ERGODIC PROPERTIES OF FUNCTION GROUPS

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Abstract. Let $\Gamma \subset PSL(2,\mathbb{C})$ be a torsion free function group. We discuss some aspects of the asymptotic relation between the geometry of the hyperbolic manifold $\mathbf{H}^3/\Gamma$ and the geometry of the boundary of its convex core. This leads to a characterization of fuchsian function groups by dynamical invariants.

1. introduction

A *function group* is a finitely generated discrete subgroup Γ of the isometry group $PSL(2,\mathbb{C})$ of hyperbolic 3-space $\mathbf{H}^3$ whose *domain of discontinuity* $\Omega \subset S^2$ contains a Γ-invariant component. Here by definition, the domain of discontinuity of Γ is the complement of the limit set $\Lambda \subset S^2$. We always assume that our function group is torsion free and not elementary, i.e. that its limit set consists of more than two points.

The *closed convex hull* $\mathrm{Conv}(\Lambda)$ of the limit set Λ is the smallest closed convex subset of $\mathbf{H}^3$ which contains all geodesics with both endpoints in the limit set. This set is Γ-invariant and projects to the *convex core* $\mathcal{C}(M)$ of the quotient manifold $M = \mathbf{H}^3/\Gamma$. The group Γ is called *convex cocompact* if the convex core $\mathcal{C}(M)$ of M is compact.

In this note we look at asymptotic invariants of a convex cocompact function group Γ. A well known such invariant is the *critical exponent* δ which is defined to be the infimum of all numbers $s > 0$ such that the series

$$\sum_{\Psi \in \Gamma} e^{-s \ \mathrm{dist}(x,\Psi x)}$$

converges, where dist denotes the distance function on $\mathbf{H}^3$ of the hyperbolic metric and x is an arbitrary point in $\mathbf{H}^3$. The δ-dimensional Hausdorff measure of Λ with respect to the spherical distance on S^2 is finite and positive [S2].

A particularly well known class of convex cocompact function groups are those *quasi-fuchsian groups* which are isomorphic to a torsion free cocompact lattice $\hat{\Gamma}$ in $PSL(2,\mathbb{R})$. The space $\mathcal{A}_0(\hat{\Gamma})$ of discrete faithful representations of $\hat{\Gamma}$ into $PSL(2,\mathbb{C})$ will be equipped

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with the *algebraic topology*. Thus a sequence of representations ρ_j converges to ρ in $\mathcal{A}_0(\hat{\Gamma})$ if and only if for every $\Psi \in \hat{\Gamma}$ the sequence $\rho_j(\Psi)$ converge in $PSL(2,\mathbb{C})$ to $\rho(\Psi)$. We identify two representations in $\mathcal{A}_0(\hat{\Gamma})$ if they are conjugate in $PSL(2,\mathbb{C})$, and we equip the quotient space $\mathcal{A}(\hat{\Gamma})$ with the quotient topology.

The group $\hat{\Gamma}$ acts as a group of homeomorphisms on the circle $S^1 = \partial\mathbf{H}^2$ at infinity of the hyperbolic plane. A *geodesic current* for $\hat{\Gamma}$ is a locally finite measure on the *space of geodesics* $S^1 \times S^1$ − diagonal of $\mathbf{H}^2$ which is invariant under the diagonal action of $\hat{\Gamma}$ and under the change of factors [B]. We denote by $\mathcal{PC}$ the projectivized space of geodesic currents, equipped with the weak*-topology.

Teichmüller space of marked hyperbolic metrics on the topological surface $\mathbf{H}^2/\hat{\Gamma}$ naturally embeds into $\mathcal{PC}$ [B]. We call this embedding the *fuchsian embedding*. It extends continuously to the Thurston boundary of Teichmüller space. We can view the fuchsian embedding as a continuous map from a subset of $\mathcal{A}(\hat{\Gamma})$ into $\mathcal{PC}$. We show.

Theorem 1. (1) *There is a natural continuous embedding Ψ of $\mathcal{A}(\hat{\Gamma})$ into the space $\mathcal{PC}$ of projective currents of $\hat{\Gamma}$ which extends the fuchsian embedding.*
(2) *Let $\rho \in \mathcal{A}(\hat{\Gamma})$ and let $f : \mathbf{H}^2/\hat{\Gamma} \to \mathbf{H}^3/\rho(\hat{\Gamma})$ be a continuous map which induces ρ. Denote by $F : \mathbf{H}^2 \to \mathbf{H}^3$ the $\hat{\Gamma} - \Gamma$-equivariant lift of f. Let λ be the Lebesgue Liouville measure on the unit tangent bundle $T^1\mathbf{H}^2$ of $\mathbf{H}^2$ and for $v \in T^1\mathbf{H}^2$ let γ_v be the geodesic in $\mathbf{H}^2$ with initial velocity v. Then for λ-almost every $v \in T^1\mathbf{H}^2$ the limit*
$$\lim_{t\to\infty} \frac{1}{t}\,\mathrm{dist}(F\gamma_v(0), F\gamma_v(t))$$
exists. Its value is independent of v and not smaller than $\frac{1}{\delta}$, with equality if and only if Γ is conjugate to $\hat{\Gamma} \subset PSL(2,\mathbb{R})$.

To describe the image of the embedding from the first part of our theorem, recall that every positive Hölder continuous function f on the unit tangent bundle T^1S of the surface S defines a unique *Gibbs equilibrium state*. If f is invariant under the *flip* $v \to -v$, then its Gibbs state defines a geodesic current. We call the currents thus obtained the *Gibbs currents* (see [H2]). The image of our embedding is then a subset of the set of projective Gibbs currents.

A partial result in direction of the inequality stated in part 2) of our theorem is the main result of the paper [BT] of Bridgeman and Taylor. The discussion of equality answers one of the questions raised in their paper in the negative.

Let now Γ be an arbitrary torsion free convex cocompact function group with invariant component Ω_0 of its domain of discontinuity. Then Γ acts freely on Ω_0 and the quotient $S = \Omega_0/\Gamma$ is a compact surface S of genus $g \geq 2$. The fundamental group G of Ω_0 is a normal subgroup of the fundamental group $\pi_1(S)$ of the surface S, and we have $\Gamma = \pi_1(S)/G$.

