THE GEOMETRY OF THE HANDLEBODY GROUPS II: DEHN FUNCTIONS

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Abstract. We show that the Dehn function of the handlebody group is exponential in any genus $g \geq 3$. On the other hand, we show that the handlebody group of genus 2 is cubical, biautomatic, and therefore has a quadratic Dehn function.

1. Introduction

This article is concerned with the word geometry of the *handlebody group* $\mathcal{H}_g$, i.e. the mapping class group of a handlebody of genus g. The core motivation to study this group is twofold. On the one hand, handlebodies are basic building blocks for three-manifolds – namely, for any closed 3–manifold M there is a g so that M can be obtained by gluing two genus g handlebodies V, V' along their boundaries with a homeomorphism φ. Any topological property of M is then determined by the gluing map φ. One of the difficulties in extracting this information is that φ is by no means unique. In fact, modifying it on either side by a homeomorphism which extends to V or V' does not change M. In this sense, the handlebody group encodes part of the non-uniqueness of the description of a 3–manifold via a Heegaard splitting.

The other motivation stems from geometric group theory, and it is the more pertinent for the current work. Identify the boundary surface of V_g with a surface Σ_g of genus g. Then there is a restriction homomorphism of $\mathcal{H}_g$ into the surface mapping class group $\mathrm{Mcg}(\Sigma_g)$, and it is not hard to see that it is injective. On the other hand, considering the action of homeomorphisms of V_g on the fundamental group $\pi_1(V_g) = F_g$ gives rise to a surjection of $\mathcal{H}_g$ onto $\mathrm{Out}(F_g)$. The handlebody group is thus immediately related to two of the most studied groups in geometric group theory.

But from a geometric perspective, neither of these relations is simple: in previous work [HH1] we showed that the inclusion of $\mathcal{H}_g$ into $\mathrm{Mcg}(\Sigma_g)$ is exponentially distorted for any genus $g \geq 2$. Furthermore, a result by McCullough [McC1] shows that the kernel of the surjection $\mathcal{H}_g \to \mathrm{Out}(F_g)$ is infinitely generated.

In particular, there is no a priori reason to expect that $\mathcal{H}_g$ shares geometric properties with either surface mapping class groups or outer automorphism groups of free groups.

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The first main result of this paper shows that the geometry of $\mathcal{H}_g$ for $g \geq 3$ seems to share geometric features with the (outer) automorphism group of a free group. In this result, we slightly extend our perspective and also consider handlebodies $V_{g,1}$ of genus g with a marked point and their handlebody group $\mathcal{H}_{g,1}$. We show.

Theorem 1.1. *The Dehn function of $\mathcal{H}_g$ and $\mathcal{H}_{g,1}$ is exponential for any $g \geq 3$.*

The Dehn function of a group is a combinatorial isoperimetric function, and it is a geometric measure for the difficulty of the word problem. See Section 2.2 for details and a formal definition. Theorem 1.1 should be contrasted with the situation in the surface mapping class group – by a theorem of Mosher [Mos], these groups are automatic and therefore have quadratic Dehn functions. On the other hand, Bridson and Vogtmann showed that $\mathrm{Out}(F_g)$ has exponential Dehn function for $g \geq 3$ [BV1, BV2, HM, HV].

Mapping class groups of small complexity are known to have properties not shared with properties of mapping class groups of higher complexity. For example, the mapping class group $\mathrm{Mcg}(\Sigma_2)$ of a surface of genus is a $\mathbb{Z}/2\mathbb{Z}$-extension of the mapping class group of a sphere with 6 punctures. This implies among others that the group virtually surjects onto $\mathbb{Z}$, a property which is not known for higher genus. On the other hand, the group $\mathrm{Out}(F_2)$ is just the full linear group $GL(2,\mathbb{Z})$. Similarly, it is known that the genus 2 handlebody group surjects onto $\mathbb{Z}$ as well [IS].

Our second goal is to add to these results by showing that the handlebody group $\mathcal{H}_2$ has properties not shared by or unknown for handlebody groups of higher genus.

Theorem 1.2. *The group $\mathcal{H}_2$ admits a proper cocompact action on a* CAT(0) *cube complex.*

As an immediate corollary, using Corollary 8.1 of [Ś] and Proposition 1 of [CMV]), we obtain among others that the genus bound in Theorem 1.1 is optimal.

Corollary. *The group $\mathcal{H}_2$ is biautomatic, in particular it has quadratic Dehn function, and it has the Haagerup property.*

In order to prove Theorem 1.1, we recall from previous work [HH2] that the Dehn function of handlebody groups is at most exponential, and it therefore suffices to exhibit a family of cycles which requires exponential area to fill (compared to their lengths). These cycles will be lifted from cycles in automorphism groups of free groups used by Bridson and Vogtmann [BV1]. This construction occupies Section 3.

The proof of Theorem 1.2 is more involved, and relies on constructing and studying a suitable geometric model for the genus 2 handlebody group. In Section 4 we describe in detail the intersection pattern of disk-bounding curves in a genus 2 handlebody. The model of $\mathcal{H}_2$ is built in two steps: in

Section 5 we construct a tree on which $\mathcal{H}_2$ acts and in Section 6 we then use this tree to build our cubical model for $\mathcal{H}_2$ and prove Theorem 1.2. We also discuss some additional geometric consequences.

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2. Preliminaries

2.1. **Handlebody Groups and Meridians.** Let V be a handlebody of genus $g \geq 2$. We identify the boundary of V once and for all with a surface Σ of genus g. Restrictions of homeomorphisms of V to the boundary then induce a restriction map

$$r : \mathrm{Mcg}(V) \to \mathrm{Mcg}(\Sigma).$$

It is an easy consequence of the fact that V is aspherical that this map is injective, and we call its image the *handlebody group* $\mathcal{H}_g$.

When we consider a handlebody with a marked point p, we will always assume that the marked point is contained in the boundary. We then get a map

$$r : \mathrm{Mcg}(V, p) \to \mathrm{Mcg}(\Sigma, p)$$

whose image is the handlebody group $\mathcal{H}_{g,1}$.

We call a curve α on the boundary of a handlebody V a *meridian* if it is the boundary of an embedded disk $D \subset V$. We will frequently use some basic properties of meridians, which are summarised in the following lemma.

Lemma 2.1. *i) $\mathcal{H}_g$ and $\mathcal{H}_{g,1}$ act transitively on the set of nonseparating meridians.*

ii) Every meridian can be extended to a pants decomposition consisting of meridians.

iii) If $\alpha_1, \ldots, \alpha_k$ are meridians so that $S - \cup\alpha_i$ consists of bordered spheres, then any curve β which is disjoint from $\cup\alpha_i$ is a meridian.

Proof. Part i) follows since a handlebody is, up to homeomorphism, determined by its boundary surface, and the handlebodies obtained by cutting at a nonseparating meridian have homeomorphic boundary surfaces. This immediately implies part ii), as there is a pants decomposition of meridians containing a nonseparating curve. Part iii) follows, since every curve on the boundary of a ball bounds a disk in that ball. □

If we consider several meridians at once, we will always assume that they are in pairwise minimal position. This can always be arranged by an isotopy (we emphasise that we only require this for the curves on ∂V, as there is no unique minimal position for disks in a handlebody).

We also need some special elements of the handlebody group. We denote by $T_\alpha \in \mathrm{Mcg}(\Sigma)$ the positive (or left) *Dehn twist about* α (compare [FM, Chapter 3]). Dehn twists T_α are elements of the handlebody group exactly if the curve α is a meridian [McC2, Oer].

We have the following standard lemma which gives rise to another important class of elements.

Lemma 2.2. *Suppose that α, β, δ are three disjoint simple closed curves on Σ which bound a pair of pants on Σ. Suppose that δ is a meridian. Then the product*

$$T_\alpha T_\beta^{-1}$$

is an element of the handlebody group.

Proof. Let P be the pair of pants with $\partial P = \alpha \cup \beta \cup \delta$, and let D be the disk bounded by δ. Then $P \cup D$ is an embedded annulus in V whose two boundary curves are α and β. By applying a small isotopy we may then assume that there is a properly embedded annulus A whose boundary curves are α, β.

Consider the homeomorphism F of V which is a twist about A. To be more precise, consider a regular neighborhood U of A of the form

$$U = [0,1] \times A = [0,1] \times S^1 \times [0,1].$$

The homeomorphism F is defined to be the standard Dehn twist on each annular slice $[0,1] \times S^1 \times \{t\} \subset U$. This map restricts to the identity on $\{0,1\} \times A$, and thus extends to a homeomorphism of V. It restricts on the boundary of V to the desired element, finishing the proof. □

The final type of elements we need are *point-push maps.* Recall (e.g. from [FM]) the *Birman exact sequence*

$$1 \to \pi_1(\Sigma, p) \to \mathrm{Mcg}(\Sigma, p) \to \mathrm{Mcg}(\Sigma) \to 1.$$

The image of $\pi_1(\Sigma, p) \to \mathrm{Mcg}(\Sigma, p)$ is the point pushing subgroup. We need three facts about these mapping classes, all of which are well-known, and are fairly immediate from the definition.

Lemma 2.3. *i) The point-pushing subgroup is contained in $\mathcal{H}_{g,1}$ (compare* [HH1, Section 3]*).*

ii) If $\gamma \in \pi_1(\Sigma, p)$ is simple, then the point push about γ is a product $T_\alpha T_\beta^{-1}$ of two Dehn twists, where α, β are the two simple closed curves obtained by pushing γ off itself to the left and right, respectively (compare [FM, Fact 4.7]*).*

iii) The point push about γ acts on $\pi_1(\Sigma, p)$ as conjugation by γ. Similarly, it acts on $\pi_1(V, p)$ as conjugation by the image of γ in $\pi_1(V, p)$ (compare [FM, Discussion in Section 4.2.1]*).*

2.2. **Dehn functions.** Consider a finitely presented group G with a fixed finite presentation $\langle S | R \rangle$. A word w in S (or, alternatively, an element of the free group $F(S)$ on the set S) is trivial in G exactly if w can be written (in $F(S)$) as a product

$$w = \prod_{i=1}^{n} x_i r_i x_i^{-1}$$

for elements $r_i \in R$ and $x_i \in F(S)$. We define the *area of* w as the minimal n for which such a description is possible. The *Dehn function* is the function

$$D(n) = \sup\{\text{area}(w) \mid l(w) = n\}$$

where $l(w)$ denotes the length of the word w (alternatively, the word norm in $F(S)$).

The Dehn function depends on the choice of the presentation, but its growth type does not (see e.g. [Alo]). We employ the convention that products in mapping class groups are compositions (i.e. the rightmost mapping classes are applied first).

2.3. **Annular subsurface projections.** In this subsection we briefly recall subsurface projections into annular regions, as defined in [MM], Section 2.4.

Let $A = S^1 \times [0,1]$ be a closed annulus. Recall that the *arc graph* $\mathcal{A}(A)$ of the annulus A is the graph whose vertices correspond to embedded arcs which connect the two boundary circles $S^1 \times \{0\}, S^1 \times \{1\}$, up to homotopy fixing the endpoints. Two such vertices are joined by an edge if the corresponding arcs are disjoint except possibly at the endpoints (up to homotopy fixing the endpoints). It is shown in Section 2.4 of [MM] that the resulting graph is quasi-isometric to the integers.

