

CLOSED TEICHMÜLLER GEODESICS IN THE THIN PART OF MODULI SPACE

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ABSTRACT. Let S be an oriented surface of genus $g \geq 0$ with $m \geq 0$ punctures where $3g - 3 + m \geq 2$. We show that if $3g - 3 + m \geq 4$ then for every compact subset K of the moduli space $\text{Mod}(S)$ for S there is a closed Teichmüller geodesic in $\text{Mod}(S)$ which does not intersect K .

1. INTRODUCTION

Let S be a *nonexceptional* oriented surface of finite type, i.e. S is a closed surface of genus $g \geq 0$ from which $m \geq 0$ points, so-called *punctures*, have been deleted, and $3g - 3 + m \geq 2$. The *Teichmüller space* $\mathcal{T}(S)$ of S is a smooth manifold diffeomorphic to $\mathbb{R}^{6g-6+2m}$. It can be represented as the quotient of the space of all complete hyperbolic metrics on S of finite volume under the action of the group of all diffeomorphisms of S which are isotopic to the identity. The group of all isotopy classes of orientation preserving diffeomorphisms of S is called the *mapping class group* $\mathcal{M}(S)$ of S . It acts smoothly and properly discontinuously on $\mathcal{T}(S)$ preserving a complete Finsler metric, the so-called *Teichmüller metric*. The quotient orbifold $\text{Mod}(S) = \mathcal{T}(S)/\mathcal{M}(S)$, equipped with the projection of the Teichmüller metric, is a complete noncompact geodesic metric space.

Even though the Teichmüller metric is not non-positively curved in any reasonable sense, it shares many properties with a Riemannian metric of non-positive curvature. For example, any two points in $\mathcal{T}(S)$ can be connected by a unique Teichmüller geodesic, and *closed* geodesics in the moduli space $\text{Mod}(S)$ are in one-to-one correspondence with the conjugacy classes of the so-called *pseudo-Anosov* elements of the mapping class group. However, unlike in the case of negatively curved Riemannian manifolds of finite volume, closed geodesics may escape into the end of $\text{Mod}(S)$. Namely, we show.

Theorem : *If $3g - 3 + m \geq 4$ then for every compact subset K of $\text{Mod}(S)$ there is a closed Teichmüller geodesic which does not intersect K .*

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The organization of this note is as follows. In Section 2 we summarize the properties of the train track complex \mathcal{TT} which are needed for our purpose. In Section 3 we use train tracks and splitting sequences to show Theorem A.

2. THE COMPLEX OF TRAIN TRACKS

In this section we summarize some results and constructions from [PH92, H09] which will be used throughout the paper (compare also [Mo03]).

Let S be an oriented surface of genus $g \geq 0$ with $m \geq 0$ punctures and where $3g - 3 + m \geq 2$. A *train track* on S is an embedded 1-complex $\tau \subset S$ whose edges (called *branches*) are smooth arcs with well-defined tangent vectors at the endpoints. At any vertex (called a *switch*) the incident edges are mutually tangent. Through each switch there is a path of class C^1 which is embedded in τ and contains the switch in its interior. In particular, the branches which are incident on a fixed switch are divided into “incoming” and “outgoing” branches according to their inward pointing tangent at the switch. Each closed curve component of τ has a unique bivalent switch, and all other switches are at least trivalent. The complementary regions of the train track have negative Euler characteristic, which means that they are different from discs with 0, 1 or 2 cusps at the boundary and different from annuli and once-punctured discs with no cusps at the boundary. We always identify train tracks which are isotopic.

A *trainpath* on a train track τ is a C^1 -immersion $\rho : [m, n] \rightarrow \tau \subset S$ which maps each interval $[k, k + 1]$ ($m \leq k \leq n - 1$) onto a branch of τ . The integer $n - m$ is then called the *length* of ρ . We sometimes identify a trainpath on S with its image in τ . Each complementary region of τ is bounded by a finite number of trainpaths which either are simple closed curves or terminate at the cusps of the region. A *subtrack* of a train track τ is a subset σ of τ which itself is a train track. Thus every switch of σ is also a switch of τ , and every branch of σ is an embedded trainpath of τ . We write $\sigma < \tau$ if σ is a subtrack of τ .

A train track is called *generic* if all switches are at most trivalent. The train track τ is called *transversely recurrent* if every branch b of τ is intersected by an embedded simple closed curve $c = c(b) \subset S$ which intersects τ transversely and is such that $S - \tau - c$ does not contain an embedded *bigon*, i.e. a disc with two corners at the boundary.

A *transverse measure* on a train track τ is a nonnegative weight function μ on the branches of τ satisfying the *switch condition*: For every switch s of τ , the sum of the weights over all incoming branches at s is required to coincide with the sum of the weights over all outgoing branches at s . The train track is called *recurrent* if it admits a transverse measure which is positive on every branch. We call such a transverse measure μ *positive*, and we write $\mu > 0$. If μ is any transverse measure on a train track τ then the subset of τ consisting of all branches with positive μ -weight is a recurrent subtrack of τ . A train track τ is called *birecurrent* if τ is recurrent and transversely recurrent.

A *geodesic lamination* for a complete hyperbolic structure on S of finite volume is a *compact* subset of S which is foliated into simple geodesics. A geodesic lamination λ is called *minimal* if each of its half-leaves is dense in λ . Thus a simple closed geodesic is a minimal geodesic lamination. A minimal geodesic lamination with more than one leaf has uncountably many leaves and is called *minimal arational*. Every geodesic lamination λ consists of a disjoint union of finitely many minimal components and a finite number of isolated leaves. Each of the isolated leaves of λ either is an isolated closed geodesic and hence a minimal component, or it *spirals* about one or two minimal components [CEG87, O96].

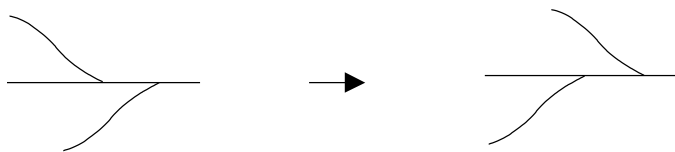
A geodesic lamination is *finite* if it contains only finitely many leaves, and this is the case if and only if each minimal component is a closed geodesic. A geodesic lamination is *maximal* if its complementary regions are all ideal triangles or once punctured monogons. A geodesic lamination λ is called *complete* if λ is maximal and can be approximated in the *Hausdorff topology* by simple closed geodesics. The space \mathcal{CL} of all complete geodesic laminations equipped with the Hausdorff topology is a compact metrizable space. Every geodesic lamination λ which is a disjoint union of finitely many minimal components is a *sublamination* of a complete geodesic lamination, i.e. there is a complete geodesic lamination which contains λ as a closed subset [H09].

A train track or a geodesic lamination σ is *carried* by a transversely recurrent train track τ if there is a map $F : S \rightarrow S$ of class C^1 which is isotopic to the identity and maps σ into τ in such a way that the restriction of the differential of F to the tangent space of σ vanishes nowhere. Note that this makes sense since a train track has a tangent line everywhere. We call the restriction of F to σ a *carrying map* for σ . Write $\sigma \prec \tau$ if the train track or the geodesic lamination σ is carried by the train track τ . If σ is a train track which is carried by τ , then every geodesic lamination λ which is carried by σ is also carried by τ . A train track τ is called *complete* if it is generic and transversely recurrent and if it carries a complete geodesic lamination. Every complete train track is recurrent. The set of all complete geodesic laminations which are carried by a complete train track τ is open and closed in \mathcal{CL} . In particular, the space \mathcal{CL} is totally disconnected [H09].

A half-branch \hat{b} in a generic train track τ incident on a switch v of τ is called *large* if every trainpath containing v in its interior passes through \hat{b} . A half-branch which is not large is called *small*. A branch b in a generic train track τ is called *large* if each of its two half-branches is large; in this case b is necessarily incident on two distinct switches, and it is large at both of them. A branch is called *small* if each of its two half-branches is small. A branch is called *mixed* if one of its half-branches is large and the other half-branch is small (for all this, see [PH92] p.118).