The convex hull of Λ has a distinguished boundary component C which is the image of Ω_0 under the nearest-point retraction of the boundary sphere S^2 into $\mathrm{Conv}(\Lambda)$. The set C with its path-metric induced by the hyperbolic structure on $\mathbf{H}^3$ is path-isometric to the planar surface Ω_0 equipped with some complete hyperbolic Γ-invariant metric. This metric projects to a hyperbolic metric on the compact surface $S = C/\Gamma$ (see [MT]).

There is also a critical exponent χ for the action of Γ on C. Namely, if we denote by dist_C the distance function on C of the above intrinsic complete hyperbolic metric, then this exponent is defined to be the infimum of all numbers $s > 0$ such that the series $\sum_{\Psi\in\Gamma} e^{-s\,\mathrm{dist}_C(x,\Psi x)}$ converges. The critical exponent χ is contained in $[0,1]$ and is not bigger than the critical exponent δ of Γ. We can relate these critical exponents χ, δ to the geometry of the quotient manifold $\mathbf{H}^3/\Gamma$ as follows.

Theorem 2. *Let Γ be a convex cocompact function group with critical exponent δ. Let C be the distinguished component of the boundary of $\mathrm{Conv}(\Lambda)$ and let χ be the critical exponent for the action of Γ on C. Let $\xi \in \Lambda$ be a typical point for the δ-dimensional Hausdorff measure on Λ and let $\gamma : [0,\infty) \to C$ be a geodesic ray with respect to the intrinsic metric which converges to ξ. Then the limit*

$$\lim_{t\to\infty} \frac{1}{t}\,\mathrm{dist}(\gamma(0),\gamma(t))$$

exists and does not depend on ξ. Its value is not bigger than $\frac{\chi}{\delta}$, with equality if and only if Γ is conjugate to a subgroup of $PSL(2,\mathbb{R})$.

A *conformal density* of dimension $d > 0$ for a Kleinian group Γ is a family $\{\nu^x\}$ $(x \in \mathbf{H}^3)$ of finite Borel measures on S^2 with the following properties.

(1) The measures ν^x are equivariant under the action of Γ and define all the same measure class.
(2) For $\xi \in S^2$ and $y, x \in \mathbf{H}^3$ we have $\frac{d\nu^y}{d\nu^x}(\xi) = e^{d(\theta(x)-\theta(y))}$, where θ is a Busemann function at ξ.

A conformal density of dimension d for Γ exists if and only if $d \geq \delta$ [S2]. If Γ is convex cocompact, then a conformal density $\{\nu^x\}$ of dimension δ is unique up to a universal constant and is supported on Λ. If we denote for $x \in \mathbf{H}^3$ by d_x the *visual distance* on $S^2 = \partial\mathbf{H}^3$ as seen from x, i.e. the projection to S^2 of the distance function of the round metric on the unit tangent sphere $T^1_x\mathbf{H}^3$ at x, then after renormalization the measure ν^x is just the δ-dimensional Hausdorff measure of Λ with respect to d_x.

Our last result shows that every conformal density for a convex cocompact function group Γ behaves near the limit set Λ like a Hausdorff measure with respect to the visual distance. For a precise statement of this fact, denote for $x \in \mathbf{H}^3$ and $\xi \in S^2, r > 0$ by $B_x(\xi, r)$ the ball of radius r about ξ with respect to the distance d_x. We show.

Theorem 3. *Let Γ be a convex cocompact function group with invariant component Ω_0 and limit set Λ. There is a constant $c > 0$ depending on Γ such that for every conformal density $\{\nu^y\}$ of dimension $d \leq 2$ for Γ which is supported on $\Omega_0 \cup \Lambda$, every $\xi \in \Lambda$, $x \in \mathrm{Conv}(\Lambda)$ and $r < 1$ we have*

$$c^{-1}\nu^x(S^2) \leq \nu^x B_x(\xi, r) r^{-d} \leq c\nu^x(S^2).$$

Since the restrictions to Ω_0 of the Lebesgue measures λ^x on the fibres $T^1_x\mathbf{H}^3$ project to a conformal density for Γ of dimension 2 supported on Ω_0, our theorem implies in particular that $\lambda^x B_x(\xi, r) \cap \Omega_0 \geq c^{-1}r^2$ for every $x \in \mathbf{H}^3$, every $\xi \in \Lambda$ and every $r < 1$.

2. Function groups and Gibbs measures

This section is devoted to the proof of Theorem 2. We use the notations from the introduction. To begin with, we consider an arbitrary torsion free convex cocompact Kleinian group Γ. The next lemma is well known (compare [BH]).

Lemma 2.1. *Let $\Gamma \subset PSL(2, \mathbb{C})$ be a torsion free convex cocompact group with limit set Λ. Then $\mathrm{Conv}(\Lambda)$ is Γ-equivariantly quasiisometric to a Cayley graph of Γ.*

Proof: The group Γ acts freely and cocompactly on $\mathrm{Conv}(\Lambda)$ as a discrete group of isometries. Fix a point $x \in \mathrm{Conv}(\Lambda)$. Then the Γ-orbit of x can naturally be identified with the group Γ itself. It suffices to show that this orbit with the restriction of the distance function of $\mathbf{H}^3$ is bilipschitz equivalent to the group Γ equipped with the word metric with respect to a suitable choice of a finite generating set.

For this let D be the closed Dirichlet fundamental polyhedron with respect to x for the action of Γ on $\mathrm{Conv}(\Lambda)$. Since $\mathrm{Conv}(\Lambda)$ is convex, D is convex and compact. Let $d > 0$ be the diameter of D and let $G = \{\Psi \in \Gamma \mid \mathrm{dist}(x, \Psi x) \leq 3d\}$. If $\Psi \in \Gamma$ is such that D and ΨD intersect, then $\Psi \in G$ and therefore G is a finite generating subset of Γ. We equip Γ with the word metric with respect to this particular generator.

Let $\mathrm{Id} \neq \Psi \in \Gamma$. Since $\mathrm{Conv}(\Lambda)$ is convex, the unique geodesic γ in $\mathbf{H}^3$ which joins x to Ψx is contained in $\mathrm{Conv}(\Lambda)$. Thus if $\ell \geq 1$ is such that $\mathrm{dist}(x, \Psi x) \leq \ell d < \mathrm{dist}(x, \Psi x) + d$ then there are isometries $\Psi_1, \dots, \Psi_\ell$ such that $\Psi_\ell = \Psi$ and that $\mathrm{dist}(\gamma(jd), \Psi_j(x)) \leq d$ for $1 \leq j \leq \ell$. Then $\mathrm{dist}(\Psi_j x, \Psi_{j-1}x) = \mathrm{dist}(x, \Psi_j^{-1}\Psi_{j-1}x) \leq 3d$ for all j and therefore the distance of Ψ to the identity in our word metric is not bigger than ℓ. This shows that the natural map from our orbit to the group Γ is Lipschitz.

