TYPICAL AND ATYPICAL PROPERTIES OF PERIODIC TEICHMÜLLER GEODESICS

URSULA HAMENSTÄDT

Abstract. Consider a component $\mathcal{Q}$ of a stratum in the moduli space of area one abelian differentials on a surface of genus g. We also show that $\mathcal{Q}$ contains only finitely many algebraically primitive Teichmüller curves, and only finitely many affine invariant submanifolds of rank $\ell \geq 2$. We also show that periodic orbits whose $SL(2,\mathbb{R})$-orbit closure equals $\mathcal{Q}$ are typical.

Contents

Date: July 23, 2018.
Keywords: Abelian differentials, Teichmüller flow, periodic orbits, Lyapunov exponents, trace fields, orbit closures, equidistribution
AMS subject classification: 37C40, 37C27, 30F60
Research supported by ERC grant "Moduli".

1. Introduction

The *mapping class group* $\mathrm{Mod}(S)$ of a closed surface S of genus $g \geq 2$ acts by precomposition of marking on the *Teichmüller space* $\mathcal{T}(S)$ of marked complex structures on S. The action is properly discontinuous, with quotient the *moduli space* $\mathcal{M}_g$ of complex structures on S.

The *Hodge bundle* $\mathcal{H} \to \mathcal{M}_g$ over moduli space is the bundle whose fibre over a Riemann surface x equals the vector space of holomorphic one-forms on x. This is a holomorphic vector bundle of complex dimension g which decomposes into *strata* of differentials with zeros of given multiplicities. Its sphere subbundle is the moduli space of area one abelian differentials on S. There is a natural $SL(2,\mathbb{R})$-action on this sphere bundle preserving any connected component $\mathcal{Q}$ of a stratum. The action of the diagonal subgroup is called the *Teichmüller flow* Φ^t.

By the groundbreaking work of Eskin, Mirzakhani and Mohammadi [EMM15], affine invariant manifolds in a component $\mathcal{Q}$ of a stratum are precisely the closures of orbits for the $SL(2,\mathbb{R})$-action. Examples of non-trivial orbit closures are arithmetic *Teichmüller curves*. They arise from branched covers of the torus, and they are dense in any component of a stratum of abelian differentials. Other examples of orbit closures different from entire components of strata can be constructed using more general branched coverings. A more exotic orbit closure was recently discovered by McMullen, Mukamel and Wright [MMW16].

Period coordinates for a component $\mathcal{Q}$ of a stratum of abelian differentials, with set $\Sigma \subset S$ of zeros, are obtained by integration of a holomorphic one-form $q \in \mathcal{Q}$ over a basis of the relative homology group $H_1(S,\Sigma;\mathbb{Z})$. Thus a tangent vector of $\mathcal{Q}$ defines a point in $H_1(S,\Sigma;\mathbb{Z})^*$. The *rank* of an affine invariant manifold $\mathcal{C}$ is defined by

$$\mathrm{rk}(\mathcal{C}) = \frac{1}{2}\dim_{\mathbb{C}}(pT\mathcal{C})$$

where p is the projection of $H_1(S,\Sigma;\mathbb{R})^*$ into $H_1(S,\mathbb{R})^* = H^1(S,\mathbb{R})$ [W15]. The rank of a component of a stratum equals g, and *Teichmüller curves*, ie closed orbits of the $SL(2,\mathbb{R})$-action, are affine invariant manifolds of rank one and real dimension three.

We establish a finiteness result for affine invariant submanifolds of rank at least two which is independently due to Eskin, Filip and Wright [EFW17].

Theorem 1. *Let $g \geq 2$ and let $\mathcal{Q}$ be a component of a stratum in the moduli space of abelian differentials. For every $2 \leq \ell \leq g$, there are only finitely many proper affine invariant submanifolds in $\mathcal{Q}$ of rank ℓ.*

Let Γ be the set of all periodic orbits for Φ^t in $\mathcal{Q}$. The length of a periodic orbit $\gamma \in \Gamma$ is denoted by $\ell(\gamma)$. Let $k \geq 1$ be the number of zeros of the differentials in $\mathcal{Q}$ and let $h = 2g - 1 + k$. As an application of [EMR12] (see also [EM11]) we showed in [H13] that

$$\sharp\{\gamma \in \Gamma \mid \ell(\gamma) \leq R\}\frac{hR}{e^{hR}} \to 1 \quad (R \to \infty).$$

Call a subset $\mathcal{A}$ of Γ *typical* if

$$\sharp\{\gamma \in \mathcal{A} \mid \ell(\gamma) \leq R\}\frac{hR}{e^{hR}} \to 1 \quad (R \to \infty).$$

Thus a subset of Γ is typical if its growth rate is maximal. The intersection of two typical subsets of Γ is typical. As an application, we obtain

Theorem 2. *Let $\mathcal{Q}$ be any component of a stratum in genus $g \geq 3$. Then the set of all $\gamma \in \Gamma$ whose $SL(2,\mathbb{R})$-orbit closure equals $\mathcal{Q}$ is typical.*

For $g = 2$, Theorem 2 is false in a very strong sense. Namely, McMullen [McM03a] showed that in this case, the orbit closure of any periodic orbit is an affine invariant manifold of rank one. If the trace field $\mathfrak{k}$ of the periodic orbit is quadratic, then $\mathfrak{k}$ defines a Hilbert modular surface in the moduli space of principally polarized abelian varieties which contains the image of the orbit closure under the *Torelli map*. Such a Hilbert modular surface is a quotient of $\mathbf{H}^2 \times \mathbf{H}^2$ by the lattice $SL(2, \mathfrak{o}_{\mathfrak{k}})$ where $\mathfrak{o}_{\mathfrak{k}}$ is an order in $\mathfrak{k}$. This insight is the starting point of a complete classification of orbit closures in genus 2 [Ca04, McM03b].

In higher genus, Apisa [Ap15] classified all orbit closures of complex dimension at least four in hyperelliptic components of strata. For other components of strata, a classification of orbit closures is not available. However, there is substantial recent progress towards a geometric understanding of orbit closures. In particular, Mirzakhani and Wright [MW16] showed that all affine invariant manifolds of maximal rank either are components of strata or are contained in the hyperelliptic locus. We refer to the work [LNW15] of Lanneau, Nguyen and Wright for an excellent recent overview of what is known and for a structural result for rank one affine invariant manifolds.

To each Teichmüller curve is associated a *trace field* which is an algebraic number field of degree at most g over $\mathbb{Q}$. This trace field coincides with the trace field of every periodic orbit contained in the curve [KS00]. The Teichmüller curve is called *algebraically primitive* if the algebraic degree of its trace field equals g.

The stratum $\mathcal{H}(2)$ of abelian differentials with a single zero on a surface of genus 2 contains infinitely many algebraically primitive Teichmüller curves [Ca04, McM03b]. Recently, Bainbridge, Habegger and Möller [BHM14] showed finiteness of algebraically primitive Teichmüller curves in any stratum in genus 3. Finiteness of algebraically primitive Teichmüller curves in strata of differentials with a single zero for surfaces of prime genus $g \geq 3$ was established in [MW15]. Our final result generalizes this to every stratum in every genus $g \geq 3$, with a different proof. A stronger finiteness result covering Teichmüller curves whose field of definition is of degree at least three over $\mathbb{Q}$ is contained in [EFW17].

Theorem 3. *Any component $\mathcal{Q}$ of a stratum in genus $g \geq 3$ contains only finitely many algebraically primitive Teichmüller curves.*

Plan of the paper and strategy of the proofs: The proofs of the above results use tools from hyperbolic and non-uniform hyperbolic dynamics, differential geometry and algebraic groups. We embark from the foundational results of Eskin, Mirzakhani and Mohammadi [EMM15] and Filip [F16], but we do not use any methods developed in these works. We also do not use methods from the theory of flat surfaces, nor from algebraic geometry, although we apply several recent results from these areas, notably a structural result of Möller [Mo06]. Instead we initiate a study of differential geometric properties of the moduli space of abelian differentials using the geometry of the moduli space of principally polarized abelian differentials and the Torelli map. We hope that such ideas together with the use of algebraic geometry will lead to a better understanding of the Schottky locus in the future.

In Section 2 we first give a geometic description of the Hodge bundle in the form used later on, and we discuss the Gauss Manin connection. We then introduce affine invariant manifolds and establish some first easy properties. In particular, we study of the so-called *absolute period foliation*. This foliation has extensively been studied for components of strata. We will only need some fairly elementary properties discovered in [McM13] (see also [H15]), and we generalize these properties to affine invariant manifolds.

In Section 3 we begin with the differential geometric analysis of the moduli space of abelian differentials. We compare the Chern connection on the Hodge bundle to the Gauss Manin connection and establish a rigidity result using the results from [H18b]. This then leads to the proof of the first part of Theorem 1 in Section 4. As a byproduct, we obtain that the Oseledets splitting of the Hodge bundle over a component of a stratum is not of class C^1, however our methods are insufficient to deduce that this splitting is not continuous.

Section 5 is based on the ideas developed in Section 3, but relies on precise information on the absolute period foliation. It contains the completion of the proof of Theorem 1.

The proofs of Theorem 2 and Theorem 3 are contained in Section 6. The proof of Theorem 3 only depends on the results in Section 2, the main algebraic result of [H18b], Section 3 and Section 6.

The article concludes with an appendix which collects some differential geometric properties of the moduli space of principally polarized abelian differentials which are used in Section 3 and Section 5.

Acknowledgement: During the various stages of this work, I obtained generous help from many collegues. I am particularly grateful to Alex Wright for pointing out a mistake in an earlier version of this work. Both Matt Bainbridge and Alex Eskin notified me about parts in an earlier version of the paper which needed clarification. Discussions with Curtis McMullen inspired me to the differential geometric approach in Section 5. This article is based on work which was supported by the National Science Foundation under Grant No. DMS-1440140 while the author was in residence at the MSRI in Berkeley, California, in spring 2015.

2. The geometry of affine invariant manifolds

The goal of this section is to collect some geometric and dynamical properties of components of strata and on affine invariant manifolds which are used throughout this article. Most if not all of the results in this section are known to the experts. We provide proofs whenever we did not find a precise reference in the literature.

2.1. **The Hodge bundle.** In this subsection we introduce the geometric setup which will be used throughout the remainder of this article.

A point in *Siegel upper half-space* $\mathfrak{D}_g = \mathrm{Sp}(2g,\mathbb{R})/U(g)$ is a principally polarized abelian variety of complex dimension g. Here as usual, $U(g)$ denotes the unitary group of rank g. Let $\omega = \sum_i dx_i \wedge dy_i$ be the standard symplectic form on the real vector space $\mathbb{R}^{2g}$. A point z in $\mathfrak{D}_g$ can be viewed as a complex structure J_z on $(\mathbb{R}^{2g},\omega)$ which is *compatible* with the symplectic structure, ie such that $\omega(\cdot, J_z\cdot)$ is an inner product on $\mathbb{R}^{2g}$. The Siegel upper half-space is a Hermitean symmetric space, in particular it is a complex manifold, and the symmetric metric is Kähler.

There is a natural rank g complex vector bundle $\tilde{\mathcal{V}} \to \mathfrak{D}_g$ whose fibre over y is just the complex vector space defining y. Thus as a real vector bundle, $\tilde{\mathcal{V}}$ is just the bundle $\mathfrak{D}_g \times \mathbb{R}^{2g}$, however the complex structure on the fibre over z depends on z. This bundle is holomorphic. This means there are local complex trivializations for $\tilde{\mathcal{V}}$ with holomorphic transition functions.

The polarization (ie the symplectic structure) and the complex structure define a Hermitean metric h on $\tilde{\mathcal{V}}$. The real part of this metric in the fibre over z is defined by $g(X,Y) = \omega(X, J_zY)$ where J_z is the complex structure on $\mathbb{R}^{2g}$ corresponding to z.

The group $Sp(2g,\mathbb{R})$ acts from the left on the bundle $\tilde{\mathcal{V}}$ as a group of holomorphic bundle automorphisms preserving the polarization and the complex structure, and hence this action preserves the Hermitean metric. Thus the bundle $\tilde{\mathcal{V}}$ projects to a holomorphic Hermitean (orbifold) vector bundle

$$\mathcal{V} \to Sp(2g,\mathbb{Z})\backslash Sp(2g,\mathbb{R})/U(g) = \mathcal{A}_g.$$

We refer to the appendix for a more detailed information on this bundle.

Let $\mathcal{M}_g$ be the moduli space of closed Riemann surfaces of genus g. This is the quotient of *Teichmüller space* $\mathcal{T}(S)$ under the action of the mapping class group $\mathrm{Mod}(S)$. The *Torelli map*

$$\mathcal{I}_g : \mathcal{M}_g \to \mathcal{A}_g = Sp(2g,\mathbb{Z})\backslash Sp(2g,\mathbb{R})/U(g)$$

which associates to a Riemann surface its Jacobian is holomorphic. The *Hodge bundle*

$$\Pi : \mathcal{H} \to \mathcal{M}_g$$

is the pullback of the holomorphic vector bundle $\mathcal{V} \to \mathcal{A}_g$ by the Torelli map. As the Torelli map is holomorphic, $\mathcal{H}$ is a g-dimensional holomorphic Hermitean vector bundle on $\mathcal{M}_g$ (in the orbifold sense). Its fibre over $x \in \mathcal{M}_g$ can be identified with the vector space of holomorphic one-forms on x. The Hermitean inner product on $\mathcal{H}$ is given by

$$(\omega,\zeta) = \frac{i}{2}\int \omega \wedge \overline{\zeta}.$$

Here the integration is over the basepoint, which is a Riemann surface. With this interpretation, the sphere bundle in $\mathcal{H}$ for the inner product $(,)$ is just the moduli space of area one abelian differentials.

As a real vector bundle, the Hodge bundle $\mathcal{H}$ has the following additional description. The action of the mapping class group $\mathrm{Mod}(S)$ on the first real cohomology group $H^1(S,\mathbb{R})$ defines the homomorphism $\Psi : \mathrm{Mod}(S) \to Sp(2g,\mathbb{Z})$. As a real vector bundle, the Hodge bundle is then the flat orbifold vector bundle

$$\mathcal{N} = \mathcal{T}(S) \times_{\mathrm{Mod}(S)} H^1(S,\mathbb{R}) \to \mathcal{M}_g \tag{1}$$

for the standard left action of $\mathrm{Mod}(S)$ on Teichmüller space $\mathcal{T}(S)$ and the right action of $\mathrm{Mod}(S)$ on $H^1(S,\mathbb{R})$ via Ψ. This description determines a flat connection on $\mathcal{N}$ which is called the *Gauss Manin* connection. We use the notation $\mathcal{N}$ to emphasize that we consider a flat real vector bundle. The bundle $\mathcal{N}$ has a natural real analytic structure induced by the complex structure on $\mathcal{M}_g$ so that the Gauss Manin connection is real analytic.

The Hodge bundle $\mathcal{H}$ is the real vector bundle $\mathcal{N}$ equipped with the following complex structure. Each point $x \in \mathcal{M}_g$ determines a complex structure J_x on $H^1(S,\mathbb{R})$. Namely, every cohomology class $\alpha \in H^1(S,\mathbb{R})$ can be represented by a unique harmonic one-form for the complex structure x, and this one-form is the real part of a unique holomorphic one-form ζ on x. The imaginary part of ζ is a harmonic one-form which represents the cohomology class $J_x\alpha$. The complex structure J_x is compatible with the symplectic structure defined by the intersection form ι on $H^1(S,\mathbb{R})$.

The assignment $x \to J_x$ defines a real analytic section J of the endomorphism bundle $\mathcal{N}^* \otimes \mathcal{N} \to \mathcal{M}_g$ of $\mathcal{N}$ which satisfies $J^2 = -\mathrm{Id}$. Thus the flat vector bundle $\mathcal{N} \otimes_{\mathbb{R}} \mathbb{C} \to \mathcal{M}_g$ can be decomposed as

$$\mathcal{N} \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{H} \oplus \overline{\mathcal{H}}$$

where the *holomorphic* bundle $\mathcal{H} = \{\alpha + iJ\alpha \mid \alpha \in \mathcal{N}\}$ admits a natural identification with the bundle of holomorphic one-forms on $\mathcal{M}_g$, ie $\mathcal{H}$ is just the Hodge bundle over $\mathcal{M}_g$. The antiholomorphic bundle $\overline{\mathcal{H}}$ is defined by $\overline{\mathcal{H}} = \{\alpha - iJ\alpha \mid \alpha \in \mathcal{N}\}$.

The Hodge bundle over $\mathcal{M}_g$ is a holomorphic vector bundle over the complex orbifold $\mathcal{M}_g$ and therefore it is naturally a complex orbifold in its own right. Denote by $\mathcal{H}_+ \subset \mathcal{H}$ the complement of the zero section in the Hodge bundle $\mathcal{H}$. This is a complex orbifold. The pull-back

$$\Pi^*\mathcal{H} \to \mathcal{H}_+$$

to $\mathcal{H}_+$ of the Hodge bundle on $\mathcal{M}_g$ is a holomorphic vector bundle on $\mathcal{H}_+$. As a real vector bundle, it coincides with the pull-back $\Pi^*\mathcal{N}$ of the flat bundle $\mathcal{N}$. The pull-back of the Gauss-Manin connection on $\mathcal{N}$ is a flat connection on $\Pi^*\mathcal{N}$ which we call again the Gauss Manin connection. In the sequel we identify the real vector bundles $\Pi^*\mathcal{N}$ and $\Pi^*\mathcal{H}$ at leisure, using mainly the notation $\Pi^*\mathcal{H}$. However, sometimes we are only interested in the flat structure of $\Pi^*\mathcal{N}$ and then we write $\Pi^*\mathcal{N}$ to avoid confusion.

2.2. **Connections.** The goal of this subsection is to summarize some well known properties of connections on vectors bundles in the form we need.

We begin with the flat connection on the bundle $\mathcal{N} = \mathcal{T}(S) \times_{\mathrm{Mod}(S)} H^1(S,\mathbb{R})$. Its holonomy group is the group $Sp(2g,\mathbb{Z})$, the image of the mapping class group under the homomorphism Ψ. As the action of the mapping class group on $\mathcal{T}(S)$ is not free, we have to be slightly careful when computing the holonomy about a closed loop. The following discussion is geared at circumventing this difficulty.

Let $\mathrm{Sing} \subset \mathcal{T}(S)$ be the $\mathrm{Mod}(S)$-invariant subvariety of surfaces with nontrivial automorphisms. The complex codimension of Sing is at least two. We will not need this fact in the sequel; all we need is that this set is closed and nowhere dense. Let $\tilde\alpha : [0,1] \to \mathcal{T}(S)$ be any smooth path with $\tilde\alpha(0), \tilde\alpha(1) \in \mathcal{T}(S) - \mathrm{Sing}$. Assume that there is an element $\varphi \in \mathrm{Mod}(S)$ so that $\varphi(\tilde\alpha(0)) = \tilde\alpha(1)$. Then φ is unique. Furthermore, $\tilde\alpha$ projects to a closed path α in $\mathcal{M}_g$. Up to conjugation, the holonomy along α for the flat connection on $\mathcal{N}$ equals the map $\Psi \circ \varphi^{-1}$ which maps the fibre of $\mathcal{N}$ over $\tilde\alpha(1)$ to the fibre of $\mathcal{N}$ over $\tilde\alpha(0)$. It only depends on the endpoints of $\tilde\alpha$. In particular, it is well defined even if the path $\tilde\alpha$ is not entirely contained in $\mathcal{T}(S) - \mathrm{Sing}$.

As $\mathrm{Sing} \subset \mathcal{T}(S)$ is closed and nowhere dense, there exists a contractible neighborhood U of $\tilde\alpha(0)$ which is entirely contained in $\mathcal{T}(S) - \mathrm{Sing}$ and such that $\eta(U) \cap U = \emptyset$ for all $\mathrm{Id} \neq \eta \in \mathrm{Mod}(S)$. Let $\gamma : [0,1] \to \mathcal{T}(S) - \mathrm{Sing}$ be any smooth path which connects a point $\gamma(0) \in U$ to a point $\gamma(1) = \varphi(\gamma(0))$ in $\varphi(U)$. The discussion in the previous paragraph shows that the holonomy of the parallel transport of $\mathcal{N}$ along the projection of γ to $\mathcal{M}_g$ is conjugate to the holomomy for parallel transport along α. In particular, the absolute values of the eigenvalues of these holonomy maps coincide. The same consideration is also valid for the holonomy of the pullback bundle $\Pi^*\mathcal{N} \to \mathcal{H}_+$ with respect to the flat pullback connection.

We use this fact as follows. Let $\mathcal{Q}_+ \subset \mathcal{H}_+$ be a component of a stratum of abelian differentials. Define the *good subset* $\mathcal{Q}_{+,\mathrm{good}}$ of $\mathcal{Q}_+$ to be the set of all points $q \in \mathcal{Q}_+$ with the following property. Let $\tilde{\mathcal{Q}}_+$ be a component of the preimage of $\mathcal{Q}_+$ in the Teichmüller space of *marked* abelian differentials and let $\tilde q \in \tilde{\mathcal{Q}}_+$ be a lift of q; then an element of $\mathrm{Mod}(S)$ which fixes $\tilde q$ acts as the identity on $\tilde{\mathcal{Q}}_+$ (compare [H13] for more information on this technical condition). Then $\mathcal{Q}_{+,\mathrm{good}}$ is precisely the subset of $\mathcal{Q}_+$ of manifold points. Lemma 4.5 of [H13] shows that the good subset $\mathcal{Q}_{+,\mathrm{good}}$ of $\mathcal{Q}_+$ is open, dense and Φ^t-invariant, furthermore it is invariant under scaling.

Consider a smooth closed curve $\alpha : [0,1] \to \mathcal{Q}_{+,\mathrm{good}}$. As before, the parallel transport along α of the bundle $\Pi^*\mathcal{N} \to \mathcal{Q}_+$ with respect to the flat pull-back connection is defined.

Definition 2.1. A closed curve $\eta : [0,a] \to \mathcal{Q}_{+,\mathrm{good}}$ *defines the conjugacy class of a pseudo-Anosov mapping class* $\varphi \in \mathrm{Mod}(S)$ if the following holds true. Let $\tilde\eta$ be a lift of η to an arc in the Teichmüller space of abelian differentials. Then $\psi\tilde\eta(0) = \tilde\eta(a)$ for some $\psi \in \mathrm{Mod}(S)$, and we require that ψ is conjugate to φ.

Using Definition 2.1, the following is now immediate from the above discussion.

Lemma 2.2. *Let $\eta \subset \mathcal{Q}_{+,\rm good}$ be a closed curve which defines the conjugacy class of a pseudo-Anosov mapping class $\varphi \in {\rm Mod}(S)$. Then the eigenvalues of the holonomy map obtained by parallel transport of the bundle $\Pi^*\mathcal{N}$ along η coincide with the eigenvalues of the map $\Psi \circ \varphi^{-1}$.*

Proof. As discussed above, if $\eta : [0,a] \to \mathcal{Q}_{+,\rm good}$ is a closed curve and if $\tilde\eta$ is a lift of η to the Teichḿuller space of marked abelian differentials, then there is a unique element $\psi \in {\rm Mod}(S)$ with $\psi(\tilde\eta(0)) = \tilde\eta(a)$. The absolute values of the eigenvalues of ψ^{-1} are precisely the absolute values of the eigenvalues of the holonomy map for parallel transport of $\Pi^*\mathcal{N}$ along η. The lemma now follows from the definition of a curve which defines the conjugacy class of φ. □

We include some remarks on more general connections used later on. Let $L \to M$ be a real analytic vector bundle of rank k over a real analytic orbifold with a connection ∇. Let $e_1,\dots,e_k$ be a local basis of L over an open set U. Then the connection ∇ can be represented by a *connection matrix* with respect to this basis. This matrix is a (k,k)-matrix $\theta = (\theta_{ij})$ whose entries are one-forms on M. For any tangent vector X of U and all i, we have

$$\nabla_X(e_i) = \sum_j \theta_{ij}(X) e_j.$$

The connection is *real analytic* if whenever the basis $e_1,\dots,e_k$ is real analytic, then the one-forms θ_{ij} are real analytic. If moreover ∇ is flat, then we can choose the basis $e_1,\dots,e_k$ in such a way that the one-forms θ_{ij} vanish identically. This is equivalent to the basis e_i being parallel.