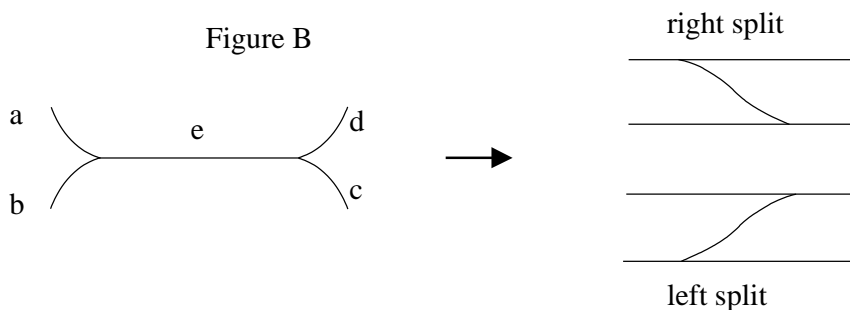
There are two simple ways to modify a complete train track τ to another complete train track. First, we can *shift* τ along a mixed branch to a train track τ' as shown in Figure A below. If τ is complete then the same is true for τ' . Moreover, a train track or a lamination is carried by τ if and only if it is carried by τ' (see [PH92] p.119). In particular, the shift τ' of τ is carried by τ . Note that there is a natural bijection of the set of branches of τ onto the set of branches of τ' .

Figure A



Second, if e is a large branch of τ then we can perform a right or left *split* of τ at e as shown in Figure B. Note that a right split at e is uniquely determined by the orientation of S and does not depend on the orientation of e . Using the labels in the figure, in the case of a right split we call the branches a and c *winners* of the split, and the branches b, d are *losers* of the split. If we perform a left split, then the branches b, d are winners of the split, and the branches a, c are losers of the split. The split τ' of a train track τ is carried by τ , and there is a natural choice of a carrying map which maps the switches of τ' to the switches of τ . The image of a branch of τ' is then a trainpath on τ whose length either equals one or two. We call this carrying map the *canonical carrying map*. It induces a natural bijection of the set of branches of τ onto the set of branches of τ' which maps the branch e to the *diagonal* e' of the split. The split of a maximal transversely recurrent generic train track is maximal, transversely recurrent and generic. If τ is complete and if $\lambda \in \mathcal{CL}$ is carried by τ , then there is a unique choice of a right or left split of τ at e with the property that the split track τ' carries λ . We call such a split a λ -*split*. The train track τ' is complete. In particular, a complete train track τ can always be split at any large branch e to a complete train track τ' ; however there may be a choice of a right or left split at e such that the resulting train track is not recurrent any more (compare p.120 in [PH92]). The reverse of a split is called a *collapse*.

Figure B



Denote by \mathcal{TT} the directed graph whose vertices are the isotopy classes of complete train tracks on S and whose edges are determined as follows. The train track $\tau \in \mathcal{TT}$ is connected to the train track τ' by a directed edge if and only if τ' can be obtained from τ by a single split. The graph \mathcal{TT} is connected [H09]. As a consequence, if we identify each edge in \mathcal{TT} with the unit interval $[0, 1]$ then this provides \mathcal{TT} with the structure of a connected locally finite metric graph. Thus \mathcal{TT} is a locally compact complete geodesic metric space. In the sequel we always assume that \mathcal{TT} is equipped with this metric without further comment. The mapping class group $\mathcal{M}(S)$ of S acts properly and cocompactly on \mathcal{TT} as a group of isometries. In particular, \mathcal{TT} is $\mathcal{M}(S)$ -equivariantly quasi-isometric to $\mathcal{M}(S)$ equipped with any word metric [H09].

3. CLOSED TEICHMÜLLER GEODESICS

In this section we use train tracks and splitting sequences to investigate closed *Teichmüller geodesics* in *moduli space* $\text{Mod}(S)$ of our nonexceptional surface S of finite type and show Theorem A from the introduction. Such closed geodesics are the projections of those Teichmüller geodesics in the Teichmüller space $\mathcal{T}(S)$ which are invariant under a pseudo-Anosov element of $\mathcal{M}(S)$. Recall that since the Euler characteristic of S is negative by assumption, $\mathcal{T}(S)$ can be identified with the space of all marked complete hyperbolic metrics on S of finite volume (with the usual identification of the elements contained in an orbit for the action of the group of diffeomorphisms of S isotopic to the identity).

A *measured geodesic lamination* is a geodesic lamination λ with a translation invariant transverse measure ν such that the ν -weight of every compact arc in S with endpoints in $S - \lambda$ which intersects λ nontrivially and transversely is positive. We say that λ is the *support* of the measured geodesic lamination. The space \mathcal{ML} of measured geodesic laminations equipped with the weak*-topology admits a natural continuous action of the multiplicative group $(0, \infty)$. The quotient under this action is the space \mathcal{PML} of *projective measured geodesic laminations* which is homeomorphic to the sphere $S^{6g-7+2m}$. Every simple closed geodesic c on S defines a measured geodesic lamination. The geometric intersection number between simple closed curves on S extends to a continuous bilinear form i on \mathcal{ML} , the *intersection form*. We say that a pair $(\lambda, \mu) \in \mathcal{ML} \times \mathcal{ML}$ of measured geodesic laminations *jointly fills up* S if for every measured geodesic lamination $\eta \in \mathcal{ML}$ we have $i(\eta, \lambda) + i(\eta, \mu) > 0$. This is equivalent to saying that every complete simple (possibly infinite) geodesic on S intersects either the support of λ or the support of μ transversely.

The unit cotangent bundle of the Teichmüller space with respect to the Teichmüller metric can naturally be identified with the space $\mathcal{Q}^1(S)$ of unit area marked holomorphic *quadratic differentials* on our surface S . Such a quadratic differential q is determined by a pair (λ^+, λ^-) of measured geodesic laminations on S which jointly fill up S and such that $i(\lambda^+, \lambda^-) = 1$ by our area normalization. The *horizontal* measured lamination λ^+ for q corresponds to the equivalence class of the horizontal measured foliation of q (compare [L83] and [K92] for a discussion of the relation between measured geodesic laminations and equivalence classes of measured foliations on S). The quadratic differential q determines a singular euclidean metric on S of unit area. For every simple closed curve c on S , the *q-length* of c is defined to be the infimum of the lengths with respect to this metric of any curve which is freely homotopic to c .

The bundle $\mathcal{Q}^1(S)$ admits a smooth action of the group $SL(2, \mathbb{R})$. The restriction of this action to the one-parameter subgroup of diagonal matrices of $SL(2, \mathbb{R})$ is the *Teichmüller geodesic flow* Φ^t on $\mathcal{Q}^1(S)$. If a quadratic differential q is given by the pair (λ^+, λ^-) of transverse measured geodesic laminations with $i(\lambda^+, \lambda^-) = 1$ then $\Phi^t q$ is given by the pair $(e^t \lambda^+, e^{-t} \lambda^-)$. The action of $SL(2, \mathbb{R})$ commutes with the natural action of the mapping class group $\mathcal{M}(S)$ on $\mathcal{Q}^1(S)$ and hence the flow Φ^t projects to a flow on the quotient $\mathcal{Q}(S) = \mathcal{Q}^1(S)/\mathcal{M}(S)$, again denoted

by Φ^t . A flow line of this flow projects to a Teichmüller geodesic in moduli space $\mathcal{T}(S)/\mathcal{M}(S) = \text{Mod}(S)$.

For every complete train track τ , the convex cone $\mathcal{V}(\tau)$ of all transverse measures on τ can naturally be identified with the set of all measured geodesic laminations whose support is carried by τ . A *tangential measure* on a complete train track τ assigns to a branch b of τ a weight $\mu(b) \geq 0$ such that for every complementary triangle T of τ with sides c_1, c_2, c_3 we have $\mu(c_i) \leq \mu(c_{i+1}) + \mu(c_{i+2})$ (here indices are taken modulo 3). Let $\mathcal{V}^*(\tau)$ be the convex cone of all tangential measures on τ . By the results from Section 3.4 of [PH92], there is a one-to-one correspondence between tangential measures on τ and measured geodesic laminations which *hit τ efficiently*, i.e. measured geodesic laminations whose support λ intersects τ transversely up to isotopy and is such that $\tau \cup \lambda$ does not contain any embedded bigon. In the sequel we often identify a measure $\mu \in \mathcal{V}(\tau)$ (or $\nu \in \mathcal{V}^*(\tau)$) with the measured geodesic lamination it defines. With this identification, the bilinear pairing $\langle, \rangle: \mathcal{V}(\tau) \times \mathcal{V}^*(\tau) \rightarrow [0, \infty)$ defined by $\langle \mu, \nu \rangle = \sum_b \mu(b)\nu(b)$ is just the restriction of the intersection form on $\mathcal{ML} \times \mathcal{ML}$ (see Section 3.4 of [PH92]).

Denote by $\mathcal{B}(\tau)$ the set of all pairs $(\lambda, \nu) \in \mathcal{V}(\tau) \times \mathcal{V}^*(\tau)$ where $\lambda \in \mathcal{V}(\tau)$ is a transverse measure on τ of total weight one and where $\nu \in \mathcal{V}^*(\tau)$ is a tangential measure with $\langle \lambda, \nu \rangle = 1$ and such that λ, ν jointly fill up S . Every pair $(\lambda, \nu) \in \mathcal{B}(\tau)$ defines a quadratic differential $q(\lambda, \nu)$ of area one.

