$ be the sum of the lengths of the hyperbolic pieces contained in S_2. Note that the hyperbolic pieces alternate between pieces in S_1 and S_2. Note furthermore that this does not depend on the grafting datum u.

Proof of Proposition 8.2. The proof is completely analogous to the proof of the Proposition 8.1 and will only be sketched.

Namely, as in the proof of Proposition 8.1, we use the Ehresmann connection constructed in Proposition 8.3 to control the derivative of the Finsler metric along the deformation defined by the connection. Together with the Theorem 6.1, the reasoning in the proof of Propostion 8.1 applies word by word and yields the required estimate. □

9 Quantitative convergence of currents

In Section 2.2 we introduced the measure of maximal entropy for Hitchin representations with respect to a Finsler metric. In this section we investigate the behavior of these measures along grafting rays in the Hitchin component. Using the geometric control established in Section 6, we compare length functions for representations obtained by Hitchin grafting rays to length functions of the corresponding abstract grafted surfaces, viewed as functions on the unit tangent bundle of the hyperbolic surface S which is the starting point for the grafting, and estimate the entropy of the reparameterized flow. This then leads to the proof of Theorem C from the introduction.

The Finsler metric on $\mathbb{X}$ used for the pressure metric is normalized in such a way that its restriction to a hyperbolic plane stabilized by an irreducible representation of $\mathrm{PSL}_2(\mathbb{R})$ coincides with the Riemannian metric of constant curvature -1.

We start with a hyperbolic metric on the closed surface S of genus $g \geq 2$ and choose a simple geodesic multicurve γ^* on S (the grafting locus). For each grafting parameter $z = (z_e)_{e \subset \gamma^*} \subset \mathfrak{a}$, denote by ρ_z the Hitchin grafting representation with datum z (see Definition 3.4).

By Proposition 2.8, for each z there exists a positive Hölder continuous function f_z on the unit tangent bundle T^1S of S with the property that for every periodic orbit γ for the geodesic flow Φ^t on T^1S, we have that

$$\ell_{f_z}(\gamma) = \int_\gamma f_z$$

equals the translation length defined by the element $\rho_z(\gamma) \in \mathrm{PSL}_d(\mathbb{R})$ with respect to the Finsler metric.

The Hölder continuous function f_z on T^1S determines a reparametrization $\Phi^t_{f_z}$ of the geodesic flow Φ^t on T^1S, whose measure of maximal entropy corresponds to a Φ^t-invariant Gibbs equilibrium state $\nu(z)$ on T^1S. There are several possible normalizations for this equilibrium state. We assume $\nu(z)$ to be normalized in such a way that

$$\int f_z d\nu(z) = 1 \text{ for all } z. \tag{34}$$

Note that this normalization only depends on the cohomology class of f_z and hence it does not depend on choices. Our main goal is to determine the possible limits of $\nu(z)$ as the cylinder height of every component z_e of z (that is, at every component of the multi-curve γ^*) tends to infinity, and to show that the intersection numbers with γ^* of the geodesic currents $\hat{\nu}(z)$ determined by the measures $\nu(z)$ decay exponentially fast.

By Section 2.3, the equilibrium measure of the function f_z can be described in terms of Patterson–Sullivan measures. Denoting as before by $\mathcal{F}$ the flag variety of $\mathrm{PSL}_d(\mathbb{R})$, recall that for $\xi, \eta \in \mathcal{F}$ and $x, y \in \mathbb{X}$, the function $b_\xi^{\mathfrak{F}}(x,y)$ denotes the Busemann cocycle and $\langle \xi | \eta \rangle_x$ denotes the Gromov product associated to the Finsler metric $\mathfrak{F}$ (see Equations 5 and 8).

For any non-trivial grafting datum z with nontrivial cylinder height, let $\Xi_z : \partial_\infty \mathbb{H}^2 \to \mathcal{F}$ be the limit map associated to the Hitchin grafting representation ρ_z. Then there exists a family of measures $(\mu_z^x)_{x\in\mathbb{X}}$ on $\partial_\infty \mathbb{H}^2$ such that for all $x, y \in \mathbb{X}$ and $\gamma \in \pi_1(S)$ we have $\mu_z^{\rho_z(\gamma)x} = \gamma_* \mu_z^x$ and

$$\frac{d\mu_z^y}{d\mu_z^x}(\xi) = e^{\delta(z) b^{\mathfrak{F}}_{\Xi_z(\xi)}(x,y)}, \tag{35}$$

where $\delta(z)$ is the *critical exponent* of the group $\rho_z(\pi_1(S))$, or, equivalently, the topological entropy of the reparameterized flow $\Phi^t_{f_z}$ on T^1S. These measure are unique up to a global multiplicative positive constant. Note that the equality 35 is immediate from the fact that the topological entropy of the reparameterized flow equals the expansion rate of the conditional measures on strong unstable manifolds for its unique measure of maximal entropy, which in turn equals the critical exponent by construction.

Finally $\nu(z)$ is the quotient under $\pi_1(S)$ of the measure

$$e^{\delta(z)\langle \Xi_z(\xi) | \Xi_z(\eta)\rangle_x} d\mu_z^x(\xi) d\mu_z^x(\eta) dt \tag{36}$$

on $\partial_\infty \mathbb{H}^2 \times \partial_\infty \mathbb{H}^2 \times \mathbb{R}$. Note that the measures μ_z^x are finite but in general they are not probability measures, instead their normalisation is determined by the normalisation of $\nu(z)$.

Our goal is to estimate the intersection $\iota(\hat{\nu}(z), \gamma^*)$ (here γ^* is viewed as a Dirac current) as the size of z tends to infinity. Since $\nu(z)$ and hence $\hat{\nu}(z)$ depends continuously (in fact, analytically) on z by Proposition 2.4, we can achieve this using continuity of the intersection form on the space of currents. However, although the space of projective currents, equipped with the weak*-topology, is compact since this is the case for the space of Φ^t-invariant Borel probability measures on T^1S where Φ^t is the geodesic flow, the family $\hat{\nu}(z)$ may not be precompact as the corresponding Φ^t-invariant measure $\nu(z)$ on T^1S in general is not a probability measure. We shall use the Patterson–Sullivan measures to estimate the total volume of $\nu(z)$ and overcome this difficulty.

9.1 The entropy of the subsurfaces

The geodesic multicurve γ^* decomposes S into (closed) complementary components $S_1, \ldots, S_k$. For each $i \le k$ we denote by $K_i \subset T^1S$ the set of all unit tangent vectors $v \in T^1S_i$ with the property that $\Phi^t v \in T^1S_i$ for all $t \in \mathbb{R}$.

Lemma 9.1. *For each i the set K_i is compact and Φ^t-invariant.*

Proof. The set K_i is clearly Φ^t-invariant and closed by continuity of Φ^t, hence it is compact. □

Since S is a closed hyperbolic surface, the geodesic flow Φ^t on T^1S is an Anosov flow and hence for each i its restriction to the compact invariant set K_i is an Axiom A flow.

The preimage of the geodesic multicurve γ^* in the universal covering $\mathbb{H}^2$ of S consists of a countable union of pairwise disjoint geodesic lines. These geodesic lines decompose $\mathbb{H}^2$ into countably many connected components which are permuted by the action of the fundamental group $\pi_1(S)$ of S. If we denote by $\Gamma \subset \pi_1(S)$ the stabilizer of one of these components $\tilde{\Sigma}$, which is a convex subsurface of $\mathbb{H}^2$ with geodesic boundary, then Γ acts properly and cocompactly on $\tilde{\Sigma}$, with quotient one of the components S_i of $S - \gamma^*$. Thus Γ is a non-elementary convex cocompact Fuchsian group.

The *limit set*, that is, the set of accumulation points of a Γ-orbit $\Gamma x \subset \mathbb{H}^2$ $(x \in \tilde{\Sigma})$ in $\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$, is a Γ-invariant Cantor subset Λ of $\partial_\infty \mathbb{H}^2$. The quotient under the action of Γ of the set of all unit tangent vectors of geodesics with both endpoints in Λ has a natural identification with the invariant set $K_i \subset T^1S$. In particular, the restriction of Φ^t to K_i is topologically transitive. Its topological entropy equals the Hausdorff dimension $\delta_i \in (0,1)$ of Λ [Sul84].

Write $K = \cup_i K_i$ and let $\delta > 0$ be the topological entropy of $\Phi^t_{|K}$. We have $\delta = \max\{\delta_i \mid i \le k\}$. Recall that $\delta(z)$ denotes the topological entropy of the reparameterized flow $\Phi^t_{f_z}$ on T^1S and equals the critical exponent of the group $\rho_z(\pi_1(S)) \subset \mathrm{PSL}_d(\mathbb{R})$.

We have bounds on $\delta(z)$. The upper bound is very general:

Theorem 9.2 (Corollary 1.4 of [PS17]). *There is a constant $m > 0$ that bounds from above the entropy of any Hitchin representation.*

The lower bound depends on the choice of the grafting locus γ^* and the hyperbolic metric on S, and its proof is classical.

Lemma 9.3 (e.g. Theorem 4.1 of [CZZ23]). *$\delta(z) \in (\delta, m]$ for all z, where $m > \delta$ is the universal constant from the above Theorem 9.2.*

Proof. By definition of a Hitchin grafting representation, the image under ρ_z of the fundamental group Γ of any component of $S - \gamma^*$ is conjugate to its image under ρ, and hence has the same critical exponent. Suppose we picked the component with largest critical exponent, namely δ.

Then $\rho_z(\Gamma)$ is also Anosov (Γ is quasi-convex in $\pi_1(S)$) and its limit set is a proper subset of that of $\rho_z(\pi_1(S))$ so by Theorem 4.1 of [CZZ23] it has a strictly smaller critical exponent. On the other hand, since Γ stabilizes a totally geodesic hyperbolic plane in $\mathbb{X}$, the critical exponent of $\rho_z(\Gamma) \subset \mathrm{PSL}_d(\mathbb{R})$ equals the critical exponent δ for the group Γ. □

Let $h_{\mathrm{top}}(\Psi^t)$ be the topological entropy of a flow Ψ^t on a compact space; thus $\delta = h_{\mathrm{top}}(\Phi^t_{|K})$. A *measure of maximal entropy* for $\Phi^t_{|K}$ is an invariant probability measure μ with $h_\mu = \delta$.

Since $\Phi^t_{|K_i}$ is a topologically transitive Axiom A flow and K_i is compact, it admits a unique measure ν_i of maximal entropy. The measure ν_i is a Gibbs equilibrium state for $\Phi^t_{|K_i}$ with respect to the constant function 1, and it can be obtained from a Patterson Sullivan construction [Sul84]. The following well known fact will be useful later on.

Lemma 9.4. *A measure of maximal entropy for $\Phi^t_{|K}$ exists. It is unique if and only if there exists a number $i \leq k$ such that $h_{\rm top}(\Phi^t_{|K_i}) > \max\{h_{\rm top}(\Phi^t_{|K_j}) \mid j \neq i\}$. In this case the measure of maximal entropy is supported in K_i.*

Proof. Write again $K = \cup_i K_i$. The function which associates to a $\Phi^t_{|K}$-invariant probability measure μ its entropy h_μ is affine: for μ, η and $s \in (0,1)$ we have $h_{s\mu+(1-s)\eta} = sh_\mu + (1-s)h_\eta$.

The topologically transitive invariant subsets $K_i \subset K$ intersect at most along a finite number of periodic orbits. As a consequence, any Φ^t-invariant probability measure μ on K can be decomposed as $\mu = \sum_i \mu_i$ where μ_i is supported in K_i. The decomposition is unique if the μ-mass of any periodic orbit for Φ^t which projects to a component of γ^* vanishes.

Since $\Phi^t_{|K_i}$ is a topologically transitive axiom A flow, it admits a unique measure ν_i of maximal entropy. Then we have $h_{\nu_i} = h_{\rm top}(\Phi^t_{|K_i})$. Let $\mu = \sum_i \mu_i$ be any Φ^t-invariant Borel probability measure on K. Let $s_i = \mu_i(K_i)$; then $\sum_i s_i = 1$ and

$$h_\mu = \sum_i s_i h_{\mu_i} \leq \sum_i s_i h_{\rm top}(\Phi^t_{|K_i}) \leq \delta$$

with equality if and only if $s_j = 0$ for all j such that $h_{\rm top}(\Phi^t_{|K_j}) < \delta$, and $\mu_j = \nu_j$ if $s_j > 0$. In particular, a measure of maximal entropy exists, and if there exists a unique $i \leq k$ such that $h_{\rm top}(\Phi^t_{|K_i}) = \delta$, then such a measure is unique and coincides with ν_i. □

9.2 The total mass of the equilibrium state

For the fixed hyperbolic metric on S with unit tangent bundle T^1S and geodesic flow Φ^t denote by $\nu^1(z)$ the Φ^t-invariant probability measure on T^1S which is a multiple of $\nu(z)$. It turns out that the two normalisations $\nu(z)$ and $\nu^1(z)$ for the equilibrium states are comparable independently of z, as soon as the grafting datum z is taken in $\ker \alpha_0$ where α_0 is the linear functional which determines the Finsler norm of the tangent of a Riemannian geodesic in X which is invariant under $\rho(\gamma^*)$ (or a component of $\rho(\gamma^*)$).

Lemma 9.5. *For any $\sigma > 0$ there exists a constant $C > 0$ such that if the length of each component of $\gamma^* \subset S$ is at most σ, then for any grafting parameter $z \subset \ker \alpha_0^\perp$,*

$$C^{-1} \leq \|\nu(z)\| = \nu(z)(T^1S) \leq C.$$

Proof. Put $\nu^1(z) = \frac{\nu(z)}{\|\nu(z)\|}$ so that $\nu^1(z)$ is a probability measure on T^1S. Then $\|\nu(z)\| = (\int f_z d\nu^1(z))^{-1}$ since by equation (34), $\nu(z)$ was normalised so that $\int f_z d\nu(z) = 1$.

By definition of the equilibrium state of $-f_z$ and the fact that the entropy of the reparameterized flow $\Phi^t_{f_z}$ equals $\delta(z)$, we have

$$\int f_z d\nu^1(z) = \frac{h(\nu^1(z))}{\delta(z)}. \tag{37}$$

Since $h(\nu^1(z)) \leq 1$ (the topological entropy of Φ^t is 1, and is greater than or equal to the entropy of any invariant measure) and $\delta(z) > \delta$ by Lemma 9.3, it holds $\int f_z d\nu^1(z) \leq \frac{1}{\delta}$. It remains to get a lower bound.