Vice versa, for every reduced word in the generating set G for Γ we can find a piecewise geodesic which passes through the images of the fundamental domain D under those elements of Γ which are determined by this word and whose length is not bigger than $3d$-times the length of the word. From this the lemma follows. $\square$

A *geodesic ray* in a Gromov hyperbolic geodesic metric space X emanating from a point $x \in X$ is a globally minimizing geodesic $\gamma : [0, \infty) \to X$. Two such rays are *equivalent* if their images in X have bounded distance in the Hausdorff metric. The set of equivalence classes of geodesic rays is the *Gromov boundary* ∂X of X.

Corollary 2.2. *The group* Γ *is Gromov-hyperbolic, and its Gromov boundary naturally coincides with the limit set* Λ *of* Γ.

Proof: Since $\mathrm{Conv}(\Lambda)$ is a convex subset of $\mathbf{H}^3$, with the restriction of the distance function of $\mathbf{H}^3$ it is a Gromov hyperbolic geodesic metric space. By Lemma 2.1 a Cayley graph of Γ is equivariantly quasiisometric to $\mathrm{Conv}(\Lambda)$ and therefore this Cayley graph is Gromov hyperbolic as well (see [BH]).

Now for every limit point $\xi \in \Lambda$ and every $x \in \mathrm{Conv}(\Lambda)$ there is a geodesic ray contained in $\mathrm{Conv}(\Lambda)$ which emanates from x and converges to ξ, and every geodesic ray contained in $\mathrm{Conv}(\Lambda)$ can be obtained in this way. This shows that the limit set Λ of Γ coincides with the Gromov boundary of $\mathrm{Conv}(\Lambda)$ and hence with the Gromov boundary of Γ as well. $\square$

Let now $\Gamma \subset PSL(2, \mathbb{C})$ be a convex cocompact torsion free function group. Then there is a cocompact lattice L in $PSL(2, \mathbb{R})$ and a normal (possibly trivial or infinitely generated) subgroup G of L with the following properties:

(1) Γ is isomorphic to $\hat{\Gamma} = L/G$.
(2) $C = \mathbf{H}^2/G$ is a planar hyperbolic surface which is $\hat{\Gamma} - \Gamma$-equivariantly path isometric to the boundary component of $\mathrm{Conv}(\Lambda)$.

The group $\hat{\Gamma}$ is called a *fuchsian model* for Γ. In the sequel we always identify C with the distinguished boundary component of $\mathrm{Conv}(\Lambda)$, and we identify $\hat{\Gamma}$ with Γ.

Corollary 2.3. *The planar surface* C *is a complete Gromov hyperbolic geodesic metric space. The inclusion of* C *into* $\mathrm{Conv}(\Lambda)$ *is a quasiisometry which extends to a* $\hat{\Gamma} - \Gamma$*-equivariant homeomorphism* f *of the Gromov boundary* ∂C *of* C *onto* Λ.

Proof: The surface C with its distance function induced from the hyperbolic metric is a complete geodesic metric space which is quasiisometric to a Cayley graph of $\hat{\Gamma} \sim \Gamma$. Thus C is Gromov-hyperbolic and $\hat{\Gamma} - \Gamma$-quasiisometric to $\mathrm{Conv}(\Lambda)$. $\square$

The *geodesic flow* Φ^t acts on the unit tangent bundle of hyperbolic space. The subset $\hat{E}$ of $T^1\mathbf{H}^3$ which consists of all unit tangent vectors of all geodesics with both endpoints in Λ is Φ^t-invariant and also invariant under the action of Γ. The quotient E of $\hat{E}$ under Γ is compact. The set E is then a compact hyperbolic set for the geodesic flow on $\mathbf{H}^3/\Gamma$. In particular, every Hölder continuous function on E admits a unique *Gibbs equilibrium state*. The Gibbs state of a constant function is the unique measure of maximal entropy for E.

Recall that the set E is invariant under the *flip* $\mathcal{F} : v \to -v$ and therefore we can talk about flip invariant flow invariant measures on E. The Gibbs state ν of a flip invariant function is itself flip invariant. Moreover it projects to a Γ-invariant locally finite Borel measure on the complement of the diagonal in $\Lambda \times \Lambda$ of the form $\varphi\nu_+ \times \nu_+$ where ν_+ is a finite Borel measure on Λ and where φ is a positive function. The measure class of ν_+ is called the *disintegration class* of the measure ν.

Consider again the group Γ acting on C with critical exponent χ. Denote by dist_C the distance function on C induced by the intrinsic metric. Choose a point $x \in C$, and for $\Psi \in \Gamma$ and for $s > \chi$ define a measure $\tilde{\mu}_{\Psi x,s}$ on $\mathrm{Conv}(\Lambda)$ by

$$\tilde{\mu}_{\Psi x,s} = \frac{1}{\sum e^{-s\ \mathrm{dist}_C(x,\Upsilon x)}} \sum_{\Upsilon\in\hat{\Gamma}} e^{-s\ \mathrm{dist}_C(\Psi x,\Upsilon x)}\delta_{\Upsilon x}$$

where δ_z is the Dirac mass at z. As $s \to \chi$, this family of measures converges to a family $\tilde{\mu}_{\Psi x}$ ($\Psi \in \Gamma$) of measures on Λ. More precisely, since the action of Γ on C is cocompact, the arguments of Sullivan in [S2] imply that every sequence $\{s_j\} \subset (\chi,\infty)$ converging to χ has a subsequence such that the measures $\tilde{\mu}_{\Psi x,s_j}$ converge weakly to finite Borel measures $\tilde{\mu}_{\Psi x}$ ($\Psi \in \Gamma$) on Λ. The measure class of the measure $\tilde{\mu}_{\Psi x}$ does not depend on Ψ and is invariant under the action of Γ. The fact that this measure class is independent of the choice of the defining sequence $\{s_j\}$ is a consequence of the next lemma and its proof.