If $\{\tau(i)\}$ is any splitting sequence, if $(\lambda, \nu) \in \mathcal{B}(\tau(0))$ and if λ is carried by $\tau(s)$ for some $s > 0$ then for every $i \in \{0, \dots, s\}$ there is a number $a(i) > 0$ such that $(a(i)\lambda, a(i)^{-1}\nu) \in \mathcal{B}(\tau(i))$. Denote by $[\lambda], [\nu] \in \mathcal{PM}\mathcal{L}$ the projective classes of λ, ν and for $i < s$ define $\rho([\lambda], [\nu])(\tau(i)) = a(i+1)/a(i)$. We call ρ a *roof function* for the pair (λ, ν) . We have.

Lemma 3.1. *Roof functions are uniformly bounded.*

Proof. Let $\eta \in \mathcal{TT}$ be obtained from τ by a single split at a large branch e and let μ be a transverse measure on η of total weight one. Via the natural carrying map $\eta \rightarrow \tau$, the measure μ defines a transverse measure μ_0 on τ . By the definition of a roof function we have to show that the total weight of μ_0 is bounded from above by universal constant. However, this is immediate from the fact that if a, c are the losing branches of the split connecting τ to η and if e' is the diagonal of the split then $\mu_0(e) = \mu(e') + \mu(a) + \mu(c)$ and $\mu_0(b) = \mu(b)$ for every branch $b \neq e$ of τ and its corresponding branch of η . This shows the lemma. \square

For $\epsilon > 0$ let $\mathcal{T}(S)_\epsilon$ be the ϵ -thick part of Teichmüller space consisting of all marked hyperbolic metrics whose *systole* (i.e. the length of the shortest closed geodesic) is at least ϵ . For sufficiently small ϵ the set $\mathcal{T}(S)_\epsilon$ is an $\mathcal{M}(S)$ -invariant contractible subset of $\mathcal{T}(S)$ on which the mapping class group acts cocompactly. Let $P: \mathcal{Q}^1(S) \rightarrow \mathcal{T}(S)$ be the canonical projection and define $\mathcal{Q}^1(\epsilon) = \{q \in \mathcal{Q}^1(S) \mid Pq \in \mathcal{T}(S)_\epsilon\}$. The sets $\mathcal{Q}^1(\epsilon)$ are invariant under the action of $\mathcal{M}(S)$. Their projections $\mathcal{Q}(\epsilon) = \mathcal{Q}^1(\epsilon)/\mathcal{M}(S) \subset \mathcal{Q}(S)$ to $\mathcal{Q}(S)$ are compact and satisfy $\mathcal{Q}(\epsilon) \subset \mathcal{Q}(\delta)$ for $\epsilon > \delta$ and $\cup_{\epsilon > 0} \mathcal{Q}(\epsilon) = \text{Mod}(S)$.

Call a finite splitting sequence $\{\tau(i)\}_{0 \leq i \leq m} \subset \mathcal{V}(\mathcal{TT})$ *tight* if the natural carrying map $\tau(m) \rightarrow \tau(0)$ maps every branch b of $\tau(m)$ onto $\tau(0)$. Note that every splitting sequence which contains a tight subsequence is tight itself. We have

Lemma 3.2. *Let $\{\tau(i)\}_{0 \leq i \leq m} \subset \mathcal{V}(\mathcal{TT})$ be any finite splitting sequence such that for some $\ell < m$, both sequences $\{\tau(i)\}_{0 \leq i \leq \ell}$ and $\{\tau(i)\}_{\ell \leq i \leq m}$ are tight. Then there is a number $\epsilon = \epsilon\{\tau(i)\} > 0$ depending on our sequence such that for all $(\lambda, \nu) \in \mathcal{B}(\tau(0))$ with the additional property that λ is carried by $\tau(m)$ the following holds.*

- (1) $q(\lambda, \nu) \in \mathcal{Q}^1(\epsilon)$.
- (2) The minimal λ -weight of every branch of $\tau(0)$ is at least ϵ .

Proof. Let $\{\tau(i)\}_{0 \leq i \leq m} \subset \mathcal{V}(\mathcal{TT})$ be a finite splitting sequence such that for some $\ell > 0$ the sequences $\{\tau(i)\}_{0 \leq i \leq \ell}$ and $\{\tau(i)\}_{\ell \leq i \leq m}$ are tight. Extending the distance function on the *curve graph* of S , define a distance d on the space \mathcal{U} of minimal geodesic laminations which do *not* fill up S (i.e. which admit a complementary component which is neither a disc nor a once punctured disc) by requiring that the distance between $\lambda \neq \mu$ equals one if and only if λ, μ can be realized disjointly. By the choice of \mathcal{U} , the distance between any two elements $\alpha, \beta \in \mathcal{U}$ is finite. Let $\lambda \in \mathcal{U}$ be a minimal geodesic lamination which is carried by $\tau(m)$ and let $\beta \in \mathcal{U}$ be such that $d(\beta, \lambda) = 1$; we claim that β is carried by $\tau(\ell)$. Namely, since our splitting sequence $\{\tau(i)\}_{\ell \leq i \leq m}$ is tight by assumption, the lamination λ fills $\tau(\ell)$, i.e. any transverse measure for λ defines a *positive* transverse measure on $\tau(\ell)$. Assume first that β is a simple closed curve on S with $d(\lambda, \beta) = 1$. Since λ is minimal there is a sequence $(\alpha_i)_i$ of simple closed curves which converge to λ in the Hausdorff topology [CEG87]. Now λ is disjoint from β and therefore we can choose the sequence $(\alpha_i)_i$ in such a way that each of the curves α_i is disjoint from β . For sufficiently large i , the curve α_i is carried by $\tau(m)$ (see Lemma 2.4 of [H09]) and fills $\tau(\ell)$, i.e. a carrying map $\alpha_i \rightarrow \tau(\ell)$ is surjective. By Lemma 4.4 of [MM99], since α_i, β are disjoint the curve β is carried by $\tau(\ell)$. Now the space of geodesic laminations carried by $\tau(\ell)$ is closed with respect to the Hausdorff topology and therefore $\tau(\ell)$ carries every geodesic lamination $\beta \in \mathcal{U}$ with $d(\lambda, \beta) = 1$.

Let again λ be a minimal geodesic lamination which is carried by $\tau(m)$ and let ν be a minimal geodesic lamination which hits $\tau(0)$ efficiently. We claim that λ and ν jointly fill up S . Namely, otherwise there is a simple closed curve c which is disjoint from both λ, ν . But this means that the distance between λ and c is at most one and hence c is carried by $\tau(\ell)$ by our above consideration. The same argument, applied to c and ν and the splitting sequence $\{\tau(i)\}_{0 \leq i \leq \ell}$, shows that ν is carried by $\tau(0)$ which contradicts our assumption that ν hits $\tau(0)$ efficiently.

Let μ be any transverse measure for $\tau(m)$ and let $m_0 = \max\{\mu(b) \mid b \text{ is a branch of } \tau(m)\}$. Let μ_0 be the transverse measure on $\tau(0)$ induced from μ by the carrying map $\tau(m) \rightarrow \tau(0)$. Since the splitting sequence $\{\tau(i)\}_{0 \leq i \leq m}$ is tight, the μ_0 -weight of a branch b of $\tau(0)$ is not smaller than m_0 , and it is not bigger than cm_0 where $c > 0$ only depends on our splitting sequence $\{\tau(i)\}_{0 \leq i \leq m}$ but not on μ . As a consequence, there is a universal constant $s > 0$ not depending on μ with the property that if the measure μ on $\tau(m)$ is normalized in such a way that the

total weight of μ_0 on $\tau(0)$ equals one, then the minimum of the weights that the measure μ_0 disposes on the branches of $\tau(0)$ is not smaller than s . This shows the second statement in our lemma. Now for any tangential measure ν on $\tau(0)$ we have $i(\mu, \nu) = \sum_b \mu_0(b)\nu(b)$ (where as before we identify the measures μ, ν with the measured geodesic laminations they define) and therefore if $i(\mu, \nu) = 1$ then the *maximal* weight disposed on a branch of $\tau(0)$ by the tangential measure ν is not bigger than $1/s$. Since $\mathcal{V}(\tau(m))$ and $\mathcal{V}^*(\tau(0))$ are closed subsets of \mathcal{ML} and the intersection form is continuous, this implies that the set \mathcal{B}_0 of pairs $(\mu, \nu) \in \mathcal{V}(\tau(m)) \times \mathcal{V}^*(\tau(0)) \subset \mathcal{V}(\tau(0)) \times \mathcal{V}^*(\tau(0))$ such that μ defines a transverse measure of total weight one on $\tau(0)$ and such that $i(\mu, \nu) = 1$ is a compact subset of $\mathcal{V}(\tau(0)) \times \mathcal{V}^*(\tau(0))$. By our above consideration, every pair $(\mu, \nu) \in \mathcal{B}_0$ jointly fills up S and hence \mathcal{B}_0 is a compact subset of $\mathcal{B}(\tau(0))$.

