

THE BOUNDARY OF THE FREE FACTOR GRAPH AND THE FREE SPLITTING GRAPH

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ABSTRACT. We show that the Gromov boundary of the free factor graph for the free group F_n with $n \geq 3$ generators is the space of equivalence classes of minimal very small indecomposable projective F_n -trees without point stabilizer containing a free factor equipped with a quotient topology. Here two such trees are equivalent if the union of their metric completions with their Gromov boundaries are F_n -equivariantly homeomorphic with respect to the observer's topology. The boundary of the cyclic splitting graph is the space of equivalence classes of trees which either are indecomposable or split as very large graph of actions. The boundary of the free splitting graph is the space of equivalence classes of trees which either are indecomposable or which split as large graphs of actions.

CONTENTS

1. Introduction	2
2. Geometric models	5
3. Skora paths	15
4. Alignment preserving maps	21
5. Trees in $\partial CV(F_n)$ without dense orbits	24
6. Boundaries and the boundary of Outer space	28
7. Indecomposable trees	35
8. Trees with point stabilizers containing a free factor	42
9. Trees which split as graphs of actions	44
10. The boundary of the free factor graph	48
11. The boundaries of the free and the cyclic splitting graph	52
Appendix A. Splitting control	57
References	58

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1. INTRODUCTION

A *free splitting* of the free group F_n with $n \geq 3$ generators is a one-edge graph of groups decomposition of F_n with trivial edge group. Thus it either is a decomposition as a free product $F_n = A * B$ where A, B is a proper subgroups of F_n , or as an HNN-extension $F_n = A*$. The subgroups A, B are *free factors* of F_n .

A vertex of the *free splitting graph* is a conjugacy class of a free splitting of F_n . Two such free splittings are connected by an edge of length one if up to conjugation, they have a common refinement. The free splitting graph is hyperbolic in the sense of Gromov [HM13b, HiHo12]. The *outer automorphism group* $\text{Out}(F_n)$ of F_n acts on the free splitting graph as a group of simplicial isometries.

A *cyclic splitting* of F_n is a one-edge graph of groups decomposition of F_n with infinite cyclic edge group. Thus it either is a decomposition as an amalgamated product $F_n = A *_{\langle w \rangle} B$ or as an HNN-extension $F_n = A *_{\langle w \rangle}$ where $\langle w \rangle$ is an infinite cyclic subgroup of F_n . A vertex of the *cyclic splitting graph* is a conjugacy class of a free or of a cyclic splitting of F_n . Two such splittings are connected by an edge of length one if up to conjugation, they have a common refinement. The cyclic splitting graph is hyperbolic in the sense of Gromov [Mn12]. The outer automorphism group $\text{Out}(F_n)$ of F_n acts on the cyclic splitting graph as a group of simplicial isometries. The free splitting graph is a coarsely dense $\text{Out}(F_n)$ -invariant subgraph of the cyclic splitting graph.

The *free factor graph* is the graph \mathcal{FF} whose vertices are conjugacy classes of free factors of F_n . Two free factors A, B are connected by an edge of length one if up to conjugation, either $A < B$ or $B < A$. The free factor graph is hyperbolic in the sense of Gromov [BF14a]. The outer automorphism group $\text{Out}(F_n)$ of F_n acts on the free factor graph as a group of simplicial isometries.

The goal of this article is to determine the Gromov boundaries of these three graphs.

Let $cv(F_n)$ be the unprojectivized *Outer space* of minimal free simplicial F_n -trees, with its boundary $\partial cv(F_n)$ of minimal *very small* F_n -trees [CL95, BF92] which either are not simplicial or which are not free. Write $\overline{cv(F_n)} = cv(F_n) \cup \partial cv(F_n)$.

A tree $T \in \partial cv(F_n)$ is called *indecomposable* if for any finite, non-degenerate arcs $I, J \subset T$, there are elements $g_1, \dots, g_r \in F_n$ so that

$$J \subset g_1 I \cup \dots \cup g_r I$$

and such that $g_i I \cap g_{i+1} I$ is non-degenerate for $i \leq r - 1$ (Definition 1.17 of [G08]). The F_n -orbits on an indecomposable tree are dense (Lemma 1.18 of [G08]). Being indecomposable is invariant under scaling and hence it is defined for projective trees.

A *graph of actions* is a minimal F_n -tree \mathcal{G} which consists of

- (1) a simplicial tree S , called the *skeleton*, equipped with an action of F_n
- (2) for each vertex v of S an \mathbb{R} -tree Y_v , called a *vertex tree*, and

- (3) for each oriented edge e of S with terminal vertex v a point $p_e \in Y_v$, called an *attaching point*.

It is required that the projection $Y_v \rightarrow p_e$ is equivariant, that for $g \in F_n$ one has $gp_e = p_{ge}$ and that each vertex tree is a minimal very small tree for the action of its stabilizer in F_n .

Associated to a graph of actions \mathcal{G} is a canonical action of F_n on an \mathbb{R} -tree $T_{\mathcal{G}}$ which is called the *dual* of the graph of actions [L94]. Define a pseudo-metric d on $\coprod_{v \in V(S)} Y_v$ as follows. If $x \in Y_{v_0}, y \in Y_{v_k}$ let $e_1 \dots e_k$ be the reduced edge-path from v_0 to v_k in S and define

$$d(x, y) = d_{Y_{v_1}}(x, p_{\bar{e}_1}) + \dots + d_{Y_{v_k}}(p_{e_k}, y).$$

Making this pseudo-metric Hausdorff gives an \mathbb{R} -tree $T_{\mathcal{G}}$. The vertex trees of the graph of actions \mathcal{G} are isometrically embedded in $T_{\mathcal{G}}$ and will be called the vertex trees of $T_{\mathcal{G}}$.

An minimal very small F_n -tree T *splits as a graph of actions* if there is a graph of actions \mathcal{G} with dual tree $T_{\mathcal{G}}$, and there is an equivariant isometry $T \rightarrow T_{\mathcal{G}}$. We call \mathcal{G} a structure of a graph of action for T . We also say that the projectivization $[T]$ of an F_n -tree T splits as a graph of actions if T splits as a graph of actions.

A tree $T \in \partial cv(F_n)$ with dense orbits *splits as a large graph of actions* if T splits as a graph of actions with the following properties.

- (1) There is a single F_n -orbit of non-degenerate vertex trees.
- (2) The stabilizer of a non-degenerate vertex tree is not contained in any proper free factor of F_n .
- (3) A non-degenerate vertex tree is indecomposable for its stabilizer.

Note that by the second requirement, the stabilizer of an edge of the skeleton of a large graph of actions is non-trivial.

We say that a tree $T \in \partial cv(F_n)$ with dense orbits *splits as a very large graph of actions* if T splits as a large graph of actions and if the stabilizer of an edge of the skeleton of the is not infinite cyclic.

If $T \in \partial cv(F_n)$ is a tree with a dense action of F_n then the union \hat{T} of the metric completion \bar{T} of T with the Gromov boundary ∂T can be equipped with an *observer's topology* [CHL07]. This topology only depends on the projective class of T .

Denote by $\partial CV(F_n)$ the boundary of projectivized Outer space, i.e. the space of projective classes of trees in $\partial cv(F_n)$. Let $\mathcal{ST} \subset \partial CV(F_n)$ be the $\text{Out}(F_n)$ -invariant subspace of projective trees which either are indecomposable or split as large graph of actions. Let \sim be the equivalence relation on \mathcal{ST} which is defined as follows. The equivalence class of the projective class $[T]$ of a tree T consists of all projective classes of trees S so that the trees \hat{S} and \hat{T} equipped with the observer's topology are F_n -equivariantly homeomorphic. We equip \mathcal{ST}/\sim with the quotient topology.

Theorem 1. *The Gromov boundary of the free splitting graph is the space \mathcal{ST}/\sim of equivalence classes of projective very small trees which either are indecomposable or which split as large graph of actions.*

Let $\mathcal{CT} \subset \mathcal{ST}$ be the $\text{Out}(F_n)$ -invariant subset of equivalence classes of projective trees which either are indecomposable or split as very large graph of actions. This subspace is saturated for the equivalence relation \sim .

Theorem 2. *The Gromov boundary of the cyclic splitting graph is the space \mathcal{CT}/\sim of equivalence classes of projective trees which either are indecomposable or which split as very large graph of actions.*

An indecomposable projective tree $[T] \in \mathcal{ST}$ is called *arational* if no nontrivial point stabilizer of $[T]$ contains a free factor. The space of arational trees is saturated for the equivalence relation \sim .

Theorem 3. *The Gromov boundary of the free factor graph is the space \mathcal{FT}/\sim of equivalence classes of arational trees equipped with the quotient topology.*

The strategy of proof for the above theorems builds on the strategy of Klarreich [K99] who determined the Gromov boundary of the curve graph of a non-exceptional surface of finite type.

As in [K99], we begin with describing in Section 2 two electrifications of Outer space which are quasi-isometric to the free factor graph and to the free splitting graph, respectively.

In Section 3 we introduce folding paths and collect those of their properties which are used later on.

Section 4 establishes a technical result on folding paths and their relation to the geometry of the free splitting graph. Section 5 gives some information on trees in $\partial cv(F_n)$ for which the orbits of the action of F_n are not dense.

In Section 6 we show that the boundary of each of the above three $\text{Out}(F_n)$ -graphs is the image of a $\text{Out}(F_n)$ -invariant subspace of the boundary $\partial CV(F_n)$ of projectivized Outer space under a continuous map. The main technical tool to this end is a detailed analysis on properties of folding paths.

In Section 7 we investigate the structure of indecomposable trees, and Section 8 contains some information on trees with point stabilizers containing a free factor. In Section 9 we show that only indecomposable projective trees can give rise to points in the boundary of the free factor graph.

The proof of Theorem 3 is contained in Section 10, and the proofs of Theorem 1 and Theorem 2 are completed in Section 11.

Theorem 3 was independently and at the same time obtained by Mladen Bestvina and Patrick Reynolds [BR12]. Very recently Horbez [Ho14] extended Theorem 2 to a more general class of groups, with a somewhat different proof.

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2. GEOMETRIC MODELS

As in the introduction, we consider a free group F_n of rank $n \geq 3$. Our goal is to introduce geometric models for the free factor graph and the free splitting graph. We also construct $n - 3$ additional $\text{Out}(F_n)$ -graphs which geometrically lie between the free factor graph and the free splitting graph; they are analogs of the graphs considered in [H11]. Although these graphs are not used for the proofs of the theorems from the introduction, they shed some light on the geometry of the free splitting graph and the structure of its boundary. They will be useful in another context.

The *free splitting graph* [KL09] is defined to be the graph whose vertices are one-edge graph of groups decompositions of F_n with trivial edge group. Two such vertices are connected by an edge if up to conjugation, they have a common refinement. It is more convenient for our purpose to use instead the first barycentric subdivision \mathcal{FS} of the free splitting graph. Its vertices are graph of groups decompositions of F_n with trivial edge groups. Two vertices Γ, Γ' are connected by an edge if Γ is a collapse or blow-up of Γ' . The outer automorphism group $\text{Out}(F_n)$ of F_n acts on \mathcal{FS} as a group of simplicial isometries.

The *cyclic splitting graph* [Mn12] is the graph whose vertices are one-edge graph of groups decompositions of F_n with trivial or infinite cyclic edge group. Two such vertices are connected by an edge if up to conjugation, they have a common refinement. The vertices of the first barycentric subdivision \mathcal{CS} of the cyclic splitting graph are graph of groups decompositions of F_n with trivial or infinite cyclic edge groups. Two vertices Γ, Γ' are connected by an edge if Γ is a collapse or a blow-up of Γ' . The group $\text{Out}(F_n)$ acts on \mathcal{CS} as a group of simplicial isometries. The graph \mathcal{FS} is an $\text{Out}(F_n)$ -invariant subgraph of \mathcal{CS} . This means that the inclusion

$$(1) \quad \Psi : \mathcal{FS} \rightarrow \mathcal{CS}$$

is an $\text{Out}(F_n)$ -equivariant one-Lipschitz embedding.

The *free factor graph* \mathcal{FF} is the graph whose vertices are free factors of F_n and where two such vertices A, B are connected by an edge of length one if and only if up to conjugation, either $A < B$ or $B < A$. As before, $\text{Out}(F_n)$ acts on \mathcal{FF} as a group of simplicial isometries. The free factor graph is related to the cyclic splitting graph as follows.

Say that a map $f : X \rightarrow Y$ between metric spaces X, Y equipped with an isometric action of a group Γ is *coarsely Γ -equivariant* if there is a number $C > 0$

such that

$$d(g(f(x)), f(gx)) \leq C$$

for all $x \in X$, all $g \in \Gamma$. The map f is called *coarsely L -Lipschitz* if

$$d(f(x), f(y)) \leq Ld(x, y) + L$$

for all $x, y \in X$.

Lemma 2.1. *There is a number $k > 1$, and there is a coarsely k -Lipschitz coarsely $\text{Out}(F_n)$ -equivariant map $\Omega : \mathcal{CS} \rightarrow \mathcal{FF}$.*

Proof. There is a second description of the cyclic splitting graph as follows. Namely, let FZ_n be the graph whose vertex set is the set of one edge free splittings of F_n . Two such splittings X, Y are connected by an edge if either

- (a) they are connected in the free splitting graph by an edge or
- (b) there exists a \mathbb{Z} -splitting T and equivariant *edge folds* $X \rightarrow T, Y \rightarrow T$.

An edge fold of a one edge two vertex free splitting $A * B$ is a splitting of the form $A *_C \langle B, C \rangle$ where C is any maximal cyclic subgroup of A . An edge fold of a one loop free splitting $A *$ is a splitting of the form $(A * C^t) *_C$ where C is a maximal cyclic subgroup of A and t is the generator of the fundamental group of the loop. We refer to [St65, BF92, Mn12] for details of this construction.

By Proposition 2 of [Mn12], the vertex inclusion of the set of vertices of ZF_n into the set of vertices of the cyclic splitting graph extends to an $\text{Out}(F_n)$ -equivariant quasi-isometry $ZF_n \rightarrow \mathcal{CS}$.

Define a map R from the vertex set of FZ_n into the vertex set of the free factor graph by associating to a vertex X of FZ_n a vertex group of the corresponding free splitting.

Since for every free splitting of F_n of the form $F_n = A * B$ the distance in \mathcal{FF} between the free factors A, B is at most three, the map R coarsely does not depend on choices and hence it is coarsely $\text{Out}(F_n)$ -equivariant. We claim that it extends to a coarsely Lipschitz map $ZF_n \rightarrow \mathcal{FF}$.

To this end let $d_{\mathcal{F}}$ be the distance in \mathcal{FF} . Since the metrics on the graphs $\mathcal{CS}, \mathcal{FF}$ are geodesic, by an iterated application of the triangle inequality it suffices to show that there is a number $L > 1$ so that $d_{\mathcal{F}}(R(X), R(Y)) \leq L$ whenever $X, Y \in ZF_n$ are vertices connected by an edge.

Consider first the case that X, Y are connected by an edge in the free splitting graph. Assume furthermore that X, Y are one edge two vertex splittings, i.e. that these splittings are of the form $X = A * B$ and $Y = C * D$. Then up to conjugation, these splittings have a common refinement, i.e. up to exchanging A, B and C, D there is a free factor E of F_n which is a subgroup of both A, C .

Then $d_{\mathcal{F}}(E, A) \leq 1, d_{\mathcal{F}}(E, C) \leq 1$, moreover the distance in \mathcal{FF} between A and B and between C and D is at most three. Thus the distance between $R(X)$ and $R(Y)$ is at most 8. The case that one or both of the splittings X, Y is a one-loop splitting follows in the same way and will be omitted (see [KR14] for details).

Now assume that X, Y are connected by an edge of type (b) above. Following Lemma 1 of [Mn12] and the discussion in the proof of Theorem 5 of [Mn12], let $\langle w \rangle$ be the edge group of the \mathbb{Z} -splitting to which X, Y fold and let A be the smallest free factor of F_n which contains $\langle w \rangle$. Then A is a subgroup of a vertex group of both X and Y and hence the distance in the free factor graph between $R(X), R(Y)$ is at most 8. \square

Let $cv(F_n)$ be the unprojectivized Outer space of all simplicial trees with minimal free isometric actions of F_n , equipped with the equivariant Gromov-Hausdorff topology. We refer to [P88] for detailed information on this topology.

An F_n -tree T is called *very small* if T is minimal and if moreover the following holds.

- (1) Stabilizers of non-degenerate segments are at most cyclic.
- (2) If g^n stabilizes a non-degenerate segment e for some $n \geq 1$ then so does g .
- (3) $\text{Fix}(g)$ contains no tripod for $g \neq 1$.

Here a tripod is a compact subset of T which is homeomorphic to a cone over three points.

The equivariant Gromov Hausdorff topology extends to the space $\partial cv(F_n)$ of minimal very small F_n -trees [BF92, CL95] which either are not simplicial or which are not free. The subspace $cv(F_n)$ is dense in $\overline{cv(F_n)} = cv(F_n) \cup \partial cv(F_n)$. The group $\text{Out}(F_n)$ acts on $\overline{cv(F_n)}$ as a group of homeomorphisms by precomposition of marking.

Let

$$(2) \quad \overline{cv(F_n)}^s \subset \overline{cv(F_n)}$$

be the $\text{Out}(F_n)$ -invariant subset of all minimal very small simplicial F_n -trees. Then $\overline{cv(F_n)}^s - cv(F_n)$ consists of simplicial F_n -trees so that the action of F_n is not free.

For each $T \in \overline{cv(F_n)}^s$ the quotient graph T/F_n defines a graph of groups decomposition of F_n with at most cyclic edge groups (we refer to [CL95] for a detailed discussion). Thus there is a natural $\text{Out}(F_n)$ -equivariant map

$$(3) \quad \Upsilon_C : \overline{cv(F_n)}^s \rightarrow \mathcal{CS}.$$

Its image is the vertex set of \mathcal{CS} .

By Lemma 2.1, the map

$$(4) \quad \Upsilon_{\mathcal{F}} = \Omega \circ \Upsilon_C : \overline{cv(F_n)}^s \rightarrow \mathcal{FF}$$

is coarsely $\text{Out}(F_n)$ -equivariant and coarsely surjective.

Let

$$(5) \quad \overline{cv(F_n)}^+ \subset \overline{cv(F_n)}^s$$

be the $\text{Out}(F_n)$ -invariant subspace of simplicial trees with at least one F_n -orbit of edges with trivial edge stabilizer. To each tree $T \in \overline{cv(F_n)}^+$ we can associate the

graph of groups decomposition $\Upsilon(T)$ with trivial edge groups obtained by collapsing all edges in T/F_n with non-trivial edge groups to a point (see [CL95] for a detailed description of this construction). Thus there is an $\text{Out}(F_n)$ -equivariant map

$$(6) \quad \Upsilon : \overline{cv(F_n)}^+ \rightarrow \mathcal{FS}.$$

Its image is the vertex set of \mathcal{FS} . Note that for $T \in \overline{cv(F_n)}^+$ the distance in \mathcal{CS} between $\Upsilon(T) \in \mathcal{FS} \subset \mathcal{CS}$ and $\Upsilon_{\mathcal{C}}(T)$ is at most one.

The following definition is taken from [A12] (Definition 3.11).

Definition 2.2. The *covolume* of a tree $T \in \overline{cv(F_n)}$ is defined as

$$\text{cvol}(T) = \inf\{\text{vol}(V) \mid V \subset T \text{ a finite forest and } F_n V = T\}.$$

If $T \in \overline{cv(F_n)}^s$ is simplicial then $\text{cvol}(T)$ is just the volume of T/F_n , i.e. the sum of the length of the edges of T/F_n .

Remark 2.3. The function which associates to a tree $T \in \overline{cv(F_n)}$ its covolume is upper semi-continuous for the equivariant Gromov Hausdorff topology. Its restriction to the dense set $cv(F_n)$ of free simplicial F_n -trees is continuous [A12]. However, it follows from [G00] that its extension to the entire space $\overline{cv(F_n)}^s$ is not continuous.

Definition 2.4. For a number $\ell > 0$, a free basis e_1, \dots, e_n of F_n is ℓ -short for a tree $T \in \overline{cv(F_n)}$ if there is $v \in T$ so that for each i the distance in T between v and $e_i v$ is at most ℓ .

A basis e_1, \dots, e_n of F_n is ℓ -short for T if and only if there is a marked rose R with n petals of length one defining the basis and with universal covering \tilde{R} , and there is an equivariant ℓ -Lipschitz map $f : \tilde{R} \rightarrow T$. If $v \in T$ is the image of a vertex of \tilde{R} , then the convex hull of the points $v, e_i v$ is a compact subtree K of T of volume at most $n\ell$, and $F_n K = T$ (see Proposition 3.13 of [A12] for a detailed proof of this last fact). In particular, the covolume of T is at most $n\ell$.

Let

$$\overline{cv_0(F_n)}^s \subset \overline{cv(F_n)}^s$$

be the $\text{Out}(F_n)$ -invariant subspace of all simplicial trees T of covolume one. Let

$$\overline{cv_0(F_n)}^+ = \overline{cv(F_n)}^+ \cap \overline{cv_0(F_n)}^s$$

and define

$$(7) \quad \overline{cv_0(F_n)}^{++} \subset \overline{cv_0(F_n)}^+$$

to be the set of all simplicial very small F_n -trees of covolume one and no non-trivial edge stabilizer.

Lemma 2.5. *Every tree $T \in \overline{cv_0(F_n)}^{++}$ admits a 3-short basis.*

Proof. If $T \in cv_0(F_n)$ then choose a vertex $v \in T/F_n$ and collapse a maximal tree in $G = T/F_n$ containing v to a point. The resulting graph is a rose R with n petals which determines a free basis e_1, \dots, e_n of F_n . In the graph G , each basis element e_i is represented by a loop based at v which passes through every edge of G at most twice. Since the volume of G equals one, the length of each such loop is at most two. This shows that the basis e_1, \dots, e_n is 2-short for T .

If T has non-trivial vertex stabilizers then by the main result of [A12], there is a sequence $T_i \subset cv_0(F_n)$, and for each i there is an equivariant $(1+1/i)$ -Lipschitz map $f_i : T_i \rightarrow T$. Now the image under f_i of a two-short basis for T_i is $2(1+1/i)$ -short for T . The lemma follows. \square

Remark 2.6. Lemma 2.5 is also immediate from the Combination Lemma 8.6 of [CL95]. We used the main result of [A12] in the proof because it better illustrates the difficulty with simplicial trees with non-trivial edge stabilizers.

Lemma 2.5 does not seem to hold for all points in $\overline{cv_0(F_n)}^+$. Namely, the main result of [A12] states that a tree $T \in \overline{cv_0(F_n)}^s$ with at least one non-trivial edge stabilizer is *not* a limit of trees $T_i \in \overline{cv_0(F_n)}^{++}$ under a sequence of Lipschitz maps with Lipschitz constant tending to one. However, there is a weaker statement which holds true.

For its formulation, define a *pure cyclic splitting* to consist of a graph of groups decomposition for F_n with all edge groups cyclic. Define a *free refinement* of a pure cyclic splitting s to be a refinement s' of s so that each edge group of an edge in $s' - s$ is trivial. We allow the refinement to be trivial. We have

Lemma 2.7. *For every pure cyclic splitting s of F_n there is a number $\ell(s) > 0$ with the following property. Every simplicial tree $T \in \overline{cv_0(F_n)}^s$ so that $\Upsilon_C(T)$ is a free refinement of s admits an $\ell(s)$ -short basis.*

Proof. Let s be a pure cyclic splitting of F_n . For some $\epsilon \ll 1/3n - 4$ let $B \subset \overline{cv_0(F_n)}^s$ be the set of all trees T with the following properties.

- (1) $\Upsilon_C(T) \in \mathcal{CS}$ is a free refinement of s .
- (2) The length of each edge of T is at least ϵ .

Then B is a closed subset of $\overline{cv_0(F_n)}^s$ (this standard fact is discussed in detail in [CL95]) which however is disconnected in general. It is invariant under the action of the stabilizer $\text{Stab}(s) < \text{Out}(F_n)$ of the pure cyclic splitting s .

As there are only finitely many topological types of quotient graphs T/F_n for trees $T \in B$, the stabilizer $\text{Stab}(s) < \text{Out}(F_n)$ of s acts on B properly and cocompactly.

Now by the definition of the equivariant Gromov Hausdorff topology [P89], if $k > 0$ and if \mathcal{A} is a k -short free basis for a tree $T \in \overline{cv_0(F_n)}^s$ then there is a neighborhood U of T in $\overline{cv_0(F_n)}^s$ such that \mathcal{A} is a $k+1$ -short free basis for every tree $S \in U$.