Now consider a surface Σ of genus at least two. Fix once and for all a hyperbolic metric on Σ. If α is any simple closed curve on Σ, let $\Sigma_\alpha \to \Sigma$ be the annular cover corresponding to α, i.e. the cover homeomorphic to an (open) annulus to which α lifts with degree 1. By pulling back the hyperbolic metric from Σ to Σ_α, we obtain a hyperbolic metric on Σ_α. This allows us to add two boundary circles at infinity which compactify Σ_α to a closed annulus $\widehat{\Sigma_\alpha}$.

If β is a simple closed curve on Σ, then any lift $\hat{\beta}$ of β to Σ_α has well-defined endpoints at infinity (for example, since this is true for lifts to the universal cover). In addition, if β has an essential intersection with α, there is at least one lift $\hat{\beta}$ of β to Σ_α which connects the two boundary circles of Σ_α. Such a lift $\hat{\beta}$ has well-defined endpoints at infinity in $\widehat{\Sigma_\alpha}$, and so it defines a vertex in $\mathcal{A}(\widehat{\Sigma_\alpha})$. We define the projection $\pi_\alpha(\beta) \subset \mathcal{A}(\widehat{\Sigma_\alpha})$ to be the set of all such lifts. Since β is simple, this is a (finite) subset of diameter one.

Observe that if β' is freely homotopic to β, then any lift of β' is homotopic to a lift of β with the same endpoints at infinity. Hence, the projection $\pi_\alpha(\beta)$ depends only on the free homotopy class of β.

If β is disjoint from α, the projection $\pi_\alpha(\beta)$ is undefined.

If β_1, β_2 are two simple closed curves which both intersect α essentially, then we define the *subsurface distance*

$$d_\alpha(\beta_1, \beta_2) = \text{diam}(\pi_\alpha(\beta_1) \cup \pi_\alpha(\beta_2)).$$

3. Exponential Dehn functions in genus at least 3

Theorem 3.1. *Let $g \geq 3$. The Dehn function of $\mathcal{H}_g$ and $\mathcal{H}_{g,1}$ is at least exponential.*

The core ingredient in the proof is the natural map

$$\mathcal{H}_{g,1} \to \mathrm{Aut}(F_g)$$

induced by the action of homeomorphisms of the handlebody V_g on the fundamental group $\pi_1(V_g) = F_g$. Our strategy will be to take a sequence of trivial words w_n in $\mathrm{Aut}(F_g)$ which have exponentially growing area and lift them into the handlebody group.

We will spend most of this section with discussing the case of $\mathcal{H}_{3,1}$ in detail; the other cases will be derived from this special case at the end of the section.

In [BV1] the following three automorphisms of F_3 are considered. Let a, b, c be a free basis of F_3. Then define automorphisms

$$A : \begin{cases} a & \mapsto & a \\ b & \mapsto & b \\ c & \mapsto & ac \end{cases}, \quad B : \begin{cases} a & \mapsto & a \\ b & \mapsto & b \\ c & \mapsto & cb \end{cases}, \quad T : \begin{cases} a & \mapsto & a^2b \\ b & \mapsto & ab \\ c & \mapsto & c \end{cases}$$

Observe that B and T^nAT^{-n} commute for all n, and therefore we have the following equation

$$T^nAT^{-n}BT^nA^{-1}T^{-n}B^{-1} = \mathrm{id}$$

in the automorphism group $\mathrm{Aut}(F_3)$. The crucial result we need is the following, which is proved in [BV1, Theorem A].

Theorem 3.2 (Bridson-Vogtmann)**.** *Consider any presentation of* $\mathrm{Aut}(F_3)$ *or* $\mathrm{Out}(F_3)$ *whose generating set contains the automorphisms A, B, T. Then, if $f : \mathbb{N} \to \mathbb{N}$ is any sub-exponential function, the loops defined by the words*

$$w_n = T^nAT^{-n}BT^nA^{-1}T^{-n}B^{-1}$$

cannot be filled with area less than $f(n)$ for large n.

In particular, since the words w_n have length growing linearly in n, the theorem immediately implies that $\mathrm{Aut}(F_3)$ and $\mathrm{Out}(F_3)$ have exponential Dehn function.

In order to show Theorem 3.1, we will realize A, B, T in a specific way as homeomorphisms of a genus 3 handlebody. This construction will be performed in several steps.

Constructing the handlebody. The first step is to give a specific construction of a genus 3 handlebody V_3 that will be particularly useful to us. We construct V_3 by attaching a single handle H to an interval-bundle V_2 over a torus S with one boundary component (which is a genus 2 handlebody).

To be more precise, denote by S a surface of genus 1 with one boundary component. We pick a basepoint $p \in \partial S$. We define

$$V_2 = S \times [0,1].$$

This is a handlebody of genus 2, and its boundary ∂V_2 has the form

$$\partial V_2 = (S \times \{0\}) \cup (\partial S \times [0,1]) \cup (S \times \{1\}).$$

In other words, the boundary consists of two tori $S_i = S \times \{i\}, i = 0, 1$ and an annulus $A = \partial S \times [0,1]$. We employ the convention that a subscript 0 or 1 attached to any object in S denotes its image in $S \times \{0,1\}$. For example, p_0 will denote the point $p \times \{0\}$.

Next, we want to attach a handle in A to form the genus 3 handlebody. To this end, choose two disjoint embedded disks D^-, D^+ in the interior of A which are disjoint from $p \times [0,1]$. Gluing D^- to D^+ (or, alternatively, attaching a 1-handle at these disks) yields our genus 3 handlebody V_3. We denote by D the image of the disks D^+, D^- in V_3.

Finally we will construct a core graph in V_3 in a way that is compatible with our construction. Begin by choosing two loops $a, b \subset S$ which intersect only in p, and which define a free basis of $\pi_1(S,p) = F_2$. Furthermore, choose points q^-, q^+ in $\partial D^-, \partial D^+$ which are identified with each other in forming V_3. Then choose embedded arcs $c^+, c^- \subset A$ from p_1 to q^+, q^- which only intersect in p_1. We denote by c the loop in V_3 formed by traversing c^+ from p_1 to q^+, then c^- from q^- back to p_1.

Then the union

$$\Gamma = a_1 \cup b_1 \cup c$$

is an embedded three-petal rose in ∂V_3, so that the inclusion $\Gamma \to V_3$ induces an isomorphism on fundamental groups (recall that $a_1 = a \times \{1\}$ and similar for b_1). By slight abuse of notation, we will denote the images of the three petals in $\pi_1(V_3)$ by a, b, c, and note that they form a free basis.

Realizing T as a bundle map. Recall that the mapping class group of a torus S with one boundary component surjects to $\mathrm{Aut}(F_2)$ [FM, Section 2.2.4 and Proposition 3.19], and therefore there is a homeomorphism t of S which restricts to the identity on ∂S, and so that the induced map $t : \pi_1(S,p) \to \pi_1(S,p)$ acts on the basis defined by the loops a, b as follows:

$$t_*(a) = a^2 b, \quad t_*(b) = ab.$$

The homeomorphism $t \times \mathrm{Id}$ of V_2 preserves S_0, S_1 setwise and restricts to the identity on A. By the latter fact, the homeomorphism $t \times \mathrm{Id}$ then defines a homeomorphism τ of V_3, which is the identity on A, and in particular fixes c pointwise.

We summarize the important properties of τ in the following lemma.

Lemma 3.3. *τ is a homeomorphism of V_3 fixing the marked point p_1 with the following properties:*

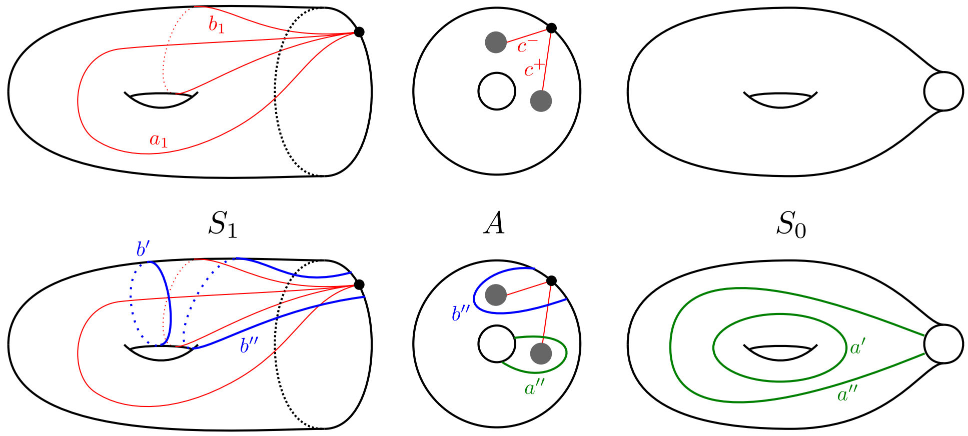

FIGURE 1. Constructing the handlebody and curves needed to construct the maps α, β, τ.

i) The support of τ restricted to the boundary ∂V_3 is $S_0 \cup S_1$, and it preserves both subsurfaces S_i set-wise.
ii) τ acts on $\pi_1(V_3, p_1)$ in the basis a, b, c as the automorphism T:

$$\tau_*(a) = a^2 b, \quad \tau_*(b) = ab, \quad \tau_*(c) = c.$$

Realizing A by a handleslide. Intuitively, α will slide the end D^+ of the handle H around the loop a_0 in the "bottom surface" S_0 of the interval bundle $V_2 \subset V_3$.

To be precise, let z be an arc which joins D^+ to $p \times \{0\}$ inside A, and is disjoint from c. Consider a small regular neighbourhood of $\partial D^+ \cup z \cup a_0$ in ∂V_3. Its boundary consists of three simple closed curves, one of which is homotopic to ∂D, and the two others we denote by a', a''. One of them, say a', is contained in S_0 and will intersect $a_0 \cup b_0$ in a single point (necessarily of b_0). The other curve a'' is disjoint from S_1, and intersects A in a single arc. Note further that a', a'' and ∂D bound a pair of pants in ∂V_3.

Let α be a homeomorphism of ∂V_3 which defines the product $T_{a''} T_{a'}^{-1}$ of Dehn twists in the mapping class group of ∂V_3 and is supported in a small regular neighbourhood of $a' \cup a''$. It extends to a homemorphism of the handlebody V_3 by Lemma 2.2, and we will denote this extension by the same symbol.

Next, we compute the action of α on the fundamental group of V_3. We will do this using the core graph

$$\Gamma = a_1 \cup b_1 \cup c$$

defined above. Since a', a'' are disjoint from S_1, we have that $\alpha(a_1) = a_1, \alpha(b_1) = b_1$ for $i = 1, 2$. Since $a' \subset S_0$, we see that c is disjoint from a'. Finally, c intersects a'' in a single point q. Thus, $\alpha(c)$ is a loop, which is formed by following c until the intersection point q, traversing a'' once, and then continuing along c. Observe that by pushing a'' first through D, and then to the “top” half S_1 of the interval bundle, this loop $\alpha(c)$ is therefore homotopic in V_3 (relative to p_1) to the concatenation of a_1 and c. We thus have the following properties of α:

Lemma 3.4. *α is a homeomorphism of V fixing p_1 with the following properties:*

i) The restriction of α to ∂V is supported in a small neighbourhood of a', a'', where $a' \subset S_0$, and a'' is disjoint from S_1 and intersects A in a single arc.
ii) α acts on $\pi_1(V_3, p_1)$ as the automorphism A:

$$\alpha_*(a) = a, \quad \alpha_*(b) = b, \quad \alpha_*(c) = ac.$$

Realizing B by a handleslide and a point push. We will realise B similar to A, by pushing D^- along the loop b of the “top” side of the interval bundle.

