HARMONIC MEASURES ARE PRODUCT MEASURES

URSULA HAMENSTÄDT

ABSTRACT. Let M be a closed negatively curved manifold with unit tangent bundle T^1M. Denote by X the geodesic spray on T^1M, i.e. the generator of the geodesic flow Φ^t. Let Δ^s be the stable Laplacian of a smooth Riemannian metric g on T^1M and let Y be a Hölder-continuous leafwise smooth section of the tangent bundle TW^s of the stable foliation which is g-dual to a leafwise closed one-form. We show that if the pressure of the function $g(X,Y)$ is negative, then there is a continuous bijection from the space of ergodic harmonic measures for the differential operator $\Delta^s + Y$, equipped with the weak*-topology, onto a space of ergodic Φ^t-invariant Borel probability measures of positive entropy on T^1M which contains all Gibbs equilibrium states of Hölder continuous functions.

1. Introduction

Let M be a closed n-dimensional Riemannian manifold of negative sectional curvature. The *geodesic flow* Φ^t is a smooth dynamical system on the unit tangent bundle T^1M of M, generated by the *geodesic spray* X. There is a Hölder-continuous, Φ^t-invariant decomposition

$$TT^1M = \mathbb{R}X \oplus TW^{su} \oplus TW^{ss}$$

where TW^{su} and TW^{ss} are the tangent bundles of the *strong unstable* and the *strong stable* foliation W^{su} and W^{ss}, respectively. The bundle $\mathbb{R}X \oplus TW^{ss} = TW^s$ is the tangent bundle of the *stable foliation* W^s. For every $v \in T^1M$ the leaf $W^s(v)$ through v of the stable foliation is a smoothly immersed submanifold of T^1M which is locally diffeomorphic to M. The stable foliation itself is in general not differentiable, but all its jet bundles are Hölder-continuous subbundles of the corresponding jet bundles of T^1M.

Given $\alpha > 0$, let g be a nonnegative bilinear form of class $C^{1,\alpha}$ on T^1M with the property that the restriction of g to the bundle TW^s is positive definite and hence defines a Riemannian metric. The restriction of this Riemannian metric to every leaf of W^s is of class C^1 and therefore g induces for every $v \in T^1M$ a Laplace operator on $W^s(v)$. Since g is of class $C^{1,\alpha}$, these leafwise Laplacians group together to form a differential operator Δ on T^1M with Hölder-continuous coefficients.

For $\alpha > 0$, call a section Y of TW^s over T^1M *of class* $C^{1,\alpha}_s$ if the restriction of Y to every leaf of W^s is of class C^1, with its leafwise differential depending Hölder-continuously on the leaf. Let Y be a section of TW^s of class $C^{1,\alpha}$ with the additional

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property that Y is g-dual to a section of T^*W^s over T^1M whose restriction to every stable manifold is a closed 1-form. Then $\Delta + Y$ is a second-order leafwise elliptic differential operator on T^1M with Hölder-continuous coefficients.

Let $\mathcal{M}$ be the compact convex space of all Φ^t-invariant Borel-probability measures on T^1M, equipped with the weak*-topology. Denote by h_ν the entropy of $\nu \in \mathcal{M}$. For a continuous function $f : T^1M \to \mathbf{R}$ the *pressure* $pr(f)$ is defined by $pr(f) = \sup \{h_\nu - \int f d\nu \mid \nu \in \mathcal{M}\}$. As in [H97a] we call the operator $\Delta + Y$ of *positive escape* if the pressure of the function $g(X,Y)$ is positive, and of *negative escape* if the pressure of the function $g(X,Y)$ is negative. The operators for which the pressure of the function $g(X,Y)$ vanishes are called *self-adjoint*. We refer to [H97a] for an explanation and justification of this terminology.

Extending earlier results of Ledrappier [L95a], we showed in [H97a] that an operator of positive escape shares many properties with an elliptic operator on a closed manifold. As an example, if $\Delta+Y$ is of positive escape then there is a unique Borel probability measure σ on T^1M which is *harmonic* for $\Delta + Y$, i.e. such that

$$\int (\Delta + Y)(f)\, d\sigma = 0$$

for every smooth function f on T^1M. Unlike in the case of an elliptic operator on a closed manifold, this measure is in general not contained in the Lebesgue measure class. Moreover, for every Hölder continuous function f on T^1M with $\int f d\sigma = 0$ there is a Hölder continuous leafwise-C^2 function h on T^1M, unique up to a constant, such that $(\Delta + Y)h = f$. The function h is Hölder continuous.

The purpose of this paper is to investigate dynamical properties of the leafwise diffusion of an operator of negative escape. We show that these properties correspond precisely to dynamical properties of the geodesic flow. For an exact statement of our main result, recall first that for every Hölder function f on T^1M there is a unique measure $\nu_f \in \mathcal{M}$ such that $pr(f) = h_{\nu_f} - \int f d\nu_f$, and this measure is called the *Gibbs equilibrium state* of f. In Section 3 we define a class of Φ^t-invariant Borel probability measures on T^1M which we call the *measures of quasi-product type.* This class contains all Gibbs equilibrium states of all Hölder continuous functions as well as all measures supported on a single closed orbit for the geodesic flow.

By the results of Garnett [G83], the space $\mathcal{H}_{\Delta+Y}$ of harmonic measures for $\Delta+Y$ is a nonempty compact convex subset of the space of all Borel probability measures on T^1M equipped with the weak*-topology. Every harmonic measure σ for $\Delta + Y$ is absolutely continuous with respect to the stable and strong unstable foliation, with conditional measures on stable manifolds in the Lebesgue measure class. The conditional measures on strong unstable manifolds define a measure class $mc(\sigma,\infty)$ on the *ideal boundary* $\partial\tilde{M}$ of the universal covering $\tilde{M}$ of M which is invariant under the natural action of the fundamental group $\pi_1(M)$ of M on $\partial\tilde{M}$. Call the harmonic measure σ *ergodic* if $mc(\sigma,\infty)$ is ergodic under the action of $\pi_1(M)$. Ergodic harmonic measures are just the extremal points of the set of all harmonic measures [H97a].

Now we can formulate the main result of this paper.

Theorem: *Let $\Delta + Y$ be as above with $pr(g(X,Y)) < 0$. Then there is a continuous bijection $\Psi(\Delta + Y)$ of the space of ergodic harmonic measures for $\Delta + Y$ onto the space of Φ^t-invariant ergodic Borel probability measures on T^1M of quasi-product type.*

The map $\Psi(\Delta + Y)$ simply assigns to an ergodic harmonic measure σ for $\Delta + Y$ a weak limit of the sequence of measures $\frac{1}{m}\sum_{i=0}^{m-1} \sigma \circ \Phi^{-i}$.

By convex combination, the map $\Psi(\Delta + Y)$ extends to an injective map from the compact space $\mathcal{H}_{\Delta+Y}$ into the space of Φ^t-invariant measures on T^1M. However, this map is not continuous. In other words, via the map $\Psi(\Delta + Y)$ we obtain a new topology on the space of Φ^t-invariant probability measures of quasi-product type so that the space equipped with this topology is compact.

The organization of the paper is as follows. In Section 2 we introduce a class of measures on T^1M containing the space $\mathcal{H}_{\Delta+Y}$. We show that equipped with the weak*-topology, this space is compact. In Section 3 we define the measures of quasi-product type and derive some first properties. Section 4 contains the proof that the space of measures of quasi-product type contains all Gibbs equilibrium states of a class of functions including all Hölder continuous functions. In Section 5 we use the measures of quasi-product type to construct measurable solutions of suitable leafwise differential operators with prescribed asymptotic behavior at infinity. The results from Section 6 are used in Section 6 to construct an injective map from the space of ergodic measures of quasi-product type into the space of ergodic harmonic measures of for $\Delta + Y$. In Section 7 we complete the proof our theorem.

As a notational convention, throughout the paper we often denote functions, measures and vector fields on T^1M and their $\pi_1(M)$-equivariant lifts to the unit tangent bundle $T^1\tilde{M}$ of $\tilde{M}$ by the same symbols.

2. k-FAMILIES

The purpose of this chapter is to define a class of measures containing all harmonic measures for a differential operator as in the introduction and to discuss some of their basic properties.

Let $\tilde{M}$ be a simply connected manifold of bounded negative sectional curvature $-\infty < -b^2 \leq K \leq -1$. Denote by dist the distance function on $\tilde{M}$ and by $T^1\tilde{M}$ the unit tangent bundle of $\tilde{M}$. There is a natural projection $\pi : T^1\tilde{M} \to \partial\tilde{M}$ of $T^1\tilde{M}$ onto the *ideal boundary* $\partial\tilde{M}$ of $\tilde{M}$ which maps a unit tangent vector to the equivalence class of the geodesic ray it defines. The restriction π_p of the map π to a fibre $T^1_p\tilde{M}$ of the fibration $P : T^1\tilde{M} \to \tilde{M}$ is a homeomorphism. Denote by λ the Lebesgue measure on $\tilde{M}$ defined by the metric.

Let Γ be a discrete torsion free cocompact group of isometries of $\tilde{M}$. Then $\tilde{M}/\Gamma = M$ is a compact smooth manifold of bounded negative curvature. We denote by T^1M the unit tangent bundle of M and by $P : T^1M \to M$ the canonical projection. We are interested in Borel measure classes on $\partial\tilde{M}$ which are invariant under the action of Γ and with some additional regularity properties which we define as follows.

Definition: For $k > 0$, a Γ-*k-family* on $\partial\tilde{M}$ is a family $\{\bar{\nu}^x\}$ $(x \in \tilde{M})$ of finite Borel measures on $\partial\tilde{M}$ with the following properties.

1. (Invariance) The measures $\{\bar{\nu}^x\}$ are equivariant under the action of Γ on $\partial\tilde{M} \times \tilde{M}$ and define all the same measure class $mc(\nu_+)$.
2. (Regularity) For $mc(\nu_+)$-almost every $\xi \in \partial\tilde{M}$ and all $x, y \in \tilde{M}$ the Radon Nikodym derivative $Q(x, y, \xi)$ of $\bar{\nu}^y$ with respect to $\bar{\nu}^x$ exists at ξ, and the function $y \to \log Q(x, y, \xi)$ is k-Lipschitz continuous. The function Q on $\tilde{M} \times \tilde{M} \times \partial\tilde{M}$ is called the *Radon Nikodym kernel* of the k-family $\{\bar{\nu}^x\}$.

The k-family is called *ergodic* if the measure class $mc(\nu_+)$ is ergodic under the action of Γ.

In the sequel we always denote by $\{\bar{\nu}^x\}$ a Γ-k-family on $\partial\tilde{M}$.

Lemma 2.1: *There is a number $k > 0$ with the following property. Let mc be a Γ-invariant measure class on $\partial\tilde{M}$; then there is a Γ-k-family $\{\bar{\nu}^x\}$ $(x \in \tilde{M})$ which defines mc.*

Proof: The *critical exponent* $\delta(\Gamma)$ of Γ is the infimum of all numbers $s > 0$ such that the *Poincaré series* $\sum_{\Psi\in\Gamma} e^{-s\,\mathrm{dist}(y,\Psi x)}$ converges for $x, y \in \tilde{M}$. Since the curvature of $\tilde{M}$ is bounded from above and below by a negative constant, the critical exponent is finite and positive and there is a number $s > \delta(\Gamma)$ such that for fixed $x \in \tilde{M}$ the value of the (convergent) series $\sum_{\Psi\in\Gamma} e^{-s\,\mathrm{dist}(y,\Psi x)}$ is bounded independent of $y \in \tilde{M}$ (this follows directly from the arguments of Sullivan [S79]).

Let mc be a Γ-invariant measure class on $\partial\tilde{M}$ and let μ be a Borel probability measure on $\partial\tilde{M}$ defining mc. Choose some $x \in \tilde{M}$ and define $\mu^x = \mu$ and $\mu^{\Psi x} = \mu \circ \Psi^{-1}$ for $\Psi \in \Gamma$.

Let $r > \delta(\Gamma)$ be such that the Poincaré series $\sum_{\Psi\in\Gamma} e^{-r\,\mathrm{dist}(y,\Psi x)}$ is bounded independent of y. For $y \in \tilde{M}$ define $\bar{\nu}^y = \sum_{\Psi\in\Gamma} e^{-r\,\mathrm{dist}(y,\Psi x)}\mu^{\Psi x}$. Then $\bar{\nu}^y$ is a Borel-measure on $\partial\tilde{M}$ which defines the measure class mc and whose total mass is bounded from above independent of y.

We claim that the measures $\bar{\nu}^y$ $(y \in \tilde{M})$ are equivariant under the action of Γ. To see this let $\zeta \in \Gamma$. Then

$$
\begin{aligned}
(1)\qquad \bar{\nu}^{\zeta y} &= \sum_{\Psi\in\Gamma} e^{-r\,\mathrm{dist}(\zeta y,\Psi x)}\mu^{\Psi x} = \sum_{\Psi\in\Gamma} e^{-r\,\mathrm{dist}(\zeta y,\zeta(\zeta^{-1}\Psi)x)}\mu\circ\Psi^{-1}\zeta\zeta^{-1} \\
&= (\sum_{\Psi\in\Gamma} e^{-r\,\mathrm{dist}(y,(\zeta^{-1}\Psi)x)}\mu\circ(\zeta^{-1}\Psi)^{-1})\circ\zeta^{-1} = \bar{\nu}^y\circ\zeta^{-1}.
\end{aligned}
$$

To show that $\{\bar{\nu}^y\}$ is indeed a Γ-k-family on $\partial\tilde{M}$ for some $k > 0$ not depending on mc it is enough to show that for mc-almost every $\xi \in \partial\tilde{M}$ and all $y, z \in \tilde{M}$ the

Radon Nikodym derivative of $\bar{\nu}^z$ with respect to $\bar{\nu}^y$ exists at ξ and is contained in $[e^{-r\,\mathrm{dist}(y,z)}, e^{r\,\mathrm{dist}(y,z)}]$. For this let $y, z \in \tilde{M}$ and let $\Psi \in \Gamma$. Then $|\mathrm{dist}(z, \Psi x) - \mathrm{dist}(y, \Psi x)| \leq \mathrm{dist}(z, y)$ and therefore the measures $\bar{\nu}^y, \bar{\nu}^x$ are absolutely continuous, with Radon Nikodym derivative contained in $[e^{-r\,\mathrm{dist}(y,z)}, e^{r\,\mathrm{dist}(y,z)}]$. Now let $\xi \in \partial\tilde{M}$ be such that for every $\Psi \in \Gamma$ the Radon Nikodym derivative $\frac{d\mu^{\Psi x}}{d\mu^x}$ exists at ξ. Then for every $\Psi \in \Gamma$ the Radon Nikodym derivative of the measure $e^{-r\,\mathrm{dist}(y,\psi x)}\mu^{\Psi x}$ with respect to $\bar{\nu}^y$ exists at ξ, and the sum of these Radon Nikodym derivatives is finite. But this just means that the Radon Nikodym derivative $\frac{d\bar{\nu}^z}{d\bar{\nu}^y}$ exists at ξ. This shows the lemma. □

For a Γ-k-family $\{\bar{\nu}^x\}$ on $\partial\tilde{M}$ the measures $\nu^x = \bar{\nu}^x \circ \pi_x$ $(x \in \tilde{M})$ on the fibres of the fibration $T^1\tilde{M} \to M$ are equivariant under the action of the group Γ and hence they project to a family of measures on the fibres of the fibration $T^1M \to M$. Define a *k-family on M* to be a family $\{\nu^p\}$ $(p \in M)$ of measures on the fibres of the fibration $T^1M \to M$ which are obtained from a Γ-k-family $\{\bar{\nu}^x\}$ on $\partial\tilde{M}$ in this way.

Recall from [GH90] that for $x \in \tilde{M}$ and $\xi, \zeta \in \partial\tilde{M}$ the *Gromov product* $(\xi \mid \zeta)_x$ of ξ and ζ as seen from x is defined by

$$(\xi \mid \zeta)_x = \lim_{y\to\xi, z\to\zeta} \frac{1}{2}(\mathrm{dist}(x, y) + \mathrm{dist}(x, z) - \mathrm{dist}(y, z)). \tag{2}$$

For $x \in \tilde{M}$ and $v \neq w \in T^1_x\tilde{M}$ define $(v \mid w) = (\pi(v) \mid \pi(w))_x$. We then have $(v \mid -v) = 0$ and $(v \mid w) > 0$ for $w \in T^1_x\tilde{M} - \{v, -v\}$. There are numbers $a > 0, \rho > 0$ and for every $x \in \tilde{M}$ there is a distance δ_x on $T^1_x\tilde{M}$ which satisfies $ae^{-\rho(v|w)} \leq \delta_x(v, w) \leq e^{-\rho(v|w)}$ for all $v, w \in T^1_x\tilde{M}$. Since the curvature of M is bounded from above by -1 by assumption we can choose $\rho = 1$ (compare e.g. [H89]). This means that there is a constant $c > 0$ with the following property. Let $v, w \in T^1\tilde{M}$ with $\delta_x(v, w) = e^{-t}$ for some $t \geq 0$. Let γ be the geodesic in $\tilde{M}$ connecting $\pi(v)$ to $\pi(w)$; then $t - c \leq \mathrm{dist}(\gamma, x) \leq t + c$.

The distances δ_x can be chosen to be invariant under the action of Γ and therefore they project to a family of distances on the fibres of the fibration $T^1M \to M$ which we denote again by $\{\delta_x\}$ $(x \in M)$. For $v \in T^1M$ and $r > 0$ we denote by $B(v, r)$ the open ball of radius r about v in $(T^1_{Pv}M, \delta_{Pv})$. The next lemma estimates the mass of these balls with respect to any k-family.

Lemma 2.2: *Let $\{\nu^p\}$ be a k-family on M. Then for every $\epsilon > 0$ the function $v \in T^1M \to \nu^{Pv}(B(v, \epsilon))$ on T^1M is bounded from above and below by a positive constant.*

Proof: Choose a number $\epsilon > 0$. Let $\{\nu^p\}$ be a k-family on M and for $v \in T^1M$ define $\beta(v) = \nu^{Pv}(B(v, \epsilon))$. Since the measure class on $\partial\tilde{M}$ defined by the k-family $\{\nu^p\}$ is invariant under the action of Γ and every Γ-orbit on $\partial\tilde{M}$ is dense, the measures ν^p on the fibres of the fibration $T^1M \to M$ have full support. Thus β is a positive measurable function on T^1M.

Let $\{v_i\}_i \subset T^1M$ be a sequence such that $\beta(v_i) \to \inf\{\beta(w) \mid w \in T^1M\}$. By passing to a subsequence we may assume that the vectors v_i converge in T^1M to a vector v. Write $\rho = \nu^{Pv}(B(v,\epsilon/2)) > 0$. Let $\tilde{v}$ be a lift of v to $T^1\tilde{M}$ and let $\{\tilde{v}_i\}_i$ be a sequence of lifts of $\{v_i\}_i$ which converges to $\tilde{v}$. Denote by $\{\bar{\nu}^x\}$ the Γ-k-family on $\partial\tilde{M}$ which corresponds to $\{\nu^p\}$. By definition of a Γ-k-family, there is a number $c > 0$ such that for every sufficiently large $i > 0$ the Radon Nikodym derivative of $\bar{\nu}^{P\tilde{v}_i}$ with respect to $\bar{\nu}^{P\tilde{v}}$ is bounded from below on $\pi B(\tilde{v},\epsilon/2)$ by c. On the other hand, for sufficiently large i the set $\pi B(\tilde{v},\epsilon/2)$ is contained in $\pi B(\tilde{v}_i,\epsilon)$. This means that $\nu^{P\tilde{v}_i}B(\tilde{v}_i,\epsilon) \geq c\rho$ and therefore β is bounded from below on T^1M by a positive constant. In the same way we obtain that β is uniformly bounded from above as well. □

Lemma 2.2 and its proof means that the measures ν^p of a k-family $\{\nu^p\}$ are *locally uniformly bounded* on the fibres of the bundle $T^1M \to M$. Moreover, for every k-family $\{\nu^p\}$ and every $p \in M$ the total mass of the measure ν^p on T^1_pM is bounded from above and below by a positive constant only depending on k. In particular, we obtain a finite Borel measure ν_L on T^1M by defining $\nu_L(A) = \int \nu^p(A \cap T^1_pM)d\lambda(p)$ where λ is the Lebesgue measure on M defined by the metric. We call ν_L the *Lebesgue extension* of the k-family. The k-family is called *normalized* if its Lebesgue extension is a probability measure on T^1M.

For $v \in T^1\tilde{M}$ we denote by $W^{ss}(v)$ the inner normal field of the horosphere at $\pi(v) \in \partial\tilde{M}$ which passes through $Pv \in \tilde{M}$. Since $\tilde{M}$ admits a cocompact isometry group by assumption, $W^{ss}(v)$ is a C^∞-submanifold of $T^1\tilde{M}$. The manifolds $W^{ss}(v)$ $(v \in T^1\tilde{M})$ define a Hölder continuous foliation W^{ss} of $T^1\tilde{M}$ which is called the *strong stable foliation*. The strong stable foliation is invariant under the action of the isometry group of $\tilde{M}$ and under the *geodesic flow* Φ^t. Its tangent bundle TW^{ss} is a Φ^t-invariant Hölder continuous subbundle of $TT^1\tilde{M}$.

For every $v \in T^1\tilde{M}$ the disjoint union $\bigcup_{t\in\mathbb{R}} \Phi^t W^{ss}(v) = W^s(v)$ of the images of $W^{ss}(v)$ under the geodesic flow is a smooth submanifold $W^s(v)$ of $T^1\tilde{M}$ which is just the preimage $\pi^{-1}(\pi(v))$ under π of the point $\pi(v) \in \partial\tilde{M}$. The canonical projection $P : T^1\tilde{M} \to \tilde{M}$ maps $W^s(v)$ diffeomorphically onto $\tilde{M}$. The manifolds $W^s(v)$ $(v \in T^1\tilde{M})$ define a Hölder continuous foliation W^s of $T^1\tilde{M}$ which is called the *stable foliation*. The stable foliation is invariant under the action of the isometry group of $\tilde{M}$ and under the geodesic flow. The foliations W^{ss} and W^s on $T^1\tilde{M}$ project to the *strong stable* foliation W^{ss} and the *stable foliation* W^s for the geodesic flow on T^1M.