Choose a compact fundamental domain K for the action of $\text{Stab}(s)$ on B . By the reasoning in the previous paragraph, for each $S \in K$ there is a number $k(S) > 0$ and an open neighborhood $U(S)$ of S in K so that each tree in $U(S)$ admits a $k(S)$ -short free basis. By compactness, K can be covered by finitely many of such sets, say the sets $U(S_1), \dots, U(S_m)$. Then any tree $T \in K$ admits a $q(s) = \max_i k(S_i)$ -short free basis. By invariance under the action of $\text{Stab}(s)$, the same holds true for any tree $T \in B$.

By the choice of ϵ , for every tree $T \in \overline{cv_0(F_n)}^s$ such that $\Upsilon_{\mathcal{C}}(T)$ is a free refinement of s there is a tree $T' \in B$, and there is an equivariant $1/\epsilon$ -Lipschitz map $h : T' \rightarrow T$. The map h preserves the topological type of the quotient graph and expands or contracts the lengths of the edges of T' . Its existence follows from the fact that the length of each edge of T is at most one, and the length of each edge of a tree $T' \in B$ is not smaller than ϵ . As T' admits a $q(s)$ -short free basis, the tree T admits a $q(s)/\epsilon$ -short free basis. This shows the lemma. \square

Fix a number $k \geq 3$. For $1 \leq \ell \leq n$ let R_ℓ be the rose with ℓ petals of equal length one and let \tilde{R}_ℓ be its universal covering. Let A be a free factor of F_n of rank ℓ and identify A with the fundamental group of R_ℓ . The free factor A is called k -short for $T \in \overline{cv(F_n)}$ if there is an equivariant k -Lipschitz map

$$F : \tilde{R}_\ell \rightarrow T.$$

By Lemma 2.5, each $T \in \overline{cv_0(F_n)}^{++}$ admits 3-short free factors of any rank $\ell \leq n-1$.

The following technical observation relates short free factors of rank $\ell \leq n-1$ in a tree $T \in \overline{cv_0(F_n)}$ to the free splitting $\Upsilon(T)$. It can be viewed as a geometric version of Lemma A.3 of [BF14c] (see also Lemma A.1). For its formulation, note that a corank one free factor $A < F_n$ determines a one-loop graph of groups decomposition of F_n with vertex group A (see Section 4.1 of [HM13a] for details). In the sequel we often view a corank one free factor of F_n as a point in \mathcal{FS} without further notice.

Lemma 2.8. *Let $A < F_n$ be a free factor of rank $\ell \leq n-1$ which is k -short for a tree $T \in \overline{cv_0(F_n)}$. Then there is a free factor $B > A$ of rank $n-1$ whose distance in \mathcal{FS} to $\Upsilon(T)$ is at most $1 + \log_2 k(3n-4)\ell$.*

Proof. Let $\ell \leq n-1$ and let $A < F_n$ be a free factor of rank ℓ . Let $T \in \overline{cv_0(F_n)}$ be such that A is k -short for T . map $\Upsilon : \overline{cv_0(F_n)} \rightarrow \mathcal{FS}$.

Let as before R_ℓ be a rose with ℓ petals of equal length one and let $f : R_\ell \rightarrow T/F_n$ be a k -Lipschitz map such that $f_*(\pi_1(R_\ell))$ is conjugate to A . As T/F_n has at most $3n-4$ edges and volume one, there is an edge $e \subset T/F_n$ of length at least $1/(3n-4)$. Then e is covered by $f(R_\ell)$ at most $k(3n-4)\ell$ times.

Assume first that the edge e of T/F_n is non-separating. Collapse a maximal forest in T/F_n not containing e to a point and let V be the resulting rose. Let $e_0 \subset V$ be the image of e under the collapsing map.

Embed V into the manifold $M = \sharp_n(S^2 \times S^1)$ to define a marked isomorphism of fundamental groups. Let $S_0 \subset M$ be an embedded sphere which intersects the interior of the petal e_0 of $V \subset M$ in a single point and has no other intersection

with V . Then S_0 defines a one-loop free splitting of F_n obtained by collapsing the complement of e_0 in V to a point. Its vertex group is up to conjugation the group of all homotopy classes of loops in M based at a point $x \in M - S_0$ which do not intersect S_0 . The distance in the graph \mathcal{FS} between $\Upsilon(T)$ and the one-loop free splitting defined by S_0 equals one.

Let $G \subset M$ be an embedded rose with ℓ petals, with vertex $x \in M - S_0$ and with fundamental group the free factor A . We may assume that the rose G intersects the sphere S_0 in at most $k(3n - 4)\ell$ points. Let $\Sigma \subset M$ be a sphere which is disjoint from the rose G . Such a sphere exists since the rank of A equals $\ell \leq n - 1$. The free factor A is a subgroup of the corank one free factor of F_n which is the vertex group B of the splitting defined by Σ .

If Σ is disjoint from S_0 then S_0, Σ define free splittings of F_n of distance two in \mathcal{FS} . The distance in the free splitting graph between B and $\Upsilon(T)$ is at most four, so the above claim follows in this case.

If Σ intersects S_0 then use Σ to surger S_0 so that the intersection number with the rose G is decreased as follows. Choose a disc component D of $\Sigma - S_0$. Its boundary divides S_0 into two discs D_1, D_2 . Assume that D_1 is the disc with fewer intersections with G . Then $D \cup D_1 = S_1$ is a sphere disjoint from S_0 with at most $k(3n - 4)\ell/2$ intersections with G . The sphere S_1 defines a free splitting of F_n whose distance in \mathcal{FS} to the splitting defined by S_0 equals two (we refer to [HiHo12] for details of this well known construction).

Repeat this construction with S_1 and Σ . After at most $\log_2 k(3n - 4)\ell$ such surgeries the resulting sphere Q is disjoint from G . The sphere Q determines a corank one free factor of F_n containing A whose distance to $\Upsilon(T)$ in \mathcal{FS} is at most $1 + \log_2 k(3n - 4)\ell$.

As a consequence, if e is non-separating then in the free splitting graph, $\Upsilon(T)$ is at uniformly bounded distance from a one-loop graph of groups decomposition $F_n = C*$ with trivial edge group and vertex group $C > A$. The same reasoning also applies if the edge e is separating- in this case the sphere S_0 is chosen to be separating as well. \square

Remark 2.9. Purists may find the simultaneous use of the sphere graph and of Outer space as a model for the free splitting graph unsatisfactory. However we felt that using spheres makes the proof of Lemma 2.18 more transparent. We invite the reader to find another proof along the lines of the proofs of Lemma A.3 and Lemma 4.1 of [BF14c]

We note a special case of Lemma 2.8 as a corollary.

Corollary 2.10. *For every $k \geq 3$ there is a number $c = c(k) > 0$ with the following property. For every $T \in \overline{cv_0(F_n)}^{++}$, the distance in \mathcal{FS} between $\Upsilon(T)$ and any corank one free factor which is k -short for T is at most c .*

Corollary 2.11. (1) *There is a number $m = m(n) > 0$, and for every tree $T \in \overline{cv_0(F_n)}^{++}$ there is a neighborhood U of T in $\overline{cv_0(F_n)}^{++}$ such that*

$$\text{diam}(\Upsilon(U)) \leq m.$$

- (2) For every simplicial tree $T \in \overline{cv(F_n)}^s$ there is a neighborhood U of T in $\overline{cv(F_n)}$ such that the diameter of $\Upsilon(U \cap \overline{cv_0(F_n)}^{++})$ is finite.

Proof. By Lemma 2.5, every tree $T \in \overline{cv_0(F_n)}^{++}$ admits a 3-short basis. Let e_1, \dots, e_n be such a basis. By the definition of the equivariant Gromov Hausdorff topology, there is a neighborhood V of T in $\overline{cv(F_n)}$, and for every $S \in V$ there is a vertex $v \in S$ so that for each i the distance in S between v and $e_i v$ is at most 4. Thus the corank one free factor with basis e_1, \dots, e_{n-1} is 4-short for each $S \in V$. The first part of the corollary for $U = V \cap \overline{cv_0(F_n)}^{++}$ is now immediate from Lemma 2.10.

To show the second part of the corollary, let $T \in \overline{cv_0(F_n)}^s$. Choose any free basis e_1, \dots, e_n for F_n . Let $v \in T$ be any vertex and let $\ell = \max\{\text{dist}(v, e_i v)\}$. Then the corank one free factor with basis e_1, \dots, e_{n-1} is ℓ -short for T . As in the first part of this proof, this factor is $\ell + 1$ -short for every tree in some neighborhood of T in $\overline{cv(F_n)}$. The second part of the corollary now follows from Lemma 2.10. \square

Remark 2.12. It follows from [A12] that the function cvol which associates to a tree $T \in \overline{cv(F_n)}$ its covolume is continuous on the subspace $\overline{cv(F_n)}^{++}$, but it is not continuous on all of $\overline{cv(F_n)}$ (see also [G00]).

Corollary 2.13. Let $T \in \overline{cv(F_n)}^s$ be a simplicial tree and let $T_i \subset \overline{cv_0(F_n)}^{++}$ be a sequence converging to T . Then $\text{diam}(\Upsilon\{T_i \mid i\}) < \infty$.

Fix once and for all a number $k \geq 3$. Quantitative versions of all statements in the sequel will depend on this choice of k , but for simplicity we will drop this dependence in our notations. For a number $\ell \leq n - 1$ call two trees $T, T' \in \overline{cv_0(F_n)}$ ℓ -tied if there is a free factor A of F_n of rank ℓ which is k -short for both T, T' .

For trees $T \neq T' \in \overline{cv_0(F_n)}$ let $d_{ng}^\ell(T, T')$ be the minimum of all numbers $s \geq 1$ with the following property. There is a sequence $T = T_0, \dots, T_s = T' \subset \overline{cv_0(F_n)}$ so that for all i the trees T_i, T_{i+1} are ℓ -tied. Define moreover $d_{ng}^\ell(T, T) = 0$ for all $T \in \overline{cv_0(F_n)}$.

Proposition 2.14. For all $1 \leq \ell \leq n - 1$, the function

$$d_{ng}^\ell : \overline{cv_0(F_n)} \times \overline{cv_0(F_n)} \rightarrow \mathbb{N}$$

is a distance on $\overline{cv_0(F_n)}$.

Proof. Symmetry of d_{ng}^ℓ is immediate from the definition, as well as the fact that $d_{ng}^\ell(T, T') = 0$ if and only if $T = T'$. The triangle inequality is built into the construction, so all we have to show is that $d_{ng}^\ell(T, T') < \infty$ for all $T, T' \in \overline{cv_0(F_n)}$. For $\ell = n - 1$ this is a consequence of Lemma 2.15 below and its proof, and the case $\ell \leq n - 2$ is established in Lemma 2.18. \square

Lemma 2.15. The map $\Upsilon : (\overline{cv_0(F_n)}, d_{ng}^{n-1}) \rightarrow \mathcal{FS}$ is a coarsely $\text{Out}(F_n)$ -equivariant quasi-isometry.

Proof. Recall that the map Υ is coarsely surjective. We begin with showing that Υ is coarsely L -Lipschitz for some $L = L(k) \geq 1$.

Let $d_{\mathcal{FS}}$ be the distance in \mathcal{FS} . By definition, if $T, T' \in cv_0(F_n)$ are $(n-1)$ -tied then there is a free factor A of F_n of rank $n-1$ which is k -short for both T, T' . By Corollary 2.10,

$$d_{\mathcal{FS}}(A, \Upsilon(T)) \leq c(k), d_{\mathcal{FS}}(A, \Upsilon(T')) \leq c(k)$$

and hence $d_{\mathcal{FS}}(\Upsilon(T), \Upsilon(T')) \leq 2c(k)$. Thus Υ is $2c(k)$ -Lipschitz by an iterated application of the triangle inequality (remember that d_{ng}^{n-1} only assumes integral values).

We are left with showing that the map Υ coarsely decreases distances by at most a fixed positive multiplicative constant. Choose for every free factor A of F_n of corank one a marked rose $R(A)$ with n petals of length $1/n$ each which represents this factor A , i.e. such that $n-1$ petals of $R(A)$ generate A . We denote by $\tilde{R}(A) \in cv_0(F_n)$ the universal covering of $R(A)$. By construction, $d_{\mathcal{FS}}(A, \Upsilon(\tilde{R}(A))) = 1$.

Let $\hat{\mathcal{FS}}$ be the complete subgraph of the free splitting graph (not of its first barycentric subdivision) whose vertex set consists of one-loop free splittings of F_n . It is well known that the vertex inclusion extends to a coarsely $\text{Out}(F_n)$ -equivariant quasi-isometry $\hat{\mathcal{FS}} \rightarrow \mathcal{FS}$ (see e.g. [H14b] for a detailed proof in the case of the curve graph which carries over word by word).

Since $\hat{\mathcal{FS}}$ is a metric graph and, in particular, a geodesic metric space, and since for all corank one free factors A the distance between A and $\Upsilon(\tilde{R}(A))$ equals one, it now suffices to show the following. Whenever two one-loop graph of groups decompositions A, C of F_n are of distance one in $\hat{\mathcal{FS}}$, i.e. if they have a common refinement, then $d_{ng}^{n-1}(\tilde{R}(A), \tilde{R}(C)) \leq 2$.

Thus let $A < F_n, C < F_n$ be corank one free factors defining one-loop graph of groups decompositions with a common refinement. Then up to replacing one of these free factors by a conjugate, the intersection $B = A \cap C$ is a free factor of F_n of rank $n-2$, and there is a free splitting $F_n = U * B * D$ with $U * B = A$ and $B * D = C$.

Let G be a metric rose with n petals of length $1/n$ which represents this splitting, with universal covering $\tilde{G} \in cv_0(F_n)$. Then both corank one free factors A, C of F_n are 2-short for \tilde{G} . This implies that the distance with respect to the metric d_{ng}^{n-1} between $\tilde{R}(A)$ and \tilde{G} is at most one, and the same holds true for the distance between \tilde{G} and $\tilde{R}(C)$. Therefore we have $d_{ng}^{n-1}(\tilde{R}(A), \tilde{R}(C)) \leq 2$ which is what we wanted to show. The lemma is proven. \square

The final goal of this section is to give a geometric interpretation of the free factor graph. We will take a slightly more general viewpoint and introduce $n-3$ additional intermediate $\text{Out}(F_n)$ -graphs.

Fix a number $\ell \leq n-2$. Define a graph \mathcal{FF}_ℓ as follows. Vertices of \mathcal{FF}_ℓ are free factors of rank $n-1$. Two such vertices A, B are connected by an edge of length one if and only if up to conjugation, the intersection $A \cap B$ contains a free factor

of rank ℓ . Note that the graphs \mathcal{FF}_ℓ all have the same set of vertices, and for each $\ell \leq n - 2$ the graph \mathcal{FF}_ℓ can be obtained from the graph $\mathcal{FF}_{\ell-1}$ by deleting some edges. In particular, the vertex inclusion extends to a one-Lipschitz map $\mathcal{FF}_\ell \rightarrow \mathcal{FF}_{\ell-1}$. The group $\text{Out}(F_n)$ acts as a group of simplicial automorphisms on each of the graphs \mathcal{FF}_ℓ . We have

Lemma 2.16. *The graph \mathcal{FF}_1 is coarsely $\text{Out}(F_n)$ -equivariantly quasi-isometric to the free factor graph.*

Proof. Associate to a free factor A in F_n a free factor $h(A) > A$ of corank one. If the free factors A, B are connected by an edge in the free factor graph, then up to conjugation and exchanging A and B we have $A < B$. But this just means that up to conjugation, $h(A)$ and $h(B)$ intersect in the free factor A and hence $h(A)$ and $h(B)$ are connected by an edge in \mathcal{FF}_1 . In other words, the map h which maps the vertex set of the free factor graph into the vertex set of the graph \mathcal{FF}_1 is one-Lipschitz with respect to the metric of \mathcal{FF} and the metric of \mathcal{FF}_1 . Moreover, h is coarsely $\text{Out}(F_n)$ -equivariant.

Now if A, B are free factors of rank $n - 1$ which are connected by an edge in \mathcal{FF}_1 then up to conjugation, they intersect in some free factor C . Then the distance between A, C and between B, C in \mathcal{FF} equals one and hence the distance between A, B in \mathcal{FF} is at most two. The lemma follows. \square

Remark 2.17. The dual of a graph G is a graph G' whose vertex set is the set of edges of G and where two distinct vertices of G' are connected by an edge if the corresponding edges of G are incident on the same vertex. For $\ell \leq n - 2$, the dual \mathcal{FF}'_ℓ of \mathcal{FF}_ℓ has a simple description. Its vertex set consists of free factors of rank ℓ , and two such free factors are connected by an edge if up to conjugation, they are subgroups of the same free factor of rank $n - 1$. Although we do not use these dual graphs directly, they are more closely related to the geometric description of the graphs \mathcal{FF}_ℓ given in Lemma 2.18 below.

For each $\ell \leq n - 2$ let

$$\Upsilon_\ell : cv_0(F_n) \rightarrow \mathcal{FF}_\ell$$

be a map which associates to a tree T a k -short corank one free factor- in fact we can take the same map for every ℓ and view its images as vertices in any one of the graphs \mathcal{FF}_ℓ .

The statement of the following lemma is only needed in the case $\ell = 1$.

Lemma 2.18. *For $\ell \leq n - 2$ the map $\Upsilon_\ell : (cv_0(F_n), d_{ng}^\ell) \rightarrow \mathcal{FF}_\ell$ is a coarsely $\text{Out}(F_n)$ -equivariant quasi-isometry.*

Proof. The proof of the lemma is similar to the proof of Lemma 2.15. Note first that the map Υ_ℓ is coarsely surjective.

Let $\ell \leq n - 2$ and let $A < F_n$ be a free factor of rank ℓ . Let $T \in cv_0(F_n)$ be such that A is k -short for T . By Lemma 2.8, there is a free factor $B > A$ of rank $n - 1$ whose distance to $\Upsilon(T)$ in the free splitting graph is bounded from above by a number only depending on k .

By the definition of the metric d_{ng}^ℓ and of the graph \mathcal{FF}_ℓ , this implies that whenever $T, T' \in cv_0(F_n)$ are such that $d_{ng}^\ell(T, T') = 1$ then the distance in \mathcal{FF}_ℓ between $\Upsilon(T)$ and $\Upsilon(T')$ is uniformly bounded. By an iterative application of the triangle inequality, this shows that the map $\Upsilon_\ell : (cv_0(F_n), d_{ng}^\ell) \rightarrow \mathcal{FF}_\ell$ is coarsely Lipschitz.

To show that the map Υ_ℓ coarsely decreases distances at most by a fixed positive multiplicative constant, note that if two corank one free factors A, A' of F_n are connected by an edge in \mathcal{FF}_ℓ then there are free bases e_1, \dots, e_n and e'_1, \dots, e'_n of F_n such that $e_i = e'_i$ for $1 \leq i \leq \ell$ and that $A = \langle e_1, \dots, e_{n-1} \rangle, A' = \langle e'_1, \dots, e'_{n-1} \rangle$ up to conjugation. The universal coverings $R, R' \in cv_0(T)$ of the marked roses with petals of length $1/n$ determined by these bases are ℓ -tied. As in the proof of Lemma 2.15, this yields the lemma. \square

3. SKORA PATHS

The goal of this section is to introduce folding paths and establish some of their properties needed later on.

A *morphism* between F_n -trees S, T is an equivariant map $\varphi : S \rightarrow T$ such that every segment of S can be subdivided into finitely many subintervals on which φ is an isometric embedding. If the tree S is simplicial then we require that the restriction of φ to any edge of S is an isometric embedding.

The following (well known) construction is taken from Section 2 of [BF14a]. Let for the moment U be an arbitrary F_n -tree. A *direction* at a point $x \in U$ is a germ of non-degenerate segments $[x, y]$ with $y \neq x$. At each interior point of an edge of U there are exactly two directions. A collection of directions at x is called a *gate* at x . A *turn* at x is an unordered pair of distinct directions at x . It is called *illegal* if the directions belong to the same gate, and it is called *legal* otherwise. A *train track structure* on U is an F_n -invariant family of gates at the points of U so that at each $x \in U$ there are at least two gates.

A morphism $\varphi : S \rightarrow T$ determines a collection of gates as follows. Define a turn in S to be illegal if it is given by two directions which are identified by the morphism φ . Otherwise the turn is defined to be legal. Two directions d, d' at the same point belong to the same gate if either $d = d'$ or if the turn d, d' is illegal. If these gates determine a train track structure on S and if moreover there is a train track structure on T so that legal turns are sent to legal turns, then φ is called a *train track map* (see p.110 of [BF14a]). The train track map $\varphi : S \rightarrow T$ is called *optimal* if it realizes the optimal Lipschitz constant between equivariant maps $S \rightarrow T$, i.e. if this optimal Lipschitz constant equals one.

Let as before $cv(F_n)$ be unprojectivized Outer space, with its boundary $\partial cv(F_n)$ of minimal very small actions of F_n on \mathbb{R} -trees which are either not simplicial or which are not free. As in Section 2, denote by $cv_0(F_n) \subset cv(F_n)$ the subspace of trees of covolume one. Let $CV(F_n)$ be the projectivization of $cv(F_n)$, with its boundary $\partial CV(F_n)$. The union

$$\overline{CV(F_n)} = CV(F_n) \cup \partial CV(F_n)$$

is a compact space. The outer automorphism group $\text{Out}(F_n)$ of F_n acts on $\overline{\text{CV}(F_n)}$ as a group of homeomorphisms. The projection map restricts to an $\text{Out}(F_n)$ -equivariant homeomorphism $cv_0(F_n) \rightarrow \text{CV}(F_n)$.

There is a natural bijection between conjugacy classes of free bases of F_n and roses (=marked metric roses with n petals of length $1/n$ each). Define the *standard simplex* of a free basis of F_n to consist of all simplicial trees

$$U \in \overline{cv_0(F_n)}^{++} \subset \overline{cv_0(F_n)}^+ \subset \overline{cv(F_n)} = cv(F_n) \cup \partial cv(F_n)$$

of covolume one whose F_n -quotients are graphs obtained from the rose R corresponding to the basis by changing the lengths of the edges. We allow that U is contained in the boundary of unprojectivized Outer space, i.e. that the rank of the fundamental group of the graph U/F_n is strictly smaller than n . A standard simplex is a compact subset of the space $\overline{cv_0(F_n)}^{++}$ of simplicial F_n -trees with trivial edge stabilizers and covolume one.

In the proof of Lemma 3.3 below and several times later on we will use the main result of [P88] which we report here for easy reference. We do not define the equivariant Gromov Hausdorff topology, but we note that this is the topology used for $\overline{cv(F_n)}$ (see [P89] for a precise statement).

Theorem 3.1. *Let (X_i) be a sequence of complete \mathbb{R} -trees. Let Γ be a countable group acting by isometries on all X_i . Suppose that for each i there exists a point $x_i \in X_i$ such that the following holds true. For every finite subset P of Γ , the closed convex hulls of the images of x_i under P admit for all $\epsilon > 0$ a covering by balls of radius ϵ of uniformly bounded cardinality. Then there exists a subsequence converging in the equivariant Gromov Hausdorff topology to an \mathbb{R} -tree.*

We use Theorem 3.1 to establish the following technical result which will be used several times in the sequel. For its formulation, we say that a sequence $f_i : S_i \rightarrow T_i$ of train track maps *converges in the equivariant Gromov Hausdorff topology* to a train track map $f : S \rightarrow T$ if $S_i \rightarrow S, T_i \rightarrow T$ in the equivariant Gromov Hausdorff topology and if the graphs of the maps f_i converge in the equivariant Gromov Hausdorff topology for the diagonal action of F_n on $S_i \times T_i$ to the graph of f .

Lemma 3.2. *Let $K \subset \overline{cv_0(F_n)}^{++}$ be a compact set, let $S_i \subset K$ and let $f_i : S_i \rightarrow T_i$ be a sequence of optimal train track maps where $T_i \in \overline{cv(F_n)}$. Then up to passing to a subsequence, the maps f_i converge to a train track map $f : S \rightarrow T$.*

Proof. Let $K \subset \overline{cv_0(F_n)}^{++}$ be a compact set. Let $\Sigma \subset \overline{cv(F_n)}$ be the set of all trees $U \in \overline{cv(F_n)}$ such that the minimum over all trees $S \in K$ of the smallest Lipschitz constant for equivariant maps $S \rightarrow U$ equals one. Since $K \subset \overline{cv_0(F_n)}^{++}$ is compact, the set Σ is compact with respect to the equivariant Gromov Hausdorff topology [H14a] and hence

$$\mathcal{P} = \{U \times V \mid U \in K, V \in \Sigma\}$$

is precompact in the equivariant Gromov Hausdorff topology for the diagonal action of F_n (in fact, compactness of Σ rather than precompactness can also easily be established from the discussion below).

