

# INVARIANT MEASURES FOR THE WEIL-PETERSSON FLOW

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ABSTRACT. Let  $\mathcal{T}(S)$  be the Teichmüller space of an oriented surface  $S$  of genus  $g \geq 0$  with  $m \geq 0$  punctures and  $3g - 3 + m \geq 2$ . Let  $\mathcal{Q}(S)$  and  $\mathcal{Q}_{WP}(S)$  be the moduli space of area one quadratic differentials and of quadratic differentials of unit norm for the Weil-Petersson metric, respectively. We show that there is a continuous injection of the space of all Borel probability measures on  $\mathcal{Q}(S)$  which are invariant under the Teichmüller flow into the space of all Borel probability measures on  $\mathcal{Q}_{WP}(S)$  which are invariant under the geodesic flow for the Weil-Petersson metric. We also give an alternative proof of the following result of Brock, Masur and Minsky: For every  $\epsilon > 0$ , a Weil-Petersson geodesic in  $\mathcal{T}(S)$  which is entirely contained in the  $\epsilon$ -thick part of  $\mathcal{T}(S)$  is uniformly fellow-traveled by a Teichmüller geodesic. Vice versa, a Teichmüller geodesic which is entirely contained in the  $\epsilon$ -thick part of Teichmüller space is uniformly fellow-traveled by a Weil-Petersson geodesic.

## 1. INTRODUCTION

Let  $S$  be an oriented surface of finite type, i.e.  $S$  is a closed surface of genus  $g \geq 0$  from which  $m \geq 0$  points, so-called *punctures*, have been deleted. We assume that  $3g - 3 + m \geq 2$ , i.e. that  $S$  is not a sphere with at most four punctures or a torus with at most one puncture. We then call the surface  $S$  *nonexceptional*.

Since the Euler characteristic of  $S$  is negative, the *Teichmüller space*  $\mathcal{T}(S)$  of  $S$  is the quotient of the space of all complete hyperbolic metrics on  $S$  of finite volume under the action of the group of diffeomorphisms of  $S$  which are isotopic to the identity. The *mapping class group*  $\text{Mod}(S)$  of all isotopy classes of orientation preserving diffeomorphisms of  $S$  acts properly discontinuously on  $\mathcal{T}(S)$ . For every  $\epsilon > 0$ , this action preserves the subset  $\mathcal{T}(S)_\epsilon$  of  $\mathcal{T}(S)$  of all hyperbolic metrics whose *systole*, i.e. the shortest length of a closed geodesic, is at least  $\epsilon$ . Moreover, the action of  $\text{Mod}(S)$  on  $\mathcal{T}(S)_\epsilon$  is cocompact.

The *Teichmüller metric* is a complete  $\text{Mod}(S)$ -invariant Finsler metric on  $\mathcal{T}(S)$  with the property that any two points can be connected by a unique geodesic. We call a geodesic for the Teichmüller metric a *Teichmüller geodesic*.

The cotangent bundle of Teichmüller space is the bundle of quadratic differentials. The unit sphere subbundle for the Teichmüller metric is the bundle  $\mathcal{Q}^1(S)$  of area one quadratic differentials. The Teichmüller metric defines a *geodesic*

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flow on  $\mathcal{Q}^1(S)$  which is equivariant with respect to the action of the mapping class group and hence projects to a (continuous) flow  $\Phi_{\mathcal{T}}^t$  on the moduli space  $\mathcal{Q}(S) = \mathcal{Q}^1(S)/\text{Mod}(S)$  of area one quadratic differentials.

Even though  $\mathcal{Q}(S)$  is non-compact, it admits many  $\Phi_{\mathcal{T}}^t$ -invariant Borel probability measures. For example, there are countably many periodic orbits for the Teichmüller flow on  $\mathcal{Q}(S)$ , and each such orbit supports a natural invariant Borel probability measure. However, for every compact set  $K \subset \mathcal{Q}(S)$  there are periodic orbits which do not intersect  $K$  [H06b] and hence the space  $\mathcal{M}_{\mathcal{T}}(\mathcal{Q}(S))$  of all  $\Phi_{\mathcal{T}}^t$ -invariant Borel probability measures on  $\mathcal{Q}(S)$  equipped with the weak\*-topology is non-compact: There are sequences of such measures supported on periodic orbits which converge weakly to the trivial measure of vanishing total mass.

The *Weil-Petersson metric* is a  $\text{Mod}(S)$ -invariant Kähler metric on  $\mathcal{T}(S)$  of negative sectional curvature which induces an incomplete distance  $d_{WP}$ . The completion  $\overline{\mathcal{T}(S)}$  of  $\mathcal{T}(S)$  with respect to  $d_{WP}$  is a  $\text{Cat}(0)$ -space. In particular, any two points in  $\overline{\mathcal{T}(S)}$  can be connected by a unique geodesic. Such a geodesic for the Weil-Petersson metric will be called a *Weil-Petersson geodesic* in the sequel. Weil-Petersson geodesic arcs with both endpoints in  $\mathcal{T}(S)$  are entirely contained in  $\mathcal{T}(S)$  [DW03]. However,  $\overline{\mathcal{T}(S)}$  is not locally compact.

The Weil-Petersson metric defines an (incomplete) geodesic flow on the space of all quadratic differentials of Weil-Petersson norm one which projects to a continuous (but incomplete) flow  $\Phi_{WP}^t$  on the moduli space  $\mathcal{Q}_{WP}(S)$  of quadratic differentials of Weil-Petersson norm one. There is a bijection between periodic orbits in  $\mathcal{Q}(S)$  for the Teichmüller flow and periodic orbits in  $\mathcal{Q}_{WP}(S)$  for the Weil-Petersson geodesic flow. Our main goal is to show that this bijection extends to a continuous injection of the space of  $\Phi_{\mathcal{T}}^t$ -invariant Borel probability measures on  $\mathcal{Q}(S)$  into the space  $\mathcal{M}_{WP}(\mathcal{Q}_{WP}(S))$  of  $\Phi_{WP}^t$ -invariant Borel probability measures on  $\mathcal{Q}_{WP}(S)$ .

For a measure  $\mu \in \mathcal{M}_{\mathcal{T}}(\mathcal{Q}(S))$  (or  $\nu \in \mathcal{M}_{WP}(\mathcal{Q}_{WP}(S))$ ) let  $h(\mu)$  be the *entropy* of  $\mu$  (or of  $\nu$ ). We show.

**Theorem 1.** *There is an injective continuous map*

$$\Theta : \mathcal{M}_{\mathcal{T}}(\mathcal{Q}(S)) \rightarrow \mathcal{M}_{WP}(\mathcal{Q}_{WP}(S)).$$

Moreover, there is a number  $c > 0$  such that

$$h(\Theta(\mu)) \geq ch(\mu) \text{ for all } \mu \in \mathcal{M}_{\mathcal{T}}(\mathcal{Q}(S)).$$

We do not know whether the map  $\Theta$  is surjective.

The entropy of a measure  $\mu \in \mathcal{M}_{\mathcal{T}}(\mathcal{Q}(S))$  is at most  $6g - 6 + 2m$  [H07b]. In contrast, Brock, Masur and Minsky constructed invariant Borel probability measures for the Weil-Petersson geodesic flow whose entropy is arbitrarily large [BMM08b]. Even though it is unclear whether these measures are contained in the image of the map  $\Theta$ , this result suggests that for  $\mu \in \mathcal{M}_{\mathcal{T}}(\mathcal{Q}(S))$  the entropy of  $\Theta(\mu)$  is not bounded from above by a fixed multiple of the entropy of  $\mu$ .

The second goal of this paper is to compare geodesics for the Teichmüller metric and the Weil-Petersson metric which are entirely contained in the thick part

of Teichmüller space. Note that by invariance under the action of the mapping class group and cocompactness, for every  $\epsilon > 0$  the restrictions to  $\mathcal{T}(S)_\epsilon$  of the Teichmüller metric and the Weil-Petersson metric are locally uniformly bilipschitz equivalent. However, on the large scale, the restrictions to  $\mathcal{T}(S)_\epsilon$  of the distances  $d_{\mathcal{T}}$  and  $d_{WP}$  induced by the Teichmüller metric and the Weil-Petersson metric, respectively, are not bilipschitz equivalent. For example, there is a sequence of points  $\{x_i\} \subset \mathcal{T}(S)_\epsilon$  and a number  $c > 0$  such that  $d_{\mathcal{T}}(x_0, x_i) \rightarrow \infty$  and  $d_{WP}(x_0, x_i) \leq c$ . The Teichmüller geodesics connecting  $x_0$  to  $x_i$  enter arbitrarily deeply into the thin part of Teichmüller space (see [Mi96, W08]).

In [BMM08b], Brock, Masur and Minsky show that on the other hand, if  $x, y \in \mathcal{T}(S)_\epsilon$  are such that either the connecting Teichmüller geodesic is entirely contained in  $\mathcal{T}(S)_\epsilon$  or that this holds true for the connecting Weil-Petersson geodesic, then the distances  $d_{\mathcal{T}}(x, y)$  and  $d_{WP}(x, y)$  are uniformly equivalent, and the Teichmüller geodesic and the Weil-Petersson geodesic connecting  $x$  to  $y$  are uniform fellow travelers. In fact, much more is true: As in the case of a Teichmüller geodesic, a simple closed curve which becomes short along a bi-infinite Weil-Petersson geodesic  $\gamma$  can be detected combinatorially.

We give an alternative proof of a partial result in this direction. For a formulation, we use the distance  $d_{\mathcal{T}}$  on  $\mathcal{T}(S)$  induced by the Teichmüller metric to define the Hausdorff-distance  $d_H(A, B) \in [0, \infty]$  between two subsets  $A, B$  of  $\mathcal{T}(S)$  as the infimum of all numbers  $r > 0$  such that  $A$  is contained in the  $r$ -neighborhood of  $B$  and  $B$  is contained in the  $r$ -neighborhood of  $A$ .