To do this, we first construct an auxiliary homeomorphism $\hat{\beta}$ of V_3 analogous to the previous step. Consider a regular neighbourhood of $\partial D \cup c^- \cup b_1$. Its boundary again consists of three curves; one of which is homotopic to ∂D, and we denote the others by b', b''. Let b' be the one which is completely contained in S_1 (and thus freely homotopic to b_1).

As above, we can choose a homeomorphism $\hat{\beta}$ which is supported in a small neighbourhood of $b' \cup b''$ and defines the mapping class $T_{b''}T_{b'}^{-1}$. By Lemma 2.2 and the fact that b', b'' and ∂D bound a pair of pants, it extends to V_3 and we denote the extension by the same symbol.

We now compute the effect of $\hat{\beta}$ on the core graph Γ. We begin with the petal a_1. It intersects both b' and b'' in one point each. Hence, we have that $\hat{\beta}(a_1)$ is homotopic on ∂V_3, relative to the basepoint p_1, to the concatenation $b_1 * a_1 * b_1^{-1}$. The loop b_1 is, by construction, disjoint from b', b'' and so we have $\hat{\beta}(b_1) = b_1$. Finally, c^+ intersects b'' in a single point, and is disjoint from b', while c^- is disjoint from both b', b''. Thus, $\hat{\beta}(c)$ is homotopic on ∂V_3 to the concatenation $b_1 * c$. In total, we see that $\hat{\beta}$ acts on our chosen basis of $\pi_1(V_3, p_1)$ as follows:

$$\hat{\beta}_*(a) = bab^{-1}, \quad \hat{\beta}_*(b) = b, \quad \hat{\beta}_*(c) = bc.$$

To define the homeomorphism β, we post-compose $\hat{\beta}$ with a point push P around b_1^{-1} (which is an element of the handlebody group by Lemma 2.3 i)). Since this point push has the effect on the level of fundamental group of conjugating by b^{-1} (Lemma 2.3 iii)), we see that therefore β will indeed realize the automorphism B as desired.

By Lemma 2.3 ii), the point push homeomorphism P can be chosen to be supported in the union of S_1 and a small neighborhood of p_1. In particular, we may assume that the support of the point push is disjoint from the arc $a'' \cap A$ occurring in Lemma 3.4. We summarize the required properties of β in the following lemma.

Lemma 3.5. *β is a homeomorphism of V fixing the marked point p with the following properties:*

(1) The restriction of β to ∂V is the product of four Dehn twists about curves $d_i \subset S_0 \cup A$, all four of which are disjoint from the curves a', a'' occurring in Lemma 3.4.
(2) β acts on $\pi_1(V_3, p_1)$ as the automorphism B:

$$\beta_*(a) = a, \quad \beta_*(b) = b, \quad \beta_*(c) = cb.$$

Completing the proof. Consider the homeomorphisms

$$\tau^n \alpha \tau^{-n}$$

of V_3. Their support is contained in a small regular neighbourhood of $\tau^n(a'), \tau^n(a'')$. Recall that τ preserves S_0 and hence $\tau^n(a') \subset S_0$. Thus $\tau^n(a')$ is disjoint from all of the curves d_i from Lemma 3.5. Furthermore, $\tau^n(a'') \cap A = a'' \cap A$ (since τ restricts to the identity on A). Hence, since the d_i from Lemma 3.5 are disjoint from S_0 and intersect A in arcs which are disjoint from a'', we have that $\tau^n(a'')$ is disjoint from all d_i for any n.

As a consequence, the homeomorphisms $\tau^n \alpha \tau^{-n}$ and β have (up to isotopy) disjoint supports, and therefore define commuting mapping classes in $\mathcal{H}_{3,1}$. Therefore we conclude the following relation in $\mathcal{H}_{3,1}$

$$1 = [\tau^n \alpha \tau^{-n}, \beta] = \tau^n \alpha \tau^{-n} \beta \tau^n \alpha^{-1} \tau^{-n} \beta^{-1},$$

where we have denoted the mapping classes defined by the homeomorphisms α, β, τ by the same symbols. In other words, if we choose a generating set of $\mathcal{H}_{3,1}$ which contains α, β, τ then

$$\omega_n = \tau^n \alpha \tau^{-n} \beta \tau^n \alpha^{-1} \tau^{-n} \beta^{-1}$$

are words in $\mathcal{H}_{3,1}$ which define the trivial element. Furthermore, under the map $\mathcal{H}_{3,1} \to \mathrm{Aut}(F_3)$ they map exactly to the words w_n occurring in Theorem 3.2. By the conclusion of that theorem, w_n cannot be filled with subexponential area in $\mathrm{Aut}(F_3)$. Since group homomorphisms coarsely decrease area, the same is therefore true for the words ω_n. But, by construction, the length of the word ω_n grows linearly in n, showing that $\mathcal{H}_{3,1}$ has at least exponential Dehn function. The same is true for $\mathcal{H}_3$, since the words w_n are also exponentially hard to fill in $\mathrm{Out}(F_3)$ by Theorem 3.2.

To extend the proof of Theorem 3.1 to any genus $g \geq 3$, we argue as follows. Consider the handlebody V of genus 3 constructed above. Take a connected sum of V with a handlebody V' of genus $g-3$ at a disk D_g in the annulus A, which is disjoint from all curves used to define α, β, to obtain a handlebody V_g of genus g. The homeomorphisms α, β, τ can then be extended to homeomorphisms of V_g which restrict to be the identity on V'. In this way the words ω_n define trivial words $\hat{\omega}_n$ in $\mathcal{H}_{g,1}$.

There is a natural map

$$\iota : \mathrm{Aut}(F_3) \to \mathrm{Aut}(F_g)$$

which maps an automorphism φ of $F\langle x_1, x_2, x_3\rangle$ to its extension to the free group $\langle x_1, \ldots, x_g\rangle$ on g generators which fixes all $x_i, i > 3$.

By construction, the words $\hat{\omega}_n$ map to the image of the words w_n under ι. Corollary 4 of [HM] (compare also [HH3]) shows that the image of ι is a Lipschitz retract of $\mathrm{Aut}(F_g)$ and $\mathrm{Out}(F_g)$, and therefore the words $\iota(w_n)$ are also exponentially hard to fill. By the same argument as above, the same is therefore true for the words $\hat{\omega}_n$.

4. Waves in genus 2

In this section we study intersection pattern between meridians in a genus two handlebody V. Recall that a *cut system* of a genus two handlebody is a pair $(\alpha_1, \alpha_2) \subset \partial V$ of disjoint meridians with connected complement. Equivalently, cut systems are the boundary curves of disjoint disks $D_1, D_2 \subset V$ so that $V - (D_1 \cup D_2)$ is a single 3–ball.

The next proposition (which is true in any genus) is well-known, see e.g. [Mas, Lemma 1.1] or [HH1, Lemma 5.2 and the discussion preceding it].

Proposition 4.1. *Suppose that (α_1, α_2) is a cut system and β is an arbitrary (multi)meridian. Either $\alpha_1 \cup \alpha_2$ and β are disjoint, or there is a subarc $b \subset \beta$, called a* wave*, with the following properties:*

i) The arc b intersects $\alpha_1 \cup \alpha_2$ only in its endpoints, and both endpoints lie on the same curve, say α_1.
ii) The arc b approaches α_1 from the same side at both endpoints.
iii) Let a, a' be the two components of $\alpha_1 \setminus b$. Then exactly one of

$$(a \cup b, \alpha_2), (a' \cup b, \alpha_2)$$

is a cut system, which we call the surgery defined by the wave b in the direction of β.
iv) The surgery defined by b has fewer intersections with β than (α_1, α_2).

We say that a sequence $\left(\alpha_i^{(n)}\right)_n$ of cut systems is a *surgery sequence in the direction of* β if each $\left(\alpha_i^{(n+1)}\right)$ is the surgery of $\left(\alpha_i^{(n)}\right)$ defined by some wave b of β. By Proposition 4.1, these exist for any initial cut system, and they end in a system which is disjoint from β.

The following lemmas describe certain symmetry and uniqueness features of waves in genus 2. They are central ingredients in our study of projection maps in the next section. The results of these lemmas are discussed and essentially proved in Section 4 of [Mas]. We include a proof for completeness and convenience of the reader.

To describe them, it is easier to take a slightly different point of view. Namely, let (α_1, α_2) be a cut system of a genus 2 handlebody V. Consider $S = \partial V - (\alpha_1 \cup \alpha_2)$. This is a four-holed sphere, with boundary components $\alpha_1^+, \alpha_1^-, \alpha_2^+, \alpha_2^-$ corresponding to the sides of the α_i. A wave b corresponds exactly to a subarc of β which joins one of the boundary components of S to itself.

At this point we want to emphasize that when considering arcs in S, we always consider them up to homotopy which is allowed to move the endpoints (in ∂S).

Lemma 4.2. *Let β be a meridian, and $b \subset S$ be a wave of β joining a boundary component α_i^* to itself. Then b separates the two boundary components α_{1-i}^+ and α_{1-i}^-.*

Proof. Let b be such a wave, joining without loss of generality α_1^+ to itself. Suppose that α_2^- and α_2^+ are contained in the same component of $S - b$. Then b and a subarc $a \subset \alpha_1$ concatenate to a curve homotopic to α_1^- on S. Continue b beyond one of its endpoints across α_1 to form a larger subarc of β. It then exits from α_1^- and, by minimal position, has its next intersection point with $\alpha_1 \cup \alpha_2$ in a again. This can be iterated, and leads to a contradiction as the curve β then cannot close up (compare Figure 2). □

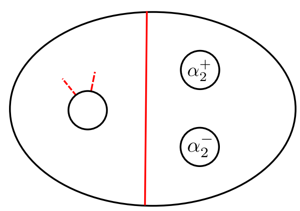

FIGURE 2. A "bad" wave – the central (red) arc cannot be part of a meridian, since both ends would have to continue into the left annulus, and are unable to close up to a simple closed curve.

Lemma 4.3. *Suppose (α_1, α_2) is a cut system of V, and β is any meridian. Suppose that β has a wave b^+ at α_i^+.*

i) There is a unique essential arc $b^- \subset S$ with both endpoints on α_i^- which is disjoint from b^+.
ii) The arc b^- from i) appears as a wave of β.
iii) Any wave of β is homotopic to either b^+ or b^-.

(The same is true with the roles of α_i^-, α_i^+ reversed)

Proof. Let b^+ be a wave as in the prerequisites, joining without loss of generality α_1^+ to itself. Denote the complementary components of b^+ in S by C_1 and C_2. The three boundary components $\alpha_1^-, \alpha_2^-, \alpha_2^+$ are contained in $C_1 \cup C_2$. By Lemma 4.2, α_2^- and α_2^+ are not contained in the same C_i. We may therefore assume that C_1 contains α_2^-, and C_2 contains α_2^+ and α_1^-. In particular, C_1 is a annulus with core curve homotopic to α_2^-, and C_2 is a pair of pants.