Let $S_i \subset K$ be a sequence, let $T_i \in \Sigma$ and let

$$f_i : S_i \rightarrow T_i$$

be an optimal train track map of Lipschitz constant one. Then $S_i \times T_i \in \mathcal{P}$ for each i and therefore up to passing to a subsequence, we may assume that $S_i \times T_i$ converges in the equivariant Gromov Hausdorff topology to $S \times T \in \mathcal{P}$ where $S \in K$ and where $T \in \overline{cv(F_n)}$ (see Theorem 3.1 and [P88]).

For each i the graph A_i of f_i is a closed F_n -invariant subset of $S_i \times T_i$. By passing to another subsequence we may assume that the sequence A_i converges in the equivariant Gromov Hausdorff topology to a closed F_n -invariant subset A of $S \times T$ (see once more Theorem 3.1 and [P88]). By compactness, we have $T \in \Sigma$.

Let d_S, d_T be the distance on S, T . For each i the set $A_i \subset S_i \times T_i$ is the graph of a one-Lipschitz map $S_i \rightarrow T_i$. Thus by continuity, for $(x_1, y_1), (x_2, y_2) \in A \subset S \times T$ we have $d_T(y_1, y_2) \leq d_S(x_1, x_2)$. In particular, for each $x \in S$ there is a unique point $f(x) \in T$ so that $(x, f(x)) \in A$, and the assignment $x \rightarrow f(x)$ is an equivariant map $S \rightarrow T$ with Lipschitz constant one. The map f is an isometry on edges since this is the case for each of the maps f_i . Thus f is a morphism which induces a collection of gates on S .

Since $T \in \Sigma$, the Lipschitz constant of f is optimal. This then implies that there are at least two gates at each vertex of S . A detailed argument to this end is contained in the end of the proof of Proposition 2.5 of [BF14a]. \square

We use Lemma 3.2 to extend a construction from Section 2 of [BF14a]. We also refer to this paper for references to earlier works where this construction is introduced. In the rest of the paper, we always denote by $[T] \in \overline{CV(F_n)}$ the projectivization of a tree $T \in \overline{cv(F_n)}$.

Lemma 3.3. *For every $[T] \in \overline{CV(F_n)}$ and every standard simplex $\Delta \subset \overline{cv_0(F_n)}^{++}$ there is a tree $U \in \Delta$ and an optimal train track map $f : U \rightarrow T$ where T is some representative of $[T]$.*

Proof. In the case that $[T] \in CV(F_n)$ is free simplicial a detailed argument is given in the proof of Proposition 2.5 of [BF14a] (see also [FM11]). A limiting argument then yields the result for trees $[T] \in \partial CV(F_n)$.

Let $[T_i] \subset CV(F_n)$ be a sequence of free simplicial projective F_n -trees which converge in $\overline{CV(F_n)}$ to a projective F_n -tree $[T]$. By Proposition 2.5 of [BF14a], for each i there is a point $S_i \in \Delta$, there is a representative $T_i \in \overline{cv(F_n)}$ of $[T_i]$ and there is an optimal train track map

$$f_i : S_i \rightarrow T_i.$$

By Lemma 3.2, up to passing to a subsequence we may assume that the maps f_i converge in the equivariant Gromov Hausdorff topology to an optimal train track map $f : S \rightarrow V$ for $S \in \Delta$ and some $V \in \overline{cv(F_n)}$. As $[T_i] \rightarrow [T]$ in $\overline{CV(F_n)}$, the tree V is a representative of $[T]$. \square

Let $S \in \overline{cv(F_n)}^s$ be a simplicial tree, let $T \in \overline{cv(F_n)}$ and let $f : S \rightarrow T$ be an optimal train track map. By the definition of a train track map, f isometrically embeds every edge.

Let $\epsilon > 0$ be half of the smallest length of an edge of S . Let e, e' be edges with the same initial vertex v which define an illegal turn. Assume that e, e' are parametrized by arc length on compact intervals $[0, a], [0, a']$ where $e(0) = e'(0) = v$. Then there is some $t \in (0, \epsilon]$ so that $fe[0, t] = fe'[0, t]$. For $s \in [0, t]$ let S_s be the quotient of S by the equivalence relation \sim_s which is defined by $u \sim_s v$ if and only if $u = ge(r)$ and $v = ge'(r)$ for some $r \leq s$ and some $g \in F_n$. The tree S_s is called a *fold* of S obtained by folding the illegal turn defined by e, e' (once again, compare the discussion in Section 2 of [BF14a]). Note that for $s > 0$ the volume of the graph S_s/F_n is strictly smaller than the volume of S/F_n . There also is an obvious notion of a maximal fold at the illegal turn defined by e, e' .

Using the terminology of the previous paragraph, the assignment $s \rightarrow S_s$ ($s \in [0, t]$) is a path in $\overline{cv(F_n)}^s$ through $S_0 = S$ which is called a *folding path*. We call the train track map $f : S \rightarrow T$ used to construct the path a *guide* for the path. The semigroup property holds for folding paths. For each $s \in [0, t]$ the train track map $f : S \rightarrow T$ decomposes as

$$f = f_s \circ \varphi_s$$

where $f_s : S_s \rightarrow T$ is a train track map and $\varphi_s : S \rightarrow S_s$ is a train track map for the train track structures on S, S_s defined by f and f_s . We refer to [HM11] and to Section 2 of [BF14a] for details of this construction. We insist that we view the initial train track map $f : S \rightarrow T$ as part of the data defining a folding path.

Repeat this construction with S_t and a perhaps different pair of edges. The path constructed in this way by successive foldings terminates if T is free simplicial (Proposition 2.2 of [BF14a]).

We next observe that with perhaps the exception of the endpoint, such folding paths are in fact contained in the subspace $\overline{cv(F_n)}^+$ of all simplicial trees with at least one orbit of edges with trivial stabilizer. It is certainly well known to the experts, in fact the stronger version Lemma 5.3 below seems to be well known as well.

Lemma 3.4. *Let $U \in \overline{cv(F_n)}^s$ be a simplicial tree and assume that there is a non-trivial folding path $(x_t) \subset \overline{cv(F_n)}$ connecting $U = x_0$ to a tree $U \neq T \in \overline{cv(F_n)}$. Then $U \in \overline{cv(F_n)}^+$. If all stabilizers of non-degenerate segments in T are trivial then $U \in \overline{cv(F_n)}^{++}$.*

Proof. Let $f : U \rightarrow T$ be a train track map which guides the folding path (x_t) . Let e_1, e_2 be two edges in U incident on the same vertex v with the property that the map f identifies two non-degenerate initial proper subsegments e'_1, e'_2 of e_1, e_2 . Assume that the stabilizers $A_1, A_2 < F_n$ of both edges e_1, e_2 are non-trivial. Since U is very small, the groups A_1, A_2 are maximal cyclic. By equivariance, the non-degenerate segments $f(e_i) \subset T$ are stabilized by A_i ($i = 1, 2$) and hence $f(e'_1) = f(e'_2) \subset f(e_1) \cap f(e_2)$ is stabilized by the subgroup of F_n generated by A_1

and A_2 . Since T is very small and hence stabilizers of non-degenerate segments in T are maximal cyclic, we conclude that $A_1 = A_2 = A$.

Fold the tree U and equivariantly identify ge'_1 and ge'_2 for all $g \in F_n$. The resulting tree U' contains a tripod which consists of the images of the identified segments e'_1, e'_2 and the images of the segments $e_i - e'_i$ ($i = 1, 2$). As both e_1, e_2 are stabilized by A , by equivariance this tripod is stabilized by A as well. But U' is very small by assumption, in particular stabilizers of tripods are trivial. This is a contradiction which implies that the stabilizer of at least one of the edges e_1, e_2 is trivial. The first part of the lemma follows.

To show the second part, note that if there is some $t \geq 0$ so that x_t has a non-trivial edge stabilizer then by equivariance and the fact that the train track map $x_t \rightarrow T$ maps edges isometrically, there is a non-degenerate segment in T with non-trivial edge stabilizer. Thus indeed $(x_t) \subset \overline{cv(F_n)}^{++}$ if all stabilizers of non-degenerate segments in T are trivial. \square

Remark 3.5. Lemma 3.4 shows in particular that the map Υ as defined in (6) of Section 2 is defined on every folding path, perhaps with the exception of its endpoint.

We can also fold all illegal turns with unit speed at once [BF14a]. The resulting path is unique (Proposition 2.2 of [BF14a]). In the same vein we can rescale all trees along the path to have covolume one and rescale the endpoint tree T accordingly while folding with unit speed all illegal turns at once. Using Proposition 2.2 and Proposition 2.5 of [BF14a] (compare [BF14a] for references), a path constructed in this way from a train track map $f : S \rightarrow T$ is unique and will be called a *Skora path* in the sequel (however, the path depends on the guiding train track map $f : S \rightarrow T$). In [FM11], these paths are called fast folding paths. If $T \in cv(F_n)$ then this path has finite length, otherwise its length may be infinite. By convention, even if the path has finite length we do not consider its endpoint to be a point on the path.

We say that a (normalized or unnormalized) folding path or Skora path $(x_t)_{0 \leq t < \xi}$ ($\xi \in [0, \infty]$) *converges* to a projective tree $[T] \in \overline{CV(F_n)}$ if as $t \rightarrow \xi$ the projective trees $[x_t]$ converge in $\overline{CV(F_n)}$ to $[T]$. If (x_t) is a Skora path then this is equivalent to stating that (x_t) is guided by a train track map $x_0 \rightarrow T$. We summarize the discussion as follows (see also Proposition 2.5 of [BF14a]).

Lemma 3.6. *For every standard simplex Δ and every tree $[T] \in \overline{CV(F_n)}$ there is a Skora path $(x_t) \subset \overline{cv_0(F_n)}^+$ with $x_0 \in \Delta$ which converges to $[T]$.*

Proof. By Lemma 3.3, there is a representative T of $[T]$, a tree $U \in \Delta$ and a train track map $f : U \rightarrow T$. This train track map then determines a unique Skora path (x_t) issuing from U . The projectivizations $[x_t]$ of the trees x_t converge as $t \rightarrow \infty$ in $\overline{CV(F_n)}$ to the projectivization $[T]$ of T (see [FM11, BF14a] as well as the dimension below). \square

Normalized Skora paths have the following geometric interpretation. Namely, for trees $T, T' \in cv_0(F_n)$ define the *one-sided Lipschitz distance*

$$d_L(X, Y) = \log \sup \left\{ \frac{\ell(g, Y)}{\ell(g, X)} \mid g \in F_n \right\}$$

where $\ell(g, U)$ is the translation length of g on the tree U [FM11]. The distance $d_L(X, Y)$ also equals the logarithm of the minimal Lipschitz constant of a marked homotopy equivalence $T/F_n \rightarrow T'/F_n$ [FM11]. Then d_L is a function on $cv_0(F_n) \times cv_0(F_n)$ which satisfies $d_L(T, T') = 0$ only if $T = T'$, moreover it satisfies the triangle inequality, but it is not symmetric. Nevertheless we can speak of geodesics for d_L . By definition, such a geodesic is a path $\gamma : [a, b] \rightarrow cv_0(F_n)$ so that for all $s < t$ we have $d_L(\gamma(s), \gamma(t)) = |t - s|$.

The one-sided Lipschitz distance is also defined for points in $\overline{cv_0(F_n)}^{++}$ (where it may be infinite). The first part of the following result follows from Theorem 5.5 of [FM11]. Its second part is a version of the Arzela Ascoli theorem for normalized Skora paths.

- Proposition 3.7.** (1) *A normalized Skora path $(x_t) \subset \overline{cv_0(F_n)}^{++}$ guided by an optimal train track map is a geodesic for d_L .*
- (2) *Let $\Delta \subset \overline{cv_0(F_n)}^{++}$ be a standard simplex and let $(x_t^i) \subset \overline{cv_0(F_n)}^{++}$ be a sequence of unnormalized Skora paths guided by optimal train track maps $f_i : S_i \rightarrow T_i$ where $S_i \in \Delta$. Then up to passing to a subsequence, the paths converge locally uniformly to a Skora path guided by a limit $f : S \rightarrow T$ of the maps f_i .*

Proof. The first part of the proposition for paths contained in $cv_0(F_n)$ is Theorem 5.5 of [FM11].

In [A12] it is shown that the space $\overline{cv_0(F_n)}^{++}$ is contained in the metric completion of $cv_0(F_n)$ with respect to the one-sided Lipschitz metric. We refer to [A12] for a discussion of the meaning of such a completion for asymmetric metrics. Moreover, by Proposition 4.1 of [A12], the completed one-sided Lipschitz distance between any two points in $\overline{cv_0(F_n)}^{++}$ can be calculated as

$$d_L(X, Y) = \log \sup \left\{ \frac{\ell(g, Y)}{\ell(g, X)} \mid g \in F_n \right\}$$

where $\ell(g, U)$ is the translation length of $g \in F_n$ in U . Then the reasoning in the proof of Theorem 5.5 of [FM11] carries over word for word and completes the proof of the first part of the proposition. We leave the details to the reader.

To show the second part of the proposition, let (x_t^i) be Skora paths as in the proposition, and for $s < t$ let $g_{st}^i : x_s^i \rightarrow x_t^i$ be the corresponding train track maps. Since the paths (x_t^i) are geodesics for the one-sided Lipschitz metric, for each i and each rational $t \in \mathbb{Q} \cap [0, \infty)$ there is an optimal equivariant e^t -Lipschitz map connecting $x_0^i \in \Delta$ to x_t^i . We therefore can apply Lemma 3.2 and rescaling and deduce that up to passing to a subsequence, the trees x_t^i converge to a tree x_t , and the optimal train track maps $g_{0t}^i : x_0^i \rightarrow x_t^i$ converge to an optimal train track map $g_t : x_0 \rightarrow x_t$ of Lipschitz constant e^t . Now the construction is compatible with

choosing further subsequences and hence the second part of the proposition follows from the standard construction in the proof of the Arzela Ascoli theorem. \square

In the sequel we will often use volume renormalization to define a Skora path $(x_t) \subset cv_0(F_n)$. However, most of the time we consider *unnormalized* Skora paths, i.e. we scale the trees along the path (x_t) in such a way that the train track maps along the path are edge isometries onto a fixed endpoint tree T . Note that our parametrization of such unnormalized Skora path does not coincide with the parametrization used in [BF14a] although the paths coincide as sets.

4. ALIGNMENT PRESERVING MAPS

For a number $L > 1$, an L -quasi-geodesic in a metric space (X, d) is a map $\rho : J \subset \mathbb{R} \rightarrow X$ such that

$$|s - t|/L - L \leq d(\rho(s), \rho(t)) \leq L|s - t| + L$$

for all $s, t \in J$. The path ρ is called a *reparametrized L -quasi-geodesic* if there is a homeomorphism $\psi : I \rightarrow J$ such that $\rho \circ \psi : I \rightarrow X$ is an L -quasi-geodesic.

Recall from (6), (3), (4) of Section 2 the definition of the maps $\Upsilon, \Upsilon_{\mathcal{C}}, \Upsilon_{\mathcal{F}}$. A *liberal folding path* is a folding path where folding of edges occurs with any speed, and there may also be rest intervals. Building on the work [HM13b], it was observed in the appendix of [BF14c] that there is a number $L > 1$ such that the image under Υ of any liberal folding path in $\overline{cv(F_n)}^{++}$ is an L -unparametrized quasi-geodesic, and the same holds true for the maps $\Upsilon_{\mathcal{C}}$ [Mn12] and $\Upsilon_{\mathcal{F}}$ [BF14a].

By Lemma 3.4, *all* folding paths (with perhaps the exception of their endpoints) are contained in $\overline{cv(F_n)}^+$ and therefore the map Υ is defined on any folding path. Our next goal is to show that the image under Υ of *any* folding path is a reparametrized L -quasi-geodesic and that the same holds true for the maps $\Upsilon_{\mathcal{C}}, \Upsilon_{\mathcal{F}}$. The main result of this section (Proposition 4.3 below) will be used several times again in later sections.

For technical reasons it will be convenient to work with folding paths or liberal folding paths where at most one fold is performed at the time. We call such folding paths *simple* folding paths. We record from the proof of Proposition 2.2 of [BF14a] a technical observation which yields that for our purpose, studying simple folding paths is not a restriction. In its formulation and later on, d denotes the distance in \mathcal{FS} .

- Lemma 4.1.** (1) *For every Skora path $(x_t) \subset \overline{cv_0(F_n)}^+$ converging to a projective tree $[T]$ there is a simple liberal folding path $(y_s) \subset \overline{cv_0(F_n)}^+$ converging to $[T]$ so that $d(\Upsilon(x_t), \Upsilon(y_t)) \leq 2$ for all t .*
- (2) *If (x_t) is any liberal folding path converging to a projective tree $[T]$ which is guided by an optimal train track map $x_0 \rightarrow T$ then there is a Skora path (y_t) converging to $[T]$ such that the Hausdorff distance between $\Upsilon(x_t)$ and $\Upsilon(y_t)$ is uniformly bounded.*

An *alignment preserving map* between two F_n -trees $T, T' \in \overline{cv(F_n)}$ is an equivariant map $\rho : T \rightarrow T'$ with the property that $x \in [y, z]$ implies $\rho(x) \in [\rho(y), \rho(z)]$. An equivariant map $\rho : T \rightarrow T'$ is alignment preserving if and only if the preimage of every point in T' is convex ([G00] and Def. 10.7 of [R10]). The map ρ is then continuous on segments.

An example of an alignment preserving map can be obtained as follows. Let G be a finite metric graph with fundamental group F_n and without univalent vertices and let G' be obtained from G by collapsing a forest. The collapsing map $G \rightarrow G'$ lifts to a one-Lipschitz alignment preserving map $T \rightarrow T'$ where T, T' is the universal covering of G, G' . An alignment preserving morphism is an equivariant isometry.

Versions of the next lemma can be found in [HM13b] and [BF14c].

Lemma 4.2. *Let $S, S' \in \overline{cv_0(F_n)}^+$ and assume that there is an alignment preserving map $\rho : S \rightarrow S'$; then $d(\Upsilon(S), \Upsilon(S')) \leq 2$.*

Proof. By equivariance, an edge in S with non-trivial stabilizer is mapped by ρ to a (perhaps degenerate) edge in S' with non-trivial stabilizer. Thus ρ induces an alignment preserving map $\rho_0 : S_0 \rightarrow S'_0$ where S_0, S'_0 is obtained from S, S' by collapsing all edges with non-trivial stabilizers to points. The map ρ_0 projects to a quotient map $\hat{\rho}_0 : S_0/F_n \rightarrow S'_0/F_n$.

Let e' be an edge in S'_0/F_n and let V be the graph obtained from S'_0/F_n by collapsing the complement of e' to a point. Let $\zeta : S'_0/F_n \rightarrow V$ be the collapsing map. Then $\zeta \circ \hat{\rho}_0 : S_0/F_n \rightarrow V$ collapses S_0/F_n to V . As S_0/F_n is the graph of groups decomposition defining $\Upsilon(S)$ and S'_0/F_n is the graph of groups decomposition defining $\Upsilon(S')$, this shows that $\Upsilon(S)$ and $\Upsilon(S')$ collapse to the same vertex in \mathcal{FS} . Thus the distance in \mathcal{FS} between $\Upsilon(S)$ and $\Upsilon(S')$ is at most two. \square

Proposition 4.3. *Let $T, T' \in \partial cv(F_n)$ and assume that there is a one-Lipschitz alignment preserving map $\rho : T \rightarrow T'$. Let $\Delta \subset \overline{cv_0(F_n)}^{++}$ be a standard simplex and let (x_t) be a simple liberal folding path connecting a point in Δ to T . Then there is a simple liberal folding path (y_t) connecting a point in Δ to T' such that for all t we have*

$$d(\Upsilon(x_t), \Upsilon(y_t)) \leq 2.$$

Proof. Let $S \in \overline{cv_0(F_n)}^{++}$, let $\varphi : S \rightarrow T$ be a train track map and let (x_t) be a simple liberal folding path guided by φ . Let $T' \in \partial cv(F_n)$ and assume that there is a one-Lipschitz alignment preserving map

$$\rho : T \rightarrow T'.$$

The map $\rho \circ \varphi : S \rightarrow T'$ is equivariant and one-Lipschitz.

We claim that there is a tree $S' \in \overline{cv(F_n)}^{++}$ which can be obtained from S by decreasing the lengths of some edges of S , there is a one-Lipschitz alignment preserving map $\alpha : S \rightarrow S'$ and there is a morphism $\varphi' : S' \rightarrow T'$ such that

$$\varphi' \circ \alpha = \rho \circ \varphi.$$

The map α may collapse some edges of S to points.

The tree S' and the map α are constructed as follows. Let e be an edge in S . By definition of a train track map, the restriction of φ to e is an isometric embedding and hence $\varphi(e)$ is a segment in T whose length equals the length $\ell(e)$ of e . The alignment preserving map ρ maps $\varphi(e)$ to a segment $\rho(\varphi(e))$ of length $\ell'(e) \leq \ell(e)$. To construct S' we reduce the length of the edge e to $\ell'(e) \geq 0$. There is a natural equivariant one-Lipschitz simplicial map $\alpha : S \rightarrow S'$ which associates to e the (possibly degenerate) edge in S' obtained in this way (compare with the discussion in [G00, HM13b, BF14c]). This map can be chosen in such a way that for each edge e of S , there is an isometry φ' of $\alpha(e)$ onto the segment $\rho \circ \varphi(e) \subset T'$ such that $\varphi' \circ \alpha|_e = \rho \circ \varphi|_e$. The thus defined map $\varphi' : S' \rightarrow T'$ is a morphism.

Let $a, b \subset S$ be subsegments of edges incident on the same vertex v which are identified by the map φ . Then the segments a, b are also identified by $\rho \circ \varphi$. This means the following. Let U be the simplicial tree obtained from S by equivariantly identifying the F_n -translates of the segments a, b . Then there is a morphism $\chi : U \rightarrow T$ which maps the identified segments a, b to a segment in T , and this segment is mapped by ρ to a segment in T' . By equivariance of the map ρ , there is a tree U' which is obtained from S' by equivariantly identifying the F_n -translates of $\alpha(a)$ and $\alpha(b)$, i.e. by a (perhaps trivial) fold, and there is a one-Lipschitz alignment preserving map $\beta : U \rightarrow U'$ and a morphism $\chi' : U' \rightarrow T'$ such that

$$\chi' \circ \beta = \rho \circ \chi.$$

Now let (x_t) be a simple liberal folding path connecting S to T . The above discussion shows that there is a (suitably parametrized) liberal folding path (y_t) connecting S' to T' with the following property. For each t , there is a one-Lipschitz alignment preserving map $\zeta_t : x_t \rightarrow y_t$, and there is a morphism $h'_t : y_t \rightarrow T'$ such that the diagram

$$\begin{array}{ccc} x_t & \xrightarrow{h_t} & T \\ \zeta_t \downarrow & & \rho \downarrow \\ y_t & \xrightarrow{h'_t} & T' \end{array}$$

commutes. As $t \rightarrow \infty$, y_t converges in the equivariant Gromov Hausdorff topology to T' . We refer to [HM13b, BF14c] for a detailed proof of this fact in the case that $T \in cv(F_n)$.

By Lemma 4.2, for each t the distance in \mathcal{FS} between $\Upsilon(x_t)$ and $\Upsilon(y_t)$ is at most two. This shows the proposition. \square

Recall the definitions of the maps $\Upsilon_C : \overline{cv(F_n)}^s \rightarrow \mathcal{CS}$ and $\Upsilon_{\mathcal{F}} = \Omega \circ \Upsilon_C : \overline{cv(F_n)}^s \rightarrow \mathcal{FF}$.