Lemma 2.4. *The measure class of $\tilde{\mu}_x$ is the positive disintegration class of the unique Gibbs equilibrium state μ of a continuous function on E.*

Proof: By construction, the measures $\tilde{\mu}_{\Psi x}$ are equivariant under the action of Γ on $\Gamma x \times \Lambda$ and define all the same measure class. More precisely, there is a number $L > 1$ and for every $\Psi \in \Gamma$ there is a measurable function β_Ψ with values in $[e^{-L\ \mathrm{dist}(x,\Psi x)}, e^{L\ \mathrm{dist}(x,\Psi x)}]$ such that $\beta_\Psi\tilde{\mu}_x = \tilde{\mu}_{\Psi x}$.

We claim that the functions β_Ψ are continuous. For this recall that $C = \mathbf{H}^2/G$ where G is a discrete subgroup of $PSL(2,\mathbb{R})$. Let $\tilde{x}$ be a lift of x to $\mathbf{H}^2$ and let $D(\tilde{x})$ be the Dirichlet fundamental polygon of G centered at $\tilde{x}$. Then $D(\tilde{x}) \subset \mathbf{H}^2$ is convex. Moreover every geodesic ray through x lifts to a geodesic ray through $\tilde{x}$ in $\mathbf{H}^2$ which is contained in the interior of $D(\tilde{x})$. Thus we obtain a Hölder continuous projection Π of the set $\partial D(\tilde{x})$ of accumulation points of $D(\tilde{x})$ in $S^1 = \partial\mathbf{H}^2$ onto the Gromov boundary ∂C of C.

For $\Psi \in \Gamma$ choose a lift $\tilde{z}$ of Ψx to $D(\tilde{x})$. Recall that there is a homeomorphism $f : \partial C = \Pi\partial D(\tilde{y}) \to \Lambda$. For $\xi \in \partial D(\tilde{x})$ let θ_ξ be the Busemann function at ξ which vanishes at $\tilde{x}$ and write

$$\kappa_\Psi(f(\Pi\xi)) = \inf\{\theta_\xi(\Upsilon\tilde{z}) \mid \Upsilon \in G\}.$$

By the construction of the measures $\tilde{\mu}_{\Psi x}$ we have

$$\frac{d\tilde{\mu}_{\Psi x}}{d\tilde{\mu}_x}(f(\Pi\xi)) = e^{-\chi\kappa_\Psi(f(\Pi\xi))}.$$

Now the function κ_Ψ is the infimum of a family of continuous functions and therefore it is upper semi-continuous. Then the function $-\log\beta_\Psi$ on Λ is upper semi-continuous as well. On the other hand, we can use the same argument after exchanging x and Ψx to show that the function $\log\beta_\Psi$ is also upper semi-continuous. Thus the function β_Ψ is in fact continuous.

We extend the measures $\tilde\mu_{\Psi x}$ to an equivariant family of finite Borel measures μ_z ($z \in \mathrm{Conv}(\Lambda)$) on Λ in the following way. First, let $\epsilon > 0$ be smaller than the injectivity radius of $\mathbf{H}^3/\Gamma$ and let $\sigma : \mathbb{R} \to [0,1]$ be a smooth function with $\sigma(0) = 1$ and with support in $(-\infty, \epsilon]$. Choose a compact fundamental domain D for the action of Γ on $\mathrm{Conv}(\Lambda)$ which contains x in its interior and such that the Lebesgue measure of its boundary vanishes. For $\Psi \in \Gamma$ and $z \in \Psi D$ define $\tilde\mu_z = \tilde\mu_{\Psi x}$, and define $\tilde\mu_z = 0$ for $z \notin \mathrm{Conv}(\Lambda)$. Let now $\mu_z = \int \sigma(\mathrm{dist}(z,w))\tilde\mu_w dw$ (integration with respect to the Lebesgue measure on $\mathbf{H}^3$). Clearly μ_z ($z \in \mathrm{Conv}(\Lambda)$) is a finite family of Borel measures on Λ which is equivariant under the action of Γ on $\mathbf{H}^3 \times \Lambda$. The function ρ on $\mathrm{Conv}(\Lambda) \times \mathrm{Conv}(\Lambda) \times \Lambda$ which assigns to (z, w, ξ) the value $\rho(z, w, \xi) = \log \frac{d\mu_z}{d\mu_w}(\xi)$ is continuous.

For $v \in \hat{E}$ let γ_v be the geodesic in $\mathbf{H}^3$ with initial velocity v and define $\tilde\varphi(v) = \rho(\gamma_v(1), \gamma_v(0), \gamma_v(\infty))$. The function $\tilde\varphi$ is invariant under Γ and projects to a function φ on E. The pressure of φ vanishes and the measures μ_y define the positive disintegration class of a Gibbs state of φ. To show that the Gibbs state of the function φ is unique, simply observe that by our explicit construction, for every fixed $\xi \in \Lambda$ the restriction of ρ to $\mathrm{Conv}(\Lambda) \times \mathrm{Conv}(\Lambda) \times \{\xi\}$ is uniformly Lipschitz continuous. Thus our function φ is admissible in the sense of [HK] and therefore a Gibbs equilibrium state for φ is unique. ☐

Let now φ be a continuous function on E of pressure zero as in the proof of Lemma 2.4 which defines uniquely the Gibbs state μ. Let moreover ν be the measure of maximal entropy for the geodesic flow on E which is the Gibbs state of the constant function δ. We have.

Lemma 2.5. *For ν-almost every $v \in E$ the limit*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \varphi(\Phi^s v) ds$$

exists. Its value is independent of v and not smaller than δ, with equality if and only if $\mu = \nu$.

Proof: By construction, the pressure of the constant function δ on E vanishes, and its Gibbs equilibrium state is induced from the δ-dimensional Hausdorff measure on Λ (see the construction in [S2] and compare [H1]). Since the pressure of the function φ vanishes as well, the lemma just follows from the Birkhoff ergodic theorem and uniqueness of a Gibbs equilibrium state for the function φ. ☐

The next lemma finishes the proof of Theorem 2 from the introduction.