Denote by $\mathcal{D}_k$ the set of all normalized k-families on M. Via the map which associates to a k-family $\{\nu^p\} \in \mathcal{D}_k$ its Lebesgue extension the weak*-topology on the space of Borel probability measures on T^1M induces a topology on $\mathcal{D}_k$. With respect to this topology, a sequence $\{\nu^p_i\}_i \subset \mathcal{D}_k$ converges to a k-family $\{\nu^p\}$ if and only if the Lebesgue extensions $\int \nu^p_i d\lambda(p)$ of the families $\{\nu^p_i\}$ converge in the weak*-topology to the Lebesgue extension $\int \nu^p d\lambda(p)$ of $\{\nu^p\}$. With this notion of convergence we have.

Lemma 2.3:

(1) *The space $\mathcal{D}_k$ of all k-families is compact.*
(2) *A sequence $\{\nu_i^p\}_i$ of k-families converges in $\mathcal{D}_k$ to a k-family $\{\nu^p\}$ if and only if for every $p \in M$ the finite Borel measures $\{\nu_i^p\}$ on $T_p^1 M$ converge weakly to the measure ν^p.*

Proof: For $x, y \in M$ and some $\rho > 0$ call two subsets A of $T_x^1 M$, B of $T_y^1 M$ *ρ–equivalent* if for every $v \in A$ there is a curve of length at most ρ in $W^s(v)$ whose endpoint is contained in B and if every point in B can be connected to a point in A in this way. By definition, for every k-family $\{\nu^p\}$, for every $\delta > 0$ and all $\log(1+\delta)/k$–equivalent nontrivial open subsets A of $T_x^1 M$, B of $T_y^1 M$ we have $\nu^x(A)/\nu^y(B) \in [(1+\delta)^{-1}, 1+\delta]$. In particular, there is a constant $c(k) > 0$ only depending on k such that $\nu^p(T_p^1 M) \in [c(k)^{-1}, c(k)]$ for every $p \in M$ (compare Lemma 2.2).

Following Margulis [M70] we denote by T the family of functions on $T^1 M$ which are supported on a fibre $T_p^1 M$ of the fibration $T^1 M \to M$ and whose restriction to this fibre is continuous. Let $\{\nu_i^p\}$ be a sequence of k-families. For $i > 0$ define a linear functional l_i on T by $l_i(f) = \int f \, d\nu_i^p$ where $p \in M$ is such that f is supported on $T_p^1 M$. By the above the functionals l_i on T $(i > 0)$ satisfy properties R1 - R3 of [M70] and hence they form a precompact subset of the product space $L = \Pi_{f \in T} X_f$ where each of the spaces X_f can naturally be identified with $\mathbb{R}$.

For $m \in \mathbb{N}$ denote by L_m the closure in L of the set $\{l_i \mid i \in [m, \infty)\}$. Then L_m is compact, $L_m \supset L_{m+1}$ and clearly $L_m \neq \emptyset$ for all $m \geq 1$. Thus $L_0 = \cap_{m \geq 1} L_m$ is a nonempty compact subset of L. Every $l_0 \in L_0$ is a functional on T which again satisfies properties R1 - R3 of [M70]; hence there is a family ν_0^p $(p \in M)$ of Borel-probability measures on the fibres $T_p^1 M$ of the fibration $T^1 M \to M$ such that $l_0(f) = \int f d\nu_0^p$ for every $f \in T$ which is supported in $T_p^1 M$. This family of measures lifts to a Γ-invariant family of measures on the fibres of the fibration $T^1 \tilde{M} \to \tilde{M}$, and these measures project to a family of measures $\bar{\nu}_0^x$ $(x \in \tilde{M})$ on $\partial \tilde{M}$. The measures $\bar{\nu}^x$ define all the same measures class, and the Radon Nikodym derivative of $\bar{\nu}_0^x$ with respect to $\bar{\nu}_0^y$ at a point $\zeta \in \partial \tilde{M}$ is bounded from above by $e^{k \ \mathrm{dist}(x,y)}$. In particular, the measures $\{\nu_0^p\}$ define again a k-family on M. Choose a family $\{A_m\}_{m \geq 0}$ of compact neighborhoods of l_0 in L_1 such that $\cap_m A_m = \{l_0\}$. For $m \geq 1$ choose $i(m) \in [m, \infty)$ such that $l_{i(m)} \in A_m$; then the sequence $\{l_{i(m)}\}_m$ converges in L to l_0.

For every $p \in M$ the measures $\nu_{i(m)}^p$ on $T_p^1 M$ converge weakly to ν_0^p as $m \to \infty$. This follows from the fact that the measures $\nu_{i(m)}^p$ converge as $m \to \infty$ to ν_0^p with respect to the weak topology if and only if for every continuous functions φ on $T_p^1 M$ we have $\int \varphi \, d\nu_{i(m)}^p \to \int \varphi \, d\nu_0^p$ $(m \to \infty)$. But this is just a consequence of $l_{i(m)} \to l_0$ in L.

Let now ν_i be the Lebesgue extension of the k-family $\{\nu_i^p\}$ and let ν be the Lebesgue extension of the k-family $\{\nu_0^p\}$. Choose any subsequence $\{m_j\}_j$ of the sequence $\{i(m)\}_m$ such that the measures ν_{m_j} converge as $j \to \infty$ weakly to a Borel-probability measure ρ on $T^1 M$. We have to show that $\rho = \nu$.

Recall that the map $F : T^1\tilde{M} \to \tilde{M} \times \partial\tilde{M}$ which is defined by $F(v) = (Pv, \pi(v))$ is a Hölder continuous homeomorphism. For $i > 0$ let ν_i' be the lift of ν_i to $T^1\tilde{M}$ and let ρ' be the lift of ρ. Since the sets $F^{-1}(V \times W)$ with $V \subset \tilde{M}$ open, $W \subset \partial\tilde{M}$ open form a basis for the topology of $T^1\tilde{M}$ it suffices to show that ρ' coincides with the lift ν' of ν to $T^1\tilde{M}$ on sets of this form. Let $\delta > 0$; by the definition of a k-family, there is a number $\beta(\delta) > 0$ only depending on k, δ such that for every open subset V of $\tilde{M}$ of diameter smaller than $\beta(\delta)$, every open subset W of $\partial\tilde{M}$, every $i > 0$ and every $x \in V$ we have

$$\nu_i'(F^{-1}(V \times W))/\lambda(V)\bar{\nu}_i^x(W) \in [(1+\delta)^{-1}, 1+\delta] \tag{3}$$

where λ is the Lebesgue measure on $\tilde{M}$ and $\{\bar{\nu}_i^x\}$ is the Γ-k-family on $\partial\tilde{M}$ defined by $\{\nu_i^p\}$.

Let now $V \subset \tilde{M}, W \subset \partial\tilde{M}$ be arbitrary open sets. Then there are countably many pairwise disjoint open sets $V_\ell \subset V(\ell \geq 1)$ of diameter smaller than $\beta(\delta)$ such that

$$\lambda(V) = \sum_{\ell=1}^{\infty} \lambda(V_\ell). \tag{4}$$

For $\ell \geq 1$ choose $x_\ell \in V_\ell$; then

$$\nu_i'(F^{-1}(V \times W)) \leq (1+\delta) \sum_{\ell=1}^{\infty} \lambda(V_\ell)\bar{\nu}_i^{x_\ell}(W) \tag{5}$$

and hence

$$\begin{aligned} \lim_{j\to\infty} \nu_{m_j}'(F^{-1}(V \times W)) &\leq (1+\delta) \sum_{\ell=1}^{\infty} \lambda(V_\ell)\bar{\nu}_0^{x_\ell}(W) \\ &\leq (1+\delta)^2 \nu_0'(F^{-1}(V \times W)). \end{aligned} \tag{6}$$

Since $\delta > 0$ was arbitrary this shows that $\rho \leq \nu$. The same argument also shows that $\rho \geq \nu$, i.e. indeed equality holds. This finishes the proof of the lemma. ☐

Define the *Hausdorff dimension* of a k-family $\{\nu^p\}$ to be the Hausdorff dimension of the measure ν^p with respect to the metric δ_p for some point $p \in M$. Recall that this is defined as follows. For a subset Z of $\partial\tilde{M}$ and a number $\alpha > 0$ let

$$m_H(Z, \alpha) = \lim_{\epsilon\to 0} \inf_{\mathcal{G}} \sum_{u\in\mathcal{G}} (\operatorname{diam} U)^\alpha$$

where the infimum is taken over all finite or countable coverings $\mathcal{G}$ of Z by open sets U of δ_p-diameter $\operatorname{diam} U$ at most ϵ. The δ_p-Hausdorff dimension of Z is the infimum of all number $\alpha > 0$ such that $m_H(Z, \alpha) = 0$. It easily follows from the definition of the metrics δ_p and the fact that the curvature of M is bounded that the δ_p-Hausdorff dimension of any subset Z of $\partial\tilde{M}$ is independent of p and finite; we call this number the Hausdorff dimension of Z. The *Hausdorff dimension* $\dim_H(\nu^p)$ of the measure ν^p is defined to be the infimum of the Hausdorff dimensions of a subset of $\partial\tilde{M}$ of full ν^p-measure; as before, this does not depend on p.

It will also be useful to introduce the upper box dimension of ν^p which is given as follows. For $\epsilon \in (0, b)$, $\rho > 0$ let $q(\epsilon, \rho)$ be the minimal number of δ_p-balls of radius at most ρ which are needed to cover a subset of T^1_pM of ν^p-mass at least $\nu^p(T^1_pM)-\epsilon$; then

$$\overline{\dim}_B(\nu^p)=\sup_{\epsilon>0}\lim\sup_{\rho\to 0}(-\log q(\epsilon,\rho)/\log\rho). \tag{7}$$

It follows from the definitions that $\dim_H(\nu^p)\leq\overline{\dim}_B(\nu^p)$.

Recall that a k-family $\{\nu^p\}$ is ergodic if it induces an ergodic Γ-invariant measure class on $\partial\tilde M$. The ergodic k-families form a measurable, but not closed subset of $\mathcal{D}_k$. Let $\{\nu^p\}$ be a k-family on M with Radon Nikodym kernel Q, i.e. $Q:\tilde M\times\tilde M\times\partial\tilde M\to\mathbb{R}$ is the Radon Nikodym kernel of the corresponding Γ-k-family $\{\bar\nu^p\}$ on $\partial\tilde M$. Then for $mc(\nu_+)$-almost every $\xi\in\partial\tilde M$ and every $v\in\pi^{-1}(\xi)$ the value $\tilde f(v)=Q(Pv,P\Phi^1v,\xi)$ is well defined. The function $v\to\tilde f(v)$ is bounded and invariant under the action of Γ and hence it projects to a bounded function f on T^1M which we call the *Radon Nikodym function* of $\{\nu^p\}$. By the definition of the Lebesgue extension ν_L of $\{\nu^p\}$, the function f is ν_L-measurable.

Lemma 2.4: *Let $\{\nu^p\}$ be an ergodic k-family with Radon Nikodym function f and Lebesgue extension ν_L. Then for ν-almost every $v\in T^1M$ we have*

$$\lim\inf_{t\to\infty}\frac{1}{t}\int_0^t f(\Phi^s v)ds\leq\dim_H(\nu^p)\leq\overline{\dim}_B(\nu^p)\leq\lim\sup_{t\to\infty}-\log\nu^{Pv}(B(v,e^{-t}))/t.$$

Proof: Let $\{\nu^p\}$ be an ergodic k-family with Lebesgue extension ν_L and Radon Nikodym function f. Define a function ρ_- on T^1M by

$$\rho_-(v)=\lim\inf_{t\to\infty}\frac{1}{t}\int_0^t f(\Phi^s v)ds; \tag{8}$$

then ρ_- is ν_L-measurable. By the definition of a k-family, ρ_- is constant along the leaves of the stable foliation and hence it defines a Γ-invariant bounded $mc(\nu_+)$-measurable function on $\partial\tilde M$. This function is constant almost everywhere by ergodicity. Let a_- be its constant value; we claim that $a_-\leq\dim_H(\nu^p)$.

To see this let $\epsilon>0$. Fix a point $p\in M$ and write $\kappa=\nu^p(T^1_pM)$. There is a closed subset $B\subset T^1_pM$ with $\nu^p(B)>\kappa-\epsilon$ and there is a number $T>0$ such that for every $p\in B$ and every $t\geq T$ we have $\int_0^t f(\Phi^s(v))ds\geq t(a_--\epsilon)$. Since the ν^q-masses of the balls $B(w,1)\subset T^1_{Pw}M$ are bounded from above by a universal constant $c>0$, any ball of radius $r<e^{-T}$ with respect to our distance function δ_p whose center is contained in B intersects B in a subset of ν^p-mass at most $cr^{a_--\epsilon}$. This implies that every measurable subset of T^1_pM of diameter at most $r<e^{-T}$ intersects B in a set of ν^p-mass at most $cr^{a_--\epsilon}$. Now if $A\subset T^1_pM$ is a set of ν^p-measure at least $\kappa-\epsilon$ then the ν^p-mass of $A\cap B$ is at least $\kappa-2\epsilon$ and therefore for every $r<e^{-T}$, for every covering $\mathcal{G}$ of A by sets of diameter at most $r<e^{-T}$ we have

$$\kappa-2\epsilon\leq\sum_{U\in\mathcal{G}}\nu^p(U)\leq c\sum_{U\in\mathcal{G}}(\mathrm{diam}(U))^{a_--\epsilon}.$$

By the definition of the Hausdorff dimension this shows that $\dim_H(\nu^p)\geq a_--\epsilon$. Since $\epsilon>0$ was arbitrary we conclude that $\dim_H(\nu)\geq a_-$.

Now for $v \in T^1M$ let $\rho_+(v) = \limsup_{t\to\infty} \frac{1}{t} - \log \nu^p(B(v, e^{-t}))$. By the definition of a k-family, ρ_+ is constant along stable manifolds and hence constant almost everywhere by ergodicity; we denote its constant value by ρ_+. Our goal is to show that ρ_+ is not smaller than the box dimension of ν^p. For this let $\tilde{p}$ be the lift of p to $\tilde{M}$. For $R > 0$ and $v \in T^1_{\tilde{p}}\tilde{M}$ let $B_0(v, e^{-R}) \subset T^1_{\tilde{p}}\tilde{M}$ be the set of all initial tangents of geodesic rays in $\tilde{M}$ which intersect the ball of radius 1 about $P\Phi^R v$. By the definition of the distance function δ_p, there is a universal constant $\chi > 0$ such that $B_0(v, \chi^{-1}r) \subset B(v, r) \subset B_0(v, \chi r)$ for all $r < 1/2$. As a consequence, it is enough to show that $\overline{\dim}_B(\nu^p) \le \limsup_{t\to\infty} \frac{1}{t} \log \nu^p(B_0(v, e^{-t})) = \rho_+$.

For $\epsilon > 0$ there is a closed subset B of T^1_pM of ν^p-mass at least $\kappa - \epsilon$ and a number $r_0 > 0$ such that $\nu^p(B_0(v, r)) \ge r^{\rho_+ + \epsilon}$ whenever $r < \rho_0$ and $v \in B$. Let $r < r_0$; by the Besicovich covering theorem, applied to a covering of $P\Phi^{-\log r}B \subset \tilde{M}$ by balls of radius 1 in $\tilde{M}$ centered at points in the distance sphere of radius $-\log r$ about $\tilde{p}$, there is a universal constant $q > 0$ only depending on the dimension n of M and the curvature bounds and for every $r < r_0$ there is a covering of B by a family $\mathcal{C} = \cup_{i=1}^q \mathcal{C}_i$ of balls $B_0(v_i, r)$ $(i > 0)$ with $v_i \in B$ such that the balls in the family $\mathcal{C}_i$ are pairwise disjoint. Now we may assume that the ν^p-mass of a ball from $\mathcal{C}$ is at least $r^{a_+ + \epsilon}$ and therefore in each of the families $\mathcal{C}_i$ there are at most $r^{-a_+ - \epsilon}$ elements. Thus for every $\rho < \rho_0$ the set B can be covered by at most $qr^{-a_+ - \epsilon}$ balls $B_0(w, r)$ $(w \in B)$. Since $\epsilon > 0$ was arbitrary we conclude that $\overline{\dim}_B(\nu^p) \le \rho_+$. □

3. Measures of quasi-product type

In this section we investigate k-families induced from Borel probability measures on T^1M which are invariant under the geodesic flow. For this recall that the *space of geodesics* on $\tilde{M}$ is the quotient of $T^1\tilde{M}$ under the action of the geodesic flow. Since every flow line on $T^1\tilde{M}$ is a properly embedded submanifold, the smooth structure on $T^1\tilde{M}$ induces a smooth structure on the space of geodesics which is invariant under the action of Γ. The ideal boundary $\partial\tilde{M}$ of $\tilde{M}$ admits a natural Γ-invariant Hölder structure. Let $\mathcal{F} : v \to -v$ be the *flip* on T^1M and $T^1\tilde{M}$. The map $\pi \times \pi \circ \mathcal{F}$ of $T^1\tilde{M}$ into $\partial\tilde{M} \times \partial\tilde{M}$ factors through a Hölder homeomorphism of the space of geodesics onto the complement of the diagonal Δ in $\partial\tilde{M} \times \partial\tilde{M}$. This homeomorphism is equivariant under the action of Γ. Every Φ^t-invariant finite Borel measure μ on T^1M lifts to a locally finite Φ^t-invariant Borel measure on $T^1\tilde{M}$, and this measure projects to a locally finite Borel measure $\hat{\mu}$ on $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ which is invariant under the product action of the fundamental group Γ of M. The measure μ is ergodic under the action of the geodesic flow if and only if the measure $\hat{\mu}$ is ergodic under the action of Γ. Vice versa, every locally finite Borel measure on $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ which is invariant under the product action of Γ induces a Φ^t-invariant finite Borel measure on T^1M.

Among the Γ-invariant measures on $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ there is a subclass which are of particularly simple form with respect to the product decomposition of $\partial\tilde{M} \times \partial\tilde{M} - \Delta$. We describe this class in the next definition.

Definition. A Φ^t-invariant Borel probability measure ν on T^1M is called *of quasi-product type* if there are finite Borel measures ν_+, ν_- on $\partial\tilde{M}$ whose measure classes are invariant under the action of Γ and such that the lift $\tilde{\nu}$ of ν to $T^1\tilde{M}$ has the following properties.

(1) For every Borel-set $B \subset \partial\tilde{M}$ we have $\nu_+(B) = 0$ if and only if $\tilde{\nu}(\pi^{-1}(B)) = 0$ and $\nu_-(B) = 0$ if and only if $(\mathcal{F} \circ \tilde{\nu})(\pi^{-1}(B)) = 0$.
(2) The projection $\hat{\nu}$ of ν to the space of geodesics is contained in the measure class of the product $\nu_+ \times \nu_-$.

The measure classes $mc(\nu_+), mc(\nu_-)$ of ν_+, ν_- are called *positive* and *negative disintegration classes* of ν. The *quasi-product type* of ν is defined to be $mc(\nu_+) \times mc(\nu_-)$. The measure ν of quasi-product type $mc(\nu_+) \times mc(\nu_-)$ is called *non-degenerate* if its projection $\hat{\nu}$ to the space of geodesics *defines* the measure class of $mc(\nu_+) \times mc(\nu_-)$, i.e. if $\hat{\nu}(A) = 0$ implies $\nu_+ \times \nu_-(A) = 0$ for every Borel subset A of $\partial\tilde{M} \times \partial\tilde{M} - \Delta$.

Property 2) means that if ν is a measure of quasi-product type $mc(\nu_+) \times mc(\nu_-)$ then for every choice ν_+, ν_- of finite Borel measures on $\partial\tilde{M}$ defining the measure classes $mc(\nu_+), mc(\nu_-)$ there is a measurable non-negative function ψ on $\partial\tilde{M} \times \partial\tilde{M}$ such that $\hat{\nu} = \psi\nu_+ \times \nu_-$. Property 1) implies that if ν is ergodic, i.e. if $\hat{\nu}$ is ergodic under the product action of Γ, then the measure classes $mc(\nu_+), mc(\nu_-)$ are ergodic under the action of Γ.

The flip $\mathcal{F} : v \to -v$ on T^1M preserves the Φ^t-invariant measures on T^1M. The image under $\mathcal{F}$ of a measure ν of quasi-product type $mc(\nu_+) \times mc(\nu_-)$ is a measure of product type $mc(\nu_-) \times mc(\nu_+)$.

First we look at ergodic measures of quasi-product type whose positive disintegration class has an atom.

Lemma 3.1: *For a Φ^t-invariant probability measure ν on T^1M the following are equivalent.*

(1) *ν is an ergodic measure of quasi-product type whose positive disintegration class has an atom.*
(2) *ν is supported on a single closed orbit of Φ^t.*

Proof: We show first that a Φ^t-invariant measure ν supported on a single closed orbit of Φ^t is of quasi-product type. For this define $mc(\nu_+)$ and $mc(\nu_-)$ to be the measure class of the sum of the (countably many) Dirac masses on the forward endpoints and backward endpoints, respectively, of all lifts of the support of ν to $T^1\tilde{M}$. Since a product of two Dirac masses is again a Dirac mass, ν is of quasi-product type $mc(\nu_+) \times mc(\nu_-)$, but clearly ν is degenerate.

Let for the moment η be any Φ^t-invariant Borel probability measure on T^1M with lift $\tilde{\eta}$ to $T^1\tilde{M}$. Assume that there is a finite Borel measure η_∞ on $\partial\tilde{M}$ with the property that $\eta_\infty(B) = 0$ if and only if $\tilde{\eta}(\pi^{-1}(B)) = 0$ for every Borel subset B of $\partial\tilde{M}$. Then the measure class of η_∞ is invariant under the action of Γ on $\partial\tilde{M}$.