Let ∇ be flat and let $\hat\nabla$ be a second connection on the same vector bundle. Let furthermore $X_1,\dots,X_n$ be a local basis of the tangent bundle TM of M. Then we have

Lemma 2.3. *If for some local basis $e_1,\dots,e_k$ of L which is parallel for ∇ and for all i,j we have $\hat\nabla_{X_j} e_j = 0$ then $\hat\nabla = \nabla$.*

Proof. By linearity, under the condition of the lemma the connection matrix for $\hat\nabla$ and the basis $e_1,\dots,e_k$ vanishes identically. This means that the basis $e_1,\dots,e_k$ is parallel for $\hat\nabla$. □

2.3. **The absolute period foliation.** Let again $\mathcal{Q}_+$ be a component of a stratum of abelian differentials on the surface S with fixed number and multiplicities of zeros. If we denote by $\Sigma \subset S$ this zero set then an abelian differential $\omega \in \mathcal{Q}_+$ determines an euclidean metric on $S - \Sigma$, given by a system of complex local coordinates z on $S - \Sigma$ for which ω assumes the form $\omega = dz$. Chart transitions are translations. The foliations of S into horizontal and vertical line segments in these coordinates are equipped with a transverse invariant measure obtained by integration of the imaginary or real part, respectively, of ω. These measured foliations are oriented, and they are called the *horizontal and vertical measured foliation* of ω. Integration of these signed (=oriented) transverse invariant measures over cycles defining classes in $H_1(S,\Sigma;\mathbb{R})$ determine points in $H_1(S,\Sigma;\mathbb{R})^*$. Period coordinates for $\mathcal{Q}_+$ on a neighborhood of ω are defined by integration of the real and imaginary part of a differential near ω over a local basis of $H_1(S,\Sigma;\mathbb{Z})$.

An *affine invariant manifold* $\mathcal{C}_+$ in $\mathcal{Q}_+$ is the closure in $\mathcal{Q}_+$ of an orbit of the $GL^+(2,\mathbb{R})$-action. Such an affine invariant manifold is complex affine in period coordinates [EMM15]. In particular, $\mathcal{C}_+ \subset \mathcal{Q}_+$ is a complex suborbifold. Period coordinates determine a projection

$$p : T\mathcal{C}_+ \to \Pi^*(\mathcal{H} \oplus \overline{\mathcal{H}})|\mathcal{C}_+ = \Pi^*\mathcal{N} \otimes_{\mathbb{R}} \mathbb{C}|\mathcal{C}_+$$

to absolute periods (see [W14] for a clear exposition). The image $p(T\mathcal{C}_+)$ is flat, ie it is invariant under the restriction of the Gauss Manin connection to a connection on $\Pi^*(\mathcal{H} \oplus \overline{\mathcal{H}})|_{\mathcal{C}_+} = \Pi^*\mathcal{N} \otimes_{\mathbb{R}} \mathbb{C}|_{\mathcal{C}_+}$.

By the main result of [F16], there is a *holomorphic* subbundle $\mathcal{Z}$ of $\Pi^*\mathcal{H}|_{\mathcal{C}_+}$ such that

$$p(T\mathcal{C}_+) = \mathcal{Z} \oplus \overline{\mathcal{Z}}.$$

We call $\mathcal{Z}$ the *absolute holomorphic tangent bundle* of $\mathcal{C}_+$. As a consequence, the bundle $p(T\mathcal{C}_+)$ is invariant under the complex structure on $\Pi^*\mathcal{N} \otimes_{\mathbb{R}} \mathbb{C}$.

As a real vector bundle, $\mathcal{Z}$ is isomorphic to $p(T\mathcal{C}_+) \cap \Pi^*\mathcal{N}|\mathcal{C}_+$. Since $\mathcal{Z}$ is invariant under the compatible complex structure J, $p(T\mathcal{C}_+) \cap \Pi^*\mathcal{N}$ is symplectic [AEM12].

Define the *rank* of the affine invariant manifold $\mathcal{C}_+$ as [W14]

$$\mathrm{rk}(\mathcal{C}_+) = \frac{1}{2}\dim_{\mathbb{C}} p(T\mathcal{C}_+) = \dim_{\mathbb{C}}\mathcal{Z}.$$

With this definition, components of strata are affine invariant manifolds of rank g.

When we investigate dynamical properties it is as before more convenient to consider the intersection $\mathcal{C}$ of an affine invariant manifold $\mathcal{C}_+ \subset \mathcal{H}_+$ with the moduli space of area one abelian differentials. This intersection $\mathcal{C}$ is invariant under the action of the group $SL(2,\mathbb{R}) < GL^+(2,\mathbb{R})$. Throughout we always denote such an affine invariant manifold by $\mathcal{C}$, and we let $\mathcal{C}_+$ be its natural extension to $\mathcal{H}_+$.

Every component $\mathcal{Q}$ of a stratum in the bundle of area one abelian differentials which consists of differentials with at least two zeros admits a foliation $\mathcal{AP}(\mathcal{Q})$ whose leaves locally consist of differentials with the same absolute periods. This foliation is called the *absolute period foliation* (we adopt this terminology from [McM13], other authors call it the relative period foliation). The leaves of this foliation admit a complex affine structure (see e.g. [McM13]).

If $\mathcal{C}_+ \subset \mathcal{Q}_+$ is an affine invariant manifold whose complex dimension is strictly bigger than twice its rank then $\mathcal{C}$ intersects the leaves of the absolute period foliation of $\mathcal{Q}$ nontrivially. This fact alone does not imply that $\mathcal{C} \cap \mathcal{AP}(\mathcal{Q})$ is a foliation of $\mathcal{C}$. The main goal of this subsection is to verify that indeed, this is always the case.

To this end we need some more detailed information on the absolute period foliation of the component $\mathcal{Q}$ of a stratum. Its tangent bundle $T\mathcal{AP}(\mathcal{Q})$ has an explicit description via so-called *Schiffer variations* [McM13] which we explain now.

Let first ω be an abelian differential with a simple zero p. Then ω defines a singular euclidean metric on S which has a cone point of cone angle 4π at p. There are four horizontal separatrices at p for this metric. In a complex coordinate z near p so that $\omega = (z/2)dz$, the horizontal separatrices are the four rays contained in the real or the imaginary axis. The restriction of ω to these rays defines an orientation on the rays. With respect to this orientation, the two rays contained in the real axis are outgoing from p, while the rays contained in the imaginary axis are incoming. The *Schiffer variation* of ω with weight one at p is the tangent at ω of the following arc of deformations of ω. For small $u > 0$ cut the surface S open along the initial subsegment of length $2u$ of the two separatrices whose orientations point towards p and refold the resulting four-gon so that the singular point p slides backwards along the incoming rays in the imaginary axis. We refer to [McM13] and [H15] for a more detailed description.

If ω has a zero of order $n \geq 2$ at p then the Schiffer variation of ω with weight one at p is defined as follows (see p.1235 of [McM13]). Choose a coordinate z near p so that $\omega = (z^n/2)dz$ in this coordinate. This choice of coordinate is unique up to multiplication with $e^{\ell 2\pi i/(n+1)}$ for some $\ell \leq n$. There are $n+1$ horizontal separatrices at p for the flat metric defined by ω whose orientations point towards p. For small $u > 0$ cut the surface S open along the initial subsegments of length $2u$ of these $n+1$ segments. The result is a $2n+2$-gon which we refold as in the case of a simple zero. The tangent at ω of this arc of deformations of ω is called the Schiffer variation of ω with weight one at p.

Consider a component $\mathcal{Q}$ of a stratum of area one abelian differentials consisting of differentials with $k \geq 2$ zeros. By passing to a finite cover $\hat{\mathcal{Q}}$ of $\mathcal{Q}$ we may assume that the zeros are *numbered*. For $\omega \in \hat{\mathcal{Q}}$ let $Z(\omega)$ be the set of numbered zeros of ω. Let moreover $V(\omega) \sim \mathbb{C}^k$ be the complex vector space freely generated by the set $Z(\omega)$. Then the tangent space $T\mathcal{AP}(\hat{\mathcal{Q}})$ of the absolute period foliation of $\hat{\mathcal{Q}}$ at ω is naturally isomorphic to the hyperplane in $V(\omega)$ of all points whose coordinates sum up to zero [McM13, H15], ie of points with zero mean.

More explicitly, let $\mathfrak{a} = (a_1, \dots, a_k) \in \mathbb{R}^k$ be any k-tuple of *real* numbers with $\sum_i a_i = 0$. Then $\mathfrak{a}$ defines a smooth vector field $X_{\mathfrak{a}}$ on $\hat{\mathcal{Q}}$ as follows. For each $\omega \in \hat{\mathcal{Q}}$, the value of $X_{\mathfrak{a}}$ at ω is the Schiffer variation for the tuple $(a_1, \dots, a_k)$ of weight parameters at the numbered zeros of ω. Thus $X_{\mathfrak{a}}$ is tangent to the absolute period foliation. The $k-1$-dimensional real subbundle of the tangent bundle of $\hat{\mathcal{Q}}$ spanned by these vector fields is the tangent bundle of the *real rel foliation* $\mathcal{R}$ of $\hat{\mathcal{Q}}$ which is the intersection of the absolute period foliation with the *strong unstable foliation* W^{su} of $\hat{\mathcal{Q}}$. Recall that the leaf of the strong unstable foliation through $q \in \hat{\mathcal{Q}}$ locally consists of all differentials with the same horizontal *measured foliation* as q. Thus in period coordinates, the local leaf of W^{su} through q is just the set of all differentials q' whose imaginary parts define the same relative cohomology class as the imaginary part of q, taken relative to the zeros of q (or q') (via integration of the transverse invariant measure over relative cycles).

Similarly, we define the *imaginary rel foliation* of $\hat{\mathcal{Q}}$ to be the intersection of the absolute period foliation with the *strong stable foliation* W^{ss} of $\hat{\mathcal{Q}}$. The leaf of the foliation W^{ss} through q locally consists of all differentials with the same vertical measured foliations as q. Exchanging the roles of the horizontal and the vertical measured foliation in the definition of the Schiffer variations identifies the tangent bundle of the imaginary rel foliation of $\hat{\mathcal{Q}}$ with the purely imaginary weight vectors of zero mean on the numbered zeros of the differentials in $\hat{\mathcal{Q}}$. In this way each k-tuple $\mathfrak{a} = (a_1, \dots, a_k) \in \mathbb{R}^k$ of real numbers with $\sum_i a_i = 0$ determines a vector field $X_{i\mathfrak{a}}$ which is tangent to the imaginary rel foliation. As the tangent bundle of the absolute period foliation is spanned by its intersection with the tangent bundle of the strong stable and the strong unstable foliation, mapping a real weight vector to its multiple with $i = \sqrt{-1}$ defines a natural almost complex structure on $T\mathcal{AP}(\hat{\mathcal{Q}})$. This almost complex structure is in fact integrable [McM13] and equals the complex structure defined by period coordinates.

The Teichmüller flow Φ^t preserves the absolute period foliation. The following is Lemma 2.2 of [H15].

Lemma 2.4. *$d\Phi^t X_{\mathfrak{a}} = e^t X_{\mathfrak{a}}$ and $d\Phi^t X_{i\mathfrak{a}} = e^{-t} X_{i\mathfrak{a}}$ for every $\mathfrak{a} \in \mathbb{R}^k$ with zero mean.*

We observe next that an affine invariant submanifold $\mathcal{C}$ of $\mathcal{Q}$ intersects the absolute period foliation of $\mathcal{Q}$ in a real analytic foliation $\mathcal{AP}(\mathcal{C})$ with complex affine leaves.

For the formulation, denote again by $\mathcal{C}_+$ the extension of $\mathcal{C}$ to a $GL^+(2,\mathbb{R})$-invariant subspace of $\mathcal{Q}_+$. We define the *deficiency* $\mathrm{def}(\mathcal{C})$ as

$$\mathrm{def}(\mathcal{C}) = \dim_{\mathbb{C}}(\mathcal{C}_+) - 2\mathrm{rk}(\mathcal{C}_+).$$

The following lemma is a concrete and global version of Remark 1.4.(ii) of [F16]. As before, $k \geq 1$ denotes the number of zeros of a differential in $\mathcal{Q}$.

Lemma 2.5. *Let $\mathcal{C}$ be an affine invariant submanifold of $\mathcal{Q}$ of deficiency $r = \mathrm{def}(\mathcal{C}) > 0$ and let $\hat{\mathcal{C}}$ be a component of the preimage of $\mathcal{C}$ in $\hat{\mathcal{Q}}$. Then $\hat{\mathcal{C}}$ intersects the real rel foliation (or the imaginary rel foliation) of $\hat{\mathcal{Q}}$ in a real analytic foliation of real dimension r. Furthermore, if $q \in \hat{\mathcal{C}}$ and if $\mathfrak{a} \in \mathbb{R}^k$ is a vector of zero mean such that $X_{\mathfrak{a}}(q) \in T\mathcal{AP}(\hat{\mathcal{C}})$, then $X_{\mathfrak{a}}(z) \in T\mathcal{AP}(\hat{\mathcal{C}})$, $X_{i\mathfrak{a}}(z) \in T\mathcal{AP}(\hat{\mathcal{C}})$ for every $z \in \hat{\mathcal{C}}$.*

Proof. Let $\hat{\mathcal{C}} \subset \hat{\mathcal{Q}}$ be an affine invariant manifold of deficiency $r = \mathrm{def}(\mathcal{C}) > 0$. Then for each $q \in \hat{\mathcal{C}}$ there is a vector $0 \neq X \in T_q\mathcal{AP}(\hat{\mathcal{Q}})$ which is tangent to $\hat{\mathcal{C}}$. By invariance of $\hat{\mathcal{C}}$ and of the absolute period foliation of $\hat{\mathcal{Q}}$ under the Teichmüller flow, we have $d\Phi^t(X) \in T\hat{\mathcal{C}} \cap T\mathcal{AP}(\hat{\mathcal{Q}})$ for all t.

A vector $X \in T\hat{\mathcal{C}} \cap T\mathcal{AP}(\hat{\mathcal{Q}})$ decomposes as $X = X^u + X^s$ where $X^u \in T\mathcal{AP}(\hat{\mathcal{Q}})$ is real (and hence tangent to the strong unstable foliation) and X^s is imaginary (and hence tangent to the strong stable foliation). We claim that we can find a vector $Y \in T\hat{\mathcal{C}} \cap T\mathcal{AP}(\hat{\mathcal{Q}})$ which either is tangent to the strong unstable or to the strong stable foliation. To this end we may assume that $X^u \neq 0$. Since this is an open condition and since the Teichmüller flow on $\hat{\mathcal{C}}$ is topologically transitive, we may furthermore assume that the Φ^t-orbit of the footpoint q of X is dense in $\hat{\mathcal{C}}$. Then there is a sequence $t_i \to \infty$ such that $\Phi^{t_i}(q) \to q$.

Choose any smooth norm $\| \|$ on $T\hat{\mathcal{Q}}$. As $X^u \neq 0$, Lemma 2.4 and its analog for imaginary vectors and the inverse $t \to \Phi^{-t}$ of the Teichmüller flow shows that $\|d\Phi^{t_i} X^u\| \to \infty$ and $\|d\Phi^{t_i} X^s\| \to 0$. Therefore up to passing to a subsequence,

$$d\Phi^{t_i}(X)/\|d\Phi^{t_i}(X)\|$$

converges to a vector $Y \in T_q\mathcal{AP}(\hat{\mathcal{Q}})$ which is tangent to the strong unstable foliation. Now $T\hat{\mathcal{C}}$ is a smooth $d\Phi^t$-invariant subbundle of the restriction of the tangent bundle of $\hat{\mathcal{Q}}$ to $\hat{\mathcal{C}}$ (this is meant in the orbifold sense) and hence we have $Y \in T\hat{\mathcal{C}} \cap T\mathcal{AP}(\hat{\mathcal{Q}})$ which is what we wanted to show.

Using Lemma 2.4 and density of the Φ^t-orbit of q, if $0 \neq \mathfrak{a} \in \mathbb{R}^k$ is a vector of zero mean such that $Y = X_{\mathfrak{a}}(q)$, then $X_{\mathfrak{a}}(u) \in T\hat{\mathcal{C}}$ for all $u \in \hat{\mathcal{C}}$.

By invariance of $T\mathcal{C}_+$ under the complex structure defined by period coordinates, if $r = 1$ then

$$T\hat{\mathcal{C}} \cap T\mathcal{AP}(\hat{\mathcal{Q}}) = \mathbb{R}X_{\mathfrak{a}} \oplus \mathbb{R}X_{i\mathfrak{a}}$$

and we are done. Otherwise there is a tangent vector $Y \in T\hat{\mathcal{C}} \cap T\mathcal{AP}(\hat{\mathcal{Q}}) - \mathbb{C}X_{\mathfrak{a}}$. Apply the above argument to Y, perhaps via replacing the Teichmüller flow by its inverse. In finitely many such steps we conclude that there is a smooth subbundle $\mathcal{B}$ of $T\hat{\mathcal{C}} \cap T\mathcal{AP}(\hat{\mathcal{Q}})$ which is tangent to the strong unstable foliation (ie real for the real structure), of real rank r, and such that $T\hat{\mathcal{C}} \cap T\mathcal{AP}(\hat{\mathcal{Q}}) = \mathbb{C}\mathcal{B}$. Moreover, if $z \in \hat{\mathcal{C}}$ and if $\mathfrak{a} \in \mathbb{R}^k$ is such that $X_{\mathfrak{a}}(z) \in \mathcal{B}$ then $X_{\mathfrak{a}}(q) \in \mathcal{B}$ for every $q \in \hat{\mathcal{C}}$.

To summarize, there exists an r-dimensional real linear subspace V of the hyperplane of $\mathbb{R}^k$ of vectors of zero mean, and for each $\mathfrak{a} \in V$ and every $q \in \hat{\mathcal{C}}$, the vectors $X_{\mathfrak{a}}(q), X_{i\mathfrak{a}}(q)$ are both tangent to $\hat{\mathcal{C}}$ at q. Then $\hat{\mathcal{C}}$ is invariant under the flows $\Lambda^t_{\mathfrak{a}}$ generated by the vector fields $X_{\mathfrak{a}}$ for $\mathfrak{a} \in V$. However, the flow lines of these flows define an affine structure on the leaves of the absolute period foliation: For $q \in \mathcal{C}$ there exists a neighborhood U of 0 in the vector space V such that the map $\mathfrak{a} \in U \to \Lambda^1_{\mathfrak{a}}(q)$ defines a system of coordinates near q. Coordinate transitions for such coordinates are affine maps. As a consequence, the intersection of $\hat{\mathcal{C}}$ with a leaf of the absolute period foliation is locally an affine submanifold of the corresponding leaf of $\mathcal{AP}(\hat{\mathcal{Q}})$. This completes the proof of the lemma. □

3. Connections on the Hodge bundle

In this section we begin the investigation of differential geometric properties of an affine invariant manifold $\mathcal{C}_+$, with tangent bundle $T\mathcal{C}_+$. We study the Gauss Manin connection on the projection $p(T\mathcal{C}_+)$ of $T\mathcal{C}_+$ to the flat bundle $\Pi^*\mathcal{N} \otimes_{\mathbb{R}} \mathbb{C}|\mathcal{C}_+$. Recall that $p(T\mathcal{C}_+)$ is a subbundle of $\Pi^*\mathcal{N} \otimes_{\mathbb{R}} \mathbb{C}|\mathcal{C}_+$ which is invariant under the Gauss Manin connection [EMM15] and invariant under the complex structure i [F16]. We establish a first rigidity result geared towards Theorem 1 from the introduction. We always assume that $g \geq 3$.

Recall that the Hodge bundle $\mathcal{H}$ on the moduli space $\mathcal{M}_g$ of curves of genus g is the pull-back under the Torelli map of the Hermitean holomorphic (orbifold) vector bundle $\mathcal{V} \to \mathcal{A}_g$ (see also the appendix).

The complement $\mathcal{H}_+$ of the zero section in $\mathcal{H}$ is a complex orbifold. Let as before $\Pi : \mathcal{H} \to \mathcal{M}_g$ be the canonical projection. The pull-back $\Pi^*\mathcal{H} \to \mathcal{H}_+$ to $\mathcal{H}_+$ of the Hodge bundle on $\mathcal{M}_g$ is a holomorphic vector bundle on $\mathcal{H}_+$. The Hermitean metric on $\mathcal{H}$ which is determined by the complex structure J on $\mathcal{H}$ and a natural symplectic structure (see the appendix for more details) pulls back to a Hermitean structure on $\Pi^*\mathcal{H}$. The bundle $\Pi^*\mathcal{H}$ splits as a direct sum

$$\Pi^*\mathcal{H} = \mathcal{T} \oplus \mathcal{L}$$

of complex vector bundles. Here the fibre of $\mathcal{T}$ over a point $q \in \mathcal{H}_+$ is just the $\mathbb{C}$-span of q, and the fibre of $\mathcal{L}$ is the orthogonal complement of $\mathcal{T}$ for the natural Hermitean metric, or, equivalently, the orthogonal complement of $\mathcal{T}$ for the symplectic form. The complex line bundle $\mathcal{T}$ is holomorphic. Via identification of $\mathcal{L}$ with the quotient bundle $\Pi^*\mathcal{H}/\mathcal{T}$, we may assume that $\mathcal{L}$ is holomorphic. Its complex dimension equals $g-1 \geq 2$.

The group $GL^+(2,\mathbb{R})$ acts on $\mathcal{H}_+$ as a group of real analytic transformations, and this action pulls back to an action on $\Pi^*\mathcal{H} \to \mathcal{H}_+$ as a group of real analytic bundle automorphisms.

Recall that the bundle $\Pi^*\mathcal{H}$ can be equipped with the flat Gauss Manin connection. We say that a subbundle of $\Pi^*\mathcal{H}$ over a subset V of $\mathcal{H}_+$ is *flat* if it is invariant under parallel transport for the Gauss Manin along paths in V.

Lemma 3.1. *The restriction of the bundle $\mathcal{L}$ to the orbits of the $GL^+(2,\mathbb{R})$-action is flat.*

Proof. Let $q \in \mathcal{H}_+$ and let $A \subset H^1(S,\mathbb{R})$ be the $\mathbb{R}$-linear span of the real and the imaginary part of q (for some some choice of marking). Then A is locally constant along the orbit $GL^+(2,\mathbb{R})q$ and hence it defines a subbundle of $\Pi^*\mathcal{H} \to GL^+(2,\mathbb{R})q$ which is locally invariant under parallel transport for the Gauss-Manin connection.

Now in a neighborhood of q in $GL^+(2,\mathbb{R})q$, the subspace A coincides with the fibre of the bundle $\mathcal{T} \to \mathcal{H}_+$, viewed as a subbundle of the bundle $\Pi^*\mathcal{N} \to \mathcal{H}_+$ which is invariant under the complex structure J on $\Pi^*\mathcal{N}$. Therefore the restriction of the bundle $\mathcal{T}$ to any orbit of $GL^+(2,\mathbb{R})$ is flat. As the Gauss Manin connection preserves the symplectic structure on $\Pi^*\mathcal{H}$, the restriction of its symplectic complement $\mathcal{L}$ to an orbit of the action of the group $GL^+(2,\mathbb{R})$ is flat as well. □

The foliation $\mathcal{F}$ of $\mathcal{H}_+$ into the orbits of the $GL^+(2,\mathbb{R})$-action is real analytic in period coordinates since the action of $GL^+(2,\mathbb{R})$ is affine in period coordinates. Furthermore, its leaves are complex suborbifolds of the complex orbifold $\mathcal{H}_+$. In particular, the tangent bundle $T\mathcal{F}$ of $\mathcal{F}$ is a real analytic subbundle of the tangent bundle of the complex orbifold $\mathcal{H}_+$.

By Lemma 3.1, the Gauss-Manin connection on the flat bundle $\Pi^*\mathcal{H} \to \mathcal{H}_+$ restricts to a real analytic flat leafwise connection ∇^{GM} on the bundle $\mathcal{L} \to \mathcal{H}_+$. Here a leafwise connection is a connection whose covariant derivative is only defined for vectors tangent to the foliation $\mathcal{F}$. In other words, a leafwise connection associates to a smoth section of $\mathcal{L}$ and a tangent vector $X \in T\mathcal{F}$ a point in $\mathcal{L}$. The leafwise connection ∇^{GM} is real analytic (compare subsection 2.2 for details), and it preserves the symplectic structure of $\mathcal{L}$ as this is true for the Gauss Manin connection. There is no information on the complex structure.

For each $k \leq g-2$, the leafwise connection ∇^{GM} extends to a flat leafwise connection on the bundle $\wedge^{2k}_{\mathbb{R}}\mathcal{L}$ whose fibre at q is the $2k$-th exterior power of the fibre of $\mathcal{L}$ at q, viewed as a real vector space. We refer to [KN63] for this standard fact.