**Theorem 2.** *For every  $\epsilon > 0$  there is a number  $R = R(\epsilon) > 0$  with the following property.*

- (1) *Let  $J \subset \mathbb{R}$  be a closed connected set and let  $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$  be a Teichmüller geodesic. Then there is a closed connected set  $J' \subset \mathbb{R}$  and there is a Weil-Petersson geodesic  $\xi : J' \rightarrow \mathcal{T}(S)$  with  $d_H(\gamma(J), \xi(J')) \leq R$ .*
- (2) *Let  $J \subset \mathbb{R}$  be a closed connected set and let  $\xi : J \rightarrow \mathcal{T}(S)_\epsilon$  be a Weil-Petersson geodesic. Then there is a closed connected set  $J' \subset \mathbb{R}$  and there is a Teichmüller geodesic  $\gamma : J' \rightarrow \mathcal{T}(S)$  with  $d_H(\xi(J), \gamma(J')) \leq R$ .*

There are two more well known families of lines in Teichmüller space, namely the *lines of minima* introduced by Kerckhoff [K92] and the geodesics for Thurston's *minimal stretch metric*. The strategy for the proof of Theorem 2 can also be used to show that a Teichmüller geodesic segment or line which is entirely contained in  $\mathcal{T}(S)_\epsilon$  is uniformly fellow traveled by a line of minima or a geodesic for Thurston's minimal stretch metric. In both cases, however, stronger results are available in the literature. Namely, Choi, Rafi and Series [CRS06] established a precise quantitative comparison between the lines of minima and Teichmüller geodesics. Choi and Rafi [CR07] showed that for every  $\epsilon > 0$  the distance function on  $\mathcal{T}(S)_\epsilon$  which is the restriction of the (symmetrized) minimal stretch metric and the Teichmüller metric are bilipschitz equivalent. They also describe the symmetrized minimal stretch metric in the thin part of Teichmüller space. The asymmetric Thurston metric was investigated by Papadopoulos and Theret [PT07].

**Definition.** A *conjugacy* of a continuous flow  $\Psi^t$  on a topological space  $X$  into a continuous flow  $\Theta^t$  on a topological space  $Y$  is an injective continuous map  $\Lambda : X \rightarrow Y$  such that there is a continuous function  $\rho : X \times \mathbb{R} \rightarrow \mathbb{R}$  with the following properties.

- (1)  $\rho(x, 0) = 0$  for all  $x \in X$ .
- (2) For each fixed  $x \in X$  the function  $\rho(x, \cdot) : s \rightarrow \rho(x, s)$  is a homeomorphism.
- (3)  $\Lambda(\Psi^t x) = \Theta^{\rho(x, t)} \Lambda(x)$  for all  $x \in X, t \in \mathbb{R}$ .

From Theorem 2 we deduce the following corollary.

**Corollary 3.** (1) *Let  $K \subset \mathcal{Q}(S)$  be any compact set which is invariant under the Teichmüller geodesic flow  $\Phi_{\mathcal{T}}^t$ . Then there is a conjugacy  $\Lambda : K \rightarrow \mathcal{Q}_{\text{WP}}(S)$  of the restriction of  $\Phi_{\mathcal{T}}^t$  to  $K$  into the geodesic flow  $\Phi_{\text{WP}}^t$  of the Weil-Petersson metric.*

(2) *Let  $C \subset \mathcal{Q}_{\text{WP}}(S)$  be a compact set which is invariant under the geodesic flow  $\Phi_{\text{WP}}^t$  for the Weil-Petersson metric. Then there is a conjugacy  $\Psi : C \rightarrow \mathcal{Q}(S)$  of the restriction of  $\Phi_{\text{WP}}^t$  to  $C$  into the Teichmüller geodesic flow.*

The proof of both parts of Theorem 2 relies on the same type of compactness argument which goes back to an idea of Mosher [Mo03]. The main task is to set up a framework to which this idea can be applied.

The proof of the second part of Theorem 2 in Section 2 uses standard properties of Teichmüller geodesics and some recent results of Brock, Masur and Minsky [BMM08a] on the Weil-Petersson geometry.

The proof of the first part of Theorem 2 is much more involved. We begin with the fairly easy observation that given  $\epsilon > 0$  and a Teichmüller geodesic  $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)_\epsilon$ , there are infinite Weil-Petersson geodesic rays  $\gamma_+, \gamma_- : [0, \infty) \rightarrow \mathcal{T}(S)$  which are uniform limits on compact sets of Weil-Petersson geodesic segments connecting  $\gamma(0)$  to points  $\gamma(T_i), \gamma(-R_i)$  for suitably chosen sequences  $T_i \rightarrow \infty, R_i \rightarrow \infty$ . Since the completion of Teichmüller space with respect to the Weil-Petersson metric is a CAT(0) geodesic metric space, the theorem now predicts the existence of a biinfinite Weil-Petersson geodesic which is forward *asymptotic* to  $\gamma_+$  and backward asymptotic to  $\gamma_-$ . However, the curvature of the Weil-Petersson metric is not bounded from above by a negative constant, and the existence of such a geodesic is not immediate.

In Section 3, with an argument based on ruled surfaces and angle comparison we derive a sufficient condition for the existence of a biinfinite Weil-Petersson geodesic which is forward and backward asymptotic to given Weil-Petersson geodesic rays. Roughly speaking, this condition is satisfied if the geodesic rays spend a sufficient amount of time in the thick part of Teichmüller space, and it is directly inspired by the work of Brock, Masur and Minsky (Section 4 of [BMM08a]).

In Section 4 we follow [BMM08b] and use Wolpert's description of the Weil-Petersson metric in the thin part of Teichmüller space [W03] to find a sufficient condition for a Weil-Petersson geodesic segment of uniformly bounded length to

pass through the thick part of Teichmüller space. This condition is a consequence of a quantitative version of the following result of Wolpert [W03]: If  $\zeta : [0, R] \rightarrow \mathcal{T}(S)$  is any Weil-Petersson geodesic of uniformly bounded length and if there is a simple closed curve  $\alpha$  which is long at both endpoints of  $\zeta$  and becomes very short along  $\zeta$ , then  $\zeta$  twists about  $\alpha$  or a curve  $\beta$  which is disjoint from  $\alpha$ . Together with some consequences of hyperbolicity of the curve graph and some of the combinatorial tools introduced by Masur and Minsky [MM00], we establish in Section 5 a sufficient condition for a Weil-Petersson geodesic segment of arbitrary length to spend a fixed proportion of time in the thick part of Teichmüller space. This is used in Section 6 to show the existence of a biinfinite Weil-Petersson geodesic which is asymptotic to rays constructed from a biinfinite Teichmüller geodesic in  $\mathcal{T}(S)_\epsilon$ . This then leads to the proof of the first part of Theorem 2. Corollary 3 is an easy consequence of Theorem 2 which is discussed in Section 2 and Section 6.

In Section 7, using some standard facts from dynamical systems we extend the arguments from Section 6 to obtain a control of Weil-Petersson geodesics connecting two far apart points on a Teichmüller geodesic which is typical for a  $\Phi_T^t$ -invariant ergodic Borel probability measure on  $\mathcal{Q}(S)$ . Theorem 1. is a fairly direct consequence.

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## 2. WEIL-PETERSSON GEODESICS IN THE THICK PART OF $\mathcal{T}(S)$

Let  $S$  be an oriented surface of genus  $g \geq 0$  with  $m \geq 0$  punctures and  $3g - 3 + m \geq 2$ . In this section we investigate geodesics for the *Weil-Petersson metric* on the Teichmüller space  $\mathcal{T}(S)$  for  $S$  which remain entirely in the thick part of Teichmüller space. We relate these geodesics to *Teichmüller geodesics*, i.e. geodesics for the Teichmüller metric.

We begin with summarizing those properties of the Weil-Petersson metric which are needed in the sequel. More information and references are contained in the survey paper [W08]. The Weil-Petersson metric on  $\mathcal{T}(S)$  is a smooth  $\text{Mod}(S)$ -invariant Riemannian metric of negative sectional curvature which induces an incomplete distance  $d_{\text{WP}}$ . The completion  $\overline{\mathcal{T}(S)}$  of  $\mathcal{T}(S)$  with respect to  $d_{\text{WP}}$  is a  $\text{Cat}(0)$  space. In particular, any two points in  $\overline{\mathcal{T}(S)}$  can be connected by a unique geodesic. If the endpoints of the geodesic are contained in  $\mathcal{T}(S)$  then the connecting geodesic is contained in  $\mathcal{T}(S)$  as well [DW03]. However,  $\overline{\mathcal{T}(S)}$  is not locally compact.

In the remainder of this section, we sometimes write WP-geodesic for a geodesic with respect to the Weil-Petersson metric. A geodesic is always parametrized by arc

length. A *WP-ray* in  $\mathcal{T}(S)$  is a WP-geodesic  $\gamma : [0, T) \rightarrow \mathcal{T}(S)$  for some  $T \in (0, \infty]$  which can not be extended, i.e. which leaves every compact set.

A *geodesic lamination* for a complete hyperbolic structure on  $S$  of finite volume is a *compact* subset of  $S$  which is foliated into simple geodesics. A geodesic lamination  $\lambda$  on  $S$  is called *minimal* if each of its half-leaves is dense in  $\lambda$ . Thus a simple closed geodesic is a minimal geodesic lamination. A minimal geodesic lamination with more than one leaf has uncountably many leaves and is called *minimal arational*. A geodesic lamination  $\lambda$  is said to *fill up*  $S$  if every simple closed geodesic on  $S$  intersects  $\lambda$  transversely. This is equivalent to stating that the complementary components of  $\lambda$  are all topological discs or once punctured topological discs.

A *measured geodesic lamination* is a geodesic lamination  $\lambda$  together with a translation invariant transverse measure. Such a measure assigns a positive weight to each compact arc in  $S$  which intersects  $\lambda$  nontrivially and transversely and whose endpoints are contained in complementary regions of  $\lambda$ . The geodesic lamination  $\lambda$  is called the *support* of the measured geodesic lamination; it consists of a disjoint union of minimal components. Vice versa, every minimal geodesic lamination is the support of a measured geodesic lamination.