By Theorem 6.2, we have

$$\int f_z \frac{d\gamma}{\ell_\rho(\gamma)} \geq \left(1 + \frac{C}{L+1}\right)^{-1},$$

for any $\gamma \in \pi_1(S)$, where $L \geq 0$ is any lower bound on the heights of the cylinders added along the components of γ^* to construct S_z (see Definition 3.1).

Then by density of the convex hull of currents supported on closed geodesics in the space of all currents, we get

$$\int f_z d\nu^1(z) \geq (1 + C)^{-1}. \qquad \square$$

9.3 The total mass of the Patterson–Sullivan measure

In this section we establish estimates on the total mass of some of the Patterson–Sullivan measures (see Equations (35) and (36)).

Proposition 9.6. *For any $\sigma > 0$ there is a constant $C > 0$ such that if the length of each component of γ^* is at most σ, then for any grafting parameter $z \subset \ker \alpha_0^\perp$, in any hyperbolic piece of $\widetilde{S}_z$ there exists a point x such that*

$$\mu_z^x(\partial_\infty \mathbb{H}^2) \leq C.$$

The strategy of the proof is as follows (see Figure 5). Assume that each component of $S - \gamma^*$ is a pair of pants. We fix one of them, say the pair of pants Σ, and its fundamental group Γ. Let $\tilde{\Sigma}$ be the universal covering of Σ. Then $\tilde{\Sigma} \subset \mathbb{H}^2$ is a convex hyperbolic surface with geodesic boundary. We find two disjoint intervals $I, J \subset \partial_\infty \mathbb{H}^2$, numbers $C_1, C_2 > 0$ and a fundamental domain for the action $\Gamma \curvearrowright \tilde{\Sigma}$, made of two right-angle hexagons $H \cup H'$ whose diameters are bounded from above by a constant only depending on σ and which depend somehow continuously on the data, so that the following holds. Let x be the center of H. First, the masses $\mu_z^x(I)$ and $\mu_z^x(J)$ are bounded from below by $C_1 \mu_z^x(\partial_\infty \mathbb{H}^2)$. Second, each geodesic connecting a point in I to a point in J intersects H in an arc of length at least C_2 and hence passes uniformly near x. We then can estimate the Gromov product and bound the product measure $\mu_z^x(I) \times \mu_z^x(J) = \mu_z^x \times \mu_z^x(I \times J)$ from above by a constant multiple of $\nu(z)(T^1S)$, which is uniformly bounded from above by Lemma 9.5.

We begin with establishing a few estimates in a more general setting involving representations of the fundamental group of a pair of pants (the free group F_2 with two generators) into $\mathrm{PSL}_2(\mathbb{R})$. Let us introduce some notations. Let P be a topological pair of pants, equipped with a fixed orientation. We fix a basepoint p_0 in P and three generators a, b, c of the fundamental group $\pi_1(P, p_0) = F_2$ such that $c \cdot b \cdot a = 1$ and each generator corresponds to one of the boundary components of P.

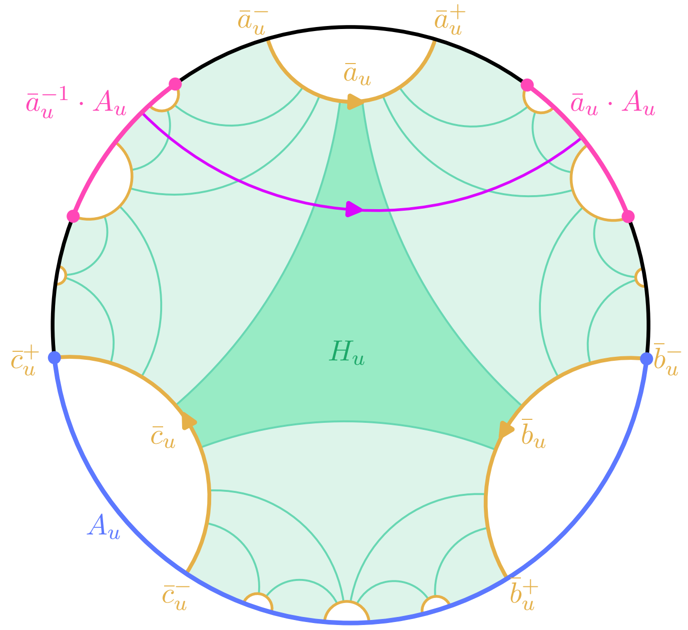

Figure 5: Control of $\nu(T^1S)$ using the measure for μ of two intervals $I = \bar{a}_u \cdot A_u$ and $J = \bar{a}_u^{-1} \cdot A_u$ in $\partial\mathbb{H}^2$. The green hexagon H_u is half of a fundamental domain of a pair of pant P_u. The purple geodesic goes from I to J and intersects H_u in an arc whose length is bounded from below.

For a set of lengths $u = (u_a, u_b, u_c) \in [0, \infty)^3$ there is a unique hyperbolic structure on P whose boundary components have these lengths on a, b, c, and up to conjugation, there is a unique representation $j_u : \pi_1(P) \to \mathrm{PSL}_2(\mathbb{R})$ associated to this hyperbolic structure which is normalised so that the following ordering assumption holds.

Put a_u, b_u, c_u instead of $j_u(a), j_u(b), j_u(c)$. Then u_i are loxodromic elements of $\mathrm{PSL}_2(\mathbb{R})$ with axes $\bar{a}_u, \bar{b}_u, \bar{c}_u \subset \mathbb{H}^2$, oriented to define the boundary orientation for the oriented pair of pants P, with endpoints $\bar{a}_u^{\pm}, \bar{b}_u^{\pm}, \bar{c}_u^{\pm} \subset \partial_\infty \mathbb{H}^2 = S^1$. We use the abuse of notation that if for instance $u_a = 0$ then $j_u(a)$ has only one fixed point on $\partial \mathbb{H}^2$ and $\bar{a}_u = \bar{a}_u^+ = \bar{a}_u^-$.

The representation j_u is chosen so that the cycle $(\bar{a}_u^-, \bar{a}_u^+, \bar{b}_u^-, \bar{b}_u^+, \bar{c}_u^-, \bar{c}_u^+)$ is oriented clockwise for the circular order on $\partial_\infty \mathbb{H}^2$. We may also assume that the center 0 of the unit disk $D = \mathbb{H}^2$ is contained in the convex hull of the limit set of j_u and that j_u varies continuously in u.

Consider the three intervals of $\partial_\infty \mathbb{H}^2$ (that is, the segment in $\partial_\infty \mathbb{H}^2$ determined by the orientation of S^1 and its endpoints)

- $A_u = [\bar{b}_u^-, \bar{c}_u^+]$,
- $B_u = [\bar{c}_u^-, \bar{a}_u^+]$,
- $C_u = [\bar{a}_u^-, \bar{b}_u^+]$.

By construction, we have $A_u \cup B_u \cup C_u = \partial_\infty \mathbb{H}^2$, so for any finite measure μ on $\partial_\infty \mathbb{H}^2$, one of the intervals has mass at least $\frac{1}{3}\mu(\partial_\infty \mathbb{H}^2)$. Put $A_u^+ = a_u \cdot A_u$, $A_u^- = a_u^{-1} \cdot A_u$, and similarly for $B_u^+, B_u^-, C_u^+, C_u^-$.

Lemma 9.7. *The intervals A_u^+ and A_u^- are disjoint. Similarly $B_u^+ \cap B_u^- = \emptyset$ and $C_u^+ \cap C_u^- = \emptyset$.*

Proof. First notice that $a_u \cdot \bar{c}_u^+$ belongs to $[\bar{a}_u^+, \bar{c}_u^+]$ since a_u is an hyperbolic element with attractive fixed point $\bar{a}_u^+$. By construction, we have $c_u \cdot b_u \cdot a_u = 1$, so that

$$a_u \cdot \bar{c}_u^+ = (b_u^{-1} \cdot c_u^{-1}) \cdot \bar{c}_u^+ = b_u^{-1} \cdot \bar{c}_u^+$$

So $a_u \cdot \bar{c}_u^+$ is included in both $[\bar{a}_u^+, \bar{c}_u^+]$ and $[\bar{c}_u^+, \bar{b}_u^-]$, whose intersection is equal to the interval $[\bar{a}_u^+, \bar{b}_u^-]$. Similarly, $a_u \cdot \bar{b}_u^-$ lies inside $[\bar{a}_u^+, a_u \cdot \bar{c}_u^+] \subset [\bar{a}_u^+, \bar{b}_u^-)$. And since $A_u = [\bar{b}_u^-, \bar{c}_u^+]$, $a_u \cdot A_u \subset [\bar{a}_u^+, \bar{b}_u^-)$, and similarly $a_u^{-1} \cdot A_u \subset (\bar{c}_u^+, \bar{a}_u^-]$, it follows that they are disjoint. $\square$

Write $\Gamma_0 = \{a, a^{-1}, b, b^{-1}, c, c^{-1}\} \subset \pi_1(P)$

Corollary 9.8. *If μ is a $j_u(\pi_1(P))$-quasi-invariant finite measure on $\partial_\infty \mathbb{H}^2$, then the measure for $\mu \times \mu$ of one of the three products $A_u^- \times A_u^+$, $B_u^- \times B_u^+$, $C_u^- \times C_u^+$ has mass at least $\frac{C^2}{9}\mu(\partial_\infty \mathbb{H}^2)^2$, where*

$$C = C_{\mu,u} = \inf\left\{ \frac{dj_u(\gamma)_*\mu}{d\mu}(\xi) : \ \xi \in \partial_\infty \mathbb{H}^2, \ \gamma \in \Gamma_0 \right\}.$$

We will also need an estimate on the lengths of the intersection of geodesics from A_u^- to A_u^+, with $H_u \subset \mathbb{H}^2$ the (possibly degenerate) right-angled hexagon adjacent to the axes of $j_u(a), j_u(b), j_u(c)$.

Lemma 9.9. *For any $\sigma > 0$ there exists L_σ such that for any $u \in [0,\sigma]^3$, for all (x,y) in $A_u^- \times A_u^+$, $B_u^- \times B_u^+$ or $C_u^- \times C_u^+$, the length of the intersection of the hexagon H_u with the geodesic from x to y is at least L_σ.*

Proof. This is a direct consequence of the following three facts. $A_u^\pm$ varies continuously with u. The length $length(\gamma \cap H_u)$ for a geodesic γ with ends in $A_u^- \times A_u^+$ is positive and continuous in the pair (u,γ). And $[0,\sigma]^3$ is compact. □

Proof of Proposition 9.6. According to Theorem A.2, we can choose a pair of pants in $S \setminus \gamma^*$, whose boundary curves have length bounded from above by a constant only depending on the genus of S and σ. Let us identify it topologically with P, and identify a convex subsurface $\widetilde{P}$ of a hyperbolic piece inside the universal covering $\widetilde{S}_z$ of the abstract grafted surface with the universal cover of P. We denote by $u = (u_a, u_b, u_c)$ the lengths of the boundary components of this pair of pants. The surface $\tilde{P}$ contains a right angled hexagon H_u whose double is a fundamental domain for the deck group $\pi_1(P)$.

Identify $\mathrm{PSL}_2(\mathbb{R})$ with a subgroup of $\mathrm{PSL}_d(\mathbb{R})$ and $\mathbb{H}^2$ with a totally geodesic subspace of $\mathbb{X}$. Up to conjugation of the Hitchin grafting representation ρ_z, we may assume that its restriction to $\pi_1(P)$ coincides with $j_u : \pi_1(P) \to \mathrm{PSL}_2(\mathbb{R})$, so that the fixed hyperbolic piece $\widetilde{P}$ of the characteristic surface is contained in $\mathbb{H}^2 \subset \mathbb{X}$. More precisely, it is the convex hull of the limit set of $j_u(\pi_1(P))$. Note that this makes sense since the boundary of $\mathbb{H}^2$ embeds naturally into the flag variety $\mathcal{F}$ as well as the visual boundary $\partial_\infty \mathbb{X}$ of $\mathbb{X}$. We choose a point x in the interior of the hexagon $H_u \subset \widetilde{P}$ as a basepoint in $\mathbb{H}^2$.

By Lemma 9.5, Corollary 9.8 and Lemma 9.9, and since the Gromov product is nonnegative, there is a constant C such that

$$C \geq \left(e^{\delta\langle\cdot|\cdot\rangle_x}\mu_z^x \times \mu_z^x \times \mathrm{Leb}\right)(T^1 H_u) \geq \frac{L_\sigma C_\delta^2}{9}\mu_z^x(\partial_\infty \mathbb{H}^2)^2$$

where $T^1 H_u$ denotes the set of unit tangent vectors in $T^1\mathbb{H}^2$ with footpoint in the hexagon H_u, and C_δ is the infimum of the constants $C_{\mu,u}$ appearing in the corollary, that is

$$C_\delta = \inf\left\{e^{\delta b_\xi^{\mathfrak{F}}(x, j_u(\gamma)x)} : \ \xi \in \mathcal{F},\ u \in [0,\sigma]^3,\ \gamma \in \Gamma_0\right\}.$$

To conclude the proof one can use Theorem 9.2, which implies $C_\delta \geq C_m$ where $C_m > 0$ is a constant that only depends on the genus of S and the choice of length function on $\mathrm{PSL}_d(\mathbb{R})$ (i.e. the choice of a linear functional α_0 on $\mathfrak{a}$). □

Remark 9.10. In the proposition 9.6, we may actually take x to be any point in ϵ-thick part of S, for $\epsilon > 0$ first fixed. To see that, modify the proof as follow. Fix $\epsilon > 0$, and instead of taking a unique representation j_u for a fixed data of $u \in [0,\sigma]^3$, take a larger compact set of representations J so that each representation in J is conjugated to one j_u as above. Furthermore, for each j_u and each point p in the ϵ-thick part of the hyperbolic pair of pants *obtained as the quotient by j_u of the convex core of j_u, there exists a representation $j_{u,p}$ in J and an isometry f of $\mathbb{H}^2$ which conjugates $j_{u,p}$ to j_u and which sends the origin of $\mathbb{H}^2$ to a preimage of p in the convex core of $j_{u,p}$. The same compactness argument holds except that the constant C my be larger. Additionally for $\epsilon > 0$ small enough, the ϵ-thick part of a surface S is equal to the union of the ϵ-thick part of the pair of pants in any pant decomposition of S.*

9.4 Estimating the intersection number of the equilibrium state with the grafting locus

Recall that γ^* is the grafting locus and $\hat{\nu}(z)$ is the geodesic current defined by the equilibrium measure $\nu(z)$ for a grafting parameter z. The following is the main result of this section.