The Hermitean holomorphic vector bundle $\Pi^*\mathcal{H} \to \mathcal{H}_+$ admits a unique *Chern connection* ∇ (see e.g. [GH78]). The Chern connection defines parallel transport of the fibres of $\Pi^*\mathcal{H}$ along smooth curves in $\mathcal{H}_+$. This parallel transport preserves the Hermitean metric. The complex structure J on $\Pi^*\mathcal{H}$ is parallel for ∇, ie ∇ commutes with J. The $GL^+(2,\mathbb{R})$-orbits on $\mathcal{H}_+$ are complex suborbifolds of $\mathcal{H}_+$. The restriction of the bundle $\mathcal{T}$ to each leaf of the foliation $\mathcal{F}$ can locally be identified with the projection $pT\mathcal{F}$ of the tangent bundle of $\mathcal{F}$, ie with the pull-back to $GL^+(2,\mathbb{R})$ of the tangent bundle of the complex homogeneous space $GL^+(2,\mathbb{R})/(\mathbb{R}^+ \times S^1) = \mathbf{H}^2$. Thus by naturality, the restriction of the Chern connection to the leaves of the foliation $\mathcal{F}$ of $\mathcal{H}_+$ into the orbits of the $GL^+(2,\mathbb{R})$-action preserves the decomposition $\Pi^*\mathcal{H} = \mathcal{T} \oplus \mathcal{L}$.

For every $k \leq g-2$, the complex structure J on $\mathcal{L}$ can be viewed as a real vector bundle automorphism of $\mathcal{L}$, and such a bundle automorphism extends to an automorphism of the real tensor bundle $\wedge^{2k}_{\mathbb{R}}\mathcal{L}$. The restriction of the connection ∇ to the orbits of the $GL^+(2,\mathbb{R})$-action extends to a leafwise connection on $\wedge^{2k}_{\mathbb{R}}\mathcal{L}$ which commutes with this automorphism.

The Hermitean metric which determines the Chern connection is defined by the polarization and the complex structure. These data are real analytic in period coordinates (recall that the Torelli map is holomorphic) and consequently the connection matrix for the Chern connection with respect to a real analytic local basis of $\Pi^*\mathcal{H}$ is real analytic.

To summarize, for every $k \leq g-2$, both the Chern connection and the Gauss Manin connection restrict to leafwise connections of the restriction on the bundle $\wedge^{2k}_{\mathbb{R}}\mathcal{L} \to \mathcal{H}_+$ to the orbits of the $GL^+(2,\mathbb{R})$-action. Thus $\nabla - \nabla^{GM}$ defines a real analytic tensor field

$$\Xi^k \in \Omega((T\mathcal{F})^* \otimes (\wedge^{2k}_{\mathbb{R}}\mathcal{L})^* \otimes \wedge^{2k}_{\mathbb{R}}\mathcal{L}) \tag{2}$$

where we denote by $\Omega((T\mathcal{F})^* \otimes (\wedge^{2k}_{\mathbb{R}}\mathcal{L})^* \otimes \wedge^{2k}_{\mathbb{R}}\mathcal{L})$ the vector space of real analytic sections of the real analytic vector bundle $(T\mathcal{F})^* \otimes (\wedge^{2k}_{\mathbb{R}}\mathcal{L})^* \otimes \wedge^{2k}_{\mathbb{R}}\mathcal{L}$. If $\mathcal{C}_+ \subset \mathcal{H}_+$ is an affine invariant manifold of rank $2 \leq \ell \leq g-1$, with absolute holomorphic tangent bundle $\mathcal{Z}$, then the restriction of the tensor field $\Xi^{\ell-1}$ to $\mathcal{C}_+$ preserves the J-invariant section of $\wedge^{2\ell-2}_{\mathbb{R}}\mathcal{L}|\mathcal{C}_+$ which is defined by $p(T\mathcal{C}_+) \cap \Pi^*\mathcal{N}$. This section associates to a point $q \in \mathcal{C}_+$ the exterior product of a normalized oriented basis of the (real) $2\ell-2$-dimensional J-invariant vector space $p(T_q\mathcal{C}_+) \cap \mathcal{L}$. Note that as $p(T\mathcal{C}_+) \cap \mathcal{L}$ is equipped with a complex structure, it also is equipped with an orientation.

The next proposition is a key step towards Theorem 1. Before we proceed we evoke a simple lemma from Linear Algebra needed for its proof and once more in the proof of Proposition 5.1.

Lemma 3.2. *Let V be an $2n$-dimensional symplectic vector space with compatible complex structure J. Let $A : V \to V$ be a symplectic transformation with only real eigenvalues different from ± 1 and for some $k < n$ let $W \subset V$ be a k-dimensional complex subspace such that for all $j \in \mathbb{Z}$, $A^j W$ is complex. Then W is a direct sum of subspaces of eigenspaces of A.*

Proof. Let V be a $2n$-dimensional vector space with symplectic structure ω and compatible complex structure J. Let $1 \leq k < n$ and let $\mathcal{P}$ be the complex Gassmannian of complex k-dimensional linear subspaces of V. Then $\mathcal{P}$ be is a compact subset of the Grassmannian $\mathrm{Gr}_{2k}(V)$ of $2k$-dimensional oriented real linear subspaces of V. The symplectic subspaces form an open subset of $\mathrm{Gr}_{2k}(V)$.

Let $A : V \to V$ be a symplectic transformation with the properties stated in the lemma. Let $a_1 > \cdots > a_s > a_s^{-1} > \cdots > a_1^{-1}$ be the different eigenvalues of A. For each eigenvalue a let $E(a)$ be the corresponding eigenspace. As A is symplectic, the eigenspaces $E(a), E(b)$ for $b \neq a^{-1}$ are orthogonal for the symplectic form. Namely, if $X \in E(a), Y \in E(b)$ then $\omega(X,Y) = \omega(AX, AY) = ab\omega(X,Y)$ which implies $\omega(X,Y) = 0$.

Let $W \subset V$ be a k-dimensional complex subspace such that $A^j W$ is complex for all $j \in \mathbb{Z}$. As $\mathcal{P}$ is compact, by passing to a subsequence we may assume that $A^j W$ converges as $j \to \infty$ to some complex subspace Z of V. We claim that Z is a direct sum of subspaces of eigenspaces of the map A.

To this end let $P(a) : V \to E(a)$ be the natural projection defined by the decomposition of V into eigenspaces. Let $i \leq n$ be the minimum of all numbers so that either $P(a_i)(W) = Z(a_i) \neq \{0\}$ or $P(a_i^{-1})(W) = Z(a_i^{-1}) \neq \{0\}$. By exchanging A and A^{-1} we may assume that $Z(a_i) \neq \{0\}$. Then $Z(a_i)$ is a linear subspace of $E(a_i)$ of positive dimension $r \geq 1$. We claim that $Z(a_i) \subset Z$.

To this end let $0 \neq X \in Z(a_i)$; then there exists a vector $\hat{X} \in W$ with $P(a_i)(\hat{X}) = X$. Thus $\hat{X} = X + \sum_j X_j$ where X_j is an eigenvector for A for an eigenvalue $b < a_i$. As a consequence, $A^j(\hat{X})/a_i^j \to X$ as $j \to \infty$ and hence $X \in Z$ as claimed.

As Z is symplectic, we also have $\dim(P(a_i^{-1})(Z)) = r$. But $P(a_j^{-1})(W) = \{0\}$ for $j < i$ and hence $P(a_j^{-1})(Z) = \{0\}$ for $j < i$ by invariance. Then the above discussion implies that $P(a_i^{-1})(W) = P(a_i^{-1})(Z) \subset W$. Reversing the roles of A and A^{-1} then yields that $P(a_i)(W) \subset W$.

The statement of the Lemma now follows by iteration of this argument. □

The following proposition is a fairly easy consequence of the main algebraic result of [H18b].

Proposition 3.3. *Let $\mathcal{C}_+$ be an affine invariant manifold of rank $\ell \geq 1$, wtih absolute holomorphic tangent bundle $\mathcal{Z}$. Let $\mathcal{C} \subset \mathcal{C}_+$ be the hyperplane of area one abelian differentials. Then for every open subset U of $\mathcal{C}$ there exists a periodic orbit γ for Φ^t through U with the following properties.*

(1) *The eigenvalues of the matrix $A = \Psi(\Omega(\gamma))|\mathcal{Z}$ are real and pairwise distinct.*
(2) *No product of two eigenvalues of A is an eigenvalue.*

Proof. By Theorem 3 of [H18b], for every small open subset U of $\mathcal{C}$, the image under Ψ of the subsemigroup $\Psi(\Omega(\Gamma_0))$ defined as in Proposition 3.11 of [H18b] by the monodromy along periodic orbits through a small open contractible subset V of U is Zariski dense in $Sp(2\ell, \mathbb{R})$. The statement of the corollary is now an immediate consequence of the main result of [Be97]. □

Denote again by $\mathcal{Q}_+ \subset \mathcal{H}_+$ a component of a stratum. Recall that the group $GL^+(2, \mathbb{R})$ acts on the bundle $\mathcal{L}$ by parallel transport for the Gauss Manin connection.

Proposition 3.4. *Let $\mathcal{C}_+ \subset \mathcal{Q}_+$ be an affine invariant manifold of rank $\ell \geq 3$, with absolute holomorphic tangent bundle $\mathcal{Z}$. Then one of the following two possibilities holds true.*

(1) *There are finitely many proper affine invariant submanifolds of $\mathcal{C}_+$ which contain every affine invariant submanifold of $\mathcal{C}_+$ of rank $2 \leq k \leq \ell - 1$.*
(2) *Up to passing to a finite cover of $\mathcal{C}_+$, the restriction of the bundle $\mathcal{L} \cap \mathcal{Z}$ to an open dense $GL^+(2, \mathbb{R})$-invariant subset of $\mathcal{C}_+$ admits a non-trivial $GL^+(2, \mathbb{R})$-invariant real analytic splitting $\mathcal{L} \cap \mathcal{Z} = \mathcal{E}_1 \oplus \mathcal{E}_2$ into two complex subbundles.*

Proof. Let $\mathcal{Q}_+ \subset \mathcal{H}_+$ be a component of a stratum and let $\mathcal{C}_+ \subset \mathcal{Q}_+$ be an affine invariant manifold of rank $\ell \geq 3$, with absolute holomorphic tangent bundle $\mathcal{Z} \to \mathcal{C}_+$. An affine invariant manifold is affine in period coordinates and hence it inherits from $\mathcal{Q}_+$ a real analytic structure. As before, there is a splitting

$$\mathcal{Z} = \mathcal{T} \oplus (\mathcal{L} \cap \mathcal{Z}).$$

The bundle

$$\mathcal{W} = \mathcal{L} \cap \mathcal{Z} \to \mathcal{C}_+$$

is holomorphic (recall that we identify $\mathcal{W}$ with the quotient of the holomorphic bundle $\mathcal{Z}$ by its holomorphic subbundle $\mathcal{T}$). It also can be viewed as a real analytic real vector bundle with a real analytic complex structure J (which is just a real analytic section of the real analytic endomorphism bundle of $\mathcal{W}$ with $J^2 = -\mathrm{Id}$).

For $1 \leq k \leq \ell - 2$ denote by $\mathrm{Gr}(2k) \to \mathcal{C}_+$ the fibre bundle whose fibre over q is the Grassmannian of oriented $2k$-dimensional real subspaces of $\mathcal{W}_q$. This is a real analytic fibre bundle with compact fibre. It contains a real analytic subbundle $\mathcal{P}(k) \to \mathcal{C}_+$ whose fibre over q is the Grassmannian of *complex* k-dimensional subspaces of $\mathcal{W}_q$ (for the complex structure J).

The real part of the Hermitean metric on $\mathcal{W}$ naturally extends to a real analytic Riemannian metric on $\wedge_{\mathbb{R}}^{2k}\mathcal{W}$. The bundle $\mathrm{Gr}(2k)$ can be identified with the set of pure vectors in the sphere subbundle of $\wedge_{\mathbb{R}}^{2k}\mathcal{W}$ for this metric. Namely, an oriented $2k$-dimensional real linear subspace E of $\mathcal{W}_q$ defines uniquely a pure vector in $\wedge_{\mathbb{R}}^{2k}\mathcal{W}_q$ of norm one which is just the exterior product of an orthonormal basis of E with respect to the inner product on $\mathcal{W}_q$. The points in $\mathcal{P}(k)$ correspond precisely to those pure vectors which are invariant under the extension of J to an automorphism of $\wedge_{\mathbb{R}}^{2k}\mathcal{W}$.

From now on, we work on the real analytic hyperplane $\mathcal{C} \subset \mathcal{C}_+$ of abelian differentials in $\mathcal{C}_+$ of area one, and we replace the action of $GL^+(2,\mathbb{R})$ by the action of $SL(2,\mathbb{R})$. The tangent bundle of the foliation of $\mathcal{C}$ into the orbits of the $SL(2,\mathbb{R})$-action is naturally trivialized by the following elements of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of $SL(2,\mathbb{R})$:

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3}$$

defining the generator X of the Teichmüller flow, the generator Y of the horocycle flow, and the generator Z of the circle group of rotations.

Let B^k (or C^k, D^k) be the contraction of the tensor field Ξ^k defined in equation (2) with the vector field X (or Y, Z). Since these vector fields are real analytic and since the bundle $\mathcal{W} \to \mathcal{C}$ is invariant under both the Gauss Manin connection and the Chern connection, B^k (or C^k, D^k) can be viewed as a real analytic section of the endomorphism bundle $(\wedge_{\mathbb{R}}^{2k}(\mathcal{W}))^* \otimes \wedge_{\mathbb{R}}^{2k}(\mathcal{W})$ of $\wedge_{\mathbb{R}}^{2k}(\mathcal{W})$.

Define a *real analytic subset* of $\mathcal{P}(k)$ to be the intersection of the zero sets of a finite or countable number of real analytic functions on $\mathcal{P}(k)$. Recall that this is well defined since $\mathcal{P}(k)$ has a natural real analytic structure. We allow such functions to be constant zero, ie we do not exclude that such a set coincides with $\mathcal{P}(k)$. The real analytic set is *proper* if it does not coincide with $\mathcal{P}(k)$. Then there is at least one defining function which is not identically zero, and the set is closed and nowhere dense in $\mathcal{P}(k)$. We do not exclude the possibility that the set is empty.

For $1 \leq k \leq \ell - 2$ and $q \in \mathcal{C}$ let

$$\mathcal{R}_0^k(q,0) \subset \mathcal{P}(k)_q$$

be the set of all k-dimensional complex linear subspaces L of $\mathcal{W}_q$ with $B^k L = 0 = C^k L = D^k L$ (here we view as before a k-dimensional complex subspace of $\mathcal{W}_q$ as a pure J-invariant vector in $\wedge_{\mathbb{R}}^{2k}(\mathcal{W})_q$). By linearity of the contractions B^k, C^k, D^k of the tensor field Ξ^k, the set $\mathcal{R}_0^k(q,0)$ can be identified with the set of all J-invariant pure vectors in $\wedge_{\mathbb{R}}^{k}(\mathcal{W})_q$ which are contained in some (perhaps trivial) linear subspace of $\wedge_{\mathbb{R}}^{k}(\mathcal{W}_q)$. This subspace is the intersection of the kernels of the endomorphisms B^k, C^k, D^k. Furthermore, by linearity, $\mathcal{R}_0^k(q,0)$ consists of all J-invariant pure vectors $V \in \wedge_{\mathbb{R}}^{2k}(\mathcal{W})_q$ with the following property. Let $\alpha : (-\epsilon, \epsilon) \to qSL(2,\mathbb{R})$ be any smooth curve through $\alpha(0) = q$. Extend V to a section $t \to V(t)$ of $\wedge_{\mathbb{R}}^{2k}(\mathcal{W})$ over α by parallel transport for ∇^{GM}; then $\frac{\nabla}{dt} V(t)|_{t=0} = 0$.

Since the tensor field Ξ^k and the vector fields X, Y, Z are real analytic, $\cup_q \mathcal{R}_0^k(q,0)$ is a real analytic subset of $\mathcal{P}(k)$, defined as the common zero set of three real analytic functions (the function which associates to a pure vector of square norm one the square norm of its image under the bundle map B^k, C^k, D^k).

Our next goal is to construct a subset of $\mathcal{P}(k)$ with properties similar to the properties of $\mathcal{R}_0^k(q,0)$ which is invariant under the action of $SL(2,\mathbb{R})$ by parallel transport with respect to the leafwise flat connection ∇^{GM}. To this end let ρ_t be the flow on $\mathcal{C}$ induced by the action of the circle group of rotations in $SL(2,\mathbb{R})$, obtained by a standard parametrization as a one-parameter subgroup of $SL(2,\mathbb{R})$. For $t\in\mathbb{R}$ define

$$\mathcal{R}_0^k(q,t)\subset \mathrm{Gr}(2k)_q$$

to be the preimage of $\mathcal{R}_0^k(\rho_t q,0)$ under parallel transport for the Gauss Manin connection along the flow line $s\to\rho_s q$ ($s\in[0,t]$). By the previous paragraph and the fact that parallel transport is real analytic, $\cup_q\mathcal{R}_0^k(q,t)$ is a real analytic subset of $\mathrm{Gr}(2k)_q$ and hence the same holds true for

$$\mathcal{A}_0^k=\cap_{t\in\mathbb{R}}(\cup_q\mathcal{R}_0^k(q,t))\subset\mathcal{P}(k)$$

(take the intersections for all $t\in\mathbb{Q}$).

By construction, the set $\mathcal{A}_0^k$ is invariant under the extension of the circle group of rotations by parallel transport with respect to the Gauss Manin connection to the fibres of the bundle $\mathrm{Gr}(2k)\to\mathcal{C}$. Here as before, we view $\mathrm{Gr}(2k)$ as a subset of the bundle $\wedge_{\mathbb{R}}^{2k}(\mathcal{W})$. It also is invariant under parallel transport with respect to the Chern connection: Namely, by definition, if $Z\in\mathcal{A}_0^k(q)$ and if $Z(t)$ is the parallel transport of $Z=Z(0)$ for the Gauss Manin connection along the orbit $t\to\rho_t(q)$ through q, then the covariant derivative of the section $t\to Z(t)$ for the Chern connection vanishes since for each t, the vector $Z(t)$ is contained in the kernel of the contraction of the tensor field Ξ^k with the generator of the flow.

Recall that the Teichmüller flow Φ^t is just the action of the one-parameter subgroup of $SL(2,\mathbb{R})$ generated by the diagonal matrix in (3). For $t\in\mathbb{R}$ define

$$\mathcal{R}_1^k(q,t)\subset\mathrm{Gr}(2k)_q$$

to be the preimage of $\mathcal{A}_0^k(\Phi^t q)$ under parallel transport for the Gauss Manin connection along the flow line $s\to\Phi^s q$ ($s\in[0,t]$) of the Teichmüller flow and let

$$\mathcal{A}_1^k=\cap_{t\in\mathbb{R}}\big(\cup_q\mathcal{R}_1^k(q,t)\big)\subset\mathcal{P}(k).$$

Then $\mathcal{A}_1^k$ is invariant under the extension of the Teichmüller flow by parallel transport both for the Gauss Manin connection and the Chern connection. Furthermore, if $\alpha:[0,1]\to SL(2,\mathbb{R})$ is any path which is a concatenation of an orbit segment of the Teichmüller flow, ie an orbit segment of the action of the diagonal group, with an orbit segment of the circle group of rotations, then for every $L\in\mathcal{A}_1^k$, the parallel transport of L along α for the Gauss Manin connection coincides with the parallel transport for the Chern connection, and it consists of points in the kernels of the tensor fields B^k, C^k, D^k.

Repeat this construction once more with the circle group of rotations to find a real analytic set

$$\mathcal{A}^k\subset\mathcal{P}(k).$$

This set is invariant under the action of $SL(2,\mathbb{R})$ defined by parallel transport for the Gauss Manin connection. Namely, let $\alpha:[0,3]\to qSL(2,\mathbb{R})$ be a path which is a concatentation of three segments $\alpha_1 \circ \alpha_2 \circ \alpha_3$, where α_1, α_3 are orbit segments of the circle group of rotations and α_2 is an orbit segment of the diagonal group (read from left to right). If $L \in \mathcal{A}^k(\alpha(0))$ then, in particular, $L \in \mathcal{A}^k_1(\alpha(0))$ and the same holds true for the image L_1 of L under parallel transport for ∇^{GM} along the path α_1 which coincides with parallel transport for the Chern connection. Now $L_1 \in \mathcal{A}^k_1(\alpha(1))$, in particular $L_1 \in \mathcal{A}^k_0$ and the same holds true for the image of L_1 under parallel transport for ∇^{GM} along the flow line of the Teichmüller flow. Thus the parallel transport of L_1 along α_2 for the Chern connection coincides with the parallel transport for the Gauss Manin connection, and its image L_2 is contained in $\mathcal{A}^k_0$. Repeating once more this reasoning shows the statement of the beginning of this paragraph.

The circle group of rotations K is a maximal compact subgroup of $SL(2,\mathbb{R})$. The Cartan decomposition of $SL(2,\mathbb{R})$ states that any element in $SL(2,\mathbb{R})$ can be written in the form $k_1 a k_2$ where $k_i \in K$ and where a is an element in the diagonal subgroup. Therefore for each $z \in \mathcal{C}$, each point on $zSL(2,\mathbb{R})$ is the endpoint of a path of the above form beginning at z. As the restriction of ∇^{GM} to the orbits of the action of $SL(2,\mathbb{R})$ is flat, via approximation of smooth paths in $SL(2,\mathbb{R})$ by a concatenation of paths of the above form, this implies the following. If $z \in \mathcal{C}$ and if $L \in \mathcal{A}^k(z)$, then the image of L under parallel transport for the Gauss Manin connection along the $SL(2,\mathbb{R})$-orbit $zSL(2,\mathbb{R})$ defines a section of the bundle $\wedge^k_{\mathbb{R}}(\mathcal{W})|zSL(2,\mathbb{R})$ which is parallel for the Chern connection and contained in $\mathcal{A}^k$.

If $\mathcal{D} \subset \mathcal{C}$ is a proper affine invariant manifold of rank $2 \leq k+1 < \ell$, then it follows from the discussion preceding this proof (see [F16]) that for every $q \in \mathcal{D}$ the projected tangent space $p(T_q\mathcal{D})$ defines a point in $\mathcal{A}^k(q) \subset \mathcal{P}(k)_q$. In particular, we have $\mathcal{A}^k \neq \emptyset$.

Let $\pi : \mathcal{P}(k) \to \mathcal{C}$ be the natural projection and let

$$M(k) = \pi(\mathcal{A}^k).$$

The fibres of π are compact and hence π is closed. Therefore $M(k)$ is a closed $SL(2,\mathbb{R})$-invariant subset of $\mathcal{C}$ which contains all affine invariant submanifolds of $\mathcal{C}$ of rank $k+1$.

There are now two possibilities. In the first case, the set $M(k)$ is nowhere dense in $\mathcal{C}$. Theorem 2.2 of [EMM15] then shows that $M(k)$ is a finite union of proper affine invariant submanifolds of $\mathcal{C}$. By construction, the union of these affine invariant submanifolds contains each affine invariant submanifold of $\mathcal{C}$ of rank $k+1$. Thus the first possibility in the proposition is fulfilled for affine invariant manifolds of rank $k+1$.

It now suffices to show the following. If there is some $k \leq \ell - 2$ such that the set $M(k)$ contains an open subset of $\mathcal{C}$, then there is a splitting of the bundle $\mathcal{W}$ over an open dense invariant subset of a finite cover of $\mathcal{C}$ as predicted in case (2) of the proposition.

Thus assume that the set $M = M(k) = \pi(\mathcal{A}^k)$ contains an open subset of $\mathcal{C}$. By invariance and topological transitivity of the action of $SL(2,\mathbb{R})$ on $\mathcal{C}$ [EMM15], M

contains an open dense invariant set. On the other hand, M is closed and hence we have $M=\mathcal{C}$. This is equivalent to stating that for every $q\in\mathcal{C}$ the set $\mathcal{A}^k(q)\subset\mathcal{P}(k)_q$ is non-empty. In particular, for every $q\in\mathcal{C}$ there is a line in $\wedge^{2k}_{\mathbb{R}}(\mathcal{W})_q$ spanned by a pure vector L which is an eigenvector for the extension of the complex structure J and which is contained in the kernel of B^k,C^k,D^k. Moreover, the same holds true for the parallel transport of L with respect to the Gauss Manin connection along the orbits of the $SL(2,\mathbb{R})$-action (compare the above discussion).

With respect to a real analytic local trivialization of the bundle $\mathcal{P}(k)$ over an open set $V\subset\mathcal{C}$, the set $\mathcal{A}^k$ is of the form $(q,\mathcal{A}^k(q))$ where $\mathcal{A}^k(q)$ is a real analytic subset of the compact projective variety $\mathcal{P}(k)_q$ of k-dimensional complex linear subspaces of $\mathcal{W}_q$ depending in a real analytic fashion on q. Even more is true: $\mathcal{A}^k(q)$ can be identified with the space of all J-invariant pure vectors which are contained in some J-invariant linear subspace R_q of $\wedge^{2k}_{\mathbb{R}}(\mathcal{W})_q$ of positive dimension depending in a real analytic fashion on q. The subspaces R_q are equivariant with respect to the action of $SL(2,\mathbb{R})$ by parallel transport for the Gauss Manin connection. Thus the set of all $q\in\mathcal{C}$ so that the dimension of R_q is minimal is an open $SL(2,\mathbb{R})$-invariant subset V of $\mathcal{C}$.