The space  $\mathcal{ML}$  of measured geodesic laminations on  $S$  can be equipped with the weak\*-topology. Its projectivization  $\mathcal{PML}$  is called the space of *projective measured geodesic laminations* and is homeomorphic to the sphere  $S^{6g-7+2m}$ . There is a continuous *length function*  $\mathcal{T}(S) \times \mathcal{ML} \rightarrow (0, \infty)$  which assigns to a hyperbolic metric  $h \in \mathcal{T}(S)$  and a measured geodesic lamination  $\mu$  the  $h$ -length  $\ell_h(\mu)$  of  $\mu$ . This length function satisfies  $\ell_h(a\mu) = a\ell_h(\mu)$  for all  $h \in \mathcal{T}(S)$ ,  $\mu \in \mathcal{ML}$  and every  $a > 0$ . For every fixed  $h \in \mathcal{T}(S)$  the set of all measured geodesic laminations of  $h$ -length one is a section of the projection  $\mathcal{ML} \rightarrow \mathcal{PML}$ .

If  $\mu$  is any measured geodesic lamination on  $S$  then for every Weil-Petersson geodesic  $\gamma : [a, b] \rightarrow \mathcal{T}(S)$  the function  $t \rightarrow \ell_{\gamma(t)}(\mu)$  which associates to  $t \in [a, b]$  the length  $\ell_{\gamma(t)}(\mu)$  of  $\mu$  with respect to the hyperbolic metric  $\gamma(t)$  is convex (see [W07a] for an excellent discussion of length functions and Weil-Petersson geometry).

There is also a continuous pairing  $i : \mathcal{ML} \times \mathcal{ML} \rightarrow [0, \infty)$ , the so-called *intersection form*, which extends the geometric intersection number between simple closed curves [Bo86]. Two measured geodesic laminations  $\mu, \nu$  *jointly fill up*  $S$  if  $i(\zeta, \mu) + i(\zeta, \nu) > 0$  for every  $\zeta \in \mathcal{ML}$ . Pairs  $(\mu, \nu)$  of measured geodesic laminations which jointly fill up  $S$  and which satisfy  $i(\mu, \nu) = 1$  correspond precisely to area one quadratic differentials for  $S$  by associating to an area one quadratic differential the pair consisting of its horizontal and its vertical measured geodesic lamination, respectively.

A *pants decomposition* for  $S$  is a collection of  $3g - 3 + m$  pairwise disjoint simple closed essential curves on  $S$  which decompose  $S$  into  $2g - 2 + m$  *pairs of pants*. Here by a pair of pants we mean a surface which is homeomorphic to a three-holed sphere. By a classical result of Bers (see [B92]), there is a number  $\chi_0 > 0$  only depending on the topological type of  $S$  such that for every complete hyperbolic metric  $h$  on  $S$  of finite volume there is a pants decomposition for  $S$  consisting of simple closed curves of  $h$ -length at most  $\chi_0$ . A number  $\chi_0 > 0$  with this property is

called a *Bers constant* for  $S$ . A pants decomposition which consists of simple closed curves of  $h$ -length at most  $\chi_0$  will be called a *Bers decomposition* for  $h$ . If  $P$  is any pants decomposition for  $S$  then the projectivization of the sum of the natural transverse counting measures on the components of  $P$  is a projective measured geodesic lamination which we denote by  $[P]$ .

We begin with recalling from [BMM08a] (Definition 2.4) the definition of an ending invariant for a WP-ray.

**Definition.** An *ending measure*  $[\mu] \in \mathcal{PML}$  for a WP-geodesic ray  $\gamma : [0, T) \rightarrow \mathcal{T}(S)$  ( $T \in (0, \infty]$ ) is any limit in  $\mathcal{PML}$  of the projective classes of any infinite family of distinct Bers decompositions for  $\gamma(t)$  as  $t \rightarrow T$ .

The following theorem combines Corollary 2.10 and Proposition 4.3 of [BMM08a]. For its formulation, call an infinite WP-ray  $\gamma : [0, \infty) \rightarrow \mathcal{T}(S)$  *recurrent* if there is some  $\epsilon > 0$  and an unbounded sequence  $(t_i) \subset [0, \infty)$  such that  $\gamma(t_i) \in \mathcal{T}(S)_\epsilon$ .

**Theorem 2.1.** *Let  $\gamma : [0, \infty) \rightarrow \mathcal{T}(S)$  be a recurrent WP-geodesic ray.*

- (1) *Any two ending measures for  $\gamma$  have the same support.*
- (2) *The support of an ending measure for  $\gamma$  is a minimal geodesic lamination which fills up  $S$ .*
- (3) *If  $\mu \in \mathcal{ML}$  is any measured geodesic lamination whose length along  $\gamma$  is bounded then the support of  $\mu$  equals the support of an ending measure for  $\gamma$ .*

Unlike in the case of Teichmüller geodesics, however, there are recurrent WP-rays so that the support of an ending measure is not uniquely ergodic, i.e. it admits more than one transverse measure up to scale (see [BMM08a] for a more precise announcement of this result of Brock).

The main tool for the proof of the second part of Theorem 2 from the introduction is the *curve graph*  $\mathcal{CG}(S)$  of  $S$ . The vertex set of this graph is the set  $\mathcal{C}(S)$  of all free homotopy classes of unoriented *essential* simple closed curves on  $S$ , i.e. simple closed curves which are neither contractible nor freely homotopic into a puncture. Two vertices are joined by an edge if and only if the corresponding free homotopy classes can be realized disjointly. Since  $3g - 3 + m \geq 2$  by assumption,  $\mathcal{CG}(S)$  is connected (see [MM99] and the references given there). In the sequel we often do not distinguish between an essential simple closed curve  $\alpha$  on  $S$  and the vertex of the curve graph defined by  $\alpha$ .

Providing each edge in  $\mathcal{CG}(S)$  with the standard euclidean metric of diameter 1 equips the curve graph with a geodesic metric  $d_C$ . However,  $\mathcal{CG}(S)$  is not locally finite and therefore the metric space  $(\mathcal{CG}(S), d_C)$  is not locally compact. Masur and Minsky [MM99] showed that nevertheless its geometry can be understood quite explicitly. Namely,  $\mathcal{CG}(S)$  is hyperbolic of infinite diameter (see also [Bw06, H07a] for alternative shorter proofs). The mapping class group naturally acts on  $\mathcal{CG}(S)$  as a group of simplicial isometries.

For every marked hyperbolic metric  $h \in \mathcal{T}(S)$ , every essential free homotopy class  $\alpha$  on  $S$  can be represented by a closed  $h$ -geodesic which is unique up to

parametrization. This geodesic is simple if the free homotopy class admits a simple representative. The  $h$ -length  $\ell_h(\alpha)$  of the class  $\alpha$  is the length of its geodesic representative. Equivalently,  $\ell_h(\alpha)$  equals the minimum of the  $h$ -lengths of all closed curves representing the class  $\alpha$ .

Define a map

$$\Upsilon_{\mathcal{T}} : \mathcal{T}(S) \rightarrow \mathcal{C}(S)$$

by associating to a complete hyperbolic metric  $h$  on  $S$  of finite volume an essential simple closed curve  $\Upsilon_{\mathcal{T}}(h) \in \mathcal{C}(S)$  whose  $h$ -length is at most  $\chi_0$ . Note that such a map is not unique. However, by Lemma 2.1 of [H08] (which is due to Masur and Minsky [MM99]) the ambiguity in its definition is uniformly controlled.

**Lemma 2.2.** *For every number  $\chi > 0$  there is a number  $a(\chi) > 0$  with the following property. Let  $h \in \mathcal{T}(S)$  and let  $\alpha, \beta$  be two simple closed curves of  $h$ -length at most  $\chi$ . Then  $d_{\mathcal{C}}(\alpha, \beta) \leq a(\chi)$ .*

Lemma 2.2 of [H08] (see also [MM99]) shows that the map  $\Upsilon_{\mathcal{T}}$  is coarsely Lipschitz with respect to the Teichmüller distance  $d_{\mathcal{T}}$  on  $\mathcal{T}(S)$  and the distance  $d_{\mathcal{C}}$  on the curve graph.

**Lemma 2.3.** *There is a number  $L_0 > 1$  such that*

- (1)  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(g), \Upsilon_{\mathcal{T}}(h)) \leq L_0 d_{\mathcal{T}}(g, h) + L_0$  for all  $g, h \in \mathcal{T}(S)$ .
- (2)  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(\varphi g), \varphi \Upsilon_{\mathcal{T}}(g)) \leq L_0$  for all  $g \in \mathcal{T}(S), \varphi \in \text{Mod}(S)$ .

Let  $J \subset \mathbb{R}$  be a closed connected set. For a number  $L > 1$ , a map  $\gamma : J \rightarrow \mathcal{CG}(S)$  is an  $L$ -quasi-geodesic if

$$|t - s|/L - L \leq d_{\mathcal{C}}(\gamma(s), \gamma(t)) \leq L|t - s| + L \text{ for all } s, t \in J.$$

A map  $\gamma : J \rightarrow \mathcal{CG}(S)$  is an *unparametrized  $L$ -quasi-geodesic* if there is a closed connected set  $I \subset \mathbb{R}$  and a homeomorphism  $\rho : I \rightarrow J$  such that  $\gamma \circ \rho : I \rightarrow \mathcal{CG}(S)$  is an  $L$ -quasi-geodesic.

The following result of Masur and Minsky (Theorem 2.3 and Theorem 2.6 of [MM99]) is essential for the entire paper.

**Theorem 2.4.** *There is a number  $L_1 > 1$  such that the image under  $\Upsilon_{\mathcal{T}}$  of every Teichmüller geodesic  $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)$  is an unparametrized  $L_1$ -quasi-geodesic in  $\mathcal{CG}(S)$ .*

We do not know whether the image under the map  $\Upsilon_{\mathcal{T}}$  of a Weil-Petersson geodesic is an unparametrized quasi-geodesic in  $\mathcal{CG}(S)$ . However, we can use ending measures to show that the image under  $\Upsilon_{\mathcal{T}}$  of a Weil-Petersson geodesic ray which is entirely contained in  $\mathcal{T}(S)_{\epsilon}$  for some  $\epsilon > 0$  advances a definite amount in bounded time.