We claim that every atom of η_∞ is necessarily supported on a fixed point for the action of a nontrivial element of Γ. To see this assume that η_∞ has an atom at a point $\xi \in \partial\tilde{M}$. By property 1) above we have $\tilde{\eta}(\pi^{-1}(\xi)) \neq 0$ and hence there is a compact subset $\tilde{B}$ of $\pi^{-1}(\xi)$ such that $\eta(\tilde{B}) > 0$. The projection B of $\tilde{B}$ to T^1M is a compact subset of a stable manifold $W^s(v)$ for some $v \in T^1M$. If ξ is not a fixed point for a nontrivial element of the fundamental group Γ of M, then $W^s(v)$ is an immersed submanifold of T^1M which is diffeomorphic to $W^{ss}(v) \times \mathbb{R}$. In particular, there are infinitely many numbers $t_i > 0$ such that the sets $\Phi^{t_i}(B) \subset T^1M$ are pairwise disjoint. Since $\eta(B) > 0$ by assumption and since η is finite and Φ^t-invariant this is impossible.

Now let ν be a Φ^t-invariant ergodic measure of quasi-product type. Assume that the positive disintegration class $mc(\nu_+)$ of ν has an atom at a point $\xi \in \partial\tilde{M}$. By the above, ξ is the attracting fixed point for a nontrivial element $\Psi \in \Gamma$. By ergodicity, $mc(\nu_+)$ is purely atomic and its atoms are precisely the points on the Γ-orbit of ξ.

Let $\hat{\nu}$ be the projection of ν to a Γ-invariant measure on $\partial\tilde{M} \times \partial\tilde{M} - \Delta$. There is a compact neighborhood B of the repelling fixed point ζ of Ψ in $\partial\tilde{M} - \{\xi\}$ such that $\hat{\nu}(\{\xi\} \times B) > 0$. We can choose B in such a way that $\Psi^{-k}B \subset \Psi^{-k+1}B$ for all $k > 0$. Then $\cap_{k\geq 0}\Psi^{-k}(B) = \{\zeta\}$ and therefore by invariance of $\hat{\nu}$ under the action of Γ, the restriction of the negative disintegration class $mc(\nu_-)$ to B has an atom at ζ. Since $mc(\nu_-)$ is Γ-invariant and ergodic we conclude that ν is supported on the closed orbit of Φ^t which defines the conjugacy class of the isometry Ψ in the fundamental group of M. □

The foliation $W^{su} = \mathcal{F}W^{ss}$ on $T^1\tilde{M}$ and T^1M is invariant under Φ^t and is called the *strong unstable foliation*. The tangent bundle TT^1M of T^1M admits a Φ^t-invariant direct decomposition $TT^1M = \mathbb{R}X \oplus TW^{ss} \oplus TW^{su}$ into the tangent bundles TW^i of the foliations W^i $(i = ss, su)$ and the line bundle spanned by the generator X of the geodesic flow.

Let d^{su}, d^{ss} be the distance functions on the leaves of W^{su}, W^{ss} induced by the lift of the Riemannian metric on M. For $v \in T^1M$ and $t \in \mathbb{R}$ define $B^{su}(v, e^{-t})$ to be the image under Φ^{-t} of the d^{su}-ball of radius 1 in $W^{su}(\Phi^t v)$. Similarly, define $B^{ss}(v, e^{-t})$ to be the image under Φ^t of the d^{ss}-ball of radius 1 in $W^{ss}(\Phi^{-t}v)$. Call a family $\{\nu^i\}$ of Borel measures on the leaves of the foliation W^i $(i = ss, su)$ *locally uniformly bounded* if for every $v \in T^1M$ the ν^i-mass of the ball $B^i(v, 1)$ about v in $W^i(v)$ is bounded from above by a constant not depending on v.

Let $\{\bar{\nu}^x\}$ $(x \in \tilde{M})$ be a Γ-k-family on $\partial\tilde{M}$. For $v \in T^1\tilde{M}$ the restriction of the projection π is a homeomorphism of $W^{su}(v)$ onto $\partial\tilde{M} - \{\pi(-v)\}$ and hence we can project $\bar{\nu}^{Pv}$ to a finite nontrivial Borel measure $\bar{\nu}^v$ on the strong unstable manifold $W^{su}(v)$. The measure class of $\bar{\nu}^v$ does not depend on v. More precisely, if Q is the Radon Nikodym kernel of the k-family $\{\bar{\nu}^x\}$, then for $w \in W^{su}(v)$ we have $\frac{d\bar{\nu}^w}{d\bar{\nu}^v}(w) = Q(Pv, Pw, \pi(w))$. Thus by Lemma 2.2 we obtain a family $\{\nu^{su}\}$ of locally uniformly bounded measures on the leaves of the foliation $W^{su} \subset T^1\tilde{M}$ by defining $\frac{d\nu^{su}}{d\bar{\nu}^v}(w) = Q(Pv, Pw, \pi(w))$. The measures ν^{su} only depend on the Γ-k-family $\{\bar{\nu}^x\}$

and they transform under the geodesic flow via $\frac{d\nu^{su}\circ\Phi^t}{d\nu^{su}}(v) = Q(Pv, P\Phi^t v, \pi(v))$. Moreover they are equivariant under the action of Γ and therefore they project to a family of locally uniformly bounded Borel measures on the leaves of $W^{su} \subset T^1M$. We call the collection $\{\nu^{su}\}$ of these measures the *strong unstable family* induced by the Γ-k-family $\{\bar{\nu}^x\}$ on $\partial\tilde{M}$.

Let dt be the 1-dimensional Lebesgue measure on the flow lines of the geodesic flow. Let ν be a measure of quasi-product type and let $\{\bar{\nu}^x\}$ be a Γ-k-family on $\partial\tilde{M}$ whose measure class $mc(\nu_+)$ on $\partial\tilde{M}$ is the positive disintegration class of ν. By Lemma 2.1, such a Γ-k-family exists always; we call the corresponding k-family on M a *k-family for ν*. The Γ-k-family $\{\bar{\nu}^x\}$ defines the strong unstable family $\{\nu^{su}\}$. There is a family $\{\nu^{ss}\}$ of Borel measures on the leaves of W^{ss} such that $d\nu = d\nu^{su} \times d\nu^{ss} \times dt$. The image of the measures ν^{ss} under the restriction of the map $\pi \circ \mathcal{F}$ are contained in the measure class $mc(\nu_-)$, but the y not necessarily define this measure class.

The next lemma gives a more intrinsic description of measures of quasi-product type which are *ergodic*, i.e. ergodic under the action of the geodesic flow.

Lemma 3.2: *Let $\{\nu^{su}\}$ be the strong unstable family induced by an ergodic Γ-k-family $\{\bar{\nu}^x\}$ on $\partial\tilde{M}$. Assume that there is a family $\{\nu^{ss}\}$ of Borel measures on strong stable manifolds such that the measure ν defined by $d\nu = d\nu^{su} \times d\nu^{ss} \times dt$ is Φ^t-invariant and finite. Then ν is an ergodic measure of quasi-product type.*

Proof: Let ν be a finite Φ^t-invariant measure on T^1M as stated in the lemma. We have to show that the projections to $\partial\tilde{M}$ of the measures ν^{ss} on strong stable manifolds are all contained in the same Γ-invariant ergodic measure class.

For this consider first the case that the measures ν^{su} have atoms. By ergodicity, the measures ν^{su} are completely atomic. By Lemma 3.1 and its proof the measure ν is necessarily supported on a single closed orbit of Φ^t and hence it is of degenerate quasi-product type. Thus we may assume that the measures ν^{su} do not have atoms.

Since the measures ν^{su} are quasi-invariant under the action of the geodesic flow Φ^t, the same is true for the measures ν^{ss}. Let $\hat{\nu}$ be the projection of ν to a Γ-invariant measure on $\partial\tilde{M} \times \partial\tilde{M} - \Delta$. Then $\hat{\nu}$ can locally be written in the form $d\hat{\nu} = d\hat{\nu}^{su} \times d\hat{\nu}^{ss}$ where the measures $\hat{\nu}^{su}$ are simply projections of the measures ν^{su} and hence define all the same measure class, and the measures $\hat{\nu}^{ss}$ are projections of the measures ν^{ss}. By our assumption on the measures ν^{su} we may assume that the measure $\hat{\nu}^{su}$ is defined on every leaf of the first factor foliation and that moreover for every fixed $\xi, \zeta \in \partial\tilde{M}$ the Jacobian of the canonical map $\partial\tilde{M} - \{\xi,\zeta\} \times \{\xi\} \to \partial\tilde{M} - \{\xi,\zeta\} \times \{\zeta\}$ for these measures is locally uniformly Hölder continuous.

Choose countably many pairwise disjoint relative compact product sets $C_i = A_i \times B_i$ with dense interior which define a partition of $\partial\tilde{M} \times \partial\tilde{M} - \Delta$. Let $\mathcal{P}$ be the partition of $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ into the relative compact sets $\{w\} \times B_i\,(w \in A_i)$ which are contained in the leaves of the second factor foliation. By construction, the partition $\mathcal{P}$ is ν-measurable, i.e. up to a set of measure zero the quotient

$\partial\tilde M \times \partial\tilde M - \Delta/\mathcal{P}$ is separated by a countable number of measurable sets. Write $\mathcal{P}(\xi,\zeta)$ for the set of the partition containing the point $(\xi,\zeta)\in\partial\tilde M\times\partial\tilde M-\Delta$.

For $\hat\nu$-almost every (ξ,ζ) the conditional measure $\hat\nu_{(\xi,\zeta)}$ of $\hat\nu$ with respect to $\mathcal{P}$ is well defined. We claim that $\hat\nu_{(\xi,\zeta)}$ is contained in the measure class of $\hat\nu^{ss}|\mathcal{P}(\xi,\zeta)$. To see this let for the moment $C=A\times B$ be any relative compact product set in $\partial\tilde M\times\partial\tilde M-\Delta$ with dense interior. Fix a point $\zeta\in B$. By the definition of the measures $\hat\nu^{su}$ there is a function f on C which is bounded from above and below by a positive constant and Hölder continuous along the leaves of the second factor foliation and such that the Jacobians of the holonomy maps of the measures $f\nu^{su}$ on the leaves of the first factor foliation of $C=A\times B$ are constant. If we denote the measures $f\nu^{su}, f^{-1}\nu^{ss}$ again by ν^{su},ν^{ss} then for every Borel subset D of C we have

$$\hat\nu(D)=\int_{A\times\{\zeta\}}\hat\nu^{ss}(D\cap\{\xi\}\times B)d\hat\nu^{su}(\xi)=\int\big(\hat\nu^{ss}(D\cap\{\xi\}\times B)/\hat\nu^{ss}(\{\xi\}\times B)\big)d\hat\nu(\xi,\zeta).$$

Since conditional measures for a partition are unique almost everywhere we conclude that for $\hat\nu^{su}$-almost every $\xi\in A$ with the additional property that $\hat\nu^{ss}(\{\xi\}\times B)>0$ the measure $\hat\nu^{ss}/\hat\nu^{ss}(\{\xi\}\times B)$ is the conditional measure of $\hat\nu$ on $\{\xi\}\times B$ with respect to the partition of C into the leaves of the second factor foliation. Since this consideration is valid for each of our product sets C_i which defines our initial partition we conclude that the measure classes of the conditional measures for $\mathcal{P}$ are contained in the measure classes of the measures $\hat\nu^{ss}$ and do not depend on the particular choice of our partition.

Choose countably many pairwise disjoint relative compact product sets $C_i=A_i\times B_i$ with dense interior which define a partition of $\partial\tilde M\times\partial\tilde M-\Delta$.

For $i>0$ choose a point $\zeta_i\in B_i$. There is a measurable nonnegative function h on A_i such that for every Borel-subset D of A_i we have $\hat\nu(D\times B_i)=h\hat\nu^{su}(D\times\{\zeta_i\})$. Since by assumption the measure class $mc(\nu_+)$ of the measures $\hat\nu^{su}$ is invariant and ergodic under the action of Γ we obtain that for $mc(\nu_+)$-almost every $\xi\in\partial\tilde M$ the $\hat\nu^{ss}$-mass of $\{\xi\}\times(\partial\tilde M-\{\xi\}$ is positive. This shows that $mc(\nu_+)$ is uniquely determined by $\hat\nu$ and the product structure of $\partial\tilde M\times\partial\tilde M-\Delta$. In particular, for $\Psi\in\Gamma$ there is a measurable nonnegative function β such that $\hat\nu\circ(\Psi,\Psi)=\hat\nu^{su}\times\beta(\hat\nu^{ss}\circ\Psi)=\hat\nu^{su}\times\hat\nu^{ss}$. Thus for $\xi\in\partial\tilde M$ we have $\hat\nu^{ss}|\{\Psi\xi\}\times\partial\tilde M-\{\Psi\xi\}=\hat\nu^{ss}|(\{\xi\}\times\partial\tilde M-\{\xi\}\circ\Psi^{-1}$. Via multiplying $\hat\nu^{ss}$ with a suitable positive function we may assume that the total mass of $\hat\nu^{ss}$ is finite.

Choose a point $x\in\tilde M$ and a number $r>0$ such that $\sum_{\Psi\in\Gamma}e^{-r\,\mathrm{dist}(x,\Psi x)}<\infty$. For a density point ξ for $mc(\nu_+)$ write $\mu(\xi)=\sum_{\Psi\in\Gamma}e^{-r\,\mathrm{dist}(x,\Psi x)}\hat\nu^{ss}(\{\Psi\xi\}\times\partial\tilde M-\{\Psi\xi\})$. By construction, we have $\mu(\Psi\xi)=\mu(\xi)\circ\Psi^{-1}$ for all $\Psi\in\Gamma$. In other words, the assignment $\xi\to(\Psi\to\mu(\Psi\xi)$ is a $mc(\nu_+)$-measurable Γ-invariant map from $\partial\tilde M$ into the space of Γ-equivariant maps from Γ into the space of finite Borel measures on $\partial\tilde M$.

Since by our assumption the measure class of $\hat\nu^{su}$ is ergodic under the action of Γ we conclude that the map $\xi\to(\Psi\to\mu(\Psi\xi))$ is $\hat\nu^{su}$-almost everywhere constant. In particular, for $\hat\nu^{s}$-almost every $\xi\in\partial\tilde M$ the measure class of the measure $\mu(\xi)$ equals a fixed class $mc(\nu_-)$. Then for ν-almost every $v\in T^1M$ the projection to $\partial\tilde M$ of the measure ν^{ss} on $W^{ss}(v)$ is contained in the class $mc(\nu_-)$. In other words, ν is a (possibly degenerate) measure of product type. Moreover ν is ergodic provided that the measure class $mc(\nu_-)$ is ergodic under the action of Γ and this in turn is equivalent to saying that the measure $\hat{\nu}$ is ergodic under the action of $\Gamma \times \Gamma$.

To see that this is indeed the case let φ be any continuous function on T^1M. By the Birkhoff ergodic theorem, for ν-almost every $v \in T^1M$ the limit

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \varphi(\Phi^s v)ds = \varphi^*(v)$$

exists. The function φ^* is constant along the flow lines of the geodesic flow. If for every continuous function φ the function φ^* is constant ν-almost everywhere on T^1M then ν is ergodic.

Now φ^* is constant along the stable manifolds and therefore it projects to a function $\bar{\varphi}$ on $\partial\tilde{M}$ which is measurable and invariant under the action of Γ. Since ν is absolutely continuous with respect to the stable foliation and since the projections to $\partial\tilde{M}$ of the conditional measures ν^{su} define a measure class $mc(\nu_+)$ on $\partial\tilde{M}$ which is ergodic under the action of Γ the function $\bar{\varphi}$ is constant $mc(\nu_+)$-almost everywhere on $\partial\tilde{M}$ and the measure ν is ergodic under the geodesic flow. This finishes the proof of our lemma. □

Let again ν be a Φ^t-invariant ergodic Borel probability measure of quasi-product type on T^1M. We denote by h_ν the entropy of ν. Let $\{\nu^p\}$ be a k-family for ν with Radon Nikodym kernel Q and let $\tilde{f}(v) = \log Q(Pv, P\Phi^1 v, \pi(v))$; the function $\tilde{f}$ projects to the Radon Nikodym function f on T^1M. As in Section 2 we use the function f to calculate the entropy h_ν of ν.

Lemma 3.3: *Let ν be an ergodic measure of quasi-product type with entropy h_ν. If $\{\nu^p\}$ is a regular k-family for ν with Radon Nikodym function f then $h_\nu = \dim_H(\nu^p) = \int f d\nu$.*

Proof: Let ν be an ergodic measure of quasi-product type and let $\{\nu^p\}$ be a k-family for ν. Let f be the Radon Nikodym function of this k-family. By the Birkhoff ergodic theorem, for ν-almost every $v \in T^1M$ the limit

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\Phi^s v)ds$$

exists and equals $\int f d\nu$.

For almost every $v \in T^1M$ there is a Borel measure ν^{ss} on $W^{ss}(v)$ such that $d\nu = d\nu^{su} \times d\nu^{ss} \times dt$ at v. By Lemma 3.2 and its proof, the measures ν^{ss} are absolutely continuous with respect to a family of conditional measures for the strong stable foliation defined by a partition of T^1M subordinate to W^{ss}. Moreover, they are quasi-invariant under the geodesic flow. Let $\epsilon > 0$ and for $v \in T^1M$ define

$$\rho_-(v) = \liminf_{t\to\infty} -\frac{1}{t}\log \nu^{ss}B^{ss}(v, e^{-t}) \quad \text{and} \tag{9}$$

$$\rho_+(v) = \limsup_{t\to\infty} -\frac{1}{t}\log \nu^{ss}B^{ss}(v, e^{-t}).$$

From the definition of the balls $B^{ss}(v,e^{-t})$ and by quasi-invariance of ν^{ss} under the geodesic flow the functions ρ_-,ρ_+ are ν-measurable and invariant under Φ^t and hence they are constant almost everywhere on T^1M. Let ρ_-,ρ_+ be these constants. We first claim that $\rho_- = \int f d\nu$.

To see this recall that the restriction of the Radon Nikodym kernel of the measures $\{\nu^p\}$ to stable manifolds is uniformly Lipschitz continuous. By the definition of the measures ν^{ss} this means that there is a number $c>0$ such that for almost every $v\in T^1M$ and all $t>0$ we have

$$\nu^{ss}(B^{ss}(v,e^{-t}))e^{\int_0^t f(\Phi^s v)ds}/\nu^{ss}(B^{ss}(\Phi^t v,1))\in[c^{-1},c]. \tag{10}$$

For $\epsilon>0$ define $A_\epsilon=\{w\in T^1M \mid \nu^{ss}(B^{ss}(w,1))\geq\epsilon\}$. Choose $\epsilon>0$ sufficiently small that $\nu(A_\epsilon)>0$. By ergodicity, for ν-almost every $v\in T^1M$ we have $\Phi^{t_i}v\in A_\epsilon$ for a sequence of number t_i converging to infinity. Then $\nu^{ss}(B^{ss}(v,e^{-t_i}))\geq e^{-\int_0^{t_i} f(\Phi^s v)ds}\epsilon/c$ for all i and hence $\rho_-\leq\int f d\nu$. From Lemma 2.4 we then conclude that $\rho_-\leq\int f d\nu\leq \dim_H(\nu^p)\leq\rho_+$.

Let ξ be a ν-measurable partition of T^1M with the additional property that for ν-almost every $v\in T^1M$ the set $\xi(v)$ of the partition containing v is an open neighborhood of v in $W^{ss}(v)$. Such a partition was for example constructed by Ledrappier and Young in [LY85a]. Then for ν-almost every v there is a probability measure ν_v on $\xi(v)$ so that the family of these measures has the following property. If $\mathcal{B}_\xi$ denotes the σ-subalgebra whose elements are unions of elements of ξ and if $A\subset T^1M$ is a measurable set then the function $v\to\nu_v(A\cap\xi(v))$ is $\mathcal{B}_\xi$-measurable and

$$\nu(A)=\int \nu_v(A\cap\xi(v))d\nu(v).$$

By the definition of a measure of quasi-product type, ν can locally be written in the form $d\nu=\psi d\nu^{su}\times d\nu^{ss}\times dt$ for a function ψ. Thus for every v the measure ν_v can be written in the form $\nu_v=\kappa_v\nu^{ss}$ for a ν^{ss}-measurable function κ_v. Moreover, at any density point v for ν the value of κ_v at v is positive. In particular, for every density point v for ν we have

$$\begin{aligned}\rho_-(v) &= \liminf_{t\to\infty} -\frac{1}{t}\log\nu_v B^{ss}(v,e^{-1}) \quad\text{and}\\ \rho_+(v) &= \limsup_{t\to\infty} -\frac{1}{t}\log\nu_v B^{ss}(v,e^{-1}).\end{aligned} \tag{11}$$

Thus the results of Ledrappier and Young [LY85b] applied to the map Φ^{-1} show that $h_\nu=\rho_-=\rho_+$ This completes the proof of the lemma. □

Remark: The above lemma gives a uniform interpretation of the results of Ledrappier and Young in our particular situation.

4. Examples of measures of quasi-product type

In this section we look at examples of measures of quasi-product type. For this let $M = \tilde{M}/\Gamma$ as before be a closed Riemannian manifold of negative sectional curvature with unit tangent bundle T^1M. Then the geodesic flow Φ^t of M is an Anosov flow and therefore a Hölder continuous function f on T^1M induces a unique *Gibbs equilibrium state* which is a Φ^t-invariant Borel probability measure on T^1M.

There is a more general class of functions on T^1M which admit a unique Gibbs equilibrium state. To define these functions, let d_m $(m \geq 0)$ be the family of metrics on T^1M defined by $d_m(v,w) = \sup_{0\leq t\leq m} \operatorname{dist}(\Phi^t v, \Phi^t w)$. We call a continuous function f on T^1M *Φ^t-regular* if there are numbers $\epsilon > 0, C > 0$ such that

$$|\int_0^m f(\Phi^s v)ds - \int_0^m f(\Phi^s w)ds| \leq C$$

for all $v, w \in T^1M$ with $d_m(v,w) < \epsilon$ and all $m \geq 0$. Our Φ^t-regular functions are just the continuous functions on T^1M which are admissible in the sense of [HK95]. We do not require here that the constant C can be chosen arbitrarily small provided that $\epsilon > 0$ is small enough, i.e. that the function satisfies Walters' condition. We refer to the paper of Bousch [Bo01] for more about functions with this stronger property.