Proposition 9.11. *For any $\sigma > 0$ there are $C, C' > 0$ such that if every component of γ^* has hyperbolic length at most σ, then for any grafting parameter z we have*

$$\iota(\hat{\nu}(z), \gamma^*) \leq C(L+1)e^{-\delta(z)L} \leq C' e^{-\delta L/2},$$

where L is the minimum of the heights of the flat cylinders in the abstract grafting (see Equation (18)), and δ is the topological entropy of the geodesic flow on $S - \gamma^*$.

The inequality $C(L+1)e^{-\delta(z)L} \leq C'e^{-\delta L/2}$ comes from the fact that $\delta(z) \geq \delta$ and that by Proposition A.3, the topological entropy δ of the geodesic flow on $T^1(S - \gamma^*)$ is bounded from below by a positive constant only depending on σ.

We will need the following result about Hitchin representations. In its formulation, $\partial\pi_1(S)$ denotes the Gromov boundary of the surface group $\pi_1(S)$.

Lemma 9.12. *Let $\rho' : \pi_1(S) \to G$ be a Hitchin representation with limit map $\Xi' : \partial\pi_1(S) \to \mathcal{F}$, and let $(\gamma_n)_n \subset \pi_1(S)$ be a sequence converging to $\xi \in \partial\pi_1(S)$. Then for any compact set $K \subset \mathbb{X}$, the accumulation points of $\rho'(\gamma_n)K$ in the visual boundary $\partial_\infty\mathbb{X}$ of $\mathbb{X}$ are contained in the interior of the Weyl Chamber $\Xi'(\xi)$.*

Proof. This is a consequence of the Anosov property discussed in Section 2.2, which is satisfied by Hitchin representations, and a characterisation of this property in terms of Cartan decompositions of the images $\rho'(\gamma)$ with $\gamma \in \pi_1(S)$.

Let $\rho'(\gamma_n) = k_n \exp(a_n)\ell_n$ be a Cartan decomposition, so that $k_n, \ell_n \in K$ (the maximal compact subgroup) and $a_n \in \mathfrak{a}^+$. By a characterisation of the Anosov property (see Theorem 4.37 of [Kas24] for more details and a history of this result), the angle formed by a_n with each wall of the Weyl Cone $\mathfrak{a}^+$ is bounded from below independently of n. In other words, denoting by $\|\cdot\|$ the Euclidean norm on $\mathfrak{a}$, we have $\alpha(a_n) \geq \mathrm{Cst}||a_n||$ for any positive root α, which means precisely that $(\exp(a_n))_n$ accumulates in the interior of the Weyl Chamber $\partial_\infty \exp(\mathfrak{a}^+) \subset \partial_\infty\mathbb{X}$ in the ideal boundary of the flat cone $\exp(\mathfrak{a}^+) \subset \mathbb{X}$.

Up to passing to a subsequence we may assume that $k_n \to k$ and $\ell_n \to \ell$. Let $\mathfrak{a}^-$ be the Weyl chamber opposite to $\mathfrak{a}^+$, with boundary $\partial_\infty \exp(\mathfrak{a}^-)$, viewed as a point in $\mathcal{F}$.

Then for any $\eta \in \mathcal{F}$ transverse to $\ell^{-1}\partial_\infty \exp(\mathfrak{a}^-)$ we have $\rho'(\gamma_n)\eta \to k\partial_\infty \exp(\mathfrak{a}^+)$. By the definition of the limit map Ξ', this implies that $\Xi'(\xi) = k\partial_\infty \exp(\mathfrak{a}^+)$ (see Definition 2.5).

Then $\rho'(\gamma_n)\mathbf{x} = k_n e^{a_n}\mathbf{x}$ only accumulates in the interior of the Weyl Chamber $\Xi'(\xi)$, and the same holds for $\rho'(\gamma_n)K$ which lies at bounded distance from $\rho'(\gamma_n)\mathbf{x}$. □

Proof of Proposition 9.11. Let γ_0^* be a component of γ^*. Let $\widetilde{\gamma}^* \subset \mathbb{H}^2$ be the preimage of γ^* and choose a component $\hat{\gamma}_0^* \subset \widetilde{\gamma}^*$ of the preimage of γ_0^*. Denote by I^- and I^+ the two connected component of $\partial_\infty\mathbb{H}^2 - \hat{\gamma}_0^*(\pm\infty)$. Recall that an oriented geodesic in $\mathbb{H}^2$ can be thought of as an ordered sets of distinct points $(\xi^-, \xi^+) \in \partial_\infty\mathbb{H}^2 \times \partial_\infty\mathbb{H}^2 - \Delta$. Then a geodesic (ξ^-, ξ^+) intersects $\hat{\gamma}_0^*$ transversely if and only if $(\xi^-, \xi^+) \in I^- \times I^+ \cup I^+ \times I^-$.

Let $g_u \in \pi_1(S)$ be a generator of the infinite cyclic subgroup $\langle g_u \rangle$ of $\pi_1(S)$ which preserves $\hat{\gamma}_0^*$ and acts on it as a group of translations. Choose a fundamental domain $\Omega^\pm$ for the action of $\langle g_u \rangle$ on $I^\pm$ of the form $\Omega^\pm = [\xi_0^\pm, g_u\xi_0^\pm) \subset I^\pm$, where $\xi_0^\pm \in I^\pm$ are taken so that the geodesic (ξ_0^-, ξ_0^+) crosses $\widetilde{\gamma}_0^*$ orthogonally at a point x.

Using these notations, it follows from the definitions of the intersection number between two geodesic currents of S (see the appendix of [Bon88] and Chapter 8.2.11 of [Mar16]) that

$$\iota(\hat{\nu}(z), \gamma_0^*) = \hat{\nu}(z)(\Omega^- \times I^+) + \hat{\nu}(z)(I^+ \times \Omega^-). \tag{38}$$

Namely, the intersection number of $\hat{\nu}(z)$ with γ_0 is just the total $\hat{\eta}(z)$-mass of all geodesics crossing transversely through a fundamental domain for the action of g_u on $\hat{\gamma}_0$. This set in turn is a fundamental domain for the action of $\langle g_u \rangle$ on the space of all geodesics crossing through $\hat{\gamma}_0^*$. As $(\Omega^- \cup I^+) \cup (I^+ \cup \Omega^-)$ is another such fundamental domain, and $\hat{\eta}(z)$ is $\langle g_u \rangle$- invariant, this yields the formula (38).

By symmetry, it suffices to bound $\hat{\nu}(z)(\Omega^- \times I^+)$ from above. Recall that $\Xi_z : \partial_\infty\mathbb{H}^2 \to \mathcal{F}$ is the limit map induced by the Hitchin grafting representation ρ_z. Our computations rely on the characterisation of $\hat{\nu}(z)$ as the product

$$\hat{\nu}(z) = e^{\delta(z)\langle \Xi_z(\xi) | \Xi_z(\eta) \rangle_p} d\mu_z^p(\xi) d\mu_z^p(\eta),$$

where $\langle \cdot | \cdot \rangle_p$ is the Gromov product based at p and p is any point in $\mathbb{X}$. The measure $\hat{\nu}(z)(\Omega^- \times I^+)$ can be bounded from above by the product of $\mu_z^p(\Omega^-) \cdot \mu_z^p(\partial I^+)$ with the maximum of the Gromov products based at p of points in $\Xi_z(\Omega^-) \times \Xi_z(I^+)$. The strategy for estimating these quantities and completing the proof is the following.

(i) Make a suitable choice of basepoint p.

(ii) Use Proposition 9.6 to find a constant C_1 only depending on σ such that $\mu_z^p(\Omega^-) \leq C_1$.

(iii) Use admissible paths and Proposition 5.10 to find a constant C_2 only depending on σ such that $\langle \Xi_z(\xi), \Xi_z(\eta) \rangle_p \leq C_2$ for all $\xi \in \Omega^-$ and $\eta \in I^+$.

(iv) Use admissible paths and Propositions 9.6 and 5.10 to find a constant C_3 only depending on σ such that

$$\mu_z^p(I^+) = \sum_n \mu_z^p(g_u^n \Omega^+) \leq C_3(L+1)e^{-\delta(z)L}.$$

The most involved part will be the last step (iv) of the above list.

First step (i). Put $\ell := \ell_S(\gamma_0^*) = \ell^{\mathfrak{F}}(\rho_z(g_u))$, which is bounded from above by σ by assumption, and let $\omega = \sinh^{-1}\left(\frac{1}{\sinh(\ell/2)}\right)$ be the size of the collar in S around γ_0^*, so that the two boundaries of the collar are in the ϵ_0-thick part of S for some universal constant ϵ_0.

The geodesic line $\hat{\gamma}_0^*$ is adjacent to two connected components $\widetilde{S}^-, \widetilde{S}^+$ of $\mathbb{H}^2 - \widetilde{\gamma}^*$. Denote by H^+, H^- the two closed half-planes of $\mathbb{H}^2$ with boundary $\hat{\gamma}_0^*$ and assume that $\tilde{S}^\pm \subset H^\pm$ and that $I^\pm \subset \partial_\infty H^\pm$. Let x^-, x^+ be the points lying in this order on the geodesic (ξ_0^-, ξ_0^+), both at distance exactly ω from the intersection point x of (ξ_0^-, ξ_0^+) with $\hat{\gamma}_0^*$. In particular, $x^\pm$ projects into the ϵ_0-thick part of S.

Recall that for the abstract grafting surface S_z there exists a natural projection map $\pi_z : S_z \to S$ which is injective outside of the flat cylinders (see Definition 3.1). Lift π_z to a $\pi_1(S)$-equivariant map $\widetilde{\pi}_z : \widetilde{S}_z \to \widetilde{S} = \mathbb{H}^2$, which is injective on the preimages $\widetilde{S}_z^\pm \subset \widetilde{S}_z$ of the components $\widetilde{S}^\pm$ of $\mathbb{H}^2 - \widetilde{\gamma}^*$. Then $x^\pm \in \mathbb{H}^2$ (but not x) have unique preimages $\widetilde{x}^\pm \in \widetilde{S}_z^\pm$. The basepoint p we were looking for is $p = \widetilde{Q}_z(\tilde{x}^-)$.

Second step (ii). Pulling the Patterson Sullivan measure μ_p based at p for the action of $\rho_z(\pi_1(S))$ back to a measure $\mu_z^{\tilde{x}^-}$ on $\partial_\infty\mathbb{H}^2$, this is an immediate application of Proposition 9.6 (and 9.10), which says that $\mu_z^{\widetilde{x}^-}(\partial_\infty\mathbb{H}^2)$ is bounded from above by a constant depending only on σ.

Third step (iii). Let $\xi \in \Omega^-$ and $\eta \in I^+$. There is a unique bi-infinite admissible path $a\colon \mathbb{R} \to \mathbb{H}^2$ from ξ to η, which is a lift of an admissible path in the hyperbolic surface S, defined with respect to the multicurve γ^*. Recall that a is a concatenation of geodesic pieces, alternating between arcs contained in $\widetilde{\gamma}^*$, called flat-type, and geodesic arcs with endpoints on $\widetilde{\gamma}^*$ and orthogonal to $\widetilde{\gamma}^*$, called hyperbolic-type.

By Observation 3.7, up to parameterization of the flat pieces, a is the image under π_z of a unique admissible path $\widetilde{a}\colon \mathbb{R} \to \widetilde{S}_z \subset \mathbb{X}$. By Lemma 9.12, $\tilde{Q}_z(\tilde{a}(t))$ accumulates as $t \to \infty$ (resp. $t \to -\infty$) in the interior of the Weyl Chamber $\Xi_z(\eta)$ (resp. $\Xi_z(\xi)$).

Using the map $\tilde{Q}_z$, which is a $\pi_1(S)$-equivariant embedding of $\tilde{S}_z$ onto the universal covering of the characteristic surface of $\rho_z(\pi_1(S))$, pull the Finsler distance $d^{\mathfrak{F}}$ back to $\tilde{S}_z$ and denote this distance by the same symbol. With this notation and by definition of the Gromov product (see (8)), we have

$$\langle \Xi_z(\eta), \Xi_z(\xi)\rangle_{x^-} = \lim_{T\to\infty} \frac{1}{2}\left(d^{\mathfrak{F}}(\widetilde{a}(-T), \tilde{x}^-) + d^{\mathfrak{F}}(\tilde{x}^-, \widetilde{a}(T)) - d^{\mathfrak{F}}(\widetilde{a}(-T), \widetilde{a}(T))\right).$$

By Proposition 5.10, the path $\tilde{Q}_z(\widetilde{a})$ is C_2-quasi-ruled for some constant C_2 depending only on σ, so $d_{\mathbb{X}}^{\mathfrak{F}}(a(-T), \widetilde{a}(t)) + d_{\mathbb{X}}^{\mathfrak{F}}(\widetilde{a}(t), \widetilde{a}(T)) \leq d_{\mathbb{X}}^{\mathfrak{F}}(\widetilde{a}(-T), \widetilde{a}(T)) + C_2$ for all $-T \leq t \leq T$. Thus, to find an upper bound on $\langle \Xi_z(\eta), \Xi_z(\xi)\rangle_{\widetilde{x}^-}$, it suffices to prove that there exists a number $R > 0$ only depending on σ such that $\tilde{a}$ intersects the ball of radius R about $\tilde{x}^-$.

