GAFA, Geom. funct. anal.
Vol. 9 (1999) 1 – 0
1016-443X/99/010001-0 $ 1.50+0.20/0

GAFA Geometric And Functional Analysis

COCYCLES, SYMPLECTIC STRUCTURES AND INTERSECTION

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1 Introduction

In the paper [BCG], Besson, Courtois and Gallot proved the following remarkable theorem: Let S be a closed rank 1 locally symmetric space of noncompact type and let M be a closed manifold of negative curvature which is homotopy equivalent to S. If S and M have the same volume and the same volume entropies (i.e. the same asymptotic growth rate of volumes of balls in their universal covers), then S and M are isometric.

One application of this result is the solution of the so-called *conjugacy problem* for locally symmetric manifolds. This problem can be stated in the following way: The *marked length spectrum* of a closed negatively curved manifold (M, g) is the function ρ_M which assigns to each conjugacy class in the fundamental group $\pi_1(M)$ of M the length of the unique g-geodesic representing the class. Two homotopy equivalent negatively curved manifolds M, N have *the same marked length spectrum* if there is an isomorphism $\Psi : \pi_1(M) \to \pi_1(N)$ such that $\rho_N \circ \Psi = \rho_M$. This is equivalent to the existence of a continuous *time preserving conjugacy* for the *geodesic flows* for M and N [H1]. Such a conjugacy is defined to be a homeomorphism Λ of the unit tangent bundle T^1M of M onto the unit tangent bundle T^1N of N which is equivariant under the action of the geodesic flows Φ^t on T^1M and T^1N [H1]. The conjugacy problem now asks whether a composition of Λ with the time-t-map of the geodesic flow on T^1N for a suitable $t \in \mathbf{R}$ is the restriction to T^1M of the differential of an isometry of M onto N. This is known to be true for surfaces $[O, C]$.

Since the volume entropy of a closed negatively curved manifold equals the topological entropy of the geodesic flow, two manifolds with time preserving conjugate geodesic flows have the same volume entropy.

This work was partially supported by Sonderforschungsbereich 256.

Let X^0 be the *geodesic spray*, i.e. the generator of the geodesic flow Φ^t. Recall that there is a Hölder continuous Φ^t-invariant decomposition $TT^1M = \mathbf{R}X^0 \oplus TW^{ss} \oplus TW^{su}$, the so called *Anosov splitting*, where TW^{ss} (or TW^{su}) is the tangent bundle of the *strong stable (or strong unstable) foliation* W^{ss} (or W^{su}) on T^1M. The bundle $TW^{ss} \oplus TW^{su}$ is smooth, and hence we obtain a smooth 1-form ω on T^1M by defining $\omega(X^0) \equiv 1$ and $\omega(TW^{ss} \oplus TW^{su}) = 0$. This form ω is called the *canonical contact form*. The differential form $\omega \wedge (d\omega)^{n-1}$ is the volume form of the so called *Sasaki metric* on T^1M, and the total mass of T^1M with respect to this volume form equals the product of the volume of M with the volume of the $(n-1)$-dimensional unit sphere in $\mathbf{R}^n$ where $n = \dim M$.

The pull-back of the canonical contact form on T^1N under a time preserving conjugacy $\Lambda : T^1M \to T^1N$ of class C^1 is the canonical contact form on T^1M. This implies that M and N have the same volume if their geodesic flows are C^1-time-preserving conjugate. Therefore the result of Besson, Courtois and Gallot gives a positive answer to the conjugacy problem if one of the manifolds under consideration is locally symmetric and if the conjugacy is assumed to be of class C^1.

One of the objectives of this note is to remove this regularity assumption on the conjugacy. In section 3 we show

Theorem A. *If M and N are closed negatively curved manifolds with the same marked length spectrum and if the Anosov splitting of TT^1M is of class C^1, then M and N have the same volume.*

Manifolds with strictly 1/4-pinched sectional curvature or locally symmetric manifolds have C^1–Anosov splitting. Thus as a corollary from [BCG] we obtain

COROLLARY. *If M has the same marked length spectrum as a closed negatively curved locally symmetric space S, then M and S are isometric.*

Recall that the *space of geodesics* $\mathcal{G}\tilde{M}$ of the universal cover $\tilde{M}$ of (M, g) is the quotient of the unit tangent bundle $T^1\tilde{M}$ of $\tilde{M}$ under the action of the geodesic flow Φ^t. Also $\mathcal{G}\tilde{M}$ can naturally be identified with $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ where $\partial\tilde{M}$ is the *ideal boundary* of $\tilde{M}$ and Δ is the diagonal. The space of geodesics is a smooth manifold, and the differential $d\omega$ of the canonical contact form projects to a smooth symplectic structure on $\mathcal{G}\tilde{M}$. The fundamental group $\pi_1(M)$ of M acts on $\mathcal{G}\tilde{M}$ as a group of symplectomorphisms.

The *cross ratio* of the metric g is a function on $\mathcal{G}\tilde{M} \times \mathcal{G}\tilde{M}$. We show in section 3 that this function can be viewed as a coarse (or rather integrated) version of the symplectic form $d\omega$ on $\mathcal{G}\tilde{M}$. This enables us to construct the measure λ_g on $\mathcal{G}\tilde{M}$ defined by the smooth volume form $(d\omega)^{n-1}$ in a purely combinatorial way from the cross ratio. The measure λ_g is usually called the *Lebesgue–Liouville current* of the metric g. Our construction is general, but unfortunately so far we are only able to show that the measure we obtain is exactly the Lebesgue–Liouville current (rather than some multiple of it) under the additional assumption that the Anosov splitting of TT^1M is of class C^1.

A *geodesic current* for M is a locally finite Borel measure on the space of geodesics which is invariant under the action of the fundamental group and under the exchange of the factors in the product decomposition of $\mathcal{G}\tilde{M} = \partial\tilde{M}\times\partial\tilde{M}-\Delta$. If M and N are two homotopy equivalent closed negatively curved manifolds, then there is a natural $\pi_1(M)=\pi_1(N)$-equivariant Hölder homeomorphism of $\partial\tilde{M}$ onto $\partial\tilde{N}$ which induces an equivariant homeomorphism of $\mathcal{G}\tilde{M}$ onto $\mathcal{G}\tilde{N}$, in particular the spaces of geodesic currents for M and N are naturally identified.

With this identification we show in section 4

Theorem B. *Let $(M,g),(N,h)$ be homotopy equivalent closed negatively curved manifolds. If $\lambda_g=\lambda_h$ and if the Anosov splittings of TT^1M and TT^1N are of class C^1, then M and N have the same marked length spectrum.*

Recall that a Hölder continuous *additive cocycle* for the geodesic flow Φ^t is a Hölder continuous function $\zeta\colon T^1M\times\mathbf{R}\to\mathbf{R}$ which satisfies $\zeta(v,s+t)=\zeta(v,s)+\zeta(\Phi^s v,t)$ for all $v\in T^1M$ and all $s,t\in\mathbf{R}$. Two cocycles ζ,ξ are called *cohomologous* if there is a Hölder continuous function β on T^1M such that $\zeta(v,t)-\xi(v,t)=\beta(\Phi^t v)-\beta(v)$ for all $v\in T^1M$ and all $t\in\mathbf{R}$. We denote by $[\zeta]$ the cohomology class of ζ. Let $\mathcal{F}\colon T^1M\to T^1M$ be the *flip* $v\to\mathcal{F}v=-v$; the flip $\mathcal{F}\zeta$ of a cocycle ζ is the cocycle $\mathcal{F}\zeta(v,t)=\zeta(\mathcal{F}\Phi^t v,t)$. Two cocycles ζ,ξ are cohomologous if and only if this is true for $\mathcal{F}\zeta$ and $\mathcal{F}\xi$. In other words, $\mathcal{F}$ induces an action on cohomology classes of Hölder cocycles which we denote again by $\mathcal{F}$. We call ζ *quasi-invariant under the flip* if the cocycles ζ and $\mathcal{F}\zeta$ are cohomologous, i.e. if $[\zeta]$ is a fix point for the action of $\mathcal{F}$.

Consider now a closed negatively curved surface M. Every flip invariant Hölder cohomology class for the geodesic flow defines a unique cross ratio [H5], and in the two-dimensional case the defining properties of a cross ratio [H5] show that we can obtain from a cross ratio a finitely additive signed measure on $\mathcal{G}\tilde{M}$ in a natural way. In other words, there is a natural linear map which associates to a flip invariant Hölder cohomology class $[\zeta]$ a signed measure ν_ζ. Now the *length cocycle* ℓ of the negatively curved manifold (M,g) is defined by $\ell(v,t)=t$ for all $v\in T^1M$ and all $t\in\mathbf{R}$. For the length cocycle ℓ of g on the surface M, ν_ℓ is just $1/2$ times the Lebesgue–Liouville current of g.

On the other hand, if the Hölder cohomology class is *positive*, i.e. if it can be represented by a cocycle $\bar\zeta$ which satisfies $\bar\zeta(v,t)>0$ for all $v\in T^1M$ and all $t>0$, then there is another more classical way to define a projective class of currents on $\mathcal{G}\tilde M$. Namely, a suitable positive multiple $[a\zeta]$ of $[\zeta]$ (here $a>0$ is a constant depending on $[\zeta]$) defines a *Gibbs equilibrium state* which is a $\pi_1(M)$-invariant measure on $\mathcal{G}\tilde M$, determined uniquely up to a constant. This measure is a geodesic current if and only if $[\zeta]=[\mathcal{F}\zeta]$, and if this is satisfied we call it a *Gibbs-current of* ζ and denote its projective class in the space of projectivized currents by $[\mu_\zeta]$. The assignment $[\zeta]\to[\mu_\zeta]$ then is a map from a subset of the projectivization of the space of flip invariant cohomology classes into the projectivization of the space of geodesic currents.

If ℓ is the length cocycle of the Riemannian metric g on M, then its Gibbs current is the *Bowen-Margulis current* of g which corresponds to the measure of maximal entropy for the geodesic flow.

Katok showed in [K] that the Lebesgue–Liouville current and the Bowen Margulis current of a negatively curved metric on a surface are equivalent if and only if the metric has constant curvature. With our above notation this result can be stated as follows: If ℓ is the length cocycle of a metric g of negative curvature on the surface M and if ν_ℓ is equal to μ_ℓ up to a constant, then g has constant curvature.

In section 2 we obtain a generalization of Katok's result to arbitrary positive Hölder cohomology classes. For its formulation denote by $[\nu]$ the class of a current in the projectivization of the space of geodesic currents. We show:

Theorem C. *Let M be a closed surface of genus $g\geq 2$. If $[\mu_\ell]=[\nu_\ell]$ for a positive flip invariant Hölder cohomology class ℓ, then ℓ is the class of the length cocycle of a metric on M of constant curvature.*

Before we proceed we will fix some more notation. All our manifolds will be closed and equipped with a fixed negatively curved metric. We denote by T^1M (or $T^1\tilde M$) the unit tangent bundle of M (or $\tilde M$). There is a natural $\pi_1(M)$-equivariant projection $\pi\colon T^1\tilde M\to\partial\tilde M$. The pre-image of a point $\xi\in\partial\tilde M$ under π equals the stable manifold in $T^1\tilde M$ of all directions pointing towards ξ. We let $W^i(v)$ be the leaf of the foliation W^i containing v ($i = \mathrm{ss}, \mathrm{su}$) and write P for the canonical projections $T^1M \to M$ and $T^1\tilde{M} \to \tilde{M}$.

We will need the following facts which were pointed out by Ledrappier [Le]: The fundamental group $\pi_1(M)$ of M acts naturally on the ideal boundary $\partial\tilde{M}$ of $\tilde{M}$ as a group of homeomorphisms. There is a natural bijection between the space of Hölder cocycles for the geodesic flow on T^1M and the space of Hölder cocycles for the action of $\pi_1(M)$ on $\partial\tilde{M}$. Such a Hölder cocycle for the action of $\pi_1(M)$ is a Hölder continuous function $\zeta_0\colon \pi_1(M) \times \partial\tilde{M} \to \mathbf{R}$ which satisfies $\zeta_0(\varphi_1\varphi_2, \xi) = \zeta_0(\varphi_1, \varphi_2\xi) + \zeta_0(\varphi_2, \xi)$ for $\varphi_1, \varphi_2 \in \pi_1(M)$ and $\xi \in \partial\tilde{M}$.

Two such functions ζ_0, η_0 are *cohomologous* if there is a Hölder continuous function β_0 on $\partial\tilde{M}$ such that $\zeta_0(\varphi, \xi) - \eta_0(\varphi, \xi) = \beta_0(\varphi\xi) - \beta_0(\xi)$ for all $\varphi \in \pi_1(M)$ and all $\xi \in \partial\tilde{M}$. This notion of equivalence or cohomology defines an equivalence relation on the space of Hölder continuous cocycles. We call the equivalence class $[\zeta_0]$ of a Hölder continuous cocycle ζ_0 with respect to this equivalence relation simply the *Hölder class* of ζ_0. If $\xi(\varphi)$ is the attracting fixed point for the action of $\varphi \in \pi_1(M)$ on $\partial\tilde{M}$, then $\zeta_0(\varphi, \xi(\varphi))$ is called the *period* of ζ_0 at φ. Two Hölder cocycles are cohomologous if and only if they have the same periods (see [Le]).

The natural map from Hölder cocycles for the geodesic flow Φ^t to Hölder cocycles for the action of $\pi_1(M)$ on $\partial\tilde{M}$ preserves the equivalence relation which defines cohomology. Moreover a Hölder class $[\zeta]$ for Φ^t is invariant under the flip if and only if the corresponding class $[\zeta_0]$ for the action of $\pi_1(M)$ satisfies $\zeta_0(\varphi, \xi(\varphi)) = \zeta_0(\varphi^{-1}, \xi(\varphi^{-1}))$ for all $\varphi \in \pi_1(M)$ (all this is discussed in [Le]). In particular, the space of Hölder classes for the geodesic flow of a negatively curved metric g on M as well as the notion of flip-invariance and positivity do not depend on the choice of the metric.

Let $\mathcal{M}$ be the space of finite Φ^t-invariant Borel measures on T^1M. For every Hölder cocycle ζ for Φ^t and every $\mu \in \mathcal{M}$ the integral $\int \zeta\, d\mu$ can be defined as follows:

Let $\zeta\colon T^1M \times \mathbf{R} \to \mathbf{R}$ be a Hölder cocycle and write $f(v) = \zeta(v, 1)$ for all $v \in T^1M$. Then f is a Hölder continuous function on T^1M and defines a Hölder cocycle ζ_f by $\zeta_f(v, t) = \int_0^t f(\Phi^s v) ds$. If $v \in T^1M$ is a periodic point for Φ^t of period $\tau > 0$ then

$$\zeta_f(v, \tau) = \int_0^\tau f(\Phi^s v) ds = \int_0^\tau \zeta(\Phi^s v, 1) ds$$

$$= \int_0^\tau \big(\zeta(v,s+1)-\zeta(v,s)\big)ds = \int_0^1 \big(\zeta(v,\tau+s)-\zeta(v,s)\big)ds = \zeta(v,\tau)$$

by the cocycle equality for ζ. But this means that ζ and ζ_f are equivalent (compare [Le]).

For a measure $\mu \in \mathcal{M}$ we then define $\int \zeta\, d\mu = \int f\, d\mu$.

To show that this integral only depends on the cohomology class of ζ let ξ be a cocycle which is equivalent (i.e. cohomologous) to ζ. Then there is a Hölder continuous function β on T^1M such that $\xi(v,t) = \zeta(v,t) + \beta(\Phi^t v) - \beta(v)$. If $g(v) = \xi(v,1)$, $f(v) = \zeta(v,1)$ then $g(v) = f(v) + \beta(\Phi^1 v) - \beta(v)$ and therefore $\int g\, d\mu = \int f\, d\mu$ for every $\mu \in \mathcal{M}$.

As a conclusion we see that every Hölder cohomology class can be represented by a Hölder continuous function f, and two functions f and g define the same Hölder cohomology class if and only if for every $\eta \in \mathcal{M}$ we have $\int f\, d\eta = \int g\, d\eta$ (this is the Livshic theorem for Hölder continuous functions).

Now $\mathcal{M}$ has a natural identification with the space of locally finite Borel-measures on the space of geodesics $\mathcal{G}\tilde{M} = \partial\tilde{M} \times \partial\tilde{M} - \Delta$ which are invariant under the action of $\pi_1(M)$. In particular $\mathcal{M}$ is independent of the choice of the negatively curved metric. Then integration defines a pairing between two spaces ($\mathcal{M}$ and the space of cohomology classes of Hölder cocycles) which are independent of the choice of the metric g.

Fix now again a negatively curved metric g on M and let $\mathcal{M}_g \subset \mathcal{M}$ be the space of Φ^t-invariant Borel-probability measures on T^1M. For every $\mu \in \mathcal{M}$ there is some number $c(g,\mu) > 0$ such that $c(g,\mu)\mu \in \mathcal{M}_g$, but $c(g,\mu)$ depends on g. For $\mu \in \mathcal{M}_g$ the *entropy* $h_{\mu,g}$ of μ with respect to the geodesic flow Φ^t of g is defined. If $[\zeta]$ is a Hölder cohomology class then the *pressure* $pr([\zeta])$ of $[\zeta]$ is defined to be the supremum of the values $h_{\mu,g} - \int \zeta\, d\mu$ where μ ranges through $\mathcal{M}_g$. In other words, the pressure of $[\zeta]$ is the pressure in the usual sense of the Hölder continuous function $f_\zeta(v) = \zeta(v,1)$ on T^1M which represents the cohomology class $[\zeta]$.

For a positive Hölder class $[\zeta]$ there is a unique number $h([\zeta]) > 0$ such that $[h([\zeta])\zeta]$ is *normalized,* i.e. that $pr([h([\zeta])\zeta]) = 0$. The function $f\colon v \to f(v) = h([\zeta])\zeta(v,t)$ then admits a unique Gibbs equilibrium state which is the unique measure μ in $\mathcal{M}_g$ which satisfies $\int f\, d\mu = h_{\mu,g}$. A *Gibbs current* μ_ζ of the flip invariant Hölder class $[\zeta]$ then is the projection of this Gibbs equilibrium state of f to $\mathcal{G}\tilde{M} = \partial\tilde{M} \times \partial\tilde{M} - \Delta$.

The number $h([\zeta])$ does not depend on the choice of g, i.e. it is an invariant of the Hölder cohomology class $[\zeta]$. We call $h([\zeta])$ the *topological* *entropy* of the positive Hölder class $[\zeta]$ (see [Le]). Similarly, the class $[\mu_\zeta]$ of the Gibbs measure μ_ζ in the projectivization $\mathcal{PM}$ of $\mathcal{M}$ is only determined by $[\zeta]$.

As shown in [Bo2], for a surface M there is a natural bilinear form, the so called *intersection form*, on the locally compact convex space $\mathcal{C} \subset \mathcal{M}$ of geodesic currents for M. This bilinear form is continuous with respect to the weak*-topology on $\mathcal{C}$. The intersection of the Lebesgue–Liouville current of a negatively curved metric g with any current β is just twice the integral of the length cocycle of g with respect to β.

One consequence of Theorem B is the following: For every closed manifold M, the map which assigns to the class of the length cocycle of a strictly 1/4-pinched negatively curved metric on M its Lebesgue–Liouville current is injective. Thus we can define the intersection between a current ν and the Lebesgue–Liouville current λ_ℓ of the length cocycle ℓ of such a metric as $2\int \ell\, d\nu$.

We conjecture that the class of the length cocycle of every negatively curved metric is determined by its Lebesgue–Liouville current, and that it is possible to extend the above function to a continuous non-symmetric bilinear form defined on the vector space spanned by all Gibbs currents.

2 Gibbs Currents for Surfaces

To begin with, let M be an arbitrary closed Riemannian manifold of negative sectional curvature. Recall that the space of cohomology classes of Hölder cocycles for the geodesic flow on T^1M is independent of the choice of a metric of negative curvature on M (compare [Le] and the introduction).

Denote by $\mathcal{H}$ the space of positive flip invariant Hölder classes (compare the introduction for the notion of positivity and flip invariance for the cohomology class of a Hölder cocycle on T^1M). We would like to discuss briefly the geometric significance of the space $\mathcal{H}$. For this let $f\colon T^1M \to (0,\infty)$ be a positive Hölder function on T^1M. The function f defines the cocycle $\zeta_f\colon (v,t) \in T^1M \times \mathbf{R} \to \int_0^t f(\Phi^s v)ds$. Since f is positive, for every fixed $v \in T^1M$ the map $t \to \zeta_f(v,t)$ is a homeomorphism of $\mathbf{R}$ and hence the inverse of this homeomorphism induces a new cocycle β by $\beta(v,\zeta_f(v,t)) = t$. Then $\Psi^t(v) = \Phi^{\beta(v,t)}v$ defines a new flow on T^1M which is a Hölder continuous time change of Φ^t (see [Pa]).

Now let Ψ^t be any (smooth) Anosov flow on T^1M. The flows Ψ^t and Φ^t are called *orbit equivalent* if there is a Hölder continuous homeomorphism

$h: T^1M \to T^1M$ which maps every orbit of Φ^t order preserving onto an orbit of Ψ^t. This means in particular that for every $v \in T^1M$ and every $t \in \mathbf{R}$ there is a unique number $\zeta(v,t) \in \mathbf{R}$ such that $h(\Phi^t v) = \Psi^{\zeta(v,t)}h(v)$. Since the orbit equivalence is order preserving the map $t \to \zeta(v,t)$ is a homeomorphism of $\mathbf{R}$ for every $v \in T^1M$, and $\zeta(v,s+t) = \zeta(v,s) + \zeta(\Phi^s v,t)$ which is just the cocycle equality for ζ. In other words, via the homeomorphism h the flow Ψ^t is just a time change of the flow Φ^t, and the space $\mathcal{H}$ just corresponds to all Hölder continuous flows on T^1M which are order preserving orbit equivalent to Φ^t via a homeomorphism which is compatible with the flip $\mathcal{F}$ and an involution $\mathcal{F}'$ for Ψ^t which satisfies $\Psi^t \circ \mathcal{F}' = \mathcal{F}' \circ \Psi^{-t}$.