The set $C^0_\Phi(T^1M)$ of all Φ^t-regular functions on T^1M is a vector subspace of the space of all continuous functions which contains all Hölder continuous functions. Moreover if $\mathcal{F} : v \to -v$ denotes again the flip on T^1M then $C^0_\Phi(T^1M)$ is invariant under the action of $\mathcal{F}$ on the space of continuous functions on T^1M. This follows from the fact that $\mathcal{F} \circ \Phi^s = \Phi^{-s} \circ \mathcal{F}$ and that therefore $d_m(v,w) < \epsilon$ if and only if $d_m(\mathcal{F}\Phi^m v, \mathcal{F}\Phi^m w) < \epsilon$ and moreover $\int_0^m (f \circ \mathcal{F})(\Phi^s v)ds = \int_0^m f(\Phi^s(\mathcal{F}\Phi^m v))ds$.

Let $f \in C^0_\Phi(T^1M)$ be a Φ^t-regular function on N. Recall that the *topological pressure* $pr(f)$ of f is defined to be the supremum of the numbers $h_\mu - \int f d\mu$ where μ ranges through all Φ^t-invariant Borel probability measures on T^1M and h_μ is the entropy of μ. Notice that the pressure of f coincides with the pressure of $f \circ \mathcal{F}$.

A *Gibbs equilibrium state* for a continuous function f on T^1M is a Φ^t-invariant Borel probability measure ν_f on T^1M which satisfies $pr(f) = h_{\nu_f} - \int f d\nu_f$. A Φ^t-regular function $f \in C^0_\Phi(T^1M)$ admits a unique Gibbs equilibrium state (this result can be found in Section 20.3 of [HK]). The following proposition is the main result of this section.

Proposition 4.1: *The unique Gibbs equilibrium state of a Φ^t-regular function is a nondegenerate measure of quasi-product type.*

We begin with an alternative characterization of the pressure of a Φ^t-regular function f. For $x \in \tilde{M}$ define a function $\tilde{\zeta}_x$ on $\tilde{M}$ as follows. For every $z \in \tilde{M} - \{x\}$ there is a unique $v \in T^1_x\tilde{M}$ and a number $t > 0$ such that $P\Phi^t v = z$; define $\tilde{\zeta}_x(P\Phi^t v) = e^{\int_0^t f(\Phi^s v)ds}$. By the definition of a Φ^t-regular function there is for every $R > 0$ a number $\beta > 0$ depending on R and f such that $\tilde\zeta_x(y)/\tilde\zeta_u(z) \leq \beta$ whenever $\mathrm{dist}(y,z) \leq R$ and $\mathrm{dist}(x,u) \leq R$.

Define the *critical exponent* of f to be the infimum of all numbers $m \in \mathbb{R}$ such that the *Poincaré series* $\sum_{\Psi\in\Gamma} \tilde\zeta_x^{-1}(\Psi x)e^{-m\,\mathrm{dist}(\Psi x,x)}$ converges. We have.

Lemma 4.2: *The critical exponent of a Φ^t-regular function f equals its topological pressure.*

Proof: For a Φ^t-regular function $f \in C^0_\Phi(T^1M)$, a finite subset E of T^1M and $t > 0$ define $S_t(E) = \sum_{v\in E} e^{-\int_0^t f(\Phi^s v)ds}$. Recall that a subset E of T^1M is (δ,m)-*separated* if for $v \neq w \in E$ we have $d_m(v,w) \geq \delta$. If $\delta > 0$ is arbitrarily fixed and if $\{E_m\}_m$ is a family of (δ,m)-separated subsets of T^1M where $m \to \infty$ then $\limsup_{m\to\infty} \frac{1}{m}\log S_m(E_m) \leq pr(f)$ (Section 20.3 of [HK]).

Fix a point $x \in \tilde M$. There is a number $\rho > 0$ such that $\mathrm{dist}(y,z) \geq \rho$ for all $y \neq z \in \Gamma x$. If $v,w \in T^1\tilde M$ and $s,t \in [k\rho/4,(k+1)\rho/4]$ are such that $P\Phi^s v = y \in \Gamma x$, $P\Phi^t w = z \in \Gamma x$ then $\mathrm{dist}(P\Phi^{k\rho/4}v, P\Phi^{k\rho/4}w) \geq \rho/2$ and therefore $d_{k\rho/4}(v,w) \geq \rho/2$. In particular, for every $\epsilon > 0$ and sufficiently large $k > 0$ we have

$$\sum\{\tilde\zeta_x^{-1}(\Psi x) \mid \Psi \in \Gamma, k\rho/4 \leq \mathrm{dist}(x,\Psi x) \leq (k+1)\rho/4\} \leq e^{k\rho(pr(f)+\epsilon)/4}.$$

This implies that for every $\epsilon > 0$ the series $\sum_{\Psi\in\Gamma} \tilde\zeta^{-1}(\Psi x)e^{-(pr(f)+\epsilon)\mathrm{dist}(\Psi x,x)}$ converges. Thus the critical exponent of f is not bigger than $pr(f)$.

To show the reverse inequality recall that a subset E of T^1M is (δ,m)- *spanning* if the d_m-balls of radius δ centered at points in E cover T^1M. Since f is Φ^t-regular there is a number $\delta > 0$ such that for every sequence $\{E_m\}_m$ of (δ,m)-spanning sets with $m \to \infty$ we have $\liminf_{m\to\infty} \frac{1}{m}\log\sum_{v\in E_m} e^{-\int_0^t f(\Phi^s v)ds} \geq pr(f)$ (Section 20.3 of [HK]).

For a small enough number $\sigma \in (0,\delta)$ let now A be a finite subset of M with the property that the balls of radius σ centered at points of A cover M. Choose a large number $R > 0$ which is larger than three times the diameter of M. Let K be a compact fundamental domain for the action of Γ on $\tilde M$ whose diameter coincides with the diameter of M.

Fix a point $x \in A \cap K$ and for $m \geq 2R$ let E_m be the set of all unit vectors which are tangent to geodesics connecting a point $y \in A \cap K$ to a point $z \in \tilde A$ with $m \leq \mathrm{dist}(x,z) \leq m + R$. For sufficiently small σ and sufficiently large m the set E_m projects to a (δ,m)- spanning subset for T^1M. Since f is Φ^t-regular there is a number $\kappa > 0$ not depending on m such that $\sum_{E_m} e^{-\int_0^m f(\Phi^s v)ds} \leq \kappa\sum_{\Psi\in\Gamma, m\leq\mathrm{dist}(\Psi x,x)\leq m+R} \tilde\zeta_x^{-1}(\Psi x)$. From this we obtain immediately that the critical exponent of f is not smaller than $pr(f)$. □

The next proposition shows that the Gibbs equilibrium state of every Φ^t-regular function on T^1M admits a k-family as a family of positive disintegration measures.

Proposition 4.3: *Let $f \in C^0_\Phi(T^1M)$ be Φ^t-regular with $pr(f)=0$. Then there is a family $\{\bar\nu_f^x\}$ $(x \in \tilde M)$ of Borel measures on $\partial\tilde M$ with the following properties.*

1. *The measures $\bar\nu_f^x$ are finite and positive on open sets.*
2. *$\bar\nu_f^{\Psi x} = \bar\nu_f^x \circ \Psi^{-1}$ for every $p \in \tilde M$ and every $\Psi \in \pi_1(M)$.*
3. *The measure class $mc(f)$ of $\bar\nu_f^x$ is independent of $x \in \tilde M$ and Γ-invariant.*
4. *There is a subset A of $\partial\tilde M$ of full measure such that for every $\xi \in A$ and all $x,y \in \tilde M$ the Radon Nikodym derivative $\frac{d\bar\nu_f^y}{d\bar\nu_f^x}(\xi) = Q(x,y,\xi)$ exists and its logarithm depends uniformly Lipschitz continuously on x,y independent of ξ. Moreover there is a constant $c>0$ such that*
$$Q(Pv, P\Phi^t v, \pi(v))/e^{\int_0^t f(\Phi^s v)ds} \in [c^{-1}, c]$$
for all $v \in \pi^{-1}(A)$ and all $t>0$.

Proof: Let f be Φ^t-regular with vanishing pressure. By Lemma 4.2 the critical exponent of f equals 0. Fix a point $x \in \tilde M$ and assume for the moment that the Poincaré series $\sum_{\Psi\in\Gamma} \tilde\zeta_x^{-1}(\Psi x)$ diverges.

For small $\epsilon>0$ and $y \in \tilde M$ write
$$\tilde\eta_{y,\epsilon} = \sum_{\Psi\in\Gamma} \zeta_y^{-1}(\Psi x) e^{-\epsilon\,\mathrm{dist}(\Psi x, y)} \delta_{\Psi x}$$
where δ_z is the Dirac mass at $z \in \tilde M$. Then $\tilde\eta_{y,\epsilon}$ is a finite Borel measure on $\tilde M$.

Define $\eta_{y,\epsilon} = \tilde\eta_{y,\epsilon}/\tilde\eta_{x,\epsilon}(\tilde M)$. Since f is Φ^t-regular the total mass of $\eta_{y,\epsilon}$ is bounded from above and below by a positive constant only depending on $\mathrm{dist}(x,y)$ and not on $\epsilon>0$. Moreover we have $\eta_{\Psi x,\epsilon} = \eta_{x,\epsilon} \circ \Psi^{-1}$.

Choose a sequence $\epsilon_i \to 0$ such that the measures η_{x,ϵ_i} converge as $i\to\infty$ weakly on the compact space $\tilde M \cup \partial\tilde M$ to a Borel measure η_x. Since the Poincaré series diverges the measure η_x is supported on $\partial\tilde M$. By Φ^t-regularity the measure class of η_x does not depend on the choice of x. The measures $\eta_{\Psi x,\epsilon_i}$ converge to the measure $\eta_x \circ \Psi^{-1} = \eta_{\Psi x}$ $(\Psi \in \Gamma)$ and η_x and $\eta_{\Psi x}$ are absolutely continuous. There is a universal constant $c>0$ such that the Radon Nikodym derivative of $\eta_{\Psi x}$ with respect to η_x is bounded from above by $e^{c\,\mathrm{dist}(x,\Psi x)}$.

Since Γ is countable there is a Borel subset A of $\partial\tilde M$ of full η_x-measure such that for every $\Psi\in\Gamma$ and every $\xi \in A$ the Radon Nikodym derivative of $\eta_{\Psi x}$ with respect to η_x exists at ξ. Moreover there is a constant $c>0$ such that whenever $v \in T^1_x\tilde M$ and $w \in T^1_{\Psi x}\tilde M$ are such that $\pi(v)=\pi(w)=\xi\in A$ and whenever $\tau\in\mathbb{R}$ is such that $\Phi^\tau w \in W^{ss}(v)$ then
$$c^{-1} \le \frac{d\eta_{\Psi x}}{d\eta_x}(\xi)/\limsup_{t\to\infty} e^{\int_0^{\tau+t} f(\Phi^s w)ds - \int_0^t f(\Phi^s v)ds} \le c.$$

The lemma now follows from the arguments in the proof of Lemma 2.1. We can also proceed as follows. Choose a number $\rho>0$ which is smaller than the injectivity radius of M. Let $\{B_i \mid i=1,\dots,r\}$ be a finite covering of M by open balls of the same radius ρ. Let x_j be the midpoint of B_j and choose a smooth partition of unity $\{\psi_j \mid j = 1, \dots, r\}$ subordinate to the covering $\{B_j\}$. Let K be a compact fundamental domain for the action of Γ on $\tilde{M}$ which contains x in its interior and such that the boundary of K does not meet the pre-image of the set $\{x_j \mid j \leq r\}$. Let $\tilde{x}_i$ be the lift of x_i which is contained in K. Define $\eta_{\tilde{x}_i} = \eta_x$ and for $\Psi \in \Gamma$ write $\eta_{\Psi\tilde{x}_i} = \eta_{\Psi x}$.

Let $\{\tilde{\psi}_i\}_i$ be the lift of the partition of unity ψ_i on M to $\tilde{M}$. Then $\tilde{\psi}_i$ is an infinite locally finite partition of unity for $\tilde{M}$, indexed by the union J of all points in the Γ-orbit of the set $\{\tilde{x}_1, \dots, \tilde{x}_r\}$.

For $y \in \tilde{M}$ define $\bar{\nu}_f^y = \sum_{u\in J} \tilde{\psi}_u(y)\eta_u$. Then $\bar{\nu}_f^y$ is a Borel probability measure on $\lambda \subset \partial\tilde{M}$ depending on y, $\bar{\nu}_f^{\Psi y} = \bar{\nu}_f^y \circ \Psi^{-1}$ and moreover the measures $\bar{\nu}_f^y$ are absolutely continuous with respect to η_x. By construction, for every $\xi \in A$ and all $x, y \in \tilde{M}$ the Radon Nikodym derivative of $\bar{\nu}_f^y$ with respect to $\bar{\nu}_f^x$ exists at ξ and its logarithm depends smoothly and uniformly Lipschitz continuously on x and y. The estimate in 4) above for this Radon Nikodym derivative is immediate from the construction.

This finishes the proof of the proposition in the case that the Poincaré series diverges. If the Poincaré series converges then we add a fast decaying exponent as in the construction of Patterson [P76] to obtain our statement. □

Remark: 1) The measures $\bar{\nu}_f^x$ $(x \in \tilde{M})$ on $\partial\tilde{M}$ which we constructed in the above proof from a Φ^t-regular function f are not uniquely determined by f. However, if we make a different choice in our above construction, then we find a new family of measures which coincide with the old ones up to a function which is measurable and bounded from above and below by a positive constant. Note that our construction was made in such a way that all the measures $\bar{\nu}_f^x$ are normalized, i.e. their total mass is one.

2) If the function f is Hölder continuous, then the measures $\bar{\nu}_f^x$ which we obtain from the construction in the proof of Proposition 4.3 have the additional property that the Radon Nikodym derivative $Q(x, y, \xi) = \frac{d\bar{\nu}_f^y}{d\bar{\nu}_f^x}(\xi)$ exists for all $\xi \in \partial\tilde{M}$ and all $x, y, \in \tilde{M}$. The function $Q : \tilde{M} \times \tilde{M} \times \partial\tilde{M} \to \mathbb{R}$ defined in this way is locally Hölder continuous. Moreover for every $\xi \in \partial\tilde{M}$ and $x \in \tilde{M}$ the function $y \to \log Q(x, y, \xi)$ is smooth and the norm of its derivative of an arbitrary fixed order is uniformly bounded. With the notations from the introduction, this shows that every Hölder continuous section of T^*W^s over T^1M is cohomologous to a Hölder continuous section whose restriction to every stable manifold is smooth, with uniformly bounded differential. That this is always possible has to our knowledge first been observed by Ledrappier [L95a].

Let again $\{\nu^p\}$ be a k-family for a nondegenerate measure ν of quasi-product type and let ν^{ss} be the induced regular family of stable measures. We lift these measure as before to measures on $T^1\tilde{M}$. For $v \in T^1M$ the measure ν^{ss} on $W^{ss}(v)$ projects to a locally finite Borel measure $\bar{\nu}_v^{ss}$ on $\partial\tilde{M} - \{\pi(v)\}$ which defines the measure class $mc(\nu_-)$. In particular, for $w \in T^1M$ sufficiently close to v the measures $\bar{\nu}_v^{ss}$

and $\bar{\nu}_w^{ss}$ are absolutely continuous and finite on a neighborhood of $\pi(-v)$ in $\partial\tilde{M}$. We call the regular family $\{\nu^p\}$ *uniformly regular* if there is a number $r > 0$ such that for all $v, w \in T^1M$ with $\text{dist}(v, w) < r$ the Radon Nikodym derivative of $\bar{\nu}_w^{ss}$ with respect to $\bar{\nu}_v^{ss}$ is bounded on $\pi \circ \mathcal{F}B^{ss}(v, r) \cup \pi \circ \mathcal{F}B^{ss}(w, r)$ by a uniform constant only depending on r but not on v, w.

With this notation we can now formulate an improved version of Proposition 4.1.

Corollary 4.4: *The unique Gibbs state ν_f of a Φ^t-regular function f on T^1M is a measure of nondegenerate quasi-product type. If $pr(f) = 0$ then there is a uniformly regular family $\{\nu_f^p\}$ of positive disintegration measures for ν_f whose Radon Nikodym kernel Q satisfies*

$$Q(Pv, P\Phi^t v, \pi(v))/e^{\int_0^t f(\Phi^s v)ds} \in [c^{-1}, c]$$

for a universal constant $c > 0$.

Proof: Let $f \in C_\Phi^0(N)$ with $pr(f) = 0$. By Proposition 3.3 there is some $k > 0$ and a k-family $\{\nu_f^p\}$ on T^1M whose Radon Nikodym kernel Q satisfies

$$Q(Pv, P\Phi^t v, \pi(v))/e^{\int_0^t f(\Phi^s v)ds} \in [c^{-1}, c]$$

for all $v \in T^1\tilde{M}, t \in \mathbb{R}$ and for a universal constant $c > 0$. The same construction for the Φ^t-regular function $f \circ \mathcal{F}$ of pressure zero gives a k-family $\{\nu_{f\circ\mathcal{F}}^p\}$ whose Radon Nikodym kernel $\mathcal{F}Q$ satisfies $\mathcal{F}Q(Pv, P\Phi^t v, \pi(v))/e^{\int_0^t f(-\Phi^s v)ds} \in [c^{-1}, c]$ whenever $v \in \pi^{-1}\Lambda \subset T^1\tilde{M}$.

The k-families $\{\nu_f^p\}, \{\nu_{f\circ\mathcal{F}}^p\}$ induce uniformly locally bounded measures ν^{su}, ν^{ss} on the leaves of $W^{su}, \mathcal{F}W^{su} = W^{ss}$. The measure $\hat{\nu}$ on T^1M defined by $d\hat{\nu} = d\nu^{su} \times d\nu^{ss} \times dt$ transforms under the geodesic flow via

$$\frac{d\hat{\nu} \circ \Phi^t}{d\hat{\nu}}(v) = Q(Pv, P\Phi^t v, \pi(v))\mathcal{F}Q(Pv, P\Phi^t v, \pi(-v)) \in [c^{-2}, c^2].$$

This means that the measures $\frac{1}{k}\sum_{i=1}^k \hat{\nu} \circ \Phi^i$ are absolutely continuous with respect to $\hat{\nu}$, with Radon Nikodym derivative in $[c^{-2}, c^2]$. In particular, there is a measurable function β with values in $[c^{-2}, c^2]$ and such that the measure $\beta\hat{\nu}$ is flow invariant. The measures $\beta\nu^{ss}$ are just the stable measures induced by the regular family $\{\nu_f^p\}$. They are locally uniformly finite on the intersection of the strong stable manifolds with the invariant set N since this is true for the measures ν^{ss}. Moreover, the Radon Nikodym derivatives of the projections of the measures ν^{ss} to Λ are locally uniformly bounded by construction. This shows that the measure $\tilde{\nu} = \beta\hat{\nu}$ admits a uniformly regular family $\{\nu_f^p\}$ of positive disintegration measures. Thus by Lemma 3.2 the measure is nondegenerate of quasi-product type.

We show next that $\beta\hat{\nu} = \tilde{\nu}$ is the Gibbs equilibrium state ν_f of the function f. For this recall the definition of the distances d_m $(m \geq 1)$ on T^1M. Observe from the explicit construction of the measure $\tilde{\nu}$ that there is for every $\epsilon > 0$ a number $c(\epsilon) > 0$ such that for $v \in T^1\tilde{M}$ and $m \geq 1$ we have

$$\tilde{\nu}\{w \mid d_m(v, w) < \epsilon\}e^{\int_0^m f(\Phi^s v)ds} \in [c(\epsilon)^{-1}, c(\epsilon)].$$

But this just means that $\tilde{\nu}$ is the unique Gibbs equilibrim state for f (see [HK]). As a consequence, $\tilde{\nu}$ is ergodic. This finishes the proof of our corollary. □

5. Solutions of leafwise elliptic equations

In this section we consider a differential operator $\Delta+Y$ of class $C^{1,\alpha}$ on T^1M as in the introduction and of negative escape. We use a measure ν of quasi-product type to construct Γ-equivariant $mc(\nu_+)$-measurable solutions of the equation $\Delta+Y=0$ on $T^1\tilde{M}$. We continue to use the assumptions and notations from Sections 1-4.

Let ν be a Φ^-t-invariant Borel probability measure on T^1M of quasi-product type. Choose a Γ-k-family $\{\bar{\nu}^x\}$ of positive disintegration measures for ν on $\partial\tilde{M}$, induced strong unstable family $\{\nu^{su}\}$ and strong stable family $\{\nu^{ss}\}$. The measures ν^{su} are well defined on *every* strong unstable manifold, but this may not be true for the measures ν^{ss}. However there is a subset A of $\partial\tilde{M}$ of full $mc(\nu_+)$-mass such that for every $w\in\pi^{-1}(A)$ the measure ν^{ss} is defined on $W^{ss}(w)$. Let $Q:\tilde{M}\times\tilde{M}\times\partial\tilde{M}\to\mathbb{R}$ be the Radon Nikodym kernel of the Γ-k-family $\{\bar{\nu}^x\}$ on $\partial\tilde{M}$ defined by the k-family $\{\nu^p\}$.

For every $v\in T^1\tilde{M}$ the projection $\pi\circ\mathcal{F}$ maps the strong stable manifold $W^{ss}(v)$ onto $\partial\tilde{M}-\pi(v)$. Let again d^{ss} be the distance function on the leaves of the strong stable foliation. Recall also the definition of the balls $B^{ss}(v,r)$ $(r>0)$ on strong stable manifolds. For $v\in T^1M$ and $w\in W^{ss}(v)$ define $\rho(v,w)=\inf\{t\geq 0\mid w\in\Phi^{-t}B^{ss}(\Phi^t v,1)\}=\inf\{t\geq 0\mid w\in B^{ss}(v,e^t)\}$. Choose a number $\chi>0$ and for $v\in\pi^{-1}(A)$ define a locally finite Borel measure η^v on $W^{ss}(v)$ by $\eta^v(w)=\min\{e^{-\chi(\rho(w,v))}Q(Pw,P\Phi^{\rho(v,w)}w,\pi(w))^{-1},1\}d\nu^{ss}(w)$. We have.