Lemma 2.6. *If $\mu=\nu$ then the group Γ is conjugate to a subgroup of $PSL(2,\mathbb{R})$.*

Proof: With the above notations, if $\mu=\nu$ then the cocycles for the geodesic flow on E defined by the constant function δ and the function φ are cohomologous. In particular, for every periodic orbit for the geodesic flow on E their integrals over this orbit coincide. By construction of these functions, this means the following. Write again $S=C/\Gamma$ and for $\Psi\in\Gamma$ denote by $\ell(\Psi)$ the length of the unique closed geodesic in $M=\mathbf{H}^3/\Gamma$ which defines the free homotopy class of Ψ, and let $\hat{\ell}(\Psi)$ be the infimum of the lengths of the closed geodesics on S which define the free homotopy classes of ΨG. Then for every $\Psi\in\Gamma$ we have $\chi\hat{\ell}(\Psi)=\delta\ell(\Psi)$. Since by definition clearly $\hat{\ell}(\Psi)\geq\ell(\Psi)$ for all $\Psi\in\Gamma$ we conclude that $\chi\leq\delta$.

Assume first that $\chi=\delta$. Then $\hat{\ell}(\Psi)=\ell(\Psi)$ for every $\Psi\in\Gamma$ and therefore every closed geodesic in $M=\mathbf{H}^3/\Gamma$ is contained in the boundary component of $\mathcal{C}(M)=\mathrm{Conv}(\Lambda)/\Gamma$ which is the quotient of our planar surface C under Γ. Since closed geodesics are dense in $\mathcal{C}(M)$, this means that in fact $\mathrm{Conv}(\Lambda)$ coincides with this boundary component and hence Γ is fuchsian.

To show that $\chi<\delta$ is impossible, recall that the boundary surface S is a *pleated surface*. Its pleating locus is a geodesic lamination. Now if this lamination contains a closed curve, then this curve is both a geodesic on S with respect to the intrinisic metric and also a geodesic in M of the same length. This contradicts our assumption that $\chi<\delta$. Otherwise our pleating lamination contains leaves which are recurrent. Such a recurrent leaf can be closed up to yield a closed curve σ on S with the property that the length of the unique closed geodesic on M which is freely homotopic to σ is approximately the length of σ up to a small constant. If we choose σ long enough, this yields again a contradiction to our assumption that $\chi<\delta$. $\square$

3. QUASIFUCHSIAN GROUPS

This section is devoted to the proof of Theorem 1 from the introduction. Let $\hat{\Gamma}\subset PSL(2,\mathbb{R})$ be a torsion free cocompact fuchsian group and let Γ be a convex cocompact Kleinian group which is isomorphic to $\hat{\Gamma}$. As in Section 2 we denote by $\hat{E}\subset T^1\mathbf{H}^3$ the set of all unit tangent vectors with the property that both endpoints of the geodesic with initial velocity v are contained in Λ. The set $\hat{E}$ is invariant under the action of Γ and the geodesic flow Φ^t and its quotient $E=\hat{E}/\Gamma$ is a hyperbolic set for the geodesic flow on $\mathbf{H}^3/\Gamma=M$.

Recall that two Hölder continuous flows Φ^t, Ψ^t on compact metric spaces X, Y are *Hölder continuous order preserving orbit equivalent* if there is a homeomorphism $F:X\to Y$ and a Hölder continuous cocycle $\alpha:X\times\mathbb{R}\to\mathbb{R}$ for Φ^t such that $\alpha(x,t)>0$ for all $t>0$ and that moreover $F(\Phi^tx)=\Psi^{\alpha(x,t)}Fx$ for all $x\in X$ and all $t\in\mathbb{R}$. We have.

Lemma 3.1. *The geodesic flows on $T^1S = T^1\mathbf{H}^2/\hat{\Gamma}$ and on E are Hölder continuous order preserving orbit equivalent.*

Proof: For $x \in \mathbf{H}^2$ let ν^x be the visual measure on S^1 as seen from x, and for $y \in \mathrm{Conv}(\Lambda)$ let μ^y be the *Patterson Sullivan measure* on Λ with basepoint y. Then ν^x and μ^y are finite Borel measures on the topological circles S^1, Λ without atoms. Their respective measure classes do not depend on $x \in \mathbf{H}^2$, $y \in \mathbf{H}^3$.

Let $P : T^1\mathbf{H}^2 \to \mathbf{H}^2$ and $\pi : T^1\mathbf{H}^2 \to \partial\mathbf{H}^2$ be the canonical projections. For $v \in T^1\mathbf{H}^2$ define $B(v,1) = \pi\{w \in T^1_{Pv}\mathbf{H}^2 \mid \angle(v,w) \leq \pi/2\}$. Recall that there is a homeomorphism $f : \partial\mathbf{H}^2 = S^1 \to \Lambda$. This homeomorphism is Hölder continuous with respect to the natural Hölder structures on S^1 and Λ induced by a spherical distance function [BH]. Consider a geodesic γ in $\hat{E}$ with endpoints $f(\pi(v)), f(\pi(-v))$. There is a unique $t \in \mathbf{R}$ such that $\mu^{\gamma(t)}(fB(v,1)) = \nu^{Pv}B(v,1)$. Define $F(v) = \gamma'(t)$. Then the map F is $\hat{\Gamma} - \Gamma$-equivariant and projects to a Hölder continuous order preserving orbit equivalence of the geodesic flows on T^1S, E as required. $\square$

By Lemma 3.1, the geodesic flow on E can be viewed as a Hölder continuous time-change of the geodesic flow on T^1S. In particular, there is a natural one-one-correspondence between Φ^t-invariant Borel probability measures on E and on T^1S. Moreover, every Hölder continuous function on E defines a unique Gibbs equilibrium state on T^1S.

The Patterson Sullivan measure of Γ induces a locally finite Γ-invariant Borel-measure on $\Lambda \times \Lambda -$ diagonal [S2] which corresponds to a finite measure μ on E which is invariant under the geodesic flow and under the flip. If $\delta > 0$ is the critical exponent of Γ, then μ is the Gibbs state on E of the constant function δ. The image of this measure under the diagonal extension of f^{-1} to $\Lambda \times \Lambda -$ diagonal is a $\hat{\Gamma}$-invariant locally finite Borel measure on $S^1 \times S^1 -$ diagonal which is a current $\mu(\Gamma)$ for $\hat{\Gamma}$. By Lemma 3.1 we have.

Lemma 3.2. *The current $\mu(\Gamma)$ is a Gibbs current.*

Since the current $\mu(\Gamma)$ only depends on the conjugacy class of Γ we can now define a map $\Psi : \mathcal{A}(\hat{\Gamma}) \to \mathcal{PC}$ by assigning to the class of a representation ρ the projective class of the Gibbs current $\mu(\rho(\hat{\Gamma}))$.