More specifically we can consider the following examples:

1. The flow Φ^t is an Anosov flow and hence structurally stable. Thus if Ψ^t is any flow on T^1M which is sufficiently close in the C^k-topology to Φ^t with respect to a suitable $k \geq 2$, then Ψ^t is order preserving orbit equivalent to Φ^t and thus defines a positive Hölder class.

2. A *reversible Finsler metric* on M is a smooth positive function F on the complement of the zero section in the tangent bundle TM of M which satisfies $F(\lambda X) = |\lambda|F(X)$ for all $\lambda \in \mathbf{R} - \{0\}$ and all $X \in TM$ and with strictly convex level sets in the fibres (in the sense that the Hessian of the restriction of F to every fibre is positive definite on the tangent spaces of the level sets). The function F induces a smooth norm on the symplectic cotangent bundle T^*M of M. The Hamiltonian flow of this function is the co-geodesic flow of the Finsler metric F. Foulon showed in [Fo1] that the corresponding geodesic flow is an Anosov flow if F has negative flag curvature. If F is a Riemannian metric of negative curvature, then F is Finsler of negative flag curvature, so we obtain here a more general class of Anosov flows than just geodesic flows of Riemannian structures. Moreover Egloff derived in [E] that the geodesic flows of any two Finsler metrics on M of negative flag curvature are order preserving orbit equivalent via a Hölder homeomorphism. In other words, every such Finsler metric defines an element of $\mathcal{H}$.

The space $\mathcal{H}$ carries a natural (non-complete) topology as follows:

Recall that a *geodesic current* for M is a $\Gamma = \pi_1(M)$-invariant locally finite Borel measure on $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ (where Δ is the diagonal in $\partial\tilde{M} \times \partial\tilde{M}$) which in addition is invariant under the natural involution of $\partial\tilde{M} \times \partial\tilde{M}$ exchanging the two factors. Geodesic currents correspond naturally to finite Φ^t-invariant Borel measures on T^1M which are invariant under the flip. We equip the space $\mathcal{C}$ of geodesic currents with the weak*-topology. With this topology, $\mathcal{C}$ is a locally compact space.

For every $\alpha \in \mathcal{H}$ and every current $\eta \in \mathcal{C}$ the integral $\int \alpha\, d\eta$ is well defined. We equip $\mathcal{H}$ with the coarsest topology such that the function $(\alpha, \eta) \in \mathcal{H} \times \mathcal{C} \to \int \alpha\, d\eta$ is continuous.

To explain this topology in a more geometric way consider the Banach space $C^0_{\mathcal{F}}(T^1M)$ of continuous flip invariant functions on T^1M with the maximums-norm. It contains the vector space $C^H(T^1M)$ of flip invariant Hölder continuous functions as a dense subspace, and the set $C^H_+(T^1M)$ of flip invariant Hölder continuous positive functions is an open cone in $C^H(T^1M)$.

Every function $f \in C^H_+(T^1M)$ defines a Hölder continuous cocycle ζ_f by $\zeta_f(v,t) = \int_0^t f(\Phi^s v)ds$. Now if $\{f_j\} \subset C^H_+(T^1M)$ is a sequence converging to some $f \in C^H_+(T^1M)$ and if $\{\eta_j\} \subset \mathcal{C}$ is a sequence of currents converging weakly to some $\eta \in \mathcal{C}$, then $\int f_j\, d\eta_j = \int \zeta_{f_j} d\eta_j \to \int f\, d\eta = \int \zeta_f\, d\eta$ and therefore the above topology on $\mathcal{H}$ is the finest topology on $\mathcal{H}$ for which the natural map $f \in C^H_+(T^1M) \to [\zeta_f] \in \mathcal{H}$ is continuous.

The above discussion shows that positive Hölder cocycles are in 1-1-correspondence to Hölder continuous time changes of the geodesic flow. If Ψ^t is such a time change of Φ^t given by a cocycle ζ, then the topological entropy of Ψ^t is defined; it only depends on the cohomology class of ζ. Since Ψ^t is a Hölder continuous Anosov flow it admits a unique measure μ of maximal entropy. The entropy of μ is the topological entropy of Ψ^t, and this entropy is one if and only if the pressure of the time-1-parameter of the flow Ψ^t vanishes.

Vanishing pressure can be defined without referring to the time change of the flow explicitly. As in the introduction we obtain the topological entropy of the modified flow only in terms of $[\zeta] \in \mathcal{H}$. Namely this topological entropy is just the unique constant $h(\zeta) > 0$ such that $h(\zeta)\zeta$ is normalized. In the introduction we called $h(\zeta)$ the *topological entropy* of ζ.

Identify the space of flip invariant normalized Hölder classes with the projectivization $\mathcal{PH}$ of $\mathcal{H}$.

Every element from $\mathcal{PH}$ determines its Gibbs current whose class in the space $\mathcal{PC} = \mathcal{C}/\mathbf{R}_+$ of *projective currents* where $\mathbf{R}_+$ acts on $\mathcal{C}$ via multiplication with a positive constant does not depend on any choices made. Thus we obtain a natural map of the space $\mathcal{PH}$ of projective flip invariant positive Hölder classes into the space $\mathcal{PC}$ of projective currents. We call the image of $\mathcal{PH}$ under this map the *space of projective Gibbs currents*. Also we call a current a *Gibbs current* if it corresponds to the Gibbs equilibrium state of a normalized flip invariant Hölder class on T^1M. Denote by $[\mu_\alpha]$ the class in $\mathcal{PC}$ defined by the Gibbs state of $h(\alpha)\alpha$ where $\alpha \in \mathcal{H}$ and as before, $h(\alpha)$ is the topological entropy of α.

Clearly $h(a\alpha) = a^{-1}h(\alpha)$ for all $a > 0$, moreover if α is the length cocycle of a negatively curved Riemannian metric g on M then $h(\alpha)$ is just the topological entropy of the geodesic flow of g. First of all we have

LEMMA 2.1. *$h(\alpha+\beta) \le h(\alpha)h(\beta)/(h(\alpha)+h(\beta))$ with equality if and only if α is a constant multiple of β.*

Proof. Fix a negatively curved metric g on M with geodesic flow Φ^t. Let α, β be normalized Hölder classes. Let μ be the unique Φ^t-invariant Borel probability measure on the unit tangent bundle for (M,g) which is a Gibbs equilibrium state for the normalization of $\alpha+\beta$ and denote by h_μ the entropy of μ. Then we have $\int \alpha\, d\mu \ge h_\mu/h(\alpha)$, $\int \beta\, d\mu \ge h_\mu/h(\beta)$ with equality if and only if $\alpha = \beta$. Then $\int(\alpha+\beta)d\mu \ge h_\mu(1/h(\alpha)+1/h(\beta))$ and hence the lemma follows from the fact that $h_\mu = h(\alpha+\beta)\int(\alpha+\beta)d\mu$. □

We equip $\mathcal{PC}$ with the topology induced from the weak*-topology on $\mathcal{C}$. As in the introduction we write $\mathcal{M}_g$ to denote the compact convex subspace of $\mathcal{M}$ which contains the Borel-probability measures on the unit tangent bundle of M of the metric g. In other words, the symbol $\mathcal{M}_g$ means that we fix a negatively curved metric g on M and we consider all measures in $\mathcal{M}$ which are normalized with respect to g. Then $\mathcal{M}_g$ can naturally be identified with the projectivization of $\mathcal{M}$.

LEMMA 2.2. i) *The topological entropy $h\colon \mathcal{H} \to (0,\infty)$ is continuous.*

ii) *The map $[\alpha] \in \mathcal{PH} \to [\mu_\alpha] \in \mathcal{PC}$ is continuous.*

Proof. Let $\{\alpha_i\} \subset \mathcal{H}$ be a sequence converging to some $\alpha \in \mathcal{H}$. For $i \ge 1$ let $\eta_i \in \mathcal{M}_g$ be the unique Gibbs equilibrium state for $h(\alpha_i)\alpha_i$. Since $\mathcal{M}_g$ is compact we may assume by passing to a subsequence that the measures η_i converge weakly to some $\eta \in \mathcal{M}_g$. If h_ν denotes again the entropy of $\nu \in \mathcal{M}_g$ then $\nu \to h_\nu$ is upper semi-continuous and hence $h_\eta \ge \lim_{i\to\infty} \sup h_{\eta_i}$.

By the definition of the topology on $\mathcal{H}$ we have

$$\int \alpha_i d\eta_i \to \int \alpha\, d\eta > 0 \quad (i \to \infty)\,.$$

Since $h_{\eta_i} = h(\alpha_i)\int \alpha_i d\eta_i$ this implies in particular that the sequence $\{h(\alpha_i)\}$ is bounded.

By passing to a subsequence we may assume that $h(\alpha_i) \to \bar{h} \geq 0$ $(i \to \infty)$. Then $h_\eta - \bar{h} \int \alpha\, d\eta \geq \limsup_{i\to\infty}[h_{\eta_i} - h(\alpha_i) \int \alpha_i d\eta_i] = 0$ and consequently $\bar{h} \leq h(\alpha)$. On the other hand, if $\bar{h} < h(\alpha)$ then there is $\nu \in \mathcal{M}_g$ with $h_\nu - \bar{h} \int \alpha\, d\nu > 0$. Then also $h_\nu - h(\alpha_i) \int \alpha_i\, d\nu > 0$ for i sufficiently large which is impossible. In other words, the function h is indeed continuous. Moreover, if $\eta \in \mathcal{M}_g$ is as above, then $h_\eta - \bar{h} \int \alpha\, d\eta = 0$ and hence the current determined by η is contained in the class of $[\mu_\alpha]$. From this the continuity of the map $[\alpha] \to [\mu_\alpha]$ is immediate. □

REMARK. Lemma 2.2 is just a version of the well known fact that the pressure of a continuous function $f \in C^0(T^1M)$ depends continuously on f.

In the sequel we will need several times a construction from [H5] which is as follows: We start with a Hölder continuous cocycle α. This cocycle defines in a unique way a Hölder continuous kernel $k_\alpha\colon \tilde{M} \times \tilde{M} \times \partial\tilde{M} \to \mathbf{R}$ which is determined by $k_\alpha(\gamma(0), \gamma(t), \gamma(\infty)) = \alpha(\gamma'(0), t)$ for every geodesic γ in $\tilde{M}$ (here we also use the symbol α for the lift of our cocycle to $T^1\tilde{M}$).

Assume now that $\alpha(v,t) = \alpha(\mathcal{F}\Phi^t v, t)$ for all $v \in T^1M$, all $t \in \mathbf{R}$. For a fixed point $x \in \tilde{M}$ and for $v \neq w \in T^1_x\tilde{M}$, we then can define $(v,w)_\alpha = -\frac{1}{2}(k_\alpha(y,x,\pi(v)) + k_\alpha(y,x,\pi(w)))$ where again $\pi\colon T^1M \to \partial\tilde{M}$ is the natural projection and y is any point on the geodesic in $\tilde{M}$ joining $\pi(w)$ to $\pi(v)$. Since α is invariant under the flip the number $(v,w)_\alpha$ does not depend on the choice of y, moreover $(v,w)_\alpha = (w,v)_\alpha$ for all $v \neq w \in T^1_x\tilde{M}$.

If α is the cocycle defined by the constant function 1 then $(v,w)_\alpha$ is just the *Gromov product* $(\pi(v) \mid \pi(w))_x$ of $\pi(v)$ and $\pi(w)$ with respect to the base-point x (see [GhHa] for the definition of the Gromov product). Similarly we can define a product $(\ ,\ )_\alpha$ for any cocycle α which is cohomologous to the cocycle defined by a flip invariant function on T^1M (see [H5]).

LEMMA 2.3. *The map $[\alpha] \in \mathcal{PH} \to [\mu_\alpha] \in \mathcal{PC}$ is injective.*

Proof. Let $\alpha \in \mathcal{H}$ be a normalized positive Hölder cocycle and let $\mu \in [\mu_\alpha]$. Since the cohomology class of α is determined by the periods of α we have to show that we can read off the periods of α from the measure μ.

For this let $\Psi \in \Gamma = \pi_1(M)$ and let a be the attracting fixed point for the action of Ψ on $\partial\tilde{M}$, b be the repelling fix point. Let γ be a geodesic in $\tilde{M}$ joining $\gamma(-\infty) = b$ to $\gamma(\infty) = a$. Recall that there is a natural shortest-distance projection of $\tilde{M} \cup \partial\tilde{M}$ onto $\gamma[-\infty,\infty]$. Denote by B the compact set of all points in $\partial\tilde{M}$ whose projection to γ is contained in $\gamma[-\infty,0]$. Then B is a compact neighborhood of b not containing a. The current μ induces a conditional measure μ_c on the open neighborhood $U = \partial\tilde{M} - B$

of a by defining $\mu_c(A) = \mu(A \times B)$. The map Ψ maps the closure of U into the interior of U and satisfies $\cap_{k \geq 0} \Psi^k(U) = \{a\}$.

Fix a point $x \in \tilde{M}$ and for $\xi \neq \zeta \in \partial\tilde{M}$ write $(\xi, \zeta) = (v, w)_\alpha$ where $v, w \in T_x^1\tilde{M}$ are such that $\pi v = \xi$, $\pi w = \zeta$ and where $(\, , \,)_\alpha$ is as above the symmetric product on pairs of distinct points in $T_x^1\tilde{M}$ defined by the cocycle α. It was shown in [H5] that for sufficiently large k and every $\xi \in \partial\tilde{M}$ contained in the boundary of U, the quantity $\mu_c(\Psi^k U)e^{(a, \Psi^k \xi)}$ is bounded from above and below by fixed positive constants. But $\lim_{k\to\infty} \frac{1}{k}(a, \Psi^k \xi)$ equals the period $\alpha(\Psi)$ of α at Ψ [H5], and therefore $\alpha(\Psi) = -\lim_{k\to\infty} \frac{1}{k} \log \mu(\Psi^k U \times B)$. In other words, the periods of α can be read off from the measure μ. □

Recall from [H5] that a *cross-ratio* for $\Gamma = \pi_1(M)$ is a Hölder continuous positive function Cr on the space of quadruples of pairwise distinct points in $\partial\tilde{M}$ with the following properties:

1) Cr is invariant under the action of Γ on $(\partial\tilde{M})^4$.
2) $Cr(a, a', b, b') = Cr(a', a, b, b')^{-1}$
3) $Cr(a, a', b, b') = Cr(b, b', a, a')$
4) $Cr(a, a', b, b')Cr(a', a'', b, b') = Cr(a, a'', b, b')$
5) $Cr(a, a', b, b')Cr(a', b, a, b')Cr(b, a, a', b') = 1$.

Property 5 above is a consequence of properties 1–4 and the fact that Cr admits a Hölder continuous extension to the space of quadruples (a, a', b, b') of points in $\partial\tilde{M}$ which satisfy $\{a, a'\} \cap \{b, b'\} = \emptyset$. This extension equals 1 for every quadruple (a, a', b, b') for which either $a = a'$ or $b = b'$ (this was communicated to me by F. Ledrappier).

We showed in Theorem A of [H5] that there is natural bijection between the space of cohomology classes of flip invariant Hölder cocyles and the space of cross ratios on $\partial\tilde{M} \times \partial\tilde{M} - \Delta$.

This bijection associates to a flip invariant Hölder class a cross-ratio as follows: Let α be a cocycle representing the class, let $x \in \tilde{M}$ be any point and let $(\, , \,)_\alpha$ be the symmetric product on distinct points in $T_x^1\tilde{M}$.

For $\xi \neq \zeta \in \partial\tilde{M}$ write $(\xi, \zeta) = (v, w)_\alpha$ where $v, w \in T_x^1\tilde{M}$ are such that $\pi(v) = \xi, \pi(w) = \zeta$. Then $(\, , \,)$ is a symmetric product on pairs of distinct points in $\partial\tilde{M}$ depending on the choice of x and α. For a quadruple (a, a', b, b') of distinct points in $\partial\tilde{M}$ define now $[a, a', b, b']_\alpha = (a, b) + (a', b') - (a, b') - (a', b)$; then $[\,]_\alpha$ does not depend on x nor on the specific Hölder cocycle representing the cohomology class of α, and $Cr = e^{[\,]_\alpha}$ is the cross ratio associated to the class of α.

We call a cross ratio Cr *positive* if it corresponds under this identification to a positive Hölder class.

LEMMA 2.4. *Let Cr be a cross ratio on $\partial\tilde{M}$. Then $Cr(a,b,c,d)+Cr(b,c,d,a) \to 1$ as $a \to b$, locally uniformly in (c,d) if and only if Cr is positive.*

Proof. Let α be a Hölder cocycle induced by the cross ratio Cr. Then α defines a Hölder continuous symmetric function $(\ ,\ )$ on $\partial\tilde{M}\times\partial\tilde{M}-\Delta$ such that $\log Cr(a,b,c,d) = -(a,d)-(b,c)+(a,c)+(b,d)$ for pairwise distinct points a,b,c,d in $\partial\tilde{M}$. If Cr is positive, then $(a,b)\to\infty$ as $a\to b$ in $\partial\tilde{M}$. Now

$$Cr(a,b,c,d)+Cr(b,c,d,a) = e^{(a,c)+(b,d)}[e^{-(a,d)-(b,c)}+e^{-(b,a)-(c,d)}]$$

and this converges to 1 as $a\to b$ if and only if $e^{-(b,a)-(c,d)}\to 0$ as $a\to b$. The formula also shows that this convergence is locally uniform in (c,d) if Cr is positive. □

From now on we specialize to the case that M is an oriented surface. The ideal boundary of its universal covering is naturally homeomorphic to S^1. Fix once and for all an orientation for the circle S^1. This orientation then determines for every ordered pair (a,b) of points $a\neq b$ in S^1 a unique half-open interval $[a,b[\,\subset S^1$ with endpoints a and b. Call a quadruple (a_1,a_2,a_3,a_4) of pairwise distinct points in S^1 *ordered* if a_i is contained in $[a_{i-1},a_{i+1}[$ for $i=1,\dots,4$.

PROPOSITION 2.5. *For every Gibbs current μ there is a unique cross ratio Cr for $\Gamma=\pi_1(M)$ such that $-\log Cr(a,b,c,d)=\mu[a,b[\times[c,d[$ for every ordered quadruple (a,b,c,d) in S^1. Moreover Cr is positive.*

Proof. Let (a,b,c,d) be an ordered quadruple of pairwise distinct points in S^1 and define $[a,b,c,d]=\mu[a,b[\times[c,d[$. Since μ is a current and hence invariant under exchange of the intervals $[a,b[$ and $[c,d[$, we have $[c,d,a,b]=\mu[c,d[\times[a,b[=[a,b,c,d]$, i.e. $Cr=e^{[\,]}$ satisfies property 3 above. Moreover, if $y\in\,]a,b[$ then (a,y,c,d) and (y,b,c,d) are ordered and

$$\mu[a,b[\times[c,d[=\mu[a,y[\times[c,d[+\mu[y,b[\times[c,d[\qquad (*)$$

which corresponds to property 4 above for Cr on ordered quadruples.

So far we have not used that μ is a Gibbs current; this now is essential to show that $[\ ]$ is Hölder continuous. Fix a base point $x\in\tilde{M}$ and view $S^1=\partial\tilde{M}$ as the unit sphere in $T_x\tilde{M}$. By formula $(*)$ it is enough to show the following: For $(c,d)\in S^1\times S^1-\Delta$ and $b\in\,]d,c[$ there are constants $\beta>0$, $\alpha>0$ such that $\mu[a,b[\times[c,d[\,\leq\beta|a-b|^\alpha$ whenever a is sufficiently close to b.

For this choose a positive Hölder function f on T^1M of vanishing pressure such that μ is the Gibbs current for f.

As above, from the choice of the base-point x and from the Hölder function f we obtain a symmetric function $(\,,\,)_f$ on $S^1 \times S^1 - \Delta$. Let $(\ |\ )$ be the Gromov product on $S^1 \times S^1 - \Delta$ with respect to the same base point x. Then $(\ |\ )$ defines the Hölder structure on $S^1 = \partial\tilde{M}$. On the other hand we showed in [H5] that there are numbers $\xi \in (0,1)$ and $\beta > 0$ such that $\beta^{-1}(a,b)_f^{1/\chi} \leq (a \mid b) \leq \beta(a,b)_f^{\chi}$ for all $a \neq b \in \partial\tilde{M}$. Moreover for a sufficiently close to b, $\mu[a,b[\times[c,d[$ is bounded from above by a constant multiple of $e^{-(a,b)_f}$. From this Hölder continuity of $[\ ]$ on ordered quadruples of points in $\partial\tilde{M}$ is immediate.

Next, if (a,b,c,d) is ordered, then we define $[b,a,c,d] = -[a,b,c,d]$; this then implies also that $[a,b,c,d] = -[a,b,d,c]$. Finally if we put $[b,a,d,c] = [a,b,c,d]$ whenever (a,b,c,d) is ordered then we obtain an extension of $[\ ]$ to all quadruples of pairwise distinct points in $\partial\tilde{M}$ which is independent of the choice of an orientation for $\partial\tilde{M}$. Moreover $Cr = e^{-[\,]}$ clearly satisfies all defining properties of a cross ratio.