Corollary 4.4. *There is a number $L > 1$ such that the image under Υ (or under $\Upsilon_C, \Upsilon_{\mathcal{F}}$) of every folding path is a reparametrized L -quasi-geodesic in \mathcal{FS} (or $\mathcal{CS}, \mathcal{FF}$).*

Proof. It suffices to show the corollary for bounded subsegments of folding paths. Now if $S, T \in \overline{cv(F_n)}^+$ and if there is a folding path (x_t) connecting S to T , then

Proposition 4.3 shows that there are trees $S', T' \in \overline{cv(F_n)}^+$ with $\Upsilon(S) = \Upsilon(S')$, $\Upsilon(T) = \Upsilon(T')$, and there is a liberal folding path (y_t) connecting S' to T' such that $d(\Upsilon(x_t), \Upsilon(y_t)) \leq \chi$ for all t . Then $(y_t) \subset \overline{cv(F_n)}^+$ by Lemma 3.4. The corollary now follows from the fact that the image under Υ (or $\Upsilon_C, \Upsilon_{\mathcal{F}}$ of a liberal folding path in $\overline{cv(F_n)}^{++}$ is an reparametrized m -quasi-geodesic for a universal number $m > 1$ [HM13b]. \square

5. TREES IN $\partial CV(F_n)$ WITHOUT DENSE ORBITS

A tree $T \in \partial cv(F_n)$ decomposes canonically into two disjoint F_n -invariant subsets T_d and T_c . Here T_d is the set of all points p such that the orbit $F_n p$ is discrete, and $T_c = T - T_d$. The set $T_c \subset T$ is closed. Each of its connected components is a subtree T' of T . The stabilizer of T' acts on T' with dense orbits. We have $T_d = \emptyset$ if and only if the group F_n acts on T with dense orbits. This property is invariant under scaling and hence it is defined for projective trees.

Let $T \in \partial cv(F_n)$ be a very small F_n -tree with $T_d \neq \emptyset$. The quotient T/F_n admits a natural pseudo-metric. Let $\widehat{T/F_n}$ be the associated metric space.

Since T is very small, by Theorem 1 of [L94] the space $\widehat{T/F_n}$ is a finite graph. Edges correspond to orbits of the action of F_n on $\pi_0(T - \overline{B})$ where $B \subset T$ is the set of branch points of T . The graph $\widehat{T/F_n}$ defines a graph of groups decomposition for F_n , with at most cyclic edge groups.

Denote as before by \mathcal{FS} and \mathcal{CS} the first barycentric subdivision of the free splitting graph and the cyclic splitting graph. The proof of the following result uses an argument which was shown to me by Vincent Guirardel.

Proposition 5.1. *Let $[T] \in \partial CV(F_n)$ be such that $T_d \neq \emptyset$. Let $(x_t) \subset \overline{cv(F_n)}^+$ be a Skora path converging to $[T]$.*

- (1) $\text{diam}(\Upsilon_C(x_t)) < \infty$.
- (2) *If T contains an edge with trivial edge stabilizer then $\text{diam}(\Upsilon(x_t)) < \infty$.*

Proof. Let $[T] \in \partial CV(F_n)$ be such that $T_d \neq \emptyset$. Let (x_t) be an unnormalized Skora path connecting a point x_0 in a standard simplex Δ to a representative T of $[T]$. By this we mean that there is a Skora path $(y_t) \subset cv_0(F_n)$ guided by a train track map $f : x_0 = y_0 \rightarrow T$ such that for each t we have $x_t = u_t y_t$. Here $u_t > 0$ is such that for $s < t$ there is a morphism

$$g_{st} : x_s \rightarrow x_t$$

and a train track map $f_t : x_t \rightarrow T$ with $f_s = f_t \circ g_{st}$.

Let $a > 0$ be the smallest length of an edge in $\widehat{T/F_n}$ and let $c \geq a$ be the volume of $\widehat{T/F_n}$. The volume of x_t/F_n is a decreasing function in t which converges as $t \rightarrow \infty$ to some $b \geq c$. Thus there is a number $t_0 > 0$ such that for each $t \geq t_0$ the volume of x_t/F_n is smaller than $b + a/8$.

Let e_0 be an edge of T . The length of e_0 is at least a . By equivariance and the fact that for each t the map f_t is an edge isometry, there is a non-degenerate subarc ρ of e_0 of length $a/2$ (the central subarc of length $a/2$) such that for each $t \geq t_0$ the preimage of ρ under f_t consists of $k(t) \geq 1$ pairwise disjoint subarcs of edges of x_t (i.e. the preimage does not contain any vertex). Namely, since f_t is an edge isometry, otherwise there are two segments in x_t of length at least $a/4$ which are identified by the unnormalized Skora path connecting x_t to T . By equivariance, this implies that there is some $u > t$ such that the volume of x_u/F_n is strictly smaller than the volume of x_t/F_n minus $a/8$. This violates the assumption on the volume of x_t/F_n .

We claim that the number $k(t)$ of preimages of ρ in x_t does not depend on $t \geq t_0$. Namely, by equivariance and the definition of a Skora path, otherwise there is some $t > t_0$ and a vertex of x_t which is mapped by f_t into ρ . By the discussion in the previous paragraph, this is impossible.

Write $k = k(t_0)$. For each $u \geq t_0$ the preimage of ρ in x_u consists of exactly k subsegments of edges of x_u . The morphism g_{t_0u} maps the k components of the preimage of ρ in x_{t_0} isometrically onto the k components of the preimage of ρ in x_u . Thus by equivariance, there is an edge h_{t_0} of x_{t_0}/F_n , and there is a subsegment ρ_{t_0} of h_{t_0} of length $a/2$ which is mapped by the quotient map $\hat{g}_{t_0u} : x_{t_0}/F_n \rightarrow x_u/F_n$ of the train track map g_{t_0u} isometrically onto a subsegment ρ_u of an edge h_u in x_u/F_n , and $\hat{g}_{t_0u}^{-1}(\rho_u) = \rho_{t_0}$.

By Lemma A.1 (see also Lemma A.3 of [BF14c]), this implies that the distance in \mathcal{CS} between $\Upsilon_{\mathcal{C}}(x_{t_0})$ and $\Upsilon_{\mathcal{C}}(x_u)$ is uniformly bounded, independent of $u \geq t_0$. As a consequence, we have $\text{diam}(\Upsilon_{\mathcal{C}}(x_t)) < \infty$ as claimed.

Now if the stabilizer of the edge e_0 of T is trivial then by equivariance, the same holds true for the stabilizer of h_u for all $u \geq t_0$. Then Lemma A.3 of [BF14c] shows that $\text{diam}(\Upsilon(x_t)) < \infty$. \square

From Lemma 2.1 and Proposition 5.1 we obtain as an immediate consequence

Corollary 5.2. *If $[T] \in \partial\text{CV}(F_n)$ is such that $T_d \neq \emptyset$ and if (x_t) is a Skora path converging to $[T]$ then $\text{diam}(\Upsilon_{\mathcal{F}}(x_t)) < \infty$.*

For the map Υ , the analog of the first part of Proposition 5.1 may not hold. To obtain information on the image under the map Υ of a Skora path converging to a very small F_n -tree T with $T_d \neq \emptyset$ we will use the fact that a non-degenerate arc with non-trivial stabilizer in a very small F_n -tree T is contained in the closure of T_d . This seems to be well known (see e.g. [A12] where it is used). As we did not find a reference in the literature we provide a proof.

Lemma 5.3. *A non-degenerate segment in a tree $T \in \overline{\text{cv}(F_n)}$ is contained in the closure of the discrete set $T_d \subset T$.*

Proof. Let $e \subset T$ be a non-degenerate segment with non-trivial stabilizer $A < F_n$.

We claim that for each $g \in F_n - A$ the intersection $ge \cap e$ consists of at most one point. Namely, otherwise $ge \cap e$ is a non-degenerate segment. Now e is stabilized by A and by equivariance, ge is stabilized by gAg^{-1} . Then $e \cap ge$ is stabilized by the free non-cyclic group $\langle A, gAg^{-1} \rangle$. Since T is very small, this is impossible.

As a consequence, if $V \subset T$ is any finite forest with $F_n V = T$ then the volume of V is at least the length of e . By Definition 2.2, this shows that the covolume of T is at least the length of e . Then $T_d \neq \emptyset$.

We are left with showing that e is contained in the closure $\overline{T_d}$ of T_d . Namely, otherwise $T_c \neq \emptyset$ and $f_t(e)$ intersects a component T' of T_c in a non-degenerate segment.

Let \hat{T} be the tree obtained from T by equivariantly collapsing all edges in T to points and let

$$\rho : T \rightarrow \hat{T}$$

be the collapsing map. We have $\hat{T}_d = \emptyset$, moreover \hat{T}_d contains a non-degenerate segment with non-trivial stabilizers. By the above discussion, this is impossible. The lemma is proven. \square

Corollary 5.4. (1) *Let $[T] \in \partial\text{CV}(F_n)$ be such that $T_d = \emptyset$. Then any Skora path (x_t) converging to $[T]$ is contained in $\overline{cv_0(F_n)^{++}}$.*
 (2) *For every tree $T \in \partial\text{CV}(F_n)$ there is a number $k([T]) > 0$ with the following property. Let $(x_t) \subset cv_0(F_n)^+$ be any normalized folding path converging to $[T]$. Then for all sufficiently large t there is a neighborhood W of x_t in $cv_0(F_n)^+$ such that $\text{diam}(\Upsilon((W \cap \overline{cv_0 F_n}^{++}) \cup x_t)) \leq k([T])$.*

Proof. The first part of the corollary is immediate from Lemma 5.3. The first part of Corollary 2.11 then yields the second part of the corollary for trees T with $T_d = \emptyset$ and a number $m > 0$ not depending on T .

Now let T be such that $T_d \neq \emptyset$ and let $(x_t) \subset \overline{cv_0(F_n)^+}$ be any normalized folding path which converges to $[T]$. For all t let $f_t : x_t \rightarrow T$ be the corresponding train track map.

As in the beginning of this Section, let $\widehat{T/F_n}$ be the graph of groups decomposition of F_n defined by T . By Lemma 5.3 and equivariance, for every t and every edge e of x_t with non-trivial stabilizer A , the segment $f_t(e) \subset T$ is contained in a finite edge path h of T_d with stabilizer A . The edge path h defines a cyclic splitting c of F_n which is a collapse of $\widehat{T/F_n}$, and this splitting coincides with the splitting defined by the edge e of x_t , i.e. which is determined by the tree obtained by equivariantly collapsing all edges in x_t/F_n which are not contained in the F_n -orbit of e to points.

As $\widehat{T/F_n}$ only has finitely many edges, we deduce that there are only finitely many possibilities for the pure cyclic splitting defined by the union of all edges of x_s with non-trivial stabilizer. Lemma 2.7 shows that for every t there is a neighborhood W of x_t in $\overline{cv_0(F_n)^+}$ so that the diameter of $\Upsilon(W \cap \overline{cv_0(F_n)^{++}})$ is uniformly bounded, independent of t (but depending on $[T]$). The construction

in the proof of Lemma 2.7 also yields that $\Upsilon((W \cap \overline{cv_0(F_n)^{++}}) \cup x_t)$ is uniformly bounded in diameter as well. \square

We use Lemma 5.3 to show

Lemma 5.5. *Let (x_t) be a Skora path connecting a point $x_0 \in \overline{cv_0(F_n)^{++}}$ to a simplicial tree $[T]$. Then $\text{diam}(\Upsilon(x_t)) < \infty$.*

Proof. Proposition 5.1 yields the lemma for projective trees $[T]$ with at least one edge with trivial stabilizer. Thus assume that $[T] \in \partial\text{CV}(F_n)$ is simplicial, and that T/F_n is a graph of groups with each edge group maximal cyclic.

Let (x_t) be an unnormalized Skora path connecting $x_0 \in \overline{cv_0(F_n)^{++}}$ to a representative T of $[T]$. Since $(x_t) \subset \overline{cv(F_n)^+}$, for each t we can consider the tree y_t obtained from x_t by equivariantly collapsing all edges with non-trivial stabilizers to points. By definition, we have $\Upsilon(x_t) = \Upsilon(y_t)$ for all t .

As T is simplicial, we have $\text{cvol}(T) > 0$. The covolumes of the trees x_t are decreasing with t and bounded from below by $\text{cvol}(T)$, and the covolumes of the trees y_t are decreasing as well. We distinguish two cases.

Case 1: The covolumes of the trees y_t are bounded from below by a number $b > 0$.

Let e_1, \dots, e_n be any free basis of F_n . Choose a vertex $v \in T$ and let $k = \sup\{d(v, e_i v) \mid i\}$. As $x_t \rightarrow T$, for sufficiently large t the basis e_1, \dots, e_n is $(k+1)$ -short for x_t and hence it is $(k+1)$ -short for y_t . If $\beta_t > 0$ denotes the covolume of x_t , then the basis e_1, \dots, e_n is $(k+1)/b$ -short for $y_t/\beta_t \in \overline{cv_0(F_n)^{++}}$. By Lemma 2.10, this means that for sufficiently large t the point $\Upsilon(x_t) = \Upsilon(y_t/\beta_t)$ is contained in a uniformly bounded neighborhood of the corank one free factor of F_n with basis e_1, \dots, e_{n-1} . This shows the lemma in this case.

Case 2: The covolumes of the trees y_t converge to zero.

Let $c > 0$ be the smallest length of an edge in T . Choose $t > 0$ sufficiently small that the covolume of y_t is smaller than $c/10$. Let $f_t : x_t \rightarrow T$ be the guiding train track map. Then for each edge e of T , there is a subsegment ρ of e (the middle subsegment of length at least $c/2$) so that $f_t^{-1}(\rho)$ consists of a finite number $k \geq 1$ of subarcs of edges (compare the proof of Proposition 5.1).

In fact, we have $k = 1$. Namely, otherwise there is an embedded edge path $\alpha \subset x_t$ with endpoints in the center of distinct edges of x_t with non-trivial stabilizer whose image under f_t is a loop in T . As T is simplicial, this means that $f_t(\alpha)$ is a compact simplicial subtree of T . If $v \in x_t$ is such that $f_t(v)$ is a leaf of this subtree then there are two proper subsegments of edges incident on v which are identified by the map f_t . Then x_t can be folded in such a way that these segments are identified. In finitely many such folding steps, we obtain a new tree \hat{x}_t and train track maps $g : x_t \rightarrow \hat{x}_t$, $h : \hat{x}_t \rightarrow T$ so that $f_t = h \circ g$ and that $g(\alpha)$ consists of a single point. Then the volume of \hat{x}_t is smaller than the volume of x_t minus the length of an edge

with non-trivial stabilizer. This violates the assumption on the volume of x_t . (A related discussion is contained in the appendix of [BF14c].)

Let z_t the tree obtained from x_t by collapsing all edges with trivial stabilizers to points. By equivariance and the above discussion, the graph of groups decomposition $\Upsilon_{\mathcal{C}}(z_t/F_n)$ coincides with $\Upsilon_{\mathcal{C}}(T)$. This means that there is a simplicial tree S with an edge with non-trivial stabilizer, and there is a one-Lipschitz alignment preserving map $S \rightarrow T$. The claim of the lemma now follows from Proposition 4.3 and Proposition 5.1. \square

6. BOUNDARIES AND THE BOUNDARY OF OUTER SPACE

This section contains the main geometric results of this work. We establish some properties which are valid for all three graphs $\mathcal{FS}, \mathcal{CS}, \mathcal{FF}$. To ease notation we will prove the results we need only for the graph \mathcal{FS} . It will be clear that the argument is also valid for any graph \mathcal{G} which can be obtained from \mathcal{FS} by a surjective coarsely $\text{Out}(F_n)$ -equivariant Lipschitz map Ψ so that the image under $\Psi \circ \Upsilon$ of any folding path is a uniform reparametrized quasi-geodesic. Recall that by Corollary 4.4, the image under Υ (or $\Upsilon_{\mathcal{C}}, \Upsilon_{\mathcal{F}}$) of any liberal folding path is a reparametrized L -quasi-geodesic for a universal number $L > 1$.

Let $\partial\mathcal{FS}$ be the Gromov boundary of \mathcal{FS} and let Θ be an isolated point. For every standard simplex Δ define a map $\varphi_{\Delta} : \partial\text{CV}(F_n) \rightarrow \partial\mathcal{FS} \cup \Theta$ as follows. For a projective tree $[T] \in \partial\text{CV}(F_n)$ choose a Skora path (x_t) connecting a point in Δ to $[T]$. If the reparametrized quasi-geodesic $t \rightarrow \Upsilon(x_t) \in \mathcal{FS}$ has finite diameter then define $\varphi_{\Delta}([T]) = \Theta$. If the diameter of the reparametrized quasi-geodesic $\Upsilon(x_t)$ is infinite then define $\varphi_{\Delta}([T])$ to be the unique endpoint in $\partial\mathcal{FS}$ of the path $\Upsilon(x_t)$.

Note that a priori, the map φ_{Δ} depends on choices since a Skora path connecting a tree S in Δ to a representative T of $[T]$ depends on the choice of a train track map $S \rightarrow T$. The next lemma shows that the map φ_{Δ} does not depend on the standard simplex Δ nor on any other choices made. It is related to a construction of Handel and Mosher [HM13b].

For the purpose of the proof, for a number $c > 0$ we say that a path $\alpha : [0, \infty) \rightarrow \mathcal{FS}$ is a c -fellow traveler of a path $\beta : [0, \infty) \rightarrow \mathcal{FS}$ if there is a nondecreasing function $\tau : [0, \infty) \rightarrow [0, \infty)$ such that for all $t \geq 0$ we have $d(\alpha(t), \beta(\tau(t))) \leq c$. We allow that the function τ has bounded image.

Proposition 6.1. *The map $\varphi_{\Delta} : \partial\text{CV}(F_n) \rightarrow \partial\mathcal{FS} \cup \Theta$ does not depend on the choice of Skora paths, nor does it depend on the standard simplex Δ . Moreover, it is equivariant with respect to the action of $\text{Out}(F_n)$ on $\partial\text{CV}(F_n)$ and on $\partial\mathcal{FS}$.*

Proof. If $[T] \in \partial\text{CV}(F_n)$ is simplicial then Lemma 5.5 shows that the diameter of the image under Υ of any Skora path converging to $[T]$ is finite. Thus it suffices to show the proposition for trees $[T] \in \partial\text{CV}(F_n)$ with $T_c \neq \emptyset$.

Let T be such a tree. As T is not simplicial, the tree T' obtained from T by equivariantly collapsing all edges of T to points is not trivial, and there is a one-Lipschitz alignment preserving map $T \rightarrow T'$. By Proposition 4.3 and Lemma 4.1,

for every Skora path (x_t) connecting a point x_0 in a standard simplex Δ to T there is a Skora path (y_t) connecting a point $y_0 \in \Delta$ to T' so that the Hausdorff distance between $\Upsilon(x_t)$ and $\Upsilon(y_t)$ is uniformly bounded. Thus it suffices to consider trees $[T]$ with $T_d = \emptyset$.

Let $[T] \in \partial CV(F_n)$ with $T_d = \emptyset$ and let $(x_t)_{t \geq 0} \subset \overline{cv_0(F_n)}$ be a Skora path guided by an optimal train track map $x_0 \rightarrow \hat{T}$ where $x_0 \in \Delta$ and where \hat{T} is a representative of $[T]$. Such a map exists by Lemma 3.3. By Lemma 5.3, the path (x_t) is contained in $\overline{cv_0(F_n)}^{++}$.

Choose another (not necessarily distinct) standard simplex Λ and a Skora path (y_s) guided by an optimal train track map $y_0 \rightarrow T$ where $y_0 \in \Lambda$ and where T is a representative of $[T]$ (in general we expect that $T \neq \hat{T}$). If the diameters of both $\Upsilon(x_t)$ and $\Upsilon(y_s)$ are finite then there is nothing to show, so by perhaps exchanging (x_t) and (y_s) we may assume that the diameter of $\Upsilon(y_s)$ is infinite.

Let $f : y_0 \rightarrow T$ be the train track map which guides the folding path (y_t) . We claim that there is a sequence $t_i \rightarrow \infty$, a sequence $S_i \subset \Lambda$ of points in the standard simplex Λ , a sequence $a_{t_i} > 0$ such that $a_{t_i} x_{t_i} \rightarrow T$ in $\overline{cv(F_n)}$ and a sequence of (not necessarily optimal) train track maps $f_i : S_i \rightarrow a_{t_i} x_{t_i}$ with the following properties.

- (1) For every $U \in \Lambda$ the Lipschitz constant of any equivariant map $U \rightarrow a_{t_i} x_{t_i}$ is at least $1/2$.
- (2) The maps f_i converge as $i \rightarrow \infty$ to the train track map f in the sense of Lemma 3.2.

Let $B \subset \overline{cv(F_n)}$ be the set of all trees V such that the minimum of the smallest Lipschitz constant for equivariant maps $U \rightarrow V$ where U runs through the points in the standard simplex Λ equals one. Then B is a compact subset of $\overline{cv(F_n)}$. For $t > 0$ let $b_t > 0$ be such that $b_t x_t \in B$. Then we have $b_t x_t \rightarrow T$ in $\overline{cv(F_n)}$ ($t \rightarrow \infty$) with respect to the equivariant Gromov Hausdorff topology.

Let $\tilde{v} \in S$ be a preimage of the unique vertex v of the graph S/F_n . Let $\{g_1, \dots, g_n\}$ be a free basis of F_n which determines the simplex Λ and such that for each i the axis of g_i passes through \tilde{v} (in particular, if g_i has a fixed point in S then this fixed point equals \tilde{v}). Let K be the compact subtree of T which is the convex hull of the points $f(\tilde{v}), g_j f(\tilde{v}), g_j^{-1} f(\tilde{v})$ ($j = 1, \dots, n$). Then for each t there is a point $v_t \in b_t x_t$, and there is a compact subtree K_t of $b_t x_t$ which is the convex hull of the points $v_t, g_j v_t, g_j^{-1} v_t$ ($j = 1, \dots, n$) and such that the trees K_t converge to K in the usual Gromov Hausdorff topology (see [P89] for details).

Since for each $U \in \Lambda$ the quotient graph U/F_n has a single vertex, for each t there is a tree $U_t \in \Lambda$, there is a number $a_t \leq b_t$ and a morphism $U_t \rightarrow a_t x_t$ which maps a preimage of a vertex of U_t/F_n to v_t . Note that in general we have $a_t \neq b_t$, however by construction, $a_t/b_t \rightarrow 1$ ($t \rightarrow \infty$). In particular, for sufficiently large t the trees $a_t x_t$ have property (1) above.

Using again the fact that U_t has a single vertex, we may in fact assume that the morphism $U_t \rightarrow a_t x_t$ is a train track map, however it may not be optimal. Namely, the elements g_i ($1 \leq i \leq n$) generate F_n and therefore if there was only one gate for

the morphism at a vertex of U_t then the tree $a_t x_t$ can not be minimal. By Lemma 3.2, there is a sequence $t_i \rightarrow \infty$ so that the sequence

$$f_i : S_i = U_{t_i} \rightarrow a_{t_i} x_{t_i}$$

of train track maps has property (2) above as well.

The subspace $\Sigma \subset \overline{cv(F_n)}$ of all trees x for which the minimum of the smallest Lipschitz constant for equivariant maps from trees $U \in \Lambda$ to x is contained in the interval $[1/2, 1]$ is compact, and it contains B . For each i connect S_i to $a_{t_i} x_{t_i}$ by a Skora path (y_s^i) guided by the train track map f_i . For sufficiently large i the corresponding unnormalized Skora paths are entirely contained in Σ .

By Proposition 3.7, up to passing to a subsequence, the paths (y_s^i) converge as $i \rightarrow \infty$ to a Skora path (y_s) . By the discussion in the previous paragraph and construction, this path is guided by the train track map f , and it connects S to $[T]$. The image under the map Υ of the family of paths $(y_s^i), (y_s)$ is a family of reparametrized L -quasi-geodesics in the hyperbolic graph \mathcal{FS} . Moreover, since $T_d = \emptyset$, Corollary 5.4 shows that the path (y_s) is contained in $\overline{cv_0(F_n)}^{++}$, and the same holds true for each of the paths (y_s^i) .

The diameter of $\Upsilon(\Lambda) \subset \mathcal{FS}$ is bounded independent of the standard simplex Λ . Let $b \geq 0$ be such that $\Upsilon(x_b)$ is a coarsely well defined shortest distance projection of $\Upsilon(\Lambda)$ into the reparametrized L -quasi-geodesic $\Upsilon(x_t)$.

By hyperbolicity of \mathcal{FS} , for every $u > b$ and every $v > u$, the image under Υ of any Skora path connecting a point in Λ to x_v passes through a uniformly bounded neighborhood of $\Upsilon(x_u)$. Thus for all $u > b$ and all i such that $t_i > u$, the reparametrized L -quasi-geodesic $\Upsilon(y_s^i)$ in \mathcal{FS} passes through a uniformly bounded neighborhood of $\Upsilon(x_u)$.