We claim that for $q\in V$, the set $\mathcal{A}^k(q)$ consists of only finitely many points. To this end choose a periodic orbit $\gamma\subset V$ for the Teichmüller flow so that for $q\in\gamma$, the restriction B to $\mathbb{R}^{2\ell}=\mathcal{Z}_q$ of the transformation $\Psi(\Omega(\gamma))\in Sp(2\ell,\mathbb{R})$ has 2ℓ distinct real eigenvalues (the notations are as in Section 2). Such an orbit exists by Corollary 3.3. The linear map B is the return map for parallel transport of $\mathcal{Z}$ along γ with respect to the Gauss Manin connection, and it preserves the decomposition $\mathcal{Z}_q=\mathcal{T}_q\oplus\mathcal{W}_q$.

We are looking for k-dimensional complex linear subspaces L of $\mathcal{W}_q$ with the property that B^jL is complex for all $j\in\mathbb{Z}$. Now the linear map B preserves the symplectic form on $\mathcal{W}_q$, and its eigenvalues are all real, nonzero and of multiplicity one. Thus by Lemma 3.2, the linear subspace L is a direct sum of eigenspaces for B. As there are only finitely many such subspaces of $\mathcal{W}_q$, the number of points in $\mathcal{A}^k(q)$ is finite (and in fact bounded from above by a number only depending on the rank of $\mathcal{C}$).

By Corollary 3.3 and the above discussion, the set of all points $q\in V$ such that $\mathcal{A}^k(q)\subset\mathcal{P}(k)_q$ is a finite set is dense in V. But $\mathcal{A}^k$ is a real analytic subset of $\mathcal{P}(k)$ and therefore by perhaps decreasing the size of V we may assume that $\mathcal{A}^k(q)$ is finite for all $q\in V$. Furthermore, the cardinality of $\mathcal{A}^k(q)$ is locally constant and hence constant on V since by decreasing the size of V further we may assume that V is connected.

As the dependence of $\mathcal{A}^k(q)$ on $q\in V$ is real analytic, any choice of a point in $\mathcal{A}^k(q)$ defines locally near q an analytic section of $\mathcal{P}(k)|V$ and hence a real analytic J-invariant subbundle of $\mathcal{W}|V$. This subbundle is invariant under parallel transport for the Gauss Manin connection along the orbits of the $SL(2,\mathbb{R})$-action. In the case that this local section is globally invariant under parallel transport for the Gauss Manin connection along the orbits of the $SL(2,\mathbb{R})$-action, it defines a real analytic J-invariant $SL(2,\mathbb{R})$-invariant subbundle of $\mathcal{W}|V$.

Otherwise parallel transport along the orbits of the $SL(2,\mathbb{R})$-action acts as a finite group of permutations on the finite set $\mathcal{A}^k(q)$. Thus we can pass to a finite cover of $\mathcal{C}$ so that on the covering space, using the same notation, the induced local subbundles of $\mathcal{W}$ are globally defined.

In other words, up to passing to a finite cover of $\mathcal{C}$, $\mathcal{A}^k$ defines a real analytic $SL(2,\mathbb{R})$-invariant complex k-dimensional vector bundle over the open dense invariant subset V of $\mathcal{C}$. By invariance of the symplectic structure under parallel transport along the orbits of the $SL(2,\mathbb{R})$-action, $\mathcal{A}^k$ then defines a splitting of the bundle $\mathcal{W}|V$ as predicted in the second part of the proposition. ☐

Remark 3.5. By Proposition 3.11 of [H18b] (see also [W14]), a real analytic splitting of the bundle $\mathcal{L}$ as stated in the second part of Proposition 3.4 can not be flat, i.e. invariant under the Gauss Manin connection. However, the second part of Proposition 3.4 does not claim the existence of a flat subbundle of the projected tangent bundle of $\mathcal{C}$. Namely, the splitting is only required to be invariant under parallel transport along the orbits of the $SL(2,\mathbb{R})$-action.

4. Invariant splittings of the lifted Hodge bundle

In this section we use information on the moduli space of principally polarized abelian varieties to rule out the second case in Proposition 3.4. We continue to use all assumptions and notations from Section 3.

Recall the splitting $\Pi^*\mathcal{H} = \mathcal{T} \oplus \mathcal{L}$ of the lifted Hodge bundle $\Pi^*\mathcal{H} \to \mathcal{H}_+$. Let $\mathcal{C}_+$ be an affine invariant manifold with absolute holomorphic tangent bundle $\mathcal{Z}$.

Consider again the intersection $\mathcal{C}$ of $\mathcal{C}_+$ with the moduli space of abelian differentials of area one. We shall argue by contradiction. As our discussion does not change by replacing $\mathcal{C}$ by a finite cover, we assume that there is an open dense $SL(2,\mathbb{R})$-invariant subset V of $\mathcal{C}$, and there is an $SL(2,\mathbb{R})$-invariant real analytic splitting $\mathcal{L} \cap \mathcal{Z}|V = \mathcal{E}_1 \oplus \mathcal{E}_2$ into complex orthogonal subbundles as in the second part of Proposition 3.4. The restriction of the splitting to an orbit of the $SL(2,\mathbb{R})$-action is invariant under the Gauss Manin connection.

By Lemma 2.5, if $r = \dim_{\mathbb{C}}(\mathcal{C}_+) - 2\mathrm{rk}(\mathcal{C}_+) > 0$ then the absolute period foliation $\mathcal{AP}(\mathcal{C})$ of $\mathcal{C}$ is defined, and it is a real analytic foliation with complex leaves of dimension r. Furthermore, as differentials contained in a leaf of this foliation locally have the same absolute periods, they define locally the same complex one-dimensional linear subspace of $\mathcal{H}$. This means that the splitting $\mathcal{Z} = \mathcal{T} \oplus \mathcal{W}$ where $\mathcal{W} = \mathcal{L} \cap \mathcal{Z}$ is invariant under the restriction of the Gauss Manin connection to the leaves of the absolute period foliation of $\mathcal{C}$ in the sense described in Section 3.

Our first goal is to show that the real analytic splitting $\mathcal{W} = \mathcal{E}_1 \oplus \mathcal{E}_2$ is invariant under the restriction of the Gauss Manin connection to the leaves of the absolute period foliation.

Lemma 4.1. *The restriction of the bundle $\mathcal{E}_i \to V$ to a leaf of $\mathcal{AP}(\mathcal{C})$ is invariant under the Gauss Manin connection.*

Proof. We may assume that the dimension r of $\mathcal{AP}(\mathcal{C})$ is positive. Furthermore, by passing to a finite cover of $\mathcal{C}$, we may assume that the zeros of the differentials in $\mathcal{C}$ are numbered. By abuse of notation, we will ignore these modifications in our notations as they do not alter the argument.

Write again $\mathcal{W}|V = \mathcal{Z} \cap \mathcal{L} = \mathcal{E}_1 \oplus \mathcal{E}_2$. By assumption, the bundles $\mathcal{E}_i \to V$ are real analytic and invariant under parallel transport for the Gauss Manin connection along the leaves of the foliation of V into the orbits of the action of $SL(2,\mathbb{R})$.

The splitting $\mathcal{Z}|V = \mathcal{T} \oplus \mathcal{E}_1 \oplus \mathcal{E}_2$ can be used to project the Gauss Manin connection ∇^{GM} on the invariant bundle $\mathcal{Z}$ to a real analytic connection $\hat{\nabla}$ on $\mathcal{Z}$ along the leaves of the absolute period foliation which preserves this decomposition. Namely, given a tangent vector $Z \in T\mathcal{AP}(\mathcal{C})$ and a local smooth section Y of $\mathcal{E}_i$, define

$$\hat{\nabla}_Z Y = P_i(\nabla^{GM}_Z Y)$$

where

$$P_i : \mathcal{W} = \mathcal{E}_1 \oplus \mathcal{E}_2 \to \mathcal{E}_i$$

is the natural projection. Recall that this makes sense since the Gauss Manin connection restricted to a leaf of $\mathcal{AP}(\mathcal{C})$ preserves the bundle $\mathcal{W} = \mathcal{L} \cap \mathcal{Z}$ and hence $\nabla^{GM}_Z Y \in \mathcal{W}$.

We now use the assumptions and notations from Section 2. Let k be the number of zeros of a differential in $\mathcal{C}$. Choose once and for all a numbering of the zeros of a differential in $\mathcal{C}$. With respect to such a numbering, every vector $\mathfrak{a} \in \mathbb{C}^k$ of zero mean defines a vector field $X_{\mathfrak{a}}$ which is tangent to the absolute period foliation of the component $\mathcal{Q}$ of the stratum containing $\mathcal{C}$.

By Lemma 2.5, there exists a complex linear subspace $\mathcal{O}$ of $\mathbb{C}^k$ of complex dimension r contained in the complex hyperplane of vectors with zero mean so that for every $\mathfrak{a} \in \mathcal{O}$, the vector field $X_{\mathfrak{a}}$ is tangent to $\mathcal{C}$ at every point of $\mathcal{C}$. Furthermore, for every $\mathfrak{a} \in \mathcal{O}$, the affine invariant manifold $\mathcal{C}$ is invariant under the flow $\Lambda^t_{\mathfrak{a}}$ generated by $X_{\mathfrak{a}}$. For every $q \in V \subset \mathcal{C}$, every $\mathfrak{a} \in \mathcal{O}$ and every $Y \in \mathcal{E}_1(q)$ we can extend Y by parallel transport for $\hat{\nabla}$ along the flow line of the flow $\Lambda^t_{\mathfrak{a}}$.

Let us denote this extension by $\hat{Y}$; then

$$\beta(X_{\mathfrak{a}}, Y) = \frac{\nabla^{GM}}{dt}\hat{Y}(\Lambda^t_{\mathfrak{a}}(q))|_{t=0} \in \mathcal{E}_2(q)$$

only depends on $X_{\mathfrak{a}}$ and Y, moreover this dependence is linear in both variables. In this way we obtain a real analytic tensor field

$$\beta \in \Omega(T\mathcal{AP}(\mathcal{C})^* \otimes \mathcal{E}_1^* \otimes \mathcal{E}_2).$$

Here as before, $\Omega(T\mathcal{AP}(\mathcal{C})^* \otimes \mathcal{E}_1^* \otimes \mathcal{E}_2)$ is the vector space of real analytic sections of the bundle $T\mathcal{AP}(\mathcal{C})^* \otimes \mathcal{E}_1^* \otimes \mathcal{E}_2$. The splitting $\mathcal{W} = \mathcal{E}_1 \oplus \mathcal{E}_2$ is invariant under the restriction of the Gauss Manin connection to the leaves of the absolute period foliation if and only if β vanishes identically.

The Teichmüller flow Φ^t acts on the bundle $\mathcal{W}$ by parallel transport with respect to the Gauss Manin connection, and by assumption, this action preserves the bundles $\mathcal{E}_i$ $(i = 1, 2)$. The Teichmüller flow also preserves the absolute period foliation of $\mathcal{C}$. Thus the tensor field β is equivariant under the action of Φ^t.

Assume to the contrary that β does not vanish identically. As β is real analytic and bilinear and the vector space $\mathcal{O}$ is invariant under the complex structure, there is then an open subset U of $V \subset \mathcal{C}$ and either a real or a purely imaginary vector $\mathfrak{a} \in \mathcal{O}$ such that the contraction of β with $X_{\mathfrak{a}}$ does not vanish on U.

Assume that $\mathfrak{a} \in \mathbb{R}^k \cap \mathcal{O}$ is real, the case of a purely imaginary vector is treated in the same way; then $d\Phi^t X_{\mathfrak{a}} = e^t X_{\mathfrak{a}}$ by Lemma 2.4. Let now $\gamma \subset \mathcal{C}$ be a periodic orbit with the properties stated in Corollary 3.3 which passes through U. Let $q \in \gamma \cap U$. The eigenvalues of the matrix $A = \Psi(\Omega(\gamma))|\mathcal{Z}_q$ (where we identify $\mathcal{Z}_q$ with the symplectic subspace of $\Pi^*\mathcal{H}_q$ it defines) are real and of multiplicity one. The largest eigenvalue of A equals $e^{\ell(\gamma)}$ where $\ell(\gamma)$ is the length of the orbit γ, and the eigenspace for this eigenvalue is contained in the fibre $\mathcal{T}_q$ of the bundle $\mathcal{T}$.

The subspace $\mathcal{W}_q$ of $\mathcal{Z}_q$ is invariant under A and hence $\mathcal{W}_q$ is a sum of eigenspaces for A (viewed as a transformation of $\mathcal{Z}_q$) for eigenvalues whose absolute values are strictly smaller than $e^{\ell(\gamma)}$. Furthermore, by invariance of the splitting of $\mathcal{W}$ under parallel transport for the Gauss Manin connection along flow lines of the Teichmüller flow, the decomposition $\mathcal{W}_q = \mathcal{E}_1(q) \oplus \mathcal{E}_2(q)$ $(i = 1, 2)$ is invariant under the map A. Then $\mathcal{E}_i(q)$ is a direct sum of eigenspaces for A.

For clarity of exposition, write for the moment $||_\gamma^{GM}$ for parallel transport along γ with respect to the Gauss Manin connection. By equivariance of the tensor field β under the action of Φ^t, for $Y \in \mathcal{E}_1(q)$ we have

$$\beta(d\Phi^{\ell(\gamma)} X_{\mathfrak{a}}, ||_\gamma^{GM} Y) = ||_\gamma^{GM} \beta(X_{\mathfrak{a}}, Y) \in \mathcal{E}_2(q).$$

Now if $Y \in \mathcal{E}_1(q)$ is an eigenvector of A for the eigenvalue $b \neq 0$, then from $d\Phi^{\ell(\gamma)} X_{\mathfrak{a}} = e^{\ell(\gamma)} X_{\mathfrak{a}}$ we obtain

$$\beta(d\Phi^{\ell(\gamma)} X_{\mathfrak{a}}, AY) = e^{\ell(\gamma)} b \beta(X_{\mathfrak{a}}, Y) = A\beta(X_{\mathfrak{a}}, Y) \in \mathcal{E}_2(p).$$

In other words, the contraction $Y \in \mathcal{E}_1(q) \to \beta(X_{\mathfrak{a}}, Y) \in \mathcal{E}_2(q)$ of β with $X_{\mathfrak{a}}$ maps an eigenspace of A contained in $\mathcal{E}_1(q)$ for the eigenvalue b to an eigenspace of A contained in $\mathcal{E}_2(q)$ for the eigenvalue $e^{\ell(\gamma)} b$.

But $e^{\ell(\gamma)}$ is an eigenvalue of the matrix A (for an eigenvector contained in the bundle $\mathcal{T}$) and by the choice of γ, no product of two eigenvalues of A is an eigenvalue. By the discussion in the previous paragraph, this implies that the contraction of β with $X_{\mathfrak{a}}$ vanishes at q, contradicting the assumption that this contraction does not vanish at q.

As a consequence, the tensor field β vanishes identically, and parallel transport for $\hat{\nabla}$ of a vector $Y \in \mathcal{E}_1$ along a path which is entirely contained in a leaf of the absolute period foliation of $V \subset \mathcal{C}$ coincides with parallel transport with respect to the Gauss Manin connection. Equivalently, the restriction of the Gauss Manin connection to a leaf of the absolute period foliation preserves the splitting $\mathcal{W} = \mathcal{E}_1 \oplus \mathcal{E}_2$. This is what we wanted to show. □

Remark 4.2. Lemma 4.1 is valid for Φ^t-invariant splittings of the bundle $\mathcal{W}$ of class C^1, but the case of a continuous splitting can not be deduced in the same way. We expect nevertheless that the lemma holds true for continuous splittings as well. A possible strategy towards this end is to use methods from hyperbolic dynamics to show that a continuous Φ^t-invariant splitting has to be of class C^1 along the leaves of the real rel foliation and the leaves of the imaginary rel foliation and then use Lemma 4.1 and its proof to deduce that it is parallel along these leaves.

It is an interesting question whether it is possible to deduce from Lemma 4.1, Proposition 4.12 of [H18b] and Moore's theorem, applied to the right action of $SL(2,\mathbb{R})$ on $Sp(2g,\mathbb{Z})\backslash Sp(2g,\mathbb{R})$, that a splitting as in the second part of Proposition 3.4 does not exist. The main difficulty is that the global structure of the absolute period foliation of an affine invariant manifold is poorly understood. Moreover, we do not know whether there is a leaf of the foliation of the bundle $\mathcal{S}\to\mathfrak{D}_g$ as described in the appendix which intersects the image of the period map in more than one component.

We saw so far that a splitting of the subbundle $\mathcal{W}$ of the bundle $\Pi^*\mathcal{H}$ over an affine invariant manifold $\mathcal{C}$ as predicted by the second part of Proposition 3.4 has to be parallel for the Gauss Manin connection along the leaves of the absolute period foliation. Our final goal is to use the curvature of the projection of the Gauss Manin connection to the bundle $\mathcal{L}$ to derive a contradiction. Note that this projected connection is not flat (see below). To compute its curvature we take advantage of the geometry of the tautological vector bundle $\mathcal{V}\to\mathcal{A}_g$. We will use some differential geometric properties of this bundle described in the appendix.

Proposition 4.3. *Let $\mathcal{Z}$ be the absolute holomorphic tangent bundle of an affine invariant manifold $\mathcal{C}_+$ of rank at least three. There is no open dense $GL^+(2,\mathbb{R})$-invariant subset V of $\mathcal{C}_+$ such that the bundle $\mathcal{Z}\cap\mathcal{L}|V$ admits a $GL^+(2,\mathbb{R})$-invariant real analytic splitting $\mathcal{Z}\cap\mathcal{L}=\mathcal{E}_1\oplus\mathcal{E}_2$ into two complex subbundles.*

Proof. As before, we write $\mathcal{W}=\mathcal{L}\cap\mathcal{Z}$. Furthermore, we restrict our attention to the intersection $\mathcal{C}$ of $\mathcal{C}_+$ with the moduli space of area one abelian differentials.

We argue by contradiction, and we assume that an open dense invariant set V and a splitting $\mathcal{W}|V=\mathcal{E}_1\oplus\mathcal{E}_2$ with the properties stated in the proposition exists. Lemma 4.1 shows that this splitting is invariant under the restriction of the Gauss Manin connection to the leaves of the absolute period foliation of $\mathcal{C}$. Furthermore, it naturally induces an invariant splitting of the bundle $\mathcal{W}\oplus\overline{\mathcal{W}}\subset p(T\mathcal{C}_+)$ into two subbundles which are complex for the flat complex structure on $H^1(S,\mathbb{C})=H^1(S,\mathbb{R})\otimes\mathbb{C}$. Namely, recall that via the identifications used earlier, the bundle $\mathcal{W}$ can be represented as $\mathcal{W}=\{X+iJX\mid X\in\mathcal{W}_\mathbb{R}\}$ where $\mathcal{W}_\mathbb{R}$ is a (real) subbundle of the flat vector bundle $\Pi^*\mathcal{N}\to\mathcal{C}_+$ which is globally invariant under the Gauss Manin connection.

Let $\mathcal{T}(S)$ be the Teichmüller space of the surface S and let $\mathcal{I}_g<\mathrm{Mod}(S)$ be the Torelli group. The group $\mathcal{I}_g$ acts properly and freely from the left on $\mathcal{T}(S)$, with quotient the *Torelli space* $\mathcal{I}_g\backslash\mathcal{T}(S)$. Let $\mathcal{D}\to\mathcal{I}_g\backslash\mathcal{T}(S)$ be the bundle of area one homology-marked abelian differentials. The *period map* F maps the bundle $\mathcal{D}$ into the sphere subbundle $\mathcal{S}$ of the tautological vector bundle $\mathcal{V}$ over the Siegel upper half-space $\mathfrak{D}_g$ (see the appendix for the notations).

By the discussion in the appendix, the composition of the map F with the projection

$$\Pi : \mathcal{S} = Sp(2g,\mathbb{R}) \times_{U(g)} S^{2g-1} \to \Omega = Sp(2g,\mathbb{R})/Sp(2g-2,\mathbb{R})$$

is equivariant for the standard right $SL(2,\mathbb{R})$-actions on $\mathcal{D}$ and on Ω. Furthermore, we have

$$\Omega = \{x + iy \mid x, y \in \mathbb{R}^{2g}, \omega(x, y) = 1\}$$

where $\omega = \sum_i dx_i \wedge dy_i$ is the standard symplectic form on $\mathbb{R}^{2g}$.

Let $\hat{\mathcal{C}}$ be a component of the preimage of $\mathcal{C}$ in the bundle $\mathcal{D} \to \mathcal{I}_g\backslash\mathcal{T}(S)$. The projection $p(T\mathcal{C}_+)$ determines a subbundle of the trivial bundle $\hat{\mathcal{C}} \times H^1(S,\mathbb{C}) \to \hat{\mathcal{C}}$ which is locally constant and invariant under the complex structure i induced from the representation $H^1(S,\mathbb{C}) = H^1(S,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. Hence $\hat{\mathcal{C}}$ determines a complex subspace $\mathbb{C}^{2\ell}$ of $H^1(S,\mathbb{C})$ whose real part is symplectic. The composition of the map $\Pi \circ F$ with symplectic orthogonal projection then defines a map

$$\Upsilon : \hat{\mathcal{C}} \to \Omega_\ell = \{x + iy \mid x, y \in \mathbb{R}^{2\ell}, \omega(x, y) = 1\}.$$

We refer to the appendix for more details of this construction.

The manifold Ω_ℓ is a real hyperplane in the open subset

$$\mathcal{O}_\ell = \{x + iy \mid x, y \in \mathbb{R}^{2\ell}, \omega(x, y) > 0\}$$

of $\mathbb{C}^{2\ell}$. By naturality (see the appendix for details), the Gauss Manin connection on the bundle $p(T\hat{\mathcal{C}}_+)$ with fibre $\mathcal{Z} \oplus \overline{\mathcal{Z}}$ is just the pull-back via Υ of the natural flat connection $\nabla^{\mathcal{O}}$ on $T\mathcal{O}_\ell$.

As a consequence, using the notations from the appendix, we obtain the following. The restriction of the tangent bundle $T\mathcal{O}_\ell$ of $\mathcal{O}_\ell$ to Ω_ℓ decomposes as

$$T\mathcal{O}_\ell|\Omega_\ell = \mathcal{T}_{SL} \oplus \mathcal{R} \oplus \mathbb{R}$$

where $\mathcal{T}_{SL}$ is the tangent bundle of the foliation of Ω_ℓ into the orbits of the right action of the group $SL(2,\mathbb{R})$, $\mathcal{T}_{SL} \oplus \mathcal{R}$ is the tangent bundle of Ω_ℓ and $\mathbb{R}$ is the normal bundle of Ω_ℓ in $\mathcal{O}_\ell$. Using this splitting, the standard flat connection $\nabla^{\mathcal{O}_\ell}$ on $T\mathcal{O}_\ell$ projects to a connection $\nabla^{\mathcal{R}}$ on $\mathcal{R}$. The restriction of $\nabla^{\mathcal{O}_\ell}$ to the foliation of Ω_ℓ into the orbits of the $SL(2,\mathbb{R})$-action preserves the bundle $\mathcal{R}$ and hence the restriction of $\nabla^{\mathcal{R}}$ to this foliation coincides with the restriction of $\nabla^{\mathcal{O}_\ell}$. The leafwise connection ∇^{GM} on the bundle $\mathcal{W} = \mathcal{L} \cap \mathcal{Z}$ is the pull-back of the restriction of the connection $\nabla^{\mathcal{O}_\ell}$ (see the appendix for more details).

By Lemma 4.1, the splitting $\mathcal{W} = \mathcal{E}_1 \oplus \mathcal{E}_2$ is real analytic, invariant under the action of $SL(2,\mathbb{R})$ and parallel with respect to the restriction of the Gauss Manin connection to the leaves of the absolute period foliation. By Lemma A.9 in the appendix, this implies that the splitting $\mathcal{W} = \mathcal{E}_1 \oplus \mathcal{E}_2$ is the pull-back by Υ of a real analytic local splitting $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ of the bundle $\mathcal{R}$ into a sum of two complex vector bundles, defined on the image of the map Υ. That this image is open follows from the description of affine invariant manifolds via period coordinates.

The curvature form Θ for the connection $\nabla^{\mathcal{R}}$ is a two-form on Ω_ℓ with values in the bundle $\mathcal{R}^* \otimes \mathcal{R}$. We claim that Θ preserves the decomposition $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ on the image of the map Υ. This means that for any point x in the image of Υ, any two tangent vectors $u, v \in T_x\Omega_\ell$ and any $Y \in \mathcal{R}_i$, we have $\Theta(u,v)(Y) \in \mathcal{R}_i$.