Consider the boundary of a regular neighborhood of $\alpha_1^+ \cup b^+$ in S. This consists of two simple closed curves δ, δ' which bound a pair of pants together with α_1^+. Up to relabeling we have that $\delta \subset C_1, \delta' \subset C_2$. By the discussion above δ is then homotopic to α_2^-, and δ' bounds a pair of pants with α_1^-, α_2^+ (compare Figure 3). In a pair of pants there is a unique isotopy class of arcs joining a given boundary component to itself, and hence assertion i) is true. In an annulus there is no (essential) arc joining a boundary component to itself, we thus also conclude that there cannot be a wave of β based at α_2^-.

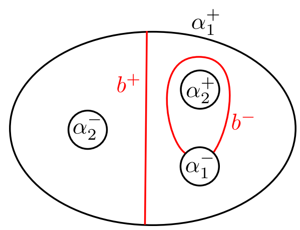

FIGURE 3. A "good" wave – the central (red) arc is a wave, and it has a partner based at α_1^-. Any other wave is necessarily homotopic to either b^+ or b^-.

Next, observe that $(\alpha_1, \alpha_2, \delta')$ is a pair of pants decomposition consisting of three non-separating curves. Let P_1, P_2 be the two components of $\partial V - (\alpha_1 \cup \alpha_2 \cup \delta')$. Both P_1 and P_2 are pairs of pants whose boundary curves are $\alpha_1, \alpha_2, \delta'$. Suppose that P_1 contains b^+. Since b^+ joins the boundary component α_1 of P_1 to itself, and β is embedded, every component of $P_1 \cap \beta$ has at least one endpoint on α_1. Since b^+ has both endpoints on α_1^+ we therefore conclude the following inequality on intersection numbers:

$$i(\alpha_1, \beta) > i(\alpha_2, \beta) + i(\delta', \beta).$$

Now consider the situation in P_2. If there would not be an arc $b^- \subset P_2$ which joins α_1 to itself, then any arc in $P_2 \cap \beta$ which has one endpoint on α_1 has the other on α_2 or δ'. This would imply

$$i(\alpha_1, \beta) \leq i(\alpha_2, \beta) + i(\delta', \beta)$$

which contradicts the inequality above. Hence, there is an arc $b^- \subset \beta$ which joins α_1 to itself in P_2. We have therefore found the desired second wave b^-, showing ii).

In order to show iii), we only have to exclude a wave based at α_2^+. However, this follows as above since α_2^+ is contained in the annulus bounded by b^- and α_2^-. □

We also observe the following corollary.

Corollary 4.4. *Let (α_1, α_2) be a cut system of a genus 2 handlebody. Let β be an arbitrary meridian and b^+, b^- be the two distinct waves guaranteed by Lemma 4.3. Then the surgeries defined by b^+ and b^- are equal.*

In particular, there is a unique surgery sequence starting in (α_1, α_2) *in the direction of* β*.*

Proof. The first claim is obvious from the fact that (in the notation of the proof of Lemma 4.3) δ' is homotopic to a boundary component of a regular neighborhood of $\alpha_1^+ \cup b^+$ and $\alpha_1^- \cup b^-$, and is therefore equal to the surgery defined by both b^+ and b^-. The second claim is immediate from the first. □

5. Meridian Graphs

The purpose of this section is to construct a tree on which the handlebody group $\mathcal{H}_2$ acts, and which is crucial for the construction of an action of $\mathcal{H}_2$ on a CAT(0) cube complex in Section 6.

5.1. The wave graph (is a tree).

We begin with a construction of a graph which will model all possibilities to change a cut system Z to a disjoint cut system Z'.

To this end, we fix in this subsection once and for all a cut system Z. We are interested in describing all curves δ in the complement of Z which are non-separating meridians on ∂V. Observe that by Lemma 2.1 iii) any curve disjoint from a cut system automatically bounds a disk in V, and we therefore only have to characterise being non-separating. The following lemma allows us to encode this in a convenient way.

Lemma 5.1. *Let* $Z = \{\alpha_1, \alpha_2\}$ *be a cut system, and* $S = \partial V - Z$ *its complementary subsurface. Fix a boundary component* ∂_0 *of* S*. Then the following are true:*

- *i) A curve* $\delta \subset S$ *is non-separating on* ∂V *exactly if for both* $i = 1, 2$*, the two boundary components of* S *corresponding to the sides of* α_i *are contained in different complementary components of* δ*.*
- *ii) Given any* δ *as in i), there is a unique embedded arc* w *in* S *with both endpoints on* ∂_0 *disjoint from* δ*, and it separates the two boundary components corresponding to the curve of* Z *that* ∂_0 *does not correspond to. We call such an arc an* admissible wave*.*
- *iii) Conversely, if* w *is any admissible wave, there is a unique curve* δ *defining it via ii).*
- *iv) Two curves* δ, δ' *as in i) intersect in two points exactly if the corresponding admissible waves* w, w' *are disjoint.*

Proof. Assertion i) is clear. Assertion ii) follows because any curve δ as in ii) separates S into two pairs of pants, and on a pair of pants there is a unique homotopy class of embedded arcs with endpoints on a specified boundary component. Assertion iii) follows since such an arc cuts S into an annulus and a pair of pants (compare the proof of Lemma 4.3). It remains to show the final Assertion iv). First observe that if w, w' are disjoint then the corresponding curves constructed in iii) indeed intersect in two points.

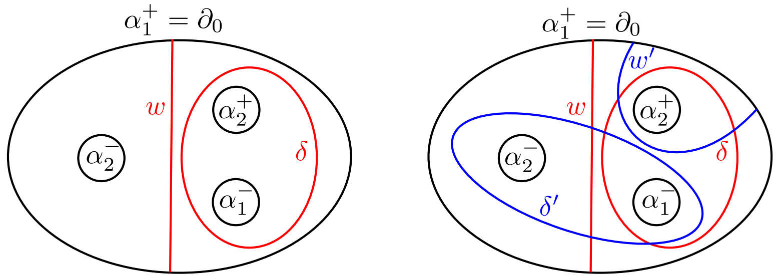

FIGURE 4. On the left: Passing from an admissible wave to a meridian and back as in Lemma 5.1 ii) and iii). On the right: Disjoint admissible waves correspond to curves intersecting twice.

Finally, suppose that δ is a curve defined by an arc w as in iii). The arc w separates S into S_1 and S_2. Without loss of generality we may assume that S_1 is an annulus containing α_2^- and S_2 is a pair of pants.

Suppose now that δ' intersects δ in two points. Then δ' will also intersect w in two points, and hence there is a subarc $d' \subset \delta'$ in S_2 with both endpoints on w. Since S_2 is a pair of pants, there is a unique arc w' in S_2 with endpoints on ∂_0 which is disjoint from d', and therefore from δ'. By the uniqueness of ii) this is the arc defining δ', and since it is contained in S_1 with endpoints on ∂_0 it is indeed disjoint from w. □

Motivated by this lemma, we make the following definition.

Definition 5.2. The *wave graph* $\mathrm{W}(Z)$ *of* Z is the graph whose vertices correspond to (isotopy classes of) admissible waves w based at ∂_0, and whose edges correspond to disjointness (up to isotopy).

For future reference, we record the following immediate corollary of Lemma 5.1:

Corollary 5.3. *The wave graph* $\mathrm{W}(Z)$ *is isomorphic to the graph whose vertices correspond to non-separating meridians* δ *which are disjoint from* Z*, and where vertices corresponding to* δ, δ' *are connected by an edge if* $i(\delta, \delta') = 2$*.*

Here, and below, intersection numbers will always mean the minimal possible intersection numbers in the respective isotopy classes.

Remark 5.4. *As a consequence of Corollary 5.3, the wave graph* $\mathrm{W}(Z)$ *can be identified with a subgraph of the Farey graph of the four-holed sphere* $\Sigma_{0,4}$*. To describe this subgraph, recall that every edge of the Farey graph is contained in two triangles. The three vertices of such a triangle correspond to the three different ways to separate the four punctures of* $\Sigma_{0,4}$ *into two sets. The condition for the meridian to be non-separating excludes one of these (again observing that by Lemma 2.1 all three curves are meridians)– so the wave graph is obtained from the Farey graph of* $\Sigma_{0,4}$ *by removing one vertex and two edges from each triangle. This point of view can be used to* *show that the wave graph is a tree (Theorem 5.9), but we will prove this theorem using different techniques which are useful later.*

We also record the following

Lemma 5.5. *The wave graph* $\mathrm{W}(Z)$ *is connected.*

Proof. This is a standard surgery argument, using induction on intersection number. Suppose w, w' are any two admissible waves corresponding to vertices in $\mathrm{W}(Z)$. If they are not disjoint, consider an initial segment $w_0 \subset w$ which intersects w' only in its endpoint $\{q\} = w_0 \cap w'$. Let w_1', w_2' be the two components of $w' \setminus \{q\}$. Then both $x_i = w_0 \cup w_i'$ are arcs which are disjoint from w' and have smaller intersection with w. It is easy to see that exactly one of them is admissible (compare also Figure 5), and hence w' is connected to a vertex $x_i \in \mathrm{W}(Z)$ which corresponds to an arc of strictly smaller intersection number with w. By induction, the lemma follows. □

In order to study $\mathrm{W}(Z)$ further, we define the following projection maps.

First, given a vertex $w \in \mathrm{W}(Z)$ corresponding to an admissible wave (which we denote by the same symbol), we define the set $\mathrm{A}(w)$ to consist of embedded arcs a in $S - w$ with one endpoint on w and the second endpoint on ∂_0, which are not homotopic into $w \cup \partial_0$ (up to homotopy of such arcs).

We observe that any arc a corresponding to a vertex in $\mathrm{A}(w)$ cuts the pair of pants $S - w$ into two annuli. From this, we obtain the following consequence (see also Figure 5):

Lemma 5.6. *i) No two arcs a, a' corresponding to different vertices in $\mathrm{A}(w)$ are disjoint.*

ii) Given an arc a corresponding to a vertex in $\mathrm{A}(w)$ there is a unique admissible wave w' which is disjoint from w and from a.

iii) No two disjoint admissible waves are disjoint from w.

iv) If y is an admissible wave disjoint from w, then there is a unique arc $a \in \mathrm{A}(w)$ which is disjoint from y.

Proof. If a' is an arc with endpoint on w and ∂_0, which is disjoint from a, then (since both components of $S-(w \cup a)$ are annuli) it is either homotopic into $\partial_0 \cup w$, or homotopic to a. This shows i). To see the second claim, note that there are two homotopy classes of arcs in S with endpoints in ∂_0 disjoint from w. Exactly one of them defines a nonseparating curve (hence, by Lemma 2.1 iii), meridian) and therefore is an admissible wave, showing ii). For the third claim, consider an admissible wave w' disjoint from w. Since it is distinct from w, it separates the two boundary components of S which are contained in $S - w$. If w'' is now any other arc with endpoints on ∂_0 which is disjoint from w and w', then it is either homotopic to one of them, or not admissible, showing iii). Finally, to prove iv), observe that y cuts the pair of pants bounded by w into two annuli, only one of which can support an arc representing a vertex of $\mathrm{A}(w)$. In that annulus there is then a unique homotopy class of arc not homotopic into $\partial_0 \cup w$. □

Given a vertex $w \in \mathrm{W}(Z)$, we now define a map

$$\pi_w : \mathrm{W}(Z) \to \mathrm{A}(w)$$

in the following way:

i) If $x \in \mathrm{W}(Z)$ is not disjoint from w, consider an initial segment $x_0 \subset x$ one of whose endpoints is on ∂_0, the other is on w, and so that its interior is disjoint from w. We then put $\pi_w(x) = x_0$ (see Figure 5). This is well-defined by Lemma 5.6 i).
ii) If y is disjoint from w, we let $\pi_w(y)$ be an arc which is disjoint from y and represents a vertex of $\mathrm{A}(w)$. This is well-defined by Lemma 5.6 iv).