Lemma 3.3. *The map Ψ is continuous.*

Proof: Let $\{\rho_j\} \subset \mathcal{A}_0(\hat{\Gamma})$ be a sequence converging to some representation ρ. By our above consideration, it is enough to show that for every $\epsilon > 0$ there is some $j(\epsilon) > 0$ such that for every $j > j(\epsilon)$ and every closed geodesic γ in the convex core $\mathcal{C}$ of $\mathbf{H}^3/\Gamma$ of length $\ell(\gamma) > 0$ the length of the corresponding geodesic in the convex core $\mathcal{C}_j$ of $\mathbf{H}^3/\Gamma_j$ is contained in $((1-\epsilon)\ell(\gamma), (1+\epsilon)\ell(\gamma))$.

For this recall that by the Marden stability theorem [M] the groups Γ_j converge in fact *strongly* to Γ (compare [MM] for a comprehensive discussion of algebraic versus geometric convergence of arbitrary geometrically finite groups) and therefore for every $\epsilon > 0$ there is some $j > 0$ and a smooth $(1-\epsilon, 1+\epsilon)$-bilipschitz map L_j of a neighborhood of the convex core $\mathcal{C}$ of $\mathbf{H}^3/\Gamma$ onto a neighborhood of the convex core $\mathcal{C}_j$ of $\mathbf{H}^3/\Gamma_j$ which induces the isomorphism $\rho_j \circ \rho^{-1}$ of Γ onto Γ_j. The image under L_j of every closed geodesic γ in $\mathcal{C}$ is a curve of length not bigger than $(1+\epsilon)\ell(\gamma)$ and hence the length of the unique closed geodesic in $\mathcal{C}_j$ which is freely homotopic to $L_j(\gamma)$ is not bigger than $(1+\epsilon)\ell(\gamma)$ as well. The same argument applied to L_j^{-1} then yields the above claim and finishes the proof of our lemma. □

The following lemma now yields the first part of our Theorem 2 from the introduction.

Lemma 3.4. *The map Ψ is injective.*

Proof: Let ρ, ρ' be representations of $\hat{\Gamma}$ such that $\Psi(\rho) = \Psi(\rho')$. Write $\Gamma = \rho(\hat{\Gamma})$ and $\Gamma' = \rho'(\hat{\Gamma})$. Let Λ, Λ' be the limit sets of Γ, Γ'. By our assumption and our above discussion, there is a Hölder continuous $\Gamma - \Gamma'$-equivariant homeomorphism of Λ onto Λ' which preserves the measure classes of the Patterson Sullivan measures. By a result of Tukia [T2], this means that as Kleinian groups, Γ and Γ' are conjugate. The lemma follows. □

Let again Γ be a convex cocompact quasifuchsian function group $\Gamma = \rho(\hat{\Gamma})$, where $\rho \in \mathcal{A}(\hat{\Gamma})$ is a group isomorphism and $\hat{\Gamma} \subset PSL(2, \mathbb{R})$ is a cocompact fuchsian group. Let $q : S = \mathbf{H}^2/\hat{\Gamma} \to M = \mathbf{H}^3/\Gamma$ be a continuous map which induces the isomorphism ρ. Then q lifts to a $\hat{\Gamma} - \Gamma$-equivariant map Q of $\mathbf{H}^2$ into $\mathbf{H}^3$. Denote by $P : T^1\mathbf{H}^2 \to \mathbf{H}^2$ the canonical projection. Let α be a Hölder continuous function on T^1S which induces the Gibbs state defining the current $\Psi(\rho)$, and let δ be the critical exponent of Γ.

Lemma 3.5. *For every Φ^t-invariant ergodic Borel probability measure η on T^1S and η-almost every $v \in T^1S$, every lift $\tilde{v}$ of v to $T^1\mathbf{H}^2$ we have*

$$\int \alpha d\eta = \delta \lim_{t\to\infty} \frac{1}{t} \operatorname{dist}(QP\tilde{v}, QP\Phi^t\tilde{v}).$$

Proof: Assume without loss of generality that the map q is smooth. For every geodesic γ in $\mathbf{H}^3$ there is a smooth closest point retraction R_γ of $\mathbf{H}^3$ onto γ. Define a function $\tilde{\beta}$ on $T^1\mathbf{H}^2$ as follows. For $\tilde{v} \in T^1\mathbf{H}^2$ let $\gamma(\tilde{v})$ be the oriented geodesic in $\mathbf{H}^3$ with endpoints $f(\pi(-\tilde{v})), f(\pi(\tilde{v}))$ and write $\tilde{\beta}(\tilde{v}) = \frac{\partial}{\partial t} R_{\gamma(\tilde{v})}(QP\Phi^t\tilde{v})|_{t=0}$. The function $\tilde{\beta}$ is continuous and invariant under the action of $\hat{\Gamma}$ and therefore it projects to a continuous function β on T^1S. The cocycle defined by β is cohomologous to α.

Let η be as in the lemma. By the Birkhoff ergodic theorem, for η-almost every $v \in T^1S$ we have $\lim_{t\to\infty} \frac{1}{t}\int_0^t \beta(\Phi^s v)ds = \int \beta d\eta = \int \alpha d\eta$. From this and the definition of β the lemma follows. □

As a corollary we obtain (compare also [BT]).

Corollary 3.6. *Let λ be the Lebesgue Liouville measure on $T^1\mathbf{H}^2$. Then for λ-almost every $v \in T^1\mathbf{H}^2$ we have $\lim_{t\to\infty} \frac{1}{t}\operatorname{dist}(QPv, QP\Phi^t v) = K \geq \frac{1}{\delta}$ with equality if and only if $\delta = 1$ and the group Γ is fuchsian.*

Proof: Denote by h_η the entropy of a Φ^t-invariant Borel probability measure η on T^1S. Let λ be the Lebesgue Liouville measure on T^1S. Since the pressure of the cocycle α vanishes, we have $\int \alpha d\lambda \geq h_\lambda = 1$ with equality if and only if λ is the unique Gibbs equilibrium state for α.

Now if this is the case, then the groups Γ_0 and Γ define the same point under the map Ψ and hence by Lemma 3.4 they are conjugate. In other words, the group Γ is fuchsian. □

4. CONFORMAL DENSITIES

In this section we prove Theorem 3 from the introduction. We continue to use the assumptions and notations from the previous sections. Recall in particular the definition of the visual distances d_x on S^2 ($x \in \mathbf{H}^3$). From now on we denote by $B_x(\xi, r)$ ($x \in \mathbf{H}^3, \xi \in S^2, r > 0$) the closed d_x-ball of radius r about ξ in S^2.