We claim that the same holds true for the reparametrized L -quasi-geodesic $\Upsilon(y_s)$. To this end recall that we assumed that the diameter of $\Upsilon(y_s)$ is infinite. Let $\tau > 0$ be arbitrary. Since $(y_s) \subset \overline{cv_0(F_n)}^{++}$, Corollary 2.11 shows that for all sufficiently large i the reparametrized quasi-geodesic $\Upsilon(y_s^i)$ connecting a point in $\Upsilon(\Lambda)$ to $\Upsilon(x_{t_i})$ passes through the L -neighborhood of $\Upsilon(y_\tau)$.

On the other hand, it also passes through a uniformly bounded neighborhood of $\Upsilon(x_u)$. Since the diameter of $\Upsilon(y_s)$ is infinite, for sufficiently large τ the point $\Upsilon(x_u)$ coarsely lies between $\Upsilon(y_0)$ and $\Upsilon(y_\tau)$. This shows that the reparametrized quasi-geodesic $\Upsilon(y_s)$ passes through a uniformly bounded neighborhood of $\Upsilon(x_u)$ as claimed above.

We use this fact to show that the diameter of $\Upsilon(x_t)$ is infinite. Namely, assume to the contrary that the diameter of $\Upsilon(x_t)$ is finite. Then by construction, for each i the diameter of $\Upsilon(y_s^i)$ is bounded from above by a number $C > 0$ not depending on i . Choose a number $t > 0$ so that the distance between $\Upsilon(\Lambda)$ and $\Upsilon(y_t)$ is at least $C + 10m$ where $m > 0$ is as in Corollary 2.11. For sufficiently large i the path $\Upsilon(y_s^i)$ passes through the m -neighborhood of $\Upsilon(y_t)$ which is a contradiction.

By symmetry, we conclude that the diameter of $\Upsilon(x_t)$ is infinite if and only if the diameter of $\Upsilon(y_t)$ is infinite. Moreover, if this holds true then there is a

number $p > 0$ only depending on Λ, Δ such that $\Upsilon(y_s)$ is a p -fellow traveller of $\Upsilon(x_t)$. As a consequence, the map φ_Δ indeed does not depend on the choice of Δ or on the choice of Skora paths. This then implies that the map φ_Δ is moreover $\text{Out}(F_n)$ -equivariant. \square

Remark 6.2. The proof of Proposition 6.1 shows more generally the following. Let Δ be a standard simplex and let $(x_t) \subset \overline{cv_0(F_n)}^+$ be any normalized folding path. If $[x_t] \rightarrow [T] \in \partial\text{CV}(F_n)$ in the equivariant Gromov Hausdorff topology then the reparametrized quasi-geodesic $\Upsilon(x_t)$ in \mathcal{FS} is of infinite diameter if and only if $\varphi_\Delta([T]) \in \partial\mathcal{FS}$, and in this case we have $\Upsilon(x_t) \rightarrow \varphi_\Delta([T])$ in $\mathcal{FS} \cup \partial\mathcal{FS}$.

Remark 6.3. The proof of Proposition 6.1 also gives some information on Skora paths whose images under Υ have finite diameter. Namely, if $\varphi_\Delta([T]) = \Theta$ then there is a point $\xi \in \mathcal{FS}$ and there is a number $r > 0$ so that for every Skora path $(x_t) \subset \overline{cv_0(F_n)}^+$ with $[x_t] \rightarrow [T]$ in $\text{CV}(F_n) \cup \partial\text{CV}(F_n)$ and for all large enough t the point $\Upsilon(x_t)$ is contained in the r -neighborhood of ξ in \mathcal{FS} .

By Proposition 6.1, we can define

$$\psi = \varphi_\Delta : \partial\text{CV}(F_n) \rightarrow \partial\mathcal{FS} \cup \Theta$$

for some (and hence every) standard simplex Δ . Similarly we define

$$\psi_{\mathcal{C}} : \partial\text{CV}(F_n) \rightarrow \partial\mathcal{CS} \cup \Theta, \psi_{\mathcal{F}} : \partial\text{CV}(F_n) \rightarrow \partial\mathcal{FF} \cup \Theta.$$

The maps $\psi, \psi_{\mathcal{C}}, \psi_{\mathcal{F}}$ do not depend on choices, and they are $\text{Out}(F_n)$ -equivariant.

Our next goal is to show that the map ψ is onto $\partial\mathcal{FS}$. As a main preparation we establish the following lemma which is motivated by the work of Klarreich (Proposition 6.4 of [K99]). For convenience of notation we extend the map ψ to $\overline{\text{CV}(F_n)}$ by defining $\psi([T]) = \Theta$ for every simplicial free projective F_n -tree $[T] \in \text{CV}(F_n)$.

Lemma 6.4. *Let $(T_i) \subset cv_0(F_n)$ be a sequence whose projectivization converges in $\text{CV}(F_n) \cup \partial\text{CV}(F_n)$ to a point $[T] \in \overline{\text{CV}(F_n)}$. If $\psi([T]) = \Theta$ then $(\Upsilon(T_i))$ does not converge to a point in $\partial\mathcal{FS}$.*

Proof. We follow the reasoning in the proof of Proposition 6.4 of [K99]. The case that $[T] \in \text{CV}(F_n)$ is immediate from Corollary 2.11, so let $([T_i]) \subset \text{CV}(F_n)$ be a sequence which converges to a point $[T] \in \partial\text{CV}(F_n)$ with $\psi([T]) = \Theta$.

For each i let $T_i \in cv_0(F_n)$ be a representative of $[T_i]$. We argue by contradiction and we assume that the sequence $(\Upsilon(T_i))$ converges to a point in the Gromov boundary of \mathcal{FS} .

For each i there is a standard simplex $\Delta_i \subset \overline{cv_0(F_n)}^{++}$ so that the distance in \mathcal{FS} between $\Upsilon(T_i)$ and $\Upsilon(\Delta_i)$ is at most 1. Such a simplex contains the covolume one rescaling of the universal covering of a rose obtained from T_i/F_n by collapsing a maximal tree to a point.

Choose a train track map $f_j : r_0^{ij} \rightarrow T_j^i$ where T_j^i is a representative of $[T_j]$ and where $r_0^{ij} \in \Delta_i$. Let (r_t^{ij}) be a Skora path connecting r_0^{ij} to T_j^i which is guided by

f_j . As the endpoints of the paths are contained in $cv_0(F_n)$, the same holds true for the entire paths.

The initial points of the paths (r_t^{ij}) ($j > i$) are contained in the compact subset Δ_i of $\overline{cv_0(F_n)}^{++}$. Hence by Proposition 3.7, applied to the Skora paths r_t^{ij} , up to passing to a subsequence we may assume that the paths (r_t^{ij}) converge as $j \rightarrow \infty$ locally uniformly in $\overline{cv(F_n)}$ to a Skora path $t \rightarrow r_t^i$ issuing from a point $r_0^i \in \Delta_i$. By the reasoning in the proof of Lemma 3.3, since $[T_j] \rightarrow [T]$ we may assume that the path (r_t^i) is defined by a train track map $r_0^i \rightarrow T$ where T is a representative of $[T]$ and hence it connects r_0^i to $[T]$.

By Remark 6.3, there is a point $\eta \in \mathcal{FS}$, and there is a number $b > 0$ only depending on T with the following property. For any Skora path $(x_t) \subset \overline{cv_0(F_n)}^+$ guided by some train track map $x_0 \rightarrow T$ where T is a representative of $[T]$ and for all large enough t , the point $\Upsilon(x_t)$ is contained in the b -neighborhood of η in \mathcal{FS} . In particular, for large enough t the point $\Upsilon(r_t^i)$ is contained in the b -neighborhood of η . Fix a number t_0 with this property.

By the second part of Corollary 5.4, for large enough j the distance between $\Upsilon(r_{t_0}^{ij})$ and $\Upsilon(r_{t_0}^i)$ is bounded by a number $k([T])$ only depending on $[T]$. As a consequence, if $(\cdot)_\xi$ is the Gromov product based at $\eta \in \mathcal{HG}$ then we have

$$(\Upsilon(T_i) \mid \Upsilon(T_j))_\eta \leq q$$

for infinitely many i, j where $q > 0$ is a constant depending on T but not on i, j . This is a contradiction to the assumption that the sequence $(\Upsilon(T_i))$ converges to a point in the Gromov boundary of \mathcal{FS} . \square

Remark 6.5. Lemma 6.4 does *not* state that if $(x_t) \subset cv_0(F_n)$ is any sequence converging to a point $[T] \in \partial CV(F_n)$ with $\psi([T]) = \Theta$ then $\Upsilon(x_t)$ is bounded. In fact, we believe that there should be sequences for which this is not true.

Next we have

Lemma 6.6. *If $[T_i] \subset CV(F_n)$ is any sequence which converges to $[T] \in \partial CV(F_n)$ with $\psi([T]) \in \partial \mathcal{FS}$ then $\Upsilon(T_i)$ converges to $\psi([T])$ in $\mathcal{FS} \cup \partial \mathcal{FS}$.*

Proof. Let Δ be a standard simplex. For each i let (r_t^i) be a Skora path connecting a point in Δ to a tree $T_i \in cv_0(F_n)$ representing the class $[T_i]$. By Proposition 3.7, up to passing to a subsequence we may assume that the paths (r_t^i) converge as $i \rightarrow \infty$ to a Skora path (x_t) connecting a point in Δ to a representative of $[T]$.

Since the paths $\Upsilon(r_t^i)$ are uniform reparametrized quasi-geodesics in \mathcal{FS} and since the path (x_t) connects a point in Δ to $[T]$, the path $\Upsilon(x_t)$ is a (reparametrized) quasi-geodesic ray connecting $\Upsilon(x_0)$ to $\psi([T])$. The lemma now follows from the second part of Corollary 5.4 and hyperbolicity of \mathcal{FS} . \square

As a corollary we obtain

Corollary 6.7. $\psi(\partial CV(F_n)) \supset \partial \mathcal{FS}$.

Proof. Since $\Upsilon(cv_0(F_n))$ equals the vertex set of \mathcal{FS} , for every $\eta \in \partial\mathcal{FS}$ there is a sequence $(T_i) \subset cv_0(F_n)$ so that $\Upsilon(T_i) \rightarrow \eta$.

By compactness of $\overline{CV(F_n)}$, by passing to a subsequence we may assume that $[T_i] \rightarrow [T]$ for some $[T] \in \overline{CV(F_n)}$. By Lemma 6.4, if $\psi([T]) = \Theta$ then the sequence $\Upsilon(T_i)$ does not converge to η . Thus $\psi([T]) = \chi \in \partial\mathcal{FS}$, and it follows from Lemma 6.6 that $\chi = \eta$. \square

Let now

$$\mathcal{FT} = \psi^{-1}(\partial\mathcal{FS}) \subset \partial CV(F_n).$$

By $\text{Out}(F_n)$ -equivariance of the map ψ , the set \mathcal{FT} is an $\text{Out}(F_n)$ -invariant subset of $\partial CV(F_n)$. We showed above that the restriction of ψ maps \mathcal{FT} onto $\partial\mathcal{FS}$. We also have

Lemma 6.8. *The restriction of the map ψ to \mathcal{FT} is continuous and closed.*

Proof. To show that the restriction to \mathcal{FT} of the map ψ is continuous, note that if $[T_i] \rightarrow [T] \in \mathcal{FT}$ then there is a sequence (r_t^i) of Skora paths starting at a point in a standard simplex Δ so that $r_t^i \rightarrow [T_i]$ ($t \rightarrow \infty$) and $r^i \rightarrow r$ locally uniformly where r is a Skora path connecting Δ to $[T]$. The images under Υ of these paths are reparametrized quasi-geodesics in \mathcal{FS} . Continuity now follows from hyperbolicity of \mathcal{FS} and Corollary 2.11.

Since the Gromov topology on $\partial\mathcal{FS}$ is metrizable, to show that the restriction of ψ is closed it suffices to show the following. If $[T_i] \subset \mathcal{FT}$ is any sequence and if $\psi([T_i]) \rightarrow \eta \in \partial\mathcal{FS}$ then up to passing to a subsequence, we have $[T_i] \rightarrow [U]$ where $\psi([U]) = \eta$.

Assume to the contrary that this is not the case. By compactness of $\partial CV(F_n)$ there is then a sequence $[T_i] \subset \mathcal{FT}$ so that $\psi([T_i]) \rightarrow \eta$ and such that $[T_i] \rightarrow [S] \in \partial CV(F_n)$ where either $[S] \notin \mathcal{FT}$ or $\psi([S]) \neq \eta$.

Now if $[S] \in \mathcal{FT}$ then by continuity of the map ψ , we have

$$\psi([S]) = \lim_{i \rightarrow \infty} \psi([T_i]) = \eta.$$

Since we assumed that $\psi([S]) \neq \eta$ this is impossible.

Thus $[S] \notin \mathcal{FT}$. However, it follows from the hypotheses that there is a sequence $(S_i) \subset cv_0(F_n)$ with $[S_i] \rightarrow [S]$ and such that $\Upsilon(S_i) \rightarrow \eta$ in $\mathcal{FS} \cup \partial\mathcal{FS}$. Here S_i can be chosen to be a point on the Skora path r_t^i which is such that $\Upsilon(S_i)$ is sufficiently close to $\Upsilon(T_i)$ in $\mathcal{FS} \cup \partial\mathcal{FS}$. This violates Lemma 6.4. The corollary follows. \square

Remark 6.9. The results in this section do not use any specific property of Outer space and can easily be formulated in a more abstract setting which is valid for example for Teichmüller space and the curve graph as in [K99] and for various disc graphs in a handlebody [H11].

Recall the definition of the hyperbolic $\text{Out}(F_n)$ -graphs $(\mathcal{CS}, \Upsilon_{\mathcal{C}})$ and $(\mathcal{FF}, \Upsilon_{\mathcal{F}})$ and of the maps

$$\psi_{\mathcal{C}} : \partial\text{CV}(F_n) \rightarrow \partial\mathcal{CS} \cup \Theta \text{ and } \psi_{\mathcal{F}} : \partial\text{CV}(F_n) \rightarrow \partial\mathcal{FF} \cup \Theta.$$

Since for any tree $T \in \overline{cv_0(F_n)}^+$ the distance in \mathcal{CS} between $\Upsilon(T) \in \mathcal{FS} \subset \mathcal{CS}$ and $\Upsilon_{\mathcal{C}}(T)$ is at most one and since $\Upsilon_{\mathcal{F}} = \Omega \circ \Upsilon$ for a coarsely Lipschitz map $\Omega : \mathcal{CS} \rightarrow \mathcal{FF}$, we obtain as an immediate consequence

Corollary 6.10. (1) *If $\psi([T]) = \Theta$ then $\psi_{\mathcal{C}}([T]) = \Theta$.*

(2) *If $\psi_{\mathcal{C}}([T]) = \Theta$ then $\psi_{\mathcal{F}}([T]) = \Theta$.*

As a consequence of the results in this section, the boundaries of \mathcal{FS} , \mathcal{CS} and \mathcal{FF} can be identified as a set with the quotient of an $\text{Out}(F_n)$ -invariant subspace of $\partial\text{CV}(F_n)$ by an equivalence relation defined by the map ψ and the map $\psi_{\mathcal{C}}$ and the map $\psi_{\mathcal{F}}$.

We complete the section with some first easy information on the fibres of ψ . From now on we only consider F_n -trees $T \in \partial cv(F_n)$ (or projective trees $[T] \in \partial\text{CV}(F_n)$) with dense orbits. For simplicity we call such a tree *dense*.

The following definition is due to Paulin (see [G00]).

Definition 6.11. A *length measure* μ on T is an F_n -invariant collection

$$\mu = \{\mu_I\}_{I \subset T}$$

of locally finite Borel measures on the finite arcs $I \subset T$; it is required that for $J \subset I$ we have $\mu_J = (\mu_I)|_J$.

The Lebesgue measure λ defining the metric on T is an example of a length measure on T with full support.

Denote by $M_0(T)$ the set of all non-atomic length measures on T . By Corollary 5.4 of [G00], $M_0(T)$ is a finite dimensional convex set which is projectively compact. Up to homothety, there are at most $3n - 4$ non-atomic ergodic length measures. In particular, $M_0(T)$ is a cone over a compact convex polyhedron with finitely many vertices. Each non-atomic length measure $\mu \in M_0(T)$ defines an F_n -tree $T_{\mu} \in \partial cv(F_n)$ as follows [G00]. Define a pseudo-metric d_{μ} on T by $d_{\mu}(x, y) = \mu([x, y])$. Making this pseudo-metric Hausdorff gives an \mathbb{R} -tree T_{μ} .

Corollary 6.12. *Let $T \in \partial cv(F_n)$ be a tree with dense orbits; then $\psi([T_{\mu}]) = \psi([T])$ for all $\mu \in M_0(T)$.*

Proof. Let ν be a length measure on T which is contained in the interior of the convex set $M_0(T)$. Let ζ be a point in $M_0(T)$ which projects to a vertex in the projectivization of $M_0(T)$; this is an ergodic measure in $M_0(T)$. Up to rescaling, there is a one-Lipschitz alignment preserving map $T_{\nu} \rightarrow T_{\zeta}$. Proposition 4.3 shows that $\psi([T_{\nu}]) = \psi([T_{\zeta}])$.

Now if $\xi \in M_0(T)$ is arbitrary then there is an ergodic measure $\beta \in M_0(T)$, and there is a one-Lipschitz alignment preserving map $T_{\xi} \rightarrow T_{\beta}$. Using once more Proposition 4.3, we deduce that $\psi([T_{\xi}]) = \psi([T_{\beta}]) = \psi([T_{\nu}])$. Since the Lebesgue measure on T defines a point in $M_0(T)$ this shows the corollary. \square

We use Proposition 4.3 and Corollary 6.12 to show

Corollary 6.13. *Let $T, T' \in \partial cv(F_n)$ be trees with dense orbits and let $\rho : T \rightarrow T'$ be alignment preserving. Then $\psi([T]) = \psi([T'])$.*

Proof. Let $T, T' \in \partial cv(F_n)$ be trees with dense orbits and assume that there is an alignment preserving map $\rho : T \rightarrow T'$.

As is shown in [G00], if μ' is a non-atomic length measure on T' then there is a length measure μ on T such that $\rho_*\mu = \mu'$. This means that for every segment $I \subset T$ we have $\mu(I) = \mu'(\rho I)$. As a consequence, there is a one-Lipschitz alignment preserving map $\hat{\rho} : T_\mu \rightarrow T_{\mu'}$. Proposition 4.3 shows that $\psi([T_\mu]) = \psi([T_{\mu'}])$. The corollary now follows from Corollary 6.12. \square

7. INDECOMPOSABLE TREES

The goal of this section is to obtain some information on the structure of indecomposable trees which is needed for the proof of the theorems from the introduction.

Let $\Delta \subset \partial F_n \times \partial F_n$ be the diagonal. The *zero lamination* $L^2(T)$ of an \mathbb{R} -tree T with an isometric action of F_n is the closed F_n -invariant subset of $\partial F_n \times \partial F_n - \Delta$ which is the set of all accumulation points of pairs of fixed points of any family of conjugacy classes with translation length on T that tends to 0. The zero lamination of a tree $T \in cv(F_n)$ is empty. For $T \in \partial cv(F_n)$, it only depends on the projective class $[T] \in \partial CV(F_n)$ of T .

In [CHL07] the following topological interpretation of the zero lamination of an \mathbb{R} -tree $T \in \partial cv(F_n)$ is given.

The union $\hat{T} = \bar{T} \cup \partial T$ of the metric completion \bar{T} of T with the Gromov boundary ∂T of T can be equipped with an *observer's topology*. With respect to this topology, \hat{T} is a compact F_n -space, and the inclusion $T \rightarrow \hat{T}$ is continuous [CHL07]. Isometries of T induce homeomorphisms of \hat{T} (see p.903 of [CHL09]).

There is an explicit description of \hat{T} as follows. Namely, let again $L^2(T)$ be the zero lamination of T . There is an F_n -equivariant [LL03] continuous (Proposition 2.3 of [CHL07]) map

$$Q : \partial F_n \rightarrow \hat{T}$$

such that $L^2(T) = \{(\xi, \zeta) \in \partial F_n \times \partial F_n - \Delta \mid Q(\xi) = Q(\zeta)\}$. The map Q determines an equivariant homeomorphism

$$\partial F_n / L^2(T) \rightarrow \hat{T}$$

(Corollary 2.6 of [CHL07]), i.e. the tree \hat{T} is the quotient of ∂F_n by the equivalence relation obtained by identifying all points $\xi, \xi' \in \partial F_n$ with $Q(\xi) = Q(\xi')$, and a pair of points (ξ, ξ') is identified if and only if it is contained in $L^2(T)$.

The *limit set* of \bar{T} is defined to be the set

$$\Omega = Q^2(L^2(T)) \subset \bar{T} \subset \hat{T}$$

where the notation Q^2 refers to applying the map Q to a pair of points with the same image.

A finitely generated subgroup $H < F_n$ is free, and its boundary ∂H is naturally a closed subset of the boundary ∂F_n of F_n . If H fixes a point in an \mathbb{R} -tree T then the set $\partial H \times \partial H - \Delta$ of pairs of distinct points in ∂H , viewed as a subset of $\partial F_n \times \partial F_n - \Delta$, is contained in $L^2(T)$. We say that a leaf $\ell \in L^2(T)$ is *carried* by a finitely generated subgroup H of F_n if it is a point in $\partial H \times \partial H - \Delta$.

Define a leaf $\ell \in L^2(T)$ to be *regular* if ℓ is not carried by the stabilizer of a point in \bar{T} (this makes sense since point stabilizers of \bar{T} are finitely generated) and if moreover there exists a sequence $\ell_n \subset L^2(T)$ of leaves converging to ℓ such that the $x_n = Q^2(\ell_n)$ are distinct. The set of regular leaves of $L^2(T)$ is the *regular sublamination* $L_r(T)$. It is F_n -invariant but in general not closed.

The space \mathcal{ML} of *measured laminations* for F_n is a closed subspace of the space of all locally finite F_n -invariant Borel measures on $\partial F_n \times \partial F_n - \Delta$, equipped with the weak*-topology. Dirac measures on pairs of fixed points of all elements in some primitive conjugacy class of F_n are dense in \mathcal{ML} [Ma95]. There is a continuous length pairing [KL09]

$$\langle \cdot, \cdot \rangle : \overline{cv(F_n)} \times \mathcal{ML} \rightarrow [0, \infty).$$

For $T \in \overline{cv(F_n)}$ and $\mu \in \mathcal{ML}$ we have $\langle T, \mu \rangle = 0$ if and only if μ is supported in $L^2(T)$ [KL09].

The following is the main result of this section.

Proposition 7.1. *Let $[T] \in \partial\text{CV}(F_n)$ be indecomposable and let μ be a measured lamination with $\langle T, \mu \rangle = 0$ and $\mu(L^2(T) - L_r(T)) = 0$. If $[S] \in \partial\text{CV}(F_n)$ is indecomposable and if $\langle S, \mu \rangle = 0$ then \hat{S} and \hat{T} equipped with the observer's topology are F_n -equivariantly homeomorphic.*

Denote by

$$\mathcal{IT} \subset \partial\text{CV}(F_n)$$

the $\text{Out}(F_n)$ -invariant subset of indecomposable projective trees. For an indecomposable projective tree $[T] \in \mathcal{IT} \subset \partial\text{CV}(F_n)$ define $\mathcal{H}([T]) \subset \mathcal{IT}$ to be the set of all indecomposable projective trees $[S]$ so that \hat{S} is F_n -equivariantly homeomorphic to \hat{T} .

Corollary 7.2. *Let $[T] \in \mathcal{IT}$ be such that there is a measured lamination μ with $\langle T, \mu \rangle = 0$ and $\mu(L^2(T) - L_r(T)) = 0$. Then $\mathcal{H}([T])$ is a closed subset of \mathcal{IT} .*

Proof. If \hat{S}, \hat{T} are F_n -equivariantly homeomorphic then for every measured lamination μ we have $\langle T, \mu \rangle = 0$ if and only if $\langle S, \mu \rangle = 0$ [CHL07, KL09]. Thus by Proposition 7.1, if $[T] \in \mathcal{IT}$ and if there is a measured lamination μ with $\mu(L^2(T) - L_r(T)) = 0$ then we have

$$\mathcal{H}([T]) = \{[S] \in \mathcal{IT} \mid \langle S, \mu \rangle = 0\}.$$

The corollary now follows from continuity of the length pairing $\langle \cdot, \cdot \rangle$ [KL09]. \square

Remark 7.3. We do not know whether every indecomposable tree $[T] \in \mathcal{IT}$ admits a measured lamination μ with $\langle T, \mu \rangle = 0$ and $\mu(L^2(T) - L_r(T)) = 0$.