As a intersects $\hat{\gamma}_0$, it contains a (possibly degenerate) piece of flat type which is a subarc of $\hat{\gamma}_0$. Choose a parameterization of a so that $a(0) \in \hat{\gamma}_0$ is the starting point of this segment. Then the piecewise geodesic ray $a|_{(-\infty,0]} : (-\infty, 0] \to H^-$ ends on $\hat{\gamma}_0^*$ with a hyperbolic-type geodesic piece of length at least ω (the collar size). By the definition of Ω^-, the shortest distance projection of ξ to $\hat{\gamma}_0^*$ is at distance at most ℓ from the shortest distance projection x of x^-. The constant R we are looking for is provided by the following lemma, whose proof (which relies on hyperbolic trigonometry) is postponed until after this proof.

Lemma 9.13. *For any $\sigma > 0$ there is $R > 0$ such that the following holds. Let $0 < \ell \leq \sigma$ and let $\omega = \sinh^{-1}\left(\frac{1}{\sinh(\ell/2)}\right) > 0$ be the collar size associated to ℓ by the hyperbolic collar lemma.*

Let $\mathcal{L} \subset \mathbb{H}^2$ be a line and $a : [0, \infty) \to \mathbb{H}^2$ an admissible path starting on $\mathcal{L}$ orthogonally to it, and with a hyperbolic-type piece of length at least ω. Suppose $a(t)$ tends as $t \to \infty$ to $\xi \in \partial\mathbb{H}^2$ whose orthogonal projection is ℓ-close to the starting point of a ray $r : [0, \infty) \to \mathbb{H}^2$ orthogonal to $\mathcal{L}$ and in the same half-plane as a. Then $d_{\mathbb{H}^2}(a(\omega), r(\omega)) \leq R$.

Lemma 9.13 exactly tells us that the point $a(-\omega)$ is contained in the ball of radius R about x^-. It remains to check that the distance between $\widetilde{a}(-\omega)$ and $\tilde{x}^-$ is at most R as well. This holds true because $a(-\omega)$ and x^- are contained in $\widetilde{S}^-$, so their preimages $\widetilde{a}(-\omega)$ and $\widetilde{x}^-$ are contained in the same hyperbolic piece $\widetilde{S}_z^- \subset \widetilde{S}_z$. As this piece is isometrically embedded in $\tilde{S}_z$ and the Finsler distance $d^{\mathfrak{F}}$ is not larger than the path distance on the grafted surface, this completes the distance estimate.

Fourth step (iv). This part of the proof is the longest and most involved. By equivariance, we have

$$\mu_z^{\widetilde{x}^-}(\xi) = e^{\delta(z) b^{\mathfrak{F}}_{\Xi_z(\xi)}(\widetilde{x}^+, \widetilde{x}^-)} d\mu_z^{\widetilde{x}^+}(\xi)$$

where $\delta(z)$ is the critical exponent of ρ_z and $b^{\mathfrak{F}}_\eta(q, q')$ is the Busemann function of (q, q') based at $\eta \in \mathcal{F}$ (for the Finsler metric), see Section 1.

Since $\mu_z^{\widetilde{x}^+}(\partial_\infty\mathbb{H}^2) \leq C_1$ for a constant $C_1 > 0$ only depending on σ by Proposition 9.6, to get the desired upper bound on $\mu_z^{\widetilde{x}^-}(I^+)$ it would suffice for $\xi \in I^+$ to bound from above the Busemann function $b^{\mathfrak{F}}_{\Xi_z}(\widetilde{x}^+, \widetilde{x}^-)$ by $-L$ plus some constant. However this is not always possible: it is possible only if $\xi \in \Omega^+$ (and another condition on z that will not matter much), using that the admissible path from $\widetilde{x}^-$ to $\Xi_z(\xi)$ is quasi-ruled and passes near $\widetilde{x}^+$, and using that $\widetilde{x}^-$ and $\widetilde{x}^+$ are at distance roughly L. If ξ is not in Ω^+, then it is contained in $g_u^n\Omega^+$ for some $n \neq 0$, and the admissible path from $\widetilde{x}^-$ to $\Xi_z(\xi)$ is still quasi-ruled but passes instead near $\rho_z(g_u^n)\widetilde{x}^+$, which allows us to bound $b^{\mathfrak{F}}_{\Xi_z(\xi)}(\rho_z(g_u^n)\widetilde{x}^+, \widetilde{x}^-)$ by minus the distance from $\widetilde{x}^-$ to $\rho_z(g_u^n)\widetilde{x}^+$, which is roughly $\max(L, n\ell) + 2\omega$. We will then be able to conclude our estimate of $\mu_z^{\widetilde{x}^-}(I^+)$ by computing

$$\mu_z^{\widetilde{x}^-}(I^+) = \sum_n \mu_z^{\widetilde{x}^-}(g_u^n\Omega^+) \leq \mathrm{Cst} \sum_n e^{-\delta(z)\max(L,n\ell) - 2\delta(z)\omega}.$$

Fix $\xi \in g_u^n\Omega^+ \subset I^+$. There exists a unique admissible path $a_\xi : [0, +\infty) \to \widetilde{S} = \mathbb{H}^2$ from x^- to ξ (lifting an admissible path of S), and it is the image under $\widetilde{\pi}_z$ of a unique admissible path $\widetilde{a}_\xi : [0, \infty) \to \widetilde{S}_z \subset \mathbb{X}$ that starts at $\widetilde{x}^-$ and accumulates in the interior of the simplex $\Xi_z(\xi)$ by Lemma 9.12. By the definition of Finsler Busemann cocycles (see Section 1), this means that we have

$$b^{\mathfrak{F}}_{\Xi_z(\xi)}(\rho_z(g_u^n)\widetilde{x}^+, \widetilde{x}^-) = \lim_{T\to\infty} d^{\mathfrak{F}}(\rho_z(g_u^n)\widetilde{x}^+, \widetilde{a}_\xi(T)) - d^{\mathfrak{F}}(\widetilde{x}^-, \widetilde{a}_\xi(T)). \tag{39}$$

Notice that the third geodesic piece of $\widetilde{a}_\xi$ (the one that leaves the flat strip $\hat{\gamma}_0^*$) is the isometric image by g_u^n of an admissible path going from $\hat{\gamma}_0^*$ to $g_u^{-n}\xi \in \Omega^+$. And therefore $\widetilde{a}_\xi$ passes within distance R of $\rho_z(g_u^n)\widetilde{x}^+$ at some time t.

By Proposition 5.10, $\widetilde{a}_\xi$ is C_2-quasi-ruled (and starts at $\widetilde{x}^-$) so

$$d^{\mathfrak{F}}(\widetilde{a}_\xi(t), \widetilde{a}_\xi(T)) - d^{\mathfrak{F}}(\widetilde{x}^-, \widetilde{a}_\xi(T)) \leq -d^{\mathfrak{F}}(\widetilde{x}^-, \widetilde{a}_\xi(t)) + C_2 \text{ for any } T \geq t.$$

This, combined with $d^{\mathfrak{F}}(\rho_z(g_u^n)\widetilde{x}^+, \widetilde{a}_\xi(t)) \leq R$ and (39) yields:

$$b^{\mathfrak{F}}_{\Xi_z(\xi)}(\rho_z(g_u^n)\widetilde{x}^+, \widetilde{x}^-) \leq -d^{\mathfrak{F}}(\rho_z(g_u^n)\widetilde{x}^+, \widetilde{x}^-) + C_2 + 2R. \tag{40}$$

We now need to estimate $d^{\mathfrak{F}}(\rho_z(g_u^n)\widetilde{x}^+, \widetilde{x}^-)$, and we also do this using that the admissible path from $\widetilde{x}^-$ to $\rho_z(g_u^n)\widetilde{x}^+$ is quasi-ruled, except that this time this path is completely explicit. The unique admissible path c in $\mathbb{H}^2$ from x^- to $g_u^n x^+$ has three geodesic pieces: first the geodesic from x^- to x, of length ω, then the geodesic from x to $g_u^n x$, of length $|n|\ell$, and finally the geodesic from $g_u^n x$ to $g_u^n x^+$, of length ω. It's image under $\widetilde{\pi}_z$ is the unique admissible path $\widetilde{c}$ from $\widetilde{x}^-$ to $\rho_z(g_u^n)\widetilde{x}^+$, which is also made of three explicit geodesic pieces. The first and last pieces are just translates of the corresponding pieces of c, and hence have length ω.

The middle piece, however, is more complicated because instead of sliding along $\hat{\gamma}_0^*$ like c, we are navigating in a flat strip that lifts the flat cylinder above $\gamma_0^* \subset \gamma^*$, and we must move diagonally in this flat strip to realise at the same time the horizontal translation prescribed by the middle piece of c and the vertical translation prescribed by the grafting parameter z. Let $z_0 \in \mathfrak{a}$ be the coordinate of z associated to the component $\gamma_0^* \subset \gamma^*$. Then the above mentioned flat strip is conjugate to the strip $\{tv_0 \pm sz_0 : t \in \mathbb{R},\ s \in [0,1]\}$ where $v_0 = d\tau\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is the special direction of $\mathfrak{a}^+$ and the sign $\pm$ depends on the choice of orientation on γ_0^* (see Section 3.1). Since we are moving horizontally a distance $|n|\ell$, the middle piece of $\widetilde{c}$ is conjugate in this strip to the geodesic segment from 0 to $n\ell v_0 \pm z_0$. As a consequence, the length of this middle piece is exactly $\mathfrak{F}(n\ell v_0 \pm z_0)$ where $\mathfrak{F}$ is the norm on $\mathfrak{a}$ defined in Equation (3). Finally, using again that $\widetilde{c}$ is C_2-quasi-ruled (Proposition 5.10), we get

$$d^{\mathfrak{F}}(\widetilde{x}^-, \rho_z(g_u^n)\widetilde{x}^+) \geq 2\omega + \mathfrak{F}(z_0 \pm n\ell v_0) - 2C_2. \tag{41}$$

Now we must estimate $\mathfrak{F}(z_0 \pm n\ell v_0)$. By the assumption, the height of the cylinder at $\widetilde{\gamma}_0^*$ is $\min_{t\in\mathbb{R}} \mathfrak{F}(z_0 + tv_0) \geq L$ (see (18)). Let t_0 be the unique point of $\mathbb{R}$ such that $z_0 + t_0 v_0 \in \ker \alpha_0$ (α_0 is the linear form which is equal to the Finsler norm in the Weyl cone that contains w_0). Then

$$\mathfrak{F}(z_0 + tv_0) \geq |\alpha_0(z_0 + tv_0)| = |\alpha_0((t - t_0)v_0)| = |t - t_0|$$

for any $t \in \mathbb{R}$. Let n_0 be the integer closest to t_0/ℓ, so that $\mathfrak{F}(z_0 + n\ell v_0) \geq |n - n_0|\ell - \ell$, which is bounded below by $|n - n_0|\ell - \sigma$. Combining this with (40) and (41) we get

$$b^{\mathfrak{F}}_{\Xi_z(\xi)}(\rho_z(g_u^n)\widetilde{x}^+, \widetilde{x}^-) \leq -2\omega - \max(|n_0 \pm n|\ell, L) + C_2 + 2R + \sigma + 2C_2.$$

(Recall that $\pm$ is just some fixed sign depending on the choice of orientation of γ_0^*.)

Recall the quasi-invariance of Patterson–Sullivan measures:

$$\mu_z^{\widetilde{x}}(g_u^n\Omega^+) = \int_{\xi\in g_u^n\Omega^+} e^{\delta(z)b^{\widetilde{\mathfrak{F}}}_{\Xi_z(\xi)}(\rho_z(g_u^n)\widetilde{y},\widetilde{x})}d\mu_z^{\rho_z(g_u^n)\widetilde{y}}(\xi).$$

Since $\mu_z^{\rho_z(g_u^n)\widetilde{y}}(\partial\mathbb{H}^2) = \mu_z^{\widetilde{y}}(\partial\mathbb{H}^2)$ is bounded from above by some constant C_1 that only depends on σ by Proposition 9.6, and $\delta(z) \le m$ for some constant m depending only on α_0 by Lemma 9.3, we get

$$\begin{aligned}\mu_z^{\widetilde{x}}(\rho_z(g_u^n)\Omega^+) &\le e^{m(2R+3C_2+\sigma)}C_1e^{-\delta(z)\max(|n_0\pm n|\ell,L)}e^{-2\delta(z)\omega}\\ &= C_3e^{-\delta(z)\max(|n_0\pm n|\ell,L)}e^{-2\delta(z)\omega},\end{aligned}$$

where C_3 only depends on σ. After some computations, and using that $e^{-\omega} \le \ell\cosh(\sigma)$ and $(1-e^{-x})^{-1} \le \frac{2}{x}+2$ for any $x>0$, we get

$$\begin{aligned}\mu_z^{\widetilde{x}}(I^+) = \sum_n \mu_z^{\widetilde{x}}(\rho_z(g_u^n)\Omega^+) &\le 2C_3\left(\frac{L}{\ell}+\frac{1}{1-e^{-\delta(z)\ell}}\right)e^{-\delta(z)L}(e^{-\omega})^{2\delta(z)}\\ &\le 2C_3\left(\frac{1}{\ell}+\frac{2}{\delta(z)\ell}+2\right)(L+1)e^{-\delta(z)L}\ell^{2\delta(z)}\cosh(\sigma)^{2m}\\ &\le C_4\max(\ell^{2\delta(z)-1},1)(L+1)e^{-\delta(z)L}.\end{aligned}$$

To obtain C_4 only depending on σ, we use that $\delta(z) \ge \delta$ and that δ is bounded from below by a constant that only depends on σ, by Proposition A.3.

By Proposition A.3, there exists $\epsilon_\sigma \le 1$ such that if $\ell \le \epsilon_\sigma$ then $\delta > \frac{1}{2}$. Thus, if on one hand $2\delta(z)-1 \ge 0$ then $\ell^{2\delta(z)-1} \le \sigma^{2\delta(z)-1} \le \sigma^{2m-1}$. On the other hand, if $2\delta(z)-1<0$ then we must have $\ell > \epsilon_\sigma$ so $\ell^{2\delta(z)-1} \le \epsilon_\sigma^{2\delta(z)-1} \le \epsilon_\sigma^{2m-1}$. In any case $\ell^{2\delta(z)-1}$ is bounded above by a constant that only depends on σ, which concludes the proof. □

We now prove the technical estimate we used in the proof.