To this end let $\gamma \subset V \subset \mathcal{C}$ be a periodic orbit for Φ^t with the properties as in Corollary 3.3 and let $\hat\gamma$ be a lift of γ to $\hat{\mathcal{C}}$. By Lemma A.9, $\Upsilon(\hat\gamma)$ is an orbit in Ω_ℓ for the action of the diagonal subgroup of $SL(2,\mathbb{R})$. This orbit is invariant under the action of an element $A \in Sp(2\ell,\mathbb{R})$ (which is the restriction of an element of $Sp(2g,\mathbb{Z})$ stabilizing the subspace $\mathbb{R}^{2\ell}$) whose eigenvalues are all real, of multiplicity one, and such that no product of two eigenvalues is an eigenvalue.

Since the local splitting $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ is invariant under the action of $SL(2,\mathbb{R})$ and is symplectic, for any choice of a point $z \in \Upsilon(\hat\gamma)$, the subspaces $(\mathcal{R}_i)_z$ are direct sums of eigenspaces for A, containing with an eigenspace for the eigenvalue a the eigenspace for a^{-1}.

We now follow the proof of Lemma 4.1. Let $\nabla^{\mathcal{R}_1}$ be the projection of the connection $\nabla^{\mathcal{R}}$ to a connection on $\mathcal{R}_1$. Then $\nabla^{\mathcal{R}} - \nabla^{\mathcal{R}_1}$ determines a real analytic (locally defined) tensor field $\beta \in \Omega(T^*\Omega_\ell \otimes \mathcal{R}_1^* \otimes \mathcal{R}_2)$. Since $\mathcal{R}_1$ and $\nabla^{\mathcal{R}}$ are invariant under the action of the diagonal flow $\Psi^t \subset SL(2,\mathbb{R})$ (we use the notation Ψ^t here to indicate that we are looking at a flow on the space Ω_ℓ), this tensor field is equivariant under the action of Ψ^t. Now no product of two eigenvalues of the matrix A is an eigenvalue and hence this implies that the restriction of β to $\Upsilon(\hat\gamma)$ vanishes (compare the proof of Lemma 4.1).

By Corollary 3.3, the set of points $q \in V \subset \mathcal{C}$ which are contained in a periodic orbit with the above properties is dense in V. Hence the image of this set under the restriction of the map Υ to a small contractible open subset of V is a dense subset of a nonempty open subset E of Ω_ℓ where the splitting $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ is defined. As the real analytic tensor field β vanishes on this dense subset of E, it vanishes identically on E. Hence the splitting $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ of $\mathcal{R}$ on E is invariant under the connection $\nabla^{\mathcal{R}}$.

As a consequence, the curvature form Θ of $\nabla^{\mathcal{R}}$ preserves the decomposition $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ on E. Using the terminology from the appendix, this means that Θ is reducible over $\mathbb{C}$. This contradicts Lemma A.5 and shows the proposition. □

Remark 4.4. The reasoning in the proof of Lemma 4.1 and Proposition 4.3 also implies that the Lyapunov filtration for the action of the Teichmüller flow on a stratum of abelian differentials is not smooth (or, less restrictive, is not of the class C^1). As we use covariant differentiation in our argument, mere continuity of the filtration can not be ruled out in this way.

Corollary 4.5. (1) *Let $\mathcal{Q}$ be a component of a stratum; then for every $2 \leq \ell \leq g-1$ there are finitely many affine invariant submanifolds of $\mathcal{Q}$ of rank ℓ which contain every affine invariant submanifold of rank ℓ.*
(2) *The smallest stratum of differentials with a single zero contains only finitely many affine invariant submanifolds of rank at least two.*

Proof. Let $\mathcal{C}$ be an affine invariant manifold of rank $k \geq 3$. By Proposition 3.4 and Proposition 4.3, there are finitely many proper affine invariant submanifolds of $\mathcal{C}$ which contain every affine invariant submanifold of $\mathcal{C}$ of rank $2 \leq \ell \leq k-1$.

An application of this fact to a component $\mathcal{Q}$ of a stratum shows that for $2 \leq \ell \leq g-1$, there are finitely many proper affine invariant submanifolds $\mathcal{C}_1, \dots, \mathcal{C}_m$ of $\mathcal{Q}$ which contain every affine invariant submanifold of $\mathcal{Q}$ of rank ℓ. The dimension of $\mathcal{C}_i$ is strictly smaller than the dimension of $\mathcal{Q}$.

By reordering we may assume that there is some $u \leq m$ such that for all $i \leq u$ the rank $\mathrm{rk}(\mathcal{C}_i)$ of $\mathcal{C}_i$ is at most ℓ, and that for $i > u$ the rank $\mathrm{rk}(\mathcal{C}_i)$ of $\mathcal{C}_i$ is bigger than ℓ. Apply the first paragraph of this proof to each of the affine invariant manifolds $\mathcal{C}_i$ $(i > u)$. We conclude that for each i there are finitely many proper affine invariant submanifolds of $\mathcal{C}_i$ of rank $r \in [\ell, \mathrm{rk}(\mathcal{C}_i))$ which contain every affine invariant submanifold of $\mathcal{C}_i$ of rank ℓ. The dimension of each of these submanifolds is strictly smaller than the dimension of $\mathcal{C}_i$. In finitely many such steps, each applied to all affine invariant submanifolds of rank strictly bigger than ℓ found in the previous step, we deduce the statement of the first part of the corollary.

Now let $\mathcal{H}(2g-2)$ be a stratum of differentials with a single zero. Period coordinates for $\mathcal{H}(2g-2)$ are given by absolute periods, and the dimension of an affine invariant manifold $\mathcal{C} \subset \mathcal{H}(2g-2)$ of rank ℓ equals 2ℓ. Thus $\mathcal{C}$ does not contain any proper affine invariant submanifold of rank ℓ.

By Proposition 3.4 and the first part of this proof, there are finitely many proper affine invariant submanifolds $\mathcal{C}_1, \dots, \mathcal{C}_s$ of $\mathcal{H}(2g-2)$ which contain every affine invariant submanifold of $\mathcal{H}(2g-2)$ of rank at most $g-1$. In particular, there are only finitely many such manifolds of rank $g-1$.

To show finiteness of affine invariant manifolds of any rank $2 \leq \ell \leq g-1$, apply Proposition 3.4 and the first part of this proof to each of the finitely many affine invariant manifolds constructed in some previous step and proceed by inverse induction on the rank. □

Remark 4.6. The proof of the second part of Corollary 4.5 immediately extends to the following statement. An affine invariant manifold $\mathcal{C}$ with trivial absolute period foliation contains only finitely many affine invariant manifolds of rank at least two.

5. Nested affine invariant submanifolds of the same rank

The goal of this section is to analyze affine invariant submanifolds of affine invariant manifolds $\mathcal{C}_+$ of the same rank $\ell \geq 2$ and to complete the proof of Theorem 1. Our strategy is a variation of the strategy used in Section 4. Namely, given an affine invariant manifold $\mathcal{C}_+$ with nontrivial absolute period foliation, we observe first that either $\mathcal{C}_+$ contains only finitely many affine invariant manifolds of the same rank, or there is a nontrivial $GL^+(2,\mathbb{R})$-invariant real analytic splitting of the tangent bundle $T\mathcal{C}_+$ of $\mathcal{C}_+$ over a $GL^+(2,\mathbb{R})$-invariant open dense subset V of $\mathcal{C}_+$ into two subbundles, where one of these subbundles is contained in the tangent bundle of the absolute period foliation.

This statement holds true for rank one affine invariant manifolds as well, however it is obvious in this case. In a second step, we use the assumption on the rank of $\mathcal{C}$ to derive a contradiction.

Denote as before by $\mathcal{AP}(\mathcal{C}_+)$ the absolute period foliation of an affine invariant manifold $\mathcal{C}_+$. By perhaps passing to a finite cover we may assume that the zeros of a differential $q \in \mathcal{C}_+$ are numbered.

The following proposition is analogous to Proposition 3.4 and carries out the first and second step of the above outline. Recall that the Teichmüller flow Φ^t acts on $T\mathcal{C}_+$ as a group of bundle automorphisms.

Proposition 5.1. *Let $\mathcal{C}_+ \subset \mathcal{Q}_+$ be an affine invariant manifold of rank $\ell \geq 1$. Then one of the following two possibilities holds true.*

1. *There are at most finitely many proper affine invariant submanifolds of $\mathcal{C}_+$ of rank ℓ.*
2. *Up to passing to a finite cover, the tangent bundle $T\mathcal{C}_+$ of $\mathcal{C}_+$ admits a non-trivial Φ^t-invariant real analytic splitting $T\mathcal{C}_+ = \mathcal{A} \oplus \mathcal{E}$ where $\mathcal{A}$ is a flat complex subbundle of $T\mathcal{AP}(\mathcal{C}_+)$ and where $\mathcal{E}$ contains the tangent bundle of the orbits of the $GL^+(2,\mathbb{R})$-action. Furthermore, the bundle $\mathcal{E}$ is integrable, and it defines a foliation of $\mathcal{C}_+$ with locally flat leaves.*

Proof. By Theorem 2.2 of [EMM15], it suffices to show the following. Let $m = \dim_{\mathbb{C}}(\mathcal{C}_+)$ and write $\ell = \mathrm{rk}(\mathcal{C}_+)$. Assume that there is a number $k \in [1, m-2\ell]$, and there is an open subset V of $\mathcal{C}_+$ such that the set of all affine invariant submanifolds of $\mathcal{C}_+$ of complex codimension k whose rank coincide with the rank of $\mathcal{C}_+$ is dense in V; then the second property in the proposition holds true.

Assume from now on that a nonempty open subset V of $\mathcal{C}_+$ with the properties stated in the previous paragraph exists. Note that we may assume that V is dense by $GL^+(2,\mathbb{R})$-invariance and topological transitivity of the action of $GL^+(2,\mathbb{R})$.

The leaves of the foliation $\mathcal{F}$ of $\mathcal{C}_+$ into the orbits of the $GL^+(2,\mathbb{R})$-action are complex suborbifolds of $\mathcal{C}_+$, ie the tangent bundle $T\mathcal{F}$ of this foliation is invariant under the complex structure i obtained from period coordinates. Let $\mathcal{Y}$ be an i-invariant $GL^+(2,\mathbb{R})$-invariant real analytic subbundle of the tangent bundle $T\mathcal{C}_+$ of $\mathcal{C}_+$ which is complementary to the bundle $T\mathcal{F}$. Using the notations from Section 3, such a bundle can be constructed as follows.

Let $\mathcal{Z}$ be the absolute holomorphic tangent bundle of $\mathcal{C}_+$ and write $\mathcal{W} = \mathcal{L} \cap \mathcal{Z}$. Let moreover $T\mathcal{C}$ be the tangent bundle of the foliation of $\mathcal{C}_+$ into the hypersurfaces of differentials with fixed area and let i be the standard complex structure in period coordinates; then we can take $\mathcal{Y} = p^{-1}(\mathcal{W} \oplus \overline{\mathcal{W}}) \cap T\mathcal{C} \cap iT\mathcal{C}$. Here as before, p denotes the projection to absolute periods. As a complex vector bundle, $\mathcal{Y}$ can be identified with the quotient of the holomorphic tangent bundle of $\mathcal{C}_+$ by the holomorphic tangent bundle of the foliation $\mathcal{F}$. In particular, we may assume that this bundle is holomorphic.

For the number $k \in \{1, \dots, m - 2\ell\}$ as specified above let $\mathcal{P} \to \mathcal{C}_+$ be the real analytic fibre bundle whose compact fibre at a point $q \in \mathcal{C}_+$ equals the Grassmannian of all *complex* subspaces of $\mathcal{Y}_q$ of complex codimension k. This is a real analytic subbundle of the fibre bundle whose fibre at q equals the Grassmannian of all oriented real linear subspaces of codimension $2k$ in $\mathcal{Y}_q$. The bundle $\mathcal{P}$ admits a natural decomposition $\mathcal{P} = \cup_{i=0}^{k} \mathcal{P}_i$ where $\mathcal{P}_i$ consists of all subspaces which intersect $T\mathcal{AP}(\mathcal{C}_+)$ in a subspace of complex codimension $k - i$. Thus $\mathcal{P}_0$ is the bundle of complex subspaces of complex codimension k which intersect $T\mathcal{AP}(\mathcal{C}_+)$ in a subspace of smallest possible dimension. In particular, $\mathcal{P}_0 \subset \mathcal{P}$ is open and $GL^+(2, \mathbb{R})$-invariant, and $\cup_{i \geq 1} \mathcal{P}_i$ is a closed nowhere dense subvariety of $\mathcal{P}$.

Our strategy is similar to the strategy used before. We begin with investigating the action of the Teichmüller flow Φ^t on the bundle $\mathcal{P}$, where for convenience of exposition, we restrict this flow to the real hypersurface $\mathcal{C}$ of differentials in $\mathcal{C}_+$ of area one, but we let its derivative act on the tangent bundle $T\mathcal{C}_+$ of $\mathcal{C}_+$.

Recall that the action of the flow Φ^t on $T\mathcal{C}_+$ preserves the bundle $\mathcal{Y}$. For $q \in \mathcal{C}$ and $t \in \mathbb{R}$ let $\rho(q, t)$ be the image of $\mathcal{P}(\Phi^t q)$ under the map $d\Phi^{-t}$. Then

$$\mathcal{R}_\infty = \cap_t \cup_q \rho(q, t)$$

is a (possibly empty) real analytic subset of $\mathcal{P}$. Here as before, a real analytic set is the common zero of a finite or countable family of real analytic functions on the real analytic variety $\mathcal{P}$. By construction, this subset is invariant under the action of Φ^t. Furthermore, the set $\mathcal{R}_\infty \subset \mathcal{P}$ is closed.

The tangent bundle of an affine invariant submanifold $\mathcal{D}_+$ of $\mathcal{C}_+$ intersects the complex vector bundle $\mathcal{Y}$ in a complex subbundle $\mathcal{Y} \cap T\mathcal{D}_+ | \mathcal{D}_+$. This subbundle is invariant under the action of the flow Φ^t. Hence if $q \in \mathcal{C}$ is contained in an affine invariant submanifold $\mathcal{D}$ of $\mathcal{C}$ of the same rank as $\mathcal{C}$ and of complex codimension k, then $\mathcal{R}_\infty \cap \mathcal{P}_0(q) \neq \emptyset$.

Thus under the assumption on the existence of a nonempty open Φ^t-invariant subset V of $\mathcal{C}$ containing a dense set of points which lie on an affine invariant submanifold of $\mathcal{C}$ of rank ℓ and complex codimension k, the real analytic subset $\mathcal{R}_\infty$ of $\mathcal{P}$ is not empty, and its image under the canonical projection $\pi : \mathcal{P} \to V$ is dense in the open set V. Since $\mathcal{R}_\infty \subset \mathcal{P}$ is closed and the canonical projection π is closed as well, this implies that the restriction of π to $\mathcal{R}_\infty$ maps $\mathcal{R}_\infty$ onto V. We refer to the proof of Proposition 3.4 for details on this construction.

Now $\mathcal{P}_0 \subset \mathcal{P}$ is an open subset of $\mathcal{P}$, and $\mathcal{R}_\infty \cap \mathcal{P}_0(q) \neq \emptyset$ for a dense set of points $q \in V$. As $\mathcal{R}_\infty$ is a real analytic set, this implies that up to perhaps decreasing the set V, we may assume that $\mathcal{R}_\infty \cap \mathcal{P}_0(q)$ is not empty for every $q \in V$. As the tangent bundle of the absolute period foliation is invariant under the action of Φ^t, the set $\mathcal{R}_\infty \cap \mathcal{P}_0$ is Φ^t-invariant as well. Thus

$$\mathcal{K} = \mathcal{R}_\infty \cap \mathcal{P}_0$$

is a real analytic subset of the (open) suborbifold $\mathcal{P}_0$ of $\mathcal{P}$ which is invariant under the natural action of the Teichmüller flow Φ^t and which projects onto an open dense Φ^t-invariant subset of $\mathcal{C}$ which we denote again by V.

For each $q\in V$, each point $z\in \mathcal{K}(q)$ is a complex linear subspace of $\mathcal{Y}_q$ of complex codimension k which intersects $T\mathcal{AP}(\mathcal{C})$ in a subspace of complex codimension k. Define

$$E(q)=\cap_{z\in \mathcal{K}(q)}z\subset \mathcal{Y}_q\subset T_q\mathcal{C}_+.$$

Then $E(q)$ is a (possibly trivial) complex linear subspace of $\mathcal{Y}_q$. As $\mathcal{K}\subset \mathcal{P}_0$ is a real analytic subset of $\mathcal{P}_0$ which projects onto V and which is invariant under the action of the Teichmüller flow Φ^t, by possibly replacing the set V by a proper open Φ^t-invariant subset we may assume that the dimension of $E(q)$ (which may be zero) does not depend on $q\in V$. If this dimension is positive, then $\cup_{q\in V}E(q)$ is a real analytic complex subbundle of $\mathcal{Y}|V$. Furthermore, if $\mathcal{D}\subset\mathcal{C}$ is an affine invariant submanifold of rank ℓ and complex codimension k which intersects the set V, then for every point $q\in V\cap\mathcal{D}$, the tangent space $T_q\mathcal{D}_+$ of $\mathcal{D}_+$ at q contains $E(q)$.

Our next goal is to show that for $q\in V$, the complex dimension of $E(q)$ is at least $\ell-1$. To this end let $\gamma\subset\mathcal{C}$ be a periodic orbit with the properties stated in Corollary 3.3 which intersects V in a point q. Let $\ell(\gamma)$ be the length of γ. The return map $d\Phi^{\ell(\gamma)}(q)$ acts on $\mathcal{Y}_q$.

Let again p be the projection of $T\mathcal{C}_+$ into absolute periods, and let $\mathcal{Z}$ be the absolute holomorphic tangent bundle of $\mathcal{C}_+$. The map $d\Phi^{\ell(\gamma)}$ commutes with p and hence it descends to a linear map A on the vector space $(\mathcal{Z}\oplus\overline{\mathcal{Z}})_q$, ie we have

$$p\circ d\Phi^{\ell(\gamma)}=A\circ p.$$

The map A is just the monodromy map obtained from parallel transport for the Gauss Manin connection on the flat bundle $\Pi^*\mathcal{N}\otimes\mathbb{C}\to\mathcal{Q}_+$.

By the choice of γ, the map A is semi-simple, with real eigenvalues, and the eigenspaces are complex lines (recall that we look here at the action of the pseudo-Anosov map $\Omega(\gamma)$ on $H^1(S,\mathbb{C})=H^1(S,\mathbb{R})\otimes\mathbb{C}$). Let $\mathcal{W}=\mathcal{L}\cap\mathcal{Z}$ be as before. Then the complex subspace $\mathcal{W}\otimes\overline{\mathcal{W}}$ is a direct sum of eigenspaces for eigenvalues whose absolute values are contained in the open interval $(e^{-\ell(\gamma)},e^{\ell(\gamma)})$.

Together with Lemma 2.4, we conclude that the restriction F of the map $d\Phi^{\ell(\gamma)}$ to $\mathcal{Y}_p$ is semi-simple. The eigenspaces for eigenvalues of absolute value contained in $(e^{-\ell(\gamma)},e^{\ell(\gamma)})$ are complex lines. The remaining eigenvalues are $e^{-\ell(\gamma)},e^{\ell(\gamma)}$. The eigenspace for the eigenvalue $e^{\ell(\gamma)}$ is the intersection of $\mathcal{Y}_q$ with the tangent space of the real rel foliation, and the eigenspace for the eigenvalue $e^{-\ell(\gamma)}$ is the intersection of $\mathcal{Y}_q$ with the tangent space of the imaginary rel foliation. Furthermore, the image under the complex structure i induced by period coordinates of an eigenvector for the eigenvalue $e^{\ell(\gamma)}$ is an eigenvector for the eigenvalue $e^{-\ell(\gamma)}$.

By definition, a point $z\in\mathcal{K}(q)$ is a complex subspace of $\mathcal{Y}_q\subset T_q\mathcal{C}_+$ of complex codimension k which is complementary to some k-dimensional complex subspace of $T_q\mathcal{AP}(\mathcal{C})$, and the image of z under the map F is complex as well. We claim that such a subspace has to contain the sum of the eigenspaces for A with respect to the eigenvalues of absolute value different from $e^{\ell(\gamma)},e^{-\ell(\gamma)}$.

To this end recall that the fibre $\mathcal{P}(q)$ of the bundle $\mathcal{P}$ at q is a closed subset of the Grassmann manifold of all oriented linear subspaces of $\mathcal{Y}_q$ of real codimension $2k$. Furthermore, $\mathcal{R}_\infty\cap\mathcal{P}(q)$ is a non-empty closed F-invariant subset containing the non-empty set $\mathcal{K}(q)$. If $z \in \mathcal{K}(q)$, then any limit of a subsequence of the sequence $F^i z$ $(i \to \pm\infty)$ is complex. By Lemma 3.2, such a limit y is a direct sum of subspaces of eigenspaces of F.

Now for $z \in \mathcal{K}(q)$, the complex dimension of the intersection of z with $T_q\mathcal{AP}(\mathcal{C}_+)$ equals $\dim_{\mathbb{C}} T_q(\mathcal{AP}(\mathcal{C}_+)) - k$. As the image of z under arbitrary iterates by the map F remains complex and the complex structure pairs an eigenvector for the eigenvalue $e^{\ell(\gamma)}$ with an eigenvector for the eigenvalue $e^{-\ell(\gamma)}$, we conclude that z contains the sum of the eigenspaces for F with respect to the eigenvalues different from $e^{\ell(\gamma)}, e^{-\ell(\gamma)}$. As this discussion is valid for each $z \in \mathcal{K}(q)$, the sum of the eigenspaces for F with respect to eigenvalues of absolute value different from $e^{\pm\ell(\gamma)}$ is contained in $\cap_{z\in\mathcal{K}(q)} z = E(q)$. In particular, we have $\dim_{\mathbb{C}} E(q) \geq \ell - 1$ and hence

$$\dim_{\mathbb{C}} E(q) \in [\ell - 1, \dim_{\mathbb{C}}(\mathcal{Y}_q) - k],$$

moreover $\mathcal{Y}_q = T\mathcal{AP}(\mathcal{C})_q + E(q)$ (this sum may not be direct). Now $E(q)$ depends in a real analytic fashion on $q \in V$ and hence the assignment $q \to E(q)$ is a real analytic Φ^t-invariant subbundle of $\mathcal{Y}|V$.

Let as before $\mathcal{F} \subset \mathcal{C}_+$ be the foliation into the orbits of the $GL^+(2,\mathbb{R})$-action and let $\hat{\mathcal{E}} \to V$ be the real analytic vector bundle whose fibre at $q \in V$ equals $T\mathcal{F} \oplus E(q)$. Clearly $\hat{\mathcal{E}}$ is invariant under the Teichmüller flow Φ^t. Moreover, if $\mathcal{D} \subset \mathcal{C}$ is an affine invariant manifold of rank ℓ and complex codimension k which intersects V, then for every $q \in \mathcal{D}$, the fibre $\hat{\mathcal{E}}(q)$ of $\hat{\mathcal{E}}$ at q is contained in the tangent space $T_q\mathcal{D}$ of $\mathcal{D}$ at q.

We use the bundle $\hat{\mathcal{E}}$ to construct a bundle $\mathcal{E}$ with the properties stated in (2) of the proposition. To this end note that as $\hat{\mathcal{E}}$ is a real analytic subbundle of the tangent bundle of $\mathcal{C}_+$, for $q \in V$ we can consider the linear subspace $\mathcal{B}(q) \supset \hat{\mathcal{E}}(q)$ of $T_q\mathcal{C}_+$ spanned by $\hat{\mathcal{E}}(q)$ and the values of all Lie brackets of sections of $\hat{\mathcal{E}}$. Let $q \in V$ be a point such that the dimension of $\mathcal{B}(q)$ is maximal, say that this dimension equals n. Then in a small neighborhood U of q, this dimension is constant and hence the assignment $u \to \mathcal{B}(u) \subset T_u\mathcal{C}_+$ defines an integrable real analytic subbundle of $T\mathcal{C}_+$ of real dimension n which contains the bundle $\hat{\mathcal{E}}$. In particular, we have $\mathcal{B} + T\mathcal{AP}(\mathcal{C}_+) = T\mathcal{C}_+$. On the other hand, for any point $q \in U$ with the property that q is contained in an affine invariant submanifold $\mathcal{D}$ of $\mathcal{C}_+$ of rank ℓ and complex codimension k, we have $T_q\mathcal{D}_+ \supset \mathcal{B}_q$. As the set of these points is dense in V by assumption, this shows that the (real) codimension of $\mathcal{B}$ is at least $2k \geq 2$.