The remainder of this section is devoted to the proof of Proposition 7.1.

Recall that a minimal subset of an F_n -space is an invariant set with each orbit dense. In the case that T is an indecomposable tree with a free action of F_n , Proposition 5.14 of [CH14] together with Proposition 3.7 and Proposition 4.11 of [CHR11] show that the regular sublamination $L_r(T)$ is minimal. The main technical tool for the proof of Proposition 7.1 is the following extension of this result to indecomposable trees which are not necessarily free.

Lemma 7.4. *Let $[T] \in \partial\text{CV}(F_n)$ be indecomposable; then the regular sublamination $L_r(T)$ of $L^2(T)$ is minimal.*

Proof. Choose a basis \mathcal{A} for F_n . Elements of F_n are reduced finite words in the alphabet \mathcal{A}^\pm . A point X in the boundary ∂F_n of F_n is an infinite reduced word in \mathcal{A}^\pm whose first letter will be denoted by X_1 . The cylinder

$$C_{\mathcal{A}}(1) = \{(X, Y) \in \partial F_n \times \partial F_n - \Delta \mid X_1 \neq Y_1\}$$

is compact, and the same holds true for the *relative limit set*

$$\Omega_{\mathcal{A}} = Q^2(L^2(T) \cap C_{\mathcal{A}}(1)) \subset \bar{T}.$$

The *compact heart* $K_{\mathcal{A}}$ of \bar{T} is the convex hull of $\Omega_{\mathcal{A}}$. It is a compact subtree of \bar{T} (see Section 5.3 of [CH14]).

The tree T is called of *Levitt type* if the limit set Ω is a totally disconnected subset of \bar{T} . That this definition is equivalent to earlier definitions in the literature is Theorem 5.11 of [CH14]. We first show the lemma for indecomposable trees T of Levitt type. For such trees the relative limit set $\Omega_{\mathcal{A}}$ is a Cantor set [CH14].

For any element $a \in \mathcal{A}^\pm$ consider the partial isometry defined by the restriction of the action of a^{-1} .

$$K_{\mathcal{A}} \cap aK_{\mathcal{A}} \rightarrow K_{\mathcal{A}} \cap a^{-1}K_{\mathcal{A}}, x \rightarrow a^{-1}x.$$

The thus defined system of partial isometries determines a directed labeled graph Γ whose vertex set is the set of components of $K_{\mathcal{A}}$ (the compact heart is connected, but we will use this construction below also for compact subsets of $K_{\mathcal{A}}$ which may be disconnected) and where an edge labeled with $a \in \mathcal{A}$ is a partial isometry connecting a component of $K_{\mathcal{A}}$ which intersects the domain of definition of a to the component containing its image. As $K_{\mathcal{A}}$ is connected, the graph Γ is a rose.

Apply the *Rips machine* (see Section 3 of [CH14]) to the system of isometries $(K_{\mathcal{A}}, \mathcal{A}^\pm)$. Its first step consists in choosing $K_{\mathcal{A}}(1)$ to be the subset of $K_{\mathcal{A}}$ of all points which are in the domain of at least two distinct partial isometries from \mathcal{A}^\pm and restricting the partial isometries from \mathcal{A}^\pm to $K_{\mathcal{A}}(1)$. This determines a new directed labeled graph Γ_1 . There is a natural morphism τ_1 from the directed graph Γ_1 into the directed graph Γ (see Section 3 of [CH14] for details).

Since T is of Levitt type, the Rips machine never stops [CH14]. The output is a sequence Γ_i ($i > 0$) of graphs without vertices of valence zero or one (Proposition

5.6 of [CH14]) so that every leaf of $L^2(T)$ can be represented by a path in each of the Γ_i . For each i there is a homotopy equivalence $\tau_{i+1} : \Gamma_{i+1} \rightarrow \Gamma_i$ [CH14].

As the translates of the cylinder $C_{\mathcal{A}}(1)$ under the diagonal action of F_n cover all of $\partial F_n \times \partial F_n - \Delta$, each F_n -orbit of a point in T with non-trivial stabilizer intersects $\Omega_{\mathcal{A}}$ and hence $K_{\mathcal{A}}$. Now if $x \in K_{\mathcal{A}}$ is such a point and if $w = a_1 \cdots a_m$ is a reduced word in \mathcal{A} such that $wx = x$, then this word defines a loop of length m in the graph Γ_m obtained in the m -th step of the Rips machine.

Define the *size* of an element of F_n to be its length with respect to the generating set \mathcal{A}^{\pm} . There are only finitely many F_n -orbits of points in T with nontrivial point stabilizers [CH14], and each point stabilizer for T is finitely generated [GL95]. The discussion in the previous paragraph shows that there are numbers $i_0 > 0, p > 0, u \geq 0$ with the following properties. For each $i \geq i_0$ the graph Γ_i contains precisely $u \geq 0$ labeled loops $\alpha_1^i, \dots, \alpha_u^i$ which define elements of F_n of size at most p . Each of these elements stabilizes a point $x \in \Omega_{\mathcal{A}}$. Up to conjugation, the stabilizers of all points $x \in \Omega_{\mathcal{A}}$ are generated by those elements of F_n which are defined by these labeled loops. The homotopy equivalence $\tau_{i+1} : \Gamma_{i+1} \rightarrow \Gamma_i$ [CH14] maps the loop α_j^{i+1} to the loop α_j^i .

By Theorem 5.3 of [CH14], there are only finitely many F_n -orbits of points in $Q^2(L^2(T))$ whose preimage under the map Q have cardinality at least three. These points contain the points with non-trivial stabilizer. Thus the union of all F_n -orbits of points in the limit set $\Omega = Q^2(L^2(T))$ whose preimage under Q have cardinality at least three is a countable subset of Ω .

We follow the proof of Lemma 3.6 of [CHR11] and we assume to the contrary that there is a closed proper subset $L_0 \subset L_r(T)$. Since $\Omega_{\mathcal{A}}$ is uncountable and since $L_r(T)$ does not have isolated points, by the previous paragraph there is a leaf $\ell \in L_r(T) - L_0$ such that $x = Q^2(\ell) \in \Omega_{\mathcal{A}}$ and that moreover the preimage of x under Q consists precisely of the two endpoints of ℓ .

The leaf ℓ is not contained in the closure of L_0 and therefore there is a compact subarc $\rho = \ell[-k, k]$ of ℓ which is not shared by any leaf in L_0 . We may assume that the size $2k$ of the arc ρ is at least $4p$ and that none of its prefixes coincides with any of the words defining a loop α_j^i or its inverse. Note that this makes sense since the loops α_j^i define only finitely many conjugacy classes in F_n .

For sufficiently large i , every (non-oriented) loop of size at most p in the graph Γ_i is one of the loops α_j^i (see the proof of Lemma 3.6 of [CHR11]). As a consequence, the initial segment of the subarc ρ of ℓ is distinct from any of these loops.

Choose now a sequence $\ell_j \subset L_r(T)$ of pairwise distinct leaves which converge to ℓ . We may assume that $\ell_j[-k, k] = \rho$ for all j . Since Ω is a Cantor set, for every u and for sufficiently large m the leaves ℓ_j ($0 \leq j \leq u$) are contained in distinct components of $K_{\mathcal{A}}(m)$, and these components are distinct from components containing a point with non-trivial stabilizer. On the other hand, there exists a number $i(k) > i_0$ such that for $i > i(k)$ the size of any loop in Γ_i which is distinct from one of the loops α_j^i is at least $3k$. This implies that there are $n > 0, m > 0$

such that $\ell_n[-k, k]$ does not cross through any vertex of Γ_m of valence strictly bigger than two (compare the proof of Lemma 3.6 of [CHR11] for details).

As in the proof of Lemma 3.6 of [CHR11] we conclude that there is an edge of Γ_m which contains a subsegment of the arc $\ell_n[-k, k]$ and which is missed by any leaf of L_0 . Hence every leaf of L_0 is carried by some free factor of F_n . Since T is indecomposable, by Theorem 4.5 of [R11], for every free factor H of F_n the minimal H -invariant subtree of T is discrete and therefore a leaf of $L^2(T)$ carried by H is carried by a point stabilizer of T . Then $L_0 \not\subset L_r(T)$ which is a contradiction. This shows the lemma in the case that the tree T is of Levitt type.

By definition, a tree T is of *surface type* if the Rips machine stops. By Proposition 5.14 of [CH14], an indecomposable tree T either is of Levitt type or of surface type. Thus we are left with showing the lemma for indecomposable trees of surface type.

Let $(\mathcal{K}, \mathcal{A})$ be a system of isometries which is not modified further. The set \mathcal{K} is a compact forest, in particular it has finitely many connected components. For each $a \in \mathcal{A}$ let $\mathcal{K}(a) \subset \mathcal{K}$ be the union of those components which intersect the domain of definition for a . We claim that there is some $a \in \mathcal{A}$ and some component $K \in \mathcal{K}(a)$ such that the domain of definition for a intersects K in a proper subtree.

Namely, otherwise each $a \in \mathcal{A}$ maps each component K of $\mathcal{K}(a)$ isometrically onto a component $a(K)$ of \mathcal{K} . Since \mathcal{K} has only finitely many components, this implies that there is a component K of \mathcal{K} , and there is a finite cyclically reduced nontrivial word $w = a_1 \dots a_s$ in \mathcal{A}^\pm such that $w(K) = K$. Then w is an isometry of the compact tree K . Since K is compact, w has a fixed point $p \in K$ and permutes the components of $K - \{p\}$. Using again compactness of K we conclude that there is some $k \geq 1$ such that w^k stabilizes a non-degenerate segment in K . Since $K \subset \bar{T}$ the tree \bar{T} has a segment with non-trivial stabilizer. This contradicts the assumption that T is indecomposable.

As a consequence, there is some $a \in \mathcal{A}^\pm$, and there is a component $K \in \mathcal{K}(a)$ containing an extreme point x of the domain of definition of a . By definition, such an extreme point x is contained in the domain of definition of a . Moreover, this domain of definition contains a segment abutting at x , the complement $K - x$ of x has at least two connected components, and there is at least one segment abutting at x whose interior is *not* contained in the domain of a .

Following the proof of Lemma 4.10 of [CHR11], we can now *split* the suspension surface by cutting K_x at x and cutting the suspension intervals containing x accordingly. We obtain a new system of isometries. Repeat this construction. Since by the above discussion the splitting process does not terminate, we obtain the conclusion of the lemma as in the proof of Lemma 4.10 of [CHR11]. \square

Let again $[T] \in \mathcal{IT} \subset \partial\mathcal{CV}(F_n)$ be indecomposable and let μ be an ergodic measured lamination with support $\text{Supp}(\mu) \subset L^2(T)$ and $\mu(L^2(T) - L_r(T)) = 0$. We call such a measured lamination *regular*. By [KL09], a measured lamination μ is regular if and only if $\langle T, \mu \rangle = 0$ and $\mu(L^2(T) - L_r(T)) = 0$.

Define an equivalence relation \sim_μ on ∂F_n as the smallest equivalence relation with the following property. The equivalence class of a point ξ contains all points ξ' with $(\xi, \xi') \in \text{Supp}(\mu)$. Let \sim be the closure of \sim_μ . By invariance of $\text{Supp}(\mu)$ under the action of F_n , the equivalence relation \sim is F_n -invariant, and its quotient $\partial F_n / \sim$ is a compact F_n -space. Note that it is unclear whether $\partial F_n / \sim$ is a tree. The action of F_n on $\partial F_n / \sim$ is minimal and dense since this holds true for the action of F_n on ∂F_n .

By Corollary 2.6 of [CHL07], the tree \hat{T} with the observer's topology is F_n -equivariantly homeomorphic to $\partial F_n / L^2(T)$. Now the zero lamination $L^2(T)$ contains the support $\text{Supp}(\mu)$ of μ and hence the compact F_n -space $\partial F_n / \sim$ admits an F_n -equivariant continuous surjection onto the tree \hat{T} .

The following observation is also used in Section 11 and completes the proof of Proposition 7.1 under the assumption that $\partial F_n / \sim$ is a tree. In its statement, we do not assume that the tree S is very small.

Lemma 7.5. *Let S be a minimal dense F_n -tree which admits an alignment preserving map onto indecomposable trees T, T' . Then \hat{T} is F_n -equivariantly homeomorphic to \hat{T}' .*

Proof. Let S be a minimal dense F_n -tree, let $T \in \partial cv(F_n)$ be indecomposable and assume that there is an alignment preserving map

$$f : S \rightarrow T.$$

A *transverse family* for an F_n -tree U with dense orbits is an F_n -invariant family $\{Y_v\}$ of non-degenerate subtrees $Y_v \subset U$ with the property that if $Y_v \neq Y_{v'}$ then $Y_v \cap Y_{v'}$ contains at most one point.

Since S is dense we can apply Theorem 12.12 of [R10]. It shows that there is an ergodic length measure ν for S so that with the notations from Section 5 we have $T = S_\nu$. Moreover, there is a transverse family \mathcal{F} for S with the following properties.

Each component of \mathcal{F} is a subtree Y of S . Its stabilizer is a vertex group of a very small splitting of F_n , and it acts on Y with dense orbits (see the explicit construction in Section 10 of [R10] for this fact which is attributed to Guirardel and Levitt). The tree T is obtained by collapsing each component of the transverse family \mathcal{F} to a point. In particular, the family \mathcal{F} is trivial (i.e. it is empty or its components are points) if and only if \hat{S} is equivariantly homeomorphic to \hat{T} .

Let now T' be a second indecomposable tree for which there is an alignment preserving map $f' : S \rightarrow T'$. Then the tree T' is obtained from S by collapsing each tree from a second transverse family \mathcal{F}' to a point. If \hat{T} is not F_n -equivariantly homeomorphic to \hat{T}' then the transverse families $\mathcal{F}, \mathcal{F}'$ do not coincide. In particular, up to exchanging T and T' we may assume that there is a component Y' of \mathcal{F}' which is not mapped to a point by the alignment preserving map $f : S \rightarrow T$.

As the stabilizer H of Y' is a vertex group of a very small splitting of F_n , it is a finitely generated subgroup of F_n [GL95] of infinite index, and it acts on Y' with

dense orbits. By equivariance and the assumption that f is alignment preserving, the image under f of the minimal H -invariant subtree Y' of S equals the minimal H -invariant subtree Z of T . Since Y' is not mapped to a point, the tree Z is non-trivial, and by equivariance, H acts on Z with dense orbits. However, by the main result of [R11], since T is indecomposable and H is a finitely generated subgroup of F_n of infinite index, the minimal H -invariant subtree of T is simplicial. This is a contradiction which shows the lemma. \square

Proof of Proposition 7.1: Let $T \in \partial cv(F_n)$ be indecomposable and let $\mu \in \mathcal{ML}$ be a regular measured lamination for T , i.e. such that $\langle T, \mu \rangle = 0$ and $\mu(L^2(T) - L_r(T)) = 0$.

Recall that the zero lamination $L^2(T)$ for T defines a closed equivalence relation on ∂F_n , and $\hat{T} = \partial F_n / L^2(T)$ (Corollary 2.6 of [CHL07]). Let \sim be the closure of the equivalence relation on ∂F_n defined by $\text{Supp}(\mu) \subset L^2(T)$. There is a natural F_n -equivariant continuous surjection

$$G : \partial F_n / \sim \rightarrow \hat{T}.$$

A leaf ℓ of $L^2(T)$ is *diagonal* over a sublamination L_0 if $\ell = (x_1, x_n)$ and if there are points $x_2, \dots, x_{n-1} \in \partial F_n$ so that $(x_1, x_2), \dots, (x_{n-1}, x_n) \in L_0$. A leaf of $L^2(T)$ which is diagonal over $\text{Supp}(\mu)$ is contained in the equivalence relation \sim . If the tree T is free then $L^2(T) - \text{Supp}(\mu)$ consists of finitely many F_n -orbits of diagonal leaves [CHR11] and therefore G is an equivariant homeomorphism. This implies the proposition in the case that the tree T is free.

Now let T be an arbitrary indecomposable tree. The zero lamination $L^2(T)$ is the union of $L_r(T)$, the sets $\partial H \times \partial H - \Delta$ where H runs through the point stabilizers of T and some isolated diagonal leaves.

Diagonal leaves are contained in the closure of the equivalence relation defined by $\text{Supp}(\mu)$ and the point stabilizers of T . This implies that the zero lamination $L^2(T)$ is determined uniquely by $\text{Supp}(\mu)$ and the point stabilizers of T .

Now let μ be a regular measured lamination for T and let $S \in \partial cv(F_n)$ be another indecomposable tree with $\langle S, \mu \rangle = 0$. For the proof of Proposition 7.1, it now suffices to show that a subgroup H of F_n stabilizes a point in T if and only if H stabilizes a point in S .

To this end recall that the stabilizer H of a point $x \in T$ is finitely generated [GL95] and therefore the Gromov boundary ∂H of H is a closed H -invariant subset of ∂F_n . By equivariance and continuity, the preimage in $\partial F_n / \sim$ of the fixed point x of H under the surjection G is a compact H -invariant subset A of $\partial F_n / \sim$. This set is just the projection of $\partial H \subset \partial F_n$ to $\partial F_n / \sim$. The union $\cup_{g \in F_n} gA$ is disjoint and defines a transverse family \mathcal{F} of compact spaces. Each component of \mathcal{F} is invariant under a conjugate of H (these spaces may be reduced to points or may not be trees).

There also is an equivariant projection $Q : \partial F_n / \sim \rightarrow \hat{S}$. If H does not fix a point in S then the transverse family \mathcal{F} of $\partial F_n / \sim$ is not collapsed in \hat{S} to a

countable set of points. However, as in the proof of Lemma 7.5, since the interior of \hat{S} is F_n -equivariantly homeomorphic to the tree S [CHL07], this implies that the minimal H -invariant subtree of S is not reduced to a point, and H acts on it with dense orbits. As in the proof of Lemma 7.5, we conclude that this is impossible. This completes the proof of Proposition 7.1. \square

8. TREES WITH POINT STABILIZERS CONTAINING A FREE FACTOR

The goal of this section is to analyze trees in $\partial\text{CV}(F_n)$ with dense orbits which have some point stabilizers containing a free factor.

Let $A < F_n$ be a free factor of rank $r \geq 1$. A *free basis extension* of A is a free basis $e_1, \dots, e_{n-r}, e_{n-r+1}, \dots, e_n$ of F_n such that e_{n-r+1}, \dots, e_n is a free basis of A . Define the *standard simplex relative to A* of a free basis extension of A to be the set

$$\Delta(A) \subset \overline{cv_0(F_n)}^{++}$$

of all trees T with volume one quotient with the following property. Either T is contained in the boundary of the standard simplex Δ for the basis and admits A as a point stabilizer, or T/F_n can be obtained as follows.

Let $R_0, R_{n-r+1}, \dots, R_n$ be roses of rank $n-r$ and 1, respectively, with petals of length $1/(n+r)$ which are marked by the subgroup of F_n generated by e_1, \dots, e_{n-r} and by e_j ($n-r \leq j \leq n$). Connect the roses R_j in an arbitrary way by r edges of length $1/(n+r)$ so that these edges form a forest in the resulting connected graph G_1 . For $j \geq n-r+1$ collapse each rose R_j in G_1 to a point and let G_2 be the resulting graph. The graph T/F_n is obtained from G_2 by changing the lengths of the remaining edges, allowing some of the edges to shrink to zero length. The set $\Delta(A) \subset \overline{cv_0(F_n)}^{++}$ is compact.

If $A < F_n$ is a free factor which fixes a point in a tree $[T] \in \partial\text{CV}(F_n)$ then we say that the action of A on $[T]$ is elliptic.

The next observation is a relative version of Lemma 3.3.

Lemma 8.1. *Let $A < F_n$ be a free factor and let $[T] \in \partial\text{CV}(F_n)$ be such that the action of A on $[T]$ is elliptic. Then for every standard simplex $\Delta(A)$ relative to A there is a tree $U \in \Delta(A)$ and an optimal train track map $f : U \rightarrow T$ where T is some representative of $[T]$.*

Proof. Let $A < F_n$ be a free factor of rank $r \leq n-1$ and let $[T] \in \partial\text{CV}(F_n)$ be a tree containing a point x which is stabilized by A . Let e_1, \dots, e_n be a free basis extension of A and let $\Delta \subset \overline{cv_0(F_n)}^{++}$ be the standard simplex for this basis.

By Lemma 3.3 there is some $S \in \Delta$, and there is an optimal train track map $f : S \rightarrow T$ where T is a representative of $[T]$. The map f is determined by the image of a point $\tilde{v} \in S$ in the preimage of the unique vertex v of S/F_n .

There are now two possibilities. In the first case, $f(\tilde{v})$ is stabilized by a conjugate of A . Now up to conjugation, for $i \geq n-r+1$ the train track map $f : S \rightarrow T$ maps

the edge of S with endpoints $\tilde{v}, e_i\tilde{v}$ isometrically onto the segment in T connecting $f(\tilde{v})$ to $e_i f(\tilde{v})$. Since $f(\tilde{v}) = e_i f(\tilde{v})$, the tree S is contained in $\overline{cv_0(F_n)} - cv_0(F_n)$, and the vertex \tilde{v} is fixed by a conjugate of A . As a consequence, we have $S \in \Delta(A)$ and we are done.

If $f(\tilde{v})$ is not stabilized by a conjugate of A then by equivariance, the vertex \tilde{v} of S is not stabilized by a conjugate of A . Thus there is a petal of S/F_n which represents an element a of A . This petal lifts to a line in S through \tilde{v} which is stabilized by a conjugate of a (but not pointwise). Let us suppose that this conjugate is just a . The edge in S with endpoints \tilde{v} and $a\tilde{v}$ is mapped by f isometrically to a segment in T .

By our assumption on T , the element a fixes a point in T . Since T is very small, the fixed point set $\text{Fix}(a)$ of a is a point or a closed segment in T . Let $\tilde{x} \in \text{Fix}(a)$ be the point of smallest distance to $f(\tilde{v})$.

Let s be the unique segment in T connecting $f(\tilde{v})$ to \tilde{x} . The segment meets $\text{Fix}(a)$ only in \tilde{x} . Then $a(s)$ is the segment in T connecting $af(\tilde{v})$ to $a(\tilde{x}) = \tilde{x}$, and the segment $s \cap a(s)$ is fixed by a . Since s meets the fixed point set of a only in \tilde{x} , we conclude that $s \cap a(s) = \tilde{x}$. Thus the geodesic segment in T connecting $f(\tilde{v})$ to $af(\tilde{v})$ passes through \tilde{x} , and the geodesic segment connecting $f(\tilde{v})$ to $a^{-1}f(\tilde{v})$ passes through \tilde{x} as well.

As a consequence, for the train track structure on S defined by f , the turn at \tilde{v} containing the two directions of the axis of a is illegal. Folding this illegal turn collapses the petal in S/F_n defining a to a single segment without changing the rest of S/F_n . After volume renormalization, the resulting tree S_1 is contained in $\overline{cv_0(F_n)}$. Its quotient S_1/F_n contains a segment which is a collapse of the petal of S/F_n defining a . There is a train track map $f_1 : S_1 \rightarrow T_1$ where T_1 is a rescaling of T .

The graph S_1/F_n has precisely two vertices, and one of these vertices is univalent. There are at most $r - 1$ loops in S_1/F_n defining elements of A . Repeat this construction with a petal of S_1/F_n defining an element of the free factor A . After a total of r such steps we obtain a simplicial tree $S_r \in \overline{cv_0(F_n)}$ and a train track map $f_r : S_r \rightarrow T_r$ where T_r is a rescaling of T . The graph S_r/F_n consists of a rose with $n - r$ petals (some of them may be degenerate to points) with a collection of edges attached. By construction, we have $S_r \in \Delta(A)$ as claimed. \square

Recall from Section 2 the definition of the map $\Upsilon_{\mathcal{F}} : \overline{cv_0(F_n)}^+ \rightarrow \mathcal{FF}$. Recall also from Section 4 the definition of the map $\psi_{\mathcal{F}} : \partial\text{CV}(F_n) \rightarrow \partial\mathcal{FF} \cup \Theta$. We use Lemma 8.1 to show

Corollary 8.2. *Let $A < F_n$ be a free factor of rank $\ell \geq 1$ and let $[T] \in \partial\text{CV}(F_n)$ be a tree such that the action of A on $[T]$ is elliptic. Then $\psi_{\mathcal{F}}([T]) = \Theta$.*

Proof. Let $\Delta(A)$ be a standard simplex relative to A . By Lemma 8.1 there is a tree $S \in \Delta(A)$ and a train track map $f : S \rightarrow T$ where T is a representative of $[T]$.