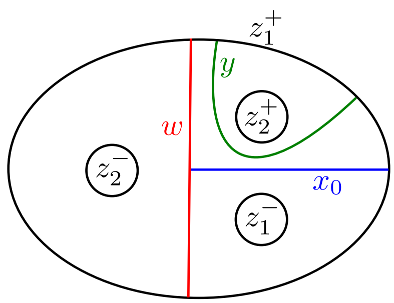

FIGURE 5. Projecting a wave into $\mathrm{A}(w)$. An arbitrary wave x intersecting w has an initial segment x_0 joining ∂_0 to w, which is its projection. For a disjoint wave y there is a unique arc as above disjoint from it.

We then get the following consequences.

Proposition 5.7. *If $x, y \in W(Z)$ are both not disjoint from w, but x, y are disjoint from each other, then $\pi_w(x) = \pi_w(y)$.*

Proof. This is immediate from the definition of projection and Lemma 5.6 i). ☐

Proposition 5.8. *If $x \in \mathrm{W}(Z)$ is not disjoint from w, and $y \in \mathrm{W}(Z)$ is disjoint from x and from w, then $\pi_w(x) = \pi_w(y)$.*

Proof. The projection arc $\pi_w(x)$ is disjoint from y. By Lemma 5.6 iv) it is therefore equal to $\pi_w(y)$. ☐

We are now ready to prove the following central result.

Theorem 5.9. *For any cut system Z, the wave graph $\mathrm{W}(Z)$ is a tree.*

Proof. We have already shown that $\mathrm{W}(Z)$ is connected (Lemma 5.5), and so it suffices to show that there are no embedded cycles. As a first step, note that by Lemma 5.6 no two vertices in the link of w are joined by an edge. Namely, if w_1, w_2 are in the link of w (i.e. disjoint from w), Lemma 5.6 iii) states that they cannot be disjoint from each other. This in particular implies that there are no cycles of length ≤ 3.

Next, observe that if $w_1, \dots, w_k$ is a path so that $w_i \neq w_0$ for all i, then $\pi_{w_0}(w_i)$ is constant. This follows by induction, applying either Proposition 5.7 or Proposition 5.8 for consecutive terms of the path (noting that from any two w_i, w_{i+1} at most one of them can be contained in the link of w_0).

Now suppose that $w_0, w_1 \dots, w_n$ is an embedded cycle of length $n \geq 4$. In particular w_1, w_n are distinct. Applying the previous obervation to $w_1, \dots, w_n$ we see that $\pi_{w_0}(w_1) = \pi_{w_0}(w_n)$. However, w_n and w_1 are assumed to be distinct, and therefore cannot have the same projection by Lemma 5.6 ii). □

5.2. **The non-separating meridional pants graph (is a tree).** We now come to the central object we will use to study $\mathcal{H}_2$.

Definition 5.10. i) The *non-separating meridional pants graph* $\mathcal{P}_2^{\mathrm{nm}}$ has vertices corresponding to pants decompositions $X = \{\delta_1, \delta_2, \delta_3\}$ so that all δ_i are non-separating meridians. We put an edge between X and X' if they intersect minimally, i.e. in two points.

ii) For any cut system Z, let $\mathcal{P}_2^{\mathrm{nm}}(Z)$ be the full subgraph corresponding to all those pants decompositions $X = \{\delta_1, \delta_2, \delta_3\}$ which contain Z.

From Corollary 5.3 and Theorem 5.9 we immediately obtain

Corollary 5.11. *For any cut system Z the subgraph $\mathcal{P}_2^{\mathrm{nm}}(Z)$ is a tree. Any two such subtrees intersect in at most a single point.*

We will use these subtrees in order to study $\mathcal{P}_2^{\mathrm{nm}}$. We begin with the following.

Lemma 5.12. *The graph $\mathcal{P}_2^{\mathrm{nm}}$ is connected.*

Proof. Let X, Y be pants decompositions corresponding to vertices of $\mathcal{P}_2^{\mathrm{nm}}$. We will construct a path joining X to Y in $\mathcal{P}_2^{\mathrm{nm}}$. Choose two curves $\{\delta_1, \delta_2\} = Z_1 \subset X$ – these will form a cut system by the definition of $\mathcal{P}_2^{\mathrm{nm}}$. Now, consider the surgery sequence (Z_i) starting in Z_1 in the direction of Y. Let n be so that Z_n is disjoint from Y. As Y is a pants decomposition, this implies that actually $Z_n \subset Y$.

Also, by definition of surgery sequences, for any i the cut systems Z_{i-1} and Z_{i+1} are contained in the complement of the cut system Z_i, and thus $Z_i \cup Z_{i-1}$ and $Z_i \cup Z_{i+1}$ correspond to vertices in the tree $\mathcal{P}_2^{\mathrm{nm}}(Z_i)$. Hence, these vertices can be joined by a path. The desired path (X_i) will now be obtained by concatenating all of these paths. To be more precise, we will have

(1) There are numbers $1 = m(0), m(1), \dots, m(n)$ so that for all $m(i-1) < j \leq m(i)$ the pants decomposition X_j contains Z_i.
(2) For all $m(i-1) < j \leq m(i)$ the pants decomposition X_j are a geodesic in $\mathcal{P}_2^{\mathrm{nm}}(Z_i)$.

From the description above it is clear that these sequences exist, showing Lemma 5.12. □

We will now define projections of $\mathcal{P}_2^{\mathrm{nm}}$ onto the subtrees $\mathcal{P}_2^{\mathrm{nm}}(Z)$. To this end, let Z be a cut system. We define a projection

$$\pi_Z : \mathcal{P}_2^{\mathrm{nm}} \to \mathcal{P}_2^{\mathrm{nm}}(Z)$$

in the following way.

i) If X is disjoint from Z, we simply put $\pi_Z(X) = X$.
ii) If X intersects Z, then there is a wave w of X with respect to Z, and we define $\pi_Z(X) = Z \cup \{\delta\}$, where δ is the surgery defined by the wave w. Corollary 4.4 implies that this is well-defined.

Proposition 5.13. *Suppose that $X, Y \in \mathcal{P}_2^{\mathrm{nm}}$ are connected by an edge, and assume that both X, Y are not disjoint from Z. Then $\pi_Z(X) = \pi_Z(Y)$.*

Proof. Since X and Y are not disjoint from Z, there are waves w_X, w'_X, w_Y, w'_Y as in Lemma 4.3. We claim that unless $\{w_X, w'_X\} = \{w_Y, w'_Y\}$ (as isotopy classes of arcs in the complement of Z), the total number of intersections between $\{w_X, w'_X\}, \{w_Y, w'_Y\}$ is at least 4, contradicting that X, Y are joined by an edge.

However, this is seen in a similar way as the argument in Lemma 4.3 in different cases (compare Figure 6). First observe that as the waves are arcs in a four-holed sphere joining the same boundary to itself, two waves are either disjoint or intersect at least in two points.

Suppose first that the waves of Y are based at the same component of Z as the ones of X, and assume that w_X, w_Y approach from the same side. If w_X and w_Y are disjoint, then by the uniqueness statement of Lemma 4.3 we have that $\{w_X, w'_X\} = \{w_Y, w'_Y\}$, and thus the claim. If w_X, w_Y are not disjoint, then w_Y also intersects w'_X (in at least two points), and we are done.

The case where the waves of X and Y are based at different components is similar, noting that each of w_Y, w'_Y needs to intersect at least one of the w_X, w'_X. □

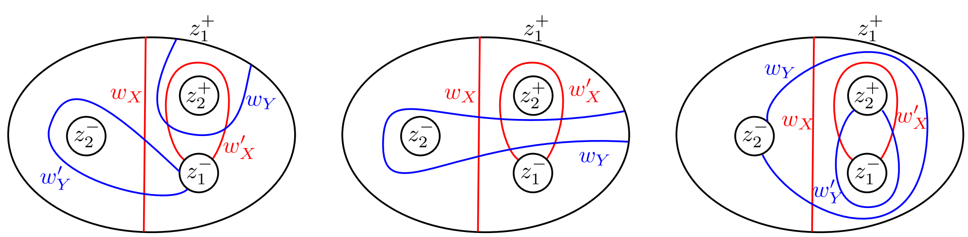

FIGURE 6. The three cases in the proof of Proposition 5.13. In any configuration, different waves generate at least four intersection points.

Proposition 5.14. *Suppose that $X, Y \in \mathcal{P}_2^{\mathrm{nm}}$ are joined by an edge, that X is not disjoint from Z, but Y is disjoint from Z. Then $\pi_Z(X) = \pi_Z(Y)$.*

Proof. Since X is not disjoint from Z, it has a pair of waves w_X, w'_X as in Lemma 4.3. Y differs from X by exchanging a single curve of X. Since Y is disjoint from Z but X is not, two curves x_1, x_2 of X are disjoint from Z, while a third one x_3 contributes the waves. The pair of curves $\{x_1, x_2\}$ which is disjoint from Z has to be distinct from Z as otherwise there could not be any waves. Hence, X and Z have precisely one curve in common, say x_1. The other curve x_2, being disjoint from one of the curves in Z and the waves, is then necessarily the surgery along that wave (see Figure 7).

The move from X to Y replaces the curve x_3 contributing the waves, and therefore keeps x_2 – which will be the projection of both X and Y. □

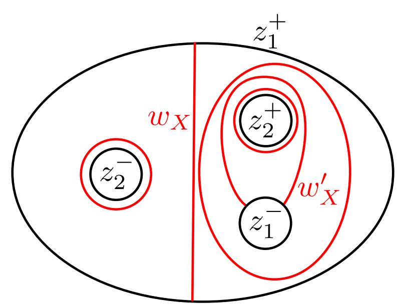

Figure 7. The situation in Proposition 5.14. X has two waves w_X, w'_X contributed by $x_3 \in X$ and one curve $x_1 = z_2$ in common. The remaining curve $x_2 \in X$ in X is then necessarily the surgery at the wave. Doing a single move to make X disjoint from Z will replace the curve x_3 contributing the wave, keeping x_2 and therefore the projection fixed.

Together, these propositions can be rephrased as saying that the projection π_Z from $\mathcal{P}_2^{\mathrm{nm}}$ to $\mathcal{P}_2^{\mathrm{nm}}(Z)$ can only change along a path while that path is actually contained within $\mathcal{P}_2^{\mathrm{nm}}(Z)$.

Theorem 5.15. *The graph $\mathcal{P}_2^{\mathrm{nm}}$ is a tree.*

Proof. The proof is very similar to the proof of Theorem 5.9. We have already seen connectivity of $\mathcal{P}_2^{\mathrm{nm}}$ in Lemma 5.12.