**Lemma 2.5.** *For every  $\epsilon > 0$  and every  $R > 0$  there is a number  $T_0 = T_0(\epsilon, R) > 0$  with the following property. Let  $b \geq T_0$  and let  $\gamma : [0, b] \rightarrow \mathcal{T}(S)_{\epsilon}$  be a Weil-Petersson geodesic. Then  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(0), \Upsilon_{\mathcal{T}}\gamma(b)) \geq R$ .*

*Proof.* Assume to the contrary that there is some  $\epsilon > 0$  and some  $R > 0$  such that there is no  $T_0(\epsilon, R) > 0$  with the properties stated in the lemma. Then there is for every  $m > 0$  a number  $T_m > m$  and a WP-geodesic  $\gamma_m : [0, T_m] \rightarrow \mathcal{T}(S)_\epsilon$  such that  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(\gamma_m(0)), \Upsilon_{\mathcal{T}}(\gamma_m(T_m))) \leq R$ .

By coarse equivariance under the action of the mapping class group (see Lemma 2.3) and cocompactness of the action of  $\text{Mod}(S)$  on  $\mathcal{T}(S)_\epsilon$ , we may assume that the WP-geodesics  $\gamma_m$  issue from the same compact set  $K \subset \mathcal{T}(S)_\epsilon$ . Since  $\gamma_m \subset \mathcal{T}(S)_\epsilon$  for all  $m$ , after passing to a subsequence we may assume that the geodesics  $\gamma_m$  converge uniformly on compact sets to a WP-ray  $\gamma : [0, \infty) \rightarrow \mathcal{T}(S)_\epsilon$ .

For  $m > 0$  let  $\alpha_m \in \mathcal{ML}$  be the measured geodesic lamination with  $\ell_{\gamma_m(0)}(\alpha_m) = 1$  which is contained in the projective class of the curve  $\Upsilon_{\mathcal{T}}(\gamma_m(T_m))$ , viewed as a projective measured geodesic lamination. Since  $\gamma_m(0) \in K \subset \mathcal{T}(S)_\epsilon$ , the shortest length of a simple closed geodesic for the metric  $\gamma_m(0)$  is at least  $\epsilon$ . Therefore the lamination  $\alpha_m$  is obtained by multiplying the simple closed curve  $\Upsilon_{\mathcal{T}}(\gamma_m(T_m))$  with a weight which is bounded from above by  $1/\epsilon$ . Now the  $\gamma_m(T_m)$ -length of the simple closed curve  $\Upsilon_{\mathcal{T}}(\gamma_m(T_m))$  is at most  $\chi_0$  and hence the  $\gamma_m(T_m)$ -length of  $\alpha_m$  does not exceed  $\chi_0/\epsilon$ . By convexity of the length function along WP-geodesics, this implies that the length of  $\alpha_m$  along  $\gamma_m[0, T_m]$  is uniformly bounded, independent of  $m$ .

The set of all measured geodesic laminations of  $x$ -length one for some  $x \in K$  is compact. Thus by passing to a subsequence we may assume that the measured geodesic laminations  $\alpha_m$  converge as  $m \rightarrow \infty$  to a measured geodesic lamination  $\alpha$ . By continuity of the length function, we have  $\ell_{\gamma(0)}(\alpha) = 1$  and the length of  $\alpha$  along  $\gamma$  is uniformly bounded. Namely, for each  $T > 0$  and each  $m$  which is sufficiently large that  $T_m > T$ , the length of  $\alpha_m$  at  $\gamma_m(T)$  is at most  $\chi_0/\epsilon$ . Since  $\alpha_m \rightarrow \alpha$  weakly and  $\gamma_m(T) \rightarrow \gamma(T)$  as  $m \rightarrow \infty$ , by continuity of the length function we have  $\ell_{\gamma(T)}(\alpha) \leq \chi_0/\epsilon$  as well. Now  $T > 0$  was arbitrary and therefore the length of  $\alpha$  is uniformly bounded on  $\gamma$ .

We claim that the support of  $\alpha$  does not fill up  $S$ . This means that there is a measured geodesic lamination  $\nu$  whose support does not coincide with the support of  $\alpha$  and that  $i(\nu, \alpha) = 0$ . To see this note that since  $\gamma_m(0) \rightarrow \gamma(0)$ , by coarse continuity of the map  $\Upsilon_{\mathcal{T}}$  (the first part of Lemma 2.3) and by our assumption that

$$d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(\gamma_m(0)), \Upsilon_{\mathcal{T}}(\gamma_m(T_m))) \leq R \text{ for all } m,$$

the distance in  $\mathcal{CG}(S)$  between  $\Upsilon_{\mathcal{T}}(\gamma_m(T_m))$  and  $c_0 = \Upsilon_{\mathcal{T}}(\gamma(0))$  is bounded independent of  $m$ . By passing to a subsequence we may assume that this distance equals a fixed number  $k \geq 0$ . Then for each  $m$  in the subsequence, there is a family  $c_m^0, \dots, c_m^k \subset \mathcal{ML}$  of weighted simple closed geodesics of (weighted) length one for the metric  $\gamma(0) \in \mathcal{T}(S)$  such that  $i(c_m^j, c_m^{j+1}) = 0$  for every  $j < k$  (here as before,  $i$  is the intersection form) and that  $[c_0] = [c_m^0]$  and  $[c_m^k] = [\alpha_m] = [\Upsilon_{\mathcal{T}}(\gamma_m(T_m))]$  for all  $m$  (where  $[\mu]$  denotes the projective class of  $\mu \in \mathcal{ML}$ ). By passing to another subsequence, we may assume that for each  $j$  the measured geodesic laminations  $c_m^j$  converge as  $m \rightarrow \infty$  to a measured geodesic lamination  $\nu_j$ . By continuity of the intersection form, we have  $i(\nu_j, \nu_{j+1}) = 0$  for all  $j$ . Since  $\nu_k = \alpha = \lim_{m \rightarrow \infty} \alpha_m$  and since  $[\nu_0] = [c_0]$ , the supports of the laminations  $\nu_j$  can only coincide with the

support of  $\alpha$  if  $[\alpha] = [c_0]$ . As a consequence, the support of the measured geodesic lamination  $\alpha$  does not fill up  $S$  (compare [MM99] for this argument of Luo).

However, the WP-ray  $\gamma$  is entirely contained in  $\mathcal{T}(S)_\epsilon$ , in particular it is recurrent. Since the length of  $\alpha$  is bounded along  $\gamma$ , this violates Theorem 2.1. The lemma follows from this contradiction.  $\square$

As in the introduction, let  $d_H$  be the Hausdorff distance for subsets of  $\mathcal{T}(S)$  with respect to the Teichmüller metric. We use Lemma 2.5 and an idea of Mosher [Mo03] to show the second part of Theorem 2 and the second part of Corollary 3 from the introduction. For its formulation, denote as in the introduction by  $\mathcal{Q}_{WP}(S)$  the moduli space of quadratic differentials of Weil-Petersson norm one.

**Proposition 2.6.** (1) *For every  $\epsilon > 0$  there is a number  $R = R(\epsilon) > 0$  with the following property. Let  $J \subset \mathbb{R}$  be a closed connected set and let  $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$  be a Weil-Petersson geodesic. Then there is a closed connected set  $J' \subset \mathbb{R}$  and there is a Teichmüller geodesic  $\xi : J' \rightarrow \mathcal{T}(S)$  with  $d_H(\gamma(J), \xi(J')) \leq R$ .*

(2) *Let  $C \subset \mathcal{Q}_{WP}(S)$  be a compact set which is invariant under the geodesic flow  $\Phi_{WP}^t$  for the Weil-Petersson metric. Then there is a conjugacy  $\Psi : C \rightarrow \mathcal{Q}(S)$  of the restriction of  $\Phi_{WP}^t$  to  $C$  into the Teichmüller geodesic flow.*

*Proof.* Let  $\epsilon > 0$ , let  $a(\chi_0) > 1$  be as in Lemma 2.2 and let  $T_0 = T_0(\epsilon, 2a(\chi_0)+3) > 0$  be as in Lemma 2.5.

Unit balls in the cotangent bundle of  $\mathcal{T}(S)$  for both the Teichmüller metric and the Weil-Petersson metric depend continuously on the basepoint. Thus by invariance under the action of the mapping class group and cocompactness of the action of  $\text{Mod}(S)$  on  $\mathcal{T}(S)_\epsilon$ , the restriction to  $\mathcal{T}(S)_\epsilon$  of the Weil-Petersson metric is locally uniformly bilipschitz equivalent to the restriction of the Teichmüller metric. Hence there is a number  $L > 1$  such that  $d_{\mathcal{T}}(\gamma(0), \gamma(b)) \leq Lb$  for any WP-geodesic  $\gamma : [0, b] \rightarrow \mathcal{T}(S)_\epsilon$  where as before,  $d_{\mathcal{T}}$  is the distance induced by the Teichmüller metric. As a consequence, for every  $b \leq T_0$ , every WP-geodesic  $\gamma : [0, b] \rightarrow \mathcal{T}(S)_\epsilon$  is entirely contained in the ball of radius  $LT_0$  about  $\gamma(0)$  for the Teichmüller metric and therefore  $d_H(\gamma[0, b], \gamma(0)) \leq LT_0$ . Thus it is enough to show the proposition for Weil-Petersson geodesics in  $\mathcal{T}(S)_\epsilon$  of length at least  $T_0$ .