We are left with showing that Cr is positive. For this we use Lemma 2.4: Namely let (a,b,c,d) be an ordered quadruple of pairwise distinct points of S^1; then $Cr(a,b,c,d) + Cr(b,c,d,a) = e^{-\mu[a,b[\times[c,d[} + e^{-\mu[b,c[\times[d,a[}$. Recall that μ is a Gibbs current and hence has no atoms (compare [H5]); this means that for $c \neq d \in S^1$ fixed we have $\mu[a,b[\,\times[c,d[\,\to 0$ as $b \to a$ and also $\mu[b,c[\,\times[d,a[\,\to \infty$ as $b \to a$ (this is true for every current by invariance under the action of Γ). From Lemma 2.4 we conclude that Cr is indeed positive. □

We call the cross ratio Cr defined as above by a Gibbs current μ the *intersection cross ratio* of μ and we write $[\ ]_\mu = -\log Cr$.

We can also ask for the reverse of the above procedure. Namely, let Cr be a cross ratio and write $[\ ] = -\log Cr$. For an ordered quadruple (a,b,c,d) of pairwise distinct points in S^1 define $\eta[a,b[\,\times[c,d[\, = [a,b,c,d]$. By our assumption, this defines a function on those subsets of $S^1 \times S^1 - \Delta$ which are products of non-empty right-half open intervals in S^1. Since $[a,y,c,d] + [y,b,c,d] = [a,b,c,d]$ for all $y \in\,]a,b[$, this function has a natural extension to a finitely additive function on the family of finite unions of products of right-half open intervals in S^1. If this function is in addition positive, then it defines a locally finite Borel measure on $S^1 \times S^1 - \Delta$ which we denote again by η. Since $[\ ]$ is invariant under the action of Γ, the measure η is in fact a geodesic current. We call η the *intersection current* of Cr. In general however the finitely additive function on finite unions of products of right-half open intervals in S^1 is not σ-additive, which means that it is not a *signed geodesic current* (the formal difference of two geodesic currents). However we always call η the intersection current of Cr.

The results of [H5] show that every Hölder class α on T^1M defines a unique intersection current ν_α. The assignment $\alpha \to \nu_\alpha$ is linear, i.e. if α, β are Hölder classes and if $a, b \in \mathbf{R}$ then $\nu_{a\alpha+b\beta} = a\nu_\alpha + b\nu_\beta$.

Clearly if the intersection current ν_α is positive, i.e. if ν_α is in fact a current, then α is a positive Hölder class. The reverse however is not true. To see this let α_1, α_2 be the Hölder classes of the length cocyles of hyperbolic metrics $g_1 \neq g_2$ on M. Since the metrics g_1, g_2 are bilipschitz equivalent there is a number $\epsilon > 0$ such that $\alpha_1 - \epsilon\alpha_2$ is a positive Hölder class.

By Corollary 2.7 the intersection current λ_i of α_i is $1/2$ times the Lebesgue–Liouville current of g_i $(i = 1, 2)$. Since the currents λ_1 and λ_2 are singular, the intersection current $\lambda_1 - \epsilon\lambda_2$ of the positive Hölder class $\alpha_1 - \epsilon\alpha_2$ is not positive.

Even if the positive Hölder class α is such that its intersection current ν_α is positive and ergodic under the action of $\Gamma = \pi_1(M)$ we do not know whether ν_α is in fact a Gibbs current.

For every geodesic current η and every Hölder class α the integral $\int \alpha\, d\eta$ is well defined. Recall that the *intersection form* i is a continuous bilinear form on the space of geodesic currents where the space of currents is equipped with the weak*-topology (see [Bo2]). Every free homotopy class $[\gamma]$ in M defines a unique geodesic current which we denote again by $[\gamma]$ as follows: Represent $[\gamma]$ by a closed geodesic in M. The lifts of this geodesic to $\tilde{M}$ define a $\Gamma = \pi_1(M)$-invariant subset of the space $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ of geodesics whose intersection with every compact subset of $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ is finite. Then $[\gamma]$ is the sum of all Dirac masses on all points of $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ corresponding to those lifts. Recall also that a free homotopy class in M is nothing else but a conjugacy class in $\pi_1(M)$.

Then we have

LEMMA 2.6. *For every positive Hölder class α for which ν_α is positive and every current β we have $\int \alpha\, d\beta = i(\beta, \nu_\alpha)$. In particular the Lebesgue–Liouville current of a negatively curved metric g on M is twice the intersection current of the length cocycle of g.*

Proof. Let γ be a current defined by a conjugacy class in $\pi_1(M)$. Let $\Psi \in \pi_1(M)$ be contained in this conjugacy class and let a, b be the fixed points of the action of Ψ on S^1. In [O] it is shown that $i(\gamma,\beta)=\beta]a,b[\times[\xi,\Psi\xi[$ for every $\xi\in S^1-\{a,b\}$ and every current β and hence by the definition of the map $\alpha\to\nu_\alpha$ (see [H5]) and the discussion after Lemma 2.2 we have $i(\gamma,\nu_\alpha)=\int\alpha\,d\gamma$. Since finite sums of weighted Dirac currents of conjugacy classes in $\pi_1(M)$ are dense in the space of all currents equipped with the weak*-topology and since the intersection form is continuous with respect to the weak*-topology ([Bo2]) the first part of the lemma follows.

The computations in [Bo1] show that the intersection of the Lebesgue–Liouville current λ_g of g with a free homotopy class γ in M is just twice the length of the closed geodesic with respect to g which represents γ. In other words, the value at γ of the Hölder cohomology class corresponding to the intersection cross ratio of λ_g equals twice the value at γ of the length cocycle of the metric g. Since a Hölder class is determined by its values on free homotopy classes, the second part of the lemma follows. □

Every Riemannian metric g on M of negative curvature defines its Lebesgue–Liouville current λ_g.

LEMMA 2.8. *The set of Lebesgue–Liouville currents of negatively curved metrics is dense in the set of currents.*

Proof. Since every current can be approximated by a sequence of Dirac currents for conjugacy classes in $\pi_1(M)$, it suffices to show that the closure of the set of all Lebesgue–Liouville currents of metrics of negative curvature in the space $\mathcal{C}$ of geodesic currents equipped with the weak*-topology contains all currents defined by conjugacy classes in $\pi_1(M)$.

For this recall first of all that a free homotopy class in M which can be represented by a simple closed curve defines the projective class of a measured geodesic lamination. On the other hand, projective measured laminations form the boundary in $\mathcal{PC}$ of the projectivizations of the Lebesgue–Liouville currents of hyperbolic metrics on M ([Bo1]). Thus we may restrict our attention to free homotopy classes which can not be represented by simple closed curves.

Fix a hyperbolic metric g on M and let γ be a closed geodesic in (M,g) with self intersections. By eventually changing the metric g we may assume that γ has only double points.

For simplicity we consider only the case that γ has exactly one double point. The construction given below is also valid in the general case. Let $\gamma\colon[0,T]\to M$ be a parametrization by arc length such that $\gamma(0)=\gamma(\tau)=\gamma(T)$ for some $\tau\in(0,T)$ (in other words, $\gamma(0)$ is the double point).

For $\epsilon > 0$ let $Q(\epsilon)$ be the ϵ-neighborhood of γ in M. For sufficiently small ϵ, $Q(\epsilon) - \gamma[0,\tau]$ has exactly two components, one of which, say Q_1, is homeomorphic to an annulus. Choose a normal vector field $t \to N(t)$ along γ which points on $(0,\tau)$ inside Q_1. Let $m > 0$ be a large number; then there are $\epsilon_1(m), \epsilon_2(m) \in (0,\epsilon)$ such that $\exp \epsilon_1(m)N(1/m) = \exp \epsilon_1(m)N(\tau - 1/m) \in Q_1$ (where exp is the exponential map of (M,g)) and $\exp(-\epsilon_2(m)N(1/m)) = \exp \epsilon_2(m)N(\tau + 1/m)$. Observe that the absolute values of $\epsilon_1(m)$ and $\epsilon_2(m)$ only depend on m and the angle between $\gamma'(0)$ and $-\gamma'(\tau)$ at $\gamma(0)$.

Thus for each sufficiently large $m \geq 1$ we obtain a closed neighborhood $A(m)$ of γ in M with the following properties:

i) $A(m) \supset A(m+1)$ and $\cap_{m\geq 1} A(m) = \gamma$.

ii) $A(m) = \cup_{i=0}^4 A_i(m)$ where the sets $A_i(m)$ are closed with piecewise smooth boundary and pairwise disjoint interior.

iii) $A_0(m)$ is a geodesic quadrangle in M containing $\gamma(0)$ in its interior, with vertices

$$\exp \epsilon_1(m)N(1/m) = x_1(m)\ , \quad \exp\big(-\epsilon_2(m)N(1/m)\big) = x_2(m)\,,$$
$$\exp\big(-\epsilon_1(m)N(T-1/m)\big) = x_3(m) \ \text{ and } \ \exp \epsilon_2(m)N(T-1/m) = x_4(m)\,.$$

iv) The boundary of $A_i(m)$ $(i = 1,\dots,4)$ contains two smooth geodesic segments of length $\epsilon_1(m)$ or $\epsilon_2(m)$ which meet at $x_i(m)$ and are subarcs of the boundary of $A_0(m)$. It also contains a subarc of γ. With respect to normal exponential coordinates based at γ the metric on $A_i(m)$ can be written in the form $(\cosh s)^2 dt^2 + ds^2 (s \in [0,\epsilon_1(m)]$ or $s \in [0,\epsilon_2(m)])$.

Now change the metric in the interior of $A(m)$ as follows: Fix $m \geq 1$ sufficiently large and for $i = 1,2$ choose a diffeomorphism $\Psi_{i,m} : [0,m] \to [0,\epsilon_i(m)]$ with the following properties:

1) $\Psi'_{i,m}(s) \in (0,1]$ for all $s \in [0,m]$.

2) $\Psi'_{i,m}(s) = 1$ for s near 0 and s near m.

3) $\frac{-\Psi''_{i,m}(s)}{(\Psi'_{i,m}(s))^2} < \frac{\cosh(\Psi_{i,m}(s))}{\sinh(\Psi_{i,m}(s))}$ for $s \in [0,m]$.

Property 3 can be fulfilled since $\frac{\cosh(s)}{\sinh(s)} \cdot s \to 1$ as $s \searrow 0$. Namely $\lim_{t\searrow 0} \int_t^1 \frac{\cosh(s)}{\sinh(s)} ds = \infty$ and consequently we can find a smooth function $f : [0,\infty) \to [0,\infty)$ which vanishes near 0 and on $[1,\infty)$ and such that $f \leq \epsilon_i(m)^2 \cosh/8m^2 \sinh$ and $\int_0^1 f(s)ds = 1 - \epsilon_i(m)/2m$.

Write $F(s) = 1 - \int_0^s f(\sigma)d\sigma$; then F is a smooth positive function on $[0,\infty)$ with values in $[\epsilon_i(m)/2m, 1]$ and such that $F(s) = \epsilon_i(m)/2m$ for all

$s \geq 1$. The function $\Psi(s) = \int_0^s F(\sigma)d\sigma$ then satisfies 3) above and moreover we have $\Psi(m) \leq 3\epsilon_i(m)/4$. But this just means that we can modify Ψ in a neighborhood of m in such a way that the resulting function $\Psi_{i,m}$ assumes the value $\epsilon_i(m)$ at m, coincides with Ψ on $[0,1]$ and such that $\Psi_{i,m}'(s) = 1$ for s near m and $\Psi_{i,m}''(s) \geq 0$ on $[1,m]$. In other words, $\Psi_{i,m}$ has the above properties 1–3.

Define now a new metric g_m on M as follows:

a) g_m coincides with the original metric g outside $A(m)$.
b) In normal exponential coordinates based at γ the restriction of g_m to $A_j(m)$ can be written in the form $dg_m = (\cosh s)^2 dt^2 + [(\Psi_{i,m}^{-1})'(s)]^2 ds^2$ ($i = 1$ or $i = 2$ and $j = 1, \dots, 4$).

This defines g_m on $M - A_0(m)$. By construction, γ is a geodesic for every metric g_m, and $(s,t) \in [0,m] \times (\epsilon_i(m), \tau - \epsilon_i(m)) \to \exp \Psi_{i,m}(s)N(t)$ define parallel coordinates for g_m along γ. Thus property 3) above guarantees that g_m has negative curvature. Moreover g_m has a natural extension to the boundary of A_0 which coincides with g near the vertices $x_j(m)$ ($j = 1, \dots, 4$). With respect to this extension A_0 is a geodesic quadrangle of side length $2m$ and such that the sum of the internal angles of the quadrangle is strictly less than 2π. This means that we can extend g_m to a metric of negative curvature on $A_0(m)$ in such a way that as $m \to \infty$ the geodesic quadrangle $(A_0(m), g_m)$ approaches an euclidean parallelogram of side length $2m$ whose angles are determined by the angle between $\gamma'(0)$ and $-\gamma'(\tau)$. Moreover we may assume that γ remains a g_m-geodesic for all m.

Let λ_m' be the Lebesgue–Liouville current of g_m and write $\lambda_m = \frac{1}{4m}\lambda_m'$. Choose a free homotopy class in M which is different from the class of γ and which is prime and represented by the g-geodesic η. The total number of intersection points, counted with multiplicities, is the intersection number between the free homotopy classes represented by η and γ.

By abuse of notation we now use the same symbol for η and its free homotopy class. After a slight perturbation of η we may assume that η does not pass through $\gamma(0)$. Then η intersects γ in finitely many points $\eta(s_1), \dots, \eta(s_q)$ where $q = i(\gamma, \eta)$. For sufficiently large $m > 1$ the intersection of η with $A(m)$ has then precisely q connected components, and the g_m-length of each of these components is contained in the interval $[2m, 2m + c]$ where $c > 0$ is a constant which depends on η but not on m. Thus $i(\lambda_m, \eta) = \frac{1}{4m} i(\lambda_m', \eta) \leq \frac{1}{2m} \cdot g_m -$ length of $\eta \to q (m \to \infty)$.

In other words, we have $\limsup_{m \to \infty} i(\lambda_m, \eta) \leq i(\gamma, \eta)$.

Since the intersection form i on $\mathcal{C} \times \mathcal{C}$ is continuous and since the set of finite weighted sums of Dirac currents of free homotopy classes different from γ is dense in $\mathcal{C}$ we conclude that the sequence $\{\lambda_m\}$ is bounded in $\mathcal{C}$. By passing to a subsequence we may therefore assume that $\{\lambda_m\}$ converges weakly to a current $\lambda \in \mathcal{C}$. Then $i(\lambda, \eta) \leq i(\gamma, \eta)$ for every free homotopy class $\eta \neq \gamma$ and therefore $\lambda \leq \gamma$. On the other hand the current γ is ergodic and hence $\lambda = a\gamma$ for some $a \geq 0$.

We are left with showing that $a \neq 0$. For this choose again a free homotopy class η with $i(\gamma, \eta) = q > 0$. Then every curve in M representing η intersects γ in at least q points, where an intersection at the point $\gamma(0)$ has to be counted twice. But this means that the g_m-length of every such curve is at least $2m$ and hence $\liminf_{m\to\infty} i(\lambda_m, \eta) = i(\lambda, \eta) > 0$. This shows $a \neq 0$ and finishes the proof of the lemma. □

To summarize the considerations at the beginning of this section we have the following situation: To every element $[\alpha]$ in the space $\mathcal{PH}$ of projective flip invariant positive Hölder classes we can associate the projective class $[\mu_\alpha]$ of the Gibbs equilibrium state defined by α and also the projective class $[\nu_\alpha]$ of the intersection current of α. This defines two injective continuous maps $[\alpha] \to [\mu_\alpha]$ and $[\alpha] \to [\nu_\alpha]$ of $\mathcal{PH}$ into the space of projective classes of finitely additive signed measures on $\mathcal{G}\tilde{M}$. The image of $\mathcal{PH}$ under the first map is just the space of projective Gibbs currents.

With this terminology, the result of Katok in [K] can be formulated as follows: If α is the Hölder cohomology class of the length cocycle of a metric on M of negative curvature, then $[\mu_\alpha] = [\nu_\alpha]$ if and only if this metric is of constant negative curvature.

The proof of Katok uses another result proved in the same paper which can be slightly generalized (for a generalization to higher dimensions see [BCG]). For the formulation of our generalization of Katok's result recall the definition of the *topological entropy* $h(\alpha)$ of a positive Hölder class α.

Theorem 2.8. *Let M be a closed surface of Euler characteristic $\chi(M) < 0$ and let α be a positive Hölder class whose intersection current ν_α is positive and hence a current. If $i(\nu_\alpha, \nu_\alpha) = \pi^2|\chi(M)|$, then $h(\alpha) \geq 1/4$ with equality if and only if α is the length cocycle of a hyperbolic metric on M.*

Proof. Let $g, \tilde{g}$ be conformally equivalent Riemannian metrics on M of negative curvature, i.e. there is a function $a\colon M \to (0,\infty)$ such that $ag = \tilde{g}$. Assume that the g-volume of M and the $\tilde{g}$-volume of M equals $(2\pi)^{-1}$. If $\lambda_g, \lambda_{\tilde{g}}$ denotes the Lebesgue–Liouville current of $g, \tilde{g}$ then this just means that $i(\lambda_g, \lambda_g) = 2 = i(\lambda_{\tilde{g}}, \lambda_{\tilde{g}})$ (see [Bo1]).

Let V be the unit tangent bundle of (M,g) and let λ be the Lebesgue–Liouville measure of g on V. For a g-unit vector $v\in V$ denote by $\rho(v)$ the $\tilde g$-norm of v. If $P\colon V\to M$ denotes the canonical projection then $\rho(v)=(a(Pv))^{1/2}$ and consequently

$$\int_V \rho\,d\lambda=\int_V (a^{1/2}\circ P)d\lambda\leq \Big(\int (a\circ P)d\lambda\Big)^{1/2}=1$$

with equality if and only if $a\equiv 1$.

Since g and $\tilde g$ are negatively curved metrics on the same surface M the identity $(M,g)\to(M,\tilde g)$ induces an order preserving orbit equivalence of the geodesic flows for g and $\tilde g$. Thus if $\tilde V$ denotes the unit tangent bundle of $(M,\tilde g)$ and $\tilde\Phi^t$ is the geodesic flow, then there is a Hölder continuous homeomorphism $\Psi\colon V\to\tilde V$ which maps the orbits of Φ^t to the orbits of $\tilde\Phi^t$ and which is homotopic to the natural homeomorphism $V\to\tilde V$ induced by the identity of M. For $v\in V$ and $t\in\mathbf{R}$ there is a unique number $\tilde\alpha(v,t)\in\mathbf{R}$ such that $\Psi(\Phi^t v)=\tilde\Phi^{\tilde\alpha(v,t)}\Psi(v)$. Then $\tilde\alpha$ is a Hölder cocycle for the geodesic flow Φ^t on V.

Now let v be a periodic point for Φ^t of period $\tau>0$. Then v is the initial velocity of a closed geodesic γ in (M,g) of length τ, and $\int_0^\tau \rho(\Phi^s v)ds$ equals the $\tilde g$-length of γ which is not larger than $\tilde\alpha(v,\tau)$, the $\tilde g$-length of the unique closed $\tilde g$-geodesic on M which is freely homotopic to γ. But this shows that for every free homotopy class in M the integral with respect to the Dirac current determined by this class of the Hölder cocycle $(w,t)\in T^1\tilde M\to\int_0^t\rho(\Phi^s w)ds$ for the geodesic flow on V is not smaller than the integral of the cocycle $\tilde\alpha$. Since weighted sums of Dirac currents of free homotopy classes in M are dense in $\mathcal{C}$ we conclude that this inequality holds for every current. Thus if $\alpha(g),\alpha(\tilde g)$ denote the Hölder classes of the length cocycles of the metrics $g,\tilde g$ then this together with the inequality $\int_V\rho\,d\lambda\leq 1$ and Lemma 2.6 shows that

$$\int \alpha(\tilde g)d\lambda_g=i(\lambda_g,\lambda_{\tilde g})/2=i(\lambda_g,\nu_{\alpha(\tilde g)})=i(\lambda_{\tilde g},\nu_{\alpha(g)})\leq 1 \qquad (*)$$

with equality if and only if $g=\tilde g$.

Assume now in addition that $\tilde g$ is a metric of constant curvature and let $h(g),h(\tilde g)$ be the topological entropy of the geodesic flow for $g,\tilde g$. Then the cocycles $\alpha(g)h(g)$ and $\alpha(\tilde g)h(\tilde g)$ are normalized. Now $\lambda_{\tilde g}=2\nu_{\alpha(\tilde g)}$, moreover the Lebesgue Liouville measure of $\tilde g$ equals the Gibbs equilibrium state for $\alpha(\tilde g)h(\tilde g)$. Therefore we obtain from $(*)$ that

$$h(g)\geq h(g)i(\lambda_{\tilde g},\nu_{\alpha(g)})=\int\alpha(g)h(g)d\lambda_{\tilde g}\geq h(\tilde g) \qquad (**)$$

with equality if and only if $g = \tilde{g}$.

Notice that $h(\tilde{g})$ does not depend on the metric $\tilde{g}$ of constant curvature with $i(\lambda_{\tilde{g}}, \lambda_{\tilde{g}}) = 2$ (compare [Bo1]). We write $h > 0$ for this common number. Now let α be a positive Hölder class such that $\nu = \nu_\alpha$ is positive and hence a current and that $i(\nu_\alpha, \nu_\alpha) = 1/2$. By Lemma 2.7, Lebesgue–Liouville currents of negatively curved metrics are dense in the set of currents. This means that there is a sequence $\{g_j\}_j$ of negatively curved metrics on M with Liouville currents λ_{g_j} and such that $\lambda_{g_j} \to 2\nu$. Since the intersection form is continuous we may assume that g_j is normalized in such a way that $i(\lambda_{g_j}, \lambda_{g_j}) = 2$.