Proof of Lemma 9.13. Parallel transport $\mathcal{L}$ along the first geodesic piece of a until time ω, to obtain $\mathcal{L}'$ at distance ω from $\mathcal{L}$. Let H be the half-plane delimited by $\mathcal{L}'$ that does not contain $\mathcal{L}$. Then by definition of admissible path one can check that $a(t)\in H$ for any $t \ge \omega$.

By a classical formula of hyperbolic trigonometry, see Theorem 7.17.1 of [Bea83], the orthogonal projection of any $x\in H$ is at distance at most $\sinh^{-1}(\frac{1}{\sinh(\omega)})$ from $a(0)$.

In particular the orthogonal projection of ξ is at distance at most $\ell' := \sinh^{-1}\left(\frac{1}{\sinh(\omega)}\right)$ from $a(0)$, and by triangle inequality a and r start at distance at most $\ell+\ell'$. Using for instance again Theorem 7.17.1 of [Bea83], one can check that $a(\omega)$ and $r(\omega)$ are at distance at most twice the following:

$$\sinh^{-1}(\sinh(\ell/2)\cosh(\omega)) + \sinh^{-1}(\sinh(\ell'/2)\cosh(\omega)) \le 2\sinh^{-1}\left(\frac{\cosh\omega}{\sinh\omega}\right),$$

which can be bounded above in terms of σ because ω can be bounded below in terms of σ (since $\ell \le \sigma$). $\square$

Note that Proposition 9.11 can be used to effectively count the number of intersections with γ^* of a closed geodesic which is typical for the measure $\nu(z)$ and shows that it is exponentially small in L as $z \to \infty$, uniformly in ϵ, σ.

9.5 Convergence of currents

Recall $\delta > 0$ is the topological entropy of $\Phi^t_{|K}$. The following is the main result of this section.

Proposition 9.14. *Let $L_i \to \infty$ and let $\rho_i = \rho_{z_i}$ be a sequence of Hitchin representations obtained by Hitchin grafting of a Fuchsian representation at the simple geodesic multicurve γ^* with cylinder heights bounded from below by L_i. Then $\delta(z_i) \to \delta$, and up to passing to a subsequence, the equilibrium measures $\nu_i = \nu(z_i)$ converge weakly to a measure of maximal entropy for $\Phi^t|K$.*

Proof. Recall that $f_i = f_{z_i}$ denotes a positive Hölder continuous potential on T^1S whose periods are the Finsler translation lengths of the elements of $\rho_i(\pi_1(S))$.

Up to passing to a subsequence, we may assume that the Φ^t-invariant probability measures $\nu_i^1 = \nu_i/||\nu_i||$ converges weakly to a Φ^t-invariant probability measure ν on T^1S. By Lemma 9.5, we may also assume that the geodesic currents $\hat{\nu}(z)$ converge weakly to a current $\hat{\nu}$ which is a positive multiple of the current defined by ν.

By Proposition 9.11, we have $\iota(\hat{\nu}, \gamma^*) = 0$ and hence the limit measure ν must be supported on K. By Lemma 9.4, we are thus left with showing that $h_\nu \ge \delta$.

From Lemma 9.5 we have $\delta(z_i) \in (\delta, m]$ for any i. Recall from (37) that

$$h_{\nu_i^1} = \delta(z_i) \int f_i d\nu_i^1. \tag{42}$$

By Theorem 6.2, it holds

$$\int f_i \frac{d\eta}{\ell_S(\eta)} \ge \left(1 + \frac{C}{L_i + 1}\right)^{-1}$$

for any $\eta \in \pi_1(S)$, and hence since the Φ^t-invariant Borel probability measures supported on closed geodesics are weak*-dense in the space of all Φ^t-invariant Borel probability measures, we get

$$\int f_i d\nu_i^1 \ge \left(1 + \frac{C}{L_i + 1}\right)^{-1},$$

and hence

$$\liminf_{i\to\infty} h_{\nu_i} \ge \liminf_{i\to\infty} \delta(z_i) \ge \delta.$$

Since the entropy function is lower semi-continuous, we conclude that $h_\nu \ge \delta$. As ν is supported in K, this implies that indeed, ν is a measure of maximal entropy for the restriction of Φ^t to K by Lemma 9.4. $\square$

Using the above results we are now ready to complete the proof of Theorem C from the introduction.

Proof of Theorem C. Part (3) of Theorem C was shown in Section 6, so we are left with showing part (1) and (2). Let $\gamma^* \subset S$ be a pair of pants decomposition of $S_1 = S - S_0$ that contains $\partial S_0 = \partial S_1$. The metric h on S_0 prescribes lengths for the components of γ^* in ∂S_0.

Since no component of S_1 is a pair of pants, every pair of pants in $S_1 - \gamma^*$ has a boundary component in $\gamma^* - \partial S_1$. By Proposition A.4, one can choose lengths large enough for each component of $\gamma^* - \partial S_1$ such that each pair of pants of $S_1 - \gamma^*$ has entropy very close to zero, and in particular strictly smaller than the entropy of S_0.

Then by Hitchin grafting along γ^* flat cylinders with bigger and bigger heights, we get a sequence $\rho_i = \rho_{z_i}$ of Hitchin representations satisfying the first two statement of Theorem C, according to Proposition 9.14. □

10 Pressure length control

Define the *entropy gap* of the pair consisting of a hyperbolic surface and a separating simple closed geodesic to be the absolute value of the difference between the entropies of the two components of $S - \gamma$. If γ is non-separating then we define the entropy gap to be one.

Consider a path $t \to \rho_{tz}$ of Hitchin representations obtained by Hitchin grafting along a single geodesic γ and grafting parameters a ray in $\mathfrak{a}$ with direction in the kernel of the linear functional defining the Finsler length of γ. The first goal of this section is to show

Theorem 10.1. *The pressure metric length of the path $t \to \rho_{tz}$ is finite and bounded from above by a constant only depending on a positive lower bound for the systole of S and for the entropy gap, and an upper bound for the length of the grafting geodesic γ.*

We also show

Theorem 10.2. *Let $(\rho_t)_{t\in[0,T]}$ be a smooth path of hyperbolic structures on S which are constant on a subsurface $S_0 \subset S$, such that for any $t \in [0, T]$ the entropy of the geodesic flow on T^1S_1 for $S_1 = S - S_0$ and the restriction of the metric ρ_t is strictly smaller than the entropy of the geodesic flow on T^1S_0 (which does not depend on t). Consider the grafting locus $\gamma^* = \partial S_0 = \partial S_1 \subset S$, and denote by $\mathrm{Gr}_z\rho_t$ the Hitchin grafting of ρ_t with parameter z.*

Then the pressure length of the smooth path $(\mathrm{Gr}_z\rho_t)_{t\in[0,T]}$ tends to zero as the cylinder height associated to z tends to infinity.

This section is subdivided into two subsections. In the first subsection, we study second derivatives of the length function along a grafting path, and in the second subsection, we investigate the second derivative of the entropy.

10.1 Second derivative of length

Resuming the notations from Section 2.2, Proposition 2.8 shows that a Hitchin grafting path $t \to \rho_{tz}$ in the Hitchin component gives rise to a real analytic family $f_t : T^1S \to (0,\infty)$ of Hölder functions defining a reparameterization of the geodesic flow on T^1S corresponding to the Finsler length of $\rho_{tz}(\pi_1(S))$. Thus we have

Lemma 10.3. *For each t_0, the first and second derivative of the Finsler length function of $t \to \rho_{tz}$ at t_0 is a cohomology class of a cocycle which can be represented by a Hölder function f'_{t_0}, f''_{t_0} on T^1S depending smoothly on t_0.*

For each t let $\nu(tz)$ be its Gibbs equilibrium state, normalized in such a way that

$$\int f_t d\nu(tz) = 1.$$

For $\eta \in \pi_1(S)$ we also put $f_t(\eta) = \int_\eta f_t$, the Finsler translation length of the element $\rho_z(\eta)$. The following is a fairly immediate consequence of Lemma 10.3 and the results from Section 8.

Corollary 10.4. *There exist numbers $\kappa > 0$ and $C > 0$ only depending on a lower bound for the systole of S and an upper bound for the length of γ^* such that*

$$\left|\frac{d}{dt}\int f_t d\nu(tz)\right|_{t=t_0} | \leq Ce^{-\kappa t_0} \text{ and } \frac{d^2}{dt^2}\int f_{z+t} d\nu(z)|_{t=t_0} \leq Ce^{-\kappa t_0}.$$

Proof. By Lemma 10.3, and exchanging derivatives and sums (which can be done by uniform boundedness and uniform convergence, using continuity of the function f'_t), we have

$$\left|\frac{d}{dt}\int f_{tz} d\nu(t_0 z)|_{t=t_0}\right| = \lim_{R\to\infty} \frac{1}{\# R_{\ell_z}(T)} \sum_{\eta \in R_{\ell_z}(T)} \frac{f'_{t_0}(\eta)}{f_{t_0}(\eta)}.$$

Proposition 8.1 shows that

$$|f'_{t_0}(\eta)| \leq C_\delta \iota(\eta,\gamma)$$

and hence since the intersection form ι is a continuous form on currents equipped with the weak topology, we conclude that

$$\left|\frac{d}{dt}\int f_t d\nu(t_0)\right|_{t=t_0} | \leq C_\delta \iota(\hat{\nu}(t_0 z),\gamma)$$

where $\hat{\nu}(t_0 z)$ is the geodesic current defined by $\nu(t_0 z)$. Thus the first part of the corollary now follows from Proposition 9.11.

The second part of the corollary follows from exactly the same argument. The details will be omitted. □

We now turn to the situation describe in Theorem 10.2. Thus assume that the geodesic γ is separating and divides S into subsurfaces S_0, S_1. Let $t \to \rho_t$ ($t \in [0, T|$) be a smooth path in the Teichmüller space of marked Riemann surfaces such that for each t the following holds true.

1. The restriction of the marked hyperbolic metric ρ_t to the subsurface S_0 does not depend on t.

2. The entropy of the geodesic flow of ρ_t restricted to the subspace of all geodesics entirely contained in S_0 is strictly larger than the entropy of the restriction of the flow to the subspace of all geodesics entirely contained in S_1.

Lemma 10.5. *Let $\mathrm{Gr}_z\rho_t$ be the Hitchin grafting of the Fuchsian representation ρ_t with grafting locus γ and grafting parameter z. Let $f_z(\rho_t)$ be a corresponding family of positive Hölder functions and let $\nu(\mathrm{Gr}_z\rho_t)$ be the corresponding measure of maximal entropy. Then for every $\epsilon > 0$ there exists a grafting height $L = L(\epsilon) > 0$ such that for z of height at least L and for all $T > 0$, we have*

$$\int_0^T \frac{d^2}{dt^2} \int f_z(\rho_t)|td\nu(\mathrm{Gr}_z\rho_t) \le \epsilon.$$

Proof. Let $\epsilon > 0$ be a lower bound on the systoles of the hyperbolic metrics ρ_t and let σ be an upper bound on the length of γ. By Theorem 6.1, there exists a number $R > 0$ only depending on σ such that the following holds.

Assume that $\mathrm{Gr}_z\rho_t$ is obtained from the Fuchsian representation ρ_t by grafting along γ with grafting datum z. Let $\tilde{Q}_{t.z} : \tilde{S}_{t,z} \to \mathbb{X}$ be the equivariant embedding defining the characteristic surface $\tilde{S}_{t,z}$ of $\mathrm{Grt}_z\rho_t$. Then for any $\psi \in \pi_1(S)$, there exists a Finsler axis for the action of ψ which is contained in the R-neighborhood of $\tilde{S}_{t,z}$.

By Proposition 8.3, there exists numbers $L = L(R, \epsilon, \sigma) > 0$ and a number $C > 0$ such that for $L > L(R, \epsilon, \sigma)$, the function which defines the derivative of length of the deformation $t \to \mathrm{Gr}_z(\rho_t)$ is trivial on the C-neighborhood of the image under the natural path isometric embedding Qt, z of the abstract grafted surface corresponding to $\mathrm{Gr}_z(\rho_t)$ into $\mathrm{Gr}_z(\rho_t)\backslash\mathbb{X}$. By the estimates in Section 6, this implies that we may assume that for large enough norm of z, that is, for large enough grafting height and all t, the restriction of the functions $fz(\rho_t)$ to the compact subset K of T^1S consisting of all unit vectors whose Φ^t-orbit remains in T^1S_0 does not depend on t. Furthermore, Proposition 8.3 also shows that

$$|\frac{d}{dt} f_z(\rho_t)| \le A f_z(\rho_t) \text{ and } \frac{d^2}{dt^2} f_z(\rho_t) \le A f_z(\rho_t)$$

pointwise up to a modification with a coboundary.

Now by assumption on $\mathrm{Gr}_z(\rho_t)$, Proposition 9.14 and the Birkhoff ergodic theorem, the proportion of time a typical orbit for $\nu(\mathrm{Gr}_z\rho_t)$ spends outside of a fixed small neighborhood of the compact invariant set $K \subset T^1S_0$ tends to zero as the grafting height tends to infinity. But this means that the integral of the derivatives of the functions $f_z(\rho_t)$ in direction of t also tend to zero. From this the lemma follows. □

10.2 Second derivative of the entropy

Our second goal is to deduce from the derivative control established in Corollary 10.4 and Lemma 10.5 an upper bound for the length with respect to the pressure metric of the Hitchin grafting path $t \to \rho_{tz}$. This is not immediate as the second derivative of length, integrated with respect to the Gibbs current $\nu(tz)$, is in general not the variance of the derivative of the length function of the path as the entropy along the path is in general not constant. The goal of this section is to overcome this difficulty and complete the proof of the main theorem.

Let $t \to h(t)$ be the function which associates to tz the entropy of the reparameterization of the geodesic flow Φ^t on T^1S by the Finsler length of the representation ρ_{tz}. The following is due to Katok, Knieper and Weiss. We refer to Proposition 2 of [Pol94] for an explicit statement. As before, the measure $\nu(tz)$ is the equilibrium state for the function f_{tz}.

Lemma 10.6. $h'(t)|_{t=t_0} = -h(t_0)\frac{\int(\frac{d}{dt}f_{tz})|_{t=t_0}d\nu(t_0z)}{\int f_{t_0z}d\nu(t_0z)}$.

Proof. For any t we have $P(-h(t)f_{tz}) = 0$, where P is the pressure function.