Lemma 5.1: *For ν-almost every $v\in T^1M$ the measure η^v on $W^s(v)$ is finite.*

Proof: We showed in the proof of Lemma 3.1 that for ν-almost every $v\in T^1M$ and every lift $\tilde{v}$ of v to $T^1\tilde{M}$ we have $\limsup_{t\to\infty}\frac{1}{t}-\log\nu^{ss}B^{ss}(v,^{-t})=\lim_{t\to\infty}\frac{1}{t}\log Q(P\tilde{v},P\Phi^t\tilde{v},\pi(\tilde{v}))$. Now let $v\in T^1M$ be such a point. By leafwise Lipschitz continuity of Q and the transformation rule for the measures ν^{ss} under the geodesic flow there is a number $T>0$ depending on v such that $\nu^{ss}(B^{ss}(\Phi^t v,1)\leq e^{\chi t/2}$ for all $t\geq T$. In other words, for $k\geq T$ the ν^v-mass of the set $\Phi^{-k}B^{ss}(\Phi^k v,1)-\Phi^{-k+1}B^{ss}(\Phi^{k-1}v,1)$ is bounded from above by a universal constant times $e^{-\chi k/2}$ and therefore the total mass of the measure ν^v is finite. □

The measure ν^v projects via the map $\pi\circ\mathcal{F}$ to a measure $\bar{\eta}^v$ on $\partial\tilde{M}$. The measures $\bar{\eta}^v$ are invariant under the action of Γ on $T^1\tilde{M}\times\partial\tilde{M}$.

For $v\in\pi^{-1}(A)$ the measure $\bar{\eta}^v$ is defined on $\partial\tilde{M}$. For $t\in\mathbb{R}$ the measures $\bar{\eta}^v$ and $\bar{\eta}^{\Phi^t v}$ define the same measure class. For $\xi\in\partial\tilde{M}-\{\pi(v)\}$ denote by $L(v,t,\xi)$ the Radon Nikodym derivative of the measure $\bar{\eta}^{\Phi^t v}$ with respect to the measure $\bar{\eta}^v$ at ξ. The function $L(v,\cdot,\xi)$ satisfies the cocycle identity $\log L(v,s+t,\xi)=$

$\log L(v,t,\xi) + \log L(\Phi^t v, s, \xi)$. By construction of our measures, for all $t > 0$, for every $v \in \pi^{-1}(A)$ and every $w \in W^{ss}(v)$, all $t > 0$ we have

$$Q(Pw, P\Phi^t w, \pi(v))^{-1} \le L(v,t,\pi(-w)) \le Q(Pw, P\Phi^t w, \pi(v))^{-1} e^{\chi t}. \tag{12}$$

The function $\log L(v,1,\cdot)$ is uniformly Hölder continuous on $\partial\tilde{M}$ with a Hölder constant only depending on k but not on v. Now for $v \in T^1\tilde{M}$ and $w \in W^{ss}(v)$ the distance between $\Phi^t v$ and $\Phi^t w$ goes to zero exponentially fast as $t \to \infty$ and therefore from this and the cocycle identity we infer that for every $v \in \pi^{-1}(A)$ and every $w \in W^{ss}(v)$ the limit

$$\lim_{t\to\infty} L(v,t,\pi(-w))/L(v,t,\pi(-v)) = \sigma(v,w) \tag{13}$$

exists. The function $\sigma(v,\cdot)$ is locally Hölder continuous on $W^{ss}(v)$. Moreover there is a universal constant $c > 0$ such that for all $v \in \pi^{-1}(A)$, all $w \in W^{ss}(v)$ and all $t > 0$ we have $\sigma(v,w)L(v,t,\pi(-v))/L(v,t,\pi(-w)) \ge c^{-1}$ and $\le c$ if $w \in B^{ss}(v,1)$ where again $c > 0$ only depends on k.

Let again Δ be the stable Laplacian of an arbitrary metric g on $TW^s \subset TT^1M$ which satisfies the regularity assumptions stated in the introduction. Let Y be a section of TW^s of class $C_s^{1,\alpha}$ which is g-dual to a leafwise closed section of T^*W^s and such that $pr(g(X,Y)) < 0$ where the pressure of the function $g(X,Y)$ is defined as in Section 4. Denote the lift of $\Delta + Y$ to an operator on $T^1\tilde{M}$ by the same symbol. For every $v \in T^1\tilde{M}$ the restriction of $\Delta + Y$ to $W^s(v)$ projects to a second order uniformly elliptic operator $\Delta_v + Y_v$ on $\tilde{M}$. By Theorem A of [H97a], our assumption on Y implies that $\Delta_v + Y_v$ is weakly coercive.

Write

$$\kappa = -\mathrm{div}(Y/2) - g(Y/2, Y/2) \tag{14}$$

where div means divergence with respect to the metric g on the leaves of W^s. Then $\Delta + \kappa$ is a second order leafwise elliptic operator on T^1M with Hölder continuous coefficients whose restriction to every stable manifold is self-adjoint with respect to g. We denote its lift to $T^1\tilde{M}$ by the same symbol. Let $\Delta_v + \kappa_v$ be the projection to $\tilde{M}$ of the restriction of $\Delta + \kappa$ to $W^s(v)$.

Since the function $g(X,Y)$ is Hölder continuous it defines a unique Hölder continuous kernel K on $\tilde{M} \times \tilde{M} \times \partial\tilde{M}$. This kernel is determined by the requirement that for $v \in T^1\tilde{M}$ and $t > 0$ we have $K(Pv, P\Phi^t v, \pi(v)) = \int_0^t g(X,Y)(\Phi^s v)ds$. For every $v \in T^1\tilde{M}$ and every $\Delta_v + Y_v$-harmonic function f on $\tilde{M}$ the function $fK(Pv,\cdot,\pi(v))^{1/2}$ is $\Delta_v + \kappa_v$-harmonic. This implies in particular that the operator $\Delta_v + \kappa_v$ is weakly coercive and therefore its *Martin kernel* K_v is defined and is a Hölder continuous function on $\tilde{M} \times \tilde{M} \times \partial\tilde{M}$. Since the constant function 1 is the unique minimal positive $\Delta_v + Y_v$-harmonic function on $\tilde{M}$ with pole at $\pi(v)$ (see [H97a]), the function $K(Pv,\cdot,\pi(v))^{1/2}$ is the unique minimal positive $\Delta_v + \kappa_v$-harmonic function on $\tilde{M}$ with pole at $\pi(v)$. In other words, we have $K_v(x,y,\pi(v)) = K(x,y,\pi(v))^{1/2}$ for all $v \in T^1\tilde{M}$ and all $(x,y) \in \tilde{M} \times \tilde{M}$.

Let ν be a measure of quasi-product type. Choose a regular k-family $\{\nu^p\}$ of positive disintegration measures for ν with Radon Nikodym kernel Q. For $v \in T^1\tilde{M}$

write $Q_v = Q(K_v)^{-1}$. Let $\{\nu^{ss}\}$ be the stable family for ν induced by the regular family $\{\nu^p\}$ and for $v \in \pi^{-1}(A)$ we define a finite measure $\bar\eta^v$ on $\partial\tilde M$ as above.

For $v \in \pi^{-1}(A)$, $t \in \mathbb{R}$ and $\xi \in \partial\tilde M - \{\pi(v)\}$ denote by $L_v(Pv, P\Phi^t v, \xi) = L(v,t,\xi)$ the Radon Nikodym derivative of the measure $\bar\eta^{\Phi^t v}$ with respect to $\bar\eta^v$ at ξ. By the results of [A1] and our assumption on the measures $\bar\eta^w$, for every $v \in \pi^{-1}(A)$ and $w \in W^{ss}(v)$ the limit

$$\lim_{t\to\infty}(L_vK_v^{-1})(Pv, P\Phi^t v, \pi(-w))/(L_vK_v^{-1})(Pv, P\Phi^t v, \pi(-v)) = \alpha_{v,\nu}(w)$$

exists and is finite and positive. The following Lemma gives an estimate for the functions $\alpha_{v,\nu}$. As before denote by K the kernel of the function $g(X,Y)$ on T^1M.

Lemma 5.2: *There is a constant $c > 0$ with the following properties: Let $v \in T^1\tilde M, w \in W^{ss}(v)$ and let $\tau = \inf\{t \geq 0 \mid w \in \Phi^{-t}B^{ss}(\Phi^t v, 1)\}$; then*

$$\alpha_{v,\nu}(w)K(Pv, P\Phi^\tau v, \pi(v))L_v(v, \Phi^\tau v, \pi(-w))/Q(Pv, P\Phi^\tau v, \pi(v)) \in [c^{-1}, c].$$

Proof: Let $v \in T^1\tilde M, w \in W^{ss}(v)$ and let $\tau = \inf\{t \geq 0 \mid w \in \Phi^{-t}B^{ss}(\Phi^t v, 1)\} \geq 0$. We first claim that there is a universal constant $c_0 > 0$ such that

$$(15)\quad K_v(Pv, P\Phi^t v, \pi(-v)) \cdot K_v(Pv, P\Phi^t v, \pi(w))/K(Pv, P\Phi^\tau v, \pi(v)) \in [c_0^{-1}, c_0]$$

for all $t \geq \tau$.

For this let $u \in T^1_{Pv}\tilde M - \{v\}$, $\chi \geq 0$ be such that the geodesic γ joining $\gamma(-\infty) = \pi(-w)$ to $\gamma(\infty) = \pi(v)$ meets the geodesic $t \to P\Phi^t u$ orthogonally at $P\Phi^\chi u$. By the definition of the balls $B^{ss}(v,r)$ the distance between $P\Phi^\chi u$ and $P\Phi^\tau w$ is bounded from above by a universal constant. Since the kernel K is uniformly Hölder continuous we conclude that there is a universal constant $c_1 > 0$ such that

$$(16)\qquad K(Pv, P\Phi^\chi u, \pi(u))/K(Pv, P\Phi^\tau v, \pi(v)) \in [c_1^{-1}, c_1].$$

Thus we may replace $K(Pv, P\Phi^t v, \pi(v))$ in our claim (15) by $K(Pu, P\Phi^\chi u, \pi(u))$.

Since the operators $\Delta_v + \kappa_v$ are self - adjoint there is a constant $c_2 > 0$ such that $K_v(Pv, P\Phi^t v, \pi(-v)) \cdot K_v(Pv, P\Phi^t v, \pi(v)) \in [c_2^{-1}, c_2]$ for all $v \in T^1\tilde M$ and all $t \geq 0$ (see [H97a]).

Moreover the function $\log K_v$ is locally uniformly Hölder continuous and therefore we have

$$(17)\qquad K_v(Pv, P\Phi^\chi u, \pi(-v)) \cdot K(Pu, P\Phi^\chi u, \pi(u))^{1/2} \in [c_3^{-1}, c_3] \quad \text{and}$$
$$K(Pu, P\Phi^\chi u, \pi(u))^{1/2}/K_v(Pv, P\Phi^\chi u, \pi(w)) \in [c_3^{-1}, c_3]$$

for a universal constant $c_3 > 0$.

From the Harnack inequality at infinity of Ancona [A1], applied to the $\Delta_v + \kappa_v$-harmonic functions $K_v(P\Phi^\tau u, \cdot, \pi(w))$ and $K_v(P\Phi^\tau u, \cdot, \pi(-v))$ on a cone in $\tilde M$ about the geodesic joining $P\Phi^\tau u$ to $\pi(v)$ we deduce that

$$(18)\qquad K_v(P\Phi^\chi u, P\Phi^t v, \pi(w))/K_v(P\Phi^\chi u, P\Phi^t v, \pi(-v)) \in [c_4^{-1}, c_4]$$

for all sufficiently large $t \geq \tau$. Since

$$K_v(Pv, P\Phi^t v, \pi(w)) = K_v(P\Phi^\chi u, Pv, \pi(w))^{-1} K_v(P\Phi^\chi u, P\Phi^t v, \pi(w)) \tag{19}$$

our claim follows.

On the other hand, for $t \geq \tau$ we have

$$L(\Phi^\tau v, t-\tau, \pi(-w)) = L_{\Phi^\tau v}(P\Phi^\tau v, P\Phi^t v, \pi(-w)) = Q(P\Phi^\tau w, P\Phi^t w, \pi(v))^{-1} \tag{20}$$

and since the function $\log Q(\cdot,\cdot,\pi(v))$ is k-Lipschitz in both variables and since $L_v(Pv, P\Phi^t v, \pi(-w)) = L_v(Pv, P\Phi^\tau v, \pi(-w)) L_v(P\Phi^\tau v, P\Phi^t v, \pi(-w))$ we obtain that

$$L_v(Pv, P\Phi^t v, \pi(-w)) L_v(Pv, P\Phi^\tau v, \pi(-v)) / L_v(Pv, P\Phi^t v, \pi(-v)) L_v(Pv, P\Phi^\tau v, \pi(-w)) \tag{21}$$

is bounded from above and below by a universal constant. From this and the definition of the function L_v the lemma follows. □

Corollary 5.3: *If*

$$\liminf_{t\to\infty} \frac{1}{t} \log(K(Pv, P\Phi^t v, \pi(v)) Q(Pv, P\Phi^t v, \pi(v))^{-1}) = a > 0$$

then $\int \alpha_{v,\nu}(\zeta) d\bar{\eta}^v(\zeta) < \infty$.

Proof: Recall from the definition that the measure $\bar{\eta}^v$ on $\partial\tilde{M}$ coincides on $\pi \circ \mathcal{F}B^{ss}(v,1)$ with the projection to $\partial\tilde{M}$ of the stable measure ν^{ss} on $W^{ss}(v)$ induced by the k-family $\{\nu^p\}$. Moreover, the total mass of $\bar{\eta}^v$ is finite.

By Lemma 5.2, for every positive integer $k > 0$ and every point $w \in B^{ss}(v, e^{k+1}) - B^{ss}(v, e^k)$ the ratio

$$\alpha_{v,\nu}(\pi(-w)) K(Pv, P\Phi^k v, \pi(v)) L_v(Pv, P\Phi^k v, \pi(-w)) / Q(Pv, P\Phi^k v, \pi(v)) \tag{22}$$

is bounded from above and below by a positive constant. Since $L_v(Pv, P\Phi^k v, \pi(-w))^{-1}$ is the Radon Nikodym derivative of the measure $\bar{\eta}^{\Phi^k v}$ with respect to the measure $\bar{\eta}^v$ at $\pi(w)$, this just means that there is a number $\beta > 0$ such that

$$\begin{aligned} &\int_{\partial\tilde{M}} \alpha_{v,\nu}(\pi(-w)) d\bar{\eta}^v(\pi(w)) \\ &\leq \sum_{k=0}^{\infty} \beta Q(Pv, P\Phi^k v, \pi(v)) / K(Pv, P\Phi^k v, \pi(v)) \eta^{\Phi^k v}(B^{ss}(\Phi^k v, 1)). \end{aligned} \tag{23}$$

By our assumption on v and the fact that $\limsup_{t\to\infty} -\frac{1}{t} \log \nu^{ss}(B^{ss}(\Phi^t v, 1)) = 0$, we conclude that the series on the right hand side of the above inequality is convergent. □

For every $v \in \pi^{-1}(A)$ we can define a function $u_{v,\nu}\colon \tilde{M} \times \tilde{M} \to (0,\infty)$ by $u_{v,\nu}(x,y) = \int K_v(x,y,\zeta) d\bar{\eta}^v(\zeta)$. For any fixed $x \in \tilde{M}$ the assignment $y \to u_{v,\nu}(x,y)$ is a positive $\Delta_v + \kappa_v$-harmonic function on $\tilde{M}$. The functions $u_{v,\nu}$ are invariant under the action of Γ on $T^1\tilde{M} \times \tilde{M} \times \tilde{M}$ and $u_{v,\nu}(Pv, Pv) = \bar{\eta}^v(\partial\tilde{M})$ for all $v \in \pi^{-1}(A)$.

Lemma 5.4: *For every $v \in \pi^{-1}(A)$ the functions*

$$u_{\Phi^t v,\nu}(P\Phi^t v, \cdot)/u_{\Phi^t v,\nu}(P\Phi^t v, Pv)$$

converge as $t \to \infty$ uniformly on compact subsets of $\tilde{M}$ to a positive $\Delta_v + \kappa_v$-harmonic function $\tilde{u}_{v,\nu}$ on $\tilde{M}$. The assignment $v \to \tilde{u}_{v,\nu}$ is equivariant under the action of the fundamental group $\pi_1(M)$ of M. If

$$\lim_{t\to\infty} \inf \frac{1}{t} \log(K(Pv, P\Phi^t v, \pi(v))Q(Pv, P\Phi^t v, \pi(v))^{-1}) > 0$$

then $\tilde{u}_{v,\nu}(y) = \int K_v(Pv, y, \zeta)\alpha_{v,\nu}(\zeta)\, d\bar{\eta}^v(\zeta)/\int \alpha_{v,\nu}(\zeta) d\bar{\eta}^v(\zeta)$.

Proof: Let $x \in \tilde{M}$ and $v \in T^1_x\tilde{M}$. For $t \geq 0$ define a positive $\Delta_v + \kappa_v$-harmonic function $a^t_{v,f}$ on $\tilde{M}$ by

$$a^t_{v,\nu}(y) = u_{\Phi^t v,\nu}(P\Phi^t v, y)/F_v(Pv, P\Phi^t v, \pi(-v))$$

where $F_v = L_v K_v^{-1}$. The Radon-Nikodym derivative of the measure $\bar{\eta}^{\Phi^t v}$ with respect to $\bar{\eta}^v$ at ζ equals $K_v(Pv, P\Phi^t v, \zeta)F_v(Pv, P\Phi^t v, \zeta)$ and consequently

$$a^t_{v,\nu}(y) = u_{\Phi^t v,\nu}(P\Phi^t v, y)/F_v(Pv, P\Phi^t v, \pi(-v)) \tag{24}$$

$$= F_v(Pv, P\Phi^t v, \pi(-v))^{-1} \int_{\partial\tilde{M}} K_v(P\Phi^t v, y, \zeta) d\bar{\eta}^{\Phi^t v}(\zeta) \tag{25}$$

$$= \int_{\partial\tilde{M}} K_v(Pv, y, \zeta) F_v(Pv, P\Phi^t v, \pi(-v))^{-1} F_v(Pv, P\Phi^t v, \zeta) d\bar{\eta}^v(\zeta) \tag{26}$$

for all $y \in \tilde{M}$. Thus if we write $\alpha^t_{v,\nu}(\zeta) = F_v(Pv, P\Phi^t v, \zeta)/F_v(Pv, P\Phi^t v, \pi(-v))$ then

$$a^t_{v,\nu}(y) = \int K_v(Pv, y, \zeta)\alpha^t_{v,\nu}(\zeta) d\bar{\eta}^v(\zeta)$$

for all $y \in \tilde{M}, v \in T^1\tilde{M}$ and $t \geq 0$.

We distinguish two cases:

(1) $\int \alpha_{v,\nu}(\zeta) d\bar{\eta}^v(\zeta) = \infty$.

We claim that then the functions $u_{\Phi^t v,\nu}(P\Phi^t v, \cdot)/u_{\Phi^t v,\nu}(P\Phi^t v, Pv)$ converge as $t \to \infty$ uniformly on compact subsets of $\tilde{M}$ to the minimal positive $\Delta_v + \kappa_v$-harmonic function $y \to K_v(x, y, \pi(v))$. For this it is enough to show the following: For $t > 0$ write $\bar{\alpha}^t_{v,\nu} = \alpha^t_{v,\nu}/\int \alpha^t_{v,\nu}(\zeta) d\bar{\eta}^v(\zeta)$; then for every $\delta > 0$ there is a number $\tau(\delta) > 0$ such that

$$\int_{\partial\tilde{M} - C(v,\delta)} \bar{\alpha}^t_{v,\nu}(\zeta) d\bar{\eta}^v(\zeta) < \delta$$

for all $t \geq \tau(\delta)$, where $C(v, \delta) = \{\pi(w) \mid \angle(v, w) < \delta\}$. To see this simply recall that the functions $\alpha^t_{v,\nu}$ converge as $t \to \infty$ uniformly on compact subsets of $\partial\tilde{M} - \{\pi(v)\}$ to $\alpha_{v,\nu}$. Since $\alpha_{v,\nu}$ is bounded on $\partial\tilde{M} - C(v, \delta)$ the integrals $\int_{\partial\tilde{M}-C(v,\delta)} \alpha^t_{v,\nu}(\zeta) d\bar{\eta}^v(\zeta)$ are bounded on $\partial\tilde{M} - C(v, \delta)$ as well and hence $\int_{C(v,\delta)} \alpha^t_{v,\nu}(\zeta) d\bar{\eta}^v(\zeta) \to \infty$ $(t \to \infty)$.

As a consequence, for every fixed $\delta > 0$ the functions $\bar{\alpha}^t_{v,\nu}$ converge to zero on $\partial\tilde{M} - C(v,\delta)$ as $t \to \infty$. From this the above claim is immediate.

Now assume that

2) $\int \alpha_{v,\nu}(\zeta) d\bar{\eta}^v(\zeta) < \infty$.

We claim that $\int \alpha^t_{v,\nu}(\zeta) d\bar{\eta}^v(\zeta) \to \int \alpha_{v,\nu}(\zeta) d\bar{\eta}^v(\zeta)(t \to \infty)$. To see this, recall from $(*)$ above that for every $v \in T^1M$ and $k > 0$, all $t > k$ the ratio $\alpha^t_{v,\nu}/\alpha_{v,\nu}$ is bounded on $\partial\tilde{M} - \pi B(v, re^{-k})$ from above and below by a universal positive constant, where $r > 0$ is as in the beginning of this section. On the other hand, by Hölder continuity of the kernel K_v there is a universal constant $a > 0$ such that

$$\int_{\pi B(v,re^{-k})} \alpha^k_{v,\nu} d\bar{\eta}^v(\zeta)/K_v^2 L_v(Pv, P\Phi^k v, \pi(-v))\bar{\eta}^{\Phi^k v}(\pi B(\Phi^k v, r)) \in [a^{-1}, a].$$

By the choice of the measures $\bar{\eta}^w$, the $\bar{\eta}^{\Phi^k v}$-mass of the set $\pi(B(v, re^{-k}) - B(v, re^{-k-1}))$ is not smaller than a constant multiple of $\bar{\eta}^{\Phi^k v} B(v, re^{-k})$. Together with the above estimates we conclude from this that there is a universal constant $\hat{a} > 0$ such that $\int \alpha^t_{v,\nu}(\zeta) d\bar{\eta}^v(\zeta) \le \hat{a} \int \alpha_{v,\nu}(\zeta) d\bar{\eta}^v(\zeta)$ for all $t > 0$. Since $\alpha^t_{v,\nu} \to \alpha_{v,\nu}$ uniformly on compact subsets of $\partial\tilde{M} - \{\pi(v)\}$ the above claim now follows from Lebesgue's theorem of dominated convergence.