Since the assignment which associates to a quadratic differential its horizontal and vertical measured foliation is injective, continuous and open with respect to the smooth topology on $\mathcal{Q}^1(S)$ and the weak topology on $\mathcal{ML} \times \mathcal{ML}$ the set $\mathcal{C} = \{q(\mu, \nu) \mid (\mu, \nu) \in \mathcal{B}_0\}$ is compact. Therefore there is some $\epsilon > 0$ such that $\mathcal{C} \subset \mathcal{Q}^1(\epsilon)$ which shows the lemma. \square

For the proof of the theorem from the introduction, we use biinfinite splitting sequences $\{\tau(i)\}_i \subset \mathcal{V}(\mathcal{TT})$ to construct closed orbits of the Teichmüller flow. For this recall that there are only finitely many orbits of complete train tracks under the action of $\mathcal{M}(S)$. Call a biinfinite splitting sequence $\{\tau(i)\}$ *periodic* if there is a number $m > 0$ and some $\varphi \in \mathcal{M}(S)$ such that for every $k \in \mathbb{Z}$ we have $\{\tau(i)\}_{km \leq i \leq (k+1)m} = \varphi^k \{\tau(i)\}_{0 \leq i \leq m}$. We then call the mapping class φ a *period map* of the sequence with *period* m . A periodic splitting sequence can easily be constructed by choosing any infinite splitting sequence $\{\eta(i)\}$ and some $i < j$ such that $\eta(j) = \varphi\eta(i)$ for some $\varphi \in \mathcal{M}(S)$; then $\tau(k(j-i) + s) = \varphi^k \eta(i + s)$ ($s < j - i, k \in \mathbb{Z}$) defines a periodic splitting sequence with period map φ . Note that a period map of any periodic splitting sequence is an element of $\mathcal{M}(S)$ of infinite order. We say that a measured geodesic lamination λ *fills up* S if for every simple closed curve c on S we have $i(c, \lambda) > 0$. The support of such a measured geodesic lamination is necessarily connected. If $\lambda \in \mathcal{ML}$ is such that its projectivization $[\lambda] \in \mathcal{PML}$ is a fixed point of a pseudo-Anosov mapping class then λ fills up S . Recall that for every complete train track τ we view $\mathcal{V}(\tau)$ as the space of all measured geodesic laminations which are carried by τ . We have

Lemma 3.3. *Let $\{\tau(i)\} \subset \mathcal{V}(\mathcal{TT})$ be a biinfinite periodic splitting sequence with period map $\varphi \in \mathcal{M}(S)$. Then φ is pseudo-Anosov if and only if there is some $\lambda \in \cap_i \mathcal{V}(\tau(i))$ which fills up S .*

Proof. Let $\{\tau(i)\}_i$ be a biinfinite periodic splitting sequence with period map $\varphi \in \mathcal{M}(S)$. Then there is a number $\ell > 0$ such that $\varphi\mathcal{V}(\tau(0)) = \mathcal{V}(\tau(\ell)) \subset \mathcal{V}(\tau(0))$ and $\varphi^{-1}\mathcal{V}^*(\tau(0)) = \mathcal{V}^*(\tau(-\ell)) \subset \mathcal{V}^*(\tau(0))$. Assume first that φ is pseudo-Anosov. Since φ acts on the Thurston boundary \mathcal{PML} of Teichmüller space with north-south-dynamics and since the projectivizations P, Q of the cones $\mathcal{V}(\tau(0)), \mathcal{V}^*(\tau(0))$ are closed disjoint subsets of \mathcal{PML} with non-empty interior, the attracting fixed point λ^+ for the action of φ on \mathcal{PML} is contained in $\cap_i \varphi^i(P)$. But λ^+ is the

projectivization of a measured geodesic lamination λ which fills up S and hence $\lambda \in \cap_i \mathcal{V}(\tau(i))$ is a lamination as required in the lemma.

Now assume that there is some $\lambda \in \cap_i \mathcal{V}(\tau(i)) = \cap_i \varphi^i \mathcal{V}(\tau(0)) = A$ which fills up S . The set A is a closed non-empty φ -invariant subset of \mathcal{ML} . By Theorem 8.5.1 of [Mo03], it can be represented as a join $\mathcal{V}_1 * \mathcal{V}_2$ where $\mathcal{V}_1 \neq \emptyset$ is the space of measured geodesic laminations whose support is contained in a fixed geodesic lamination ζ on S and where \mathcal{V}_2 is a space of measured geodesic laminations whose support is carried by a train track on S contained in the complement of ζ . A measured geodesic lamination whose support contains more than one connected component does not fill up S . But $\lambda \in \mathcal{V}_1 * \mathcal{V}_2$ fills up S by assumption and therefore the set \mathcal{V}_2 is empty and A is the space of measured geodesic laminations supported in the single geodesic lamination ζ . Moreover, ζ is minimal and connected, with complementary components which are topological discs or once punctured topological discs. Every mapping class which preserves the set A also preserves the geodesic lamination ζ and hence it is pseudo-Anosov by Thurston's classification of elements of $\mathcal{M}(S)$. But A is invariant under the period map φ and hence φ is pseudo-Anosov as claimed. \square

Let $p > 0$ be the number of branches of a complete train track on S . Let $\tau \in \mathcal{V}(\mathcal{TT})$ and number the branches of τ with numbers $1, \dots, p$. If τ' can be obtained from τ by a single split at a large branch e , then the numbering of the branches of τ naturally induces a numbering of the branches of τ' (compare the discussion in Section 5 of [H09]). In other words, every splitting sequence $\{\tau(i)\}$ together with a numbering of the branches of $\tau(0)$ determines uniquely a *numbered splitting sequence*, i.e. a splitting sequence together with a consistent numbering of the branches of the train tracks in the sequence which is determined by $\{\tau(i)\}$ up to a permutation of the numbering of the branches of $\tau(0)$. Let e_1, \dots, e_p be the standard basis of \mathbb{R}^p . A numbering of the branches of $\tau(0)$ then defines an embedding of $\mathcal{V}(\tau(0))$ onto a closed convex cone in \mathbb{R}^p determined by the switch conditions by associating to a measure $\mu \in \mathcal{V}(\tau(0))$ the vector $\sum_i \mu(i)e_i \in \mathbb{R}^p$ where we identify a branch of $\tau(0)$ with its number. If τ' is obtained from τ by a split at the large branch e with number k and if i, j are the losing branches of the split connecting τ to τ' then the transformation $\mathcal{V}(\tau') \rightarrow \mathcal{V}(\tau)$ induced by the canonical carrying map $\tau' \rightarrow \tau$ is just the restriction to $\mathcal{V}(\tau')$ of the unique linear map $A : \mathbb{R}^p \rightarrow \mathbb{R}^p$ which satisfies $A(e_i) = e_i + e_k$, $A(e_j) = e_j + e_k$ and $A(e_s) = e_s$ for $s \neq i, j$. A change of the numbering of the branches of $\tau(0)$ results in replacing A by its conjugate under a *permutation map*, i.e. a linear isometry of \mathbb{R}^p corresponding to a permutation of the standard basis vectors. Similarly, for every finite splitting sequence $\{\tau(i)\}_{0 \leq i \leq m}$ the canonical map $\mathcal{V}(\tau(m)) \rightarrow \mathcal{V}(\tau(0))$ corresponds to a (p, p) -matrix $A(\tau(m))$ with non-negative entries which is uniquely determined by the sequence up to conjugation with a permutation matrix.

By the Perron Frobenius theorem, a (p, p) -matrix A with positive entries admits up to a multiple a unique eigenvector with positive entries. The corresponding eigenvalue α is positive, and its absolute value is bigger than the absolute value of any other eigenvalue of A . Moreover, the generalized eigenspace for α is one-dimensional. We call an eigenvector with positive entries for the eigenvalue α a *Perron Frobenius eigenvector*. We have.