We are going to differentiate this inequality, using the fact due to Parry–Pollicott, see Propositions 4.10-11 of [PP90], and Ruelle [Rue78], that for any $\mathcal{C}^1$ one-parameter family of Hölder functions $(g_t)_t$ we have $\frac{d}{dt}P(g_t) = \int(\frac{d}{dt}g_t)d\nu_t$ where ν_t is the equilibrium state associated to g_t, such that $h(\nu_t) - \int g_t d\nu_t = P(g_t)$. This yields

$$0 = \int \left((h'(t)f_{tz} + h(t)\left(\frac{d}{dt}f_{tz}\right)\right) d\nu(tz)$$

which concludes the proof. □

We use this to show

Corollary 10.7. $|h'(mz+s) - h'(mz)| \leq Ce^{-\kappa s}$ *for all* m *and all* $s \leq 1$.

Proof. It follows from Corollary 10.4 that $|\frac{d}{dt}\log h(t)| \leq Ce^{-\kappa m}$ for universal constants $C, \kappa > 0$ and all $t \in [m, m+1]$. As $h(t)$ is bounded from above and below by a universal positive constant, this implies that up to changing the constant C, this also holds true for $|h'(t)|$ for all $t \in [m, m+1]$. □

We are now ready to show

Proposition 10.8. *Let* $\rho : t \to \rho_t = \rho(tz)$ *be a Hitchin grafting path. Then the length of* ρ *for the pressure metric is bounded from above by a constant only depending on the length of the grafting geodesic* γ *and the diameter of* S.

Proof. We show that for every $m \geq 0$, the length of the restriction of the path to the interval $[m, m+1]$ is at most $Ce^{-\kappa m}$ for some $C > 0, \kappa > 0$.

To this end recall that

$$\mathbf{J}(\rho_L, \rho_{L+t}) = \frac{h(L+t)}{h(L)}\mathbf{I}(\rho_\mathbf{L}, \rho_\mathbf{L+t}).$$

Since

$$h''/h = \frac{d^2}{dt^2}\log h - (h'/h)^2,$$

differentiating twice in t we get

$$\begin{aligned}\frac{d^2}{dt^2}_{|t=0}\mathbf{J}(\rho_L,\rho_{L+t}) =& \frac{h''(L)}{h(L)} + 2\left(\frac{h'(L)}{h(L)}\right)\frac{d}{dt}_{|t=0}\mathbf{I}(\rho_L,\rho_{L+t}) + \frac{d^2}{dt^2}_{|t=0}\mathbf{I}(\rho_L,\rho_{L+t})\\ =& \frac{d^2}{dt^2}(\log h)(L) - \left(\frac{h'(L)}{h(L)}\right)^2 + 2\left(\frac{h'(L)}{h(L)}\right)\frac{d}{dt}_{|t=0}\mathbf{I}(\rho_L,\rho_{L+t}) + \frac{d^2}{dt^2}_{|t=0}\mathbf{I}(\rho_L,\rho_{L+t}).\end{aligned}$$

Now Corollary 10.4 shows that

$$\left|\frac{d}{dt}\mathbf{I}(\rho_L,\rho_{L+t})|_{t=0}\right| \le Ce^{-\kappa L}, \quad \left|\frac{d^2}{dt^2}\mathbf{I}(\rho_L,\rho_{L+t})|_{t=0}\right| \le Ce^{-\kappa L},$$

and Lemma 10.6 together with Corollary 10.4 yields

$$\left|\frac{d}{dt}\log h(L+t)|_{t=0}\right| \le Ce^{-\kappa L}.$$

Thus using the Cauchy-Schwarz inequality, for any m we have

$$\begin{aligned}\int_{L=m}^{m+1}\left(\frac{d^2}{dt^2}_{|t=0}\mathbf{J}(\rho_L,\rho_{L+t})\right)^{1/2} dL &\le \left(\int_{L=m}^{m+1}\frac{d^2}{dt^2}_{|t=0}\mathbf{J}(\rho_L,\rho_{L+t})dL\right)^{1/2}\\ &\le \left(\int_{L=m}^{m+1}\frac{d^2}{dt^2}\log h + C'e^{-\kappa L}dL\right)^{1/2}\\ &\le \left(\int_{m}^{m+1}\frac{d^2}{dt^2}\log h + C''e^{-\kappa m}dL\right)^{1/2}\\ &\le \left(h'(m+1) - h'(m) + C''e^{-\kappa m}\right)^{1/2}\end{aligned}$$

.

Using Corollary 10.7, this yields an upper bound of $C'''e^{-\kappa m/2}$ for the integral, from which the proposition follows. □

11 Distortion

The restriction of the pressure metric to the Fuchsian locus is a multiple of the Weil Petersson metric on Teichmüller space [BCLS15] and hence its intrinsic large scale geometric properties are well understood. Moreover, by [PS17], the Fuchsian locus can be characterized as the set of Hitchin representations whose critical exponent for the symmetric metric as well as for the Hilbert metric (and other sufficiently well behaved Finsler metrics) assumes a maximum. This intrinsic geometric characterization of the Fuchsian locus does however not reveal information on its significance for the large scale geometry of the Hitchin component.

In fact, the pressure metric for the *Hilbert length*, which by definition is induced from the Hilbert metric for convex domains in projective space, is degenerate and hence *not* a Finsler metric for the Hitchin component. Namely, the contragredient involution of $\mathrm{PSL}_d(\mathbb{R})$ acts isometrically on the character variety equipped with the pressure metric. If $d = 2m$ is even, then this involution is just conjugation with the standard symplectic form, with fixed point set the symplectic group $\mathrm{PSp}(2m, \mathbb{R})$. It turns out that the pressure metric for the Hilbert length is degenerate on the normal bundle of the space of representations with image in $\mathrm{PSp}(2m, \mathbb{R})$. Note that since the involution is an isometry for the pressure metric, the locus of representations into $\mathrm{PSp}(2m, \mathbb{R})$ is totally geodesic.

In the case $n = 3$, the fixed point set of the involution equals the image $\mathrm{PSO}(2, 1)$ of $\mathrm{PSL}_2(\mathbb{R})$ under the irreducible representation and hence the Fuchsian locus is totally geodesic for the pressure metric for $n = 3$ (see e.g. [Dai23]). However, in spite of recent refined information on the restriction of the pressure metric to the Fuchsian locus [LW15], the following seems to be an open question.

Question 4. For $d \geq 4$, is the Fuchsian locus totally geodesic for the pressure metric for representations into $\mathrm{PSL}_d(\mathbb{R})$?

On purpose, we leave the specification of the length function defining the pressure metric open.

The main goal of this section is to show that from a global geometric perspective, the Fuchsian locus is distorted for the pressure distance on the Hitchin component for $n \geq 3$ and genus $g \geq 3$, where the pressure distance is taken with respect to the Finsler length considered in the previous sections. We believe that similar arguments should lead to corresponding results for all variants of the pressure metric.

11.1 Regions of finite diameter

Let Σ_1 be a compact surface of genus $g_1 \geq 1$ and connected boundary and let $\mathcal{T}(\Sigma_1, \ell)$ be the Teichmüller space of marked hyperbolic metrics on Σ_1 with fixed boundary length ℓ and one marked point on the boundary. For a number $\epsilon < \ell$ we also consider the ϵ-thick part of $\mathcal{T}(\Sigma_1, \ell)$, which is the subspace $\mathcal{T}_\epsilon(\Sigma_1, \ell)$ of $\mathcal{T}(\Sigma_1, \ell)$ of marked hyperbolic metrics whose *systole*, that is, the length of the shortest closed geodesic, is at least ϵ. The mapping class group $\mathrm{Mod}(\Sigma_1)$ of all isotopy classes of diffeomorphisms of Σ_1 which fix the boundary pointwise acts properly discontinuously and cocompactly on $\mathcal{T}_\epsilon(\Sigma_1, \ell)$. Moreover, $\mathcal{T}_\epsilon(\Sigma_1, \ell)$ is path connected for sufficiently small ϵ.

Let $g_0 \geq 1$ and let Σ be a closed surface of genus $g = g_0 + g_1$; then any point $S_0 \in \mathcal{T}(\Sigma_0, \ell)$ determines an embedding $\mathcal{T}(\Sigma_1, \ell) \to \mathcal{T}(S)$ by gluing a surface $S_1 \in \mathcal{T}(\Sigma_1, \ell)$ to X_0 along the boundary matching marked points. Moving the marked point on the boundary of Σ_1 results in gluing the two component surfaces with a twist and not changing the marked hyperbolic metric on Σ otherwise.

The following is the main result of this section.

Theorem 11.1. *For $g \geq 3$, $\ell > 0$ and $\epsilon > 0$, there exists a number $C = C(g, \ell, \epsilon)$ with the following property. Let Σ_0, Σ_1 be compact surfaces of genus $2, g - 2$, respectively,* *and connected boundary. Let Σ be a closed surface of genus g. Then the diameter for the pressure distance of the image of an embedding $\mathcal{T}(\Sigma_1, \ell) \to \mathcal{T}(\Sigma)$ constructed from $S_0 \in \mathcal{T}_\epsilon(\Sigma_0, \ell)$ is at most C.*

Question 5. *Is the diameter for the pressure metric of the Fuchsian locus finite? Is the diameter of the Hitchin component for the pressure metric finite?*

To set up the proof, consider the function $h : \mathcal{T}(\Sigma_1, \ell) \to (0, 1)$ which associates to a hyperbolic metric its entropy, that is, the critical exponent. The following results is an easy consequence of Lemma A.7 from the appendix and the work of Wolpert [Wol86].

Lemma 11.2. *If $g(\Sigma_0) \geq 2$ then for every $\nu > 0$ and any $S_0 \in \mathcal{T}(\Sigma_0, \ell)$ there exists a hyperbolic metric $S_0' \in \mathcal{T}(\Sigma_0, \ell)$ with $\delta(S_0') > 1 - \nu$ whose Weil Petersson distance to S_0 is at most m for a universal constant $m > 0$.*

Proof. By Lemma A.7, for $\nu > 0$ and the fixed upper bound σ for the boundary length ℓ, there exists a number $\epsilon > 0$ such that any hyperbolic metric on S_0 with boundary of length at most σ which contains a pair of pants all of whose geodesic boundary components are of length at most ϵ has entropy at least $1 - \nu$.

By Theorem *A.2*, the surface S_0 contains a pair of pants whose boundary lengths are bounded from above by a constant only depending on $\sigma \geq \ell$. Shrinking each of these simple closed geodesics to a length of at most ϵ can be realized by a path in $\mathrm{T}(S_0, \ell)$ of uniformly bounded length [Wol86]. □

Proof of Theorem 11.1. Let Σ_0 be a surface of genus 2 with connected boundary. Let $\epsilon > 0, \ell > \epsilon$ and let $X_0 \in \mathcal{T}_\epsilon(\Sigma_0, \ell)$ be a marked hyperbolic metric in the ϵ-thick part of Teichmüller space. Choose a marked point on $\gamma^* = \partial\Sigma_0$. The metric S_0 and the marked point on γ^* determine an embedding $\mathcal{T}(\Sigma_0, \ell) \to \mathcal{T}(S)$. To prove the theorem, we concatenate six paths of Hitchin grafting, illustrated in Figure 6.

The restriction of the pressure metric to the Fuchsian locus equals a multiple of the Weil–Petersson metric on $\mathcal{T}(\Sigma)$. Thus any point in $\mathcal{T}(\Sigma)$ can be connected to a point in the ϵ-thick part $\mathcal{T}_\epsilon(\Sigma)$ of $\mathcal{T}(\Sigma)$ by a path of uniformly bounded length [Wol86]. As a consequence, via pre-and postcomposition with a path of uniformly bounded length, e.g. a Weil–Petersson geodesic segment, it suffices to show the following. Let $X_1, Y_1 \in \mathcal{T}_\epsilon(\Sigma_1, \ell)$ and let $X, Y \in \mathcal{T}(\Sigma)$ be the images of X_1, Y_1 under the embedding $\mathcal{T}(\Sigma_1, \ell) \to \mathcal{T}(\Sigma)$. Then X can be connected to Y by a path of uniformly bounded pressure metric length.

For sufficiently small ϵ, the set $\mathcal{T}_\epsilon(\Sigma_1, \ell)$ is connected. Thus we can connect X_1 to Y_1 by a smooth path $\alpha(t) \subset \mathcal{T}_\epsilon(\Sigma_1, \ell)$ with the following properties.

1. $\alpha(0) = X_1, \alpha(m) = Y_1$ for some $m > 0$.

2. For all s, t there exists a diffeomorphism $\theta_{s,t} : \alpha(s) \to \alpha(t)$ with $\theta_{s,s} = \mathrm{Id}$ and the property that $\theta_{s,t}^*(g(t))$ depends smoothly on t, and such that $\left\|\frac{d}{dt}\theta_{s,t}^*(g(t))\right\|_{t=s} \leq 1$ and $\left\|\frac{d^2}{dt^2}\theta_{s,t}^*(g(t))\right\|_{t=s} \leq 1$ pointwise. Here $g(t)$ is the hyperbolic metric on $\alpha(t)$, and norms at $t = s$ are taken with respect to the metric $g(s)$.

3. With the natural identification, $\theta_{s,t} = \mathrm{Id}$ on $\partial\Sigma_1 = \gamma^*$ for all s, t.

Namely, any smooth path in $\mathcal{T}_\epsilon(\Sigma_1, \ell)$ can be thought of as a path of smooth metric deformations. Adjusting the speed of such a deformation connecting X_1 to Y_1 in an appropriate way results in a path with the above properties.

By Lemma A.7, applied to $\mathcal{T}_\epsilon(\Sigma_1, \ell)$, there exists a number $a > 0$ only depending on ϵ and ℓ such that the topological entropy of each of the hyperbolic metrics $\alpha(t)$ is at most $1 - a$.

Recall that the genus of Σ_0 is 2. By Lemma 11.2, applied to the surface Σ_0, there exists a hyperbolic metric $X(a)$ on Σ_0 which coincides with X on Σ_1 and such that the topological entropy of the restriction of the hyperbolic metric $X(\xi)$ to Σ_0 is larger than $1 - a/2$. Furthermore, X can be connected to $X(a)$ by a path of Weil-Petersson length at most $C_0 > 0$, where C_0 is a constant only depending on a (and the genus of Σ).