Now if we write again $\bar{\alpha}^t_{v,\nu} = \alpha^t_{v,\nu}/\int \alpha^t_{v,\nu}(\zeta)\, d\bar{\eta}^v(\zeta)$ and $\bar{\alpha}_{v,\nu} = \alpha_{v,\nu}/\int \alpha_{v,\nu}(\zeta)\, d\bar{\eta}^v(\zeta)$, then $\bar{\alpha}^t_{v,\nu} \to \bar{\alpha}_{v,\nu}$ in the space of $\bar{\eta}^v$-integrable functions on $\partial\tilde{M}$. Since

$$u_{v,\nu}(P\Phi^t v, y)/u_{v,\nu}(P\Phi^t v, Pv) = \int K_v(x,y,\zeta)\bar{\alpha}^t_{v,\nu}(\zeta) d\bar{\eta}^v(\zeta)$$

this means that these functions converge as $t \to \infty$ uniformly on compact subsets of $\tilde{M}$ to $y \to \tilde{u}_{v,\nu}(y) = \int K_v(x,y,\zeta)\bar{\alpha}_{v,\nu}(\zeta) d\bar{\eta}^v(\zeta)$ as claimed in the lemma. □

Let f be a Hölder continuous function on T^1M and denote its lift to $T^1\tilde{M}$ by the same symbol. Recall that $\partial\tilde{M}$ admits a natural Hölder structure and hence the same is true for $\tilde{M} \times \tilde{M} \times \partial\tilde{M}$. There is a unique locally uniformly Hölder continuous function $F : \tilde{M} \times \tilde{M} \times \partial\tilde{M} \to (0,\infty)$ such that for every $v \in T^1\tilde{M}$ we have $F(Pv, P\Phi^t v, \pi(v)) = e^{\int_0^t f(\Phi^s v) ds}$. We call the function F the *kernel* of f. The kernel F of f satisfies $F(x,y,\zeta)F(y,z,\zeta) = F(x,z,\zeta)$ for all $x,y,z \in \tilde{M}, \zeta \in \partial\tilde{M}$. For example, the kernel C of the constant function 1 on T^1M is given as follows: For $\xi \in \partial\tilde{M}$ let θ_ξ be a *Busemann function* at ξ. Then $C(x,y,\xi) = e^{\theta_\xi(x) - \theta_\xi(y)}$ (compare [H97a]). If $\mathcal{F} : T^1M \to T^1M$ is again the flip $v \to -v = \mathcal{F}v$ on T^1M, then the function $f \circ \mathcal{F}$ is Hölder continuous and induces a kernel $\mathcal{F}F$.

Choose a point $x \in \tilde{M}$ and for $v \in T^1_x\tilde{M}$ and $t > 0$ write $\tilde{\zeta}_x(P\Phi^t v) = F(x, P\Phi^t v, \pi(v))$. Let λ be the Lebesgue measure on $\tilde{M}$ and define the *pressure* $pr(f)$ of f to be the infimum of all numbers $m \in \mathbb{R}$ such that

$$\int \zeta_x^{-1}(y) e^{-m\,\mathrm{dist}(x,y)} d\lambda(y) < \infty.$$

Note first that this does not depend on the choice of $x \in \tilde{M}$. Note also that the so-defined pressure coincides with the usual topological pressure of f.

Consider again the kernel F of the Hölder continuous function f. The function F is invariant under the action of Γ. Since for $v \in T^1\tilde{M}$ and $w \in W^{ss}(v)$ the distance in $\tilde{M}$ between $P\Phi^t v$ and $P\Phi^t w$ converges to zero exponentially fast, for $v \in T^1\tilde{M}$ and $w \in T^1_{Pv}\tilde{M} - \{v\}$ the limit

$$\alpha_f(v,w) = \frac{1}{2}\lim_{t\to\infty}[\log F(Pv, P\Phi^t v, \pi(w)) - \log F(Pv, P\Phi^t v, \pi(-v))] \tag{27}$$

exists (compare [H97a] for the definition of α_f) and is a Hölder continuous function on the complement of the diagonal in $T^1_{Pv}\tilde{M} \times T^1_{Pv}\tilde{M}$.

Lemma 5.5: *Let $p \in \tilde{M}$, $v, w \in T^1_p\tilde{M}$ and let γ be a geodesic in $\tilde{M}$ joining $\gamma(-\infty) = \pi(w)$ to $\gamma(\infty) = \pi(v)$. Then we have*

$$\begin{aligned} \alpha_f(v,w) &= \alpha_{f\circ\mathcal{F}}(w,v) \quad \text{and} \\ \alpha_f(v,w) &= \frac{1}{2}[\log F(Pv,\gamma(t),\pi(w)) + \log \mathcal{F}F(Pv,\gamma(t),\pi(v))] \end{aligned} \tag{28}$$

for all $t \in \mathbb{R}$.

Proof: Let $v \in T^1\tilde{M}$, $w \in T^1_{Pv}\tilde{M} - \{v\}$ and let γ be as in the lemma. By the definition of $\mathcal{F}F$ we have

$$\begin{aligned} \log \mathcal{F}F(\gamma(s),\gamma(t),\pi(v)) &= \int_s^t f\circ\mathcal{F}(\gamma'(\tau))d\tau = \int_0^{t-s} f(-\Phi^\tau\gamma'(s))d\tau \\ &= \int_0^{t-s} f\big(\Phi^\tau(-\gamma'(t))\big)d\tau = \log F(\gamma(t),\gamma(s),\pi(w)) = -\log F(\gamma(s),\gamma(t),\pi(w)) \end{aligned} \tag{29}$$

for all $s, t \in \mathbb{R}$.

Assume now that γ is parametrized in such a way that $\gamma'(0) \in W^{ss}(v)$; then $\mathrm{dist}(P\Phi^t v, \gamma(t)) \to 0$. This means that if we denote by $z_1(t), z_2(t)$ the unit vectors with foot point $P\Phi^t v, \gamma(t)$ and which are contained in the stable manifold $W^s(w)$ through w then $\lim_{t\to\infty}\mathrm{dist}(z_1(t), z_2(t)) = 0$ as well.

For $t \in \mathbb{R}$ let now $\tau_1(t)$ be such that $z_1(t) \in W^{ss}(\Phi^{\tau_1(t)}w)$. By construction we have

$$\log F(Pv, P\Phi^t v, \pi(w)) = \lim_{s\to\infty}\Big(\int_0^{s-\tau_1(t)} f(\Phi^\sigma z_1(t))d\sigma - \int_0^s f(\Phi^\sigma w)d\sigma\Big).$$

Similarly, if $\tau_2(t) \in \mathbb{R}$ is such that $z_2(t) \in W^{ss}(\Phi^{\tau_2(t)}w)$ then

$$\log F(Pv,\gamma(t),\pi(w)) = \lim_{s\to\infty}\Big(\int_0^{s-\tau_2(t)} f(\Phi^\sigma z_2(t))d\sigma - \int_0^s f(\Phi^\sigma w)d\sigma\Big). \tag{30}$$

Since f is Hölder continuous and since the distance between $z_1(t), z_2(t) \in W^s(w)$ converges with $t \to \infty$ exponentially fast to 0 we obtain that

$$\limsup_{t\to\infty}\limsup_{s\to\infty}\Big|\int_0^{s-\tau_1(t)} f(\Phi^\sigma z_1(t))d\sigma - \int_0^{s-\tau_2(t)} f(\Phi^\sigma z_2(t))d\sigma\Big| = 0.$$

This then shows that

$$\lim_{t\to\infty} \log F(Pv, P\Phi^t v, \pi(w)) - \log F(Pv, \gamma(t), \pi(w)) = 0,$$

$$\lim_{t\to\infty} \log F(Pv, P\Phi^t v, \pi(-v)) - \log \mathcal{F}F(Pv, \gamma(t), \pi(v)) = 0.$$

and yields the lemma. □.

Lemma 5.5 shows in particular that for the constant function 1 on T^1M and $v, w \in C$, $\alpha_1(v, w)$ is just the *Gromov product*

$$(v|w) = (\pi(v)|\pi(w))_{Pv}$$

of $\pi(v), \pi(w) \in \partial\tilde{M}$ as seen from Pv (see [GH90] for the definition of the Gromov product and compare [H97a]).

Let as before $\{\nu^p\}$ be a regular family of positive disintegration measures for a measure ν of quasi-product type. Denote by η^v ($v \in T^1M$) the family of finite Borel measures on the strong stable manifolds as above. Since via the projection π for every $x \in \tilde{M}$ the bundle $\{(v, w) \mid v \in T^1_x\tilde{M}, w \in W^{ss}(v)\}$ naturally embeds into $T^1_x\tilde{M} \times T^1_x\tilde{M}$ we can view η^v as a measure on $T^1_{Pv}\tilde{M}$. Define a measure η^p on $T^1_p\tilde{M} \times T^1_p\tilde{M}$ by $\eta^p(A) = \int \nu^v(A \cap T^1_p\tilde{M} \times \{v\})d\nu^p(v)$. The next proposition is an important ingredience for the proof of our theorem in the introduction.

Proposition 5.6: *If $f : T^1M \to \mathbb{R}$ is Hölder continuous and $pr(-f - f\circ\mathcal{F}) < 0$ then the function $p \to \iint e^{2\alpha_f(v,w)}\sigma(v, w)d\eta^p_\nu(v, w)$ on M is locally bounded.*

Proof: Choose a number $\chi > 0$ such that for every $v \in T^1M$ the balls $B(v, \chi)$ and $B(-v, \chi)$ are disjoint. Let $\delta > 0$ be smaller than the constant $r > 0$ which appears in the definition of the measures η^v.

For a unit tangent vector $v \in T^1\tilde{M}$ define a compact subset $C(v)$ of the complement of the diagonal in $\partial\tilde{M} \times \partial\tilde{M}$ by

$$C(v) = \{(\pi(z), \pi(w)) \mid z \in B(v, \delta/4) - B(v, \delta e^{-1}/4), w \in B(-v, \delta/4) - B(-v, \delta e^{-1}/4)\}. \tag{31}$$

Clearly the set $C(v)$ has nonempty interior and depends continuously on v. Moreover for every isometry $\Psi \in \pi_1(M)$ we have $C(d\Psi(v)) = \Psi \times \Psi(C(v))$.

Fix now a point $x \in \tilde{M}$ and for $R > 0$ let $S(x, R)$ and $B(x, R)$ be the distance sphere of radius R and the closed ball of radius R about x in $\tilde{M}$. Let N be the unit vector field on $\tilde{M} - \{x\}$ whose restriction to $S(x, R)$ just equals the outer normal field of $S(x, R)$. For $y \in \tilde{M} - \{x\}$ define $C(y) = C(N(y)) \subset \partial\tilde{M} \times \partial\tilde{M}$.

Write $\Omega = \{(v, w) \in T^1_x\tilde{M} \times T^1_x\tilde{M} \mid v \neq w\}$. Then Ω is an open subset of $T^1_x\tilde{M} \times T^1_x\tilde{M}$ whose complement has measure zero with respect to η^x_ν.

Let $(v, w) \in \Omega$ and let γ be a geodesic joining $\gamma(-\infty) = \pi(w)$ to $\gamma(\infty) = \pi(v)$. Then the function $t \to \mathrm{dist}(x, \gamma(t))$ is strictly convex and hence it assumes its minimum in a unique point. Denote by $\gamma_{v,w}$ the unique reparametrization of γ with the property that

$$\operatorname{dist}(\gamma_{v,w}(0),x)=\min\{\operatorname{dist}(\gamma_{v,w}(t),x)\mid t\in\mathbb{R}\}. \tag{32}$$

For $R\geq 0$ define $\Omega(R)=\{(v,w)\in\Omega\mid \operatorname{dist}(x,\gamma_{v,w}(\mathbb{R}))=R\}$. Then Ω is a disjoint union of the sets $\Omega(R)$ and for each fixed $r\geq 0$ the subset $\bigcup_{R\leq r}\Omega(R)$ of Ω is compact. Moreover $\Omega(R)$ is naturally homeomorphic to the unit tangent bundle $T^1S(x,R)$ of the distance sphere $S(x,R)$ by mapping $(v,w)\in\Omega(R)$ to $\gamma_{v,w}'(0)\in T^1S(x,R)$.

From the definition of the sets $C(y)$ $(y\in\tilde M)$ we also see that there is a universal constant $\tau>0$ such that

$$\cup\{C(y)\mid \operatorname{dist}(x,y)\geq r+\tau\}\subset\cup_{R\leq r}\Omega(R)\subset\cup\{C(y)\mid\operatorname{dist}(x,y)\geq r-\tau\}. \tag{33}$$

Denote by Q the Radon Nikodym kernel of the k-family $\{\nu^p\}$ and recall that for $\xi\in\partial\tilde M$ and $y\in\tilde M$ the Radon-Nikodym derivative of $\bar\nu^y$ with respect to $\bar\nu^x$ at ξ equals $N(x,y,\xi)$.

Let $v\in T^1_x\tilde M$ be such that $\pi(v)=\xi\in A$. For $t\in\mathbb{R}$ and $\zeta\in\partial\tilde M-\{\xi\}$ define $L_\xi(x,P\Phi^tv,\zeta)=L(v,t,\zeta)$ to be the Radon Nikodym derivative of the measure $\bar\eta^{\Phi^tv}$ with respect to $\bar\eta^v$ at ζ. By definition,

$$L(x,y,\xi,\zeta)=Q(x,y,\xi)L_\xi(x,y,\zeta)$$

is the Radon Nikodym derivative of η_ν^y with respect to η_ν^x at (ξ,ζ). Since

$$L(v,t,\pi(-v))^{-1}=Q(Pv,P\Phi^tv,\pi(v))$$

we conclude from 4) in our above list of properties of the measures η^v that for sufficiently large $T>0$, $v\in T^1_x\tilde M$ and $w\in B(v,e^{-T})-B(v,e^{-T-\tau})$ the value of the function σ at (v,w) roughly coincides with the Radon Nikodym derivative of $\bar\eta_\nu^{P\Phi^Tv}$ with respect to $\bar\eta_\nu^x$ at $(\pi(v),\pi(w))$. This together with the fact that the total mass of the measures $\bar\eta_\nu^x$ is uniformly bounded then implies that for all $v\in T^1M$ and all sufficiently large $t>0$ the integral

$$\int_{\{(w,z)\in T^1_x\tilde M\times T^1_x\tilde M\mid(\pi(w),\pi(z))\in C(P\Phi^tv)\}}\sigma(w,z)d\eta_\nu^x(w,z)$$

is bounded from above by a universal constant.

Let now $f:T^1M\to\mathbb{R}$ be Hölder continuous and let F be the kernel of f, $\mathcal{F}F$ be the kernel of $f\circ\mathcal{F}$. Write $G=F\cdot\mathcal{F}F$. Since the kernels $\log F,\log\mathcal{F}F$ are uniformly Hölder continuous there is a universal constant $c_3>0$ such that

$$F(x,y,\pi(v))\mathcal{F}F(x,y,\pi(w))/G(x,y,\pi(N(y)))\in[c_3^{-1},c_3]$$

whenever $y\in\tilde M$ and $(\pi(v),\pi(w))\in C(y)$.

Thus if we define a function ρ on $\tilde M$ by $\rho(y)=G(x,y,\pi(N(y)))^{-1}$ and denote by $\lambda_{\tilde M}$ the Lebesgue measure on $\tilde M$, then we conclude from Lemma 2.8 and the above consideration that there is a constant $c_4 > 0$ such that for every $y \in \tilde{M} - B(x, R_0)$ we have

$$(34) \qquad \int_{\{(w,z)\in T^1_x\tilde{M}\times T^1_x\tilde{M} \mid (\pi(w),\pi(v))\in C(y)\}} e^{2\alpha_f(v,w)}\sigma(v,w)d\eta^x_\nu(v,w) / \int_{B(y,1)} \rho(z)d\lambda_{\tilde{M}}(z) \le c_4.$$

In other words, if $\int_{\tilde{M}} \rho d\lambda_{\tilde{M}} < \infty$ then the integral $\int e^{2\alpha_f}\sigma d\eta^x_\nu$ is finite. By definition of the pressure, this then holds whenever $pr(-f - f\circ\mathcal{F}) < 0$ which proves the proposition. □

Remark: With slightly more careful estimates in our above proof we can also show that for convex cocompact groups and every Hölder continuous function f on T^1M with $pr(-f - f\circ\mathcal{F}) \ge 0$ the integral $\int\int e^{2\alpha_f(v,w)}\sigma(v,w)d\eta^p_\nu(v,w)$ is infinite. However we do not need this in the sequel.

6. Harmonic measures for the operator $\Delta + Y$

In this section we continue the investigation of an operator $\Delta + Y$ of negative escape. Let again λ be the Lebesgue measure on M. For a measure ν of quasi-product type with regular family $\{\nu^p\}$ of positive disintegration measures we denote by ν_L the Lebesgue extension of $\{\nu^p\}$ defined by $\nu_L(A) = \int \nu^p(A\cap T^1_pM)d\lambda(p)$. We show that the measure class of every such measure ν_L contains a unique harmonic measure for $\Delta + Y$ and we relate its entropy to the entropy of ν.

We use the results and notations of Sections 1-5. Let again ν be a measure of quasi-product type, with a regular k-family $\{\nu^p\}$ of positive disintegration measures, induced family $\{\nu^{ss}\}$ of stable measures and Radon Nikodym kernel Q which defines the Radon Nikodym function f on T^1M. Recall that for ν-almost every $v \in T^1M$ we obtain $f(v)$ by choosing a lift $\tilde{v}$ of v to $T^1\tilde{M}$ and by defining $f(v) = Q(P\tilde{v}, P\Phi^1\tilde{v}, \pi(\tilde{v}))$. For a subset A of $\partial\tilde{M}$ of full $mc(\nu_+)$-mass and every $v \in \pi^{-1}(A)$ define a finite Borel measure $\bar{\eta}^v$ on $\partial\tilde{M}$ as in Section 5.

By Lemma 3.3 and the Birkhoff ergodic theorem, for ν-almost every $v \in T^1M$ the limit $\lim_{t\to\infty}\frac{1}{t}\int_0^t f(\Phi^s v)ds$ exists and equals the entropy of ν. Since the pressure of the function $g(X,Y)$ is negative by assumption, Proposition 5.6 shows that via replacing A by a subset of full $mc(\nu_+)$-mass and which we denote again by A we may assume that for every $v \in \pi^{-1}(A)$ we have

$$\beta(v) = \int_{\partial\tilde{M}} \alpha_{v,\nu}(\zeta)d\bar{\eta}^v(\zeta) < \infty$$

and the function $\tilde{u}_{v,\nu}$ from Section 5 is defined by

$$\tilde{u}_{v,\nu}(y) = (\beta(v))^{-1}\int_{\partial\tilde{M}} \alpha_{v,\nu}(\zeta)K_v(Pv,y,\zeta)\,d\bar{\eta}^v(\zeta).$$

Note that $\tilde{u}_{v,\nu}(Pv) = 1$ for all v.

Recall that the function $\alpha_{v,\nu}$ is continuous on $\partial\tilde{M} - \pi(v)$, non-negative and satisfies $\alpha_{v,\nu}(\pi(-v)) = 1$. The function β projects to a function on T^1M which is measurable with respect to the Lebesgue extension ν_L of ν. We have.

Lemma 6.1: $\int \beta d\nu_L < \infty$.

Proof: By the definition of β, for $p \in M$ we have

$$\int \beta d\nu^p = \int\int \alpha_{v,\nu}(\zeta) d\bar{\eta}^v(\zeta) d\nu^p(v).$$

Since $pr(g(X,Y)) < 0$ this is locally uniformly bounded from above on M by Lemma 5.3 and Proposition 4.3. From this the lemma follows. □

Define a family σ^{ss} of locally finite Borel measures on the leaves of the stable foliation by $d\sigma^{ss}(v) = \beta(v) d\nu^{ss}(v)$. Using the notations from Section 5, for $v \in T^1M$ and $y \in \tilde{M}$ write

$$L(Pv, y, \pi(v)) = \tilde{u}_{v,\nu}(y)/\tilde{u}_{v,\nu}(Pv).$$

Lemma 6.2: *The measures σ_ν^{ss} transform under the geodesic flow via*

$$d\sigma^{ss} \circ \Phi^t(v) = K_v(Pv, P\Phi^t v, \pi(-v)) L(Pv, P\Phi^t v, \pi(v))^{-1} d\sigma^{ss}(v).$$

Proof: For $v \in T^1M$ and $t > 0, y \in \tilde{M}$ we have

$$\begin{aligned} L(Pv, y, \pi(v)) &= \beta(v)^{-1} \int K_v(Pv, y, \zeta) \alpha_{v,\nu}(\zeta) d\bar{\eta}^v(\zeta) \\ &= \beta(v)^{-1} \int K_v(P\Phi^t v, y, \zeta) K_v(Pv, P\Phi^t v, \zeta) \alpha_{v,\nu}(\zeta) d\bar{\eta}^v(\zeta). \end{aligned} \tag{35}$$

This simply means the following: Represent the function $L(Pv, \cdot, \pi(v))$ on $\tilde{M}$ as a Poisson integral based at $P\Phi^t v$ and based at Pv. The Radon Nikodym derivative at $\pi(-v)$ of the measures defined by these representations is just $K_v(Pv, P\Phi^t v, \pi(-v))$.