Let α_j be the Hölder class determined by the length cocycle of the metric g_j. Since the intersection form i is continuous on the space of geodesic currents we obtain from Lemma 2.6 and the fact that $\nu_{\alpha_j} \to \nu_\alpha$ weakly in $\mathcal{C}$ that $\alpha_j \to \alpha$ in $\mathcal{H}$. Lemma 2.2 then shows that $h(\alpha_j) \to h(\alpha)$. Since by uniformization every Riemannian metric on M is conformally equivalent to a hyperbolic metric, the above consideration shows that $h(\alpha) \geq h(\tilde{g}) = h$ where $\tilde{g}$ is a metric of constant curvature with $i(\lambda_{\tilde{g}}, \lambda_{\tilde{g}}) = 2$.

Assume now that $h(\alpha) = h$. Let $\tilde{g}_j$ be a metric of constant curvature on M which is conformally equivalent to g_j and such that $i(\lambda_{\tilde{g}_j}, \lambda_{\tilde{g}_j}) = 2$. By our assumptions we have $h(g_j) - h(\tilde{g}_j) \to 0$ or equivalently $h(g_j)/h \to 1\,(j \to \infty)$. Dividing inequality $(**)$ above by $h = h(\tilde{g}_j)$ then shows that $i(\lambda_{\tilde{g}_j}, \nu_{\alpha(g_j)}) = \frac{1}{2} i(\lambda_{g_j}, \lambda_{\tilde{g}_j}) \to 1 (j \to \infty)$.

Recall that the space of projective currents $\mathcal{PC}$ is compact; hence by passing to a subsequence we may assume that the projective classes $[\lambda_{\tilde{g}_j}]$ of $\lambda_{\tilde{g}_j}$ converge weakly to a projective current $[\beta]$. Then $\{\tilde{g}_j\}$ is an unbounded sequence in Teichmüller space if and only if β is a geodesic lamination, i.e. it satisfies $i(\beta, \beta) = 0$ [Bo1].

On the other hand, $\lambda_{g_j} \to \nu_\alpha$ and ν_α is the intersection current of a positive Hölder class, in particular by Lemma 2.6 its intersection with every current is positive. If $b_j > 0$ is such that $b_j \lambda_{\tilde{g}_j} \to \beta \neq 0$ then $i(\lambda_{g_j}, b_j \lambda_{\tilde{g}_j}) = b_j i(\lambda_{g_j}, \lambda_{\tilde{g}_j}) \to i(\nu_\alpha, \beta) > 0\,(j \to \infty)$, again by continuity of the intersection form. But $i(\lambda_{g_j}, \lambda_{\tilde{g}_j}) \to 2 (j \to \infty)$ and thus $\{b_j\}$ is bounded from below by a positive number. This implies in turn that $\{\tilde{g}_j\}$ is bounded in Teichmüller space and hence that $[\lambda_{\tilde{g}_j}] \to [\lambda_{\tilde{g}}]$ for some metric $\tilde{g}$ of constant curvature.

If we normalize $\lambda_{\tilde{g}}$ in such a way that $i(\lambda_{\tilde{g}}, \lambda_{\tilde{g}}) = 2$ then $\lambda_{\tilde{g}_j} \to \lambda_{\tilde{g}}$ and also $\lambda_{g_j} \to \lambda_{\tilde{g}}$ and consequently $\nu_\alpha = \frac{1}{2} \lambda_{\tilde{g}}$. This finishes the proof of the theorem. □

REMARK. Theorem 2.8 can be slightly extended as follows: Let μ be any geodesic current such that the function which assigns to every ordered quadruple (a,b,c,d) of pairwise distinct points in $\partial\tilde{M}$ the value $\mu[a,b[\times[c,d[$ is continuous and that moreover the intersection of μ with every nontrivial current is positive.

The orientation of $\partial\tilde{M} \sim S^1$ induces an orientation for $\tilde{M}$. For every unit vector $v \in T^1\tilde{M}$ there is a unique unit vector $Q(v) \in T^1\tilde{M}$ with the same foot-point which is orthogonal to v and such that $\{v, Q(v)\}$ defines the given orientation.

For $v \in T^1\tilde{M}$ and $t \geq 0$ define now $\alpha(v,t) = \mu[\pi(-v),\pi(v)[\times [\pi Q(v), \pi Q(\Phi^t v)[$; then α is the restriction to $T^1\tilde{M} \times [0,\infty)$ of a continuous positive cocycle for the geodesic flow which we denote again by α. We then have $\int \alpha d\eta = i(\mu,\eta)$ for every current $\eta \in \mathcal{C}$, i.e. μ is the intersection current of α.

Write $f(v) = \alpha(v,1)$; then f is a positive continuous function on T^1M. As before, we can define the topological entropy of the cohomology class $[\alpha]$ of α to be the unique number $h(\alpha) > 0$ such that the pressure of $h(\alpha)f$ vanishes. Since f can be approximated in the cone of continuous positive functions on T^1M by Hölder continuous functions we can apply Theorem 2.8 (and its proof) to the approximating cohomology classes $[\alpha_{f_i}]$ and conclude that if $i(\mu,\mu) = \pi^2|\chi(M)|$ then $h(\alpha) \geq 1/4$ with equality if and only if 2μ is the Lebesgue–Liouville current of a hyperbolic metric on M.

Now let α be a positive Hölder class whose intersection current ν_α is a Gibbs current (for example we can choose the class of a length cocycle of a negatively curved metric). Then we can define the *metric entropy* $h_m(\alpha)$ of α as follows: By Lemma 2.3 there is a unique normalized positive Hölder class β whose Gibbs current is contained in the projective class of ν_α, i.e. such that $[\mu_\beta] = [\nu_\alpha]$. We then define $h_m(\alpha) = \int \beta \, d\nu_\alpha / i(\nu_\alpha,\nu_\alpha) = i(\nu_\alpha,\nu_\beta)/i(\nu_\alpha,\nu_\alpha)$.

If α is the length cocycle of a metric g on M of negative curvature, then $h_m(\alpha)$ equals the metric entropy of the geodesic flow of g. Moreover $h_m(a\alpha) = a^{-1}h_m(\alpha)$, and if $[\nu_\alpha] = [\mu_\alpha]$ then $h(\alpha) = h_m(\alpha)$.

In analogy to Theorem 2.8 we can also estimate $h_m(\alpha)$:

Theorem 2.9. *Let α be a positive Hölder class for the surface M such that $i(\nu_\alpha,\nu_\alpha) = \pi^2|\chi(M)|$ and that ν_α is a Gibbs current. Then $h_m(\alpha) \leq 1/4$*

with equality if and only if α is the length cocycle of a hyperbolic metric on M.

Proof. As in the proof of Theorem 2.8, let $g, \tilde{g}$ be conformally equivalent Riemannian metrics with Lebesgue–Liouville currents $\lambda_g, \lambda_{\tilde{g}}$ and length cocycles $\alpha(g), \alpha(\tilde{g})$ and such that

$$i(\lambda_g, \lambda_g) = i(\lambda_{\tilde{g}}, \lambda_{\tilde{g}}) = 2$$

and that $\tilde{g}$ is of constant curvature. Since the cocycle $h(\alpha(\tilde{g}))\alpha(\tilde{g})$ is normalized we have $\int h(\alpha(\tilde{g}))\alpha(\tilde{g})d\lambda_g = h(\alpha(\tilde{g}))i(\lambda_g, \nu_{\alpha(\tilde{g})}) \geq h_m(\alpha(g)) \cdot \frac{1}{2}i(\lambda_g, \lambda_g) = h_m(\alpha(g))$. But $i(\lambda_g, \nu_{\alpha(\tilde{g})}) \leq 1$ with equality only if $g = \tilde{g}$ by the proof of Theorem 2.8. Thus we obtain the statement of the theorem in the case that α is the class of a length cocycle for a negatively curved Riemannian metric. The general case now follows as in the proof of Theorem 2.8.

Namely, let α be such that ν_α is a Gibbs current and $i(\nu_\alpha, \nu_\alpha) = 1/2$. Let α_j be the Hölder class of the length cocycle of a metric g_j on M such that $i(\nu_{\alpha_j}, \nu_{\alpha_j}) = 1/2$ and $\nu_{\alpha_j} \to \nu_\alpha$. Choose a metric $\tilde{g}_j$ on M of constant curvature which is conformally equivalent to g_j and satisfies $i(\lambda_{\tilde{g}_j}, \lambda_{\tilde{g}_j}) = 2$; then $i(\nu_{\alpha_j}, \lambda_{\tilde{g}_j}) \leq 1$ for all j.

Now if $\tilde{g}_j$ is unbounded in Teichmüller space, then there is a sequence $\{b_j\} \subset [0, \infty)$ such that $b_j \to 0$ and that $\lambda_{\tilde{g}_j} b_j$ converges weakly to a measured lamination β. Then $b_j \geq i(\nu_{\alpha_j}, b_j\lambda_{\tilde{g}_j}) = b_j i(\nu_{\alpha_j}, \lambda_{\tilde{g}_j}) \to i(\nu_\alpha, \beta) > 0$ which is impossible. Thus the sequence $\{\lambda_{\tilde{g}_j}\}$ is bounded and by passing to a subsequence we may assume that $\lambda_{\tilde{g}_j} \to \lambda_{\tilde{g}}$ $(j \to \infty)$ where $i(\lambda_{\tilde{g}}, \lambda_{\tilde{g}}) = 2$ and $\tilde{g}$ is a metric of constant curvature. Then $i(\nu_{\alpha_j}, \lambda_{\tilde{g}_j}) \to i(\nu_\alpha, \lambda_{\tilde{g}}) \leq 1$ and hence $h_m(\alpha) \leq h(\alpha(\tilde{g}))$ by the above argument. The discussion of equality is exactly the same as the discussion of the equality case in the proof of Theorem 2.8. □

REMARK. As before, Theorem 2.9 can be extended as follows. Let ν be a geodesic current such that the function which assigns to every ordered quadruple (a, b, c, d) of pairwise distinct points in $\partial\tilde{M}$ the value $\nu[a, b[\times[c, d[$ is continuous and that moreoever the intersection of ν with every nontrivial current is positive. Use ν to define a continuous cocycle α for Φ^t as in the remark following Theorem 2.8. Write $f(v) = \alpha(v, 1)$ for all $v \in T^1M$. If μ is a Gibbs current for the normalization $h(\alpha)f$ of f and such that $i(\mu, \mu) = \pi^2|\chi(M)|$ then $i(\mu, \nu) \leq \frac{\pi}{4}^2|\chi(M)|$ with equality if and only if 2μ is the Lebesgue-Liouville current of a hyperbolic metric on M.

Now if again α is a positive Hölder class such that $[\nu_\alpha] = [\mu_\alpha]$, then in particular ν_α is a Gibbs current and hence Theorem 2.8 and 2.9 combined show:

COROLLARY 2.10. *$[\mu_\alpha] = [\nu_\alpha]$ for a positive Hölder class α if and only if α is the class of the length cocycle of a metric on M of constant negative curvature.*

REMARK. 1) In his paper [K], Katok showed Corollary 2.10 for length cocycles of negatively curved Riemannian metrics. The arguments given here are more formal and technically much easier, but follow the same principal idea as the argument of Katok. Foulon obtained in [Fo2] a generalization of the result of Katok to smooth contact Anosov flows on compact 3-manifolds. His approach is purely dynamical and uses the fact that a Radon measure without atoms on the line defines a parametrization of the line.

2) If M is a compact rank 1 locally symmetric space, then the analogue to Theorem 2.8 for length cocycles of negatively curved metrics is due to Besson, Courtois and Gallot ([BCG]). However our simple estimate using conformal equivalence of Riemannian metrics on M is not valid any more and the generalization in our Theorem 2.8 for arbitrary Hölder classes is not true in higher dimensions (compare section 3). Moreover, the analogue of Theorem 2.9 fails even in the case that the dimension of M equals 3, as was pointed out by Flaminio ([F]). This is not so surprising given the fact that with the notation of the proof of Theorem 2.9 we have $\int_V \rho \, d\lambda > \int_V d\lambda$ whenever $g \neq \tilde{g}$ and g and $\tilde{g}$ are metrics with the same volume form. However the analogue of Corollary 2.10 may well be true for manifolds which carry a locally symmetric metric.

Recall now that the geodesic flow of a reversible Finsler metric of negative flag curvature on the surface M is a Hamiltonian flow and hence has naturally an invariant measure in the Lebesgue measure class. Call the measure entropy of this measure the *metric entropy* of the flow. As an immediate consequence of Corollary 2.10 we obtain (compare [Fo2]):

COROLLARY 2.11. *If the metric entropy of the geodesic flow of a reversible Finsler metric of negative flag curvature on a closed surface M equals its topological entropy then the Finsler metric is a Riemannian metric of constant curvature.*

We want to give another application of our above considerations. For this we recall the following result of Bonahon [Bo1]: If μ is a geodesic current on a closed surface of negative Euler characteristic such that

$$e^{-\mu[a,b[\times[c,d[} + e^{-\mu[b,c[\times[d,a[} = 1$$

for every ordered quadruple (a,b,c,d) then μ is the Lebesgue–Liouville current of a hyperbolic metric. This gives a purely algebraic description of cross ratios "of maximal entropy". We can use the above considerations to give a slightly sharper version of this result.

COROLLARY 2.12. *Let μ be a geodesic current such that there is a number $\rho > 1$ with $\rho^{-1}e^{-\mu[a,b[\times[c,d[} \le 1 - e^{-\mu[b,c[\times[d,a[} \le \rho e^{-\mu[a,b[\times[c,d[}$ for every ordered quadruple (a,b,c,d) in $\partial\tilde{M}$. Then μ is the Lebesgue–Liouville current of a metric of constant negative curvature.*

Proof. Let μ be a geodesic current which satisfies the assumptions in the corollary. For two points $a \neq b \in S^1$ we then can define a measure μ_{ab} on $]b,a[\subset S^1$ by $\mu_{ab}[x,y[=\mu[a,b[\times[x,y[$. We want to show that the measure class of μ_{ab} does not depend on $a \neq b$.

For this choose $c \in]a,b[$, let $x \in]b,a[$ and let $y \in]b,a[$ be a point near x. By our assumption there is a neighborhood U of x in $]b,a[$ such that

$$1 - e^{-\mu_{cb}[x,y[} = 1 - e^{-\mu[c,b[\times[x,y[} \ge \rho^{-1}e^{-\mu[b,x[\times[y,c[}$$

for all $y \in U - \{x\}$ and moreover

$$e^{-\mu[b,x[\times[y,c[} = e^{-\mu[b,x[\times[y,a[}e^{-\mu[b,x[\times[a,c[} \ge \rho^{-1}(1 - e^{-\mu[x,y[\times[a,b[})e^{-\mu[b,x[\times[a,c[}.$$

For y sufficiently close to x we have

$$\begin{aligned} 1 - e^{-\mu_{cb}[x,y[} &= \mu_{cb}[x,y[+ o\big(\mu_{cb}[x,y[\big), \\ 1 - e^{-\mu_{ab}[x,y[} &= \mu_{ab}[x,y[+ o\big(\mu_{ab}[x,y[\big) \end{aligned}$$

and consequently the measures μ_{cb} and μ_{ab} are absolutely continuous at x with Radon Nikodym derivative

$$1 \ge \frac{d\mu_{cb}}{d\mu_{ab}}(x) \ge \rho^{-2}e^{-\mu[b,x[\times[a,c[}.$$

This means that the current μ, viewed as a Φ^t-invariant Borel measure on the unit tangent bundle T^1M of M, is absolutely continuous with respect to the stable foliation, with locally bounded Radon Nikodym derivative.

In particular μ has no atoms and defines a positive continuous cocycle α (see the remark after Theorem 2.8). If $\Psi \in \pi_1(M)$ has the attracting fixed point a and the repelling fixed point b for its action on S^1 then for every $\xi \in]b,a[$ and every $k \ge 1$ the period $\alpha(\Psi,a)$ of α at Ψ is given by $\alpha(\Psi,a) = \frac{1}{k}\mu[a,b[\times[\xi,\Psi^k\xi[$.

On the other hand, $\lim_{k\to\infty}\frac{1}{k}\log\mu[b,\xi[\times[\Psi^k\xi,a[$ is just the asymptotic logarithmic decay of the conditional measure $\mu_{b\xi}$ at the attracting fixed point a under the action of Ψ. For sufficiently large $k\ge 1$ we conclude that

$$\begin{aligned}\rho^{-1}e^{-k\alpha(\Psi,a)} &\le 1-e^{-\mu[b,\xi[\times[\Psi^k\xi,a[}\\ &=\mu[b,\xi[\times[\Psi^k\xi,a[+o\big(\mu[b,\xi[\times[\Psi^k\xi,a[\big)\le\rho e^{-k\alpha(\Psi,a)}\end{aligned}$$

and therefore $-\alpha(\Psi,a)=\lim_{k\to\infty}\frac{1}{k}\log\mu[b,\xi[\times[\Psi^k\xi,a[$. But this just means that μ is a Gibbs equilibrium state of the cocycle α. The corollary now follows from Theorem 2.8, Theorem 2.9 and the remarks thereafter. □

3 Contact Cocycles and Lebesgue Measures in Higher Dimensions

In section 2 we looked at signed geodesic currents for a closed hyperbolic surface defined by the cross ratio of a Hölder class. This assignment can be considered as associating to the (de Rham) cohomology class of a Hölder continuous 1-form on the unit tangent bundle of this surface its derivative, viewed as a volume form on the space of geodesics.

In this section we discuss an analogue of this in higher dimensions: Namely if the Anosov splitting is of class C^1 then the Lebesgue–Liouville current of a negatively curved metric g on a closed manifold M can be obtained in a purely combinatorial way from the length cocycle of g. As a corollary we obtain Theorem A from the introduction.

The basic idea is that the cross ratio of the length cocycle can be viewed as a coarse version of the symplectic structure on the space of geodesics. For this we need some basic facts about smooth symplectic forms ρ on the vector space $\mathbf{R}^{2n}=\mathbf{R}^n\times\mathbf{R}^n$. Assume that the form ρ is such that for all fixed $x,y\in\mathbf{R}^n$ the sets $\{x\}\times\mathbf{R}^n$ and $\mathbf{R}^n\times\{y\}$ are Lagrangian submanifolds of $(\mathbf{R}^{2n},\rho)$. We define a function φ on $\mathbf{R}^{2n}=\mathbf{R}^n\times\mathbf{R}^n$ as follows: For $x,y\in\mathbf{R}^n$ let $c_x\colon[0,1]\to\mathbf{R}^n$ be a smooth curve joining $c_x(0)=0$ to $c_x(1)=x$ and choose similarly a curve $c_y\colon[0,1]\to\mathbf{R}^n$ joining 0 to y. Define a map $\Psi\colon[0,1]^2\to\mathbf{R}^{2n}$ by $\Psi(s,t)=(c_x(s),c_y(t))$ and let $\varphi(x,y)=\int_{\Psi[0,1]^2}\rho$.

LEMMA 3.1. *The function φ does not depend on the choice of the curves c_x,c_y.*

Proof. Choose a 1-form ζ such that $d\zeta=\rho$. Let $c_x,\tilde c_x$ and $c_y,\tilde c_y$ be curves joining 0 to x and y and let $\Psi,\tilde\Psi\colon[0,1]^2\to\mathbf{R}^{2n}$ be the corresponding maps defined as above.

By Stokes' theorem we have $\int_{\Psi[0,1]^2} \rho = \int_{\partial\Psi[0,1]^2} \zeta$ and $\int_{\tilde{\Psi}[0,1]^2} \rho = \int_{\partial\tilde{\Psi}[0,1]^2} \zeta$. Now the oriented boundary $\partial\Psi - \partial\tilde{\Psi}$ consists of 4 loops (one of them is $c_x \circ \tilde{c}_x^{-1}$) contained entirely in a Lagrangian submanifold for ρ. Another application of Stokes' theorem then shows that

$$\int_{\partial\tilde{\Psi}[0,1]^2} \zeta = \int_{\partial\Psi[0,1]^2} \zeta . \qquad \square$$

If ρ is the standard symplectic form $\rho_0 = \sum dx_i \wedge dy_i$ on $\mathbf{R}^n \times \mathbf{R}^n$, then we write φ_0 for the function as in Lemma 3.1. Notice that $\varphi_0(x,y) = \sum x_i y_i$. Thus $\varphi_0(x,y)$ is just the euclidean scalar product $\langle x,y\rangle$ of $x,y \in \mathbf{R}^n$.

Consider now $\mathbf{R}^{2n} = \mathbf{R}^n \times \mathbf{R}^n$ with the standard form ρ_0 and the corresponding function φ_0. For a product set $Q_1 \times Q_2 \subset \mathbf{R}^n \times \mathbf{R}^n$ define the *symplectic diameter* $\delta_0(Q_1 \times Q_2)$ by $\delta_0(Q_1 \times Q_2) = \sup\{|\varphi_0(x,y)| \mid x \in Q_1, y \in Q_2\}$ and for a compact subset K of $\mathbf{R}^n$ and $r \geq 0$ let $B(K,r) = \{y \in \mathbf{R}^n \mid |\varphi_0(x,y)| \leq r \text{ for all } x \in K\}$. Clearly we have $B(K,r) = rB(K,1)$.