As Φ^t acts on $\mathcal{C}$ as a group of diffeomorphisms, the set U constructed above is invariant under Φ^t and hence it is open and dense in $\mathcal{C}$ by topological transitivity of the action of Φ^t. To facilitate the notation we assume that in fact $U = V$.

Let $\hat{\mathcal{B}} = \mathcal{B} + i\mathcal{B}$; then for each $q \in V$, $\hat{\mathcal{B}}(q)$ is a complex subspace of $T_p\mathcal{C}_+$, and the above reasoning shows that its complex codimension is at least k. There exists an open Φ^t-invariant subset U of V such that the complex dimension of $\hat{\mathcal{B}}(q)$ is maximal for every $q \in U$. Then the restriction of $\hat{\mathcal{B}}$ to U is a real analytic complex subbundle of $T\mathcal{C}_+$ containing $\mathcal{B}$. Using again the fact that $\mathcal{B}$ is tangent to each affine invariant submanifold $\mathcal{D}_+$ of $\mathcal{C}_+$ of rank ℓ and complex codimension k, the complex codimension of $\hat{\mathcal{B}}$ is at least k.

Now if $\hat{\mathcal{B}} = \hat{\mathcal{E}}$ in U then $\hat{\mathcal{E}}$ is integrable and we put $\mathcal{E} = \hat{\mathcal{E}}$. Otherwise the complex dimension of $\hat{\mathcal{B}}$ is strictly larger than the complex dimension of $\hat{\mathcal{E}}$. Repeat the above construction with the bundle $\hat{\mathcal{B}}$ instead of $\hat{\mathcal{E}}$. In finitely many such steps, the complex dimension of the bundles constructed in this way has to stabilize. As a consequence, in finitely many such steps we find an integrable subbundle $\mathcal{E} \subset T\mathcal{C}_+$, defined on an open Φ^t-invariant subset V of $\mathcal{C}$, of complex codimension at least k, and such that for each affine invariant manifold $\mathcal{D} \subset \mathcal{C}$ of rank ℓ and complex codimension k which intersects V and each point $q \in \mathcal{D} \cap V$, a local integral manifold of $\mathcal{E}$ through q is contained in $\mathcal{D}$.

Recall that the bundle $\mathcal{E}$ contains the tangent bundle $T\mathcal{F}$ of the foliation of $\mathcal{C}$ into the orbits of the action of $GL^+(2,\mathbb{R})$. This means that its integral manifolds are locally saturated for the foliation of $\mathcal{C}$ into the orbits of the action of the group $GL^+(2,\mathbb{R})$. Then $\mathcal{E}$ is invariant under the action of $GL^+(2,\mathbb{R})$.

We show next that we can choose the bundle $\mathcal{E}$ in such a way that its the integral manifolds are locally affine. Note that as $\mathcal{E}$ is tangent to each of the affine invariant manifolds of rank ℓ and complex codimension k which intersects V and such affine invariant manifolds are affine in period coordinates, the integral manifolds of $\mathcal{E}$ are locally affine if the complex codimension of $\mathcal{E}$ in $T\mathcal{C}_+|V$ equals k.

Otherwise let $q \in V$ be a point which is contained in an affine invariant manifold $\mathcal{D}$ of rank ℓ and complex codimension k. Define $\mathcal{G}(q) \subset T\mathcal{C}_+$ to be the intersection of $T\mathcal{D}$ with all limits $T_{q_i}\mathcal{D}_i$ as $i \to \infty$ where q_i is a point on an affine invariant manifold $\mathcal{D}_i$ of rank ℓ and complex codimension k and $q_i \to q$. Since $\mathcal{E}$ is a real analytic vector bundle and since $\mathcal{E}(q_i) \subset T_{q_i}\mathcal{D}_i$ for all i, we have $\mathcal{G}(q) \supset \mathcal{E}(q)$. Furthermore, in the case that $\dim_{\mathbb{C}}(\mathcal{G}(q) \cap T_q\mathcal{D}) = \dim_{\mathbb{C}}\mathcal{E}(q)$ then in period coordinates, the local leaf M through q of the local foliation of $\mathcal{C}$ into integral manifolds of the bundle $\mathcal{E}$ equals the intersection of $\mathcal{D}$ with a collection of local limits of affine invariant manifolds $\mathcal{D}_i$ and hence this local leaf is affine.

It now suffices to observe that via perhaps decreasing the set V, we may assume that there exists a real analytic complex vector bundle $\mathcal{G} \supset \mathcal{E}$ whose fibre at a dense set of points $q \in V$ lying on an affine invariant manifold $\mathcal{D}$ as above coincides with the complex vector space constructed in the previous paragraph. To this end choose q so that the dimension of the complex vector space $\mathcal{G}(q) \supset \mathcal{E}(q)$ is minimal. As before, locally near q there exists a vector bundle $\mathcal{G} \supset \mathcal{E}$ with fibre $\mathcal{G}_q$ at q such that for a dense set of points z in a neighborhood of q, $\mathcal{G}_z$ is tangent to an affine submanifold of $\mathcal{C}$. Thus via perhaps replacing the bundle $\mathcal{E}$ by the bundle $\mathcal{G}$, we may assume that the local integral manifolds of $\mathcal{E}$ are affine.

We are left with showing that there is a flat subbundle of $T\mathcal{AP}(\mathcal{C})$ which is complementary to $\mathcal{E}$. Namely, let m be the number of zeros of a differential in $\mathcal{C}$. Let $q \in V$ and let $\mathfrak{a}_1, \dots, \mathfrak{a}_{m-1} \in \mathbb{R}^m$ be linearly independent with zero mean such that for some $u \leq m-1$, the vector fields $X_{\mathfrak{a}_1}, \dots, X_{\mathfrak{a}_u}$ are tangent to $\mathcal{C}_+$ and such that moreover their complex span is a linear subspace of $T\mathcal{AP}(\mathcal{C}_+)$ complementary to $\mathcal{E}(q)$. By invariance and Lemma 2.5, the complex span of these vector fields defines a flat invariant complex subbundle of $T\mathcal{AP}(\mathcal{C})|V$ which is complementary to the bundle $\mathcal{E}$. This is what we wanted to show. ◻

Remark 5.2. Proposition 5.1 is valid for affine invariant manifolds $\mathcal{C}_+$ of rank one, but in this case, property (2) just states that $\mathcal{C}_+$ is foliated into the orbits of the action of $GL^+(2,\mathbb{R})$, and these leaves are flat. Thus for rank one affine invariant manifolds, property (2) above always holds true for straightforward reason.

Remark 5.3. Lemma A.8 discusses real analytic $SL(2,\mathbb{R})$-invariant splittings of the tangent bundle of the sphere subbundle of the tautological vector bundle $\mathcal{V}$ over the moduli space $\mathcal{A}_g$ of principally polarized abelian differentials. In contrast to the statement of Proposition 5.4, such splittings can explicitly be constructed. This witnesses the fact that orbits of the action of $SL(2,\mathbb{R})$ on the Teichmüller space of abelian differentials project to Kobayashi geodesics which in general do not map to totally geodesic complex curves in the Siegel upper half-space, equipped with the symmetric metric. In other words, in spite of Lemma A.9, the actions of $SL(2,\mathbb{R})$ on the moduli space of abelian differentials and on the sphere subbundle of $\mathcal{V}$ are not compatible in any obvious geometric way.

Our final goal is to show that for $\ell \geq 2$, an affine invariant manifold $\mathcal{C}_+$ of rank ℓ does not admit a nontrivial $GL^+(2,\mathbb{R})$-invariant foliations into locally affine leaves which is transverse to the absolute period foliation. We refer to Theorem 5.1 of [W14] for a related result.

Proposition 5.4. *Let $\mathcal{C}_+$ be an affine invariant manifold of rank $\ell \geq 2$; then there is no nontrivial Φ^t-invariant real analytic splitting $T\mathcal{C}_+ = \mathcal{A} \oplus \mathcal{E}$ over an open dense Φ^t-invariant subset of $\mathcal{C}$ with property (2) of Proposition 5.1.*

Proof. We proceed as in the proof of Lemma 4.1 and Proposition 4.3. Let $\mathcal{C} \subset \mathcal{C}_+$ be the hyperplane of area one differentials. Assume to the contrary that there is an open dense Φ^t-invariant set $V \subset \mathcal{C}$, and there is a Φ^t-invariant real analytic splitting $T\mathcal{C}_+|V = \mathcal{A} \oplus \mathcal{E}$ as in the statement of the proposition. As before, we pass to a finite cover $\hat{\mathcal{C}}$ of $\mathcal{C}$ such that the zeros of a differential in this cover are numbered. Our goal is to show that the bundle $\mathcal{E}$ is flat; this then contradicts Theorem 5.1 of [W14].

An affine invariant manifold is locally defined by real linear equations in period coordinates (see [W14]). The affine structure of $\mathcal{C}_+$ defines a flat connection $\nabla^{\mathcal{C}}$ on $T\mathcal{C}_+$ which is invariant under affine transformations. In particular, this connection is invariant under the $GL^+(2,\mathbb{R})$-action. The bundle $\mathcal{A} \subset T\mathcal{AP}(\mathcal{C}_+)$ is flat, ie invariant under parallel transport for $\nabla^{\mathcal{C}}$. Namely, it is trivialized by globally defined vector fields $X_{\mathfrak{a}_i}$ where $\mathfrak{a}_i \in \mathbb{C}^m$ (compare the proof of Proposition 5.1), and these vector fields are parallel for $\nabla^{\mathcal{C}}$ (compare [W14]).

Recall from the proof of Proposition 5.1 that there is a real analytic complex subbundle $\mathcal{Y} \subset T\mathcal{C}_+$ which is invariant under the $GL^+(2,\mathbb{R})$-action and transverse to the tangent bundle $T\mathcal{F}$ of the foliation $\mathcal{F}$ of $\mathcal{C}_+$ into the orbits of the $GL^+(2,\mathbb{R})$-action. Let $\mathcal{K} = \mathcal{E} \cap \mathcal{Y}$. Since the rank ℓ of $\mathcal{C}_+$ is at least two, $\mathcal{K}$ is a complex subbundle of $\mathcal{Y}$ of positive dimension.

By passing to a finite cover, assume that the zeros of the differentials in $\mathcal{C}$ are numbered. Let $k \geq 2$ be the number of these zeros. Using the notation from the proof of Lemma 4.1, let $\mathcal{O} \subset \mathbb{C}^k$ be the complex vector space of vectors $\mathfrak{a}$ with zero mean which are tangent to $\mathcal{C}_+$. For $\mathfrak{a} \in \mathcal{O}$ let $X_\mathfrak{a} \subset T\mathcal{AP}(\mathcal{C})$ be the vector field defined by the Schiffer variation with weight $\mathfrak{a}$. Then for each $\mathfrak{a} \in \mathcal{O}$, the affine invariant manifold $\mathcal{C}$ is invariant under the flow $\Lambda^t_\mathfrak{a}$ generated by $X_\mathfrak{a}$ (Lemma 2.5). Furthermore, the bundle $\mathcal{A}$ is defined by a linear subspace of $\mathcal{O}$ which is invariant under the complex structure.

We claim that the bundle $\mathcal{K}$ is invariant under the flows generated by the vector fields $X_\mathfrak{a}$ for $\mathfrak{a} \in \mathcal{A}$. This is equivalent to stating that for all $\mathfrak{a} \in \mathcal{A}$ and every $q \in \mathcal{C}$, the Lie derivative $L_{X_\mathfrak{a}} Y(q)$ of every local section Y of $\mathcal{K}$ near q in direction of $X_\mathfrak{a}$ is contained in $\mathcal{K}$ at the point q.

We proceed as in the proof of Lemma 4.1. Use the $SL(2,\mathbb{R})$-invariant decomposition $T\mathcal{C}_+ = T\mathcal{F} \oplus \mathcal{Y}$ to project the flat connection $\nabla^\mathcal{C}$ on $T\mathcal{C}_+$ to a connection $\nabla^\mathcal{Y}$ on $\mathcal{Y}$. Let $q \in V$, let $Y \in \mathcal{K}$ and let $\hat{Y}$ be the vector field along the flow line of the flow $\Lambda^t_\mathfrak{a}$ obtained by parallel transport of Y for the connection $\nabla^\mathcal{Y}$. Then the Lie derivative $L_{X_\mathfrak{a}}(\hat{Y})$ is defined at q, and we have to show that $L_{X_\mathfrak{a}}(\hat{Y}) \in \mathcal{Y}$.

To this end define $\beta(X_\mathfrak{a}, Y) \in T\mathcal{F} \oplus \mathcal{A}$ to be the component of $L_{X_\mathfrak{a}}\hat{Y}$ in $T\mathcal{F} \oplus \mathcal{A}$ with respect to the decomposition $T\mathcal{C}_+ = T\mathcal{F} \oplus \mathcal{A} \oplus \mathcal{K}$. Then β is a real analytic section of $\mathcal{A}^* \otimes \mathcal{Y}^* \otimes (T\mathcal{F} \oplus \mathcal{A})$. By invariance of the decomposition of $T\mathcal{C}_+$ under the Teichmüller flow and equivariance of the flat connection $\nabla^\mathcal{C}$, the tensor field β is equivariant under the action of the Teichmüller flow.

As in the proof of Lemma 4.1, it now suffices to show that β vanishes at any point $q \in \mathcal{C}$ contained in a periodic orbit γ for Φ^t with the properties stated in Corollary 3.3. Let F be the differential of the map $d\Phi^{\ell(\gamma)}$; then the fibre $\mathcal{K}_q$ can be represented in the form

$$\mathcal{K}_q = L_q \oplus (\mathcal{K}_q \cap T\mathcal{AP}(\mathcal{C}_+)$$

where L_q is a direct sum of eigenspaces of the map F for eigenvalues which are different from $e^{\ell(\gamma)}, e^{-\ell(\gamma)}, \pm 1$.

Now if $Z \in T\mathcal{AP}(\mathcal{C}_+) \cap \mathcal{K}_q$ then $\beta(\cdot, Z) = 0$ as a leaf of the absolute period foliation is flat. On the other hand, $(T\mathcal{F} \oplus \mathcal{A})_q$ is a direct sum of eigenspaces of F for the eigenvalues $e^{\ell(\gamma)}, e^{-\ell(\gamma)}, \pm 1$ and hence vanishing of $\beta(\cdot, Z)$ for $Z \in L_q$ follows as in the proof of Lemma 4.1.

We showed so far that the bundle $\mathcal{K}$ is invariant under each of the flows $\Lambda^t_\mathfrak{a}$ generated by a vector field $X_\mathfrak{a} \subset \mathcal{A}$. Then the (locally defined) bundle $\hat{\mathcal{K}}$ generated by $\mathcal{K}$ and all Lie brackets of sections of $\mathcal{K}$ is invariant under such a flow as well (compare the proof of Proposition 5.1 for details of this construction). As $\mathcal{K}$ is a subbundle of the integrable bundle $\mathcal{E}$, the bundle $\hat{\mathcal{K}}$ is a subbundle of $\mathcal{E}$ as well. In particular, $\hat{\mathcal{K}}$ is a *non-trivial* subbundle of $T\mathcal{C}_+$.

On the other hand, $\mathcal{K}$ projects to a complex subbundle of rank at least one in the bundle $\mathcal{W}$. Hence by Lemma A.3 in the appendix, the bundle $\hat{\mathcal{K}}$ contains the generator Y of the Teichmüller flow.

However, this contradicts the fact that for each $\mathfrak{a} \in \mathcal{O} \cap \mathbb{R}^k$ we have

$$L_{X_\mathfrak{a}}(Y) = [X_\mathfrak{a}, Y] = -L_Y(X_\mathfrak{a}) = -X_\mathfrak{a}$$

by Lemma 2.4. This is a contradiction which completes the proof of the proposition. ∎

As an immediate consequence of Proposition 3.4 and Lemma 5.4, we obtain

Corollary 5.5. *An affine invariant manifold $\mathcal{C}$ of rank at least two contains only finitely many affine invariant submanifolds of the same rank.*

Theorem 1 from the introduction is now an immediate consequence of Proposition 4.5 and Corollary 5.5.

6. Algebraically primitive Teichmüller curves

A point in the moduli space of area one abelian differentials on a closed surface S of genus $g \geq 2$ defines an euclidean metric on S whose singularities are cone points of cone angle a multiple of 2π at the zeros of the differential. Such a metric is called a *translation structure* on S. An *affine automorphism* of such a translation structure (X, ω) is a homeomorphism $f : S \to S$ which takes singularities of (X, ω) to singularities and is locally affine in the nonsingular part of S. Let Γ be the group of affine automorphisms of (X, ω). The function which takes an affine automorphism f to its derivative Df gives a homomorphism from Γ into $GL(2, \mathbb{R})$. The image $D(\Gamma)$ is called the *Veech group* of the translation surface. It is contained in the subgroup $SL^{\pm}(2, \mathbb{R})$ of all elements with determinant ± 1.

If the affine automorphism group of the translation surface (X, ω) contains a pseudo-Anosov element φ, then the trace field of φ is defined. Recall that φ acts on $H^1(S, \mathbb{R})$ as a Perron Frobenius automorphism, and if $\mu > 1$ is the leading eigenvalue for this action, then the trace field of φ equals $\mathbb{Q}[\mu + \mu^{-1}]$.

By Theorem 28 in the appendix of [KS00], the trace field of φ coincides with the so-called *holonomy field* of (X, ω). The holonomy field is defined for any translation surface, however we will not make use of this fact in the sequel. Instead we refer to the appendix of [KS00] for more information. By Lemma 2.10 of [LNW15], if $\mathcal{C}$ is a rank one affine invariant manifold then for all $(X, \omega) \in \mathcal{C}$, the holonomy field of (X, ω) equals the *field of definition* of $\mathcal{C}$ [W14]. In particular, the trace field of a pseudo-Anosov element whose conjugacy class corresponds to a periodic orbit in a rank one affine invariant manifold $\mathcal{C}$ only depends on $\mathcal{C}$ but not on the periodic orbit. As we will not use any other information on the field of definition, we will not define it here. Instead we refer to [W14] for more details.

For the proof of Theorem 2 we have a closer look at rank one affine invariant manifolds $\mathcal{C}$ whose field of definition $\mathfrak{k}$ is of degree g over $\mathbb{Q}$. Then $\mathfrak{k}$ is a totally real [F16] number field of degree g, with ring of integers $\mathcal{O}_{\mathfrak{k}}$. Via the g field embeddings $\mathfrak{k} \to \mathbb{R}$, the group $SL(2, \mathcal{O}_{\mathfrak{k}})$ embeds into $G = SL(2, \mathbb{R}) \times \cdots \times SL(2, \mathbb{R}) < Sp(2g, \mathbb{R})$ and in fact, $SL(2, \mathcal{O}_{\mathfrak{k}})$ is a lattice in G. The trace field of every periodic orbit γ in $\mathcal{C}$ equals $\mathfrak{k}$ and hence the image of a corresponding pseudo-Anosov element $\Omega(\gamma)$ under the homomorphism $\Psi : \mathrm{Mod}(S) \to Sp(2g, \mathbb{R})$ is contained in a conjugate of $SL(2, \mathcal{O}_{\mathfrak{k}})$.

The following observation is immediate from Theorem 3 of [H18b] and [G12]. For its formulation, define the *extended local monodromy group* of an open contractible subset U of $\mathcal{C}$ to be the subgroup of $SL(2,\mathcal{O}_{\mathfrak{k}})$ which is generated by the monodromy of those (parametrized) periodic orbits for Φ^t in $\mathcal{C}$ which pass through U. Compare with Theorem 3 of [H18b].

Lemma 6.1. *For a rank one affine invariant manifold $\mathcal{C}$ whose field of definition is of degree g over $\mathbb{Q}$, the extended local monodromoy group of any open set is Zariski dense in $SL(2,\mathbb{R})\times\cdots\times SL(2,\mathbb{R})$.*

Proof. By Theorem 3 of [H18b], the projection of the extended local monodromy group of an open set $U\subset\mathcal{C}$ to the first factor $SL(2,\mathbb{R})$ of $G=SL(2,\mathbb{R})\times\cdots\times SL(2,\mathbb{R})$ is Zariski dense in $SL(2,\mathbb{R})$ and hence it is non-elementary. Moreover, by definition and [KS00, LNW15], for every pseudo-Anosov element $\varphi\in\mathrm{Mod}(S)$ whose conjugacy class defines a periodic orbit for the Teichmüller flow in $\mathcal{C}$, the holonomy field of φ equals the field of definition of $\mathcal{C}$. This also holds true if $\varphi=\psi^2$ for a pseudo-Anosov element ψ. As a consequence, $\mathfrak{k}$ coincides with the *invariant trace field* of the extended local monodromy group which by definition is generated by the traces of the squares of elements in the extended local monodromy group. Corollary 2.2 of [G12] now shows that the extended local monodromy group of U is Zariski dense in G. □

In the statement of the next corollary, the affine invariant manifold $\mathcal{B}_+$ may be a component of a stratum. As before, we put a lower index $+$ whenever we do not normalize the area of a holomorphic differential.

Corollary 6.2. *Let $\mathcal{C}_+$ be a rank one affine invariant manifold whose field of definition is of degree g over $\mathbb{Q}$. Assume that $\mathcal{C}_+$ is properly contained in an affine invariant manifold $\mathcal{B}_+$ of rank at least three. Let $\mathcal{Z}\to\mathcal{B}_+$ be the absolute holomorphic tangent bundle of $\mathcal{B}_+$; then $\mathcal{Z}|\mathcal{C}_+$ splits as a sum of holomorphic line bundles which are invariant under both the Chern connection and the Gauss Manin connection.*

Proof. Following Theorem 1.5 of [W14], for a rank one affine invariant manifold $\mathcal{C}_+$ whose field of definition is of degree g over $\mathbb{Q}$, there exists a direct $SL(2,\mathbb{R})$-invariant decomposition

$$\Pi^*\mathcal{H}|\mathcal{C}_+=\oplus_g V_g \tag{4}$$

where each V_k is a flat complex line bundle. Here V_1 is the bundle corresponding to the orbits of the $SL(2,\mathbb{R})$-action, and each of the bundles V_i is a Galois conjugate of V_1. Furthermore, by the main result of [F16], this decomposition is compatible with the Hodge decomposition (see also the footnote in [W14]). Therefore $\mathcal{C}_+$ parameterizes translation surfaces whose Jacobians admit real multiplication by the trace field of $\mathcal{C}$. However, in the moduli space $\mathcal{A}_g=Sp(2g,\mathbb{Z})\backslash\mathfrak{D}_g$, the Hodge decomposition over the locus of such curves defines a splitting of the bundle $\mathcal{V}$ into a direct sum of *holomorphic* line bundles, and this is the splitting which pulls back to the decomposition (4).

As a consequence, in the case that the rank of $\mathcal{B}_+$ equals g (and hence $\mathcal{Z} = \Pi^*\mathcal{H}|\mathcal{B}_+$), the statement of the lemma is immediate from the discussion in the previous paragraph. Thus assume that the rank of $\mathcal{B}_+$ is at most $g-1$.

Since $\mathcal{C}_+ \subset \mathcal{B}_+$, the restriction of $\Pi^*\mathcal{H}$ to $\mathcal{C}_+$ has two splittings which are invariant under the extended local monodromy of $\mathcal{C}_+$. Here it is important that the splittings are flat, but we do not care whether or not the splittings are holomorphic. The first splitting is the splitting into g flat line bundles discussed in the beginning of this proof. The second splitting is the splitting into the absolute holomorphic tangent bundle $\mathcal{Z}$ of $\mathcal{B}_+$ (which is a holomorphic subbundle of $\Pi^*\mathcal{H}|_{\mathcal{B}_+}$ whose complex rank $\ell \geq 1$ equals the rank of $\mathcal{B}_+$) and its symplectic complement.

By Lemma 6.1 the extended local monodromy group of $\mathcal{C}_+$ is Zariski dense in $G = SL(2,\mathbb{R}) \times \cdots \times SL(2,\mathbb{R})$. On the other hand, this local monodromy group also is contained in the group $Sp(2\ell,\mathbb{R}) \times Sp(2g-2\ell,\mathbb{R})$ which is determined by the symplectic decomposition of $\Pi^*\mathcal{H}$ as

$$\Pi^*\mathcal{H} = \mathcal{Z} \oplus \mathcal{Z}^\perp.$$

Then the Zariski closure of the extended local monodromy group is contained in this product group as well. By Zariski density of the local monodromy group in the group G, this implies that up to reordering, the group $Sp(2\ell,\mathbb{R}) \times \{\mathrm{Id}\}$ intersects G in a subgroup of the form $SL(2,\mathbb{R}) \times \cdots \times SL(2,\mathbb{R}) \times \{\mathrm{Id}\}$ (ℓ factors). Equivalently, the bundle $\mathcal{Z}|\mathcal{C}_+$ is a direct sum of some of the invariant *holomorphic* line bundles V_k. ☐

Corollary 6.3. *For a component $\mathcal{Q}_+$ of a stratum in genus $g \geq 3$, all affine invariant submanifolds of rank one whose fields of definition are of degree g over $\mathbb{Q}$ are contained in a finite collection of affine invariant submanifolds of rank at most two.*

Proof. Let $\mathfrak{C}$ be the collection of all rank one affine invariant submanifolds of $\mathcal{Q}$ whose field of definition is a number field of degree g over $\mathbb{Q}$. Recall the invariant decomposition $\Pi^*\mathcal{H} = \mathcal{T} \oplus \mathcal{L}$. For each $\mathcal{C}_+ \in \mathfrak{C}$, the restriction of the bundle $\mathcal{L}$ to $\mathcal{C}_+$ splits as a sum of holomorphic line bundles which are invariant under the Gauss-Manin connection in the sense discussed in Section 3. Thus by Proposition 3.4 and its proof, there exists a finite collection of affine invariant submanifolds of $\mathcal{Q}$ of rank at most $g-1$ which contain each element of $\mathfrak{C}$.