Any point on a Skora path (x_t) guided by f contains a point stabilized by a fixed free factor contained in A . By Lemma 2.18, this implies that for every $t > 0$ the free factor $\Upsilon_{\mathcal{F}}(x_t)$ is contained in a uniformly bounded neighborhood of A . In particular, the diameter of $\Upsilon_{\mathcal{F}}(x_t)$ is finite. The claim of the lemma now follows from Proposition 6.1. \square

9. TREES WHICH SPLIT AS GRAPHS OF ACTIONS

In this section we investigate the structure of trees in $\partial\text{CV}(F_n)$ with dense orbits which resemble trees T with $T_d \neq \emptyset$. The description of such trees is as follows [G08, L94].

Definition 9.1. A *graph of actions* is a minimal F_n -tree which consists of

- (1) a simplicial tree S , called the *skeleton*, equipped with an action of F_n
- (2) for each vertex v of S an \mathbb{R} -tree Y_v , called a *vertex tree*, and
- (3) for each oriented edge e of S with terminal vertex v a point $p_e \in Y_v$, called an *attaching point*.

It is required that the projection $Y_v \rightarrow p_e$ is equivariant, that for $g \in F_n$ one has $gp_e = p_{ge}$ and that each vertex tree is a minimal very small tree for its stabilizer.

Associated to a graph of actions \mathcal{G} is a canonical action of F_n on an \mathbb{R} -tree $T_{\mathcal{G}}$ which is called the *dual* of the graph of actions [L94]. Define a pseudo-metric d on $\coprod_{v \in V(S)} Y_v$ as follows. If $x \in Y_{v_0}, y \in Y_{v_k}$ let $e_1 \dots e_k$ be the reduced edge-path from v_0 to v_k in S and define

$$d(x, y) = d_{Y_{v_1}}(x, p_{\bar{e}_1}) + \dots + d_{Y_{v_k}}(p_{e_k}, y).$$

Making this pseudo-metric Hausdorff gives an \mathbb{R} -tree $T_{\mathcal{G}}$. Informally, the tree $T_{\mathcal{G}}$ is obtained by first inserting the vertex trees into the skeleton S and then collapsing each edge of the skeleton to a point.

We say that an F_n -tree T *splits as a graph of actions* if either $T_d \neq \emptyset$ or if there is a graph of actions \mathcal{G} with dual tree $T_{\mathcal{G}}$, and there is an equivariant isometry $T \rightarrow T_{\mathcal{G}}$. We also say that the projectivization $[T]$ of an F_n -tree T splits as a graph of actions if T splits as a graph of actions.

A transverse family for an F_n -tree T with dense orbits is an F_n -invariant family $\{Y_v\}$ of non-degenerate subtrees $Y_v \subset T$ with the property that if $Y_v \neq Y_{v'}$ then $Y_v \cap Y_{v'}$ contains at most one point. The transverse family is a *transverse covering* if any finite segment $I \subset T$ is contained in a finite union $Y_{v_1} \cup \dots \cup Y_{v_r}$ of components from the family. The vertex trees of a graph of actions with dual tree $T_{\mathcal{G}}$ define a transverse covering of $T_{\mathcal{G}}$. More precisely, by Lemma 4.7 of [G04], a minimal very small F_n -tree T admits a transverse covering if and only if T splits as a graph of actions.

A dense tree T splits as a *large graph of actions* if it is equivariantly isometric to the dual tree of a graph of actions \mathcal{G} with the following additional properties.

- (1) There is a single F_n -orbit of vertex trees.

- (2) A stabilizer of a vertex tree is not contained in a proper free factor of F_n .
- (3) A vertex tree is indecomposable for its stabilizer.
- (4) The skeleton of \mathcal{G} does not have an edge with trivial stabilizer.

Note that there is some redundancy in the above definition.

The tree T splits as a *very large graph of actions* if it splits as a large graph of actions and if moreover the skeleton does not have an edge with infinite cyclic stabilizer.

The following result is Corollary 11.2 of [R10].

Proposition 9.2. *Let $T \in \overline{cv(F_n)}$ have dense orbits, and assume that T is neither indecomposable nor splits as a graph of actions. Then there is an alignment preserving map $f : T \rightarrow T'$ such that either T' is indecomposable or T' splits as a graph of actions.*

We use Proposition 9.2 to show

- Proposition 9.3.**
- (1) *Let $[T] \in \partial CV(F_n)$ be such that $\psi([T]) \in \partial \mathcal{FS}$. Then T admits an alignment preserving map onto a tree which either is indecomposable or splits as a large graph of actions.*
 - (2) *Let $[T] \in \partial CV(F_n)$ be such that $\psi_C([T]) \in \partial \mathcal{CS}$. Then T admits an alignment preserving map onto a tree which either is indecomposable or splits as a very large graph of actions.*

Proof. Let $[T] \in \partial CV(F_n)$ be such that $\psi([T]) \in \partial \mathcal{FS}$ (or $\psi_C([T]) \in \partial \mathcal{CS}$). By Corollary 5.5, $[T]$ is not simplicial. A tree which is neither dense nor simplicial admits a one-Lipschitz alignment preserving map onto a dense tree. By Proposition 4.3, we may therefore assume without loss of generality that T is dense. By Proposition 9.2 and Corollary 6.13, it now suffices to show the following. If T splits as a graph of actions, then T admits an alignment preserving map onto a tree which splits as a large (or very large) graph of actions.

Thus assume that T is the dual tree of a graph of actions, with skeleton S . Let U be the tree obtained from S by insertion of the vertex trees. The tree U may not be very small, but it admits an alignment preserving map onto T obtained by equivariantly collapsing each edge of U to a point.

If there is an edge in S with trivial (or either trivial or infinite cyclic) stabilizer then there is an edge e in U with trivial (or infinite cyclic) stabilizer. The tree V obtained from U by equivariantly collapsing those edges of U to points which are not contained in the orbit of e is very small, and $V_d \neq \emptyset$. There is an edge in V with trivial (or either trivial or infinite cyclic) stabilizer. Moreover, V admits an alignment preserving map onto T . We then have $\psi([T]) = \Theta$ (or $\psi_C([T]) = \Theta$) by Corollary 6.13 and Proposition 5.1.

Assume not that there is no edge e in the skeleton S with trivial (or either trivial or infinite cyclic) stabilizer. If there is more than one F_n -orbit of vertex trees for this graph of actions then we can collapse the vertex trees in all but one of these orbits to points. We obtain a tree W which is dual to a graph of actions with a

single orbit of vertex trees. The edge stabilizer of each edge of the skeleton is not trivial (or neither trivial nor infinite cyclic).

Let W_v be one of the vertex trees of the F_n -tree W . Its stabilizer is a finitely generated subgroup H of F_n (and hence a finitely generated free group) which acts on W_v with dense orbits. By Proposition 9.2, the H -tree W_v either admits an alignment preserving map onto an indecomposable tree or onto a tree which splits as a graph of actions.

Assume first that W_v admits an alignment preserving map onto an indecomposable H -tree. Then W and hence T admits an alignment preserving map onto a tree Z which splits as a graph of actions, with a single orbit of indecomposable vertex trees. Moreover, Z is dual to a graph of actions whose skeleton does not have edges with trivial (or with trivial or infinite cyclic) stabilizer. In particular, Z splits as a large (or as a very large) graph of actions provided that the group H is not contained in a proper free factor of F_n .

To see that the latter property indeed holds true collapse all but one orbit of edges in the skeleton of Z to a point. This defines a one-edge splitting of F_n . Assume first that this splitting is a one-edge two-vertex splitting. Then this splitting is of the form $H *_C B$ where C is a nontrivial free subgroup of H (or a free subgroup of H of rank at least two).

Let $A < F_n$ be the smallest free factor of F_n containing H . Assume to the contrary that A is a proper subgroup of F_n . As $C < H < A$, there is then a free splitting $F_n = A * B_1$ where $B_1 < B$, and there is a refinement $(H *_C B_0) * B_1$ of $H *_C B$. Since there is a unique orbit of indecomposable vertex trees for Z whose stabilizers are conjugate to H , the group B fixes a vertex in Z , and the tree Z can be represented as a graph of actions with an edge with trivial edge group. Proposition 5.1 now shows that $\psi([Z]) = \psi([T])\Theta$ which contradicts the assumption on T . Thus we have $A = F_n$ and Z splits as a large (or as a very large) graph of actions.

If the splitting of F_n defined by an edge in the skeleton of Z is a one-edge one-vertex splitting and if the vertex group H is contained in a proper free factor A of F_n then the vertex group is a free factor of rank $n - 1$ (since F_n is generated by A and a single primitive element) and the splitting is a one-loop free splitting of F_n which violates once more the assumption that $\psi([T]) \in \partial\mathcal{FS}$ (or $\psi_C([T]) \in \partial\mathcal{CS}$). This completes the proof of the proposition in the case that the vertex tree W_v admits an alignment preserving map onto an indecomposable tree.

We are left with the case that a vertex tree W_v of the graph of actions with dual tree W admits an alignment preserving map onto a tree Y_v which splits as a graph of actions for the stabilizer of T_v . Then the tree W admits an alignment preserving map onto the tree Y which is obtained by collapsing each tree in the orbit of W_v to a tree in the orbit of Y_v .

Iterating the above argument, in the case that Y_v has more than one orbit of vertex trees then all but one of these orbits can be collapsed to points. Extending the collapsing map equivariantly yields an alignment preserving map of Y onto a tree Y' which splits as a graph of actions, with a single orbit of vertex trees. The

sums of the ranks of conjugacy classes of point stabilizers in Y' is strictly bigger than those of Y .

Since the sum of the ranks of conjugacy classes of point stabilizers in a very small F_n -tree is uniformly bounded [GL95], we find in finitely many such steps an F_n -tree Q which splits as a graph of actions with a single F_n -orbit of vertex trees, and each such vertex tree is indecomposable for its stabilizer. Moreover, there is an alignment preserving map $T \rightarrow Q$. The skeleton for the structure of a graph of actions for Q is an F_n -tree with finite quotient. Stabilizers of edges are not trivial (or neither trivial nor cyclic). In other words, Q splits as a large (or as a very large) graph of actions. This completes then proof of the proposition. \square

The following example was shown to me by Camille Horbez.

Example: Let T be an indecomposable F_n -tree with a point stabilizer which contains a free factor H of F_n of rank $k \geq 3$. Let B be a free group of rank $\ell < k$ and choose a finite index subgroup H' of B which is a free group of rank k . Identifying H and H' defines a one-edge two vertex graph of groups decomposition $F_n *_H B$ for a free group F_m where $m = n + \ell - k < n$. Let the group B act trivially on a point and define a graph of actions for F_m with a single orbit of edges in its skeleton containing T as a vertex tree. This graph of actions is very large.

We can also iterate this construction by beginning with an indecomposable F_n -tree T with more than one F_n -orbit of points with point stabilizer containing a free factor of rank $k \geq 3$ and constructing from T a very large graph of actions defined by a tree whose skeleton has more than one orbit of edges.

The final goal of this section is to show

Proposition 9.4. *If $[T] \in \partial\text{CV}(F_n)$ splits as a graph of actions then $\psi_{\mathcal{F}}([T]) = \Theta$.*

Proof. By Corollary 6.10, Corollary 6.13 and Proposition 9.3, it suffices to show that $\psi_{\mathcal{F}}([T]) = \Theta$ for every projective tree $[T]$ which splits as a very large graph of actions.

By Corollary 8.2, this indeed holds true if any tree which splits as a very large graph of actions admits a point stabilizer which contains a free factor of F_n . That this is indeed the case is due to Reynolds (Proposition 10.3 of [R12]). \square

Proposition 5.1, Corollary 9.4, Proposition 9.2 and Corollary 6.13 immediately imply

Corollary 9.5. *Let $[T] \in \partial\text{CV}(F_n)$ be such that $\psi_{\mathcal{F}}([T]) \in \partial\mathcal{FF}$. Then T admits an alignment preserving map onto an indecomposable tree.*

10. THE BOUNDARY OF THE FREE FACTOR GRAPH

In this section we complete the proof of Theorem 3 from the introduction.

An indecomposable tree T is called *arational* if no point stabilizer for T contains a free factor. We begin with collecting some additional information on arational trees. Part of what we need is covered by the main result of [R12] which shows that an arational tree T either is free or dual to a measured lamination of an oriented surface with a single boundary component. We will not use this information.

A closed F_n -invariant subset C of $\partial F_n \times \partial F_n - \Delta$ intersects a free factor if there is a proper free factor H of F_n so that $C \cap \partial H \times \partial H - \Delta \neq \emptyset$.

The following lemma is a simple consequence of Theorem 4.5 of [R11].

Lemma 10.1. *Let $T \in \partial cv(F_n)$ be indecomposable. If the zero lamination $L^2(T)$ of T intersects a proper free factor of F_n then there is a point stabilizer for the action of F_n on T which contains a proper free factor.*

Proof. The intersection of $L^2(T)$ with a proper free factor $H < F_n$ is contained in the zero lamination of the minimal H -invariant subtree T_H of T . By Theorem 4.5 of [R11], since T is indecomposable the action on T of any proper free factor H is discrete and hence T_H is simplicial. Thus the intersection of $L^2(T)$ with H is non-empty if and only if there is a leaf of $L^2(T)$ which is carried by a point stabilizer for the action of H on T_H .

Since T is indecomposable, the stabilizer of any non-degenerate segment of T is trivial. As a consequence, the tree T_H is very small simplicial, with trivial edge stabilizers. Therefore T_H/H defines a graph of groups decomposition of H with trivial edge groups, i.e. it defines a free splitting of H . Each vertex group is a free factor of H . Hence if the action of H on T_H is not free, then there is a point stabilizer of T_H which is a proper free factor of H and hence of F_n . The lemma follows. \square

We use Lemma 10.1 to show that only arational trees can give rise to points in the boundary of the free factor graph \mathcal{FF} .

Say that a measured lamination μ is *supported* in a finitely generated subgroup H of F_n if $\text{Supp}(\mu) = F_n(\text{Supp}(\mu) \cap \partial H \times \partial H - \Delta)$. Our next goal is to understand measured laminations whose supports are contained in the point stabilizer of an indecomposable tree. For this we use the following consequence of Theorem 49 of [Ma95].

Lemma 10.2. *Let $H < F_n$ be a finitely generated subgroup of infinite index which does not intersect a free factor. Then $\partial H \times \partial H - \Delta$ does not support a measured lamination for F_n .*

Proof. Assume to the contrary that there is a measured lamination ν for F_n supported in $\partial H \times \partial H - \Delta$. Let $\ell \in \partial H \times \partial H - \Delta$ be a density point for ν . Then there is no proper free factor A of F_n so that $\ell \in \partial A \times \partial A - \Delta$.

Choose a free basis \mathcal{A} for F_n and represent ℓ by a biinfinite word (w_i) in that basis. Since $H < F_n$ is finitely generated of infinite index, H is quasiconvex and hence we may assume that there is a sequence $n_i \rightarrow \infty$ such that each of the prefixes (w_{n_i}) of the word (w_i) represents an element of H .

By Theorem 49 of [Ma95] (which is attributed to Bestvina), for each i there is a free basis \mathcal{B}_i for F_n such that the Whitehead graph of w_{n_i} with respect to \mathcal{B}_i is connected and does not have a cut vertex. Since ℓ is a density point for ν , the Whitehead graph of ν for the basis \mathcal{B}_i contains the Whitehead graph of w_{n_i} , in particular it does not have a cut vertex and is connected. On the other hand, since ν is a measured lamination, Proposition 21 of [Ma95] shows that the Whitehead graph of ν with respect to \mathcal{B}_i has a cut vertex or is disconnected. This is a contradiction and shows the lemma. \square

In Section 7 we defined a measured lamination $\mu \in \mathcal{ML}$ to be regular for an indecomposable tree T if $\langle T, \mu \rangle = 0$ and $\mu(L^2(T) - L_r(T)) = 0$. By Lemma 10.1 and Lemma 10.2, if T is arational then every measured lamination μ with $\langle T, \mu \rangle = 0$ is regular for T .

Example: Let $n = 2g \geq 4$ and let S be an oriented surface of genus g with non-empty connected boundary. The fundamental group of S is the group F_n . Every measured lamination μ on S is dual to an F_n -tree T . If the support of μ is minimal and decomposes S into ideal polygons, then this tree is arational. The free homotopy class of the boundary circle defines the unique conjugacy class in F_n whose elements have fixed points in T . The point stabilizers of T do not support a measured lamination.

Denote as before by $\mathcal{IT} \subset \partial\text{CV}(F_n)$ the set of indecomposable projective trees. Let

$$\mathcal{FT} \subset \mathcal{IT} \subset \partial\text{CV}(F_n)$$

be the set of arational trees. Corollary 9.5, Corollary 8.2 and Lemma 10.1 show that $\psi(\mathcal{FT}) \supset \partial\mathcal{FF}$. Our next task is to show that $\mathcal{FT} \subset \psi_{\mathcal{F}}^{-1}(\partial\mathcal{FF})$. To this end recall that each conjugacy class of a primitive element $g \in F_n$ determines a measured lamination which is the set of all Dirac masses on the pairs of fixed points of the elements in the class. The measured lamination is called dual to the conjugacy class.

Recall from Section 2 that a primitive conjugacy class α in F_n is *short* for a tree $T \in \text{cv}_0(F_n)$ if it can be represented by a loop on T/F_n of length at most 3. Recall that the length pairing $\langle \cdot, \cdot \rangle$ on $\overline{\text{cv}(F_n)} \times \mathcal{ML}$ is continuous.

We have

Lemma 10.3. *Let $[T_i] \subset \text{CV}(F_n)$ be a sequence converging to $[T] \in \partial\text{CV}(F_n)$. For each i let $T_i \in \text{cv}_0(F_n)$ be a representative of $[T_i]$ and let α_i be a primitive short conjugacy class on T_i with dual measured lamination μ_i . If $[T]$ is dense then up to passing to a subsequence, there is a sequence $b_i \subset (0, 1]$ such that the measured laminations $b_i\mu_i$ converge weakly to a measured lamination μ with $\langle T, \mu \rangle = 0$. Moreover, either up to scaling μ is dual to a primitive conjugacy class, or $b_i \rightarrow 0$.*

Proof. For each i let $T_i \in \text{cv}_0(F_n)$ be a representative of $[T_i]$, let T be a representative of $[T]$ and let $a_i \in (0, \infty)$ be such that $a_i T_i \rightarrow T$. Since the F_n -orbits on T are dense, we have $a_i \rightarrow 0$ ($i \rightarrow \infty$).

Fix some tree $S \in \text{cv}_0(F_n)$. Then the set

$$\Sigma = \{\zeta \in \mathcal{ML} \mid \langle S, \zeta \rangle = 1\}$$

defines a continuous section of the projection $\mathcal{ML} \rightarrow \mathcal{PML}$ for the weak*-topology. In particular, the space Σ is compact.

There is a number $\epsilon > 0$ so that $\langle S, \zeta \rangle \geq \epsilon$ whenever ζ is dual to any primitive conjugacy class. Let μ_i be the measured lamination dual to a primitive short conjugacy class α_i on T_i . If $b_i > 0$ is such that $b_i\mu_i \in \Sigma$ then the sequence (b_i) is *bounded*. Since Σ is compact, by passing to a subsequence we may assume that $b_i\mu_i \rightarrow \mu$ for some measured lamination $\mu \in \Sigma$.

Now $\langle a_i T_i, \mu_i \rangle \leq k a_i$ where $k \geq 2$ is as in the definition of a short conjugacy class (see Section 2) and hence since $a_i \rightarrow 0$ ($i \rightarrow \infty$) and since the sequence (b_i) is bounded, we have

$$\langle a_i T_i, b_i \mu_i \rangle \rightarrow 0 \text{ (} i \rightarrow \infty \text{)}.$$

The first part of the lemma now follows from continuity of the length pairing. Moreover, either $b_i \rightarrow 0$ or the length on S/F_n of the conjugacy classes α_i is uniformly bounded. However, there are only finitely many conjugacy classes of primitive elements which can be represented by a loop on S/F_n of uniformly bounded length. Thus either $b_i \rightarrow 0$, or the sequence (α_i) contains only finitely many elements and hence there is some primitive conjugacy class α so that $\alpha_i = \alpha$ for infinitely many i . Then clearly μ is a multiple of the dual of α . \square

The following proposition is the main remaining step towards the proof of Theorem 3. For its formulation, define two trees $[S], [T] \in \mathcal{IT}$ to be *equivalent* if the trees \hat{S}, \hat{T} are F_n -equivariantly homeomorphic with respect to the observer's topology.

Proposition 10.4. (1) *Let $[T] \in \mathcal{IT}$; then $\psi_{\mathcal{F}}([T]) \in \partial\mathcal{FF}$ if and only if $[T] \in \mathcal{FT}$.*
 (2) *If $[T], [T'] \in \mathcal{FT}$ then $\psi_{\mathcal{F}}([T]) = \psi_{\mathcal{F}}([T'])$ if and only if $[T], [T']$ are equivalent.*

Proof. Let $\Upsilon_{\mathcal{F}} = \Omega \circ \Upsilon : \text{cv}_0(F_n) \rightarrow \mathcal{FF}$ be the map constructed in Section 2. By the discussion preceding Lemma 10.3, it suffices to show that $\psi([T]) \neq \Theta$ for every $[T] \in \mathcal{FT}$. Since Skora paths map to uniform unparametrized quasi-geodesics in \mathcal{FF} , this holds true if for every $[T] \in \mathcal{FT}$ the image under $\Upsilon_{\mathcal{F}}$ of a Skora path converging to $[T]$ is unbounded.

We show more generally the following. If $[T_i] \subset \text{CV}(F_n)$ is any sequence which converges to $[T] \in \mathcal{FT}$ and if $T_i \in \text{cv}_0(F_n)$ is a representative of $[T_i]$ then the sequence $\Upsilon_{\mathcal{F}}(T_i) \subset \mathcal{FF}$ is unbounded. For this we use a variant of an argument of Luo as explained in [MM99].

We argue by contradiction and we assume that after passing to a subsequence, the sequence $\Upsilon_{\mathcal{F}}(T_i)$ remains in a bounded subset in \mathcal{FF} .

Since by Lemma 2.18 the map $\Upsilon_{\mathcal{F}}$ is a quasi-isometry for the metric $d_{ng} = d_{ng}^1$ on $\text{cv}_0(F_n)$ which only assumes integral values, after passing to another subsequence we may assume that for all $i \geq 1$ the distance between T_i and T_0 in $(\text{cv}_0(F_n), d_{ng})$ equals m for some $m \geq 0$ which does not depend on i .

By the definition of the metric d_{ng} , this implies that for all $i \geq 1$ there is a sequence $(T_{j,i})_{0 \leq j \leq m} \subset \text{cv}_0(F_n)$ with $T_{0,i} = T_0$ and $T_{m,i} = T_i$ so that for all $j < m$ the trees $T_{j,i}$ and $T_{(j+1),i}$ are one-tied. In particular, for each $j < m$ there is a primitive conjugacy class $\alpha_{j,i}$ which can be represented by a curve of length at most 3 on both $T_{j,i}/F_n$ and $T_{(j+1),i}/F_n$. Let $\mu_{j,i}$ be the measured lamination which is dual to $\alpha_{j,i}$.

By assumption, we have $[T_{m,i}] \rightarrow [T]$ ($i \rightarrow \infty$) in $\overline{\text{CV}(F_n)}$. Since T is dense, Lemma 10.3 implies that up to passing to a subsequence, there is a bounded sequence (b_i) such that the measured laminations $b_i \mu_{m-1,i}$ converge as $i \rightarrow \infty$ to a measured lamination ν_{m-1} supported in the zero lamination of T . Since $[T] \in \mathcal{FT}$, by Lemma 10.1 the support of ν_{m-1} does not intersect a free factor and hence $b_i \rightarrow 0$ by Lemma 10.3.

By passing to another subsequence, we may assume that the projective trees $[T_{(m-1),i}]$ converge as $i \rightarrow \infty$ to a projective tree $[U_{m-1}]$. Choose a representative U_{m-1} of $[U_{m-1}]$. Since $b_i \mu_{m-1,i} \rightarrow \nu_{m-1}$ for a sequence $b_i \rightarrow 0$ and since $\langle T_{m-1,i}, \mu_{m-1,i} \rangle \leq 3$ for all i , Lemma 10.3 shows that $\langle U_{m-1}, \nu_{m-1} \rangle = 0$. In particular, ν_{m-1} is supported in the zero lamination of U_{m-1} . Proposition 7.1 now shows that U_{m-1} admits an alignment preserving map onto T . Moreover, there is a subsequence of the sequence $[\mu_{m-2,i}]$ which converges as $i \rightarrow \infty$ to a projective measured lamination supported in the zero lamination of $[U_{m-1}]$ and hence of $[T]$. (Since $[T]$ is arational, it can easily be seen that $[U_{m-1}] \in \mathcal{FT}$, a fact which is immediate from [R12]).