Now write $\tilde{K} = K \cup (-K)$; then $\tilde{K}$ is a compact subset of $\mathbf{R}^n$ which is invariant under reflection at the origin. For $y \in \mathbf{R}^n$ let $H(y) = \max\{\langle x,y\rangle \mid x \in \tilde{K}\}$; then $H(\lambda y) = \lambda H(y)$ for $\lambda \geq 0, H(y+z) \leq H(y) + H(z)$ and therefore $H\colon \mathbf{R}^n \to \mathbf{R}$ is the support function of a compact convex subset of $\mathbf{R}^n$, namely the intersection of the half-spaces $\langle x,y\rangle \leq H(y)(x \in \tilde{K})$ (see [BuZ, p. 157]; I am grateful to the referee for pointing out this reference to me).

With our above notation this compact convex set is just $B(K,1)$. Since $\tilde{K}$ is invariant under reflection at the origin the same is true for $B(K,1)$.

We summarize this observation in the following:

LEMMA 3.2. *Let $K \subset \mathbf{R}^n$ be a compact neighborhood of* 0 *with dense interior. Then $B(K,1)$ is a compact convex body in $\mathbf{R}^n$ which is reflection-symmetric at the origin. For every $x \in \mathbf{R}^n$ contained in the boundary of $B(K,1)$ a hyperplane $H \subset \mathbf{R}^n$ through x is supporting for $B(K,1)$ if and only if there is $z \in K$ with $\varphi_0(z, H-x) = 0$ and such that $|\varphi_0(z,x)| = \max\{|\varphi_0(y,x)| \mid y \in K\}$.*

If $K \subset \mathbf{R}^n$ is a compact convex body with dense interior which is reflection symmmetric at the origin, then $B(K,1)$ is just the polar of K (see [BuZ]) and $B(B(K,1),1) = K$. In this case we call $(K, B(K,1)) \subset \mathbf{R}^n \times \mathbf{R}^n$ a *dual pair of convex bodies* in $\mathbf{R}^n$. If K is an ellipsoid, i.e. if K is the image of the unit ball in $\mathbf{R}^n$ (equipped with the standard inner product) under a linear isomorphism of $\mathbf{R}^n$, then we call $(K, B(K,1))$ an *euclidean dual pair*.

Now recall that the n-th exterior power ρ_0^n of ρ_0 is a volume form on $\mathbf{R}^n$.

Denote by $a(n) > 0$ the euclidean volume of the euclidean unit ball in $\mathbf{R}^n$. The following is known as the *Blaschke-Santaló inequality*, it can be found in [HC] (see also [MP]). (I am grateful to V. Bangert for this reference).

PROPOSITION 3.3. 1) *$\int_{B_1 \times B_2} \rho_0^n \le a(n)^2$ for every dual pair (B_1, B_2) of convex bodies in $\mathbf{R}^n$, with equality if and only if (B_1, B_2) is an euclidean dual pair.*

2) *If Q_1, Q_2 are compact neighborhoods of 0 with dense interior then $\int_{Q_1 \times Q_2} \rho_0^n \le a(n)^2 \delta_0(Q_1 \times Q_2)^n$ with equality if and only if $(Q_1, \delta_0(Q_1 \times Q_2)^{-1} Q_2)$ is an euclidean dual pair.*

View $\mathbf{R}^n$ as an euclidean vector space, equipped with the inner product $\langle\, ,\rangle$. For $r > 0$ let $B(r)$ be the closed ball of radius r about 0 in $\mathbf{R}^n$; then $B(r) \times B(r) = D(r)$ is a neighborhood of 0 in $\mathbf{R}^{2n} = \mathbf{R}^n \times \mathbf{R}^n$. For $c \ge 1$ define a *c-quasisymplectic r-ball* in $\mathbf{R}^{2n}$ to be the image of $D(r)$ under a continuous embedding $\beta\colon D(r) \to \mathbf{R}^{2n}$ with the following properties:

1) There are embeddings $\beta_i\colon B(r) \to \mathbf{R}^n$ $(i = 1, 2)$ such that $\beta = (\beta_1, \beta_2)$.
2) $|\varphi_0(x, y) - \varphi_0(\beta_1 x, \beta_2 y)| \le (c-1)r^2$ for all $x, y \in B(r)$.

The above discussion indicates that a 1-quasisymplectic 1-ball in $\mathbf{R}^{2n}$ is an *euclidean dual pair*, i.e. there is a linear isomorphism $L\colon \mathbf{R}^n \to \mathbf{R}^n$ such that $\beta_1 = L$ and $\beta_2 = (L^t)^{-1}$ where L^t is the transpose of L with respect to the isomorphism $\mathbf{R}^n \to (\mathbf{R}^n)^*$ defined by the symplectic form ρ_0.

The next lemma contains a more precise statement of the relation between quasisymplectic balls and euclidean dual pairs.

PROPOSITION 3.4. *For every $n \ge 1$, every $\epsilon \in (0, 2^{-3n})$, every $r > 0$ and every $(1+\epsilon)$-quasisymplectic r-ball D in $\mathbf{R}^{2n}$ there is an euclidean dual pair $(B_1, B_2) \subset \mathbf{R}^n \times \mathbf{R}^n$ such that $(1-\epsilon 2^{3n}) r(B_1, B_2) \subset D \subset (1+\epsilon 2^{3n}) r(B_1, B_2)$.*

Proof. The statement of the proposition is invariant under scaling, so it is enough to show the proposition for quasisymplectic 1-balls. For this we will use an auxiliary concept. We define a *c-quasisymplectic map* β of $D(1)$ into $\mathbf{R}^{2n}$ to be a continuous map $\beta\colon D(1) \to \mathbf{R}^{2n}$ with the following properties:

1′) There are continuous maps $\beta_i\colon B(1) \to \mathbf{R}^n$ $(i = 1, 2)$ such that $\beta = (\beta_1, \beta_2)$.
2′) $|\varphi_0(x, y) - \varphi_0(\beta_1 x, \beta_2 y)| \le c - 1$ for all $x, y \in B(1)$.

Thus a c-quasisymplectic 1-ball is a c-quasisymplectic map of $D(1)$ into $\mathbf{R}^{2n}$ which is a homeomorphism onto its image.

Our goal will be to show the following:

(∗) For every $n \geq 1$, every $\epsilon \in (0, 2^{-3n})$ and every $(1+\epsilon)$-quasisymplectic map $\beta = (\beta_1, \beta_2)\colon D(1) \to \mathbf{R}^{2n}$ there is a linear isomorphism $L\colon \mathbf{R}^n \to \mathbf{R}^n$ such that for all $x, y \in B(1)$ the distance between $L\beta_1(x)$ and x and the distance between $(L^t)^{-1}\beta_2(y)$ and y is not larger than $\epsilon 2^{3n}$.

The statement of the proposition follows easily from (∗). Namely let

$$\beta = (\beta_1, \beta_2)\colon D(1) \to \mathbf{R}^{2n}$$

be a $(1+\epsilon)$-quasisymplectic 1-ball for $\epsilon \in (0, 2^{-3n})$. By (∗) above we can find a linear isomorphism $L\colon \mathbf{R}^n \to \mathbf{R}^n$ such that the Hausdorff distance between $B(1)$ and $L\beta_1 B(1)$ is not larger than $\epsilon 2^{3n}$. Then $L\beta_1 B(1) \subset (1+\epsilon 2^{3n})B(1)$ and hence $\beta_1 B(1) \subset (1+\epsilon 2^{3n})L^{-1}B(1)$. In the same way we obtain that $\beta_2 B(1) \subset (1+\epsilon 2^{3n})L^t B(1)$. In other words, if we write $B_1 = L^{-1}B(1)$ and $B_2 = L^t B(1)$, then (B_1, B_2) is an euclidean dual pair and $\beta D(1) \subset (1+\epsilon 2^{3n})(B_1, B_2)$.

Similarly, since $L\beta_1 B(1)$ is homeomorphic to $B(1)$ and its Hausdorff distance to $B(1)$ is not larger than $\epsilon 2^{3n}$ we find that $L\beta_1 B(1)$ contains $(1-\epsilon 2^{3n})B(1)$ and $\beta D(1) \supset (1-\epsilon 2^{3n})(B_1, B_2)$.

We have now to show (∗) above. For this we proceed by induction on $n = \dim \mathbf{R}^{2n}/2$.

The case $n = 1$ is immediate. Thus assume that the proposition is known for $n-1 \geq 1$ and let $\beta = (\beta_1, \beta_2)\colon D(1) \to \mathbf{R}^{2n}$ be a $(1+\epsilon)$-quasisymplectic map where $\epsilon \leq 2^{-3n}$.

Let $x_1 \in \partial B(1)$ be a point from the boundary of $B(1)$ and write $y_1 = \beta_1(x_1)$. Let $x_2 \in \partial B(1)$ be the unique point which satisfies $\varphi_0(x_1, x_2) = 1$ and write $y_2 = \beta_2(x_2)$. By our assumption we have $1+\epsilon \geq \varphi_0(y_1, y_2) \geq 1-\epsilon$.

This means in particular that $X = \{z \in \mathbf{R}^{2n} \mid \rho_0(z, (x_1, 0)) = \rho_0(z, (0, x_2)) = 0\}$ and $Y = \{z \in \mathbf{R}^{2n} \mid \rho_0(z, (y_1, 0)) = \rho_0(z, (0, y_2)) = 0\}$ are symplectic linear subspaces of $\mathbf{R}^{2n}$ of dimension $2n-2$.

More precisely, if we write $X_1 = \{x \in \mathbf{R}^n \mid \varphi_0(x, x_2) = 0\}$ and $X_2 = \{x \in \mathbf{R}^n \mid \varphi_0(x_1, x) = 0\}$ then we have $X = X_1 \times X_2$ and similarly $Y = Y_1 \times Y_2$ where $Y_1 = \{y \in \mathbf{R}^n \mid \varphi_0(y, y_2) = 0\}$ and $Y_2 = \{y \in \mathbf{R}^n \mid \varphi_0(y_1, y) = 0\}$.

The subspace Y_i of $\mathbf{R}^n$ is transversal to the line $\mathbf{R}y_i$ and hence we obtain a canonical projection $P_i\colon \mathbf{R}^n = \mathbf{R}y_i \oplus Y_i \to Y_i$. Define $\bar\beta_i = P_i \circ \beta_i \mid X_i$. Then $(\bar\beta_1, \bar\beta_2)$ is a map from the symplectic ball $D(1) \cong D(1) \cap X_1 \times X_2$ of the symplectic vector space $(X_1 \times X_2, \rho_0)$ into the symplectic vector space $(Y_1 \times Y_2, \rho_0)$. Moreover the restriction of ρ_0 to $X_1 \times X_2$ is the standard symplectic form on $\mathbf{R}^{2n-2}$, and the same is true for $\rho_0|_{Y_1 \times Y_2}$.

We claim that $(\bar\beta_1, \bar\beta_2)$ is a $1+2\epsilon$-quasisymplectic map of $D(1)$ into $\mathbf{R}^{2n-2}$.

To see this let $z_1 \in B(1) \cap X_1$ and write $\beta_1(z_1) = a_1y_1 + \bar\beta_1(z_1)$. Then $\varphi_0(z_1, x_2) = 0$ and

$$\left|\varphi_0(\beta_1 z_1, \beta_2 x_2)\right| = \left|\varphi_0(a_1y_1 + \bar\beta_1(z_1), y_2)\right| \geq |a_1|(1-\epsilon)\,.$$

Since β is an $(1+\epsilon)$-quasisymplectic map we conclude that $|a_1| \leq \epsilon/1-\epsilon$. The same argument applied to $z_2 \in B(1) \cap X_2$ shows that $\beta_2(z_2) = a_2y_2+\bar\beta_2(z_2)$ with $|a_2| \leq \epsilon/1-\epsilon$. Then

$$\left|\varphi_0(\bar\beta_1(z_1), \bar\beta_2(z_2)) - \varphi_0(\beta_1(z_1), \beta_2(z_2))\right| = \left|a_1a_2\varphi_0(y_1, y_2)\right| \leq \epsilon^2(1+\epsilon)/(1-\epsilon)^2$$

and hence

$$\left|\varphi_0(z_1, z_2) - \varphi_0(\bar\beta_1(z_1), \bar\beta_2(z_2))\right| \leq \epsilon + \epsilon^2(1+\epsilon)/(1-\epsilon)^2 \leq 2\epsilon\,.$$

We apply now the induction hypothesis to $(\bar\beta_1, \bar\beta_2)$. This means that there is a linear isomorphism $L_1 \colon Y_1 \cong \mathbf{R}^{n-1} \to X_1 \cong \mathbf{R}^{n-1}$ such that for every $x \in X_1 \cap B(1)$ the distance between $L_1\bar\beta_1(x)$ and x is not larger than $(2\epsilon)2^{3n-3}$ and for every $x \in X_2 \cap B(1)$ the distance between $(L_1^t)^{-1}\bar\beta_2(x)$ and x is not larger than $(2\epsilon)2^{3n-3}$.

Define a linear map $L\colon \mathbf{R}^n \to \mathbf{R}^n$ by $L|_{X_1} = L_1$ and $Lx_1 = y_1$.

Let $x = a_1x_1 + z_1 \in \mathbf{R}x_1 \oplus X_1 \cap B(1)$ and write $\beta_1(x) = b_1y_1 + \bar z_1 \in \mathbf{R}y_1 \oplus Y_1$. Then $\bar z_1 = P_1\beta_1(x)$ and we have $\max\{\varphi_0(x, w) \mid w \in X_2 \cap B(1)\} = \|z_1\| \leq 1$.

Now $a_1 = \varphi_0(x, x_2)$ and $|\varphi_0(x, x_2) - \varphi_0(\beta_1 x, \beta_2 x_2)| \leq \epsilon$ implies that

$$b_1 = \varphi_0(\beta_1 x, \beta_2 x_2) \in [a_1 - \epsilon, a_1 + \epsilon]\,.$$

Since moreover $|\varphi_0(y_1, \beta_2(w) - \bar\beta_2(w))| \leq \epsilon/1-\epsilon$ for all $w \in B(1) \cap X_2$ by the above consideration we conclude that $|\varphi_0(\beta_1 x, \beta_2 w) - \varphi_0(\beta_1 x, \bar\beta_2 w)| \leq |b_1|\epsilon/(1-\epsilon) \leq 2\epsilon$ and consequently $|\varphi_0(x, w) - \varphi_0(\beta_1 x, \bar\beta_2 w)| \leq 3\epsilon$ for all $w \in X_2 \cap B(1)$.

We claim that the distance between $L_1\bar z_1$ and z_1 is not larger than $\epsilon 2^{3n-1}$. To see this choose $u \in X_2 \cap B(1)$ such that $\varphi_0(L_1\bar z_1 - z_1, u) = \|L_1\bar z_1 - z_1\|$. Since the distance between u and $(L_1^t)^{-1}\bar\beta_2(u)$ is not larger than $\epsilon 2^{3n-2}$ we conclude that $\varphi_0(L_1\bar z_1 - z_1, (L_1^t)^{-1}\bar\beta_2(u)) \geq (1-\epsilon 2^{3n-2})\|L_1\bar z_1 - z_1\| \geq 3\|L_1\bar z_1 - z_1\|/4$. On the other hand we have

$$\begin{aligned}\varphi_0\big(L_1\bar z_1 - z_1, (L_1^t)^{-1}\bar\beta_2(u)\big) &= \varphi_0\big(\bar z_1, \bar\beta_2(u)\big) - \varphi_0\big(z_1, (L_1^t)^{-1}\bar\beta_2(u)\big)\\ &\leq \varphi_0(z_1, u) + 3\epsilon - \varphi_0(z_1, u) + \epsilon 2^{3n-2} = 3\epsilon + \epsilon 2^{3n-2} \leq 2^{3n-1}\epsilon\,.\end{aligned}$$

From this the above claim follows.

Therefore, if $\| \ \|_{L^{-1}}$ is the norm of the image of the euclidean scalar product under L^{-1}, then $\|\beta_1 x - x\|_{L^{-1}}^2 = (b_1 - a_1)^2 + \|\bar z_1 - z_1\|_{L^{-1}}^2 \leq \epsilon^2 + (\epsilon 2^{3n-1})^2 \leq (\epsilon 2^{3n})^2$ from which we obtain the claim $(*)$ above. This finishes the proof of our proposition. $\square$

From Proposition 3.4 the following is immediate:

COROLLARY 3.5. *For $\epsilon \leq 2^{-3n}$ the volume of every $(1+\epsilon)$-quasisymplectic r-ball in $(\mathbf{R}^{2n}, \rho_0)$ is contained in the interval* $[(1-\epsilon 2^{3n})^{2n} r^{2n} a(n)^2, (1+\epsilon 2^{3n})^{2n} r^{2n} a(n)^2]$.

Now again let $\partial\tilde{M}$ be the ideal boundary of the universal covering of a negatively curved manifold M of dimension n. Denote by $[\,]$ the logarithm of the cross ratio of the length cocycle of the metric on M.

Using the fact that $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ is a smooth symplectic manifold and that for every $\xi \in \partial\tilde{M}$ the subsets $\partial\tilde{M} \times \{\xi\}$ and $\{\xi\} \times \partial\tilde{M}$ are Lagrangian we can define for every fixed $(\xi, \eta) \in \partial\tilde{M} \times \partial\tilde{M} - \Delta$ a function φ as in Lemma 3.1. Our goal is to relate this function to the cross ratio $[\,]$. First, for a compact set $K \subset \partial\tilde{M}$ with dense interior and non-empty complement define a function ρ_K on $(\partial\tilde{M} - K)^2$ by $\rho_K(\xi, \eta) = \sup\{[\xi, \eta, \zeta, \bar{\zeta}] \mid \zeta, \bar{\zeta} \in K\}$.

Then we have:

LEMMA 3.6. *ρ_K is a distance on $\partial\tilde{M} - K$.*

Proof. First of all we clearly have $\rho_K(\xi, \xi) = 0$ for all $\xi \in \partial\tilde{M} - K$. Second, since $[\xi, \eta, \zeta, \bar{\zeta}] = -[\eta, \xi, \zeta, \bar{\zeta}] = [\eta, \xi, \bar{\zeta}, \zeta]$ the function ρ_K is symmetric and non-negative.

To show the triangle inequality let $\xi, \eta \in \partial\tilde{M} - K$ and let $\zeta, \bar{\zeta} \in K$ be such that $[\xi, \eta, \zeta, \bar{\zeta}] = \rho_K(\xi, \eta)$; such points exist by continuity of $[\,]$ and compactness of K. If $\omega \in \partial\tilde{M} - K$ is arbitrary, then $\rho_K(\xi, \eta) = [\xi, \omega, \zeta, \bar{\zeta}] + [\omega, \eta, \zeta, \bar{\zeta}] \leq \rho_K(\xi, \omega) + \rho_K(\omega, \eta)$.

We are left with showing that $\rho_K(\xi, \eta) > 0$ for $\xi \neq \eta$; for this we need the fact that the interior U of K is non-empty. Choose $\zeta \in U$, let γ be a geodesic joining $\gamma(-\infty) = \xi$ to $\gamma(\infty) = \eta$ and let $\bar{\gamma}$ be the geodesic in $\tilde{M}$ with end-point $\bar{\gamma}(\infty) = \zeta$ which meets γ orthogonally in $\bar{\gamma}(0)$. As $t \to \infty$, the angle between $\bar{\gamma}'(t)$ and the geodesics in $\tilde{M}$ joining $\bar{\gamma}(t)$ to ξ and η converges to π. Since ζ is an interior point of K this implies that there is some $t > 0$ such that the second endpoint ρ, ρ' of the geodesic joining $\bar{\gamma}(t)$ to ξ, η is contained in K.

Let $(\mid)$ be the Gromov product on $\partial\tilde{M}$ based at $\bar{\gamma}(t)$ (see [GhHa], [H5] and section 2). For $\omega \neq \omega' \in \partial\tilde{M}$ we have $(\omega \mid \omega') \geq 0$ with equality if and only if the geodesic in $\tilde{M}$ joining ω to ω' passes through $\bar{\gamma}(t)$.

But this means $(\xi \mid \rho) = (\eta \mid \rho') = 0$ and $(\xi \mid \rho') > 0$, $(\eta \mid \rho) > 0$ and therefore $[\xi, \eta, \rho', \rho] = (\xi \mid \rho') + (\eta \mid \rho) - (\xi \mid \rho) - (\eta \mid \rho') > 0$ which finishes the proof of our lemma. $\square$

Recall now from [H3] that $\partial\tilde{M}$ admits a C^1-structure if there is a differentiable structure for $\partial\tilde{M}$ for which the projection $\pi : T^1\tilde{M} \to \partial\tilde{M}$ is a C^1-submersion. This is the case if the Anosov splitting of TT^1M is of class C^1 and therefore holds for rank–1 locally symmetric manifolds or manifolds with strictly 1/4–pinched sectional curvature. Then for every $v \in T^1\tilde{M}$ the restriction to $W^{su}(v)$ of the natural projection $\pi\colon T^1\tilde{M} \to \partial\tilde{M}$ is a C^1-diffeomorphism onto $\partial\tilde{M} - \pi(-v)$. In particular we can equip $\partial\tilde{M}$ with a continuous Riemannian metric, and every two such metrics are globally bilipschitz-equivalent. If ρ is any distance function on $\partial\tilde{M}$ which is locally bilipschitz equivalent to one (and hence every) continuous Riemannian metric on $\partial\tilde{M}$ we say that ρ is *locally equivalent* to a Riemannian structure. Then we have

PROPOSITION 3.7. *If $\partial\tilde{M}$ admits a C^1-structure then the distance functions ρ_K on $\partial\tilde{M} - K$ are locally equivalent to a Riemannian structure.*

Proof. Let $\emptyset \neq U \subset \partial\tilde{M}$ be open and connected with closure K and let $\zeta \in U$ be arbitrarily fixed. If $\eta, \xi \in \partial\tilde{M} - K = \Omega$ and if $\zeta_1, \zeta_2 \in K$ are such that $\rho_K(\xi,\eta) = [\xi,\eta,\zeta_1,\zeta_2]$ then $[\xi,\eta,\zeta_1,\zeta_2] = [\xi,\eta,\zeta_1,\zeta] + [\xi,\eta,\zeta,\zeta_2]$ shows that $\rho_K(\xi,\eta) \leq 2\sup\{|[\xi,\eta,\zeta,\bar\zeta]| \mid \bar\zeta \in K\} = 2\bar\rho(\xi,\eta)$.