Now if $\mathcal{B}_+$ is an affine invariant manifold of rank contained in $[3, g-1]$ which contains some $\mathcal{C}_+ \in \mathfrak{C}$ then by Corollary 6.2, the above reasoning can be applied to $\mathcal{B}_+$. In finitely many steps we find finitely many proper affine invariant manifolds $\mathcal{C}_1, \ldots, \mathcal{C}_k \subset \mathcal{Q}_+$ of rank at most two which contain every $\mathcal{C}_+ \in \mathfrak{C}$. This is the statement of the corollary. ☐

Now we are ready to complete the proof of Theorem 2.

Corollary 6.4. *For $g \geq 3$ the $SL(2,\mathbb{R})$-orbit closure of a typical periodic orbit in any component of a stratum is the entire stratum.*

Proof. Let $\mathcal{Q}$ be a component of a stratum and let $U \subset \mathcal{Q}$ be a non-empty open set. Then a typical periodic orbit for Φ^t passes through U [H13]. Thus by Theorem 1 (see also [MW15, MW16]), the $SL(2,\mathbb{R})$-orbit closure of a typical periodic orbit either equals the entire stratum, or it is an affine invariant manifold of rank one.

By Theorem 1 of [H18b], the trace field of a typical perodic orbit γ is totally real and of degree g over $\mathbb{Q}$. If the rank of the $SL(2,\mathbb{R})$-orbit closure $\mathcal{C}$ of γ equals one then this trace field is the field of definition of $\mathcal{C}$ [LNW15]. Thus the corollary follows from Corollary 6.3. □

We complete the main body of this article with the proof of Theorem 3. We begin with

Proposition 6.5. *Let $g \geq 3$ and let $\mathcal{B}_+ \subset \mathcal{Q}_+$ be a rank two affine invariant manifold. Then the union of all algebraically primitive Teichmüller curves which are contained in $\mathcal{B}_+$ is nowhere dense in $\mathcal{B}_+$.*

Proof. Let $\mathcal{B}_+ \subset \mathcal{Q}_+$ be a rank two affine invariant manifold. We argue by contradiction, and we assume that the closure of the union of all algebraically primitive Teichmüller curves $\mathcal{C}_+ \subset \mathcal{B}_+$ contains some open subset V of $\mathcal{B}_+$.

Let $\mathcal{Z} \to \mathcal{B}_+$ be the absolute holomorphic tangent bundle of $\mathcal{B}_+$. Let U be a small contractible subset of V so that there is a trivialization of the Hodge bundle over U defined by the Gauss Manin connection. The extended local monodromy group of U preserves $\mathcal{Z}$. Let $\mathcal{C}_i \subset \mathcal{B}_+$ be a sequence of algebraically primitive Teichmüller curves which pass through U and whose closures contain a compact subset of U with non-empty interior W.

Let $\Pi : \mathcal{Q}_+ \to \mathcal{M}_g$ be the canonical projection and let $\mathcal{I}_g : \mathcal{M}_g \to \mathcal{A}_g$ be the Torelli map. The image under Π of the curve $\mathcal{C}_i$ is an algebraic curve (see [F16]) which admits a *modular embedding*. Namely, by the main result of [Mo06], there is a totally real number field K_i of degree g over $\mathbb{Q}$, there is an order $\mathfrak{o}_{K_i}$ in K_i, and there is an embedding

$$SL(2,\mathfrak{o}_{K_i}) \to SL(2,\mathbb{R}) \times \cdots \times SL(2,\mathbb{R}) \to Sp(2g,\mathbb{R})$$

which maps $SL(2,\mathfrak{o}_{K_i})$ into $Sp(2g,\mathbb{Z})$ and such that the image of $\mathcal{C}_i$ under the Torelli map is contained in the Hilbert modular variety $H(\mathfrak{o}_{K_i})$. This Hilbert modular variety is the quotient of $\mathbf{H}^2 \times \cdots \times \mathbf{H}^2$ under the lattice $SL(2,\mathfrak{o}_{K_i})$ in a Lie subgroup G_i of $Sp(2g,\mathbb{R})$ which is isomorphic to $SL(2,\mathbb{R}) \times \cdots \times SL(2,\mathbb{R})$.

We claim that $G_i = G_j = G$ for all i. Namely, assume otherwise. Then there are algebraically primitive Teichmüller curves $\mathcal{C}_i, \mathcal{C}_j$ which intersect U and for which the groups G_i, G_j are distinct. By Lemma 6.1, the extended local monodromy groups of $\mathcal{C}_i \cap U$ and $\mathcal{C}_j \cap U$ are Zariski dense in G_i, G_j. Therefore the Zariski closure in $Sp(2g,\mathbb{R})$ of the extended local monodromy group of $U \subset \mathcal{B}_+$ contains $G_i \cup G_j$. But as $G_i \neq G_j$, a subgroup of $Sp(2g,\mathbb{R})$ which contains $G_i \cup G_j$ can not preserve the subspace $\mathcal{Z}$. This is a contractiction and implies that indeed, $G_i = G_j = G$ for all i.

Write $SL(2,\mathfrak{o}) = SL(2,\mathfrak{o}_{K_i})$. The Hilbert modular variety $H(\mathfrak{o}) = H(\mathfrak{o}_{K_i}) \subset \mathcal{A}_g$ consists of abelian varieties with real multiplication with the field $K = K_i$. The image of $\mathcal{C}_i$ under the map $\mathcal{I}_g \circ \Pi$ is contained in $H(\mathfrak{o})$. As a consequence, the set of points in $\mathcal{B}_+$ which are mapped by the composition of the foot-point projection $\Pi : \mathcal{B}_+ \to \mathcal{M}_g$ with the Torelli map $\mathcal{I}_g$ into $H(\mathfrak{o})$ contains a dense subset of the open set W. But $H(\mathfrak{o})$ is a complex submanifold of $\mathcal{A}_g$ and this composition map is holomorphic and therefore the image of $\mathcal{B}_+$ is contained in $H(\mathfrak{o})$.

We showed so far that each point in $\mathcal{B}_+$ is an abelian differential whose Jacobian has real multiplication with K. Now a point on an algebraically primitive Teichmüller curve is mapped to an eigenform for real multiplication [Mo06] and hence the closure of the set of differentials in $\mathcal{B}_+$ which are mapped to eigenforms for real multiplication with K contains an open set. This implies as before that each point in $\mathcal{B}_+$ corresponds to such an eigenform and hence $\mathcal{B}_+$ is a rank one affine invariant manifold, contrary to our assumption. The proposition follows □

Proof of Theorem 3:

Let $\mathcal{Q}$ be a component of a stratum in genus $g \geq 3$. By Corollary 6.3, there are finitely many affine invariant submanifolds $\mathcal{B}_1, \dots, \mathcal{B}_k$ of rank two which contain all but finitely many algebraically primitive Teichmüller curves.

Let $\mathcal{B}_i$ be such an affine invariant manifold of rank two. Assume that its dimension equals r for some $r \geq 4$. By Proposition 6.5, the closure of the union of all algebraically primitive Teichmüller curves which are contained in $\mathcal{B}_i$ is nowhere dense in $\mathcal{B}_i$. As this closure is invariant under the action of $GL(2,\mathbb{R})$, it consists of a finite union of affine invariant manifolds. The dimension of each of these invariant submanifolds is at most $r-1$.

If there are submanifolds of rank two in this collection then we can repeat this argument with each of these finitely many submanifolds. By inverse induction on the dimension, this yields that all but finitely many algebraically primitive Teichmüller curves are contained in one of finitely many affine invariant manifolds of rank one. The field of definition of such a manifold coincides with the field of definition of the Teichmüller curve, in particular it is of degree g [LNW15].

By the main result of [LNW15], a rank one affine invariant manifold with field of definition of degree g over $\mathbb{Q}$ only contains finitely many Teichmüller curves. Thus the number of algebraically primitive Teichmüller curves in $\mathcal{Q}$ is finite as promised.

Appendix A. Structure of the homogeneous space $Sp(2g,\mathbb{Z})\backslash Sp(2g,\mathbb{R})$

In this appendix we collect some geometric properties of the Siegel upper half-space $\mathfrak{D}_g = Sp(2g,\mathbb{R})/U(g)$ and its quotient $\mathcal{A}_g = Sp(2g,\mathbb{Z})\backslash\mathfrak{D}_g$ which are either directly or indirectly used in the proofs of our main results.

The tautological vector bundle

$$\tilde{\mathcal{V}} \to \mathfrak{D}_g$$

over the Hermitean symmetric space $\mathfrak{D}_g = Sp(2g,\mathbb{R})/U(g)$ is obtained as follows.

Via the right action of the unitary group $U(g)$, the symplectic group $Sp(2g,\mathbb{R})$ is an $U(g)$-principal bundle over $\mathfrak{D}_g$. The bundle $\tilde{\mathcal{V}}$ is the associated vector bundle

$$\mathcal{V} = Sp(2g,\mathbb{R}) \times_{U(g)} \mathbb{C}^g$$

where $U(g)$ acts from the right by $(x,y,\alpha) \to (x\alpha, \alpha^{-1}y)$. The bundle $\tilde{\mathcal{V}}$ is holomorphic. This means that there is a covering $\mathcal{U} = \{U_i \mid i\}$ of $\mathfrak{D}_g$ by open sets and complex trivializations of $\tilde{\mathcal{V}}$ over the sets U_i such that transitions functions for these trivializations are holomorphic. Furthermore, the action of $Sp(2g,\mathbb{R})$ on $\mathfrak{D}_g$ naturally extends to an action of $Sp(2g,\mathbb{R})$ on $\tilde{\mathcal{V}}$ as a group of biholomorphic bundle automorphisms.

In the remained of this appendic we will mainly consider *complex* vector bundles over real or complex manifolds. As usual, a complex vector bundle is a (real) vector bundle E equipped with a smooth section J so that $J^2 = \mathrm{Id}$.

An $U(g)$-invariant hermitean inner product on $\mathbb{C}^g$ induces a hermitean metric on $\mathbb{C}^g$. As $U(g)$ acts transitively on the unit sphere in $\mathbb{C}^g$ for this inner product, with isotropy group $U(g-1)$, the associated sphere bundle

$$\mathcal{S} = Sp(2g,\mathbb{R}) \times_{U(g)} S^{2g-1}$$

in $\tilde{\mathcal{V}} \to \mathfrak{D}_g$ can naturally be identified with the homogeneous space

$$\mathcal{S} = Sp(2g,\mathbb{R})/U(g-1)$$

(Proposition I.5.5 of [KN63]).

The group $Sp(2g-2,\mathbb{R})$ is the isometry group of Siegel upper half-space

$$\mathfrak{D}_{g-1} = Sp(2g-2,\mathbb{R})/U(g-1).$$

Since the action of $Sp(2g-2,\mathbb{R})$ on $\mathfrak{D}_{g-1}$ is transitive, with isotropy group $U(g-1)$, the bundle $\mathcal{S} = Sp(2g,\mathbb{R})/U(g-1) \to \mathfrak{D}_g$ can also be identified with the associated bundle

$$\mathcal{S} = Sp(2g,\mathbb{R}) \times_{Sp(2g-2,\mathbb{R})} \mathfrak{D}_{g-1}$$

where $Sp(2g-2,\mathbb{R})$ acts via

$$(g,x)h = (gh, h^{-1}(x)).$$

The first factor projection then defines a projection

$$\Pi : \mathcal{S} \to Sp(2g,\mathbb{R})/Sp(2g-2,\mathbb{R}).$$

Let $\omega = \sum_i dx_i \wedge dy_i$ be the standard symplectic form on $\mathbb{R}^{2g}$. The standard representation of $Sp(2g,\mathbb{R})$ on $(\mathbb{R}^{2g},\omega)$ naturally extends to an action of $Sp(2g,\mathbb{R})$ on $\mathbb{R}^{2g} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}^{2g}$ by complex linear transformations. The open subset

$$\mathcal{O} = \{x + iy \mid x,y \in \mathbb{R}^{2g}, \omega(x,y) > 0\} \subset \mathbb{C}^{2g}$$

is $Sp(2g,\mathbb{R})$-invariant. It contains the invariant hypersurface

$$\Omega = \{x + iy \in \mathbb{C}^{2g} \mid \omega(x,y) = 1\}.$$

Lemma A.1. *Ω can naturally and $Sp(2g,\mathbb{R})$-equivariantly be identified with the homogeneous space*

$$Sp(2g,\mathbb{R})/Sp(2g-2,\mathbb{R}).$$

Proof. Observe that the diagonal action of the group $Sp(2g,\mathbb{R})$ on Ω is transitive. Namely, a pair of points $(x,y) \in \mathbb{R}^{2g}$ with $\omega(x,y) = 1$ can be extended to a symplectic basis $B = \{x, y, b_3, \ldots, b_{2g}\}$ of $\mathbb{R}^{2g}$. If $(x', y') \in \Omega$ is any other such point and if $B' = \{x', y', b'_3, \ldots, b'_{2g}\}$ is another such symplectic basis, then there exists an element $A \in Sp(2g,\mathbb{R})$ which maps B to B'. Then $A(x+iy) = x' + iy'$.

This reasoning also shows that the stabilizer in $Sp(2g,\mathbb{R})$ of a point $x+iy \in \Omega$ is isomorphic to a standard embedded

$$\mathrm{Id} \times Sp(2g-2,\mathbb{R}) < Sp(2,\mathbb{R}) \times Sp(2g-2,\mathbb{R}) < Sp(2g,\mathbb{R}).$$

□

For a more explicit description of Ω we use the standard basis $(x_1, y_1, \cdots, x_g, y_g)$ of the symplectic vector space $\mathbb{R}^{2g}$. With respect to this basis, the symplectic form ω is given by the matrix

$$\mathfrak{J} = \begin{pmatrix} & 1 & & & \\ -1 & & & & \\ & & \ddots & & \\ & & & & 1 \\ & & & -1 & \end{pmatrix}$$

The Lie algebra $\mathfrak{sp}(2g,\mathbb{R})$ of $Sp(2g,\mathbb{R})$ is then the algebra of $(2g,2g)$-matrices A with $A\mathfrak{J} + \mathfrak{J}A = 0$. The Lie algebra $\mathfrak{h}$ of the subgroup $Sp(2,\mathbb{R}) \times Sp(2g-2,\mathbb{R})$ consists of matrices in block form

$$\begin{pmatrix} A & \\ & B \end{pmatrix}$$

where $A \in \mathfrak{sl}(2,\mathbb{R})$ and $B \in \mathfrak{sp}(2g-2,\mathbb{R})$.

Let $\mathfrak{p}$ be the linear subspace of $\mathfrak{sp}(2g,\mathbb{R})$ of matrices whose only non-trivial entries are entries a_{ij} with $i = 1,2$ and $3 \leq j \leq 2g$ or $j = 1,2$ and $3 \leq i \leq 2g$. This subspace can explicitly be described as follows. Let ι be the complex structure on $\mathbb{R}^{2g}$ defined informally by $\iota x_i = y_i, \iota y_i = -x_i$; then a matrix in $\mathfrak{p}$ is of the form

$$\left[\begin{array}{cc|c} 0 & 0 & x \\ 0 & 0 & -\iota x \\ \hline & & \\ y & -\iota y & \end{array}\right]$$

where $x, y \in \mathbb{R}^{2g-2} \subset \mathbb{R}^{2g}$ are vectors with vanishing first and second coordinate. Thus $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{b}$ where $\mathfrak{a}, \mathfrak{b}$ are abelian subalgebras of dimension $2g-2$. Here $\mathfrak{a}$ is the intersection with $\mathfrak{p}$ of the vector space of matrices whose only non-zero entries are contained in the first and second line. Elementary matrix multiplication shows that this vector space is in fact stable under the Lie bracket, and this Lie bracket vanishes identically on $\mathfrak{a}$. Similarly, $\mathfrak{b}$ is the intersection with $\mathfrak{p}$ of the vector space of matrices whose only non-zero entries are contained in the first and second row. Note that the transpose of a matrix in the subspace $\mathfrak{a}$ is contained in $\mathfrak{b}$. Denote by $\mathfrak{sp}(2,\mathbb{R})$ the Lie subalgebra of $\mathfrak{sp}(2g,\mathbb{R})$ of matrices whose only non-zero entries are contained in the left upper $(2,2)$-block which is the intersection of the first two lines with the first two rows.

The group $Sp(2g,\mathbb{R})$ acts on the Lie algebra $\mathfrak{sp}(2g,\mathbb{R})$ by the adjoint representation.

Lemma A.2. *The vector spaces $\mathfrak{a},\mathfrak{b}$ and the subalgebra $\mathfrak{sp}(2,\mathbb{R})$ are invariant under the adjoint representation of the subgroup $Sp(2g-2,\mathbb{R})$.*

Proof. It suffices to show that $[\mathfrak{a},\mathfrak{sp}(2g-2,\mathbb{R})]\subset\mathfrak{a}$ and that the corresponding property holds for $\mathfrak{b}$ as well. However, this is an easy standard calculation. The subalgebra $\mathfrak{sp}(2,\mathbb{R})$ commutes with $\mathfrak{sp}(2g-2,\mathbb{R})$. □

The group $SL(2,\mathbb{R})=Sp(2,\mathbb{R})$ acts from the right on Ω. Namely, the real and imaginary part of a point $x+iy\in\Omega$ define the basis of a two-dimensional symplectic subspace V of $\mathbb{R}^{2g}$. The group $SL(2,\mathbb{R})$ acts by basis transformation on this subspace, preserving the symplectic form. Furthermore, this action of $SL(2,\mathbb{R})$ fixes pointwise the symplectic complement $V^\perp$ of the symplecitc subspace V of $\mathbb{R}^{2g}$.

If we identify the Lie algebra $\mathfrak{sp}(2g,\mathbb{R})$ with the vector space of left invariant vector fields on $Sp(2g,\mathbb{R})$, then the subspace

$$\mathfrak{u}=\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{p}$$

defines a subbundle of the tangent bundle of $Sp(2g,\mathbb{R})$ which is invariant under the right action of $Sp(2g-2,\mathbb{R})$ by Lemma A.2. In particular, for any $A\in Sp(2g,\mathbb{R})$, the fibre $\mathfrak{u}_A$ of this subbundle is transverse to the kernel at A of the differential of the canonical projection $Sp(2g,\mathbb{R})\to\Omega$.

As the subbundles $\mathfrak{a},\mathfrak{b},\mathfrak{sp}(2,\mathbb{R})$ are invariant under the right action of $Sp(2g-2,\mathbb{R})$ as well, they define an $Sp(2g,\mathbb{R})$-invariant decomposition of $T\Omega$. The decomposition $\mathfrak{u}=\mathfrak{sp}(2,\mathbb{R})\oplus\mathfrak{p}$ is just the splitting

$$T\Omega=\mathcal{T}\oplus\mathcal{R}$$

where for a point $x+iy\in\Omega$ we have

$$\mathcal{R}_{x+iy}=\{u+iv\mid\omega(u,x)=\omega(u,y)=\omega(v,x)=\omega(v,y)=0\} \tag{5}$$

and where $\mathcal{T}$ is tangent to the orbits of the right action of $SL(2,\mathbb{R})$. Note that by invariance, this fact just has to be verified at a single point, and for the projection of the identity, this verification is straighforward.

It follows from the above discussion that the subbundle $\mathcal{R}$ of $T\Omega$ specified in equation (5) is invariant under the standard complex structure on $T\mathbb{C}^n$ obtained from multiplication wtih i. We next describe the complex structure on $\mathcal{R}$ using the trivialization of $\mathcal{R}$ defined by the linear subspace $\mathfrak{p}$ of $\mathfrak{sp}(2g,\mathbb{R})$. Let J be the complex structure on $\mathfrak{p}$ defined by $JA=A^t$ for $A\in\mathfrak{a}$ and $JB=-B^t$ for $B\in\mathfrak{b}$. Using $[\mathfrak{sp}(2g-2,\mathbb{R}),\mathfrak{a}]\subset\mathfrak{a}$ and $[\mathfrak{sp}(2g-2,\mathbb{R}),\mathfrak{b}]\subset\mathfrak{b}]$, an easy calculation shows that the complex structure J on $\mathfrak{p}$ is invariant under the adjoint representation of the subgroup $Sp(2g-2,\mathbb{R})$ of $Sp(2g,\mathbb{R})$. Thus it induces an $Sp(2g,\mathbb{R})$-invariant complex structure on the bundle $\mathcal{R}$ with the property that the invariant subbundles $\mathfrak{a},\mathfrak{b}$ are totally real.

From the explicit description we infer that in the above identification of $T\Omega$ with $\mathfrak{u}$, the complex structure i on $\mathbb{C}^{2g}\supset\Omega$ restricts to the complex structure J on $\mathfrak{p}$, viewed as the subbundle $\mathcal{R}$ of $T\Omega$. Furthermore, the complex structure i on $T\mathbb{C}^n$

pairs the matrix $A \in \mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{sp}(2g,\mathbb{R})$ with entries $a_{12} = 1$ and $a_{ij} = 0$ otherwise (ie the generator of the horocycle flow) with its transpose A^t.

A *CR-hypersurface* in $\mathbb{C}^n$ is a smooth real hypersurface H in $\mathbb{C}^n$ with the following property. Let $E \subset TH$ be the maximal complex subbundle of TH, ie $E = TH \cap iTH$. If θ is a smooth local one-form on H with $\theta(E) \equiv 0$, then $\theta \wedge \theta^{n-1}$ is a local volume form on H.

Let $X \in \mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{sp}(2g,\mathbb{R})$ be the matrix given by $x_{11} = 1, x_{22} = -1$ and $x_{ij} = 0$ otherwise. Then for any $0 \neq A \in \mathfrak{a}$ we have $[A, JA] = aX$ for some $a \neq 0$. The above Lie algebra computation now shows

Lemma A.3. *$\Omega \subset \mathbb{C}^{2g}$ is a CR-hypersurface: If θ is the one-form on Ω with $\theta(X) = 1$ and $\theta(T\Omega \cap iT\Omega) = 0$ then $\theta \wedge d\theta^{2g-1}$ is a volume form on Ω.*

Proof. All we need to show is that $d\theta(Y, iY) \neq 0$ for $0 \neq Y \in T\Omega \cap iT\Omega$. As the restriction of the one-form θ to an orbit of the action of the group $SL(2,\mathbb{R})$ (which acts freely on Ω) is the standard contact form on $SL(2,\mathbb{R})$, viewed as the unit tangent bundle of the hyperbolic plane, it suffices to show that $d\theta(Y, iY) \neq 0$ for all $0 \neq Y \in \mathfrak{a}$ (here as before, $\mathfrak{a}$ is viewed as a subbundle of $\mathcal{R}$).

Thus let $\mathfrak{y}$ be the local section of $T\Omega \cap iT\Omega$ obtained by the right action of the one-parameter subgroup of $Sp(2g,\mathbb{R})$ generated by the element in $\mathfrak{a}$ which projects to Y. As $[\mathfrak{y}, J\mathfrak{y}] = aX$ for some $a > 0$ we conclude that indeed $d\theta(Y, iY) = -\theta([\mathfrak{y}, J\mathfrak{y})]) < 0$. ∎

The left action of $Sp(2g,\mathbb{R})$ on the open subset $\mathcal{O}$ of $\mathbb{C}^n$ is the restriction of a linear action on $\mathbb{C}^{2g}$. Therefore the tangent bundle of $\mathcal{O}$ admits an $Sp(2g,\mathbb{R})$-invariant flat connection. Namely, $\mathcal{O}$ is a contractible subset of $\mathbb{C}^n$ and hence we can write $T\mathcal{O} = \mathcal{O} \times \mathbb{C}^{2g}$. For each $x \in \mathbb{C}^n$, declare the tangent field of the one-parameter group of translations $(y,t) \in \mathbb{C}^n \times \mathbb{R} \to y + tx$ to be parallel for this connection. This is possible as these tangent fields define the standard trivialization of $T\mathbb{C}^n$. As the complex structure i is linear, a vector field X is parallel if and only if this is true for iX.