For each t let $A(t) \in \mathcal{T}(\Sigma)$ be the hyperbolic metric which is the image of $\alpha(t)$ under the embedding $\mathcal{T}(\Sigma_1, \ell) \to \mathcal{T}(\Sigma)$, defined by gluing $\alpha(t)$ to $X(\xi)|\Sigma_0$. By Lemma A.7, the entropy gap of $A(t)$ is at least $a/2$. Furthermore, there exists a universal constant $\kappa > 0$ only depending on ϵ and ℓ with the property that $A(t) \in \mathcal{T}_\kappa(\Sigma)$ for all t.

For $s \in [0, m]$ and $L > 0$ let $\Theta(s, L)$ be a representation obtained from $A(s)$ by Hitchin grafting along $\gamma^* = \partial\Sigma_0 = \partial\Sigma_1$ with cylinder height L. By the choice of $X(\xi)|\Sigma_0$ and Proposition 10.8, the length of the path $L \to \Theta(s, L)$ is bounded from above by a constant only depending on κ and ℓ but not on m.

By Theorem 10.2, there exists a number $L_0 > 0$ so that for all $L > L_0$ the length of the path connecting $\Theta(0, L)$ to $\Theta(m, L)$ is at most 1 provided that $L > L_0$.

Recall that the pressure length of the grafting path connecting $X(\xi)$ to its image $\Theta(0, L)$ under Hitchin grafting along γ^* of height L is bounded from above by a constant only depending on ℓ and κ, and the same holds true for the pressure length of the path connecting $\Theta(m, L)$ to the surface $Y(a)$ obtained by replacing $Y|\Sigma_0$ by $X(a)|\Sigma_0$. Since the Weil–Petersson distance between X to $X(a)$ and between Y to $Y(a)$ is uniformly bounded, together this yields that X can be connected to Y by a path of uniformly bounded length for the pressure metric. This shows the theorem. □

11.2 Length comparison with a separating curve graph

Let $g \geq 5$ and let $\mathcal{SCG}(S)$ be the graph whose vertices are separating simple closed curves which decompose S into a surface of genus 2 and a surface of genus $g - 2$ and where two such curves are connected by an edge if they can be realized disjointly. We have

Lemma 11.3. *The graph $\mathcal{SCG}(S)$ is connected.*

Proof. The mapping class group $\mathrm{Mod}(S)$ of S clearly acts transitively on the vertices of $\mathcal{SCG}(S)$. Thus to check connectedness, we can apply a trick due to Putman [Put08]: Choose a vertex c of $\mathcal{SCG}(S)$ and a generating set $\psi_1, \ldots, \psi_k$ of $\mathrm{Mod}(S)$. If for each j the vertex c can be connected to $\psi_j(c)$ by an edge path in $\mathcal{SCG}(S)$, then the graph is connected.

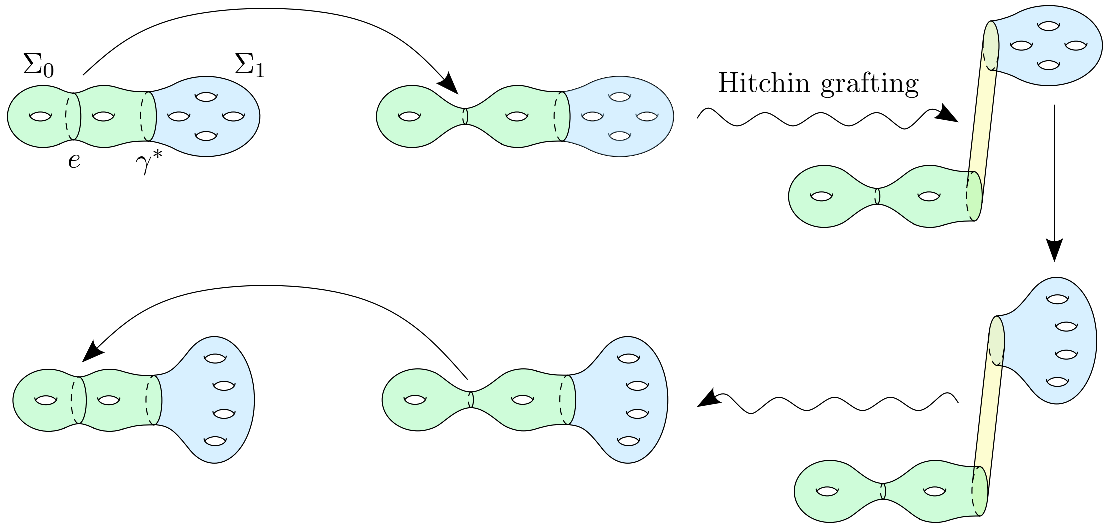

Figure 6: Bounded path of Hitchin representations for the pressure metric. Each path is bounded by a constant that depends only on the length of γ^* and on the systole of Σ_0.

To see that this condition is satisfied we choose the Humphries generating set $\psi_1, \dots, \psi_{2g+1}$ of $\mathrm{Mod}(S)$ consisting of Dehn twists about the non-separating simple closed curves $a_1, \dots, a_g, c_1, \dots, c_{g-1}, m_1, m_2$ in S as shown in Figure 4.5 of [FM11]. That these elements generate $\mathrm{Mod}(S)$ is explained in Theorem 4.14 of [FM11]. Let furthermore c be the separating simple closed curve which intersects the simple closed curve c_2 in precisely two points and is disjoint from any of the curves a_i, c_j, m_u for $j \neq 2$. Then $\psi_s(c) = c$ for $s \neq g+2$, moreover both $c, \psi_{g+2}(c)$ are disjoint from the vertex b of $\mathcal{SCG}(S)$ which intersects c_{g-2} in precisely two points and is disjoint from the remaining curves. Thus $c, b, \psi_{g+2}(c)$ is an edge path connecting c to $\psi_{g+2}(c)$, which suffices for the proof of the lemma. □

Let $\Upsilon : \mathcal{T}(S) \to \mathcal{SCG}(S)$ be a map which associates to $X \in \mathcal{T}(S)$ a point in $\mathcal{SCG}(S)$ whose length is minimal among the lengths of all separating geodesics which cut S into a surface of genus 2 and a surface of genus S_2. We use Lemma 11.3 to show

Theorem 11.4. *For any $d \geq 3$ there exists a number $C(d) > 0$ with the following property. Let $X, Y \in \mathcal{T}(S)$ be any two points in the Fuchsian locus of the Hitchin component of representations $\pi_1(S) \to \mathrm{PSL}_d(\mathbb{R})$. Then the pressure metric distance between X, Y is at most $C(d)d(\Upsilon(X), \Upsilon(Y)) + C(d)$.*

Proof. In this proof, distances between Hitchin representations are always taken with respect to the path metric defined by the pressure metric.

The restriction of the pressure metric to the Fuchsian locus is a multiple of the Weil-Petersson metric. Moreover, the length of a Weil Petersson geodesic which shrinks the length of some moderate length simple closed curves to an arbitrarily small number is bounded from above by a universal constant. Thus for every sufficiently small $\epsilon > 0$, any point in the Fuchsian locus is at uniformly bounded distance from a point in $\mathcal{T}_\epsilon(S)$. Hence it suffices to show the statement of the theorem for surfaces $X, Y \in \mathcal{T}_\epsilon(S)$.

By invariance under the mapping class group, continuity and cocompactness, the following holds true. For every $\epsilon > 0$ there exists a number $\ell = \ell(\epsilon) > 0$, and for $X \in \mathcal{T}_\epsilon(S)$ there exists a simple closed geodesic c on X whose length is contained in the interval $[\ell^{-1}, \ell]$ and which decomposes X into a surface S_0 of genus $g_0 = g - 2$ and a surface of genus 2. In other words, for $X \in \mathcal{T}_\epsilon(S)$, the length of $\Upsilon(X)$ is bounded from above by a universal constant.

Let $X, Y \in \mathcal{T}_\epsilon(X)$. Connect $c_X = \Upsilon(X)$ to $c_Y = \Upsilon(Y)$ by a geodesic in $\mathcal{SCG}(S)$, say the geodesic $(c_i)_i$ $(c_0 = c_X, c_m = c_Y)$. For each i choose a hyperbolic surface $X_i \in \mathcal{T}_\epsilon(S)$ $(i \geq 1)$ with the property that the X_i-lengths of the disjoint simple closed curves c_{i-1}, c_i are both contained in the interval $[\ell^{-1}, \ell]$. Such a surface exists because the lengths of two disjoint simple closed curves can be prescribed arbitrarily for a hyperbolic surface. More precisely, for $i = 1$ we require that the length of curve c_0 coincides with its length in X and that the length of c_n coincides with the Y-length of c_Y. Furthermore, for $i \geq 2$ we assume that the X_i and X_{i+1}-lengths of c_i coincide. construct by induction a path connecting X to Y whose length is bounded from above by a fixed multiple of $d(\Upsilon(X), \Upsilon(Y))$ as follows.

Choose a surface $Y_1 \in \mathcal{T}_\epsilon(S)$ with the property that $Y_1 \cap S_0$ is isometric to $X \cap S_0$ and that $Y_1 \cap S_1$ is isometric to $X_1 \cap S_1$. This is possible because the lengths of the boundary curve $c_X = c_0$ of S_0 coincide in X, X_1. By Theorem 11.1, the pressure distance between the Fuchsian points X, Y_1 is bounded from above by a universal constant.

As the intersection $Y_1 \cap S_1$ is isometric to $X_1 \cap S_1$ and we have $c_2 \subset S_1$, the above argument can be applied to X_1, Y_2 and shows that the distance between X_1 and Y_1 is uniformly bounded. Proceeding inductively, in m such steps we deduce that the pressure metric distance between X and Y is at most $C(n)m$ where $C(n) > 0$ is a universal constant. □

The mapping class group $\mathrm{Mod}(S)$ of S acts by precomposition of markings on the Hitchin component preserving the Fuchsian locus and the pressure metric. Thus it also acts on the metric completion of the Hitchin component for the pressure metric. This metric completion contains the metric completion of the Teichmüller space, equipped with the Weil-Petersson metric, which is a stratified space. A stratum is defined by a simple geodesic multicurve $c \subset S$, and it consists of the Teichmüller space of all marked complete finite volume hyperbolic metrics on $S - c$. By this we mean that each component of $S - c$ is an essential subsurface of S of negative Euler characteristic, and hence it determines a Teichmüller space of marked complete finite volume hyperbolic metrics on the component. The stratum of $S - c$ then is the product of these Teichmüller spaces.

The action of the mapping class group $\mathrm{Mod}(S)$ of S on boundary points for the metric completion of $\mathcal{T}(S)$ projects to the action of the mapping class group on the curve complex, thought of as remembering the nodes (or cusps) of the completion points. Dehn multitwists have global fixed points acting on this boundary: if T_c is a Dehn twist about c, then any surface with node at c is fixed by T_c. However, there is no subgroup of the mapping class group containing a free group with two generators which acts with a global fixed point.

In contrast, the action of the outer automorphism group of the free group F_k with $k \geq 3$ generators on the metric completion of Outer space of marked graphs with fundamental group F_k, equipped with an analog of the pressure metric, has a global fixed point (see [ACR22]).

Our final result shows that a weaker but related statement holds true for the action of the mapping class group $\mathrm{Mod}(S)$ on the metric completion of the Hitchin component, equipped with the pressure metric, provided that the genus of S is at least 3. For the formulation of our result, recall that for every essential subsurface S_1 of the surface S with connected boundary, the mapping class group $\mathrm{Mod}(S_1)$ of S_1 embeds into the mapping class group $\mathrm{Mod}(S)$ of S as a group of isotopy classes of homeomorphisms of S which fix $S - S_1$ pointwise.

Theorem 11.5. *Let S be a closed surface of genus $g \geq 3$ and let $S_1 \subset S$ be an essential subsurface of genus $g-2$ with connected boundary. Then the action of $\mathrm{Mod}(S_1) \subset \mathrm{Mod}(S)$ on the metric completion of $\mathrm{Hit}(S)$ with respect to the pressure metric has a global fixed point.*

Proof. Let $g \geq 3$ and let $S_1 \subset S$ be an essential subsurface of genus $g-2$ with connected boundary c. For $\epsilon > 0, \ell > \epsilon$ choose $X \in \mathcal{T}_\epsilon(S)$ such that the simple closed curve c on X has length ℓ. Let $S_0 = S - S_1$ and assume furthermore that the surface S_0 contains a separating curve e of length ℓ for the metric X which cuts S_0 into a one holed torus T_0 and a two-holed torus T_1 containing c in its boundary.

We follow the proof of Theorem 11.1. Namely, for $j > 0$ let $X_j \in \mathcal{T}(S)$ be a point obtained from X by preserving the hyperbolic structure on S_1 and shrinking the length of e to $1/j$.

There exists a number $c_0 = c_0(S) > 0$ not depending on j such that the Weil-Petersson geodesic connecting X to X_j has length at most c_0 [Wol86]. Since up to a constant the restriction of the pressure metric to the Fuchsian locus is the Weil-Petersson metric [BCLS15], by adjusting c_0 we may assume that the distance between X and X_j for the pressure metric is at most c_0, independent of j.

For each j, L let $\rho(j, L)$ be the Hitchin representation obtained by Hitchin grafting X_j along c with height L. By Theorem 10.8, there exists a number $c_1 = c_1(j) > 0$ so that the length for the pressure metric of the path $L \to \rho(j, L)$ is at most $c_1(j)$. In particular, for each j the sequence $k \to \rho(j, k)$ is a Cauchy sequence for the pressure metric, converging to a point Y_j in the metric completion of the Hitchin component.

The mapping class group $\mathrm{Mod}(S_1)$ of the subsurface S_1 of S is finitely generated. Let $\phi_1, \dots, \phi_k$ be a generating set. For each $u \leq k$ connect the surface with boundary S_1, equipped with the restriction X_1 of the hyperbolic metric X, to $\phi_u(X_1)$ by a smooth path $\alpha_u : [0, m_u] \to \mathcal{T}_\epsilon(S_1, \ell)$. We assume that this path satisfies the properties (1)-(3) in the proof of Theorem 11.1. In particular, the entropies of the surfaces $\alpha_u(t)$ are bounded from above by a universal constant $1 - a < 1$.