By Lemma 2.5, if  $\zeta : [0, b] \rightarrow \mathcal{T}(S)_\epsilon$  is a WP-geodesic of length  $b \geq T_0$  then  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\zeta(0), \Upsilon_{\mathcal{T}}\zeta(b)) \geq 2a(\chi_0) + 3$  and hence by Lemma 2.2, the  $\zeta(0)$ -length of any simple closed curve  $\alpha \in \mathcal{C}(S)$  with  $\ell_{\zeta(b)}(\alpha) \leq \chi_0$  is bigger than  $\chi_0$ . In particular, by convexity of length functions along Weil-Petersson geodesics, we have  $\ell_{\zeta(t)}(\alpha) \leq \ell_{\zeta(0)}(\alpha)$  for every  $t \in [0, b]$ .

Let  $b - a \geq T_0$  and let  $\gamma : [a, b] \rightarrow \mathcal{T}(S)_\epsilon$  be a WP-geodesic. We say that the projective measured geodesic lamination  $[\alpha]$  defined by a simple closed curve  $\alpha \in \mathcal{C}(S)$  is *realized* at the right endpoint  $b$  of the parameter interval  $[a, b]$  if the  $\gamma(b)$ -length of  $\alpha$  does not exceed  $\chi_0$ . By the above, we then have  $\ell_{\gamma(t)}(\alpha) \leq \ell_{\gamma(a)}(\alpha)$  for every  $t \in [a, b]$ . If  $\gamma : [0, \infty) \rightarrow \mathcal{T}(S)_\epsilon$  is an infinite WP-geodesic ray then the projectivization  $[\lambda] \in \mathcal{PML}$  of a measured geodesic lamination  $\lambda$  is *realized* at the

right endpoint  $\infty$  if the length of  $\lambda$  along  $\gamma[0, \infty)$  assumes its maximum at  $\gamma(0)$ . By Theorem 2.1, Lemma 2.5 and the above discussion, an ending measure for  $\gamma$  is realized at the right endpoint of  $J = [0, \infty)$ , moreover any projective measured geodesic lamination which is realized at  $\infty$  is supported in the support of an ending measure.

Using an idea of Mosher [Mo03], define  $\Gamma_\epsilon$  to be the set of all triples  $(\gamma : J \rightarrow \mathcal{T}(S)_\epsilon, \mu_+, \mu_-)$  with the following properties.

- (1)  $J \subset \mathbb{R}$  is a closed connected set of diameter at least  $T_0$  containing 0.
- (2)  $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$  is a Weil-Petersson geodesic.
- (3)  $\mu_+, \mu_- \in \mathcal{ML}$  are measured geodesic laminations of  $\gamma(0)$ -length one whose projectivizations  $[\mu_+], [\mu_-]$  are realized at the right and left endpoint of  $J$ , respectively.

We equip  $\Gamma_\epsilon$  with the product topology, using the weak\*-topology on  $\mathcal{ML}$  for the second and third factor and the compact open topology for the arcs  $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$ . The mapping class group  $\text{Mod}(S)$  naturally acts diagonally on  $\Gamma_\epsilon$ .

We follow Mosher (Proposition 3.17 of [Mo03]) and show that the action of  $\text{Mod}(S)$  on  $\Gamma_\epsilon$  is cocompact. Since  $\text{Mod}(S)$  acts isometrically and cocompactly on  $\mathcal{T}(S)_\epsilon$ , for this it is enough to show that the subset of  $\Gamma_\epsilon$  consisting of all triples with the additional property that  $\gamma(0)$  is contained in a fixed compact subset  $A$  of  $\mathcal{T}(S)_\epsilon$  is compact. Now the topology of  $\Gamma_\epsilon$  is metrizable and hence this follows if every sequence in  $\Gamma_\epsilon$  contained in the subset  $\{(\gamma : J \rightarrow \mathcal{T}(S), \mu_+, \mu_-) \in \Gamma_\epsilon \mid \gamma(0) \in A\}$  has a convergent subsequence.

By the Arzela Ascoli theorem (or simply by properties of geodesics for the Weil-Petersson metric), the set of geodesic arcs  $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$  where  $J \subset \mathbb{R}$  is a closed connected subset containing 0 and such that  $\gamma(0) \in A$  is compact with respect to the compact open topology. Moreover, the length function is continuous and hence it is enough to show that the following holds. Let  $\gamma_i : J_i \rightarrow \mathcal{T}(S)_\epsilon$  ( $i > 0$ ) be a sequence of Weil-Petersson geodesics which converge locally uniformly to  $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$ . For each  $i$  let  $\mu_i$  be a measured geodesic lamination of  $\gamma_i(0)$ -length one whose projectivization  $[\mu_i]$  is realized at the right endpoint of  $J_i$ . If  $\mu_i \rightarrow \mu \in \mathcal{ML}$ , then the projectivization  $[\mu]$  of  $\mu$  is realized at the right endpoint of  $J$ .

Assume first that  $J \cap [0, \infty) = [0, b]$  for some  $b \in (0, \infty)$ . Then for sufficiently large  $i$  we have  $J_i \cap [0, \infty) = [0, b_i]$  with  $b_i \in (0, \infty)$  and  $b_i \rightarrow b$ . Thus  $\gamma_i(b_i) \rightarrow \gamma(b)$  ( $i \rightarrow \infty$ ) and therefore by continuity of length functions and the collar lemma, there is only a *finite* number of simple closed curves  $\alpha \in \mathcal{C}(S)$  of length at most  $\chi_0$  with respect to one of the metrics  $\gamma_i(b_i), \gamma(b)$ . Hence by passing to a subsequence we may assume that there is a curve  $\alpha \in \mathcal{C}(S)$  such that  $[\mu_i] = [\alpha] = [\mu]$  for all sufficiently large  $i$ . The  $\gamma_i(b_i)$ -length of  $\alpha$  is at most  $\chi_0$  for all sufficiently large  $i$  and hence by continuity of length functions, the same is true for the  $\gamma(b)$ -length of  $\alpha$ . In other words, the limit  $[\mu] \in \mathcal{PM}\mathcal{L}$  of the sequence  $[\mu_i]$  is realized at the right endpoint  $b$  of  $J$ .

The same argument is also valid if the right endpoint of  $J$  is infinite. Namely, assume first that  $J_i \cap [0, \infty) = [0, b_i]$  for some  $b_i > 0$  with  $b_i \rightarrow \infty$  ( $i \rightarrow \infty$ ). By

the above discussion, for sufficiently large  $i$  (namely, for all  $i$  such that  $b_i > T_0$ ) the length of  $\mu_i$  along  $\gamma_i[0, b_i]$  assumes its maximum at  $\gamma_i(0)$ . Thus if  $T > 0$  is arbitrary and if  $i > 0$  is sufficiently large that  $b_i > \max\{T_0, T\}$  then  $\ell_{\gamma_i(T)}(\mu_i) \leq \ell_{\gamma_i(0)}(\mu_i)$ . Since  $\gamma_i(0) \rightarrow \gamma(0)$  and  $\gamma_i(T) \rightarrow \gamma(T)$  and  $\mu_i \rightarrow \mu$ , continuity of the length function implies that  $\ell_{\gamma(T)}(\mu) \leq \ell_{\gamma(0)}(\mu)$ . Now  $T > 0$  was arbitrary and therefore the length of  $\mu$  along  $\gamma$  assumes its maximum at  $\gamma(0)$ . However, this just means that the projectivization  $[\mu]$  is realized at the right infinite endpoint of  $J$ . The case that  $b_i = \infty$  for infinitely many  $i$  follows in the same way.

Any two simple closed curves  $\alpha, \beta \in \mathcal{C}(S)$  with  $d_{\mathcal{C}}(\alpha, \beta) \geq 3$  jointly fill up  $S$ . Thus by Theorem 2.1, Lemma 2.5 and the choice of  $T_0$ , for any  $(\gamma : J \rightarrow \mathcal{T}(S), \mu_+, \mu_-) \in \Gamma_\epsilon$  the measured geodesic laminations  $\mu_+, \mu_-$  jointly fill up  $S$ . These laminations then determine up to parametrization a Teichmüller geodesic  $\eta(\mu_+, \mu_-)$  whose vertical and horizontal projective measured geodesic laminations are just the classes  $[\mu_+], [\mu_-]$ .

Let  $\sigma(\gamma, \mu_+, \mu_-)$  be the unique point on  $\eta(\mu_+, \mu_-)$  which is the foot-point of the quadratic differential with vertical and horizontal measured geodesic laminations  $\mu_+/\sqrt{i(\mu_+, \mu_-)}, \mu_-/\sqrt{i(\mu_+, \mu_-)}$ . By continuity of the length function and the intersection form, the map taking  $(\gamma : J \rightarrow \mathcal{T}(S)_\epsilon, \mu_+, \mu_-) \in \Gamma_\epsilon$  to  $(\gamma(0), \sigma(\gamma, \mu_+, \mu_-)) \in \mathcal{T}(S) \times \mathcal{T}(S)$  is continuous. Moreover by construction, this map is equivariant with respect to the natural diagonal action of  $\text{Mod}(S)$  on  $\Gamma_\epsilon$  and on  $\mathcal{T}(S) \times \mathcal{T}(S)$ . Since the action of  $\text{Mod}(S)$  on  $\Gamma_\epsilon$  is cocompact, the same is true for the action of  $\text{Mod}(S)$  on the image of this map. Thus the Teichmüller distance between  $\gamma(0)$  and  $\sigma(\gamma, \mu_+, \mu_-)$  is bounded from above by a universal constant  $b > 0$ .

Let again  $(\gamma : J \rightarrow \mathcal{T}(S)_\epsilon, \mu_+, \mu_-) \in \Gamma_\epsilon$ . For each  $s \in J$  define

$$a_-(s) = \frac{1}{\ell_{\gamma(s)}(\mu_-)}, \quad a_+(s) = \frac{1}{\ell_{\gamma(s)}(\mu_+)}$$

where as before,  $\ell_{\gamma(s)}(\mu_\pm)$  is the  $\gamma(s)$ -length of  $\mu_\pm$ . These are continuous functions of  $s \in J$ . For  $s \in J$  define the shift  $\gamma^s(t) = \gamma(t + s)$ ; then the ordered triple  $(\gamma^s, a_+(s)\mu_+, a_-(s)\mu_-)$  lies in the  $\text{Mod}(S)$ -cocompact set  $\Gamma_\epsilon$  and hence the distance between  $\gamma(s)$  and a suitably chosen point on the geodesic  $\eta([\mu_+], [\mu_-])$  is at most  $b$ . As a consequence, the arc  $\gamma$  is contained in the  $b$ -neighborhood of the geodesic  $\eta([\mu_+], [\mu_-])$ . Now  $s > 0$  was arbitrary and the ordered triple  $(\gamma^s, a_+(s)\mu_+, a_-(s)\mu_-)$  depends continuously on  $s$  with respect to the topology of  $\Gamma_\epsilon$ . Hence the Hausdorff distance between  $\gamma(J)$  and a suitably chosen subarc of  $\eta([\mu_+], [\mu_-])$  is uniformly bounded. The first part of the proposition is proven.