Now the measure σ^{ss} on $W^{ss}(\Phi^t v)$ is determined at $\pi(-v)$ by the Poisson formula for $L(Pv, P\Phi^t v, \pi(v))^{-1} L(Pv, \cdot, \pi(v))$ from which the lemma follows. □

Recall from [H97a] that there is a Hölder continuous function ψ on T^1M such that

$$\psi(v)^{-1} K_v(Pv, P\Phi^t v, \pi(-v)) \psi(\Phi^t v) = K_v(Pv, P\Phi^t v, \pi(v))^{-1}$$

for all $v \in T^1M$ and all $t > 0$ and consequently the measures $\psi\sigma^{ss} = \psi\beta^{-1}\nu^{ss}$ transform under the geodesic flow via

$$d(\psi\sigma^{ss}) \circ \Phi^t(v) = K_v(Pv, P\Phi^t v, \pi(v))^{-1} L(Pv, P\Phi^t v, \pi(v))^{-1} d(\psi\sigma^{ss})(v).$$

Let $\{\nu^{su}\}$ be the projection of the k-family $\{\nu^p\}$ to strong unstable manifolds. Then $d\nu = dt \times d\nu^{ss} \times d\nu^{su}$ and the measures $\sigma^{su} = \beta\psi^{-1}\nu^{su}$ on the leaves of the strong unstable foliation of T^1M transform under Φ^t via

$$d\sigma^{su} \circ \Phi^t(v) = K_v(Pv, P\Phi^t v, \pi(v)) L(Pv, P\Phi^t v, \pi(v)) d\sigma^{su}(v).$$

Let λ^s be the family of Lebesgue measures on stable manifolds defined by the lift $<,>$ of the Riemannian metric on M. In [H3] it is shown that the Lebesgue extension ν_L of $\{\nu^p\}$ coincides with the measure on T^1M which is given with respect to a local product structure by the formula $d\bar{\nu} = d\lambda^s \times d\nu^{su}$.

Let η^s be the family of Lebesgue measures on the leaves of W^s induced by the metric g. Since $\eta^s = \chi\lambda^s$ for a Hölder continuous function χ we obtain from Lemma 6.1 that the Borel-measure $\rho(\Delta + Y)(\nu) = \sigma$ on T^1M defined by $\sigma = \beta\psi^{-1}\chi\bar{\nu}$ is locally finite; moreover $d\sigma = d\eta^s \times d\sigma^{su}$. Recall from [H97a] the definition of the g-*gradient* of a Borel measure ρ on T^1M which is absolutely continuous with respect to the stable and strong unstable foliation, with conditionals on stable manifolds in the Lebesgue measure class. Namely, let $\tilde{\rho}$ be the lift of ρ to $T^1\tilde{M}$ and let $\tilde{\rho}(\infty)$ be a Borel-probability measure on $\partial\tilde{M}$ which defines the measure class of the projections of the conditionals of $\tilde{\rho}$ on strong unstable manifolds. For $v \in T^1\tilde{M}$ we can represent $\tilde{\rho}$ near v in the form $d\tilde{\rho} = \alpha d\eta^s \times d\tilde{\rho}(\infty)$ where $\alpha : T^1\tilde{M} \to (0,\infty)$ is a Borel function and we identify $\tilde{\rho}(\infty)$ with its projections to the leaves of W^{su} via the map π. For $(v,w) \in D = \{(u,z) \in T^1\tilde{M} \times T^1\tilde{M} \mid z \in W^s(u)\}$ define $l(v,w) = \alpha(w)/\alpha(v)$. Then the function $l : D \to (0,\infty)$ is independent of the choice of $\tilde{\rho}(\infty)$. If for $\tilde{\rho}$-almost every $v \in T^1\tilde{M}$ the function $l_v : W^s(v) \to (0,\infty), w \to l_v(w) = l(v,w)$ is differentiable, then we obtain a measurable section $\tilde{Z}$ of TW^s over $T^1\tilde{M}$ by assigning to $v \in T^1\tilde{M}$ the gradient at v of $\log l_v$ with respect to the Riemannian metric g on $W^s(v)$. This section of TW^s over $T^1\tilde{M}$ is equivariant under the action of $\pi_1(M)$ and hence it projects to a measurable section Z of TW^s over T^1M which we call the g-gradient of ρ. We then have $\int(\mathrm{div}(Q) + g(Z,Q))d\rho = 0$ for every leafwise differentiable section Q of TW^s (see [H97a]). Write again $\kappa = -\mathrm{div}(Y/2) - g(Y/2mY/2)$. The above discussion then shows:

Proposition 6.3: *The g-gradient of σ equals $Y/2 + E$ where $Y/2 \neq E$ and* $\mathrm{div}(E) + g(E,E) + \kappa = 0$.

On the other hand we have the following (see [H97a]):

Lemma 6.4: *A Borel-probability measure ρ on T^1M is harmonic for the operator $\Delta + Y$ if and only if ρ has the following properties:*

(1) *ρ is absolutely continuous with respect to the stable and strong unstable foliation, with conditionals on stable manifolds in the Lebesgue measure class.*
(2) *The g-gradient Z of ρ is defined ρ-almost everywhere on T^1M and satisfies*
$$\mathrm{div}(Z - Y) + g(Z, Z - Y) = 0.$$

Now if Z is the g-gradient of σ then $Z - Y = E - Y/2$ and therefore $\mathrm{div}(Z - Y) + g(Z, Z - Y) = \mathrm{div}(E) - \mathrm{div}(Y/2) + g(E + Y/2, E - Y/2) = \mathrm{div}(E) + g(E,E) + \kappa$. From Proposition 6.3 and Lemma 6.4 we thus obtain:

Corollary 6.5: *σ is a harmonic measure for $\Delta + Y$.*

A harmonic measure σ for $\Delta + Y$ induces an invariant measure for the leafwise diffusion process of the operator $\Delta + Y$. Namely, for every $v \in T^1M$ the restriction of $\Delta + Y$ to the smooth manifold $W^s(v)$ is a second order elliptic operator without zero order terms and hence it induces a probability measure P^v on the space Ω_+ of paths $\omega : [0, \infty) \to T^1M$ such that $P^v\{\omega \mid \omega(0) = v \text{ and } \omega(t) \in W^s(v) \text{ for all } t \geq 0\} = 1$. Recall that the semigroup $\{T^t \mid t \geq 0\}$ of *shift transformations* acts on Ω_+ via $T^t\omega(s) = \omega(t + s)$. Since σ is harmonic for $\Delta + Y$, the measure P on Ω_+ which is defined by $P(A) = \int P^v(A)\, d\sigma(v)$ is invariant under the shift. Moreover ergodicity of $mc(\sigma, \infty)$ under the action of $\pi_1(M)$ is equivalent to ergodicity of the measure P under the shift (Lemma 2.1 of [H97a]). This immediately implies.

Corollary 6.6: *There is an injective map $\rho(\Delta + Y)$ from the space of ergodic measures of quasi-product type on T^1M into the space of ergodic harmonic measures for $\Delta + Y$ which maps a measure ν to the unique harmonic measure for $\Delta + Y$ in the measure class of the Lebesgue extension of ν.*

Proof: Let ν_L be the Lebesgue extension of a k-family $\{\nu^p\}$ for an ergodic measure of regular product type. We constructed above a harmonic measure μ for $\Delta + Y$ in the measure class of ν_L. Since ν is ergodic by assumption, the measure class $mc(\nu_+)$ on $\partial\tilde{M}$ defined by the k-family $\{\nu^p\}$ is ergodic under the action of Γ. Then μ is an ergodic harmonic measure for $\Delta + Y$ [H97a] and hence unique in its measure class. Thus every harmonic measure for $\Delta + Y$ [H97a] which is absolutely continuous with respect to μ coincides with μ up to a constant and therefore there is a well defined map $\rho(\Delta + Y)$ which associates to an ergodic measure ν of quasi-product type the unique harmonic measure for $\Delta + Y$ in the measure class of the Lebesgue extension of ν.

The map $\rho(\Delta + Y)$ is injective if and only if for every ergodic k-family $\{\nu^p\}$ on M there is at most one measure ν of quasi-product type whose positive disintegration class coincides with $mc(\nu_+)$. However this is immediate from Lemma 3.2. □

Given an ergodic harmonic measure σ for $\Delta + Y$ we can reverse the time of the diffusion to obtain a new leafwise diffusion process on T^1M. The next lemma is a direct application of Lemma 2.12 of [H97a] to our situation:

Lemma 6.7: *Let $Y/2 + E$ be the g-gradient of the harmonic measure σ for $\Delta + Y$. Then the reversal of time of the diffusion induced by $\Delta + Y$ and σ is the diffusion induced by $\Delta + 2E$ and σ.*

Assume now again that σ is an ergodic harmonic measure for $\Delta + Y$, with g-gradient $Y/2 + E$. Then σ is an invariant measure for the diffusion on T^1M induced by $\Delta + 2E$ and hence the *Kaimanovich entropy* h_σ of this diffusion is well defined. We call h_σ simply the *entropy* of σ. We have $h_\sigma = 0$ if and only if for σ-almost every $v \in T^1M$ the space of bounded $\Delta + 2E$-harmonic functions on $W^s(v)$ is 1-dimensional (see [K2]).

The entropy h_σ of σ can be computed as follows.

Lemma 6.8: *The entropy of an ergodic harmonic measure σ for $\Delta + Y$ equals* $h_\sigma = \int \big(g(Y/2, Y/2) - g(E,E)\big)\, d\sigma \geq 0$.

Proof: Let $v \in T^1\tilde{M}$ be the lift of a typical point for the harmonic measure σ on T^1M. Write again $\kappa = -\mathrm{div}(Y/2) - g(Y/2, Y/2)$ and let $\Delta_v + 2E_v$ and $\Delta_v + \kappa_v$ be the operator on $\tilde{M}$ which is the projection of the restriction of $\Delta + 2E$ and $\Delta + \kappa$ to $W^s(v)$.

Let φ be the function on $\tilde{M}$ which is normalized to be 1 at Pv and such that the g-gradient of its logarithm equals E_v. Define similarly a function ψ from the vector field $(Y/2)_v$. Denote as before by K the kernel of $g(X,Y)$.

By construction, if α is any positive $\Delta_v + 2E_v$-harmonic function on $\tilde{M}$ then $\alpha\varphi$ is a positive $\Delta_v + \kappa_v$-harmonic function. Since $\Delta_v + \kappa_v$ is self-adjoint with respect to the metric g and its Martin kernel K_v satisfies $K_v(x, y, \pi(v)) = K(x, y, \pi(v))^{-1/2}$ we obtain from the appendix of [H97a] that for every $z \in T^1_{Pv}\tilde{M} - \{v\}$ the product

$$K_v(Pv, P\Phi^t z, \pi(z))K(Pv, P\Phi^t z, \pi(v))^{1/2}$$

is bounded from above and below by a positive constant depending on z but not on $t \geq 0$. In other words, the unique minimal positive $\Delta_v + 2E_v$-harmonic function on $\tilde{M}$ with pole at $\pi(z) \neq \pi(v)$ and which is normalized to be 1 at Pv coincides along the geodesic $t \to P\Phi^t z$ up to a bounded factor with the restriction of the function $\psi^{-1}\varphi^{-1}$.

Let now ω be a typical path for the diffusion induced by $\Delta_v + 2E_v$, with initial point $\omega(0) = Pv$. Then ω converges to a point $\xi \in \partial\tilde{M} - \{\pi(v)\}$. By the Birkhoff ergodic theorem and the above considerations, the Kaimanovich entropy h of the diffusion induced by $\Delta + 2E$ and σ on T^1M equals

$$\lim_{t\to\infty} \frac{1}{t}(-\log\varphi - \log\psi)(\omega(t)).$$

From Ito's formula and the Birkhoff ergodic theorem we conclude that

$$h_\sigma = \int \mathrm{div}(-E - Y/2) + 2g(E, -E - Y/2)d\sigma = \int g(Y/2 - E, Y/2 + E)d\sigma.$$

□

As in the introduction let now $\bar{X}$ be the section of TW^s over T^1M which is g-dual to the section α of T^*W^s defined by $\alpha(X) \equiv 1$ and $\alpha|TW^{ss} \equiv 0$. Recall from the introduction that the *signed escape rate* $\ell_\sigma = \ell_\sigma(\Delta + Y)$ of the diffusion induced by $\Delta + Y$ and σ equals $-\int \mathrm{div}(\bar{X}) + g(Y, \bar{X})\, d\sigma$. Note that the absolute value of this signed escape rate is bounded from above by a universal constant. As a corollary from Lemma 5.8 we obtain.

Corollary 6.9: *Let ν be an ergodic measure of quasi-product type on T^1M; then* $h_\nu = -h_{\rho(\Delta+Y)(\nu)}/\ell_{\rho(\Delta+Y)(\nu)}$.

Proof: Let ν be a measure of quasi-product type and write $\sigma = \rho(\Delta + Y)(\nu)$. The g-gradient of σ is of the form $Y/2 + E$ for a section E of TW^s.

Let ω be a typical path for the diffusion induced by $\Delta + Y$ and σ and let $\tilde{\omega}$ be a lift of ω to $T^1\tilde{M}$. By Ito's formula, the entropy

$$h_\sigma = \int \operatorname{div}(Y/2 + E) + g(Y, Y/2 + E)d\sigma = \int g(Y/2 - E, Y/2 + E)d\sigma$$

can be computed as follows: Choose a function f on $W^s(\tilde{\omega}(0))$ whose g-gradient equals $Y/2 + E$; then

$$h_\sigma = \lim_{t\to\infty} \frac{1}{t} f(\tilde{\omega}(t)).$$

On the other hand, the lift of $\bar{X}$ to $W^s(v) \sim \tilde{M}$ is the negative of the g-gradient of a Busemann function on $W^s(\tilde{\omega}(0)) \sim \tilde{M}$ at $\pi(\tilde{\omega}(0)) \in \partial\tilde{M}$. If dist denotes the distance on $W^s(v)$ defined by the lift of the Riemannian metric of $\tilde{M}$, then the (non-signed) escape rate $\lim_{t\to\infty} \frac{1}{t} \operatorname{dist}(\tilde{\omega}(t), \tilde{\omega}(0))$ is defined and equals $-\ell_\sigma$ (see [H97a]). Thus h_σ/ℓ_σ is the average growth of the function f along geodesics in $\tilde{M}$ which converge to a typical point for the positive disintegration class of ν [H97a]. Together with Lemma 3.3 and its proof and the Birkhoff ergodic theorem this shows that the entropy of ν equals h_σ/ℓ_σ. □

Corollary 7.10: *The entropy of an ergodic measure of quasi-product type which is not supported on a closed orbit of Φ^t is positive.*

Proof: Let ν be an ergodic measure of quasi-product type which is not supported on a single closed geodesic. By Lemma 3.1, the positive and negative disintegration classes of ν do not have atoms. Let $Y/2 + E$ be the g-gradient of the harmonic measure $\sigma = \rho(\Delta + Y)(\nu)$. The Kaimanovich entropy h_σ of the diffusion on T^1M induced by $\Delta + 2E$ and $\rho(\Delta + Y)(\nu)$ vanishes if and only if for σ-almost every $v \in T^1M$ there are no nonconstant bounded positive $\Delta + 2E$-harmonic function on $W^s(v)$. By the explicit construction of σ the defining measure on $\partial\tilde{M}$ for the vector field E has no atoms. This means that the entropy h_σ of σ is necessarily positive and therefore by Lemma 5.8 and Lemma 5.9 the entropy of ν is positive as well.

7. Proof of the theorem

Consider again a closed Riemannian manifold $M = \tilde{M}/\Gamma$ of negative sectional curvature. Let as before $\Delta + Y$ be an operator of negative escape. We showed in Section 5 that there is an injective map $\rho(\Delta + Y)$ of the space of ergodic measures of quasi-product type into the space of ergodic harmonic measures for $\Delta + Y$. The first goal of this section is to show that this map is surjective, i.e. that it is in fact a bijection.

We continue to use the assumptions and notations from the previous sections. Recall in particular from Section 2 the definition of a k-family $\{\nu^p\}$ for T^1M and its Lebesgue extension ν_L. Every harmonic measure σ for $\Delta + Y$ is the Lebesgue extension of a k-family $\{\nu^p\}$. The signed escape rate $\ell_\sigma(\Delta + Y)$ for the harmonic measure σ for $\Delta + Y$ is well defined. We know from [H97a] that this escape rate is bounded from above by a negative constant $-b < 0$ not depending on σ. This observation is used to show.

Lemma 7.1: *Let σ be an ergodic harmonic measure for $\Delta + Y$. Then there exists a measure ν of quasi-product type such that $\sigma = \rho(\Delta + Y)(\nu)$.*

Proof: Let σ be an ergodic harmonic measure for $\Delta+Y$ with g-gradient $Y/2+E$. Then σ is a harmonic measure for $\Delta + 2E$. The (non-signed) escape rate of the $\Delta + 2E$–diffusion coincides with the escape rate of the diffusion induced by $\Delta + Y$ and σ [H97a] and therefore this escape rate is positive. This implies that for σ–almost every $v \in T^1\tilde{M}$ the exit boundary of the diffusion on $W^s(v)$ induced by the restriction of $\Delta + 2E$ to $W^s(v)$ is well defined and does not have an atom at $\pi(v)$ (see [H97a]).

We show next that the exit boundary of this diffusion can only have an atom if the entropy h_σ of σ vanishes. For this we assume that there is a subset A of T^1M of positive σ-mass such that for every $v \in A$ the exit measure η_v for the diffusion on $W^s(v)$ has an atom. Let $v \in T^1\tilde{M}$ be a lift of some point in A. Consider the diffusion on $\tilde{M}$ induced by the projection $\Delta_v + 2E_v$ to $\tilde{M}$ of the restriction of $\Delta + 2E$ to $W^s(v)$. Let $\xi \in \partial\tilde{M}$ be an atom for the exit measure of this diffusion. By the Poisson formula, if ψ denotes the minimal $\Delta_v + 2E_v$-harmonic function on $\tilde{M}$ with pole at ξ, then the function ψ is bounded along a geodesic which is asymptotic to ξ. But this means that for every path ω which is typical for the diffusion induced by $\Delta_v + 2E_v$ and initial probability the Dirac mass at Pv and which converges to ξ the limit $\lim_{t\to\infty} \frac{1}{t}\log\psi(\omega(t))$ is equal to 0. By ergodicity and the definition of the Kaimanovich entropy we conclude from this that the entropy h_σ of σ indeed vanishes.

The measures η_v are well defined for σ-almost every v and equivariant under the action of Γ on $T^1\tilde{M} \times \partial\tilde{M}$. For $w \in W^s(v)$ the measures η_v and η_w on $\partial\tilde{M}$ define the same measure class.

Let $v \in T^1\tilde{M}$ be a typical point for σ. We can project the measures η_w ($w \in W^{ss}(v)$) to a Borel probability measure η_0^{ss} on $W^{ss}(v)$. As before, let ψ be a Hölder continuous function on T^1M such that $\log\psi$ defines an equivalence between the cocycle of $g(X, Y/2)$ and the cocycle of the function $\frac{d}{dt}K_v(Pv, P\Phi^t v, \pi(-v))^{-1} \mid_{t=0}$. The arguments in the proof of Lemma 5.2 then show that the measures $\eta^{ss} = \psi\eta_0^{ss}$ transform under the geodesic flow via $\frac{d}{dt}\eta^{ss} \circ \Phi^t|_{t=0} = -g(X, Y/2 + E)$.

Let ν^p be the k-family whose Lebesgue extension equals σ and denote by ν^{su} the induced family of measures on strong unstable manifolds. Fix a number $r > 0$ and look at balls of radius r in the leaves of W^{ss} with respect to the restriction of g. From the infinitesimal Harnack inequality for positive $\Delta_v + \kappa_v$–harmonic functions on $\tilde{M}$ and the definition of the measures η_0^{ss} we conclude that the η_0^{ss}–mass of any such ball is bounded from above by a universal constant. Thus we obtain a finite Φ^t–invariant measure η on T^1M by defining $d\nu = dt \times d\eta^{ss} \times d\nu^{su}$. We may assume that ν is normalized in such a way that $\nu(T^1M) = 1$. Lemma 3.2 then shows that the measure ν is of quasi-product type. □

We showed so far that for every operator $\Delta + Y$ of negative escape there is a bijective map $\Psi(\Delta + Y)$ of the space of ergodic harmonic measures for $\Delta + Y$ onto the space of Φ^t-invariant Borel probability measures on T^1M of regular product type. We can now complete the proof of our theorem from the introduction by showing that this map is continuous.

Let now again $\Delta + Y$ be a differential operator of negative escape, where as in the introduction Δ is the leafwise Laplacian of a Riemannian metric g on TW^s. By the infinitesimal Harnack inequality, there is some $k > 0$ only depending on $\Delta + Y$ such that every harmonic measure ν for $\Delta + Y$ is the Lebesgue extension of a k-family $\{\nu^p\}$. Moreover ν is ergodic if and only if the k-family $\{\nu^p\}$ is ergodic. We equip the space $\mathcal{H}_{\Delta+Y}$ of harmonic measures for $\Delta + Y$ with the weak*-topology. With this topology, $\mathcal{H}_{\Delta+Y}$ is a compact space.

Lemma 7.2: *There is a constant $b > 0$ with the following property: Let σ be a harmonic measure for $\Delta + Y$, with g-gradient $Y/2 + E^\sigma$. Then $\int g(Y/2, Y/2 - E^\sigma)d\sigma \geq b$.*

Proof: As in the lemma, for a harmonic measure σ for $\Delta + Y$ let E^σ be the section of TW^s such that $Y/2 + E^\sigma$ is the g-gradient of σ. We have to show that

$$b = \inf\{\int g(Y/2, Y/2 - E^\sigma)d\sigma \mid \sigma \in \mathcal{H}_{\Delta+Y}\} > 0.$$

For this recall that by assumption the pressure of the Hölder continuous function $g(X, Y)$ is negative and therefore there is a number $a > 0$ such that $\int g(X, Y)d\mu \geq a$ for all $\mu \in \mathcal{M}$. On the other hand, by Lemma 3.13 of [H97a] the signed escape rate of the diffusion on T^1M induced by $\Delta + Y$ and any harmonic measure for $\Delta + Y$ is bounded from above by a number $-\ell < 0$. This together with Ito's formula and the Birkhoff ergodic theorem implies that $\int \mathrm{div}(Y) + g(Y, Y)d\sigma \geq a/\ell > 0$ for every $\sigma \in \mathcal{H}_{\Delta+Y}$. Since $\int \mathrm{div}(Y) + g(Y, Y/2 + E^\sigma)d\sigma = 0$ the lemma follows. □

Let again σ be an ergodic harmonic measure for $\Delta + Y$. For σ-almost every $v \in T^1M$ the g-gradient $Y/2 + E^\sigma$ is defined on $W^s(v)$. We showed in Section 5 that the restriction E^σ_v of E^σ to $W^s(v) \sim \tilde{M}$ is the gradient of the logarithm of a positive $\Delta_v + \kappa_v$-harmonic function f_v on $W^s(v)$ which we assume to be normalized in such a way that $f_v(v) = 1$. Since the operator $\Delta_v + \kappa_v$ is weakly coercive there is a unique Borel probability measure $\bar{\eta}^v$ on $\partial\tilde{M}$ such that f_v can be represented as a Poisson integral with respect to this measure. Using the notations from Section 2 we have.