Thus it suffices to show that the function $\bar\rho$ which is defined by the above equation is locally equivalent to a Riemannian structure.

For this let $\xi \in \Omega$ again be arbitrary and let γ be a geodesic joining $\gamma(-\infty) = \zeta$ to $\gamma(\infty) = \xi$. Let d^{ss} (or d^{su}) be the distances on the leaves of W^{ss} (or W^{su}) induced by the restriction of the Sasaki metric and assume that γ is parametrized in such a way that for $v(\xi) = v = \gamma'(0)$ we have $\sup\{d^{ss}(v,w) \mid \pi(-w) \in K\} = 1$.

For $\eta \in K$ let $\varphi_{v,\eta}\colon [0,1] \to W^{ss}(v)$ be the d^{ss}-geodesic which is parametrized on $[0,1]$ and joins v to the unique point $w \in W^{ss}(v)$ for which $\pi(-w) = \eta$. Let $B \subset W^{su}(v)$ be a compact neighborhood of v such that $\pi B \subset \Omega$. Then for every $\eta \in K$ and every $w \in B$ there is a unique curve $\varphi_{w,\eta}\colon [0,1] \to W^{ss}(w)$ of class C^1 such that $\pi\mathcal{F}\varphi_{w,\eta} = \pi\mathcal{F}\varphi_{v,\eta}$. The assignment $(w,\eta,t) \in B \times K \times [0,1] \to \varphi'_{w,\eta}(t)$ is continuous and hence uniformly bounded in norm with respect to the Sasaki metric by a constant $\bar\chi > 0$.

Now let $w \in B$ and let $Y_0 \in T_wW^{su}$ be a unit tangent vector with respect to the Sasaki metric. Then Y_0 has a natural extension to a section Y of TW^{su} over $W^{ss}(w)$ such that $d\pi(Y) = d\pi(Y_0)$. This vector field depends continuously on Y. Thus there is a constant $\chi \geq \bar\chi$ not depending on Y_0 such that the norm of Y on $\{z \in W^{ss}(w) \mid \pi \circ \mathcal{F}(z) \in K\}$ is bounded from above by χ.

Now if $t \to w(t) \in B$ is a smooth curve through $w(0) = w$ which satisfies $w'(0) = Y_0$, then we obtain from Lemma 1 of [H2] that

$$\frac{d}{dt}[\zeta, \eta, \pi w, \pi w(t)]\big|_{t=0} = \int_0^1 d\omega\big(\varphi'_{w,\eta}(\sigma), Y(\varphi_{w,\eta}(\sigma))\big)d\sigma$$

for every $\eta \in K$ and therefore by the above estimates we have

$$\left|\frac{d}{dt}[\zeta, \eta, \pi w, \pi w(t)]_{|t=0}\right| \leq \chi^2$$

for all $\eta \in K$. But this implies that the ρ_K-length of any smooth curve in B is not larger than $2\chi^2$-times the d^{su}-length of the curve. In other words, $\rho_K \circ \pi \leq 2\chi^2 d^{su}$ on B which shows one half of our proposition.

We are left with showing that there is a constant $\kappa > 0$ such that $\rho_K \circ \pi \geq \kappa d^{su}$ on B. For this let Y_0 and Y be as above and observe that for a smooth curve w in B through $w'(0) = Y_0$ we have $\bar{\rho}(\pi w, \pi w(t)) \geq t\beta + o(t)$ where

$$\beta = \max\left\{\left|\int_0^1 d\omega\big(\varphi'_{w,\eta}(\sigma), Y(\varphi_{w,\eta}(\sigma))\big)d\sigma\right| \;\Big|\; \eta \in K\right\}$$

is bounded from below by a universal positive constant not depending on the unit tangent vector Y_0. □

REMARK. Proposition 3.7 shows in particular that $\partial\tilde{M}$ admits a C^1-structure only if the distances ρ_K ($K = \bar{U}$ for some $\emptyset \neq U \subset \partial\tilde{M}$ open) are locally equivalent on the intersections of their domain of definition and locally equivalent to the length metric they induce. Since the distances ρ_K are defined just by the length cocycle of the metric this shows that obstructions to existence of a C^1-structure on the geometric boundary $\partial\tilde{M}$ of M equipped with a fixed metric g can be read off immediately from the cohomology class of the length cocycle. It seems to be reasonable to believe that local equivalence of the distances ρ_K implies the existence of a Lipschitz structure on $\partial\tilde{M}$.

Moreover, if $\partial\tilde{M}$ has a C^1-structure, then for every C^1-curve $c\colon S^1 \to \partial\tilde{M}$ the restriction of [ ] to quadruples of pairwise distinct points in $c(S^1)$ is a signed Borel measure, i.e. is σ-additive and finite. It is not hard to see that rectifiable curves in $\partial\tilde{M}$ with respect to the C^1-structure are characterized by this property.

Denote by $\mathcal{V}$ the *vertical bundle* which is tangent to the fibres of the fibration $T^1\tilde{M} \to \tilde{M}$. Let g^v be the restriction to $\mathcal{V}$ of the Sasaki metric. Then for every $x \in \tilde{M}$ the fibre $T^1_x\tilde{M}$ equipped with the metric g^v is just the $(n-1)$-dimensional round sphere of constant curvature 1.

Let g^{ss} be the Riemannian metric on TW^{ss} induced from the Riemannian metric on M. The bundles TW^{ss} and $\mathcal{V}$ are Lagrangian with respect to the symplectic form $d\omega$ and transversal, and g^{ss} is just the metric on TW^{ss} which is dual to g^v with respect to $d\omega$. This means that for every g^v-orthonormal basis $X_1, \dots, X_{n-1}$ of $\mathcal{V}_v$ the basis $Y_1, \dots, Y_{n-1}$ of T_vW^{ss} which satisfies $d\omega(X_i, Y_j) = \delta_{ij}$ is orthonormal with respect to g^{ss}.

Denote by $\exp^{ss}$ the exponential map of the strong stable foliation equipped with the metric g^{ss} and let $\exp^v$ be the exponential map of the vertical foliation equipped with the metric g^v.

For $v \in T^1\tilde{M}$ define a map

$$E_v \colon \mathcal{V}_v \oplus T_vW^{ss} \to \partial\tilde{M} \times \partial\tilde{M}$$

by $E_v(X, Y) = (\pi(\exp^v X), \pi(-\exp^{ss}(Y)))$. Then E_v is a homeomorphism of an open neighborhood U of 0 in $\mathcal{V}_v \oplus T_vW^{ss}$ onto a neighborhood of $(\pi(v), \pi(-v))$ in $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ which is absolutely continuous with respect to the volume element on $\mathcal{V}_v \oplus T_vW^{ss}$ induced by the symplectic form $d\omega$ and the Lebesgue–Liouville current λ, with locally uniformly Hölder continuous Jacobian whose value at $(0, 0) \in \mathcal{V}_v \oplus T_vW^{ss}$ equals 1.

Corollary 2.10 of [H3] then shows that for every $\epsilon > 0$ there is a number $r(\epsilon) > 0$ such that for every $X \in \mathcal{V}_v$ with $\|X\| \le r(\epsilon), Y \in T_vW^{ss}$ with $\|Y\| \le r(\epsilon)$ the following is satisfied:

i) The Jacobian of E_v at $X + Y$ is contained in $[1 - \epsilon, 1 + \epsilon]$.
ii) $|[\pi(\exp^v X), \pi(v), \pi(-\exp^{ss} Y), \pi(-v)] - d\omega(X, Y)| \le \epsilon r(\epsilon)^2$.

We now investigate maps of $(\mathbf{R}^{2n-2}, \rho_0)$ into $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ which are compatible with respect to the integrated symplectic structures determined by the function φ_0 on $\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}$ and the cross ratio on $\partial\tilde{M} \times \partial\tilde{M} - \Delta$. Namely, for $c \ge 1$ define a *c-quasisymplectic embedding* of $D(r)$ into $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ to be a homeomorphism β of $D(r)$ onto a compact subset of $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ with the following properties:

1) There are continuous embeddings $\beta_i \colon B(r) \to \partial\tilde{M}$ $(i = 1, 2)$ such that $\beta = (\beta_1, \beta_2)$.
2) $|[\beta_1(x), \beta_1(0), \beta_2(y), \beta_2(0)] - \varphi_0(x, y)\ | \le (c - 1)r^2$ for all $x \in B(r)$, $y \in B(r)$.

We call the image of $D(r)$ under a $(1+\epsilon)$-quasisymplectic embedding β a *$(1+\epsilon)$-quasisymplectic r-ball* and call $\beta(0)$ the *center* of the quasisymplectic ball.

By the above (i.e. Corollary 2.10 of [H3]), for every $\epsilon > 0$, every sufficiently small $r > 0$ and every $\xi \ne \eta \in \partial\tilde{M}$ there is a $1 + \epsilon$-quasisymplectic

r-ball in $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ with center (ξ, η). Namely let $v \in T^1\tilde{M}$ be tangent to the geodesic joining η to ξ, let $X_1, \ldots, X_{n-1}$ be a g^v-orthonormal basis of $\mathcal{V}_v$ and let $Y_1, \ldots, Y_{n-1}$ be the g^{ss}-orthonormal basis of T_vW^{ss} such that $d\omega(X_i, Y_j) = \delta_{ij}$. Then for $r \le r(\epsilon)$ where $r(\epsilon) > 0$ is as above, the map of $D(r)$ into $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ defined by $(x_1, \ldots, x_{n-1}; y_1, \ldots y_{n-1}) \to (\pi \exp^v \sum x_i X_i, \pi \circ \mathcal{F} \exp^{ss} \sum y_i Y_i)$ is a $(1+\epsilon)$-quasisymplectic embedding.

For $\epsilon > 0$ denote by $\mathcal{Q}(\epsilon)$ the family of all $(1+\epsilon)$-quasisymplectic r-balls for arbitrary $r > 0$ in $\partial\tilde{M} \times \partial\tilde{M} - \Delta$. Then $\mathcal{Q}(\epsilon) \subset \mathcal{Q}(\delta)$ if $\epsilon \le \delta$.

Fix a distance d on $\partial\tilde{M}$ which induces the natural topology and let $\operatorname{diam}(B)$ be the $d \times d$-diameter of a set $B \subset \partial\tilde{M} \times \partial\tilde{M}$. For closed disjoint sets $A, B \in \partial\tilde{M}$ and points $a \in A, b \in B$ define

$$\delta(A \times B; a, b) = \sup\left\{ |[\xi, a, \zeta, b]| \;\middle|\; \xi \in A, \zeta \in B \right\}$$

and $\delta(A \times B) = \inf\{\delta(A \times B; a, b) \mid a \in A, b \in B\}$.

For $\epsilon > 0$ and a Borel subset C of $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ define

$$\mathcal{S}_\epsilon(C) = \inf\left\{ \sum_{i=1}^{\infty} a(n-1)^2 \delta(A_i \times B_i)^{n-1} \;\middle|\; A_i \times B_i \in \mathcal{Q}(\epsilon), C \subset \bigcup_{i=1}^{\infty} A_i \times B_i,\ \operatorname{diam}(A_i \times B_i) \le \epsilon \right\}$$

and $\mathcal{S}(C) = \limsup_{\epsilon \to 0} \mathcal{S}_\epsilon(C)$.

Denote by $\lambda = \lambda_g$ the Lebesgue–Liouville measure on $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ induced by g. Then we have:

Proposition 3.8. $\mathcal{S} \le \lambda$.

Proof. Let again $\exp^{ss}$ be the exponential map of the foliation W^{ss} with respect to the Riemanian metric g^{ss}, and let $\exp^v$ be the vertical exponential map. Denote by $\|\ \|$ the norm of TW^i induced by g^i $(i = ss, v)$ and for $v \in T^1\tilde{M}$ let $E_v : \mathcal{V}_v \oplus T_vW^{ss} \to \partial\tilde{M} \times \partial\tilde{M}$ as before be defined by $E_v(X, Y) = (\pi(\exp^v X), \pi(-\exp^{ss} Y))$.

Fix a number $\epsilon > 0$ and let $r(\epsilon) > 0$ be as above such that i) and ii) above are satisfied.

Let $\delta \le r(\epsilon)$, let $v \in T^1\tilde{M}$ and let B^{ss} be the ball of radius $r < \delta$ about 0 in (T_vW^{ss}, g^{ss}), B^v be the ball of radius r about 0 in $(\mathcal{V}_v, g^v)$. By i) above we then have $\lambda(\pi(\exp^v B^v), \pi(-\exp^{ss} B^{ss})) \in a(n-1)^2 r^{2n-2}[1-\epsilon, 1+\epsilon]$, and ii) shows that the restriction of E_v to $B^v \times B^{ss}$ is a $(1+\epsilon)$–quasisymplectic embedding.

Now if $C \subset \partial\tilde{M} \times \partial\tilde{M} - \Delta$ is any compact set, then for every $\epsilon > 0$ there are points $v_1, \ldots, v_k \in T^1\tilde{M}$ and balls $B_j^{ss} \subset T_{v_j}W^{ss}$, $B_j^v \subset \mathcal{V}_{v_j}$ of radius $r_j \le r(\epsilon)$ such that

$$\operatorname{diam}\big(E_{v_j}(B_j^v \times B_j^{ss})\big) \le \epsilon\,,\ C \subset \cup_{j=1}^k E_{v_j}(B_j^v \times B_j^{ss}) \text{ and } \sum_{j=1}^k \lambda\big(E_{v_j}(B_j^v \times B_j^{ss})\big) \le \lambda(C) + \epsilon\,.$$

But this means

$$\begin{aligned}\lambda(C) &\ge (1-\epsilon)\sum_{j=1}^k a(n-1)^2 r_j^{2n-2} - \epsilon \\ &\ge (1-\epsilon)\sum_{j=1}^k a(n-1)^2 \delta(B_j^v \times B_j^{ss})^{n-1}(1-\epsilon)^{n-1} - \epsilon\,.\end{aligned}$$

From this we obtain that $\mathcal{S}_\epsilon(A) \le (\lambda(A)+\epsilon)/(1-\epsilon)^n$.

Since $\epsilon > 0$ was arbitrary and $\mathcal{S}_\delta \ge \mathcal{S}_\epsilon$ for $\delta \le \epsilon$ this shows that $\mathcal{S} \le \lambda$ as claimed. □

For the remainder of this section assume now that the Anosov splitting of TT^1M is of class C^1. Our goal is to study more precisely the measure $\mathcal{S}$.

For this recall the definition of the *Kanai connection* ∇ on T^1M which is defined as follows: Let $\mathcal{J}$ be the $(1,1)$-tensor field on T^1M defined by $\mathcal{J}(Y) = -Y$ for $Y \in TW^{ss}$, $\mathcal{J}Y = Y$ for $Y \in TW^{su}$ and $\mathcal{J}(\mathbf{R}X^0) = 0$. The indefinite metric h on T^1M defined by $h(Y,Z) = d\omega(Y,\mathcal{J}Z) + \omega \oplus \omega(Y,Z)$ is of class C^1 and hence there is a unique affine connection ∇ on T^1M such that

i) h is parallel with respect to ∇
ii) The torsion of ∇ is $d\omega \otimes X^0$.

The bundles TW^{su}, TW^{ss} are invariant under ∇, and the restriction of ∇ to every leaf of W^{su} or W^{ss} is flat. In other words, if we denote the lift of ∇ to $T^1\tilde{M}$ by the same symbol, then for every $v \in T^1\tilde{M}$ there is a basis of $TW^{su} \mid W^{su}(v)$ consisting of ∇-parallel vector fields.

The space of geodesics in $\tilde{M}$ is $\partial\tilde{M} \times \partial\tilde{M} - \Delta = T^1\tilde{M}/\mathbf{R}$ where $\mathbf{R}$ acts on $T^1\tilde{M}$ as the geodesic flow. Equipped with the projection of $d\omega$ this is a smooth symplectic manifold.

According to Darboux's theorem, every point $\xi \in \partial\tilde{M} \times \partial\tilde{M} - \Delta$ admits a neighborhood U and local coordinates $(x_1,\dots,x_{n-1},y_1,\dots,y_{n-1})$ such that in these coordinates the symplectic form $d\omega$ is the standard symplectic form $\sum dx_i \wedge dy_i$ on $\mathbf{R}^{2n-2}$. In the sequel we call such coordinates *symplectic coordinates* for $\partial\tilde{M} \times \partial\tilde{M} - \Delta$.

We want to use the Kanai-connection ∇ to construct particular such coordinates which are adapted to the product structure of $\partial\tilde{M} \times \partial\tilde{M} - \Delta$.

For this let $v \in T^1\tilde{M}$ be arbitrary and let $L_v^{su}\colon T_vW^{su} \to W^{su}(v)$ and $L_v^{ss}\colon T_vW^{ss}(v) \to W^{ss}(v)$ be the exponential map of the restriction of ∇ to $W^{su}(v)$ and $W^{ss}(v)$ respectively. By compactness we can find a number $r_0 > 0$ such that for every $w \in T^1\tilde{M}$ the projections to $\partial\tilde{M}$ of the images under L_w^{su} and $\mathcal{F} \circ L_w^{ss}$ of the closed balls of radius r_0 about 0 in T_wW^{su} and T_wW^{ss} are disjoint. If we denote by $A^{su}(w)$ and $A^{ss}(w)$ the interiors of these balls, then for every $w \in A^{su}(v)$ and $z \in A^{ss}(v)$ the intersection $W^{ss}(w) \cap W^u(z)$ consists of a unique point $[w, z]$. Then $H(v) = H = \{[w, z] \mid w \in A^{su}(v), z \in A^{ss}(v)\}$ is a local hypersurface in $T^1\tilde{M}$ of class C^1 transversal to the geodesic flow and hence $(H, d\omega)$ is a symplectic manifold which can naturally be identified with a neighborhood of $(\pi(v), \pi(-v))$ in $\partial\tilde{M} \times \partial\tilde{M} - \Delta$. For each point $z \in H$ the intersection $W^{ss}(z) \cap H$ is an open neighborhood of z in $W^{ss}(z)$.

Let $Y_1, \dots, Y_{n-1}$ be a basis of T_vW^{ss} which is orthonormal with respect to the lift of the metric on $\tilde{M}$, $X_1, \dots, X_{n-1}$ be the basis of T_vW^{su} which is dual with respect to $d\omega$ (i.e. such that $d\omega(X_i, Y_j) = \delta_{ij}$) and define local coordinates Ψ on H by $\Psi(x_1,\dots,x_{n-1},y_1,\dots,y_{n-1}) = [L_v^{su}(\sum x_iX_i), L_v^{ss}(\sum y_jY_j)]$. By the above we may assume that the domain of Ψ equals $D(r_0)$ independent of $v \in T^1\tilde{M}$.

Via these coordinates the symplectic structure on H induces a function φ as in Lemma 3.1.

LEMMA 3.9. *Let $x \in A^{su}(v), y \in A^{ss}(v)$. Then $[\pi(x), \pi(v), \pi(-y), \pi(-v)] = \varphi(\Psi^{-1}[x, y])$.*

Proof. Let $\beta\colon [0,1]^2 \to H$ be a map of class C^1 with $\beta(0,0) = v, \beta(s,0) \in A^{su}, \beta(0,t) \in A^{ss}$ and $\beta(s,t) = [\beta(s,0), \beta(0,t)]$. For each fixed $s \in [0,1]$ the curve $t \to \beta(s,t)$ is contained in a leaf of W^{ss}, and $d\pi(\frac{\partial}{\partial s}\beta(s,t))$ is independent of $t \in [0,1]$. In other words, $\frac{\partial}{\partial s}\beta$ is parallel along the curves $t \to \beta(s,t)$.