Lemma 3.4. *Let $\{\tau(i)\}$ be a periodic splitting sequence with period map φ . Assume that $k > 0$ is such that $\varphi(\tau(0)) = \tau(k)$ and that $\{\tau(i)\}_{0 \leq i \leq k}$ is tight; then φ is pseudo-Anosov and there is some $\ell \geq 1$ such that its attracting fixed point is the projectivization of a Perron Frobenius eigenvector of the matrix $A(\tau(k\ell))$.*

Proof. Let $\{\tau(i)\}_{0 \leq i \leq k}$ be a tight splitting sequence and assume that there is some $\varphi \in \mathcal{M}(S)$ such that $\varphi(\tau(0)) = \tau(k)$. We obtain a biinfinite splitting sequence $\{\tau(i)\}$ by defining $\tau(mk + \ell) = \varphi^m(\tau(\ell))$ for $m \in \mathbb{Z}$ and $\ell < k$. By our above discussion, with respect to some numbering of the branches of $\tau(0)$ the self-map $\mathcal{V}(\tau(0)) \rightarrow \mathcal{V}(\tau(0))$ which is the composition of the map $\mathcal{V}(\tau(0)) \rightarrow \mathcal{V}(\tau(k))$ induced by φ and the transformation $\mathcal{V}(\tau(k)) \rightarrow \mathcal{V}(\tau(0))$ induced by the canonical carrying map $\tau(k) \rightarrow \tau(0)$ can be represented by the restriction of a linear map $\mathbb{R}^p \rightarrow \mathbb{R}^p$ given with respect to the standard basis by a matrix C . Since $\{\tau(i)\}_{0 \leq i \leq k}$ is tight by assumption, the entries of the matrix C are positive and therefore by the Perron Frobenius theorem, our map uniformly contracts the projectivization of the closed cone of vectors with nonnegative entries into its interior. As a consequence, the intersection $\cap_i \mathcal{V}(\tau(i))$ consists of a single ray spanned by the positive measure on $\tau(0)$ which corresponds to a Perron Frobenius eigenvector of C (compare with the beautiful argument in [K85]). This ray defines a projective measured geodesic lamination $\lambda^+ \in \mathcal{PML}$ which is *uniquely ergodic* and fills up S . Similarly, the intersection $\cap \mathcal{V}^*(\tau(i))$ consists of a single ray which defines a projective measured geodesic lamination $\lambda^- \neq \lambda^+$ which fills up S . By Lemma 5.3, the mapping class φ is pseudo-Anosov and its attracting fixed point equals λ^+ , its repelling fixed point equals λ^- .

Recall that a numbering of the branches of $\tau(0)$ induces a numbering of the branches of $\tau(k)$. There is a second numbering of the branches of $\tau(k)$ induced from the numbering of the branches of $\tau(0)$ via the map $\varphi : \tau(0) \rightarrow \tau(k)$. These two numberings differ by a permutation σ of $\{1, \dots, p\}$. Let $\ell \geq 1$ be such that $\sigma^\ell = 1$; then $\varphi^\ell \tau(0) = \tau(k\ell)$ as numbered train tracks. It is then immediate from our above discussion that the attracting fixed point of φ corresponds to the projectivization of a Perron Frobenius eigenvector for the matrix $A(\tau(k\ell))$ which shows the lemma. \square

In a Riemannian manifold M of bounded negative curvature and finite volume with fundamental group Γ , every closed geodesic intersects a fixed compact subset of M . Moreover, we can construct an infinite sequence (γ_i) of pairwise distinct closed geodesics which intersect every compact subset K of M in arcs of uniformly bounded length as follows. Choose a hyperbolic element φ in Γ and a parabolic element ψ ; then up to replacing ψ by a conjugate, the elements $\varphi\psi^k$ are hyperbolic for all $k > 0$ and they define an infinite family of pairwise distinct closed geodesics with the required properties. In the next lemma, we point out that a similar construction for a pseudo-Anosov element of $\mathcal{M}(S)$ and a Dehn twist about a suitably chosen simple closed curve (viewed as a parabolic element in $\mathcal{M}(S)$) yields sequences of closed geodesics in $\text{Mod}(S)$ with similar properties.

Lemma 3.5. *There is a sequence (γ_i) of pairwise distinct closed Teichmüller geodesics in moduli space $\text{Mod}(S)$ with the following properties.*

- (1) *There is a fixed compact subset K_0 of $\text{Mod}(S)$ which is intersected by γ_i for every i .*
- (2) *The geodesics γ_i intersect every compact subset $K \subset \text{Mod}(S)$ in arcs of uniformly bounded length.*

Proof. Recall from Section 3 the definition of a train track σ in special standard form for a framing F of our surface S with pants decomposition P . We first claim that there is a train track τ which is shift equivalent to a such a train track in special standard form and there is a tight splitting sequence $\{\tau(i)\}_{0 \leq i \leq k}$ issuing from $\tau(0) = \tau$ such that $\tau(k) = \varphi\tau(0)$ for some $\varphi \in \mathcal{M}(S)$.

For this let η be any train track which is shift equivalent to a train track in special standard form for a framing F of S with pants decomposition P . Let λ_0 be the unique special geodesic lamination for P which is carried by η and let $\{\zeta(i)\}_{0 \leq i \leq m}$ be any tight splitting sequence issuing from $\zeta(0) = \eta$. Since every orbit for the action of the mapping class group on the space \mathcal{CL} of complete geodesic laminations on S is dense and since the set of all complete geodesic laminations on S which are carried by $\zeta(m)$ is open in \mathcal{CL} , there is an element $\tilde{\varphi}_0 \in \mathcal{MS}$ such that the special geodesic lamination $\tilde{\varphi}_0\lambda_0$ for the pants decomposition φ_0P is carried by $\zeta(m)$. By the considerations in Section 4, the train track $\zeta(m)$ is splittable to a train track η_1 which is shift equivalent to the train track $\varphi_0\eta$ for some $\varphi_0 \in \mathcal{M}(S)$. Note that every splitting sequence connecting η to η_1 is tight and therefore if $\eta_1 = \varphi_0\eta$ then such a splitting sequence has the desired properties. Otherwise we apply this construction to η_1 and find a tight splitting sequence connecting η_1 to a train track η_2 which is shift equivalent to $\varphi_1\eta$ for some $\varphi_1 \in \mathcal{M}(S)$. Since there are only finitely many train track in the shift equivalence class of η , after finitely many steps we find some $0 \leq i < j$ such that $\eta_j = (\varphi_j \circ \varphi_i^{-1})\eta_i$ and hence η_i is splittable to $(\varphi_j \circ \varphi_i^{-1})\eta_i$ with a tight splitting sequence.

Now let $\{\tau(i)\}_{0 \leq i \leq k}$ be a tight splitting sequence as above with $\tau(k) = \varphi(\tau(0))$ for some $\varphi \in \mathcal{M}(S)$ and such that $\tau(0)$ is shift equivalent to a train track in special standard form for a framing F with pants decomposition P . By Lemma 5.4, φ is pseudo-Anosov. For $k \leq i \leq 2k$ define $\tau(i) = \varphi(\tau(i-k))$. By the definition of a train track in special standard form, every component α of P defines an embedded simple closed trainpath on $\tau(0)$, and there is a number $s \geq 2$ (depending on α) and a Dehn twist ψ about α with the property that $\psi\tau(0)$ is splittable to $\tau(0)$ with a splitting sequence of length s . Each split of this sequence is a split at a large branch contained in α . In other words, for every $u > 0$ the train track $\psi^u\tau(0)$ is splittable to $\varphi^2(\tau(0)) = \tau(2k)$ with a tight splitting sequence $\{\tau(i)\}_{-su \leq i \leq 2k}$ of length $2k+su$. By Lemma 5.4, the mapping class $\zeta_u = \varphi^2\psi^{-u}$ is pseudo-Anosov. Its axis is the Teichmüller geodesic which is defined by a quadratic differential whose horizontal foliation is contained in the ray $\cap_i \zeta_u^i \mathcal{V}(\tau(0))$ and whose vertical foliation is contained in the ray $\cap_i \zeta_u^{-i} \mathcal{V}^*(\tau(0))$.

For $u \geq 1$ let $(\lambda_u, \nu_u) \in \mathcal{B}(\tau(0))$ be such that the quadratic differential $q_u = q(\lambda_u, \nu_u)$ is a cotangent vector of the axis of ζ_u . For $i \in [-su, 2k]$ let $a_u(i) > 0$ be such that $(a_u(i)\lambda_u, a_u(i)^{-1}\nu_u) \in \mathcal{B}(\tau(i))$, i.e. that the total weight of the transverse measure on $\tau(i)$ which corresponds to the measured geodesic lamination $a_u(i)\lambda_u$ equals one. Then the function $i \rightarrow a_u(i)$ is increasing, and $\log a_u(2k) - \log a_u(-su)$

is the length of the closed Teichmüller geodesic γ_u in moduli space which is the projection of the axis of ζ_u .