Next, suppose that $Q_1, \ldots, Q_k$ is a path in $\mathcal{P}_2^{\mathrm{nm}}$ so that for some Z we have that $Q_i \in \mathcal{P}_2^{\mathrm{nm}}(Z)$ exactly for $i = 1, k$. Then, by Proposition 5.14 we have $Q_1 = \pi_Z(Q_1) = \pi_Z(Q_2)$ and $\pi_Z(Q_{k-1}) = \pi_Z(Q_k) = Q_k$. Inductively applying Proposition 5.13 we then see that $\pi_Z(Q_i)$ is constant and $Q_1 = Q_k$.

Now let $P_1, \ldots, P_n$ be an embedded cycle in $\mathcal{P}_2^{\mathrm{nm}}$, and let $Z = P_1 \cap P_2$. Observe that Z is a cut system, and that P_1, P_2 are two distinct points in $\mathcal{P}_2^{\mathrm{nm}}(Z)$. Since we know that $\mathcal{P}_2^{\mathrm{nm}}(Z)$ is a tree, the embedded cycle (P_i) cannot be completely contained in $\mathcal{P}_2^{\mathrm{nm}}(Z)$. Hence, there is a smallest index r so that $P_r \notin \mathcal{P}_2^{\mathrm{nm}}(Z)$. If there would be a index $r < s \leq n$ so that $P_s \in \mathcal{P}_2^{\mathrm{nm}}(Z)$, then the argument of the previous paragraph, applied to $P_{r-1}, P_r, \dots, P_s$ (for the smallest such s) would yield $P_{r-1} = P_s$, which contradicts that the cycle is embedded.

But otherwise, we can apply the argument to the path $P_2, \dots, P_n, P_1$ to conclude $P_1 = P_2$, which also contradicts that the cycle is embedded.

This shows that $\mathcal{P}_2^{\mathrm{nm}}$ is connected but admits no embedded cycles and therefore is a tree. □

5.3. **Controlling Twists.** In this subsection we study how subsurface projections to annuli around non-separating meridians α behave along geodesics in $\mathcal{P}_2^{\mathrm{nm}}$. We begin with the following lemma, which is likely known to experts.

Lemma 5.16. *i) Suppose that $Y \subset \partial V$ is a subsurface which is not an annulus or pair of pants, and that $\alpha \subset Y$ is an essential (in Y) simple closed curve.*

Suppose that β_1, β_2 are two curves which intersect ∂Y, and suppose further that there is an arc b in Y which intersects α and so that there are subarcs $b_i \subset Y \cap \beta_i$ which are isotopic to b . Then

$$d_\alpha(\beta_1, \beta_2) \leq 5$$

(here, the subsurface distance d_α is seen as curves on S, not Y).

ii) Suppose that $Y \subset \partial V$ is a four-holed sphere, and w is an arc joining one boundary ∂_0 of Y to itself. Suppose that $\alpha \subset Y$ is disjoint from w and is not homotopic to a boundary of Y.

Suppose that β_1, β_2 are two curves (on S) which intersect α and ∂_0, and so that there are subarcs $b_i \subset \beta_i$ with endpoints on w, ∂_0 and interior disjoint from w, ∂_0 which are homotopic as arcs of this form. Then

$$d_\alpha(\beta_1, \beta_2) \leq 5$$

(here, the subsurface distance d_α is seen as curves on S, not Y).

Proof. i) Up to isotopy we may assume that the curves β_i both actually contain b (and are in minimal position with respect to themselves, α and ∂Y).

Let $S_\alpha \to \partial V$ be the annular cover corresponding to α, and let $\hat{\alpha}$ be the unique closed lift of α. Suppose that b joins components δ_1, δ_2 of ∂Y.

Consider a lift $\hat{b}$ of b which intersects $\hat{\alpha}$. Its endpoints are contained in lifts $\hat{\delta}_i$ of the curves δ_i. Observe that both the lifts $\hat{\delta}_i$ ($i = 1, 2$) do not connect different boundary components of the annulus S_α (as the curves δ_i are disjoint from α), and therefore $\hat{\delta}_i$ bounds a disk $D_i \subset S_\alpha$ whose closure in the closed annulus $\overline{S_\alpha}$ intersects the boundary of $\overline{S_\alpha}$ in a connected subarc.

Now consider lifts $\hat{\beta}_j$ which contain the arc $\hat{b}$. These are concatenations of an arc in D_1, the arc $\hat{b}$, and an arc in D_2. As the arcs in D_i can intersect in at most one point (otherwise, minimal position of

β_1, β_2 would be violated!), this implies that there are two lifts of β_i which intersect in at most 2 points. This shows part i).

ii) Assume as before that β_1, β_2 are in minimal position with respect to ∂_0, w and that $b \subset \beta_i$ is a common arc with endpoints on ∂_0, w as in the prerequisite. Note that $b = b_0 \cup b_1 \cup b_2$ where b_0 joins ∂_0 to α, and b_1 joins α to itself, and b_2 joins α to w.

Consider the annular cover corresponding to α, and let $\hat{\alpha}$ be the unique closed lift of α. Let $\hat{b} = \hat{b}_0 \cup \hat{b}_1 \cup \hat{b}_2$ be a lift of b which intersects $\hat{\alpha}$. Then observe that $\hat{b_0}$ joins a lift $\hat{\partial}_0$ of ∂_0 to $\hat{\alpha}$. Also, by minimal position $\hat{b}_1$ joins $\hat{\alpha}$ to a different lift $\hat{\alpha}'$ of α, which is then necessarily not closed. Arguing as above this implies that there are lifts $\hat{\beta}_1, \hat{\beta}_2$ which have endpoints in the disjoint disks bounded by $\hat{\partial}_0$, $\hat{\alpha}'$ and the common arc $\hat{b}$. As above, this implies that the twist is uniformly small.

□

We can use this lemma to prove the following result how subsurface projections π_α into annuli around meridians change along $\mathcal{P}_2^{\mathrm{nm}}$–geodesics if these geodesics never involve the curve α. This should be seen as the direct analog of the bounded geodesic projection theorem and its variants for hierarchies which are developed in [MM].

Proposition 5.17. *Suppose that X_i is a geodesic in $\mathcal{P}_2^{\mathrm{nm}}$, and that α is a non-separating meridian. Suppose none of the pants decompositions X_i contain α. Then the subsurface projection $\pi_\alpha(X_i)$ is coarsely constant along X_i: there is a constant K independent of α and the sequence, so that*

$$d_\alpha(X_i, X_j) \leq K.$$

Proof. First observe that as none of the X_i contain α, actually all X_i intersect α, as the X_i are pants decompositions. Let Z be a cut system completing α to a pants decomposition.

Observe that since $\mathcal{P}_2^{\mathrm{nm}}(Z)$ is a subtree of $\mathcal{P}_2^{\mathrm{nm}}$, the intersection of the geodesic X_i with $\mathcal{P}_2^{\mathrm{nm}}(Z)$ is a path, say $X_i, k \leq i \leq k'$.

Since X_1 intersects α, we may assume that there is a wave w of X_1 with respect to the cut system Z, and this wave w intersects α. For $i = 1, \ldots, k-1$, we have $X_i \notin \mathcal{P}_2^{\mathrm{nm}}(Z)$, and by Proposition 5.13 the pants decompositions X_i will therefore also all have the same wave w with respect to Z. By Lemma 5.16 i) this implies that the subsurface projection π_α is coarsely constant for $X_1, \ldots, X_{k-1}$ where k is the first index so that $X_k \in \mathcal{P}_2^{\mathrm{nm}}(Z)$. The projections of X_{k-1} and X_k are uniformly close since X_{k-1}, X_k are disjoint and both intersect α. Similarly, the projections of $X_{k'}, X_{k'+1}$ are uniformly close, and arguing with Proposition 5.13 and Lemma 5.16 i) as above, we see that the subsurface projection π_α is also coarsely constant for $i \geq k'$.

Hence, it suffices to show the statement of the proposition for paths which are completely contained in $\mathcal{P}_2^{\mathrm{nm}}(Z)$.

So, consider a path X_j in $\mathcal{P}_2^{\mathrm{nm}}(Z)$ which is never disjoint from a curve $\alpha \subset \partial V - Z$. Let w be a wave corresponding to α, in the sense that one component of $Z - w$ has boundary homotopic to α. Then, since all X_i are assumed to intersect α, they all intersect w, and so the projection $\pi_w(X_i)$ is constant by Proposition 5.7. This means that there is a arc a with one endpoint on w and the other on the curve of Z on which w ends, with the following property: all X_i contain a subsegment a_i which is homotopic to a among such arcs.

Now, using Lemma 5.16 ii) we conclude that that $\pi_\alpha(X_i)$ is coarsely constant as claimed. □

Finally, we study the case where α does appear as one of the curves along a $\mathcal{P}_2^{\mathrm{nm}}$–geodesic.

Corollary 5.18. *Let α be a non-separating meridian, and X_i be a geodesic in $\mathcal{P}_2^{\mathrm{nm}}$, which does become disjoint from α. Then there are $i_0 \leq i_1$ so that the following holds:*

i) For $i < i_0$ the subsurface projection $\pi_\alpha(X_i)$ is coarsely constant.
ii) For $i_0 \leq i \leq i_1$, the curve α is contained in X_i.
iii) For $i > i_0$ the subsurface projection $\pi_\alpha(X_i)$ is coarsely constant.

Proof. In light of the previous Proposition 5.17 the only thing that remains to show is that an interval $i_0 \leq i_1$ exists with the property that X_i contains α exactly for $i_0 \leq i \leq i_1$. This follows since the set $\mathcal{P}_2^{\mathrm{nm}}(\alpha)$ of non-separating meridional pants decompositions containing α is the union of $\mathcal{P}_2^{\mathrm{nm}}(Z)$ for Z a cut system containing α, which is a connected union of subtrees, hence itself a subtree. Therefore, a geodesic in $\mathcal{P}_2^{\mathrm{nm}}$ intersects $\mathcal{P}_2^{\mathrm{nm}}(\alpha)$ in a path. □

6. A geometric model for $\mathcal{H}_2$

In this section we define a geometric model for the handlebody group (which is very similar to the one employed in [HH2]) and use the results from Section 5 in order to study the geometry of the genus 2 handlebody group. A first step is the following lemma.

Lemma 6.1. *Suppose that $X \in \mathcal{P}_2^{\mathrm{nm}}$ is a pants decomposition, and $X = \{\delta_1, \delta_2, \delta_3\}$. Given $i \in \{1, 2, 3\}$ there is a curve β_i with the following properties:*

i) β_i is a non-separating meridian.
ii) δ_i and β_j are disjoint for $i \neq j$.
iii) δ_i and β_i intersect in exactly two points.

Furthermore, the curve β_i is uniquely defined by these properties up to Dehn twist about the curve δ_i.