Lemma 1 of [H2] then shows that

$$\begin{aligned}
&\frac{d}{ds}[\pi\beta(s,0), \pi(v), \pi(-\beta(0,1)), \pi(-v)] \\
&\qquad = \frac{d}{ds}\big([\pi\beta(s,0), \pi(v), \pi(-\beta(0,1)), \pi(-\beta(s,1))] \\
&+ [\pi\beta(s,0), \pi(v), \pi(-\beta(s,1)), \pi(-v)]\big) = \int_0^1 d\omega\left(\frac{\partial}{\partial s}\beta(s,t), \frac{\partial}{\partial t}\beta(s,t)\right) dt
\end{aligned}$$

since $\pi(-\beta(0,1)) = \pi(-\beta(s,1))$ for all s. From this the lemma is immediate. $\square$

On the other hand, X_i, Y_j can be extended to ∇-parallel vector fields along $W^{su}(v)$ which we denote by the same symbol. Define local coordinates $\Psi_0\colon \mathbf{R}^{n-1} \times \mathbf{R}^{n-1} \to H$ by

$$\Psi_0(x_1,\dots,x_{n-1},y_1,\dots y_{n-1}) = L^{ss}\Big(\sum_j y_jY_j\Big(L_v^{su}\Big(\sum x_iX_i\Big)\Big)\Big).$$

LEMMA 3.10. *$d\omega = \sum_{i=1}^{n-1} dx_i \wedge dy_i$ in the coordinates Ψ_0.*

Proof. Let $c\colon(-\epsilon,\epsilon) \to W^{su}(v)$ be the integral curve of a ∇-parallel vector field along $W^{su}(v)$, and for some ∇-parallel section Y of TW^{ss} over $W^{su}(v)$ let $\beta(s,t) = L^{ss}tY(c(s))$. We want to show that $d\omega(\frac{\partial}{\partial s}\beta, \frac{\partial}{\partial t}\beta)$ is independent of s and t. Since $d\omega$ is ∇-parallel we conclude that this function is constant along the curve $s \to \beta(s,0)$. On the other hand, for every fixed $s \in (-\epsilon,\epsilon)$ we have $\nabla_{\partial/\partial t}\frac{\partial}{\partial t}\beta = 0$ and consequently (formally)

$$\frac{\partial}{\partial t}d\omega\left(\frac{\partial}{\partial s}\beta, \frac{\partial}{\partial t}\beta\right) = d\omega\left(\frac{\nabla}{\partial t}\frac{\partial}{\partial s}\beta, \frac{\partial}{\partial t}\beta\right) =$$
$$d\omega\left(\frac{\nabla}{\partial s}\frac{\partial}{\partial t}\beta, \frac{\partial}{\partial t}\beta\right) = \frac{1}{2}\frac{\partial}{\partial s}d\omega\left(\frac{\partial}{\partial t}\beta, \frac{\partial}{\partial t}\beta\right) = 0$$

(recall that the torsion of ∇ equals $d\omega \otimes X^0$). From this the lemma is immediate. $\square$

The symplectic coordinates Ψ_0 on the symplectic manifold H (which we can canonically identify with an open subset of $\partial\tilde M \times \partial\tilde M - \Delta$) are particularly well adapted to the product structure of $\partial\tilde M\times\partial\tilde M-\Delta$. Namely, for every fixed $y \in \mathbf{R}^{n-1}$ the set $\Psi_0(\{y\} \times \mathbf{R}^{n-1})$ is contained in a leaf of the strong stable foliation. However, it is not true that $\Psi_0(\mathbf{R}^{n-1} \times \{y\})$ is contained in a leaf of the unstable foliation. For otherwise the manifold $\partial\tilde M \times \partial\tilde M - \Delta$ equipped with the Kanai connection would be flat which is impossible.

Symplectic coordinates with the above additional properties are by no means unique. We can choose them compatible with a symplectic submanifold of $\partial\tilde M \times \partial\tilde M - \Delta$ which is a product.

More precisely, let $v \in T^1\tilde M$ and let $A \subset W^{su}(v)$, $B \subset W^{ss}(v)$ be k-dimensional embedded submanifolds containing v in their interior. Assume that the restriction of $d\omega$ to $C = \{[x,y] \mid x \in A, y \in B\}$ is non-degenerate. Let $X_1,\dots,X_{n-1}$ be a basis of T_vW^{su} such that $X_1,\dots,X_k$ is a basis of T_vA and that $X_{k+1},\dots,X_{n-1}$ is a basis of the annihilator of T_vB in the dual of T_vW^{ss}, where we identify this dual space with T_vW^{su} via $d\omega$. Let $Y_1,\dots,Y_{n-1}$ be the basis of T_vW^{ss} which is dual to the basis $X_1,\dots,X_{n-1}$; then $Y_1,\dots,Y_k$ is a basis of T_vB. Extend $Y_1,\dots,Y_{n-1}$ by parallel transport to sections of TW^{ss} over $W^{su}(v)$. For every $x\in A$ and every $j\le k$ the vector $Y_j(x)$ is tangent to C and hence $Y_1,\dots,Y_k\mid A$ defines a trivialization of the dual of TA which induces a trivialization of TA. Similarly, by the dual procedure, for each $x\in A$ we obtain a trivialization of $[x,B]$. Using the exponential map of these trivializations we obtain as before symplectic coordinates for C which can be extended to symplectic coordinates for H (at least in a neighborhood of v).

Next we like to understand how the symplectic form $d\omega$ on $\partial\tilde M\times\partial\tilde M-\Delta$ looks like in our coordinates Ψ which are suitable for the study of the cross ratio. To avoid confusion we now denote by ρ the symplectic form $d\omega$ in these special coordinates. Thus ρ is the symplectic form on $D(r_0)$ defined by $\rho(X,Y)=d\omega(d\Psi(X),d\Psi(Y))$. We list a few properties of ρ:

1) ρ coincides with the standard symplectic form ρ_0 at every point in $\{0\}\times B(r_0)\cup B(r_0)\times\{0\}$.

For this recall first of all that $\rho=\rho_0$ at 0 by the definition of Ψ. The restriction to $W^{ss}(v)$ of the Kanai connection ∇ is a flat torsion-free connection on $W^{ss}(v)$ and therefore the exponential map of the restriction of ∇ to $W^{ss}(v)$ commutes with parallel transport.

Thus if we denote by

$$\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_{n-1}},\frac{\partial}{\partial y_1},\dots,\frac{\partial}{\partial y_{n-1}}$$

the trivialization of $T\mathbf{R}^{2n-2}$ given by the coordinate functions, then for every

$$y=(y_1,\dots,y_{n-1})\in\mathbf{R}^{n-1}$$

and every $j\in\{1,\dots,n-1\}$ the image of $\frac{\partial}{\partial y_j}(0,y)$ under $d\Psi$ is just the parallel transport of Y_j to $\Psi(0,y)$.

Similarly, the Kanai connection preserves the restriction of TW^{su} to $W^{ss}(v)$ and induces a flat connection on this bundle. By construction of our coordinates, for every $i\in\{1,\dots,n-1\}$ the image of $\frac{\partial}{\partial x_i}(0,y)$ under $d\Psi$ can be written in the form $X_i(\Psi(0,y))+T$ where $X_i(\Psi(0,y))$ is the image of $X_i=d\Psi\big(\frac{\partial}{\partial x_i}(0,0)\big)$ under parallel transport along $W^{ss}(v)$ and T is a multiple of the geodesic spray. Since $d\omega$ is parallel with respect to ∇ and $d\omega(T,\cdot)=0$ we conclude that $d\omega\big(d\Psi\big(\frac{\partial}{\partial x_i}(0,y)\big),d\Psi\big(\frac{\partial}{\partial y_j}(0,y)\big)\big)=dw(X_i,Y_j)=\delta_{ij}$. In other words, ρ coincides with the standard symplectic form at every point in $\{0\} \times B(r_0)$. Exchanging strong stable and strong unstable manifolds in the above argument then shows that the same is true for ρ on $B(r_0) \times \{0\}$.

2) For $x, y \in B(r_0)$ the submanifolds $\{x\} \times B(r_0)$ and $B(r_0) \times \{y\}$ are Lagrangian with respect to ρ.

By construction, for every $x \in B(r_0)$ the set $\Psi(\{x\} \times B(r_0))$ is contained in a strong stable manifold and therefore in a Lagrangian submanifold for $d\omega$. Similarly, for every $y \in B(r_0)$ the set $\Psi(B(r_0) \times \{y\})$ is contained in an unstable manifold, i.e. a submanifold of $T^1\tilde{M}$ on which $d\omega$ vanishes identically.

By 2) above we can now define the function φ as in the beginning of this section. By Lemma 3.9, φ is just a version of the cross ratio in these coordinates.

3) For every $0 \neq x \in B(r_0)$ the function $y \in B(r_0) \to \varphi(x, y) \in \mathbf{R}$ does not have critical points and its differential at 0 coincides with the differential at 0 of the function $y \to \varphi_0(x, y)$. Here as before, φ_0 is defined by the standard symplectic form ρ_0.

To show 3) recall from Theorem A of [H3] that the cross ratio is a function of class C^1 on the smooth manifold $(\partial\tilde{M} \times \partial\tilde{M} - \Delta)^2$. Since Ψ is a map of class C^1 the function φ is of class C^1 as well. The analysis in [H2] and [H3] (compare the description of φ under 1) above) implies that 0 is the only critical point for φ in $U \times U$.

More precisely, for $0 \neq x \in U$ the restriction of $\varphi(x, \cdot)$ to U does not have critical points. This is a consequence of the following computation: First of all, for fixed $y \in U$ and an arbitary $z \in U$, we have

$$\begin{aligned}\varphi(x, z) = \big[\pi\Psi(x, 0), \pi(v), \pi(-\Psi(0, z)), \pi(-v)\big] \\ = \big[\pi\Psi(x, 0), \pi(v), \pi(-\Psi(0, z)), \pi(-\Psi(0, y))\big] \\ + \big[\pi\Psi(x, 0), \pi(v), \pi(-\Psi(0, y)), \pi(-v)\big] \\ = \varphi(x, y) + \big[\pi\Psi(x, 0), \pi(v), \pi(-\Psi(0, z)), \pi(-\Psi(0, y))\big] .\end{aligned}$$

Now let $\bar{v} \in T^1\tilde{M}$ be the unique vector which is contained in the strong unstable manifold of $\Psi(0, y)$ and such that $\pi(\bar{v}) = \pi(v)$. Then

$$\big[\pi\Psi(x,0),\pi(v),\pi(-\Psi(0,z)),\pi(-\Psi(0,y))\big] = \big[\pi\Psi(x,0),\pi(\bar{v}),\pi(-\Psi(0,z)),\pi(-\bar{v})\big]$$

and therefore we can use our above analysis for the differential of $\varphi(x, \cdot)$ at 0 at the point y after replacing v by $\bar{v}$.

As a consequence, the set $A(x) = \{y \in B(r_0) \mid \varphi(x, y) = 0\}$ is a C^1-hypersurface in $B(r_0)$ whose tangent space at 0 coincides with the tangent space at 0 of the hypersurface $A_0(x) = \{y \in \mathbf{R}^{n-1} \mid \varphi_0(x,y) = 0\}$. Also, for $y \in B(r_0)$ the set $C(y) = \{x \in B(r_0) \mid \varphi(x,y) = 0\}$ is a C^1-hypersurface in $B(r_0)$ whose tangent space at 0 coincides with the tangent space at 0 of the hypersurface $C_0(y) = \{x \in \mathbf{R}^{n-1} \mid \varphi_0(x,y) = 0\}$.

Moreover the above analysis also shows:

4) For every $\epsilon>0$ there is a number $\kappa=\kappa(\epsilon)>0$ such that $|\varphi(x,y)-\varphi_0(x,y)| \leq \epsilon\|x\|\,\|y\|$ for all $x, y \in \mathbf{R}^{n-1}$ with $\|x\| < \kappa$, $\|y\| < \kappa$.

These properties of our function φ in the coordinates Ψ can be used now to show

Lemma 3.11. *If $\partial\tilde{M}$ has a C^1-structure, then for every $\epsilon > 0$ there is a number $\chi = \chi(\epsilon) > 0$ with the following property: For every $v \in T^1\tilde{M}$, every $r > 0$ and every $(1+\chi)$-quasisymplectic r-ball Q with* $\operatorname{diam}(Q) \leq \chi$ *the Lebesgue measure $\lambda(Q)$ of Q is contained in $[(1-\epsilon)a(n-1)^2r^{2n-2}, (1+\epsilon)a(n-1)^2r^{2n-2}]$.*

Proof. The statement of the lemma is local and will be proved by a local argument. Let ρ be a continuous symplectic form on $\mathbf{R}^n \times \mathbf{R}^n$ which satisfies the above properties 1)–4). Define a function φ on $\mathbf{R}^n \times \mathbf{R}^n$ as in 2) above and use the function φ to define the notion of a $(1+\epsilon)$-quasisymplectic embedding $\beta : D(r) \to \mathbf{R}^n \times \mathbf{R}^n$. We want to show by induction on n that there is a constant $k(n) > 0$ only depending on n and for every sufficiently small $\epsilon > 0$ there is a number $\chi = \chi(n,\epsilon) > 0$ such that every $(1+\epsilon)$-quasisymplectic embedding β of $D(r)$ into $(D(\chi),\varphi)$ is a $(1+k(n)\epsilon)$–quasisymplectic embedding as a map of $D(r)$ into $(D(\chi),\varphi_0)$. The statement of the lemma then follows from this and Proposition 3.4.

For the proof of the above statement observe first of all that in the case $n = 1$ there is nothing to show. Thus assume that the claim is known for some $n-1 \geq 1$. Let $\epsilon > 0$ and let $\kappa = \kappa(\epsilon) > 0$ be as in 4) above. Let $\beta = (\beta_1,\beta_2) : D(r) \to (D(\kappa(\epsilon)),\varphi)$ be a $(1+\epsilon)$–quasisymplectic embedding and let x be a point in the boundary $\partial B(r)$ of $B(r)$ such that $\nu_1 = \|\beta_1(x)\|$ is maximal.

Choose a point $y \in \partial B(r)$ such that $\beta_2(y)$ is contained in the half-line through 0 spanned by $\beta_1(x)$; such a point exists since $\beta_2 B(r)$ is a neighborhood of 0. If ν_2 denotes the norm of $\beta_2(y)$ then we have $\varphi(\beta_1(x),\beta_2(y))/\nu_1\nu_2 \geq 1-\epsilon$ by the choice of $\kappa(\epsilon)$ and moreoever $\varphi(\beta_1(y),\beta_2(y)) \geq r^2(1-\epsilon)$ by the definition of a $(1+\epsilon)$-quasisymplectic map.

Since $\varphi(\beta_1(y),\beta_2(y)) \leq (1+\epsilon)\|\beta_1(y)\|\|\beta_2(y)\|$ and $\|\beta_1(y)\| \leq \nu_1$ we conclude that $\nu_2 \geq r^2(1-\epsilon)/\nu_1(1+\epsilon)$ and $\varphi(\beta_1(x),\beta_2(y)) \geq r^2(1-\epsilon)^2/(1+\epsilon)$.

From the definition of a $(1+\epsilon)$-quasisymplectic r-ball we then see that the cosine of the angle between x and y is not larger than $(1-\epsilon)^2/(1+\epsilon)-\epsilon$, and thus the angle is not larger than 5ϵ provided ϵ is sufficiently small.

Now $\mathbf{R}\beta_1(x)$ is transversal to $C_0(\beta_2(y))$, $\mathbf{R}\beta_2(y)$ is transversal to $A_0(\beta_1(x))$ and hence we obtain a natural projection $P : \mathbf{R}^n \times \mathbf{R}^n \to A_0(\beta_2(y)) \times C_0(\beta_1(x))$ defined by $P=(P_1, P_2)$ where $P_1 : \mathbf{R}\beta_1(x) \times C_0(\beta_2(y)) \to C_0(\beta_2(y))$ and $P_2 : \mathbf{R}\beta_2(y) \times C_0(\beta_2(y)) \to C_0(\beta_2(y))$ are the natural projections. By the above, the restriction β' of $P \circ \beta$ to $D(r) \cap C_0(y) \times A_0(x)$ is a $(1+\tau\epsilon)$-quasisymplectic map where $\tau > 0$ is a universal constant. From our induction hypothesis we then obtain that β' is a $(1+\tau'\epsilon)$-quasisymplectic map into $(C_0(\beta_2 y) \times A_0(\beta_1 x), \varphi_0)$ where again $\tau' > 0$ is a universal constant only depending on $n-1$. From this and the fact that $|\varphi(x,y) - \varphi_0(x,y)| \leq \epsilon \|x\| \|y\|$ we then obtain that β is $1+\tau''$-quasisymplectic as a map into $(D(\chi(\epsilon)), \varphi_0)$. □

As an immediate corollary of Proposition 3.8 and Lemma 3.11 we obtain

COROLLARY 3.12. *If the Anosov splitting of TT^1M is of class C^1, then $\mathcal{S} = \lambda$.*

Proof. Assume that the Anosov splitting of TT^1M is of class C^1 and let $\epsilon > 0$. Choose $\chi = \chi(\epsilon) > 0$ as in Lemma 3.11. Then by Lemma 3.11, for every $Q \in \mathcal{Q}(\chi)$ with $\operatorname{diam}(Q) \leq \chi$ we have $\lambda(Q) \leq (1+\epsilon)\delta(Q)^{n-1}a(n-1)^2$ for every $Q \in \mathcal{Q}(\chi)$ where as before λ is the Lebesgue–Liouville current.

From this and the definition of $\mathcal{S}_\nu$ we obtain that $\mathcal{S}_\nu$ is not smaller than $(1+\epsilon)^{-1}\lambda$ for every $\nu \leq \chi$. This then implies that $\mathcal{S} \geq (1+\epsilon)^{-1}\lambda$. But $\epsilon > 0$ was arbitrary as well and therefore $\mathcal{S} \geq \lambda$.

Now by Proposition 3.8 the inequality $\mathcal{S} \leq \lambda$ always holds; together we then obtain the statement of the corollary. □

We want to give a second description of the Lebesgue–Liouville current as a packing measure. For this consider first again an arbitrary closed Riemannian manifold M of negative curvature. As before, let [ ] be the logarithm of the cross ratio of the length cocycle of M on the ideal boundary $\partial\tilde{M}$ of the universal covering $\tilde{M}$ of M.

For $\epsilon > 0$ and an open subset U of $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ write

$$\mathcal{P}_\epsilon(U) = \sup\Big\{ \sum_{i=1}^{\infty} \delta(Q_i)^{n-1} a(n-1)^2 \;\Big|\; Q_i \in \mathcal{Q}(\epsilon)\,, \quad \operatorname{diam}(Q_i) \leq \epsilon\,,\ Q_i \subset U\,,\ Q_i \cap Q_j = \emptyset \Big\}$$

and let $\mathcal{P}(U) = \liminf_{\epsilon\to 0} \mathcal{P}_\epsilon(U)$. If $A \subset \partial\tilde{M} \times \partial\tilde{M} - \Delta$ is any Borel set, then we define $\mathcal{P}(A) = \inf\{\mathcal{P}(U) \mid U \supset A\}$. From the arguments in the proof of Proposition 3.8 and Lemma 3.11 we infer

PROPOSITION 3.13. a) $\lambda \le \mathcal{P}$. b) *If the Anosov splitting of TT^1M is of class C^1 then $\lambda = \mathcal{P} = \mathcal{S}$.*

Proof. By the above, for a given $\epsilon > 0$ there is a number $r(\epsilon) > 0$ such that for every $v \in T^1\tilde{M}$ and every $r \le r(\epsilon)$ the image under the map $E_v = (\pi \circ \exp^v, \pi \circ \mathcal{F} \circ \exp^{ss})$ of the product of the balls of radius r in $\mathcal{V}_v$ and T_vW^{ss} is a $(1+\epsilon)$-quasisymplectic r^2-ball whose Lebesgue–Liouville measure is not larger than $(1+\epsilon)a(n-1)^2r^{2n}$. Since sets of this form define a Vitali relation for the Lebesgue–Liouville measure λ, every open subset U of $\partial\tilde{M}\times\partial\tilde{M}-\Delta$ contains a collection of countably many pairwise disjoint subsets of this kind which cover λ-almost all of U. Thus $\mathcal{P}(U) \ge (1+\epsilon)^{1-2n}\lambda(U)$, and since $\epsilon > 0$ was arbitrary the first part of the lemma follows.

To show the second part of the lemma recall from Lemma 3.11 that if $\partial\tilde{M}$ has a C^1–structure then for sufficiently small $\chi > 0$ the Lebesgue measure of every $(1+\chi)$-quasisymplectic r-ball in $\partial\tilde{M}\times\partial\tilde{M}-\Delta$ of diameter less than χ is not smaller than $(1-\epsilon)a(n-1)^2r^{2n-2}$ where $\epsilon > 0$ is a fixed constant. From this we obtain $\mathcal{P}_\chi \le (1-\epsilon)^{-1}\lambda$. □

COROLLARY 3.14. *Let M, N be compact negatively curved manifolds which have the same marked length spectrum. If the Anosov splitting of TT^1M is of class C^1, then M and N have the same volume.*

REMARK. A more natural way to obtain a current from a positive Hölder class would be to drop in the definition of $\mathcal{S}$ the requirement that the covering family consists of quasisymplectic balls. More precisely, if we define

$$\bar{\mathcal{S}}_\epsilon(A) = \inf\left\{\sum_{i=1}^\infty a(n-1)^2\delta(A_i\times B_i)^{n-1} \;\middle|\; A\subset\cup_{i=1}^\infty A_i\times B_i, \operatorname{diam}(A_i\times B_i)\le\epsilon\right\}$$

and $\bar{\mathcal{S}}(A) = \limsup_{\epsilon\to 0}\bar{\mathcal{S}}_\epsilon(A)$, then clearly $\bar{\mathcal{S}} \le \mathcal{S}$. The Blaschke-Santaló inequality for $\mathbf{R}^n$ indicates that probably $\bar{\mathcal{S}} = \mathcal{S}$ always. The only case where I was able to check this equality is the case of a hyperbolic 3-manifold.

4 Intersection in Higher Dimensions

In this section we consider again a closed negatively curved manifold M. As before, we denote by $\partial\tilde{M}$ the ideal boundary of the universal covering $\tilde{M}$ of M.

The Riemannian metric g on M lifts to a Riemannian metric g^s on the leaves of W^s. This metric defines a family of Lebesgue measures λ^{ss} on the leaves of W^{ss}. The measures λ^{ss} are quasi-invariant under the action of the geodesic flow and they transform via

$$\frac{d}{dt}\lambda^{ss}\circ\Phi^t\Big|_{t=0}\equiv -\mathrm{tr}\,U\,.$$

Here for every $v\in T^1M$, $\mathrm{tr}\,U(v)$ is just the trace of the second fundamental operator $U(v)$ at Pv of the horosphere $PW^{ss}(v)$.

The function $v\to\mathrm{tr}\,U(v)$ is Hölder continuous, positive and of pressure zero and consequently it defines a positive normalized Hölder cohomology class which does not depend on the particular choice of the family of smooth measures on strong unstable manifolds. Its Gibbs equilibrium state is just the Lebesgue–Liouville measure λ on T^1M. We call the class defined by $\mathrm{tr}U$ the *Lebesgue-class*.