Lemma 7.3: *There is a number $\delta > 0$ such that $\int \bar{\eta}^v(\partial\tilde{M} - \pi B(v, \delta))d\sigma(v) \geq \delta$ for all $\sigma \in \mathcal{H}_{\Delta+Y}$.*

Proof: Let again K be the kernel of the function $g(X, Y)$, and for $x \in \tilde{M}$ and $r > 0$ let $B(x, r)$ be the ball of radius r about x with respect to the distance induced by the Riemannian metric. Let $R > 0$ be the diameter of M.

Recall that the coefficients of the operators $\Delta_v + \kappa_v$ depend Hölder continuously on $v \in T^1\tilde{M}$. Moreover $\Delta_v + \kappa_v$ is uniformly elliptic and uniformly weakly coercive, and its Martin kernel K_v satisfies $K_v(x, y, \pi(v))^2 = K(x, y, \pi(v))$.

The function $\log K_v$ is locally uniformly Hölder continuous on $\tilde{M} \times \tilde{M} \times \partial\tilde{M}$, and for $x \in \tilde{M}$ the gradient of the $\Delta_v + \kappa_v$-harmonic function $y \to \log K_v(x, y, \xi)$ depends uniformly Hölder continuously on $\xi \in \partial\tilde{M}$. This means that for every $\epsilon > 0$ there is a number $\rho(\epsilon) > 0$ only depending on ϵ and $\Delta + Y$ such that the following holds: Let $v \in T^1\tilde{M}$ and let $\bar{\eta}$ be any Borel probability measure on $\partial\tilde{M}$ such that $\bar{\eta}(\pi B(v, \rho(\epsilon))) > 1 - \rho(\epsilon)$. Denote by h the positive solution of $\Delta_v + \kappa_v = 0$ defined by $h(y) = \int K_v(x, y, \xi) d\bar{\eta}(\xi)$, and let $\hat{h}$ be the lift of h to $W^s(v)$. Then for every $w \in W^s(v) \cap P^{-1}B(Pv, R)$ we have $|g(Y/2, Y/2 - \nabla^g(\log \hat{h}))(w)| \leq \epsilon$, where $\nabla^g(\log \hat{h})$ denotes the gradient of $\log \hat{h}$ with respect to the restriction of g to $W^s(v)$.

Let now $\{\nu^p\}$ be a k-family whose Lebesgue extension σ is harmonic for $\Delta + Y$. Choose a point $x \in \tilde{M}$. Let $\Omega \subset \tilde{M}$ be a compact fundamental domain for the action of Γ which is contained in the ball in $\tilde{M}$ of radius R about x.

Let E^σ be such that $Y/2 + E^\sigma$ is the g-gradient of σ. By the infinitesimal Harnack inequality, $g(Y/2, Y/2 - E^\sigma)$ is pointwise uniformly bounded in absolute value by a constant $q > 0$ only depending on $\Delta + Y$. For $v \in T^1\tilde{M}$ denote by $\bar{\eta}^v$ the defining measure for the $\Delta_v + \kappa_v$-harmonic function f_v on $W^s(v) \sim \tilde{M}$ which is normalized by $f_v(v) = 1$ and such that $\nabla^g(\log f_v) = E^\sigma | W^s(v)$.

For $\epsilon > 0$ let $\rho(\epsilon) > 0$ be as above and define $A_\epsilon = \{v \in T^1_x\tilde{M} \mid \bar{\eta}^v(\pi B(v, \rho(\epsilon))) > 1 - \rho(\epsilon)\}$. For $v \in A_\epsilon$ and $w \in W^s(v) \cap P^{-1}\Omega$ we have $|g(Y/2, Y/2 - E^\sigma)(w)| < \epsilon$. Recall moreover that $\bar{\nu}^y(\pi(A)) \leq ke^R\bar{\nu}^x(\pi(A))$ for every Borel subset A of $T^1_x\tilde{M}$ and every $y \in \Omega$. This implies that

$$\int g(Y/2, Y/2 - E^\sigma) d\sigma \tag{36}$$

$$\leq \int_{P^{-1}\Omega \cap \pi^{-1}(\pi A_\epsilon)} g(Y/2, Y/2 - E^\sigma) d\sigma + \int_{P^{-1}\Omega \cap \pi^{-1}(\partial\tilde{M} - \pi A_\epsilon)} g(Y/2, Y/2 - E^\sigma) d\sigma \tag{37}$$

$$\leq \epsilon + q\sigma\big(P^{-1}\Omega \cap \pi^{-1}(\partial\tilde{M} - \pi A_\epsilon)\big) \leq \epsilon + ke^R q\lambda_M(M)\nu^x(T^1_x\tilde{M} - A_\epsilon). \tag{38}$$

On the other hand, by Lemma 7.2 the integral on the left hand side of this inequality is bounded from below by a universal constant $b > 0$ not depending on σ. From this we conclude that for $\epsilon < b/2$ the ν^x-mass of $T^1_x\tilde{M} - A_\epsilon$ is bounded from below by $b/2kR\lambda_M(M)$. This shows the lemma. □

Every ergodic k-family whose Lebesgue extension σ is harmonic for $\Delta + Y$ and of positive Kaimanovich entropy is a family of positive disintegration measures for a unique Φ^t-invariant Borel probability measure $\Psi(\Delta + Y)$ of quasi-product type. Thus we obtain an injective map $\Psi(\Delta + Y)$ of the set of ergodic harmonic measure for $\Delta + Y$ into the set of ergodic Φ^t-invariant Borel probability measures on T^1M. Since every harmonic measure σ for $\Delta + Y$ admits a unique integral decomposition into ergodic components, we can extend $\Psi(\Delta + Y)$ by convex linearity to an injective map of the convex space $\mathcal{H}_{\Delta+Y}$ of harmonic measures for $\Delta + Y$ into the convex space $\mathcal{M}$ of Φ^t-invariant Borel probability measures on T^1M. Now we can show.

Lemma 7.4: *The restriction of the map $\Psi(\Delta + Y)$ to the set of extreme points of $\mathcal{H}_{\Delta+Y}$ is continuous.*

Proof: Let σ_i be a sequence of ergodic harmonic measures for $\Delta + Y$ converging weakly to an ergodic harmonic measure σ. Denote by $\{\nu_i^{su}\}$ the family of measure on strong unstable manifolds induced by the unique k-family whose Lebesgue extension is σ_i. By Lemma 6.2 we may assume that the measures ν_i^{su} converge weakly to a family $\{\nu^{su}\}$ of uniformly locally finite Borel measures on the leaves of W^{su}. Moreover $\{\nu^{su}\}$ is the projection to strong unstable manifolds of a regular family of positive disintegration measures for the measure $\Psi(\Delta+Y)(\sigma) = \nu$ of quasi-product type. By the Birkhoff ergodic theorem, for every continuous function f on T^1M, every $v \in T^1M$ and ν^{su}-almost every $w \in W^{su}(v)$ the integral $\int f d\nu$ just coincides with the limit $\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\Phi^s w)ds$.

For $i > 0$ denote by μ_i the measure on T^1M which can be written locally in the form $\mu_i = dt \times d\nu_i^{su} \times d\mu_i^{ss}$ where $\{\mu_i^{ss}\}$ is a family of measures on strong stable manifolds which define as in Section 5 via the Poisson formula and renormalization the g-gradient of σ_i. The measures μ_i^{ss} are locally uniformly bounded independent of i, in particular the total mass of the measures μ_i is bounded by a universal constant. Moreover there is a measurable nonnegative function ψ_i on T^1M not depending on i such that $\psi\mu_i$ is Φ^t-invariant and a positive multiple of $\nu_i = \Psi(\Delta + Y)(\sigma_i)$.

By Lemma 7.3 the total mass of the measures μ_i is bounded from below by a universal positive constant as well. Thus by passing to a subsequence we may assume that the measures $\psi_i\mu_i$ converge weakly to a nonzero measure μ.

The measure $\hat{\mu}$ is Φ^t-invariant and satisfies $\hat{\mu}(\cup_{v\in A}W^s(v)) = 0$ for every Borel subset $A \subset W^{su}(v)$ of vanishing ν^{su}-mass. By our assumption the measures ν^{su} are quasiinvariant and ergodic under canonical maps and therefore $\hat{\mu}$ is absolutely continuous with respect to the stable foliation, with conditional measures in the measure class of ν^{su}. But this means that for ν^{su}-almost every $v \in T^1M$ and every continuous function $f : T^1M \to \mathbb{R}$ the integral of f with respect to $\hat{\mu}$ equals $\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\Phi^s v)ds$. In other words, $\hat{\mu} = \Psi(\Delta + Y)(\sigma)$. This shows that the restriction of $\Psi(\Delta + Y)$ to the set of extreme points of $\mathcal{H}_{\Delta+Y}$ is continuous. □

The map $\Psi(\Delta + Y) : \mathcal{H}_{\Delta+Y} \to \mathcal{M}$ however is *not* continuous. To see this choose a Φ^t-invariant ergodic measure $\eta \in \mathcal{M}$ which is not supported on a single closed orbit of Φ^t and whose entropy vanishes. Such a measure can for example be constructed using symbolic dynamics. Lemma 7.1 then shows that η is not of quasi-product type. Since the set of all measures supported on single closed orbits is dense in $\mathcal{M}$, the measure η can be approximated in the weak*-topology by a sequence of measures (ζ_i) of quasi-product type. Thus if $\sigma_i = \Psi(\Delta + Y)^{-1}(\zeta_i)$ then no accumulation point of (σ_i) can be ergodic.

Recall that for every measure $\eta \in \mathcal{H}_{\Delta+Y}$ the g-gradient $Y/2+E^\eta$ is well defined and is a measurable bounded section of the bundle $TW^s \to T^1M$. For a sequence of measures $(\eta_i) \subset \mathcal{H}_{\Delta+Y}$ converging to a measure η and a sequence of η_i-measurable bounded sections (V_i) of $TW^s \to T^1M$ we say that (V_i) *converges weakly* to an η-measurable bounded section V of TW^s if for every *continuous* section Z of TW^s we have $\int g(V_i,Z)d\eta_i \to \int g(V,Z)d\eta$. We have.

Lemma 7.5: *Let $(\eta_i) \subset \mathcal{H}_{\Delta+Y}$ be a sequence converging to some $\eta \in \mathcal{H}_{\Delta+Y}$. Then the g-gradients V_i of η_i converge weakly to the g-gradient V of η.*

Proof: Let $(\eta_i) \subset \mathcal{H}_{\Delta+Y}$ be a sequence converging to some $\eta \in \mathcal{H}_{\Delta+Y}$. By assumption, for every section Z of TW^s of class $C_s^{1,\alpha}$ and every $i>0$ we have

$$0=\int \mathrm{div} Z + g(Z,V_i)d\eta_i = \int \mathrm{div} Z + g(Z,V)d\eta.$$

Since on the other hand the function $\mathrm{div}(Z)$ on T^1M is continuous the sequence $\int \mathrm{div}(Z)d\eta_i$ converges to $\int \mathrm{div}(Z)d\eta$ by the definition of the topology on $\mathcal{H}_{\Delta+Y}$. Then necessarily also $\int g(Z,V_i)d\eta_i \to \int g(Z,V)d\eta$ which shows the lemma. □

Let $(\eta_i) \subset \mathcal{H}_{\Delta+Y}$ be a sequence converging to some η and let (V_i) be a sequence of bounded η_i-measurable sections of TW^s. We say that the sequence (V_i) *converges strongly* to a bounded η-measurable section V of TW^s if the sequence converges weakly to V and if in addition we have $\int \|V_iVert^2 d\eta_i \to \int \|V\|^2 d\eta$. We have.

Corollary 7.6: Let $(\eta_i) \subset \mathcal{H}_{\Delta+Y}$ be a sequence of ergodic measures converging to some ergodic measure η. Then the sequence (V_i) of g-gradients of η_i converges strongly to the g-gradient of η if and only if $h_{\Psi(\Delta+Y)\eta_i} \to h_{\Psi(\Delta+Y)\eta}$.

Proof: Let η_i be a sequence as in the lemma. By Lemma 7.5 we have $\ell_{\eta_i} \to \ell_\eta$ and hence Lemma 6.9 shows that $h_{\Psi(\Delta+Y)\eta_i} \to h_{\Psi(\Delta+Y)\eta}$ if and only if the g-gradients of the measures η_i converge strongly to the g-gradient of g. □

We conclude this section with a few final remarks about the duality between cohomology classes of Hölder continuous 1-forms on T^1M and Gibbs measures. For this recall that a *Hölder continuous cocycle* for Φ^t is a Hölder continuous map $\zeta : T^1M \times \mathbb{R} \to \mathbb{R}$ such that $\zeta(v,t+s)=\zeta(v,s)+\zeta(\Phi^s v,t)$. Two such cocycles ζ,ψ are *cohomologous* if there is a Hölder continuous function β on T^1M such that $\zeta(v,t)=\beta(\Phi^t v)+\psi(v,t)-\beta(v)$. For every Hölder cocycle there is a Hölder continuous function f on T^1M such that ζ is cohomologous to the cocycle ζ_f defined by $\zeta_f(v,t)=\int_0^t f(\Phi^s v)ds$. If $\mu \in \mathcal{M}$ is a Φ^t-invariant Borel probability measure, and if the cocycle ζ is cohomologous to ζ_f for some continuous function f then we can define $\int \zeta d\mu = \int f d\mu$; this does not depend on the choice of f (compare e.g. [H4]). In other words, the space of cohomology classes of Hölder continuous cocycles can be viewed as a linear subspace of the dual $\mathcal{M}^*$ of $\mathcal{M}$.

There is a natural open cone in the space of cohomology classes of Hölder continuous cocycles. Namely, define a class to be *positive* if it can be represented by a positive function. The next lemma gives a description of the set of positive cohomology classes.

Lemma 7.6: *A cohomology class ζ is positive if and only if $\zeta(\mu) > 0$ for every $\mu \in \mathcal{M}$.*

Proof: Clearly $\zeta(\mu) > 0$ for all $\mu \in \mathcal{M}$ if ζ is positive. Thus let ζ be a cohomology class such that $\zeta(\mu) > 0$ for all $\mu \in \mathcal{M}$. Since ζ is continuous and $\mathcal{M}$ is compact this means that there is a number $\epsilon > 0$ such that $\zeta(\mu) \geq \epsilon$ for all μ. Let f be any function representing ζ. For $t > 0$ define a new function f_f on T^1M by $f_t(v) = \frac{1}{t}\int_0^t f(\Phi^s v)ds$. Clearly the cocycle induced by f_t is cohomologous to ζ. Thus it is enough to show that $f_t > 0$ for all sufficiently large t.

For this we argue by contradiction and we assume that there is for every $k > 0$ some $v_k \in T^1M$ such that $f_k(v_k) \leq 0$. Define a Borel probability measure μ_k on T^1M by $\mu_k(A) = \frac{1}{k}\int_0^k \chi_A(\Phi^s v_k)ds$; then $f_k(v_k) = \int f d\mu_k$. By passing to a subsequence we may assume that the measures μ_k converge weakly to a measure μ. Then $\int f d\mu \leq 0$ by continuity of f, on the other hand μ is necessarily Φ^t-invariant. This is a contradiction and shows the lemma. □

If we equip $\mathcal{M}^*$ with the dual topology, i.e. the weakest topology for which the map $\zeta \times \mu \to \zeta(\mu)$ is continuous, then the pressure is a continuous function on the space $\mathcal{CH} \subset \mathcal{M}^*$ of all cohomology classes of Hölder cocycles. We call a class whose pressure vanishes *normalized*. A normalized class $[\zeta]$ is positive. Namely, by the definition of the pressure the class is necessarily nonegative (i.e. we have $\zeta(\mu) \geq 0$ for all $\mu \in \mathcal{M}$). However, since the entropy of a Gibbs equilibrium state of every Hölder continuous function is positive, the class is in fact positive. The cone over the space of normalized Hölder cohomology classes then just equals the set of all positive classes.

There is a natural map from the space $\mathcal{N}$ of normalized Hölder classes into the space $\mathcal{M}$ of Φ^t-invariant Borel probability measures which assigns to a normalized positive Hölder cohomology class $[\zeta]$ the unique Gibbs equilibrium state of $[\zeta]$. Our main theorem implies that this duality can be expressed in terms of solutions of differential equations. Namely, if $\Delta + Z$ is an operator of *positive* escape, then its formal adjoint $\Delta + Y$ with respect to the unique harmonic measure σ for $\Delta + Z$ is an operator of negative escape. The correspondence between normalized Hölder classes and Gibbs states (which however is not a convex linear map) can be viewed as a bijection between a subspace of the space of $\Delta + Z$-harmonic 1-forms and a subspace of the space of $\Delta + Y$-harmonic measures. It would be interesting to know whether this bijection can be extended to a bijection of the space of all harmonic 1-forms for $\Delta + Z$ onto the space of all harmonic measures for $\Delta + Y$.

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References

[A87] A. Ancona, *Negatively curved manifolds, elliptic operators and the Martin boundary,* Ann. Math. 125 (1987), 495–536.

[A89] A. Ancona, *Sur les fonctions propres positive des variétés de Cartan-Hadamard,* Comm. Math. Helv. 64 (1989), 62–83.

[Bo01] T. Bousch, *La condition de Walters,* Ann. Sci. Ecole Norm. Sup. 34 (2001), 287–311.

[G83] L. Garnett, *Foliations, the ergodic theorem and Brownian motion,* J.Funct.Anal.51 (1983), 285–311.

[GH90] E. Ghys, P. de la Harpe, *Sur les groupes hyperboliques d'après Mikhael Gromov,* Birkhäuser, Boston 1990.

[Gu87] Y. Guivarch, *Propriétés ergodiques, en mesure infinie, de certains systèmes dynamiques fibrés,* Erg. Th. & Dyn. Sys. 7 (1987),

[H89] U. Hamenstädt, *A new description of the Bowen-Margulis measure,* Ergodic Theory Dynam. Systems 9 (1989), no. 3, 455–464.

[H96] U. Hamenstädt, *Harmonic measures, Hausdorff measures, and positive eigenfunctions,* J. Diff. Geo. 44 (1996), 1–31.

[H97a] U. Hamenstädt, *Harmonic measures for compact negatively curved manifolds,* Acta Math. 178 (1997), 39–107.

[H97b] U. Hamenstädt, *Cocycles, Hausdorff measures and Cross Ratios,* Erg. Th. & Dyn. Sys. 17 (1997), 1061–1081.

[H99] U. Hamenstädt, *Cocycles, symplectic structures and intersection,* GAFA 9 (1999), 90–140.

[HK95] B. Hasselblatt, A. Katok, *Introduction to the modern theory of dynamical systems,* Cambridge University Press 1995.

[K86] V.A. Kaimanovich, *Brownian motion and harmonic functions on covering manifolds. An entropy approach,* Soviet Math. Dokl. 33 (1986), 812–816.

[K88] V.A. Kaimanovich, *Brownian motion on foliations: Entropy, invariant measures, mixing,* Funct. Anal. Appl. 22 (1988), 326–328.

[K94] V.A. Kaimanovich, *Ergodic theory of harmonic invariant measures for the geodesic flow on hyperbolic spaces,* Journal für die reine und angewandte Mathematik 455 (1994), 57–103.

[L92] F. Ledrappier, *Ergodic properties of the stable foliations,* in "Ergodic theory and related topics", III (Güstrow 1990), 131-145, Lecture Notes in Math. 1514, Springer 1992.

[L95a] F. Ledrappier, *A renewal theorem for the distance in negative curvature,* Proc. Sympos. Pure Math. 57 (1995), 351–360.

[L95b] F. Ledrappier, *Central limit theorem in negative curvature,* Ann. Probab. 23 (1995), 1219–1233.

[LY85a] F. Ledrappier, L. S. Young, *The metric entropy of diffeomorphism Part I. Characterization of measures satisfying Pesin's entropy formula,* Ann. of Math. 122 (1985), 509–539.

[LY85b] F. Ledrappier, L. S. Young, *The metric entropy of diffeomorphism. II. Relations between entropy, exponents and dimension,* Ann. of Math. 122 (1985), 540–574.

[LS84] T: Lyons, D. Sullivan, *Function theory, random paths and covering spaces,* J. Diff. Geom. 19 (1984), 299-323.

[M70] G.A. Margulis, *Certain measures associated with U-flows on compact manifolds,* Funct. Anal. Appl. 4 (1970), 55–67.

[P76] S.J. Patterson, *The limit set of Fuchsian groups,* Acta Math. 136 (1976), 241–273.

[Pr75] J.J. Prat, *Movement Brownien sur une variété à courbure négative,* C.R. Acad. Sci. Paris Sér. I Math. 280 (1975), 1539–1542.

[R81] M. Rees, *Checking ergodicity of some geodesic flows with infinite Gibbs measure,* Erg. Th. & Dyn. Sys. 1 (1981), 107-133.

[S79] D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions,* Publ. Math. IHES 50 (1979), 171-202.

[S77] D. Sullivan, *On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions,* Princeton Univ. Press

[S87] D. Sullivan, *Related aspects of positivity in Riemannian geometry,* J. Diff. Geo. 25 (1987), 327–351.

[Y] C. Yue, *The ergodic theory of discrete isometry groups on manifolds of variable negative curvature,* Transactions AMS

Mathematisches Institut der Universität Bonn
Beringstrasse 1
53115 Bonn, Germany e-mail: ursula@math.uni-bonn.de

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