By the above discussion, if  $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)_\epsilon$  is any biinfinite Weil-Petersson geodesic and if  $[\mu_+], [\mu_-]$  are ending measures for  $\gamma_+ = \gamma[0, \infty), \gamma_- = \gamma(-\infty, 0]$  then the Teichmüller geodesic  $\eta([\mu_+], [\mu_-])$  defined by a quadratic differential with vertical measured geodesic lamination in the class  $[\mu_+]$  and horizontal measured geodesic lamination in the class  $[\mu_-]$  is a uniform fellow-traveler of  $\gamma$ , measured with respect to the Teichmüller metric. In particular, there is a number  $\kappa > 0$  only depending on  $\epsilon$  such that  $\eta([\mu_+], [\mu_-])$  is contained in  $\mathcal{T}(S)_\kappa$ . Hence by a result of Masur [M82], the projective measured geodesic laminations  $[\mu_+], [\mu_-]$  are uniquely ergodic. Theorem 2.1 then implies that a measured geodesic lamination

$\mu_+(\gamma), \mu_-(\gamma)$  of  $\gamma(0)$ -length one whose projective class defines an ending measure for  $\gamma_+, \gamma_-$  is unique.

We claim that  $\mu_+(\gamma)$  and  $\mu_-(\gamma)$  depend continuously on  $\gamma$ . Namely, assume that  $\gamma_i \rightarrow \gamma$  uniformly on compact sets and that  $\beta$  is a weak limit of the sequence  $(\mu_+(\gamma_i))$ . Then by continuity of length functions we have  $\ell_{\gamma(0)}(\beta) = 1$ , moreover we conclude as above that the length of  $\beta$  is bounded along  $\gamma[0, \infty)$ . However, these two properties determine the measured geodesic lamination  $\mu_+(\gamma)$  uniquely from which continuous dependence of  $\mu_+(\gamma)$  on  $\gamma$  is immediate. Continuous dependence of  $\mu_-(\gamma)$  on  $\gamma$  follows in the same way.

Now let  $C \subset \mathcal{Q}_{WP}(S)$  be a compact set which is invariant under the geodesic flow  $\Phi_{WP}^t$  for the Weil-Petersson metric. Let  $\tilde{C}$  be the preimage of  $C$  in the space of quadratic differentials of Weil-Petersson norm one. By compactness of  $C$  there is a number  $\epsilon > 0$  such that for every  $q \in \tilde{C}$  the Weil-Petersson geodesic  $\gamma_q$  with initial velocity  $q$  is entirely contained in  $\mathcal{T}(S)_\epsilon$ .

For  $q \in \tilde{C}$  define  $\tilde{\Psi}(q) \in \mathcal{Q}^1(S)$  to be the unique area one quadratic differential with vertical and horizontal measured geodesic lamination

$$\mu_+(\gamma_q)/\sqrt{i(\mu_+(\gamma_q), \mu_-(\gamma_q))}, \quad \mu_-(\gamma_q)/\sqrt{i(\mu_+(\gamma_q), \mu_-(\gamma_q))},$$

respectively. Then  $q \rightarrow \tilde{\Psi}(q)$  is continuous and equivariant with respect to the action of the mapping class group and hence it projects to a map  $\Psi : C \rightarrow \mathcal{Q}(S)$ . By convexity of length functions along WP-geodesics, the length of  $\mu_+(\gamma_q)$  is strictly decreasing along  $\gamma_q$ , and the length of  $\mu_-(\gamma_q)$  is strictly increasing. This implies that the restriction of  $\tilde{\Psi}$  to the unit cotangent line  $\gamma'_q$  of the WP-geodesic  $\gamma_q$  is a homeomorphism onto the unit cotangent line of the Teichmüller geodesic  $\tilde{\gamma}$  with initial velocity  $\tilde{\gamma}'(0) = \tilde{\Psi}(q)$ . Therefore the map  $\Psi$  defines a conjugacy of the Weil-Petersson flow on  $C$  into the Teichmüller flow as defined in the introduction. The second part of the proposition follows.  $\square$

As a corollary, we obtain.

**Corollary 2.7.** *For every  $\epsilon > 0$  there is a number  $\chi(\epsilon) > 0$  such that the image under  $\Upsilon_{\mathcal{T}}$  of every Weil-Petersson geodesic  $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$  is a  $\chi(\epsilon)$ -quasi-geodesic.*

*Proof.* By Proposition 2.6, if  $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$  is any WP-geodesic then there is a Teichmüller geodesic  $\xi : I \rightarrow \mathcal{T}(S)$  whose Hausdorff distance to  $\gamma$  with respect to the Teichmüller metric is at most  $R$  where  $R = R(\epsilon) > 0$  is a constant only depending on  $\epsilon$ . By Lemma 3.1 of [W79], there is a number  $\kappa < \epsilon$  only depending on  $\epsilon$  and for every  $s \in J$  there is a some  $\rho(s) \in I$  such that  $\gamma(s)$  can be connected to  $\xi(\rho(s))$  by a Teichmüller geodesic segment of length at most  $R$  which is entirely contained in  $\mathcal{T}(S)_\kappa$ .

Now there is a number  $L > 1$  such that the restrictions to  $\mathcal{T}(S)_\kappa$  of the Weil-Petersson metric and the Teichmüller metric are locally uniformly  $L$ -bilipschitz equivalent (compare the second paragraph in the proof of Proposition 2.6). This implies that for any subinterval  $[a, b] \subset I$  the length of  $\xi[a, b]$  with respect to the Weil-Petersson metric does not exceed  $L|b - a|$ , and for every subinterval  $[s, t] \subset J$

the length of  $\gamma[s, t]$  with respect to the Teichmüller metric does not exceed  $L|t - s|$ . Since Teichmüller geodesics and Weil-Petersson geodesics are globally length minimizing, we conclude that for a map  $\rho : J \rightarrow I$  as in the previous paragraph we have  $d_{WP}(\gamma(s), \xi(\rho(s))) \leq LR$  for all  $s \in I$  and

$$|t - s|/L - 2R \leq |\rho(t) - \rho(s)| \leq L|t - s| + 2LR.$$

In other words, the map  $\rho : J \rightarrow I$  is an  $LR$ -quasi-isometry.

By the results of Masur and Minsky [MM99] (see [H08] for a precise statement and a proof) the assignment  $s \rightarrow \Upsilon_{\mathcal{T}}(\xi(s))$  is an  $L'$ -quasi-geodesic for a number  $L' > 0$  only depending on  $\kappa$  and hence on  $\epsilon$ . Since by Lemma 2.3 the map  $\Upsilon_{\mathcal{T}}$  is coarsely Lipschitz, the corollary follows.  $\square$

### 3. ASYMPTOTIC RAYS FOR THE WEIL-PETERSSON METRIC

In this section we begin with the proof of the first part of Theorem 2 from the introduction. For this we first collect some more results from [BMM08a] which motivate our analysis. We continue to use the assumptions and notations from Section 2.

The completion  $\overline{\mathcal{T}(S)}$  of  $\mathcal{T}(S)$  for the Weil-Petersson metric is a CAT(0)-space. Call two infinite Weil-Petersson geodesic rays  $\gamma : [0, \infty) \rightarrow \mathcal{T}(S)$ ,  $\xi : [0, \infty) \rightarrow \mathcal{T}(S)$  *asymptotic* if the function

$$t \rightarrow d_{WP}(\gamma(t), \xi(t))$$

is bounded. Since  $\overline{\mathcal{T}(S)}$  is not locally compact and since  $\overline{\mathcal{T}(S)}$  is *not* hyperbolic in the sense of Gromov (see [W08] for an overview and for references), it is difficult to find out whether or not for two given infinite non-asymptotic WP-rays  $\gamma_1, \gamma_2$  there is a biinfinite WP-geodesic which is forward asymptotic to  $\gamma_1$  and backward asymptotic to  $\gamma_2$ . It seems likely that such a geodesic exists for all pairs of non-asymptotic rays.

Brock, Masur and Minsky [BMM08a] found a sufficient condition for the existence of a biinfinite WP-geodesic which is forward and backward asymptotic to two given WP-rays. Namely, as in Section 2, call a WP-ray  $\gamma : [0, \infty) \rightarrow \mathcal{T}(S)$  recurrent if there is a number  $\epsilon > 0$  and there is a sequence of numbers  $t_i \rightarrow \infty$  such that  $\gamma(t_i) \in \mathcal{T}(S)_\epsilon$  for all  $i$ . In other words, a geodesic ray is recurrent if its projection to moduli space returns to a fixed compact set for arbitrarily large times. Brock, Masur and Minsky showed that for every recurrent WP-geodesic ray  $\gamma$  and for *every* WP-geodesic ray  $\xi$  which is not asymptotic to  $\gamma$  there is a biinfinite WP-geodesic which is forward asymptotic to  $\gamma$  and backward asymptotic to  $\xi$  [BMM08a].

The goal of this section is to establish a modified form of this condition which can be proven with a similar argument. The main idea is to use Cat(0)-comparison together with the fact that for every  $\epsilon > 0$  the sectional curvature of the Weil-Petersson metric is bounded from above on  $\mathcal{T}(S)_\epsilon$  by a negative constant.