Choose as before a number $j > 0$ which is sufficiently small that the entropy of the bordered surface S_0, equipped with the restriction of the metric X_j, is bigger than $1 - a/2$.

Such a number exists since the entropies of the bordered surfaces $X_j \cap S_0$ converge to 1 as $j \to \infty$. Denote by Y_0 the restriction of the metric X_j to S_0, viewed as a hyperbolic surface with geodesic boundary.

For each $u \leq k$ and each $s \leq m_u$ let $Y(u, s)$ be the surface obtained by gluing Y_0 to the surface $\alpha_\ell(s)$ along c matching marked points. Let $Y(u, s, L)$ be the representation obtained from Hitchin grafting of $Y(u, s)$ along c with height L (in the direction of a fixed z in the kernel of the form defining the Finsler length of γ). Note that as grafting and deforming X_1 to $\alpha_u(s)$ commutes, this representation can be obtained from $Y(u, 0, L)$ by deforming along the path α_u.

By the discussion in the proof of Theorem 11.1, as $L \to \infty$, the length for the pressure metric of the path $s \to Y(u, s, L)$ (here L is fixed) tends to zero. Now the endpoint of this path is the image of $Y(u, 0, L)$ under the map ϕ_ℓ, and as $L \to \infty$, the points $Y(\ell, 0, L)$ converge to a point Z in the completion of the Hitchin component for the pressure metric. Since the action of $\mathrm{Mod}(S)$ on the Hitchin component is isometric, this implies that Z is a fixed point for ϕ_ℓ for every $\ell \leq k$.

As a consequence, the point Z is fixed by a set of generators of the subgroup $\mathrm{Mod}(S_1)$ of $\mathrm{Mod}(S)$ and hence it is fixed by the entire group. This is what we wanted to show. □

A Entropy of hyperbolic surfaces with boundary

The goal of this appendix is to collect some basic results on the entropy of hyperbolic surfaces with boundary. We give proofs for the ones we did not find in the literature, although they should be well known by the experts. Some of the following statements are consequences of more general theorems.

Consider a compact surface Σ, of genus g, with at least one boundary component. Let S be a hyperbolic surface obtained by equipping Σ with a hyperbolic metric, so that its boundary is geodesic, that is, S belongs to the Teichmüller space $\mathcal{T}(\Sigma)$ for Σ. Denote by $h(S)$ the topological entropy of the geodesic flow on T^1S. We also denote by $\delta(S)$ the critical exponent of any representation $\pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{R})$ representing the metric S.

Proposition A.1. *The following holds true:*

1. $h(S) = \delta(S)$ *(see* [Sul79]*).*
2. *The function* $\delta(S)$ *is real analytic in* S *and invariant under the action of* $\mathrm{Mod}(\Sigma)$ *(see* [Rue78]*).*
3. $h(S) < 1$.
4. *Take a pants decomposition of* Σ*. When sending to zero the lengths of all boundary curves of a fixed pair of pants, the entropy goes to one.*

Proof. Statement 3. It follows from Proposition 5 of [PS98] that the Poincaré series $P(\delta(S))$ is diverging at the critical exponent $\delta(S)$. Consider a closed hyperbolic surface Σ_d obtained by doubling Σ along its boundary, equipped with the double S_d of the given hyperbolic metric S. It follows from Proposition 2 of [DOP00] that we have $\delta(S) < \delta(S_d)$ (it uses as hypothesis that $P(\delta(S))$ is diverging). The latter is known to be equal to one. Notice that for a hyperbolic metric S_d without boundary and with finite volume, the limit set of $\pi_1(\Sigma)$ in $\partial_\infty \mathbb{H}^2$, that is the accumulation points of the orbit $\pi_1(\Sigma) \cdot x$ for any $x \in \mathbb{H}^2$, is equal to the all $\partial_\infty \mathbb{H}^2$. It follows from Theorem 1.1 of [BJ97] that the critical exponent of S_d is one.

Statement 4. Take a compact hyperbolic surface with boundary and pinch all boundary components. The critical exponent of Kleinian groups is lower semi-continuous for the so called algebraic convergence, see Theorem 2.4 of [BJ97]. It implies that when decreasing the lengths of the boundary curves to zero, the limit inferior of the critical exponents is at least the critical exponent of the surface obtained by pinching the boundary curves. That is one according to the proof of statement (3). □

We mention a result of Hugo Parlier, which is a neat improvement of results already known previously.

Theorem A.2 ([Par23]). *Let S be a hyperbolic surface, possibly with boundary, and with finite volume. Then S admits a pant decomposition for which the length of each curve is at most* $\max(length(\partial S), area(S))$.

Proposition A.3. *There exists a function f_1 depending on Σ ($\partial\Sigma \neq \emptyset$) such that the following holds. If every boundary component has length at most σ and at least one of them has length at most $\epsilon \leq \sigma$ then $\delta(S) \geq f_1(\sigma, \epsilon) > 0$ with $\liminf_{\epsilon \to 0} f_1(\sigma, \epsilon) > \frac{1}{2}$ for fixed σ.*

Proof. Denote by S_n a sequence of metrics as in the item, so that all boundary components of Σ has length at most σ. Using Theorem A.2, S_n admits a decomposition into hyperbolic pairs of pants $P_1^{(n)}, \cdots, P_r^{(n)}$ so that the decomposing curves have a length bounded from above by some constant $C(\Sigma)$, and so that the shortest boundary component of S_n is in $P_1^{(n)}$.

Suppose by contradiction that $\delta(S_n) \to 0$. Then $\delta(P_1^{(n)}) \to 0$ since it is bounded above by $\delta(S_n)$. Up to extraction we may assume that the boundary lengths of $P_1^{(n)}$ converge, which imply $P_1^{(n)}$ converge to some hyperbolic pair of pants P, possibly with cusps. By lower semicontinuity of δ (see Theorem 2.4 of [BJ97]) we get that $0 = \lim_n \delta(P_1^{(n)}) = 0$, which is absurd. Thus the critical exponents are bounded away from zero.

Let us now prove the second part of the statement. Suppose by contradiction that the shortest boundary curve of S_n has length tending to zero, but $\liminf_n \delta(S_n) \leq 1/2$. Then $\liminf_n \delta(P_1^{(n)}) \leq 1/2$. Once again, up to extracting we may assume $P_1^{(n)} \to P$, with P having a cusp (since the shortest boundary of $P_1^{(n)}$, which is that of S_n, is pinched to zero). By lower semicontinuity of δ (see Theorem 2.4 of [BJ97]) we get that $\liminf_n \delta(P_1^{(n)}) \geq \delta(P)$. This is absurd as $\delta(P) > 1/2$ by Proposition 2 of [DOP00], since the critical exponent of a neighbourhood of a cusp is $1/2$, with a diverging Poincaré series at the critical exponent. □

Hyperbolic pairs of pants. Here suppose that Σ is a sphere with three boundary components, and $S_{a,b,c}$ is the metric of a hyperbolic pair of pants with boundary length a, b and c.

Proposition A.4. *There exists a function f_2 depending on Σ ($\partial\Sigma \neq \emptyset$) with the following property. If Σ is a pair of pants, two boundary components of S have length at least $\sigma > 0$ and the third at least $\ell \geq \sigma$, then $\delta(S) \leq f_2(\sigma, \ell)$ with $f_2(\sigma, \ell) \to 0$ for fixed $\sigma > 0$ as $\ell \to \infty$.*

We use the notations from [MZ19], where the authors give some control on the entropy of a hyperbolic surface using the length of the small curves on the surface. Denote by $L(S)$ the systole of S, that is the length of the smallest geodesic inside S. Denote by $K(S)$ the length of the smallest geodesic on $S \setminus \partial S$ ($K(S)$ is more complicated to define when S is not a pair of pants). Also denote by $\delta(S)$ the critical exponent of S.

Theorem A.5 (Particular case of Theorem 1.4 of [MZ19]). *There exists a constant $C > 0$ for which we have*

$$\frac{1}{4}\log(2) \leq \delta(S)K(S) \leq C\left(\log(4) + 1 + \log\left(1 + \frac{1}{x_0}\right)\right)$$

where x_0 is the unique positive solution of the equation $(1+x)^{\left\lceil \frac{K(S)}{L(S)} - 1 \right\rceil} x = 1$.

Lemma A.6. *Let S be a pair of pants with boundary lengths a, b, c. Then $K(S) \geq \max(a, b, c)$.*

Proof. Up to reordering we may assume $\max(a, b, c) = c$.

The surface S is obtained by gluing with itself a right-angled hyperbolic hexagon H along three nonadjacent sides, such that the three other sides have lengths $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$. In particular there is a natural projection $\pi : S \to H$. Let $\bar{A}, \bar{B}, \bar{C}$ be the sides of H which are glued, so that the hyperbolic distance from $\bar{B}$ to $\bar{C}$ is $a/2$, the distance from $\bar{C}$ to $\bar{A}$ is $b/2$, and the distance from $\bar{A}$ to $\bar{B}$ is $c/2$.

Let γ be a closed geodesic of $S \setminus \partial S$, and let us check it has length at least c. Note that $\pi(\gamma) \subset H$ is a concatenation of geodesics between the sides $\bar{A}, \bar{B}, \bar{C}$. This path has to intersect all these sides, for if it was alternating between only two sides then γ would be freely homotopic to a power of the boundary component of S between these two sides.

Say γ starts on the side $\bar{A}$ at some point x, then travels until it hits $\bar{B}$ at some point y (maybe bouncing off $\bar{C}$ and $\bar{A}$ in between), and then comes back to x. The first part of the path from x to y must have length at least the distance from $\bar{A}$ to $\bar{B}$, which is $c/2$, and similarly the second part has length at least $c/2$ too, so in total γ has length at least c. □

Proof of Proposition A.4. Let $(a_n)_n, (b_n)_n, (c_n)_n$ be three sequences in $\mathbb{R}^+$ so that a_n and b_n are bounded away from zero, and c_n tends to to infinity with n. Let $S_n = S_{a_n,b_n,c_n}$ be the pair of pants with boundary lengths a_n, b_n, c_n. By Lemma A.6, $K(S_n)$ tends to infinity with n.

By assumption, $L(S_n)$ is bounded away from zero. So up to passing to a subsequence, we can assume that $\frac{K(S_n)}{L(S_n)}$ converges to $y \in (0, +\infty]$. If $y < +\infty$, then the solutions x_n of $(1+x)^{\left\lceil \frac{K(S_n)}{L(S_n)} - 1 \right\rceil} x = 1$ remain bounded away from zero. So $C\left(\log(4) + 1 + \log\left(1 + \frac{1}{x_n}\right)\right)$ is bounded, and $\delta(S_n) \leq \frac{cste}{K(S_n)}$ goes to zero.

If $y = +\infty$, then x_n goes to zero, and a simple analysis yields that $-\frac{\log(x_n)}{x_n}$ is equivalent to $\frac{K(S_n)}{L(S_n)}$. It follows that

$$\delta(S_n)K(S_n) \leq C\left(\log(4) + 1 + \log\left(1 + \frac{1}{x_n}\right)\right) \tag{43}$$

$$\leq \mathrm{Cst} \cdot x_n \frac{K(S_n)}{L(S_n)} \tag{44}$$

$$\text{and hence} \qquad \delta(S_n) \leq \mathrm{Cst} \cdot \frac{x_n}{L(S_n)} \xrightarrow[n \to 0]{} 0 \tag{45}$$

□

Surfaces with one boundary component. Assume now that the surface Σ is of genus genus $g = g(\Sigma)$, with exactly one boundary component. Let $\mathcal{T}(\Sigma, \ell)$ be the Teichmüller space of marked hyperbolic structures on Σ with geodesic connected boundary of length ℓ. Denote also by $\mathcal{T}_\epsilon(\Sigma, \ell) \subset \mathcal{T}(\Sigma, \ell)$ the subset of structures whose systole is at least ϵ.

Lemma A.7. *The following holds true:*

1. *If $g = 1$ and $S \in \mathcal{T}(\Sigma, \ell)$, then $\delta(S)$ is bounded from above by some $b(\ell) < 1$.*
2. *If $g \geq 2$ then for all $\nu, \ell > 0$ there exists a surface $S \in \mathcal{T}(\Sigma, \ell)$ with $\delta(S) > 1 - \nu$.*
3. *If $S \in \mathcal{T}_\epsilon(\Sigma, \ell)$, then $\delta(S)$ is bounded from above by some $b(\epsilon, \ell) < 1$*

Proof. Statement 1. Note that the critical exponent is invariant under the action of the mapping class group. Let $S_i \subset \mathcal{T}(\Sigma, \ell)$ be a sequence so that

$$\delta(S_i) \xrightarrow[i \to +\infty]{} \sup\{\delta(Z) \mid Z \in \mathcal{T}(\Sigma, \ell)\}$$

Up to passing to a subsequence, we may assume that the projections of the marked surfaces S_i to the moduli space $\mathrm{Mod}(S)\backslash\mathcal{T}(\Sigma, \ell)$ converge in the Deligne–Mumford compactification of the moduli space to a surface Z with connected geodesic boundary of length ℓ, of genus $g' \leq 1$, possibly with one node. Either Z is smooth and $\delta(Z) < 1$ (see point 3 of Proposition A.1). Or the surface obtained by removing the node is a sphere with 3 punctures. In this case the entropies of the surfaces S_i converge to the *metric* entropy $\delta(Z)$ of the geodesic flow on the surface Z, equipped with the normalized Liouville measure, which is also less than 1.

Statement 2. It follows from the statement 4 of Proposition A.1. Find a pair of pants decomposition of S, take one pair of pants disjoint from ∂S and shrink all its boundary components. The critical exponent of the resulting metrics goes to one.

Statement 3. This part of the lemma follows from invariance under the mapping class group and compactness. Namely, let us assume that $S_i \subset \mathcal{T}_\epsilon(\Sigma, \ell)$ is a sequence of marked metrics so that the entropy

$$h(S_i) \to \sup\{h(S) \mid S_i \in \mathcal{T}_\epsilon(\Sigma, \ell)\}.$$

By adjusting with elements of the mapping class group, we may assume that $S_i \to S$ in $\mathcal{T}_\epsilon(\Sigma, \ell)$. Then $h(S_i) \to h(S)$, on the other hand we have $h(S) < 1$. This completes the proof of the lemma. □

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