By Lemma 3.1, the numbers $\log a_u(2k)$ are bounded independent of u , and by Lemma 3.2 there is a number $\epsilon_0 > 0$ such that $q_u \in \mathcal{Q}^1(\epsilon_0)$ for every $u > 0$. The projection $\mathcal{Q}(\epsilon_0)$ of $\mathcal{Q}^1(\epsilon_0)$ to $\mathcal{Q}(S)$ is compact and therefore the geodesics γ_u have property 1) stated in the lemma. Moreover, since $\{\tau(i)\}_{0 \leq i \leq k}$ is tight by assumption, by Lemma 5.2 there is a number $\delta > 0$ such that for *every* $u > 0$ the λ_u -weight of *every* branch b of $\tau(0)$ is at least δ ; then the maximal weight that the tangential measure ν_u disposes on a branch of $\tau(0)$ is at most $1/\delta$.

To show that the geodesics γ_u also satisfy property 2) stated in the lemma, we show that $a_u(-su) \rightarrow 0$ ($u \rightarrow \infty$) and that moreover for every $\epsilon > 0$ there is a number $\beta(\epsilon) > 0$ which is independent of u and such that for all $t \in [-\log a_u(-su) + \beta(\epsilon), -\beta(\epsilon)]$ the $\Phi^t q_u$ -length of the curve α is not bigger than ϵ . To show that such a constant exists, denote by b_1, \dots, b_r the branches of $\tau(0)$ which are incident on a switch in the embedded trainpath α but which are not contained in α . Note that $r \geq 2$, i.e. there are at least two such branches. By the considerations in the proof of Lemma 2.5 of [H06] we have $i(\mu, \alpha) = \frac{1}{2} \sum_i \mu(b_i)$ for every $\mu \in \mathcal{V}(\tau(0))$. In particular, the intersection number $i(\lambda_u, \alpha)$ is bounded from above independent of u . Choose a numbering of the branches of $\tau(0)$ so that b_i is the branch with number i and such that the branches with numbers $r+1, \dots, r+\ell$ are precisely the branches contained in the embedded trainpath α . Then the linear self-map of \mathbb{R}^p which defines the natural map $\mathcal{V}(\tau(0)) \rightarrow \mathcal{V}(\tau(-s))$ with respect to this numbering maps for each $i \leq r$ the i -th standard basis vector e_i of \mathbb{R}^p to $e_i + e_{r+1} + \dots + e_{r+\ell}$, and for $i > r$ it maps e_i to itself. For $\delta > 0$ as above, the sum of the λ_u -weights of the branches b_i ($i \leq r$) is bounded from below by 2δ and hence for every $m \leq u$ we have $a_u(-sm) \leq 1/(1 + 2m\delta)$ independent of u . In particular, $a_u(-su)$ tends to zero as $u \rightarrow \infty$. However, $i(c\lambda_u, \alpha) = ci(\lambda_u, \alpha)$ for all $c > 0$ and therefore for every $\epsilon > 0$ there is a number $\beta_1(\epsilon) > 0$ not depending on u such that for $t \in [\log a_u(-su), -\beta_1(\epsilon)]$ the intersection with α of the horizontal measured geodesic lamination of $\Phi^t q_u$ does not exceed $\epsilon/4$.

We observed above that the maximal weight that the tangential measure ν_u disposes on a branch of $\tau(0)$ is bounded from above by $1/\delta$. Since the natural transformation $\mathcal{V}^*(\tau(0)) \rightarrow \mathcal{V}^*(\tau(2k))$ can be represented by a fixed linear map (with the interpretation discussed above) and the functions a_u are non-decreasing, there is then a number $\chi > 0$ with the property that for *every* $u > 0$ the maximal weight that the measure $a_u(2k)^{-1}\nu_u$ disposes on a branch of $\tau(2k)$ is bounded from above by χ . However, the tangential measure $a_u(-su)^{-1}\nu_u$ on $\tau(-su)$ is the image of the tangential measure $a_u(2k)^{-1}\nu_u$ on $\tau(2k)$ under the transformation ζ_u^{-1} which identifies $\tau(2k)$ with $\tau(-su)$ and therefore the maximal weight that $a_u(-su)^{-1}\nu_u$ disposes on a branch of $\tau(-su)$ is bounded from above by χ . Since α is an embedded trainpath in $\tau(-su)$ this just means that the intersection with α of the vertical geodesic lamination of $\Phi^{\log a_u(-su)} q_u$ is bounded from above by a universal constant not depending on u . In other words, there is a constant $\beta_2(\epsilon) > 0$ not depending on u such that for every $t \in [-\log a_u(-su) + \beta_2(\epsilon), 0]$, the intersection with α of the vertical measured geodesic lamination of $\Phi^t q_u$ does not exceed $\epsilon/4$. The $\Phi^t q_u$ -length of α is bounded from above by twice the sum of

the intersection of α with the horizontal and the vertical geodesic lamination of $\Phi^t q_u$ (compare e.g. [R05]) and hence for every sufficiently large u and for every $t \in [-\log a_u(-su) + \beta_2(\epsilon), -\beta_1(\epsilon)]$ the $\Phi^t q_u$ -length of α does not exceed ϵ . Now for every $\delta > 0$ there is a number $\rho(\delta) > 0$ such that the set $\mathcal{Q}^1(\delta)$ is contained in the set of all quadratic differentials q with the property that the q -length of every essential simple closed curve on S is at least $\rho(\delta)$. This then yields that the geodesics γ_u intersect a fixed compact subset K of $\text{Mod}(S)$ in arcs of uniformly bounded length, independent of u . The lemma follows. \square

We are now ready to show the Theorem from the introduction.

Proposition 3.6. *If $3g - 3 + m \geq 4$ then for every $\epsilon > 0$ there is a closed orbit of the Teichmüller geodesic flow which does not intersect $\mathcal{Q}(\epsilon)$.*

Proof. Assume that $3g - 3 + m \geq 4$ (which excludes a sphere with at most 6 punctures, a torus with at most 3 punctures and a closed surface of genus 2) and let P be a pants decomposition of S . We require that if the genus of S is at least 2 then P contains two non-separating simple closed curves γ_1, γ_2 such that the surface S_0 obtained by cutting S open along γ_1, γ_2 is connected and that moreover the bordered surface $S - (P - \gamma_1 - \gamma_2)$ consists of $2g - 6 + m$ pairs of pants and 2 fourth punctured spheres containing γ_1, γ_2 as essential curves in their interior. If the genus of S is at most one then S contains at least 4 punctures and we choose P in such a way that it contains two separating simple closed curves γ_1, γ_2 with the property that the surface obtained by cutting S open along γ_1, γ_2 consists of three connected components, where two of these components are pairs of pants. In other words, $S - (\gamma_1 \cup \gamma_2)$ contains a unique connected component S_0 of Euler characteristic at most -2 , and if we replace the boundary circles of this component by cusps then the resulting surface is nonexceptional.

Let τ be a train track in standard form for a framing with pants decomposition P and with only twist connectors. Then the curves γ_1, γ_2 are carried by τ and define embedded simple closed trainpaths on τ . We require that a branch of τ which is contained in the closure of S_0 , which is incident on a switch contained in $\gamma_1 \cup \gamma_2$ and which is *not* contained in $\gamma_1 \cup \gamma_2$ is a small branch. By our choice of P and the discussion on p.147-48 of [PH92], such a complete train track exists. After removing all branches of τ which are incident on a switch contained in $\gamma_1 \cup \gamma_2$ or which are contained in $S - S_0$ we obtain a train track σ_0 on the subsurface S_0 . If we identify S_0 with the surface of finite type obtained by replacing the boundary circles by cusps then σ_0 is a complete train track on S_0 which is in standard form for a framing with pants decomposition $P \cap S_0$. Similarly, for $i = 1, 2$ let $S_i \supset S_0$ be the connected component of $S - \gamma_i$ which contains S_0 as a subsurface. The train track σ_i on S_i which we obtain from τ by removing all branches which are incident on a switch in γ_i or which are not contained in S_i is complete (in the interpretation as above). Note that we have $\tau = \sigma_1 \cup \sigma_2$, i.e. every branch of τ is either a branch of σ_1 or a branch of σ_2 . Moreover, a branch of τ which is incident on a switch in σ_i and which is not contained in σ_i is a small branch contained in a once punctured monogon component C of $S_i - \sigma_i$, and it is the unique branch of τ_i in this component.