We begin with an observation which is a consequence of the Gauß-Bonnet formula for ruled surfaces as in [BMM08a]. For its formulation, for  $\epsilon > 0$  let

$$(1) \quad 2b(\epsilon) = \inf\{d_{WP}(x, y) \mid x \in \mathcal{T}(S)_\epsilon, y \in \overline{\mathcal{T}(S)} - \mathcal{T}(S)\}.$$

By invariance of  $\mathcal{T}(S)_\epsilon$  under the action of  $\text{Mod}(S)$  and cocompactness, we have  $b(\epsilon) > 0$  (see also the papers of Wolpert for more precise information). Moreover, the sectional curvature of the Weil-Petersson metric on the  $b(\epsilon)$ -neighborhood of  $\mathcal{T}(S)_\epsilon$  is bounded from above by a negative constant.

A *geodesic quadrangle* in  $(\mathcal{T}(S), d_{WP})$  consists of four WP-geodesic segments connecting four distinct points in  $\mathcal{T}(S)$ . We always assume that a geodesic quadrangle  $Q$  is non-degenerate, i.e. that no vertex of  $Q$  is contained in the interior of any side of  $Q$ . Two sides  $\alpha, \beta$  of such a quadrangle are *opposite* if they do not share a common vertex.

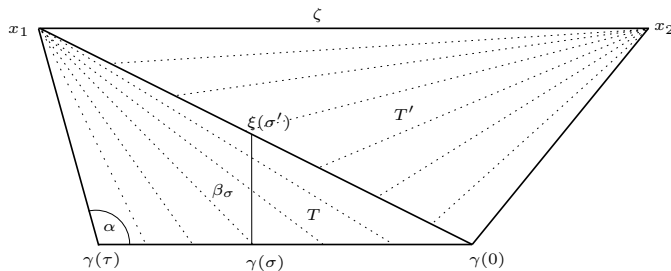
For a Weil-Petersson geodesic segment  $\gamma : [0, \tau] \rightarrow \mathcal{T}(S)$  and  $\epsilon > 0$  let  $\ell_{\epsilon\text{-thick}}(\gamma)$  be the length of the intersection of  $\gamma$  with  $\mathcal{T}(S)_\epsilon$  (in other words,  $\ell_{\epsilon\text{-thick}}(\gamma)$  is the Lebesgue measure of the set  $\{t \in [0, \tau] \mid \gamma(t) \in \mathcal{T}(S)_\epsilon\}$ ).

In the remainder of this section, distances are always distances with respect to the Weil-Petersson metric. Moreover, angles are always unoriented angles with respect to the Weil-Petersson inner product. We have.

- Lemma 3.1.** (1) *For every  $\epsilon > 0$  and every  $\alpha > 0$  there is a number  $k = k(\epsilon, \alpha) > 0$  with the following property. Let  $\tau \geq k$  and let  $\gamma : [0, \tau] \rightarrow \mathcal{T}(S)$  be a Weil-Petersson geodesic segment with  $\ell_{\epsilon\text{-thick}}(\gamma) \geq k$ . Assume that  $\gamma$  is a side of a geodesic quadrangle  $Q$  with angles at least  $\alpha$  at  $\gamma(0), \gamma(\tau)$ . Then the side of  $Q$  which is opposite to  $\gamma$  passes through the  $b(\epsilon)$ -neighborhood of  $\gamma[0, \tau] \cap \mathcal{T}(S)_\epsilon$ .*
- (2) *For every  $\epsilon > 0, \alpha > 0$  and every  $\theta > 0$  there is a number  $m = m(\epsilon, \alpha, \theta) > 0$  with the following property. Let  $\tau \geq m$  and let  $\gamma : [0, \tau] \rightarrow \mathcal{T}(S)$  be a Weil-Petersson geodesic segment with  $\ell_{\epsilon\text{-thick}}(\gamma) \geq m$ . Assume that  $\gamma$  is a side of a geodesic triangle  $T$  with angle at least  $\alpha$  at  $\gamma(\tau)$ . Then the angle of  $T$  at  $\gamma(0)$  is at most  $\theta$ .*

*Proof.* The idea of the proof is taken from [BMM08a] (and has been used before by other authors, notably by Bonahon [Bo86] and Canary [C93]). Namely, let for the moment  $\tau > 0$  be arbitrary and let  $\gamma : [0, \tau] \rightarrow \mathcal{T}(S)$  be a WP-geodesic segment. Let  $Q$  be a WP-geodesic quadrangle with vertices  $\gamma(0), \gamma(\tau), x_1, x_2$  and such that  $\gamma(\tau)$  is connected to  $x_1$  by a side.

The vertices  $\gamma(0), \gamma(\tau), x_1$  determine a WP-geodesic triangle which can be filled by WP-geodesic segments issuing from the vertex  $x_1$  and connecting  $x_1$  to the opposite side  $\gamma$ . The thus obtained *ruled triangle*  $T$  is an embedded subsurface of  $\mathcal{T}(S)$  which is smooth in its interior, with piecewise geodesic boundary. The intrinsic distance in  $T$  between any two points  $x, y \in T$  is not smaller than  $d_{WP}(x, y)$ . Moreover, the (intrinsic) Gauß curvature of  $T$  with respect to the restriction of the Weil-Petersson metric at a point  $x \in T$  does not exceed the maximum of the sectional curvatures of the Weil-Petersson metric at  $x$ . In particular, the Gauß curvature of  $T$  is negative, and for every  $\epsilon > 0$  there is a constant  $\kappa(\epsilon) > 0$  only depending on  $\epsilon$  such that the Gauß curvature at every point in  $T$  whose distance to  $\mathcal{T}(S)_\epsilon$  is at most  $b(\epsilon)$  is bounded from above by  $-\kappa(\epsilon)$  (see [BMM08a] for this construction of Bonahon [Bo86] and Canary [C93] and for references).



Even though the ruled triangle  $T$  is not naturally an embedded subsurface with piecewise geodesic boundary of a smooth simply connected surface  $U$  with a Riemannian metric, the intrinsic angle of  $T$  is defined at every vertex of  $T$ . At the vertex  $x_1$ , this angle is just the length of the arc in the unit tangent sphere for the Weil-Petersson metric which consists of the directions of all geodesics joining  $x_1$  to  $\gamma$ . We claim that the intrinsic angles of  $T$  at the vertices  $\gamma(0), \gamma(\tau)$  coincide with the angles for the Weil-Petersson metric. Namely, by slightly extending the WP-geodesic  $\gamma$  and the geodesics defining the ruling of  $T$  we obtain a smooth subsurface of  $\mathcal{T}(S)$  containing a neighborhood of  $\gamma(0), \gamma(\tau)$  in its interior. The intrinsic angle at  $\gamma(0), \gamma(\tau)$  of the triangle  $T$  is just the angle of  $T$  at  $\gamma(0), \gamma(\tau)$  in this subsurface equipped with the restriction of the Weil-Petersson metric.

Let  $\epsilon > 0$  and let  $\kappa = \kappa(\epsilon) > 0$  be as in the second paragraph of this proof. Consider first the case that the angle of the quadrangle  $Q$  at the vertex  $\gamma(\tau)$  is not smaller than  $\pi/2$ . Then by the consideration in the previous paragraph, the intrinsic angle at  $\gamma(\tau)$  of the triangle  $T$  is at least  $\pi/2$ . By the Gauß-Bonnet formula, the integral of the Gauß curvature over the triangle  $T$  equals the sum of the angles of  $T$  minus  $\pi$ . Thus if  $\theta_1, \theta_2$  are the angles of  $T$  at  $\gamma(0), x_1$  then this curvature integral is not smaller than  $-\pi/2 + \theta_1 + \theta_2 > -\pi/2$ .

Denote by  $\xi$  be the side of  $T$  connecting  $\gamma(0) = \xi(0)$  to  $x_1$ . The triangle  $T$  is negatively curved and therefore intrinsic distance functions in  $T$  are convex. Since the angle of  $T$  at  $\gamma(\tau)$  is not smaller than  $\pi/2$ , for each  $t \in [0, \tau]$  the endpoint of the intrinsic geodesic arc  $\beta_t$  in  $T$  which issues from  $\gamma(t)$  and is perpendicular to  $\gamma$  at  $\gamma(t)$  is contained in the side  $\xi$ . The length of  $\beta_t$  equals the intrinsic distance between its endpoint on  $\xi$  and the side  $\gamma$  of  $T$  and hence by convexity of the distance function, this length is increasing with  $t$ . Thus if there is a number  $\sigma \in [0, \tau]$  so that the length of  $\beta_\sigma$  is not smaller than  $b(\epsilon)/2$ , then for every  $s \in [\sigma, \tau]$  the length of  $\beta_s$  is not smaller than  $b(\epsilon)/2$ . Then  $T$  contains an embedded strip  $A_0$  of width  $b(\epsilon)/2$  and length  $\tau - \sigma$  which consists of all points in  $T$  on the initial subsegments of length  $b(\epsilon)/2$  of the geodesics  $\beta_s$  for all  $s \in [\sigma, \tau]$ .

Assume that there is such a point  $\sigma \in [0, \tau]$  such that moreover  $\ell_{\epsilon\text{-thick}}(\gamma[\sigma, \tau]) \geq \pi/\kappa b(\epsilon)$ . Let  $A \subset A_0$  be the closed subset of the embedded strip  $A_0 \subset T$  which consists of the union of all initial subarcs of length  $b(\epsilon)/2$  of those of the geodesic arcs  $\beta_s$  which issue from a point in  $\gamma[\sigma, \tau] \cap \mathcal{T}(S)_\epsilon$ . Since  $\ell_{\epsilon\text{-thick}}(\gamma[\sigma, \tau]) \geq \pi/\kappa b(\epsilon)$  and the curvature of  $T$  is negative, comparison with the euclidean plane shows that the area of  $A$  is at least  $\pi/2\kappa$ . Now the Gauß curvature of  $T$  at every point of  $A$  does not exceed  $-\kappa$  and therefore the integral of the Gauß curvature over  $T$

is smaller than  $-\pi/2$  which contradicts the above observation that by the Gauß Bonnet formula, this curvature integral is bigger than  $-\pi/2$ . This shows that if  $\sigma \in [0, \tau]$  is such that  $\ell_{\epsilon\text{-thick}}(\gamma[\sigma, \tau]) \geq \pi/\kappa b(\epsilon)$  then for  $s \in [0, \sigma]$  the length of the geodesic arc  $\beta_s$  does not exceed  $b(\epsilon)/2$ .