We begin with a well known remark about the balls $B_x(\xi, r)$ which is an immediate consequence of the fact that in the upper half space model for $\mathbf{H}^3$, euclidean dilation with a factor $\lambda > 0$ is a hyperbolic isometry of $\mathbf{H}^3$ with translation length $|\log\lambda|$ along the euclidean ray joining 0 to ∞ as its axis.

Lemma 4.1. *There is a constant $a_1 > 0$ with the following property. Let $x \in \mathbf{H}^3$, let $\xi \in S^2$ and let γ be the geodesic in $\mathbf{H}^3$ connecting $x = \gamma(0)$ to $\xi = \gamma(\infty)$. Then for every $r < \pi/2$ there is a number $\tau \in [-\log r - a_1, -\log r + a_1]$ such that $B_x(\xi, r)$ equals the set of all points $\zeta \in S^2$ whose nearest point projection to γ is contained in $\gamma[\tau, \infty]$.*

The following lemma is equally well known.

Lemma 4.2. *For every $\epsilon > 0$ there is a constant $b(\epsilon) > 0$ with the following property. Let Γ be a torsion free Kleinian group without parabolic elements, and let $\gamma : [0, \tau] \to M = \mathbf{H}^3/\Gamma$ be a geodesic loop. If the angle between $\gamma'(0)$ and $\gamma'(\tau)$ is not bigger than $\pi - \epsilon$, then the length of the unique closed geodesic in M which is freely homotopic to γ is not smaller than $\tau - b(\epsilon)$ and contains γ in its $b(\epsilon)$-tubular neighborhood.*

Proof: Consider a lift $\tilde{\gamma}$ of γ to $\mathbf{H}^3$. By assumption there is a hyperbolic isometry $\Psi \in \Gamma$ such that $\Psi\tilde{\gamma}(0) = \tilde{\gamma}(\tau)$. Since the angle between $\gamma'(0)$ and $\gamma'(\tau)$ is not bigger than $\pi - \epsilon$, hyperbolic trigonometry shows that $\tilde{\gamma}$ is contained in a tubular neighborhood about the axis of Ψ of a fixed radius $a > 0$. From this the lemma is immediate. □

We use now Lemma 4.1 and Lemma 4.2 to show that for convex cocompact Kleinian groups Γ the axes of hyperbolic isometries fill the closed convex hull of the limit set Λ in a uniform way.

Lemma 4.3. *Let Γ be a torsion free convex cocompact Kleinian group with limit set Λ. Then there is a constant $a_2 > 0$ and for every geodesic segment $\gamma : [0,\tau] \to \mathrm{Conv}(\Lambda)$ there is an isometry $\Psi \in \Gamma$ with the following properties.*

(1) *γ is contained in the tubular neighborhood of radius a_2 about the axis ζ of Ψ.*
(2) *The translation length of Ψ on ζ is contained in the interval $[\tau - a_2, \tau + a_2]$.*

Proof: Since Γ is convex cocompact, the convex core $\mathcal{C}(M)$ of the quotient manifold $M = \mathbf{H}^3/\Gamma$ is compact. Thus it suffices to show the lemma for geodesic segments in $\mathrm{Conv}(\Lambda)$ which project to geodesic loops at a fixed point $x \in \mathcal{C}(M)$.

For this choose a number $R > 0$ such that the angle sum of every geodesic triangle in the hyperbolic plane with one given angle $\alpha \geq 3\pi/5$ and adjacent sides of length at least R is not bigger than $\alpha + \pi/10$. It sufficies to show the lemma for arcs of length $\tau \geq R$.

Let again $\hat{E}$ be the set of all unit tangent vectors in $T^1\mathbf{H}^3$ which are tangent to geodesics with both endpoints in Λ. The projection E of $\hat{E}$ to T^1M is a compact hyperbolic set for the geodesic flow. Choose geodesic loops σ_1, σ_2 of length $\tau_1, \tau_2 \geq R$ at x such that the angle between $\sigma_i'(0)$ and $\sigma_i'(\tau_i)$ is not smaller than $9\pi/10$ and that moreover the angle between $-\sigma_1'(0)$ and $\sigma_2'(0)$ is not bigger than $\pi/10$. The existence of such loops follows from the fact that the geodesic flow on E has a dense orbit. Namely, if we choose a point $w \in E$ whose orbit is dense and if we pick a vector $v \in E$ which projects to x, then the orbit of w contains arbitrarily long pieces whose endpoints are arbitrarily close to v and $-v$ respectively. Connect the endpoints of the projection to M of such pieces of orbit of length at least R to the basepoint x by geodesic arcs of minimal length. The resulting closed curves are homotopic with fixed endpoints to unique geodesic loops at x. Hyperbolic trigonometry in the universal covering $\mathbf{H}^3$ of M shows that the distance between v and the outgoing tangent and $-v$ and the incoming tangent of these loops tends to zero with the distance between the endpoints of our pieces of orbit and the vectors v, $-v$.

Let now $\Theta \neq \mathrm{Id} \in \Gamma$ and let $\gamma : [0,\tau] \to \mathrm{Conv}(\Lambda)$ be the geodesic segment which joins x to Θx. Assume that $\tau \geq R$. Consider the projection γ_0 of γ to M. If the angle between $\gamma_0'(0)$ and $\gamma_0'(\tau)$ is not bigger than $9\pi/10$ then we are done by Lemma 4.2. Otherwise we choose one of the fixed loops σ_1, σ_2, say the loop σ for abbreviation, such that the angle between $\gamma_0'(\tau)$ and $\sigma'(0)$ is not bigger than $3\pi/5$.

Every lift $\gamma \circ \tilde{\sigma}$ of the composed loop $\gamma_0 \circ \sigma$ to $\mathbf{H}^3$ consists of two geodesic segments of length at least R which meet at a lift of x with an angle not smaller than $3\pi/5$. By our choice of R, the unique geodesic segment η whose endpoints coincide with the endpoints of $\gamma \circ \tilde{\sigma}$ makes at its two endpoints an angle not bigger than $\pi/10$ with $\gamma \circ \tilde{\sigma}$. In particular, η contains $\gamma \circ \tilde{\sigma}$ in a tubular neighborhood of a given size $c > 0$. Its projection to M is a geodesic loop with one breakpoint at x with exterior angle not bigger than $9\pi/10$. The lemma now follows from a second application of Lemma 4.2. $\square$

The next lemma is again valid for an arbitrary convex cocompact Kleinian group Γ.