Similarly, the image of λ^{ss} under the flip $\mathcal{F}$ is a family λ^{su} of Lebesgue measures on strong unstable manifolds. If we view $U(v)$ as a linear automorphism of the orthogonal complement $v^\perp$ of v, then $f(v)=\det(U(v)+U(-v))$ is a positive Hölder continuous function on T^1M, and $\lambda=dt\times d\lambda^{ss}\times f d\lambda^{su}$ where dt is the 1-dimensional Lebesgue measure on the flow-lines of the geodesic flow (see Lemma 2.7 of [H3]).

The measure λ is invariant under the flip $\mathcal{F}\colon v\to -v$ and hence the cocycle ζ defined by $\mathrm{tr}\,U$ is equivalent to the cocycle $\zeta\circ\mathcal{F}$ defined by $(\mathrm{tr}\,U)\circ\mathcal{F}$ where an equivalence is given by the function $\log f$.

Following [H5] (see also the discussion after Lemma 2.2 in this paper), we use the cocycle $\zeta\circ\mathcal{F}$ to define a Hölder continuous kernel $k\colon\tilde{M}\times\tilde{M}\times\partial\tilde{M}\to\mathbf{R}$. Since every weak stable manifold $W^s(v)$ is smoothly foliated by strong stable manifolds $W^{ss}(\Phi^t v)$ ($t\in\mathbf{R}$), the restriction of $\mathrm{tr}\,U$ to every weak stable manifold is smooth. Now for $v\in T^1\tilde{M}$ and $w\in W^{ss}(v)$ we have $k(Pv,Pw,\pi(v))=\lim_{t\to\infty}\int_0^t\mathrm{tr}\,U(\Phi^s v)ds-\int_0^t\mathrm{tr}\,U(\Phi^s w)ds$ and $k(Pv,P\Phi^t v,\pi(v))=\int_0^t\mathrm{tr}\,U(\Phi^s v)ds$ ($t>0$) and this shows that for every fixed $\xi\subset\partial\tilde{M}$ the function $(x,y)\in\tilde{M}\times\tilde{M}\to k(x,y,\xi)$ is smooth. Moreover $k(x,x,\xi)=0$, and if $v\in T^1_x\tilde{M}$ is such that $\pi(v)=\xi$, then there is a smooth function $\beta\colon W^{ss}(v)\to\mathbf{R}$ such that $k(x,\cdot,\xi)^{-1}(0)=\{P\Phi^{\beta(w)}w\mid w\in W^{ss}(v)\}$.

From the kernel k in turn we obtain the Lebesgue cross ratio (see [H5]). For this fix a point $x \in \tilde{M}$ and for $v \neq w \in T_x^1\tilde{M}$ choose a geodesic γ in $\tilde{M}$ joining $\gamma(-\infty) = \pi(w)$ to $\gamma(\infty) = \pi(v)$ and write $\alpha(v,w) = \frac{1}{2}(\log f(\gamma'(0)) - k(\gamma(0), x, \pi(v)) - k(\gamma(0), x, \pi(w)))$. Then $\alpha(v,w)$ does not depend on the parametrization of γ. Moreover by Lemma 1.2 of [H5] and the above considerations, for fixed $\xi \neq \eta \in \partial\tilde{M}$ the function $x \in \tilde{M} \to \alpha(\pi^{-1}(\xi) \cap T_x^1\tilde{M}, \pi^{-1}(\eta) \cap T_x^1\tilde{M})$ is smooth.

The *Lebesgue-cross ratio* is then the function [ ] on the space of quadruples of pairwise distinct points in $\partial\tilde{M}$ defined as follows: Choose $x \in \tilde{M}$, let $v_i \in T_x^1\tilde{M}$ be such that $\pi(v_i) = \xi_i$ and write $Cr(\xi_1, \xi_2, \xi_3, \xi_4) = \exp(\alpha(v_1, v_3) + \alpha(v_2, v_4) - \alpha(v_1, v_4) - \alpha(v_2, v_3))$. This function admits a continuous extension to the space of quadruples $(\xi_1, \xi_2, \xi_3, \xi_4)$ with $\xi_1 \neq \xi_3$ and $\xi_2 \neq \xi_4$ and hence can be viewed as a function on the space of pairs of oriented geodesics in $\tilde{M}$, where we identify $(\xi_1, \xi_2, \xi_3, \xi_4)$ with the pair (γ_1, γ_2) of geodesics with endpoints $\gamma_1(\infty) = \xi_1, \gamma_1(-\infty) = \xi_3, \gamma_2(\infty) = \xi_2$ and $\gamma_2(-\infty) = \xi_4$. Since the space $\mathcal{G}\tilde{M}$ of geodesics in $\tilde{M}$ is naturally a smooth manifold (via the identification $\mathcal{G}\tilde{M} = T^1\tilde{M}/\mathbf{R}$ where $\mathbf{R}$ acts as the geodesic flow), the cross ratio is therefore a function on a smooth manifold. Notice that the Lebesgue cross ratio is in general different of the cross ratio which we used in section 3.

In the next lemma we investigate the regularity of $[\,] = \log Cr$.

Lemma 4.1. *If the Anosov splitting of TT^1M is of class C^k for some $k \geq 1$, then [ ] is of class C^k.*

Proof. If the Anosov splitting is of class C^k, then the functions $\operatorname{tr} U$ and $f\colon v \to \det(U(v) + U(-v))$ are of class C^k and $\partial\tilde{M}$ admits a C^k-structure in the sense of [H3]. Then the kernel $k\colon \tilde{M} \times \tilde{M} \times \partial\tilde{M} \to \mathbf{R}$ is of class C^k as well and the same is true for the function α. From this and the definition of [ ] the lemma follows. □

The following proposition is an analogue of Corollary 2.13 of [H3]:

Proposition 4.2. *If [ ] is of class C^k for some $k \geq 1$, then $\partial\tilde{M}$ admits a C^k-structure.*

Proof. Define a cocycle $\beta(v,t)$ for the geodesic flow on T^1M and $T^1\tilde{M}$ by $\beta(v, \zeta(v,t)) = t$. Then $\Psi^t(v) = \Phi^{\beta(v,t)}v$ defines a Hölder continuous times change for Φ^t which is smooth along the leaves of the stable foliation. The resulting flow is Anosov in the following sense:

1) For every $v \in T^1M$ the set $\bar{W}^{ss}(v) = \{w \in T^1M \mid d(\Psi^t v, \Psi^t w) \to 0 (t \to \infty)\}$ is a smoothly embedded submanifold of $W^s(v)$ which can

be realized as the graph of a smooth function on $W^{ss}(v)$.

2) For every $v \in T^1M$ the set $\bar{W}^{su}(v) = \{w \in T^1M \mid d(\Psi^t v, \Psi^t w) \to 0(t \to -\infty)\}$ is a Hölder submanifold of $W^u(v)$ which can be realized as the graph of a Hölder continuous function on $W^{su}(v)$.

Now let A_1, A_2 be open, relative compact subsets of $\partial\tilde{M}$ whose closures do not intersect. Write $\Omega = \{v \in T^1\tilde{M} \mid \pi(v) \in A_1, \pi(-v) \in A_2\}$. By 1) and 2) above, for all $v, w \in \Omega$ the intersection $\bar{W}^{ss}(v) \cap W^u(w)$ consists of a unique point $q(v,w)$. Let $\bar{\sigma}(v,w) \in \mathbf{R}$ be the unique number $\tau \in \mathbf{R}$ such that $q(v,w) \in \Psi^\tau \bar{W}^{su}(w)$. Since $q(v, \Psi^t w) = q(v,w)$ and $q(\Psi^t v, w) = \Psi^t q(v,w)$ we have $\bar{\sigma}(\Psi^t v, w) = \bar{\sigma}(v,w) + t, \bar{\sigma}(v, \Psi^t w) = \bar{\sigma}(v,w) - t$ and $\bar{\sigma}(v,w) + \bar{\sigma}(w,v) = [\pi(v), \pi(w), \pi(-v), \pi(-w)]$ (this formula is just the same as the formula for the usual length cocycle).

For a fixed point $v \in \Omega$ and $w_1 \in \bar{W}^{ss}(v)$, $w_2 \in \bar{W}^{su}(v)$ we have $\bar{\sigma}(w_1, w_2)=0$ and consequently $[\pi(w_1), \pi(w_2), \pi(-w_1), \pi(-w_2)]=\bar{\sigma}(w_2, w_1)$.

Assume from now on that the cross ratio is a function of class C^k on $\mathcal{G}\tilde{M} \times \mathcal{G}\tilde{M}$ for some $k \geq 1$.

We follow the proof of Corollary 2.13 of [H3] and use the cross ratio [ ] to construct C^k-coordinates for the manifolds $\bar{W}^{ss}$.

First, let $U \subset T^1\tilde{M}$ be a nontrivial open subset. For $v, w \in T^1\tilde{M}$ write $\Sigma(v,w) = \bar{\sigma}(v,w) + \bar{\sigma}(w,v) = [\pi(v), \pi(w), \pi(-v), \pi(-w)]$. We determine inductively points

$$w_1, \dots, w_i \in T^1\tilde{M} \qquad (1 \leq i \leq n-1)$$

and nontrivial open sets $B_1 \supset \cdots \supset B_i$ of U such that for every $w \in B_i$ and every $j \leq i$ the following is satisfied:

i) The restriction of $f_j = \Sigma(\cdot, w_j)$ to $B_i \cap \bar{W}^{ss}(w)$ does not have critical points.

ii) $\cap_{j=1}^i f_j^{-1}(f_j(w)) \cap \bar{W}^{ss}(w) \cap B_i$ is a C^k-embedded connected submanifold of $\bar{W}^{ss}(w) \cap B_i$ of codimension i.

Let $0 \leq i \leq n-1$ and assume that B_i and $w_1, \dots, w_i$ are already determined. Let $w \in B_i$ and let $c\colon (-\epsilon, \epsilon) \to \cap_{j=1}^i f_j^{-1}(f_j(w)) \cap \bar{W}^{ss}(w) \cap B_i$ be an embedded curve of class C^k through $c(0) = w$ with nowhere vanishing tangent. Write

$$\xi = \pi(w)\,, \quad \eta_1 = \pi(-w)\,, \quad \eta_2 = \pi\big(-c(\epsilon/2)\big)\,;$$

the points $\xi, \eta_1, \eta_2 \in \partial\tilde{M}$ are pairwise distinct. We claim that there are $\beta_1 \neq \beta_2 \in \partial\tilde{M} - \{\xi, \eta_1, \eta_2\}$ such that $[\xi, \beta_1, \eta_1, \beta_2] \neq [\xi, \beta_1, \eta_2, \beta_2]$. For otherwise we obtain from continuity as $\beta_1 \to \eta_1$ that $[\xi, \eta_1, \eta_2, \beta] = 0$ for all

$\beta \in \partial\tilde{M}-\{\xi,\eta_1,\eta_2\}$. Then also $[\xi,\eta_1,\beta_1,\beta_2]=[\xi,\eta_1,\beta_1,\eta_2]+[\xi,\eta_1,\eta_2,\beta_2]=0$ for all β_1,β_2 which is by the explicit construction of $[\,,]$ impossible (compare the proof of Lemma 3.6).

Choose $w_{i+1} \in T^1\tilde{M}$ in such a way that

$$[\xi,\pi(w_{i+1}),\eta_1,\pi(-w_{i+1})] \neq [\xi,\pi(w_{i+1}),\eta_2,\pi(-w_{i+1})]\,.$$

Then $\Sigma(w,w_{i+1}) \neq \Sigma(c(\epsilon/2),w_{i+1})$, and since the function $f_{i+1}\colon z \to \Sigma(z,w_{i+1})$ is of class C^k it can be used as a coordinate function for the leaves of $\bar{W}^{ss}$ on an nontrivial open subset B_{i+1} of B_i (compare the proof of Lemma 2.12 in [H3]).

The arguments in the proof of Corollary 2.13 of [H3] (which only use the cocycle identities) now show that canonical maps for the strong stable manifolds $\bar{W}^{ss}$ for Ψ^t are of class C^k. Since for every $v \in T^1\tilde{M}$ and every $w \in W^s(v)$ the leaf of $\bar{W}^{ss}$ containing w a is smooth submanifold of $W^s(v)$ we conclude as in the proof of Corollary 2.13 of [H3] that canonical maps for the strong stable manifolds W^{ss} for Φ^t are of class C^k as well. This is equivalent to the statement of the proposition. □

Remark. As a conclusion of Proposition 4.2, if the Lebesgue cross ratio is of class C^k for some $k \geq 1$ then the strong stable foliation of T^1M is of class C^k in the sense of [H3] and in particular the function $\operatorname{tr} U$ is of class C^{k-1}. On the other hand, Yue observed that the arguments of Foulon and Labourie in [FoL] show that the stable foliation is smooth if this is true for the function $\operatorname{tr} U$. This raises the following questions:

1) If the bundle TW^{ss} if of class C^k, then the strong stable foliation of T^1M is of class C^k in the sense of [H3]. Is the reverse also true?
2) Is the Lebesgue-cross ratio of class C^k if and only if the function $\operatorname{tr} U$ is of class C^k?

Now let M,N be closed negatively curved manifolds with isomorphic fundamental groups $\pi_1(M)=\pi_1(N)=\Gamma$. Recall that every isomorphism of $\pi_1(M)$ onto $\pi_1(N)$ induces a Γ-equivariant homeomorphism $f\colon \partial\tilde{M} \to \partial\tilde{N}$. The Lebesgue–Liouville measure on T^1M (or T^1N) induces a Γ-invariant measure class λ_M (or λ_N) on $\partial\tilde{M}$ (or ∂N), and similarly the Bowen Margulis measure induces an invariant measure class μ_M (or μ_N). As an immediate consequence of Proposition 4.2 we obtain:

Corollary 4.3. *Assume that the Anosov splittings of T^1M and T^1N are of class C^1. Then every Γ-equivariant homeomorphism $f\colon \partial\tilde{M} \to \partial\tilde{N}$ which satisfies $f\{\lambda_M,\mu_M\} \cap \{\lambda_N,\mu_N\} \neq \emptyset$ is a C^1-diffeomorphism.*

Proof. Assume for example that $f\lambda_M = \lambda_N$. Then f^4 maps the Lebesgue–Liouville cross ratio on $\partial\tilde{M}$ to the Lebesgue–Liouville cross ratio on $\partial\tilde{N}$ up to a constant factor. Since the C^1-structures on $\partial\tilde{M}$ and $\partial\tilde{N}$ are determined by coordinate functions constructed from the cross ratios (compare the proof of Proposition 4.2) we conclude that f is a C^1-diffeomorphism. □

Recall that an *orbit-equivalence* of the geodesic flows on T^1M and T^1N is a homeomorphism $\Lambda\colon T^1M \to T^1N$ which maps every orbit of the geodesic flow on T^1M order preserving to an orbit of the geodesic flow on T^1N. Thus for every $v \in T^1M$ there is a homeomorphism $\alpha(v,\cdot)\colon \mathbf{R} \to \mathbf{R}$ such that $\Lambda(\Phi^t v) = \Phi^{\alpha(v,t)}\Lambda(v)$ for all $t \in \mathbf{R}$. If M and N have isomorphic fundamental groups, then their geodesic flows are orbit equivalent, moreover the orbit equivalence can be chosen to be smooth along the flow lines of the geodesic flow.

COROLLARY 4.4. *Under the hypothesis of Corollary 4.3 there is an orbit equivalence $\Lambda\colon T^1M \to T^1N$ of the geodesic flows which is a C^1-diffeomorphism.*

Proof. Assume that the hypothesis of Corollary 4.3 is satisfied; then $f\lambda_M = \lambda_N$ (after normalization). Let λ_M^{su} (or λ_N^{su}) be the family of Lebesgue measures on strong unstable manifolds which is induced from the lift of the Riemannian metric on M (or N). For $v \in T^1\tilde{M}$ let $\Lambda(v)$ be the unique vector in $T^1\tilde{N}$ such that $f(\pi(v)) = \pi(\Lambda(v))$, $f(\pi(-v)) = \pi(-\Lambda(v))$ and such that the Jacobian of f at $\pi(v)$ with respect to the projections of $\lambda_M^{su}\mid_{W^{su}(v)}$ and $\lambda_N^{su}\mid_{W^{su}(\Lambda(v))}$ equals 1. Then $v \to \Lambda(v)$ is a Γ-equivariant map which projects to an orbit equivalence of the geodesic flows on T^1M and T^1N.

We claim that Λ is in fact a C^1-diffeomorphism. To see this, recall that by our assumption the function $\operatorname{tr} U$ on T^1M and T^1N is of class C^1 and hence for every $v \in T^1\tilde{M}$ the set

$$\bar{W}^{ss}(v) = \bigl\{w \in W^s(v) \mid d(\pi\lambda_M^{su}|_{W^{su}(v)})/d(\pi\lambda_M^{su}|_{W^{su}(w)})(\pi v) = 1\bigr\}$$

is a C^1-submanifold of $W^s(v)$. The restriction of $\pi \circ \mathcal{F}$ to $\bar{W}^{ss}(v)$ is a C^1-diffeomorphism of $\bar{W}^{ss}(v)$ onto $\partial\tilde{M} - \pi(v)$. But $\Lambda|_{\bar{W}^{ss}(v)} = (\pi \circ \mathcal{F}|_{\bar{W}^{ss}(\Lambda v)})^{-1} \circ f \circ (\pi \circ \mathcal{F}|_{\bar{W}^{ss}(v)})$ and hence Λ maps every stable manifold in T^1M C^1-diffeomorphically onto a stable manifold in T^1N. Similarly we also see that Λ maps every unstable manifold in T^1M C^1-diffeomorphically onto an unstable manifold in T^1N. In short, Λ is a C^1-diffeomorphism. □

Recall that the geodesic flows on T^1M and T^1N are *homothetic* if there is a homeomorphism $\Lambda\colon T^1M \to T^1N$ and a number $a > 0$ such that $\Lambda(\Phi^t v) = \Phi^{at}(\Lambda v)$ for all $v \in T^1M$ and all $t \in \mathbf{R}$. In other words, the metric on N can be rescaled in such a way that after this rescaling the geodesic flows on T^1M and T^1N are time preserving conjugate.

Next we observe:

LEMMA 4.5. *Let M, N be compact negatively curved manifolds. Assume that the Anosov splitting of T^1M and T^1N is of class C^1. If the geodesic flows on T^1M and T^1N are orbit equivalent with an orbit equivalence of class C^1, then they are homothetic.*

Proof. By our assumption, there is a $\Gamma = \pi_1(M) = \pi_1(N)$-equivariant diffeomorphism $f\colon \partial\tilde{M} \to \partial\tilde{N}$ of class C^1, and $f\times f$ is a C^1-diffeomorphism of the space $\mathcal{G}\tilde{M}$ of geodesics in $\tilde{M}$ onto the space $\mathcal{G}\tilde{N}$ of geodesics in $\tilde{N}$.

The symplectic form $d\omega_0$ on $T^1\tilde{N}$ is invariant under the geodesic flow and projects to a smooth symplectic form η_0 on $\mathcal{G}\tilde{N}$. Then $(f \times f)^*\eta_0$ is a continuous, $\pi_1(M)$-invariant non-degenerate 2-form on $\mathcal{G}\tilde{M}$ which can naturally be pulled back to a continuous, $\pi_1(M)$ invariant 2-form η on $T^1\tilde{M}$ which is invariant under the geodesic flow.

Now if $\Lambda\colon T^1M \to T^1N$ is an orbit equivalence of class C^1 induced as above by f, then naturally $\eta = \Lambda^* d\omega_0$ and hence the continuous form η is exact in the sense of distributions. By Theorem A of [H4], this means that necessarily $\eta = r d\omega$ for some $r \neq 0$, where $d\omega$ is the symplectic form on T^1M. The discussion in section 3 of this paper then shows that $f \times f$ preserves the (usual) cross ratio on $\partial\tilde{M}$ and $\partial\tilde{N}$ up to a constant factor and hence the geodesic flows on T^1M and T^1N are homothetic. □

As a corollary, we obtain Theorem C from the introduction:

COROLLARY 4.6. *Let M, N be closed negatively curved manifolds with isomorphic fundamental groups and Anosov splitting of class C^1. Then the natural boundary map $\partial\tilde{M} \to \partial\tilde{N}$ preserves the Lebesgue measure classes on $\partial\tilde{M}$ and $\partial\tilde{N}$ if and only if the geodesics flows on T^1M and T^1N are homothetic.*

Let L be the set of length cocycles of smooth metrics of negative curvature on M with Anosov splitting of class C^1. By Corollary 4.6 there is an injective map β of L into the space of geodesic currents $\mathcal{C}$ of M which associates to a length cocycle α of a metric g the Lebesgue–Liouville current of g. For every $\ell \in L$ and every geodesic current γ on M we then can define the *intersection* between γ and $\beta(\ell)$ by $i(\gamma, \beta(\ell)) = 2\int \ell\, d\gamma$.

Notice that this definition is an extension of the definition in the 2-dimensional case, however i is clearly not symmetric, i.e. in general we have $i(\beta(\ell), \beta(\ell')) \neq i(\beta(\ell'), \beta(\ell))$ for $\ell, \ell' \in L$.

We conjecture that this intersection can be extended to a continuous function on the product of the space of currents on M with the subspace of Gibbs currents, in particular Corollary 4.6 should hold without any regularity assumptions.

Acknowledgement. I am grateful to the referee for carefully reading the manuscript and for his many suggestions to improve the overall readability.

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Submitted: October 1997
Revised version: July 1998