For $i = 0, 1, 2$ choose a tight splitting sequence $\{\sigma_i(j)\}_{0 \leq j \leq s_i}$ issuing from $\sigma_i(j) = \sigma_i$ with the property that there is a pseudo-Anosov element φ_i of $\mathcal{M}(S_i)$ such that $\sigma_i(s_i)$ is shift equivalent to $\varphi_i(\sigma_i)$. The existence of such a splitting sequence follows from the arguments in the proof of Lemma 3.5. We view φ_i as a reducible element of $\mathcal{M}(S)$ which can be represented by a diffeomorphism of S fixing γ_i (or γ_1 and γ_2 for $i = 0$) pointwise. Define $\tau_i = \varphi_i(\tau)$; then τ_i is *carried* by τ and contains $\varphi_i\sigma_i$ as a subtrack. Moreover, there is a carrying map $\tau_i \rightarrow \tau$ which maps every branch b of $\varphi_i\sigma_i < \tau_i$ *onto* σ_i and which maps $\tau_i - \varphi_i\sigma_i$ bijectively onto $\tau - \sigma_i$. In the sequel we always assume that our carrying maps have these properties.

For $k > 0$ define $\zeta(k) = \varphi_1 \circ \varphi_0^{2k} \circ \varphi_2 \circ \varphi_0^{2k}$ (where \circ means composition, i.e. $a \circ b$ represents the mapping class obtained by applying b first followed by an application of a). We claim that for every $k > 0$ the mapping class $\zeta(k)$ is pseudo-Anosov. For this note first that $\zeta(k)\tau$ is carried by τ for all k . Thus it follows as in Lemma 3.3 that $\zeta(k)$ is pseudo-Anosov if and only if $\cap_i \zeta(k)^i \mathcal{V}(\tau)$ consists of a single ray which fills up S . By the considerations in the proof of Lemma 5.4, this is the case if a carrying map $\zeta(k)^2\tau \rightarrow \tau$ maps every branch of $\zeta(k)^2\tau$ *onto* τ (compare also [K85]).

Now let b be any branch of $\zeta(k)^2\tau$; then $b \in \zeta^2(k)\sigma_i$ for $i = 1$ or $i = 2$. Assume first that $b \in \zeta(k)^2\sigma_2$. Since the splitting sequence $\{\sigma_2(j)\}_{0 \leq j \leq s_2}$ is tight by assumption, the image of b under a carrying map $\zeta(k)^2\tau \rightarrow \zeta(k)\varphi_1\varphi_0^{2k}\tau$ equals the subtrack $\zeta(k)\varphi_1\varphi_0^{2k}\sigma_2$ of $\zeta(k)\varphi_1\varphi_0^{2k}\tau$. In particular, it contains the subgraph $\zeta(k)\varphi_1\varphi_0^{2k}(\tau - \sigma_1)$. On the other hand, for the same reason every branch of $\zeta(k)\varphi_1\varphi_0^{2k}\sigma_0 < \zeta(k)\varphi_1\varphi_0^{2k}\sigma_2$ is mapped by a carrying map $\zeta(k)\varphi_1\varphi_0^{2k}\tau \rightarrow \zeta(k)\tau$ *onto* $\zeta(k)\sigma_1$, and it maps $\zeta(k)\varphi_1\varphi_0^{2k}(\tau - \sigma_1)$ bijectively onto $\zeta(k)(\tau - \sigma_1)$. It follows that a carrying map $\zeta(k)^2\tau \rightarrow \zeta(k)\tau$ maps b *onto* $\zeta(k)\tau$. Now if $b \in \zeta(k)^2\sigma_1$ then the same argument yields that the image of b under a carrying map $\zeta(k)^2\tau \rightarrow \zeta(k)\tau$ contains the subtrack $\zeta(k)\sigma_1$ of $\zeta(k)\tau$. In particular, it contains a branch $c \in \zeta(k)\sigma_0 < \zeta(k)\sigma_2$ which is mapped by a carrying map $\zeta(k)\tau \rightarrow \tau$ *onto* τ . As before, we conclude that a carrying map $\zeta(k)^2\tau \rightarrow \tau$ maps every branch b of $\zeta(k)^2\tau$ *onto* τ and consequently each of the maps $\zeta(k)$ is pseudo-Anosov.

To establish our proposition it is enough to show that for every $\epsilon > 0$ there is a number $k(\epsilon) > 0$ with the following property. For every $k \geq k(\epsilon)$, the periodic orbit for the Teichmüller geodesic flow on $\mathcal{Q}(S)$ which corresponds to the conjugacy class of $\zeta(k)$ is entirely contained in the subset of $\mathcal{Q}(S)$ of all quadratic differentials q which admit an essential simple closed curve of q -length at most ϵ (compare the proof of Lemma 3.5).

By Lemma 3.2, since for $i = 1, 2$ the splitting sequence $\{\sigma_i(j)\}_{0 \leq j \leq s_i}$ is tight there is a number $c > 0$ with the following property. Let μ be a measured geodesic lamination on S_i which is carried by $\sigma_i(s_i)$ and which defines the transverse measure $\mu_0 \in \mathcal{V}(\sigma_i(0))$; then $\mu_0(b_1)/\mu_0(b_2) \leq c$ for any two branches b_1, b_2 of σ_i . Now the preimage of σ_i under any carrying map $\varphi_i\tau \rightarrow \tau$ equals the subtrack $\varphi_i\sigma_i$ of $\varphi_i\tau$ and the restriction of a suitably chosen carrying map to $\varphi_i\tau - \varphi_i\sigma_i$ is injective. This implies that for *every* measured geodesic lamination μ which is carried by $\varphi_i\tau$ and defines a transverse measure on $\varphi_i\tau$ which is not supported in $\varphi_i\tau - \varphi_i\sigma_i$, the measure μ_0 on τ induced from μ by a carrying map $\varphi_i\tau \rightarrow \tau$ satisfies $\mu_0(b_1)/\mu_0(b_2) \leq c$ for all $b_1, b_2 \in \sigma_i$.

For $\epsilon > 0$ and $i = 1, 2$ let $C_i(\epsilon)$ be the closed subset of $\mathcal{V}(\tau)$ containing all transverse measures ν with the following properties.

- (1) The total weight of ν is one and the sum of the ν -weights over all branches of τ which are *not* contained in σ_i is at most ϵ .
- (2) For any two branches b_1, b_2 of $\sigma_i < \tau$ we have $\nu(b_1)/\nu(b_2) \leq c$.

For a transverse measure μ on τ denote as before by $\omega(\tau, \mu)$ the total weight of μ . We claim that for every $\epsilon > 0$ the transformation $\rho_i^k : \mathcal{V}(\tau) - \{0\} \rightarrow \mathcal{V}(\tau)$ which is the composition of the map $\varphi_i \varphi_0^{2k} : \mathcal{V}(\tau) \rightarrow \mathcal{V}(\varphi_i \varphi_0^{2k} \tau)$ with the map $\mathcal{V}(\varphi_i \varphi_0^{2k} \tau) \rightarrow \mathcal{V}(\tau)$ induced by a carrying map $\varphi_i \varphi_0^k \tau \rightarrow \tau$ and with the normalization map $\mu \rightarrow \mu/\omega(\tau, \mu)$ maps $C_{i+1}(\epsilon)$ into $C_i(\epsilon)$ provided that k is sufficiently large (indices are taken modulo 2).

We show our claim for the map ρ_1^k , the claim for ρ_2^k follows in exactly the same way. Note first that by the choice of c , the image under ρ_1^k of *every* transverse measure on τ which does not vanish on σ_1 satisfies property 2) above for $i = 1$. Thus since a measure from the set $C_2(\epsilon)$ does not vanish on $\sigma_0 < \sigma_1$, property 2) holds for all measures in $\rho_1^k C_2(\epsilon)$ and all $k > 0$. To establish property 1) for sufficiently large k , let $\nu \in C_2(\epsilon)$, viewed as a measured geodesic lamination, let $k > 0$ and let $\mu = \varphi_1 \varphi_0^{2k}(\nu)$. For $0 \leq i \leq 2k$ let $\mu(i)$ be the transverse measure on $\varphi_1 \varphi_0^i(\tau)$ of total weight one which defines a multiple of the measured geodesic lamination μ . Then there is some $a(i) \leq 1$ such that $\mu(i) = a(i)\mu$ as measured geodesic laminations. By the definition of the set $C_2(\epsilon)$, there is a number $r > 0$ only depending on ϵ and c such that the total μ -weight of the subtrack $\varphi_1 \varphi_0^{2k}(\sigma_0)$ of $\varphi_1 \varphi_0^{2k}(\tau)$ is not smaller than r . Now the splitting sequence $\{\sigma_0(j)\}_{0 \leq j \leq s_0}$ is tight and therefore a carrying map $\sigma_0(s_0) \rightarrow \sigma_0(0)$ strictly increases the total weight of a transverse measure by at least a factor $L_0 > 1$. Hence for every $s \leq 2k$ the total μ -weight of $\varphi_1 \varphi_0^{2k-s}(\sigma_0) < \varphi_1 \varphi_0^{2k-s}(\tau)$ is not smaller than $1 - r + L_0^s r$. Moreover, the sum of the μ -weights of the branches of $\varphi_1 \varphi_0^{2k-s}(\tau)$ which are not contained in $\varphi_1 \varphi_0^{2k-s}(\sigma_0)$ is independent of $s \leq 2k$. Thus the sum of the $\mu(2k - s)$ -weights of the branches of $\varphi_1 \varphi_0^{2k-s}(\tau)$ which are *not* contained in $\varphi_1 \varphi_0^{2k-s}(\sigma_0)$ is smaller than $(1 - r)/(1 - r + L_0^s r)$. As a consequence, if $(1 - r)/(1 - r + L_0^k r) < \epsilon$ then the $\mu(k)$ -weight of $\varphi_1 \varphi_0^k(\tau - \sigma_0)$ does not exceed ϵ and the same holds true for the $\rho_1^k(\nu)$ -weight of $\tau - \sigma_1$ which shows our above claim.