Proof. Assume without loss of generality that $i = 3$. Consider the surface S obtained by cutting ∂V at δ_1, δ_2 as in Section 4. The curve δ_3 defines an admissible wave w as in Lemma 5.1, and by the same lemma any curve β_3 with the desired properties will be defined by an admissible wave w' which

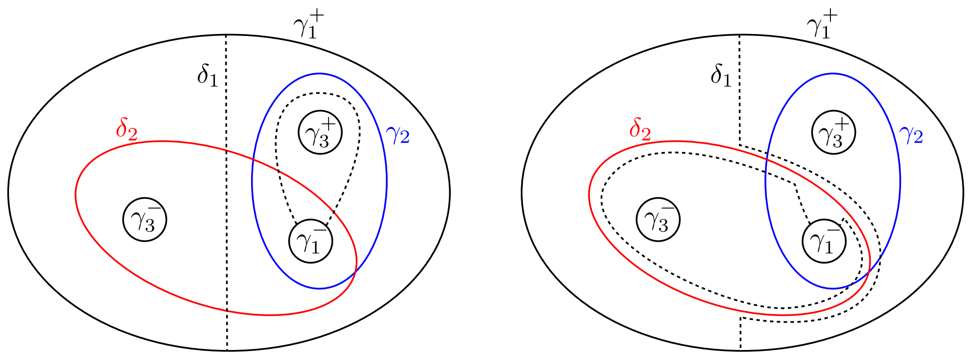

FIGURE 8. The cleanup move for dual curves in a switch.

is disjoint from w. Arguing as in Lemma 5.6, such an admissible wave w' exists and is unique up to Dehn twist in δ_3. This shows both claims of the lemma. ☐

Definition 6.2. If $X \in \mathcal{P}_2^{\mathrm{nm}}$, we call a curve β_i *dual to* $\gamma_i \in X$ if it satisfies the conclusion of Lemma 6.1. A set $\Delta = \{\beta_1, \beta_2, \beta_3\}$ containing a dual to each $\gamma_i \in X$ is called a *dual system* to X.

Since the handlebody group acts transitively on pants decompositions consisting of non-separating meridians, we see that the handlebody group also acts transitively on pairs (X, Δ) where $X \in \mathcal{P}_2^{\mathrm{nm}}$ and Δ is a dual system to X.

Next, we will describe a procedure to canonically modify the dual system when changing the pants decomposition X to an adjacent one X' in $\mathcal{P}_2^{\mathrm{nm}}$. Suppose that $X \in \mathcal{P}_2^{\mathrm{nm}}$ is a pants decomposition, and that Δ is a dual system. Suppose that $\delta \in \Delta$ is the dual curve to a curve $\gamma \in X$. Then the system $X' = X \cup \{\delta\} \setminus \{\gamma\}$ obtained by swapping γ for δ is also a pants decomposition consisting of non-separating meridians, and defines a vertex X' adjacent to X in $\mathcal{P}_2^{\mathrm{nm}}$. We say that X' is obtained from (X, Δ) by *switching* γ.

The curve γ is dual to δ in X' in the sense of Lemma 6.1. However, the other curves δ_1, δ_2 of $\Delta \setminus \{\delta\}$ are not dual to any curve in X' – each of them will intersect δ in four points.

The following lemma will allow us to clean the situation up in a controlled way.

Lemma 6.3. *Suppose that X is a pants decomposition, and Δ is a dual system. Let $\gamma \in X$ be given, and suppose X' is obtained from (X, Δ) by switching γ.*

Let $\gamma' \in X$ be distinct from γ, and $\delta' \in \Delta$ its dual. Then there is a dual curve $c(\delta')$ to γ' for the system X', and the assignment $\delta' \mapsto c(\delta')$ commutes with Dehn twists about γ'.

Proof. Suppose that $X = \{\gamma_1, \gamma_2, \gamma_3\}$, and that $\Delta = \{\delta_1, \delta_2, \delta_3\}$, so that δ_i is dual to γ_i. Assume that we switch γ_2. Consider the surface S (as in Section 5) obtained as the complement of the cut system $\{\gamma_1, \gamma_3\}$. Then both γ_2, δ_2 are contained in S and intersect in two points.

The dual curve δ_1 defines two waves w, w' with respect to X. Consider w, and note that it intersects δ_2 in two points (compare Figure 8). There are two ways of surgering w in the direction of δ_2, i.e. replacing a subarc of w by a subarc of δ_2. Exactly one of them yields an essential wave, which we denote by v. Note that v has the same endpoints as w. We define v' similarly for the other wave w'. We define $c(\delta_1)$ to be the curve $v \cup v'$. It has two (essential) intersection points with γ_1 by construction, and is indeed nonseparating since it defines admissible waves (compare Lemma 5.1).

To see that the map c commutes with Dehn twists about γ_1, it suffices to note that such a Dehn twist can be supported in a small neighbourhood of γ_1, and therefore the assignment of v, v' to δ_1 commutes with Dehn twists by construction. □

We emphasise that the assignment $\delta' \to c(\delta')$ guaranteed by Lemma 6.3 is not unique – one could e.g. modify it by Dehn twists and keep all desired properties. To make choices well-defined in the sequel, we fix assignments $c = c_{X,X'}$ as in Lemma 6.3 (for all choices of X, X' as in that lemma). for the rest of the section. By slight abuse of notation we denote all of these assignments by the same symbol c.

Definition 6.4. Suppose that $X \in \mathcal{P}_2^{\mathrm{nm}}$ is a pants decomposition, that Δ is a dual system, and $\gamma \in X$ is given. We then say that (X', Δ') *is obtained from* (X, Δ) *by switching* γ if the following hold:

- i) $X' = X \cup \{\delta\} \setminus \{\gamma\}$.
- ii) The dual curve to δ is γ.
- iii) The dual curves to both other $\delta' \in X'$ are obtained from the dual curves in Δ by the map c from Lemma 6.3.

We record the following immediate corollary of the uniqueness statement in Lemma 6.3, which states that twisting about a curve different from γ commutes with switching γ.

Corollary 6.5. *Suppose that $X \in \mathcal{P}_2^{\mathrm{nm}}$ is given, γ, γ' are two curves in X. If (X', Δ') is obtained from (X, Δ) by switching γ, then $(X', T_{\gamma'}\Delta')$ is obtained from $(X, T_{\gamma'}\Delta)$ by switching γ.*

We can now define our geometric model for the handlebody group of genus 2.

Definition 6.6. The graph $\mathcal{M}_2^{\mathrm{nm}}$ has vertices corresponding to pairs (X, Δ), where X is a vertex of $\mathcal{P}_2^{\mathrm{nm}}$, and Δ is a dual system to X. There are two types of edges:

Twist: Suppose that $X = \{\gamma_1, \gamma_2, \gamma_3\}$ is a vertex of $\mathcal{P}_2^{\mathrm{nm}}$, and Δ is a dual system. Then we join, for any i

$$(X, \Delta) \quad \text{and} \quad (X, T_{\gamma_i}^{\pm 1}\Delta)$$

by edges $e_i^{\pm}$. We call these *twist edges* and say that γ_i is *involved in* $e_i^{\pm}$.

If γ is a curve that is involved in two oriented twist edges e, e' we say that it is *involved with consistent orientation* if the corresponding Dehn twist has the same sign in both cases.

Switch: Suppose that $X \in \mathcal{P}_2^{\mathrm{nm}}$ is a vertex, and Δ is a dual system to X. Suppose further that (X', Δ') is obtained from (X, Δ) by switching some $\gamma \in X$.

We then connect (X, Δ) and (X', Δ') by an edge e. We say that e is a *switch edge*, and that it *corresponds to the edge between* X *and* X' *in* $\mathcal{P}_2^{\mathrm{nm}}$.

Proposition 6.7. *The handlebody group* $\mathcal{H}_2$ *acts on* $\mathcal{M}_2^{\mathrm{nm}}$ *properly discontinuously and cocompactly.*

Proof. The quotient of $\mathcal{M}_2^{\mathrm{nm}}$ by the handlebody group is finite, since $\mathcal{H}_2$ acts transitively on the vertices of $\mathcal{P}_2^{\mathrm{nm}}$, and the group generated by Dehn twists about X act transitively on dual systems of X. Since for a vertex (X, Δ) the union $X \cup \Delta$ cuts the surface into simply connected regions, the stabilizer of any vertex of $\mathcal{M}_2^{\mathrm{nm}}$ is finite. ☐

6.1. **Cubical Structure.** In this section we will turn $\mathcal{M}_2^{\mathrm{nm}}$ into the 1–skeleton of a CAT(0) cube complex.

In order to do so, we will glue in two types of cubes into $\mathcal{M}_2^{\mathrm{nm}}$. For the first, fix some $X \in \mathcal{P}_2^{\mathrm{nm}}$, and consider the subgraph $\mathcal{M}_2^{\mathrm{nm}}(X)$ spanned by those vertices whose corresponding pair has X as its first entry. By definition, any edge in $\mathcal{M}_2^{\mathrm{nm}}(X)$ is a twist edge, and in fact $\mathcal{M}_2^{\mathrm{nm}}(X)$ is isomorphic as a graph to the standard Cayley graph of $\mathbb{Z}^3$. We call the subgraphs $\mathcal{M}_2^{\mathrm{nm}}(X)$ *twist flats*. We then glue standard Euclidean cubes to make $\mathcal{M}_2^{\mathrm{nm}}(X)$ the 1-skeleton of the standard integral cube complex structure of $\mathbb{R}^3$. We call these cubes *twist cubes*.

The second kind of cubes will involve switch edges, and to describe them we first need to understand all the switch edges corresponding to a given edge between X, X' in $\mathcal{P}_2^{\mathrm{nm}}$. Let $\{\alpha_1, \alpha_2\} = X \cap X'$ be the two curves that the two pants decompositions have in common, and suppose γ is switched to γ'. Then the possible switch edges will join vertices

$$\big((\alpha_1, \alpha_2, \gamma), (\delta_1, \delta_2, \gamma')\big) \quad \text{to} \quad \big((\alpha_1, \alpha_2, \gamma'), (c(\delta_1), c(\delta_2), \gamma)\big)$$

Note that since δ_1, δ_2 are unique up to Dehn twists about α_1, α_2 (Lemma 6.1), and the map c commutes with twists (Lemma 6.3), we conclude that the switch edges corresponding to the edge between X, X' are exactly the edges between

$$\big((\alpha_1, \alpha_2, \gamma), (T_{\alpha_1}^{n_1}\delta_1, T_{\alpha_2}^{n_2}\delta_2, \gamma')\big) \quad \text{and} \quad \big((\alpha_1, \alpha_2, \gamma'), (T_{\alpha_1}^{n_1}c(\delta_1), T_{\alpha_2}^{n_2}c(\delta_2), \gamma)\big)$$

for any n_1, n_2. Hence, in $\mathcal{M}_2^{\mathrm{nm}}$, there is a copy of the 1–skeleton of a 3–cube with vertices

$$(X, \Delta), (X, T_{\alpha_1}\Delta), (X, T_{\alpha_2}\Delta), (X, T_{\alpha_1}T_{\alpha_2}\Delta),$$
$$(X', \Delta'), (X', T_{\alpha_1}\Delta'), (X', T_{\alpha_2}\Delta'), (X', T_{\alpha_1}T_{\alpha_2}\Delta'),$$

and we glue in a *switch cube* at this 1–skeleton. Similarly, we glue in three more switch cubes for the different possibilities of replacing T_{α_1} and/or T_{α_2} by their inverses.

For later reference, observe that by construction any square in our cube complex has either only twist edges, or exactly two nonadjacent switch edges in its boundary. This fairly immediately implies the following.