Lemma 4.4. *There is a constant $a_3 > 0$ such that $\nu^x B_x(\xi, \pi/2) \geq a_3 \nu^x S^2$ for every conformal density $\{\nu^x\}$ of dimension $d \leq 2$ on S^2, every $x \in \mathrm{Conv}(\Lambda)$ and every $\xi \in \Lambda$.*

Proof: We use the notations from the proof of Lemma 4.3. Let $v \in \hat{E}$ and let x be the projection of v to $\mathrm{Conv}(\Lambda)$. The projections ξ of v, ζ of $-v$ to Λ are not isolated in Λ. Since $\max\{\nu^x B_x(\xi, \pi/2), \nu^x B_x(\xi, \pi/2)\} \geq \nu^x(S^2)$ for every conformal density ν^x on S^2, by the transition properties of conformal densities it suffices to find finitely many isometries $\Psi_1, \dots, \Psi_k \in \Gamma$ such that for every $\eta \in \Lambda$ the ball $B_x(\eta, \pi/2)$ contains one of the sets $\Psi_i B_x(\xi, \pi/2)$ and one of the sets $\Psi_j B_x(\zeta, \pi/2)$ $(i, j \in \{1, \dots, k\})$.

For this recall that the Γ-orbit of the point $\xi \in \Lambda$ is dense in Λ. More precisely, for every open subset U of Λ which does not contain ξ and every neighborhood V of ξ there is some isometry $\Psi \in \Gamma$ which maps $\Lambda - V$ into U. From this the existence of finitely many isometries as above is immediate.

The above shows our lemma for the fixed choice of a point $x \in \mathrm{Conv}(\Lambda)$. To obtain an estimate which is independent of $x \in \mathrm{Conv}(\Lambda)$, just observe that the action of Γ on $\mathrm{Conv}(\Lambda)$ is cocompact, and that the constant found for an individual point can be chosen to vary continuously with the point and to be invariant under the action of Γ. $\square$

Now we are ready to show Theorem 3 from the introduction. For this let again Γ be a convex cocompact function group with invariant domain Ω_0. By the work of Sullivan [S2], our theorem is known for the (essentially unique) conformal density whose dimension equals the critical exponent δ of Γ. Thus it is enough to consider conformal densities of dimension $d > \delta$.

We begin with constructing explicitly such a density. For this choose a point $\xi \in \Omega_0$ and let $x \in \mathrm{Conv}(\Lambda)$ be the nearest-point projection of ξ to $\mathrm{Conv}(\Lambda)$. Let θ_ξ be the Busemann function at ξ which is normalized to vanish at x. For every $\Psi \in \Gamma$ the function $\theta_\xi \circ \Psi^{-1}$ is the Busemann function at $\Psi\xi$ which vanishes at Ψx. For $y \in \mathbf{H}^3$ define

$$\nu^y = \sum_{\Psi \in \Gamma} e^{-d(\theta_\xi \circ \Psi^{-1})(y)} \delta_{\Psi\xi}$$

where δ_z denotes the Dirac mass at z.

By our explicit construction, the measures ν^x define a conformal density for Γ of dimension d provided that the series $\sum e^{-d(\theta_\xi \circ \Psi^{-1})(x)}$ is convergent. To see that this is indeed the case, simply recall that $\xi \in \Omega$ and therefore there is a totally geodesic hyperbolic plane in $\mathbf{H}^3$ whose boundary circle separates ξ from Λ. This means that for all points in $\mathrm{Conv}(\Lambda)$ the function θ_ξ coincides with the distance from x up to a bounded error. This is in particular the case for all points in the Γ-orbit of x. Since d is bigger than the critical exponent of Γ our claim follows.

By Lemma 4.4, for every $x \in \mathrm{Conv}(\Lambda)$ we have $\chi = \min\{\nu^x(B_x(\xi,\pi/2)\cap\Omega_0) \mid \xi \in \Lambda\} \geq a_2\nu^x(S^2)$. Choose a point $\zeta \in \Lambda$ and a number $r < 1$. Lemma 4.3 and Lemma 4.1 imply that there is a universal constant $\kappa > 0$ and an isometry $\Psi \in \Gamma$ whose axis γ passes through the κ-neighborhood of x, whose translation length σ along its axis is not bigger than $-\log r + \kappa$ and such that $\Psi B_x(\zeta,\pi/2) \subset B_x(\zeta,r)$.

Assume that γ is parametrized in such a way that $\gamma(0)$ is the closest point on γ to x and that $\gamma(\infty)$ is the attracting fix point for Ψ. Then the nearest point projection of the ball $B_x(\zeta,\pi/2)$ to γ is contained in $\gamma[-c,\infty]$ for a constant $c > 0$ not depending on Ψ. Let now $\Upsilon \in \Gamma$ be such that $\Upsilon\xi \in B_x(\zeta,\pi/2)$, and let $\tau \in [-c,\infty]$ be such that $\gamma(\tau)$ is the nearest point projection of $\Upsilon\xi$ to γ. Then the nearest point projection of $\Psi\Upsilon\xi$ to γ equals $\gamma(\tau+\sigma)$. Hyperbolic geometry shows that there is a constant $b > 0$ such that $|\theta_\xi \circ \Upsilon^{-1}(\gamma(0)) - \tau| \leq b$ and $|\theta_\xi \circ (\Psi\Upsilon)^{-1}(\gamma(0)) - \tau - \sigma| \leq b$. Since this holds for every $\Upsilon \in \Gamma$ with $\Upsilon\xi \in B_x(\xi,\pi/2)$ we conclude from the definition of the density $\{\nu^x\}$ that $\nu^x B_x(\zeta,r) \geq \nu^x\Psi B_x(\zeta,\pi/2) \geq \mathrm{const}\, r^d \nu^x B_x(\zeta,\pi/2)$. This gives the desired estimate from below, and the estimate from above follows in exactly the same way.

On the other hand, every conformal density of dimension d which is supported on Ω_0 is necessarily a generalized convex combination of nonnegative multiples of densities as above. Together with our above consideration, our theorem follows.

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