Together we deduce the existence of a number $k(\epsilon) > 0$ such that for $k \geq k(\epsilon)$ the set $C_1(\epsilon)$ is invariant under the map ρ which assigns to a measured geodesic lamination $0 \neq \mu \in \mathcal{V}(\tau)$ the normalized image of $\zeta(k)\mu \in \mathcal{V}(\zeta(k)\tau)$ under a carrying map $\mathcal{V}(\zeta(k)\tau) \rightarrow \mathcal{V}(\tau)$. Now if $k > k(\epsilon)$ and if $(\lambda_k, \nu_k) \in \mathcal{B}(\tau)$ is such that $q(\lambda_k, \nu_k) = q_k$ is a cotangent vector of the axis of $\zeta(k)$ then λ_k spans the ray $\cap_i \zeta(k)^i \mathcal{V}(\tau)$ and therefore necessarily $\lambda_k \in C_1(\epsilon)$.

Let $k \geq k(\epsilon)$ and let $a_k(1) > 1$ be such that the total weight of the transverse measure on $\varphi_1 \varphi_0^k(\tau)$ defined by the measured geodesic lamination $a_k(1)\lambda_k$ equals one. Recall from Lemma 2.5 of [H06] that the intersection of $a_k(1)\lambda_k$ with the embedded trainpath γ_1 on $\varphi_1 \varphi_0^k(\tau)$ is bounded from above by the sum of the $a_k(1)\lambda_k$ -weights of the branches of $\varphi_1 \varphi_0^k(\tau) - \varphi_1 \varphi_0^k(\sigma_1)$. By our above consideration, we may assume that for $k \geq k(\epsilon)$ this weight is bounded from above by ϵ .

As a consequence, for every $t \leq \log a_k(1)$ the intersection of γ_1 with the horizontal measured geodesic lamination of the quadratic differential $\Phi^t q_k$ is at most ϵ .

Similarly, let $a_k(2) < a_k(3) < a_k(4) < a_k(5)$ be such that the total weight of $a_k(2)\lambda_k$ on $\varphi_1\varphi_0^{2k}(\tau)$ equals one, that the total weight of $a_k(3)\lambda_k$ on $\varphi_1\varphi_0^{2k}\varphi_2(\tau)$ equals one, that the total weight of $a_k(4)\lambda_k$ on $\varphi_1\varphi_0^{2k}\varphi_2\varphi_0^k(\tau)$ equals one and that the total weight of $a_k(5)\lambda_k$ on $\zeta(k)\tau$ equals one. Note that $\log a_k(5)$ is the length of the periodic orbit of the Teichmüller flow defined by the conjugacy class of $\zeta(k)$. Using once more our above consideration we conclude that for $k \geq k(\epsilon)$ and every $t \leq \log a_k(4)$ the intersection of the horizontal measured geodesic lamination of the quadratic differential $\Phi^t q_k$ with the curve $\zeta(k)\gamma_2$ is bounded from above by ϵ .

Our proposition now follows if for sufficiently large k we can control the intersections of the curves $\gamma_1, \zeta(k)\gamma_2$ with the vertical measured geodesic laminations for the quadratic differentials $\Phi^t q_k$. For this let again $\nu_k \in \mathcal{V}^*(\tau)$ be such that $(\lambda_k, \nu_k) \in \mathcal{B}(\tau)$ and $q_k = q(\lambda_k, \nu_k)$. Since a carrying map $\varphi_1\varphi_0^{2k}\varphi_2(\tau) \rightarrow \varphi_1\varphi_0^{2k}(\tau)$ maps every branch b of $\varphi_1\varphi_0^{2k}\varphi_2(\sigma_2)$ onto $\varphi_0\varphi_0^{2k}(\sigma_2)$ and since $a_k(2)\lambda_k \in \varphi_1\varphi_0^{2k}C_2(\epsilon)$ by our above consideration, the $a_k(2)\lambda_k$ -weight of every branch of $\varphi_1\varphi_0^{2k}(\tau)$ which is contained in $\varphi_1\varphi_0^{2k}(\sigma_2)$ is bounded from below by a universal constant $\delta > 0$ not depending on k . Then the $a_k(2)^{-1}\nu_k$ -weight of every such branch is bounded from above by $1/\delta$ and hence there is a number $\chi > 0$ not depending on k which bounds from above the $a_k(3)^{-1}\nu_k$ -weight of every branch of $\varphi_1\varphi_0^{2k}\varphi_2(\sigma_2) < \varphi_1\varphi_0^{2k}\varphi_2(\tau)$. Now the curve $\varphi_1\varphi_0^{2k}\varphi_2(\gamma_1)$ is an embedded train-path in the train track $\varphi_1\varphi_0^{2k}\varphi_2(\sigma_2) < \varphi_1\varphi_0^{2k}\varphi_2(\tau)$ and hence our upper bound for the values of $a_k(3)^{-1}\nu_k$ on the branches of $\varphi_1\varphi_0^{2k}\varphi_2(\sigma_2)$ implies that the intersection between $\varphi_1\varphi_0^{2k}\varphi_2(\gamma_1)$ and $a_k(3)^{-1}\nu_k$ is uniformly bounded. As we increase k , the ratios $a_k(4)/a_k(3)$ tend to infinity (compare the above consideration) and hence after possibly increasing $k(\epsilon)$ we may assume that for every $k \geq k(\epsilon)$ we have $i(a_k(4)^{-1}\nu_k, \varphi_1\varphi_0^{2k}\varphi_2\varphi_0^k(\gamma_1)) < \epsilon$. By invariance of the intersection form under the action of $\mathcal{M}(S)$ we conclude that for every $k \geq k(\epsilon)$ and every $t \geq \log(a_k(4)/a_k(5))$ the intersection between the vertical measured geodesic lamination of $\Phi^t q_k$ and γ_1 is bounded from above by ϵ . This then shows that for every $t \in [\log(a_k(4)/a_k(5)), \log a_k(1)]$ the sum of the intersection numbers between γ_1 and the vertical and the horizontal measured geodesic lamination of $\Phi^t q_k$ does not exceed 2ϵ . In other words, for every such t the $\Phi^t q_k$ -length of γ_1 is bounded from above by 4ϵ .

The same argument shows that after possibly increasing $k(\epsilon)$ once more we may assume that for $k \geq k(\epsilon)$ and every $t \in [\log a_k(1), \log a_k(4)]$ the $\Phi^t q_k$ -length of $\zeta(k)(\gamma_2)$ is bounded from above by 4ϵ . By periodicity, we conclude that for $k \geq k(\epsilon)$ the periodic orbit of Φ^t which corresponds to the conjugacy class of $\zeta(k)$ is entirely contained in the set of quadratic differentials q which admits an essential simple closed curve of q -length at most 4ϵ . This completes the proof of our proposition. \square

Remark: 1) A pseudo-Anosov element φ acts as an isometry on the curve graph $(\mathcal{C}(S), d)$ of S . By a result of Bowditch [Bw03], for every $c \in \mathcal{C}(S)$ the limit $\lim_{k \rightarrow \infty} d(\varphi^k c, c)/k$ exists and is independent of c . This limit is called the *stable*

length for this action. The stable length of each of the (infinitely many) pseudo-Anosov elements $\zeta(k)$ constructed in the proof of Proposition 5.6 is at most 2. Moreover, let γ_k be the Teichmüller geodesic in $\mathcal{T}(S)$ which is invariant under $\zeta(k)$. Then there is a number $\epsilon > 0$ such that for sufficiently large k , the set of essential simple closed curves c on our surface S for which the minimum of the hyperbolic lengths of c along γ_k is smaller than ϵ is precisely an orbit under the action of $\zeta(k)$ of a pair of disjoint essential simple closed curves on S .

2) We believe that Proposition 3.6 is valid for *every* nonexceptional surface of finite type, with pseudo-Anosov mapping classes which can be constructed as the once in the proof our proposition. However we did not attempt to carry out the details.

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