Proposition 6.8. *The link of any vertex in the cube complex $\mathcal{M}_2^{\mathrm{nm}}$ is a flag simplicial complex.*

Proof. Since the link is a 2-dimensional simplicial complex, we only have to check that any boundary of a triangle in the 1–skeleton of the link bounds a triangle in the link. Vertices in the link correspond to edges in $\mathcal{M}_2^{\mathrm{nm}}$ and are therefore of twist or switch type. Edges in the link are due to squares in the cubical structure, and by the remark above any edge has either both endpoints of twist type, or exactly one of switch type. Thus, there are only two types of triangles in the link: those were all three vertices are of twist type, and those where exactly one of the vertices is of switch type. But, in both of these cases, the three corresponding edges in $\mathcal{M}_2^{\mathrm{nm}}$ are part of a common twist or switch cube, and therefore the desired triangle in the link exists. □

Proposition 6.9. *The cube complex $\mathcal{M}_2^{\mathrm{nm}}$ is connected and simply-connected.*

Proof. The fact that $\mathcal{M}_2^{\mathrm{nm}}$ is connected is an easy consequence of the fact that the tree $\mathcal{P}_2^{\mathrm{nm}}$ is connected, and each twist flat $\mathcal{M}_2^{\mathrm{nm}}(X)$ is connected as well.

Now, suppose that $g(i) = (X_i, \Delta_i)$ is a simplicial loop in $\mathcal{M}_2^{\mathrm{nm}}$. Then, the path X_i is a loop in $\mathcal{P}_2^{\mathrm{nm}}$, and since the latter is a tree, it has backtracking. Thus we can write g as a concatenation

$$g = g_1 * \sigma_1 * \tau * \sigma_2 * g_2$$

where σ_1, σ_2 are two switch edges corresponding to an edge from some vertex X to another vertex X', and from X' back to X, respectively, and τ is a path consisting only of twist edges. In fact, in order for $\sigma_1 * \tau * \sigma_2$ to be a path, the total twisting about the curve $\{\alpha\} = X' \setminus X$ has to be zero. Since the twist flat $\mathcal{M}_2^{\mathrm{nm}}(X')$ is homeomorphic to $\mathbb{R}^3$ in our cubical structure, we may therefore homotope the path so that τ does not twist about α at all.

Next, consider the first twist edge t_1 in τ. Then, $\sigma_1 * t_1$ are two sides of a square in a switch cube, and thus g is homotopic to a path

$$g = g_1 * t_1' * \sigma_1' * \tau' * \sigma_2 * g_2$$

where now τ' has strictly smaller length than τ, and σ_1' and σ_2 still correspond to opposite orientations of the same edge in $\mathcal{P}_2^{\mathrm{nm}}$. By induction, we can reduce the length of τ' to zero, in which case g will have backtracking. An induction on the length of g then finishes the proof. □

Hence, using Gromov's criterion (e.g. [BH, Chapter II, Theorem 5.20]) we conclude:

Corollary 6.10. *$\mathcal{M}_2^{\mathrm{nm}}$ is a* CAT(0) *cube complex.*

Remark 6.11. *By our construction, the genus* 2 *handlebody group acts by semisimple isometries on a complete* CAT(0)*-cube complex of dimension* 3*. This should be contrasted to the main result of* [Bri] *which shows that any action of the mapping class group of a closed surface of genus $g \geq 2$ on a complete* CAT(0)*-space of dimension less than g fixes a point.*

Corollary 6.12. *The genus* 2 *handlebody group $\mathcal{H}_2$ is biautomatic, and has a quadratic isoperimetric inequality.*

Proof. From [Ś, Corollary 8.1] we conclude biautomaticity, since $\mathcal{H}_2$ acts properly discontinuously and cocompactly on the CAT(0) cube complex $\mathcal{M}_2^{\mathrm{nm}}$ (Proposition 6.7 and Corollary 6.10). This implies that $\mathcal{H}_2$ has at most quadratic Dehn function [BGSS, ECH$^+$]. Observe that since $\mathcal{H}_2$ contains copies of $\mathbb{Z}^2$ (generated by Dehn twists about disjoint meridians) it is not hyperbolic, and therefore its Dehn function cannot be sub-quadratic [Gro]. □

Using Proposition 1 of [CMV] we also conclude

Corollary 6.13. *The genus* 2 *handlebody group has the Haagerup property.*

6.2. **Other geometric consequences.** The geometric model $\mathcal{M}_2^{\mathrm{nm}}$ for the genus 2 handlebody group can also be used to conclude other facts about $\mathcal{H}_2$. For example, we have the following distance estimate in $\mathcal{M}_2^{\mathrm{nm}}$, which should be compared to the Masur-Minsky distance formula for the surface mapping class group from [MM].

Proposition 6.14. *There are constants $c, C > 0$ so that for all pairs of vertices $(X, \Delta_X), (Y, \Delta_Y) \in \mathcal{M}_2^{\mathrm{nm}}$ we have*

$$d_{\mathcal{M}_2^{\mathrm{nm}}}((X, \Delta_X), (Y, \Delta_Y)) \simeq_c d_{\mathcal{P}_2^{\mathrm{nm}}}(X, Y) + \sum_{\alpha} [d_\alpha(X \cup \Delta_X, Y \cup \Delta_Y)]_C$$

where the sum is taken over all non-separating meridians α. Here, $\simeq_c$ means that the equality holds up to a (uniform) multiplicative and additive constant c, and $[\cdot]_C$ means that the term only appears if the argument is at least C.

Proof. Consider a geodesic $g : [0, l] \to \mathcal{M}_2^{\mathrm{nm}}$ joining (X, Δ_X) to (Y, Δ_Y) in $\mathcal{M}_2^{\mathrm{nm}}$. We need to estimate the length l of g. First, we claim that the projection of g to $\mathcal{P}_2^{\mathrm{nm}}$ is a path without backtracking in the tree $\mathcal{P}_2^{\mathrm{nm}}$ (but possibly with intervals on which it is constant).

Namely, suppose that this is not the case. Then, as the projection X_i of g backtracks, we can write g as a concatenation

$$g = g_1 * \sigma * \tau * \sigma' * g_2$$

where σ, σ' are two switch edges corresponding to opposite orientations of the same edge in $\mathcal{P}_2^{\mathrm{nm}}$, and τ is a path just consisting of twist edges. If σ switches a curve γ, note that we may assume that τ does not twist about γ. Namely, the total twisting about γ has to be zero in order for σ' to be able to follow τ, and therefore any twists about γ can be canceled without changing the length or the projection to $\mathcal{P}_2^{\mathrm{nm}}$ of g.

However, now the twists τ can be moved to the end of g_1 by Corollary 6.5 without changing the length of g or its endpoints. However, after this modification g has backtracking, which contradicts the fact that it is a geodesic. Similarly, arguing as above, we see that in a geodesic g all the twist edges involving a given curve α need to have consistent orientation, as otherwise the geodesic could be shortened.

Using Corollary 6.5 again, we may also assume that all twist edges involving the same curve α are adjacent in g, and appear immediately after α has become a curve in one of the X_i.

Now, the number of switch edges in g is exactly $d_{\mathcal{P}_2^{\mathrm{nm}}}(X, Y)$. It therefore suffices to argue that the number of twist edges can be estimated by the right-hand side of the equality in the proposition.

Fix some non-separating meridian α. If α never appears in X_i, then by Proposition 5.17 the projection into the annulus around α is coarsely constant. Hence, by choosing C large enough, these projections will not contribute to the sum in the statement of the proposition.

If α does appear in X_i, then by Corollary 5.18, it appears exactly for $i_0 \le i \le i_1$, and the projection before i_0 and after i_1 is coarsely constant. If α appears at i_0, and g performs n twists about α at this time, the projection π_α changes by a distance of n. For any subsequent switch edge corresponding to $X_{i_0+1}, \dots, X_{i_1}$, the projection can only change by a uniformly small amount in each cleanup move (given by Lemma 6.3) as the corresponding dual curves intersect in uniformly few points. In conclusion, we have that $d_\alpha(X \cup \Delta_X, Y \cup \Delta_Y)$ differs from n, where n is the length of the twist segment in g corresponding to α, by at most the length $e_\alpha = i_1 - i_0$. However, observe that

$$\sum_{\alpha \in X_i, i=1,\dots,k} e_\alpha \le 3k$$

since any vertex in the geodesic (X_i) can be in at most three of the intervals whose lengths are counted as the e_α. Hence, the sum of the error terms e_α is bounded by $d_{\mathcal{P}_2^{\mathrm{nm}}}(X, Y)$, showing the proposition. □

We can use the description of (quasi)geodesics in $\mathcal{H}_2$ obtained in the previous theorem to gain some qualitative understanding of the geometry of $\mathcal{H}_2$.

Corollary 6.15. *The stabilizer of a nonseparating meridian* δ *in* V_2 *is undistorted in* $\mathcal{H}_2$.

Proof. By the Svarc-Milnor theorem, the corollary amounts to showing that the subgraph $\mathcal{M}_2^{\mathrm{nm}}(\delta)$ of $\mathcal{M}_2^{\mathrm{nm}}$ formed by all those vertices (X, Δ) so that $\delta \subset X$ is undistorted in $\mathcal{M}_2^{\mathrm{nm}}$. Let g be a geodesic in $\mathcal{M}_2^{\mathrm{nm}}$ between two points in $\mathcal{M}_2^{\mathrm{nm}}(\delta)$. The discussion of geodesics in the proof of Proposition 6.14 shows that the projection of g to $\mathcal{P}_2^{\mathrm{nm}}$ is a geodesic $\overline{g}$ connecting two cut systems containing δ. Since $\mathcal{P}_2^{\mathrm{nm}}$ is a tree (Theorem 5.15) and the subgraph $\mathcal{P}_2^{\mathrm{nm}}(\delta)$ spanned by pants decompositions constaining δ is connected (e.g. by the proof of Lemma 5.12), this implies that every point on the path $\overline{g}$ also contains δ. This shows that g is already contained in $\mathcal{M}_2^{\mathrm{nm}}(\delta)$, which implies undistortion. □

Corollary 6.16. *There is a quasi-isometric embedding of* $\mathcal{H}_2$ *into a product of quasi-trees.*

Proof. Choose a pants decomposition P corresponding to a vertex of $\mathcal{P}_2^{\mathrm{nm}}$. Define a map

$$\Psi : \mathcal{H}_2 \to \mathcal{P}_2^{\mathrm{nm}} \times \mathrm{Mcg}(\Sigma_2)$$

sending

$$f \mapsto (fP, f)$$

where in the second coordinate we interpret $\mathcal{H}_2$ as a subgroup of $\mathrm{Mcg}(\Sigma_2)$. By Proposition 6.14, and the Masur-Minsky distance formula [MM] for the mapping class group it follows that Ψ is a quasi-isometric embedding for the product metric on $\mathcal{P}_2^{\mathrm{nm}} \times \mathrm{Mcg}(\Sigma_2)$. By Theorem 5.15, the factor $\mathcal{P}_2^{\mathrm{nm}}$ is already a tree. By [BBF2, Theorem 1.2] (compare also [Hum]), we can embed the second factor isometrically into a product of quasi-trees, showing the corollary. □

Remark 6.17. *In fact, using the projection complex machinery developed in* [BBF1], $\mathcal{H}_2$ *embeds into* $\mathcal{P}_2^{\mathrm{nm}} \times Y_1 \times \cdots \times Y_k$, *where each* Y_k *is a quasi-tree of metric spaces coming from applying the main construction of* [BBF1] *to the set of all non-separating meridians, and projections into annuli around them. Since we do not need this precise result, we skip the proof.*

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