As the action of $A \in Sp(2g,\mathbb{R})$ on $\mathbb{C}^n$ is linear, the action of A preserves the vector space of parallel vector fields and hence it perserves the connection on $\mathbb{C}^n$ for which these vector fields are parallel. This flat connection restricts to a left $Sp(2g,\mathbb{R})$-invariant flat connection ∇^{GM} on $T\mathcal{O}|\Omega$.

The bundle $T\mathcal{O}|\Omega$ splits as a sum

$$T\mathcal{O}|\Omega = T\Omega \oplus \mathbb{R}$$

where the trivial line bundle $\mathbb{R}$ is the tangent bundle of the orbits of the one-parameter group of deformations $((x+iy),t) \to e^t x + ie^t y$ transverse to Ω. Note that the trivial line bundle $\mathbb{R}$ is not invariant under the connection ∇^{GM} and hence the splitting is not flat (see also Lemma A.3). However, as Ω is the level-one hypersurface of the real analytic function $(x,y) \to \omega(x,y)$ and the exponential function is real analytic, the splitting is real analytic.

Recall that the subspace $\mathfrak{p}$ of $\mathfrak{sp}(2g,\mathbb{R})$ is invariant under the adjoint representation of the subgroup $Sp(2g-2,\mathbb{R})$, and the same holds true for the subspace $\mathfrak{sp}(2,\mathbb{R})$. Therefore the subbundles $\mathcal{R}$ and $\mathcal{T}$ of $T\Omega$ are invariant under both the left action of $Sp(2g,\mathbb{R})$ and the right action of $SL(2,\mathbb{R})$. In particular, the flat left $Sp(2g,\mathbb{R})$-invariant connection ∇^{GM} on $T\mathcal{O}|\Omega$ projects to a left $Sp(2g,\mathbb{R})$-invariant right $SL(2,\mathbb{R})$-invariant connection $\nabla^{\mathcal{R}}$ on $\mathcal{R}$ defined as follows. Let

$$P : T\mathcal{O}|\Omega = \mathcal{R}\oplus\mathcal{T}\oplus\mathbb{R}\to\mathcal{R}$$

be the canonical projection, and for $X\in T\Omega$ and a local section Y of $\mathcal{R}$ define $\nabla^{\mathcal{R}}_X Y = P\nabla^{GM}_X(Y)$. We summarize this in the following lemma.

Lemma A.4. *The flat left $Sp(2g,\mathbb{R})$-invariant connection on $T\mathcal{O}$ projects to a connection $\nabla^{\mathcal{R}}$ on $\mathcal{R}$ which is invariant under both the left $Sp(2g,\mathbb{R})$ action and the right $SL(2,\mathbb{R})$ action.*

The curvature of the connection $\nabla^{\mathcal{R}}$ is a two-form on Ω with values in the Lie algebra $\mathfrak{sp}(2g-2,\mathbb{R})$ of $Sp(2g-2,\mathbb{R})$, acting as an algebra of transformation on $\mathcal{R}$. The restriction of this two-form to the tangent bundle of the orbits of the $SL(2,\mathbb{R})$-action vanishes. Namely, each such orbit $\mathcal{F}$ is contained in an invariant linear subspace $\mathbb{C}^2\subset\mathbb{C}^n$, and the splitting $T\mathbb{C}^n|\mathcal{F} = T\mathbb{C}^2\oplus\mathcal{R}|\mathcal{F}$ is parallel for ∇GM. Moreover, the two-form is equivariant with respect to the left action of $Sp(2g,\mathbb{R})$ and the right action of $SL(2,\mathbb{R})$.

We say that the curvature form Θ for a connection ∇ on a complex vector bundle $E\to M$ *is reducible over* $\mathbb{C}$ if there is a nontrivial Θ-invariant decomposition $E = E_1\oplus E_2$ as a Whitney sum of two complex vector bundles. This means that for any $x\in M$ and any two vectors $Y,Z\in T_xM$ the map $\Theta(Y,Z)$ preserves the decomposition $E = E_1\oplus E_2$. Note that in this definition, there is no requirement that the bundles E_i are holomorphic, even if the full bundle E is holomorphic. We do not insist in relating this notion of reducibility to properties of the holonomy group of the connection. Third, the property captured in the definition is an infinitesimal property of the connection rather than a global property of the bundle. The bundles we are considering here do admit nontrivial splittings as complex vector bundles. A curvature form which is not reducible over $\mathbb{C}$ is called *irreducible over* $\mathbb{C}$.

Since Ω is not locally affine, the curvature form of the connection $\nabla^{\mathcal{R}}$ on $\mathcal{R}$ does not vanish identically. Now recall that the stabilizer in $Sp(2g,\mathbb{R})$ of a point $z\in\Omega$ can be identified with the subgroup $Sp(2g-2,\mathbb{R})$, which acts on the fibre $\mathcal{R}_z$ of $\mathcal{R}$ at z (which is a $2g-2$-dimensional complex subspace of $T_z\mathbb{C}^n$) via the standard representation of $Sp(2g-2,\mathbb{R})$ on $\mathbb{C}^{2g-2}$, which is the complex linear extension of the standard representation on $\mathbb{R}^{2g-2}$. Since the standard representation of $Sp(2g-2,\mathbb{R})$ on the complex vector space $\mathbb{C}^{2g-2}$ is irreducible and since the curvature form of $\nabla^{\mathcal{R}}$ is equivariant with respect to the left action of $Sp(2g,\mathbb{R})$, by equivariance we have the following analog of Lemma A.3.

Lemma A.5. *The curvature form of $\nabla^{\mathcal{R}}$ is irreducible over $\mathbb{C}$.*

Proof. For each $z\in\Omega$, the curvature form Θ of the connection $\nabla^{\mathcal{R}}$ does not vanish. In particular, there exist tangent vectors $X,Y\in T_z\Omega$ such that $\Theta(X,Y)$

is a non-trivial element $A \in \mathfrak{sp}(2g-2,\mathbb{R})$. By invariance, the sub-vector space $V = \{\Theta(Z,W) \mid Z,W \in T_z\Omega\}$ of $\mathfrak{sp}(2g-2,\mathbb{R})$ contains the orbit of A under the adjoint representation of $Sp(2g-2,\mathbb{R})$. As the Lie group $Sp(2g-2,\mathbb{R})$ is simple, the adjoint representation of $Sp(2g-2,\mathbb{R})$ is irreducible. Thus we have $V = \mathfrak{sp}(2g-2,\mathbb{R})$. On the other hand, the standard representation of $\mathfrak{sp}(2g-2,\mathbb{R})$ on $\mathbb{R}^{2g-2}$ is irreducible. This yields the lemma. □

Remark A.6. Lemma A.5 is a statement about the connection $\nabla^{\mathcal{R}}$ and not a statement about the complex vector bundle $\mathcal{R}$ (which can easily seen to split as a sum of complex vector bundles if $g \geq 3$).

In the remainder of this appendix we give variation of these viewpoints which is less directly used in the proofs of the main results of this work, but which is better adapted to the understanding of the analog of the absolute period foliation on the bundle $\mathcal{S}$. To this end note that the complement of the zero section $\tilde{\mathcal{V}}_+ \subset \tilde{\mathcal{V}}$ of the bundle $\tilde{\mathcal{V}} \to \mathcal{D}_g$ is a complex manifold. The fibration $\mathcal{S} \to \Omega$ extends to a holomorphic fibration $\tilde{\mathcal{V}}_+ \to \mathcal{O}$ of complex manifolds. The fibres of the fibration define a foliation $\mathcal{U}$ of $\tilde{\mathcal{V}}_+$.

Recall that a foliation $\mathcal{U}$ of a complex manifold M is *holomorphic* if every point in M admits a neighborhood U so that there is a holomorphic map $f : U \to \mathbb{C}^p$ for some $p \geq 1$ such that local leaves of $\mathcal{U}$ are preimages under f of points in $\mathbb{C}^p$. The following is immediate from the definition of the complex structure on $\mathcal{V}_+$ and on $\mathcal{O} \subset \mathbb{C}^{2g}$.

Lemma A.7. *The foliation $\mathcal{U}$ is holomorphic. A leaf is biholomorphic to $\mathfrak{D}_{g-1}$.*

Proof. By equivariance, all we need to rule out is that the curvature of $\nabla^{\mathcal{R}}$ vanishes identically. That this is not the case was observed above. □

The foliation $\mathcal{U}$ on $\tilde{\mathcal{V}}_+$ can be viewed as the analog of the absolute period foliation on the bundle $\mathcal{H}_+ \to \mathcal{M}_g$ which is the pull-back of the bundle $\mathcal{V}_+ = Sp(2g,\mathbb{Z})\backslash\tilde{\mathcal{V}}_+$ by the Torelli map. Recall that the restriction of the absolute period foliation to any component of a stratum has a complex affine and hence a complex structure. However, this affine structure is singular at the boundary points of the strata (which are contained in lower dimensional strata).

Let for the moment G be an arbitrary Lie group. A *G-connection* for a G-principal bundle $X \to Y$ is given by an $\mathrm{Ad}(G)$-invariant subbundle of the tangent bundle of X which is transverse to the tangent bundle of the fibres. Such a bundle is called *horizontal*.

The following observation contrasts the case of the absolute period foliation on $\mathcal{H}_+$ and reflects the fact that the right $SL(2,\mathbb{R})$-action on the bundle $\mathcal{S}$ does not pull back to the $SL(2,\mathbb{R})$-action on $\mathcal{H}_+$. Namely, orbits of the $SL(2,\mathbb{R})$-action on $\mathcal{H}_+$ define *orientable Teichmüller curves* which are mapped by the Torelli map to geodesics in $\mathfrak{D}_g$ for the Kobayashi metric. However, these Kobayashi geodesics are in general not totally geodesic for the symmetric metric. We refer to [BM14] for more and for references.

The group $Sp(2g,\mathbb{R})$ is an $Sp(2g-2,\mathbb{R})$-principal bundle over Ω. In the statement of the following Lemma, the type $(2g,2g-1)$ stems from the fact that Ω is a hypersurface in the manifold $\mathcal{O}$ with invariant indefinite metric of type $(2g,2g)$.

Lemma A.8. *The $Sp(2g-2,\mathbb{R})$-principal bundle $Sp(2g,\mathbb{R})\to\Omega$ admits a natural real analytic $Sp(2g-2,\mathbb{R})$-connection which is invariant under the left action of $Sp(2g,\mathbb{R})$ and the right action of $SL(2,\mathbb{R})$. The horizontal bundle $\mathcal{Z}_0$ contains the tangent bundle $\mathcal{T}$ of the orbits of the $SL(2,\mathbb{R})$-action, and it admits an $SL(2,\mathbb{R})$-invariant $Sp(2g,\mathbb{R})$-invariant pseudo-Riemannian metric h of type $(2g,2g-1)$. The h-orthogonal complement $\mathcal{Y}_0$ of $\mathcal{T}$ in $\mathcal{Z}_0$ is a real analytic $SL(2,\mathbb{R})$-invariant $Sp(2g,\mathbb{R})$invariant bundle.*

Proof. The fibre containing the identity induces an embedding of Lie algebras

$$\mathfrak{sp}(2g-2,\mathbb{R})\to\mathfrak{sp}(2g,\mathbb{R}).$$

The restriction of the Killing form B of $\mathfrak{sp}(2g,\mathbb{R})$ to the Lie algebra $\mathfrak{sp}(2g-2,\mathbb{R})$ is non-degenerate. Thus the B-orthogonal complement $\mathfrak{z}$ of $\mathfrak{sp}(2g-2,\mathbb{R})$ is a linear subspace of $\mathfrak{sp}(2g,\mathbb{R})$ which is complementary to $\mathfrak{sp}(2g-2,\mathbb{R})$ and invariant under the restriction of the adjoint representation Ad of $Sp(2g,\mathbb{R})$ to $Sp(2g-2,\mathbb{R})$. The restriction to $\mathfrak{z}$ of the Killing form is a non-degenerate bilinear form of type $(2g,2g-1)$.

The group $Sp(2g,\mathbb{R})$ acts by left translation on itself, and this action commutes with the right action of $Sp(2g-2,\mathbb{R})$. Hence $Sp(2g,\mathbb{R})$ acts as a group of automorphisms on the principal bundle $Sp(2g,\mathbb{R})\to\Omega$.

Define a $\mathfrak{sp}(2g-2,\mathbb{R})$-valued one-form θ on $Sp(2g,\mathbb{R})$ by requiring that $\theta(e)$ equals the canonical projection

$$T_eSp(2g,\mathbb{R})=\mathfrak{z}\oplus\mathfrak{sp}(2g-2,\mathbb{R})\to\mathfrak{sp}(2g-2,\mathbb{R})$$

and

$$\theta(g)=\theta\circ dg^{-1}.$$

Then for every $h\in Sp(2g-2,\mathbb{R})$ we have

$$\theta(gh)=\mathrm{Ad}(h^{-1})\circ\theta(g)$$

and hence this defines an $Sp(2g,\mathbb{R})$-invariant connection on the $Sp(2g-2,\mathbb{R})$-principal bundle $Sp(2g,\mathbb{R})\to\Omega$. Denote by $\mathcal{Z}_0$ the horizontal bundle. It is invariant under the left action of $Sp(2g,\mathbb{R})$ and the right action of $Sp(2g-2,\mathbb{R})$, and it is equipped with an invariant pseudo-Riemannian metric of type $(2g,2g-1)$.

Now $\mathfrak{sp}(2,\mathbb{R})\subset\mathfrak{z}$, and hence the tangent bundle for the right action of $Sp(2,\mathbb{R})$ is contained in the horizontal bundle $\mathcal{Z}_0$. Thus the subbundle $\mathcal{Y}_0$ of $\mathcal{Z}_0$ defined by the B-orthogonal complement $\mathfrak{y}$ in $\mathfrak{z}$ of the Lie algebra $\mathfrak{sp}(2,\mathbb{R})$ is invariant as well. The lemma follows. □

Since $\mathcal{S}=Sp(2g,\mathbb{R})\times_{Sp(2g-2,\mathbb{R})}\mathfrak{D}_{g-1}$ and since the subgroups $SL(2,\mathbb{R})$ and $Sp(2g-2,\mathbb{R})$ commute, the right action of $SL(2,\mathbb{R})$ on $Sp(2g,\mathbb{R})$ descend to an action of $SL(2,\mathbb{R})$ on $\mathcal{S}$. The action of the unitary subgroup $U(1)$ of $Sp(2,\mathbb{R})$ is just the standard circle action on the fibres of the sphere bundle $\mathcal{S}\to\mathfrak{D}_g$ given by multiplication with complex numbers of absolute value one. The connection $\mathcal{Z}_0 = \mathcal{T} \oplus \mathcal{Y}_0$ induces a real analytic splitting

$$T\mathcal{S} = T\mathcal{U} \oplus \mathcal{Z} = T\mathcal{U} \oplus \mathcal{T} \oplus \mathcal{Y}$$

where $T\mathcal{U}$ denotes the tangent bundles of the fibres of the fibration $\mathcal{S} \to \Omega$, the *horizontal bundle* $\mathcal{Z}$ is the image of $\mathcal{Z}_0 \times T\mathfrak{D}_{g-1}$ under the projection $Sp(2g, \mathbb{R}) \times \mathfrak{D}_{g-1} \to \mathcal{S}$ and as before, $\mathcal{T}$ is the tangent bundle of the orbits of the $SL(2, \mathbb{R})$-action.

Lemma A.9. *The right action of $SL(2, \mathbb{R})$ on $\mathcal{S}$ projects to the standard action of $SL(2, \mathbb{R})$ on Ω.*

Proof. This follows as before from naturality and bi-invariance of the Killing form. □

The group $Sp(2g, \mathbb{Z})$ acts properly discontinuously from the left on the bundle $\mathcal{S} \to \Omega$ as a group of real analytic bundle automorphisms. In particular, it preserves the real analytic splitting of the tangent bundle of $\mathcal{S}$ into the tangent bundle of the leaves of the foliation $\mathcal{U}$ and the complementary bundle. Thus this splitting descends to an $SL(2, \mathbb{R})$-invariant real analytic splitting of the tangent bundle of the quotient. This quotient is just the sphere bundle of the quotient vector bundle (in the orbifold sense) over the locally symmetric space

$$\mathcal{A}_g = Sp(2g, \mathbb{Z}) \backslash Sp(2g, \mathbb{R}) / U(g).$$

References

[Ap15] P. Apisa, *$GL_2\mathbb{R}$ orbit closures in hyperelliptic components of strata*, arXiv:1508.05438.

[ABEM12] J. Athreya, A. Bufetov, A. Eskin, M. Mirzakhani, *Lattice point asymptotic and volume growth on Teichmüller space*, Duke Math. J. 161 (2012), 1055–1111.

[Au15] D. Aulicino, *Affine manifolds and zero Lyapunov exponents in genus 3*, Geom. Func. Anal. 25 (2015), 1333–1370.

[AEM12] A. Avila, A. Eskin, and M. Möller, *Symplectic and isometric $SL(2, \mathbb{R})$-invariant subbundles of the Hodge bundle*, arXiv:1209.2854, to appear in J. reine and angew. Math.

[AMY16] A. Avila, C. Mattheus, J.C. Yoccoz, *Zorich conjecture for hyperelliptic Rauzy-Veech groups*, arXiv:1606.01227.

[AV07] A. Avila, M. Viana, *Simplicity of Lyapunov spectra: Proof of the Kontsevich-Zorich conjeccture*, Acta Math. 198 (2007), 1–56.

[BHM14] M. Bainbridge, P. Habegger, M. Möller, *Teichmüller curves in genus three and just likely intersections in $G_m^n \times G_a^n$*, Publ. Math. Inst. Hautes Études Sci. 124 (2016), 1–98.

[BM14] M. Bainbridge, M. Möller, *The locus of real multiplication and the Schottky locus*, J. Reine Angew. Math. 686 (2014), 167–186.

[Be97] Y. Benoist, *Proprietes asymptotiques des groupes lineaires*, Geom. Funct. Anal. 7 (1997), 1–47.

[Bw73] R. Bowen, *Symbolic dynamics for hyperbolic flows*, Amer. J. Math. 95 (1973), 429–460.

[Ca04] K. Calta, *Veech surfaces and complete periodicity in genus 2*, J. Amer. Math. Soc. 17 (2004), 871–908.

[EFW17] A. Eskin, S. Filip, A. Wright, *The algebraic hull of the Kontsevich-Zorich cocycle*, arXiv:1702.02074.

[EM11] A. Eskin, M. Mirzakhani, *Counting closed geodesics in moduli space*, J. Mod. Dynamics 5 (2011), 71–105.

[EMR12] A. Eskin, M. Mirzakhani, K. Rafi, *Counting closed geodesics in strata*, arXiv:1206.5574.

[EMM15] A. Eskin, M. Mirzakhani, A. Mohammadi, *Isolation theorems for $SL(2,\mathbb{R})$-invariant submanifolds in moduli space*, Ann. Math. 182 (2015), 673-721.

[FM12] B. Farb, D. Margalit, *A primer on mapping class groups*, Princeton Univ. Press, Princeton 2012.

[F16] S. Filip, *Splitting mixed Hodge structures over affine invariant manifolds*, Ann. of Math. 183 (2016), 681–713,

[F14] S.Filip, *Zero Lyapunov exponents and monodromy of the Kontsevich Zorich cocycle*, arXiv:1402.2129.

[G12] S. Geninska, *Examples of infinite covolume subgroups of $PSL(2,\mathbb{R})^r$ with big limit sets*, Math. Zeit. 272 (2012), 389–404.

[GH78] P. Griffith, J. Harris, *Principles of algebraic geometry*, John Wiley & Sons, 1978.

[GR17] R. Guttierez-Romo, *Zariski density of the Rauzy-Veech group: proof of the Zorich conjecture*, arXiv:1706.04923.

[Hl08] C. Hall, *Big symplectic or orthogonal monodromy modulo ℓ*, Duke Math. J. 141 (2008), 179–203.

[H10] U. Hamenstädt, *Dynamics of the Teichmüller flow on compact invariant sets*, J. Mod. Dynamics 4 (2010), 393–418.

[H13] U. Hamenstädt, *Bowen's construction for the Teichmüller flow*, J. Mod. Dynamics 7 (2013), 489–526.

[H15] U. Hamenstädt, *Dynamical properties of the absolute period foliation*, arXiv:1511.08055, to appear in Israel J. Math.

[H16] U. Hamenstädt, *Counting periodic orbits in the thin part of strata*, preprint, January 2016.

[H18] U. Hamenstädt, *Typical properties of periodic Teichḿuller geodesics: Lyapunov exponents*, preprint, April 2018.

[H18b] U. Hamenstädt, *Typical properties of periodic Teichmüller geodesics: Stretch factors*, preprint, July 2018.

[KS00] R. Kenyon, J. Smillie, *Billiards on rational-angled triangles*, Comm. Math. Helv. 75 (2000), 65–108.

[KN63] S. Kobayashi, K. Nomizu, *Foundationsn of differential geometry*, Wiley Interscience 1963.

[KZ03] M. Kontsevich, A. Zorich, *Connected components of the moduli space of Abelian differentials with prescribed singularities*, Invent. Math. 153 (2003), 631–678.

[L08] E. Lanneau, *Connected components of the strata of the moduli space of quadratic differentials*, Ann. Sci. Ec. Norm. Sup. 41 (2008), 1–56.

[LNW15] E. Lanneau, D.-M. Nguyen, A. Wright, *Finiteness of Teichmüller curves in non-arithmetic rank 1 orbit closures*, arXiv:1504.03742, to appear in Amer. J. Math.

[LM14] E. Looijenga, G. Mondello, *The fine structure of the moduli space of abelian differentials in genus 3*, Geom. Dedicata 169 (2014), 109–128.

[Lu99] A. Lubotzky, *One for almost all: generation of SL(n,p) by subsets of SL(n,Z)*, in "Algebra, K-theory, groups and education", T. Y. Lam and A. R. Magid, Editors, Contemp. Math. 243 (1999).

[Ma04] G.A. Margulis, *On some aspects of the theory of Anosov systems*, Springer monographs in math., Springer Berlin Heidelberg New York 2004.

[M82] H. Masur, *Interval exchange transformations and measured foliations*, Ann. Math. 115 (1982), 169–200.

[MW15] C. Matheus, A. Wright, *Hodge-Teichmüller planes and finiteness results for Teichmüller curves*, Duke Math. J. 164 (2015), 1041–1077.

[McM03a] C. McMullen, *Teichmüller geodesics of infinite complexity*, Acta Math. 191 (2003), 191–223.

[McM03b] C. McMullen, *Billiards and Teichmüller curves on Hilbert modular surfaces*, J. Amer. Math. Soc. 16 (2003), 857–885.

[McM13] C. McMullen, *Navigating moduli space with complex twists*, J. Eur. Math. Soc. 5 (2013), 1223–1243.

[MMW16] C. McMullen, R. Mukamel, A. Wright, *Cubic curves and totally geodesic subvarieties of moduli space*, preprint 2016.

[MW15] M. Mirzakhani, A. Wright, *The boundary of an affine invariant submanifold*, arXiv:1508.01446.

[MW16] M. Mirzakhani, A. Wright, *Full rank affine invariant submanifolds.* arXiv:1608.02147.

[Mo06] M. Möller, *Variations of Hodge structures of Teichmüller curves*, J. Amer. Math. Soc. 19 (2006), 327–344.

[O68] V.I. Oseledets, *A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc. 19 (1968), 197–231.

[R08] I. Rivin, *Walks on groups, counting reducible matrices, polynomials, and surface and free group automorphisms*, Duke Math. J. 142 (2008), 353–379.

[St15] B. Strenner, *Algebraic degrees of pseudo-Anosov stretch factors*, arXiv:1506.06412.

[Th88] W. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. 19 (1988), 417–431.

[V86] W. Veech, *The Teichmüller geodesic flow*, Ann. Math. 124 (1990), 441–530.

[VV02] P. Viana, P. M. Veloso, *Galois theory of reciprocal polynomials*, Amer. Math. Monthly 109 (2002), 466–471.

[W14] A. Wright, *The field of definition of affine invariant submanifolds of the moduli space of abelian differentials.* Geom. Top. 18 (2014), 1323–1341.

[W15] A. Wright, *Cylinder deformations in orbit closures of translation surfaces*, Geom. Top. 19 (2015), 413–438.

[Z99] A. Zorich, *How do leaves of a closed 1-form wind around a surface?*, Pseudoperiodic topology, 135–178, Amer. Math. Soc. Transl. Ser. 2, 197, Amer. Math. Soc., Providence, RI, 1999.

MATHEMATISCHES INSTITUT DER UNIVERSITÄT BONN
ENDENICHER ALLEE 60, 53115 BONN, GERMANY
e-mail: ursula@math.uni-bonn.de