Repeat this argument with the sequence $[T_{(m-2),i}]$ and the tree $[U_{m-1}]$. After m steps we conclude that T_0 admits an alignment preserving map onto T which is impossible. The first part of the proposition is proven.

The second part of the proposition is shown in the same way. By Corollary 6.13, we only have to show that if $[T], [T'] \in \mathcal{FT}$ are such that $\psi_{\mathcal{F}}([T]) = \psi_{\mathcal{F}}([T'])$ then $[T], [T']$ are equivalent. Let $(x_t), (y_t)$ be Skora paths connecting a point in a standard simplex Δ to $[T], [T']$. Since the paths $\Upsilon_{\mathcal{F}}(x_t), \Upsilon_{\mathcal{F}}(y_t)$ are uniform reparametrized quasi-geodesics in \mathcal{FF} , by hyperbolicity there is a number $m \geq 0$ and for each $t > 0$ there is a number $s(t) > 0$ so that $d_{ng}(\Upsilon_{\mathcal{F}}(x_t), \Upsilon_{\mathcal{F}}(y_{s(t)})) \leq m$.

As above, by passing to a subsequence (and perhaps changing the constant m) we may assume that there is a sequence $t_i \rightarrow \infty$ so that $d_{ng}(\Upsilon_{\mathcal{F}}(x_{t_i}), \Upsilon_{\mathcal{F}}(y_{s(t_i)})) =$

m for all i . For each i let $u_i \in cv_0(F_n)$ be such that $d_{ng}(x_{t_i}, u_i) = 1$ and $d_{ng}(u_i, y_{s(t_i)}) = m - 1$. By the above discussion, up to passing to a subsequence the sequence $[u_i]$ converges to a point $[U] \in \partial CV(F_n)$ so that U admits an alignment preserving map onto T . Repeat with the sequence (u_i) . After m steps we conclude that $[T']$ is equivalent to $[T]$ which is what we wanted to show. \square

Remark 10.5. The proof of Proposition 10.4 shows that for every $[T] \in \mathcal{FT}$ there is a measured lamination μ with $\langle T, \mu \rangle = 0$. Thus Corollary 7.2 applies to arational trees.

We showed so far that the map $Y = \psi_{\mathcal{F}}|_{\mathcal{FT}}$ is a continuous closed surjection of \mathcal{FT} onto $\partial \mathcal{FF}$. Each fibre consists of the closed set of all trees in a fixed equivalence class for the relation \sim . Thus the Gromov boundary of \mathcal{FF} is homeomorphic to \mathcal{FT}/\sim . Theorem 3 is proven.

We complete this section with an easy consequence which will be useful in other context. For its formulation, following [H14a] we call a pair $(\mu, \nu) \in \mathcal{ML} \times \mathcal{ML}$ *positive* if for every tree $T \in cv(F_n)$ we have $\langle T, \mu \rangle + \langle T, \nu \rangle > 0$.

Corollary 10.6. *Let $\mu, \nu \in \mathcal{ML}$ be measured laminations which are supported in the zero lamination of trees $[T], [S] \in \mathcal{FT}$. If $Y([T]) \neq Y([S])$ then (μ, ν) is a positive pair.*

Remark 10.7. The topology for the boundary of the free factor graph can also be described as a measure forgetful topology in the following sense. The projection to $\partial \mathcal{FF}$ of a sequence $[T_i] \subset \mathcal{FT}$ converges to the projection of $[T]$ if and only if the following holds true. For each i let β_i be a measured lamination with $\langle T_i, \beta_i \rangle = 0$. Assume that the measured laminations converge to a measured lamination β ; then $\langle T, \beta \rangle = 0$.

11. THE BOUNDARIES OF THE FREE AND THE CYCLIC SPLITTING GRAPH

In this section we identify the Gromov boundary of the free and of the cyclic splitting graph.

We continue to use all assumptions and notations from the previous sections. In particular, we use the maps

$$\psi : \partial CV(F_n) \rightarrow \partial \mathcal{FS} \cup \Theta \text{ and } \psi_{\mathcal{C}} : \partial CV(F_n) \rightarrow \partial \mathcal{CS} \cup \Theta$$

defined in Section 6.

As in the introduction, let $\mathcal{ST} \subset \partial CV(F_n)$ be the $\text{Out}(F_n)$ -invariant set of projective dense trees which either are indecomposable or split as large graphs of actions. It contains the $\text{Out}(F_n)$ -invariant subspace \mathcal{CT} of projective trees which either are indecomposable or split as very large graph of actions. By Corollary 6.7, Corollary 6.13 and Proposition 9.3, we have

$$\psi(\mathcal{ST}) \supset \partial \mathcal{FS} \text{ and } \psi_{\mathcal{C}}(\mathcal{CT}) \supset \partial \mathcal{CS}.$$

We have to show that $\psi^{-1}(\partial \mathcal{FS}) = \mathcal{ST}$ and $\psi^{-1}(\partial \mathcal{CS}) = \mathcal{CT}$. We begin with an extension of Lemma 7.5. To this end we call as before two dense trees $T, T' \in$

$\partial cv(F_n)$ equivalent if the unions \hat{T}, \hat{T}' of their metric completions with their Gromov boundaries are equivariantly homeomorphic.

Lemma 11.1. *Let $[T] \in \mathcal{ST}$, let $T' \in \partial cv(F_n)$ be dense and assume that there is a tree $S \in \partial cv(F_n)$ which admits an alignment preserving map onto both T, T' . Then T' admits an alignment preserving map onto a tree which is equivalent to T .*

Proof. By Proposition 9.2 we may assume that T' is either indecomposable or splits as a graph of actions.

We argue as in the proof of Lemma 7.5. Namely, if T splits as a large graph of actions then let $V \subset T$ be a vertex tree of the transverse covering of T defined by the structure of a large graph of actions. If T is indecomposable then we let $V = T$. Let $H < F_n$ be the stabilizer of V . Then H is finitely generated, and V is indecomposable for the action of H . Let S_H be the minimal H -invariant subtree of S . By equivariance, there is a one-Lipschitz alignment preserving map $S_H \rightarrow V$, and there also is an alignment preserving map $S_H \rightarrow T'_H$ where T'_H is the minimal H -invariant subtree of T' .

Using again Proposition 9.2 we may assume that T'_H either is indecomposable or splits as a graph of actions. Now S_H is a minimal very small H -tree and therefore by the results in [R11], there is no alignment preserving map from S_H onto both an indecomposable tree and a tree which splits as a graph of actions. Since V is indecomposable, this shows that T'_H is indecomposable.

An application of Lemma 7.5 to S_H and T_H, T'_H now shows that \hat{T}'_H is equivariantly homeomorphic to V . Since there is a single F_n -orbit of vertex trees for the structure of a large graph of actions on T we conclude that T' admits an alignment preserving map onto a tree which is equivalent to T . This is what we wanted to show. \square

The next proposition is the main remaining step for the proof of Theorem 1 and Theorem 2. It entirely relies on the work of Handel and Mosher [HM13b] and its variation due to Bestvina and Feighn [BF14c]. Recall that two trees T, T' with dense orbits are equivalent if the union of their metric completions with their Gromov boundaries are equivariantly homeomorphic.

Proposition 11.2. *Let $\Delta \subset \overline{cv_0(F_n)}$ be a standard simplex, let $[T] \in \mathcal{ST}$ and let (x_t) be a Skora path connecting a point $x_0 \in \Delta$ to $[T]$.*

- (1) $\Upsilon(x_t) \subset \mathcal{FS}$ is unbounded.
- (2) If $[T] \in \mathcal{CT}$ then $\Upsilon_{\mathcal{C}}(x_t) \subset \mathcal{CS}$ is unbounded.

Proof. Let $(x_t) \subset \overline{cv(F_n)}$ be an unnormalized Skora path connecting a point $x_0 \in \Delta$ to a representative T of $[T]$.

For the proof of the first part of the proposition we argue by contradiction and we assume that $\Upsilon(x_t) \subset \mathcal{FS}$ is bounded. Let d be the distance in \mathcal{FS} . Since the

distance between two vertices in \mathcal{FS} assumes only integral values, there is some $k \geq 0$ and a sequence $t_i \rightarrow \infty$ such that

$$d(\Upsilon(x_0), \Upsilon(x_{t_i})) = k$$

for all i .

For each i let $y_i(k)$ be the tree obtained from x_{t_i} by equivariantly rescaling edges in such a way that the lengths of all edges of the quotient $y_i(k)/F_n$ coincide. This can be arranged in such a way that there is a one-Lipschitz alignment preserving map $y_i(k) \rightarrow x_{t_i}$ and that the volume of $y_i(k)/F_n$ is bounded from above by a uniform multiple of the volume of x_{t_i}/F_n . Thus after passing to a subsequence, we may assume that the trees $y_i(k)$ converge to a tree $T_k \in \overline{cv(F_n)}$. By Proposition 5.5 of [G00], there is an alignment preserving map $T_k \rightarrow T$.

For each i let $\Gamma_i(j)$ ($0 \leq j \leq k$) be a geodesic in \mathcal{FS} connecting $\Upsilon(x_0)$ to $\Upsilon(x_{t_i})$. Then for all i , the vertex $\Gamma_i(k-1)$ can be obtained from x_{t_i} by either an expansion or a collapse. By passing to a subsequence, assume first that for all i , $\Gamma_i(k-1)$ is obtained from $\Gamma_i(k)$ by a collapse.

Let $y_i(k-1) \in \overline{cv(F_n)}$ be a simplicial tree whose quotient $y_i(k-1)/F_n$ has edges of equal length and defines the splitting $\Gamma_i(k-1)$. As $\Gamma_i(k-1)$ is obtained from $\Gamma_i(k)$ by a collapse, we may assume that the volume of $y_i(k-1)$ is bounded from below by a uniform multiple of the volume of $y_i(k)$ and that there is a one-Lipschitz alignment preserving map $y_i(k) \rightarrow y_i(k-1)$. By passing to a subsequence, we may moreover assume that the trees $y_i(k-1)$ converge to a tree $T_{k-1} \in \overline{cv(F_n)}$. Since T_k is dense, by equivariance the same holds true for T_{k-1} . By Proposition 5.5 of [G00], there is an alignment preserving map $T_k \rightarrow T_{k-1}$. Lemma 11.1 then shows that T_{k-1} admits an alignment preserving map onto a tree T'_{k-1} which is equivalent to T .

In the case that $\Gamma_i(k-1)$ is obtained from $\Gamma_i(k)$ by an expansion for all but finitely many i we construct the trees $y_i(k-1)$ as before, and we find by passing to a subsequence that $y_i(k-1)$ converges to a tree T_{k-1} which admits an alignment preserving map onto T_k and hence onto a tree equivalent to T .

Repeat this construction with the sequence $y_i(k-1)$ and trees $y_i(k-2)$ which define the splitting $\Gamma_i(k-2)$. In finitely many steps we conclude that x_0 admits an alignment preserving map onto a tree which is equivalent to T . This is a contradiction which shows the first part of the proposition.

The argument for the second part of the proposition is identical and will be omitted. \square

As an immediate consequence of Lemma 6.4, Proposition 11.2 and Lemma 6.8 we obtain

Corollary 11.3. *The map ψ (or ψ_C) restricts to a continuous closed $\text{Out}(F_n)$ -equivariant surjection $\mathcal{ST} \rightarrow \partial\mathcal{FS}$ (or a surjection $\mathcal{CT} \rightarrow \partial\mathcal{CS}$).*

We are left with calculating the fibres of the map. This is done in the next lemma.

- Lemma 11.4.** (1) $\psi([T]) = \psi([S])$ for trees $[S], [T] \in \mathcal{ST}$ if and only if $[T]$ and $[S]$ are equivalent.
 (2) $\psi_{\mathcal{C}}([T]) = \psi_{\mathcal{C}}([S])$ for trees $[S], [T] \in \mathcal{CT}$ if and only if $[T]$ and $[S]$ are equivalent.

Proof. If \hat{S}, \hat{T} are F_n -equivariantly homeomorphic then there exists an alignment preserving map $T \rightarrow S$ [CHL07] and therefore $\psi([S]) = \psi([T])$ by Corollary 6.13.

Now let $[S], [T] \in \mathcal{ST}$ be such that $\psi([S]) = \psi([T])$. Let Δ be a standard simplex and let $(x_t), (y_t)$ be Skora paths connecting a point in Δ to $[S], [T]$. By Corollary 11.3, the paths $\Upsilon(x_t), \Upsilon(y_t)$ are uniform unparametrized quasi-geodesics in \mathcal{FS} of infinite diameter with the same endpoint. The distance between their starting points is uniformly bounded.

By hyperbolicity, there is a number $m > 0$ such that for every $t \geq 0$ the free splitting $\Upsilon(x_t)$ is contained in the m -neighborhood of the quasi-geodesic ray $(\Upsilon(y_s))_{s \geq 0}$ in \mathcal{FS} . As a consequence, for each $t > 0$ there is some $s(t) > 0$ such that the distance between $\Upsilon(x_t)$ and $\Upsilon(y_{s(t)})$ is at most m .

For each i and each $k = 0, \dots, m$ let $\beta_i(k)$ be a graph of groups decomposition for F_n with trivial edge group with $\beta_i(0) = x_i/F_n, \beta_i(m) = y_{s(i)}/F_n$ such that for each $j \leq m/2$, the splitting $\beta_i(2j)$ collapses to both $\beta_i(2j - 1)$ and $\beta_i(2j + 1)$. For $1 \leq k \leq m - 1$ the splitting $\beta_i(k)$ defines a simplicial tree $\hat{\beta}_i(k) \in \overline{cv_0(F_n)}$ which is unique if we require that all edges have the same length.

Choose a sequence $(i_\ell)_{\ell \geq 0}$ so that for each $k \leq m$ the projectivizations $[\hat{\beta}_{i_\ell}(k)]$ of the trees $\hat{\beta}_{i_\ell}(k)$ converge in $\overline{CV(F_n)}$ to a tree $[U_k]$. Apply Proposition 5.5 of [G00] and conclude that for each j there is an alignment preserving map $U_{2j} \rightarrow U_{2j-1}$ and $U_{2j} \rightarrow U_{2j+1}$.

As in the proof of Proposition 11.2, we can now successively apply Lemma 11.1 and deduce that \hat{S} and \hat{T} are homeomorphic which is what we wanted to show. \square

Example: We give an example which shows that the Gromov boundary of the free splitting graph does not coincide with the Gromov boundary of the cyclic splitting graph.

Namely, let S be a compact surface of genus $g \geq 2$ with connected boundary ∂S . The *arc graph* of S is defined as follows. Vertices are embedded arcs in S with both endpoints on ∂S . Two such arcs are connected by an edge of length one if they are disjoint. The arc graph of S is quasi-isometrically embedded in the free splitting graph of the free group $\pi_1(S) = F_{2g}$ [HH11]. The Gromov boundary of the arc graph has been determined in [H11].

Let $c \subset S$ be a non-separating simple closed curve and let φ be a pseudo-Anosov mapping class of $S - c$. It follows from the results in [MS13], [H11] and [HH11] that ψ acts as a hyperbolic isometry on the free splitting graph of F_{2g} . However,

ψ preserves the cyclic splitting $A*_{\langle c \rangle}$ and therefore it acts as an elliptic element on the cyclic splitting graph.

We conclude this section with some remarks and open questions.

Question 1: An element $\varphi \in \text{Out}(F_n)$ acts on the free factor graph as a hyperbolic isometry if and only if it is irreducible with irreducible powers. An irreducible element with irreducible powers φ acts on $\partial\text{CV}(F_n)$ with north-south dynamics [LL03], fixing precisely two points. It was shown in [BFH97] (see also [KL11]) that the stabilizer in $\text{Out}(F_n)$ of such a fixed point of φ is virtually cyclic.

A fixed point $[T]$ of φ defines a point in the boundary of the free factor graph and hence is an arational tree. In analogy of the mapping class group action on the Thurston boundary of Teichmüller space, it can be shown that $[T]$ is uniquely ergodic, i.e. there is a unique projective measured lamination μ with $\langle T, \mu \rangle = 0$ [HH14].

More generally, in [H14a] we defined an F_n -invariant set \mathcal{UT} of projective trees in $\partial\text{CV}(F_n)$ as follows. If $[T] \in \mathcal{UT}$ and if $\mu \in \mathcal{ML}$ is such that $\langle T, \mu \rangle = 0$ then the projective class of μ is unique, and $[T]$ is the unique projective tree with $\langle T, \mu \rangle = 0$. It follows immediately from this work that $\mathcal{UT} \subset \mathcal{FT}$ (see also [R12, HH14]).

Define the ϵ -thick part $\text{Thick}_\epsilon(F_n)$ of $cv_0(F_n)$ to consist of simplicial trees with quotient of volume one which do not admit any essential loop of length smaller than ϵ . In analogy to properties of the curve graph and Teichmüller space, we conjecture that whenever (r_t) is a normalized Skora path in $cv_0(F_n)$ with the property that $r_{t_i} \in \text{Thick}_\epsilon(F_n)$ for a sequence $t_i \rightarrow \infty$ and some fixed number $\epsilon > 0$ then $[r_t]$ converges as $t \rightarrow \infty$ to a tree $[T] \in \mathcal{UT}$ (see [Ho12] and [HH14] for results in this direction). Moreover, if $(r_t) \subset \text{Thick}_\epsilon(F_n)$ for all t then the path $t \rightarrow \Upsilon(r_t)$ is a parametrized L -quasi-geodesic in \mathcal{FS} for a number $L > 1$ only depending on ϵ .

Question 2: Let $\varphi \in \text{Out}(F_n)$ be a reducible outer automorphism which acts as a hyperbolic isometry on the free splitting graph \mathcal{FS} . It follows from Theorem 1 that φ fixes a tree $[T] \in \mathcal{ST}$. It is true that up to scale there is a unique measured lamination μ with $\langle T, \mu \rangle = 0$ and $\mu(L^2(T) - L_r(T)) = 0$? What can we say about the subgroup of $\text{Out}(F_n)$ which fixes $[T]$? Recent work of Handel and Mosher [HM13c] indicates that this group may be quite large (i.e. it may not be infinitely cyclic).

Vice versa, given a map $\varphi \in \text{Out}(F_n)$, it is known that φ can be represented by a relative train track map. It was observed by Handel and Mosher that φ defines a hyperbolic automorphism of the free splitting complex if the support of the top stratum of the relative train track is all of F_n . Theorem 1 shows that the converse is also true: If the action of φ on \mathcal{FS} is hyperbolic, then the support of the top stratum of a relative train track for φ is all of F_n .

Note that Handel and Mosher showed [HM13c] that there are non-torsion elements $\varphi \in \text{Out}(F_n)$ which act with bounded orbits on the free splitting graph, but for which there is no $k \geq 1$ so that φ^k fixes a point in \mathcal{FS} .

APPENDIX A. SPLITTING CONTROL

In this appendix we collect some information from the work of Handel and Mosher [HM13b].

Consider a free group F_n of rank $n \geq 3$. Denote by \mathcal{FS} and \mathcal{CS} the first barycentric subdivision of the free splitting graph and the cyclic splitting graph of F_n , respectively. Vertices of \mathcal{FS} are graphs of groups decompositions of F_n with trivial edge groups. Such a graph of groups decomposition is a finite graph G whose vertices are labeled with (possibly trivial) free factors of F_n . Two vertices G, G' in \mathcal{FS} are connected by an edge if G' is either a collapse or an expansion of G . Vertices of \mathcal{CS} are graph of groups decompositions of F_n with at most cyclic edge groups. Two such vertices G, G' are connected by an edge if G' is either a collapse or an expansion of G .

Throughout this appendix we follow the appendix of [BF14c].

A simplicial F_n -tree $T \in \overline{cv(F_n)}$ with at least one trivial edge stabilizer projects to a finite metric graph T/F_n which defines a graph of groups decomposition for F_n with trivial edge groups, i.e. it defines a vertex in \mathcal{FS} . Namely, an arbitrary simplicial tree $T \in \overline{cv(F_n)}$ defines a vertex in \mathcal{CS} .

A finite *folding path* is a path $(x_t) \subset \overline{cv(F_n)}$ ($t \in [0, L]$) which is guided by an optimal morphism $f : x_0 \rightarrow x_L$ as introduced in Section 3. The points along the path are obtained from x_0 by identifying directions which are mapped to the same direction in x_L . We allow folding at any speed, and we also allow rest intervals. Such paths are called *liberal* folding paths in [BF14c].

An optimal morphism projects to a map of the quotient graphs which minimizes the Lipschitz constant in its homotopy class. There are induced optimal maps

$$f_t : x_t \rightarrow x_L$$

for all $t \in [0, L]$.

Let $G_t \subset \overline{cv(F_n)}$ ($0 \leq t \leq L$) be any folding path and let $F_L \subset G_L$ be a proper equivariant forest. Then for each t the preimage $F_t \subset G_t$ of F_L under the map $f_t : G_t \rightarrow G_L$ is defined, and $G'_t = G_t/F_t$ is an F_n -tree. By Proposition 4.4 of [HM13b] (see also Lemma A.1 of [BF14c]), the path $t \rightarrow G'_t$ is a liberal folding path, and we obtain a commutative diagram

$$(8) \quad \begin{array}{ccc} \bullet^{G'_0} & \longrightarrow & \bullet^{G'_L} \\ \uparrow & & \uparrow \\ \bullet^{G_0} & \longrightarrow & \bullet^{G_L} \end{array}$$

Similarly, if G_t ($0 \leq t \leq L$) is a folding path and if $G'_L \rightarrow G_L$ is a collapse map then there is a folding path G'_t so that the diagram

$$(9) \quad \begin{array}{ccc} G'_0 & \longrightarrow & G'_L \\ \bullet & & \bullet \\ \downarrow & & \downarrow \\ G_0 & \longrightarrow & G_L \\ \bullet & & \bullet \end{array}$$

commutes (Lemma A.2 of [BF14c]). Here one may have to insert rest intervals into the folding path G_t .

The following lemma is a version of Lemma A.3 of [BF14c].

Lemma A.1. *There is a number $M > 0$ with the following property. Let $G, G' \in \overline{cv(F_n)}$ be such that $G/F_n, G'/F_n$ are finite metric graphs defining a graph of groups decomposition for F_n . Let $f : G \rightarrow G'$ be an optimal morphism. Assume that there is a point $y \in G'$ such that the cardinality of $f^{-1}(y)$ is at most p . Then the distance in \mathcal{CS} between G, G' is at most Mp . If the edge groups of $G/F_n, G'/F_n$ are trivial then the same holds true for the distance in \mathcal{FS} .*

Proof. The proof of Lemma 4.1 of [BF14c] is valid without modification. A variation of the argument is used in Section 11, so we provide a sketch.

Connect G to G' by a folding path G_t guided by f , with $G_0 = G$. For each t there are optimal morphisms $\varphi_t : G \rightarrow G_t$ and $f_t : G_t \rightarrow G'$ so that $f = f_t \circ \varphi_t$. The number of preimages of y under f_t decreases with t .

Let $t > 0$ be such that this number coincides with the number of preimages of y in G . Then there is point $z_t \in G_t/F_n$ whose preimage under the quotient of φ_t consists of a single point z . If z_t is a vertex then the preimage of a nearby interior point of an adjacent edge consists of a single point as well, so may assume that the points z, z_t are contained in the interior of edges e, e_t . A loop in G/F_n not passing through the projection of z is mapped by the quotient of φ_t to a loop in G_t/F_n not passing through z_t . Thus collapsing the complement of the edges e, e_t in $G/F_n, G_t/F_n$ to a single point yields two identical graph of groups decompositions containing A as a vertex group. In particular, G/F_n and G_t/F_n collapse to the same vertex in $\mathcal{CS}(A)$, and if the edge group of e is trivial then G/F_n and G_t/F_n collapse to the same point in \mathcal{FS} .

Let $t_0 > 0$ be the first point so that the number of preimages of y in G_{t_0} is strictly smaller than the number of preimages of y in G_0 . For $t < t_0$ sufficiently close to t_0 , the graphs $G_t/F_n, G_{t_0}/F_n$ collapse to the same graph of groups decomposition of F_n . Hence the lemma now follows by induction. \square

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