Now assume more restrictively that the angles of the quadrangle  $Q$  at the vertices  $\gamma(0), \gamma(\tau)$  are not smaller than  $3\pi/4$ . Then the angle of the triangle  $T$  at the vertex  $\gamma(\tau)$  is not smaller than  $3\pi/4$ . Since the triangle  $T$  is negatively curved, the sum of its angles is smaller than  $\pi$ . In particular, the Weil-Petersson angle at  $\gamma(0)$  between  $\gamma$  and the WP-geodesic  $\xi$  connecting  $\gamma(0)$  to  $x_1$  does not exceed  $\pi/4$ . As a consequence, the angle at  $\gamma(0)$  of the ruled triangle  $T'$  with vertices  $\gamma(0), x_1, x_2$  which is obtained by connecting  $x_2$  to the opposite side  $\xi$  by geodesic arcs is at least  $\pi/2$ .

Assume that  $\ell_{\epsilon\text{-thick}}(\gamma) \geq 2\pi/\kappa b(\epsilon)$  and let  $\sigma \in [0, \tau]$  be such that

$$\ell_{\epsilon\text{-thick}}(\gamma[0, \sigma]) \geq \pi/\kappa b(\epsilon) \text{ and that } \ell_{\epsilon\text{-thick}}(\gamma[\sigma, \tau]) \geq \pi/\kappa b(\epsilon).$$

Let  $\sigma' > \sigma$  be such that  $\xi(\sigma')$  is the endpoint of the geodesic segment  $\beta_{\sigma'}$  in the ruled triangle  $T$  issuing perpendicularly from  $\gamma(\sigma)$ . Since  $\ell_{\epsilon\text{-thick}}(\gamma[\sigma, \tau]) \geq \pi/\kappa b(\epsilon)$ , the above consideration shows that the distance between  $\gamma(\sigma)$  and  $\xi(\sigma')$  does not exceed  $b(\epsilon)/2$ . Since the distance function for the Weil-Petersson metric is convex, we conclude that  $d_{WP}(\xi(t), \gamma(t\sigma/\sigma')) \leq b(\epsilon)/2$  for all  $t \in [0, \sigma']$ .

Apply the above discussion to the ruled triangle  $T'$ , the side  $\xi$  of  $T'$  and the side  $\zeta$  connecting  $x_1$  to  $x_2$ . Note that  $\zeta$  is the side of the quadrangle  $Q$  opposite to  $\gamma$ . By the assumption that  $\ell_{\epsilon\text{-thick}}(\gamma[0, \sigma]) \geq \pi/\kappa b(\epsilon)$ , if the distance between  $\zeta$  and  $\gamma[0, \sigma] \cap \mathcal{T}(S)_{\epsilon}$  is at least  $b(\epsilon)$  then there is an embedded strip in  $T'$  with curvature integral smaller than  $-\pi/2$ . This strip consists of all points on geodesic arcs in  $T'$  of length  $b(\epsilon)/2$  issuing perpendicularly from a point  $\xi(t)$  for some  $t \in [0, \sigma']$  such that  $\gamma(t\sigma/\sigma') \in \mathcal{T}(S)_{\epsilon}$ . Since the angle of  $T'$  at  $\gamma(0)$  is bounded from below by  $\pi/2$ , this is impossible. As a consequence, the side  $\zeta$  intersects the  $b(\epsilon)$ -neighborhood of  $\gamma[0, \sigma] \cap \mathcal{T}(S)_{\epsilon}$ . This shows the first part of the lemma for  $\alpha = 3\pi/4$  with  $k(\epsilon, \alpha) = 2\pi/\kappa b(\epsilon)$ .

The first part of the lemma for an arbitrary angle  $\alpha > 0$  is now a consequence of the second part of the lemma for  $\epsilon, \alpha$  and  $\theta = \pi/4$ .

Namely, assume that the second part of the lemma holds true and let  $Q$  be a geodesic quadrangle with a side  $\gamma : [0, \tau] \rightarrow \mathcal{T}(S)$  and angles at least  $\alpha$  at the vertices  $\gamma(0), \gamma(\tau)$ . Let  $x_1, x_2$  be the vertices of  $Q$  which are distinct from  $\gamma(0), \gamma(\tau)$  and assume that  $\ell_{\epsilon\text{-thick}}(\gamma) \geq 2\pi/\kappa b(\epsilon) + 2m$  where  $m = m(\epsilon, \alpha, \pi/4) > 0$  is as in the second part of the lemma. Let  $0 < t_1 < t_2 < \tau$  be such that

$$\ell_{\epsilon\text{-thick}}(\gamma[0, t_1]) \geq m, \ell_{\epsilon\text{-thick}}(\gamma[t_1, t_2]) \geq 2\pi/\kappa b(\epsilon), \ell_{\epsilon\text{-thick}}(\gamma[t_2, \tau]) \geq m$$

and let  $Q'$  be the quadrangle with vertices  $\gamma(t_1), \gamma(t_2), x_1, x_2$ . The side  $\zeta$  of  $Q'$  opposite to  $\gamma[t_1, t_2]$  coincides with the side of  $Q$  opposite to  $\gamma$ . By the choice of  $t_1, t_2$  and by the second part of the lemma, applied to the triangle with vertices  $\gamma(0), \gamma(t_1), x_2$  and an angle at least  $\alpha$  at  $\gamma(0)$  and the triangle with vertices  $\gamma(t_2), \gamma(\tau), x_1$  and an angle at least  $\alpha$  at  $\gamma(\tau)$ , the angle of  $Q'$  at the vertices  $\gamma(t_1), \gamma(t_2)$  is at least  $3\pi/4$ .

Therefore the first part of the lemma for  $\alpha = \pi/4$  this proof implies that  $\zeta$  passes through the  $b(\epsilon)$ -neighborhood of  $\gamma[t_1, t_2] \cap \mathcal{T}(S)_\epsilon$ . This is what we had to prove.

To establish the angle estimate in the second part of the lemma, let  $\alpha > 0, \theta > 0$  and let  $T$  be a triangle in  $(\mathcal{T}(S), d_{WP})$  with a side  $\zeta : [0, \sigma] \rightarrow \mathcal{T}(S)$  of length  $\sigma > 0$ , an angle  $\theta_0 \geq \theta$  at  $\zeta(0)$  and an angle  $\alpha_0 \geq \alpha$  at  $\zeta(\sigma)$ . Let  $x_1$  be the vertex of  $T$  opposite to the side  $\zeta$  and assume that  $T$  is ruled by WP-geodesics connecting  $x_1$  to the points on  $\zeta$ . The Gauss curvature of  $T$  is negative, in particular we have  $\alpha_0 + \theta_0 < \pi$ .

Let  $\hat{T}$  be a comparison triangle in the euclidean plane  $\mathbb{R}^2$  with a side  $\hat{\zeta} : [0, \sigma] \rightarrow \mathbb{R}^2$  of length  $\sigma$ , an angle  $\theta$  at  $\hat{\zeta}(0)$  and an angle  $\alpha$  at  $\hat{\zeta}(\sigma)$ . Note that the angles of  $\hat{T}$  at  $\hat{\zeta}(0), \hat{\zeta}(\sigma)$  may both be smaller than the corresponding angles of  $T$ . By CAT(0)-comparison (see [BH99]), for every  $s \in [0, \sigma]$  the length of the intrinsic geodesic  $\beta_s$  in the ruled triangle  $T$  which is orthogonal to  $\zeta$  at  $\zeta(s)$  and which ends on one of the two sides different from  $\zeta$  is not smaller than the length of the geodesic  $\hat{\beta}_s$  in  $\hat{T}$  which is orthogonal to  $\hat{\zeta}$  at  $\hat{\zeta}(s)$  and ends on one of the sides of  $\hat{T}$  distinct from  $\hat{\zeta}$ .

But this just means that there is a number  $t(\alpha, \theta) > 0$  only depending on  $\alpha$  and  $\theta$  such that for  $s \in [t(\alpha, \theta), \sigma - t(\alpha, \theta)]$  the length of the geodesic arc  $\beta_s$  in  $T$  is not smaller than  $b(\epsilon)/2$ . In particular, by comparison, the ruled triangle  $T$  contains an embedded strip of area at least  $b(\epsilon)(\sigma - 2t(\alpha, \theta))/2$  which is the union of the initial subsegments of length  $b(\epsilon)/2$  of the geodesic arcs  $\beta_s$  issuing from points  $\zeta(s)$  where  $s \in [t(\alpha, \theta), \sigma - t(\alpha, \theta)]$ .

Now following the reasoning in the beginning of this proof, if  $\rho = \ell_{\epsilon\text{-thick}}(\zeta)$  then the integral of the Gauss curvature over this strip is at most  $-\kappa b(\epsilon)(\rho - 2t(\alpha, \theta))/2$ , and hence the sum of the angles of the triangle  $T$  is at most  $\pi - \kappa b(\epsilon)(\rho - 2t(\alpha, \theta))/2$ . On the other hand, the angle sum of  $T$  is at least  $\alpha + \theta$  by assumption which implies that

$$\rho \leq 2(\pi - \alpha - \theta)/\kappa b(\epsilon) + 2t(\alpha, \theta).$$

As a consequence, if  $T$  is a WP-geodesic triangle in  $\mathcal{T}(S)$  with a side  $\zeta : [0, \sigma] \rightarrow \mathcal{T}(S)$  such that  $\ell_{\epsilon\text{-thick}}(\zeta) \geq 2(\pi - \alpha - \theta)/\kappa b(\epsilon) + 2t(\alpha, \theta)$  and if the angle of  $T$  at  $\zeta(\sigma)$  is not smaller than  $\alpha$ , then the angle of  $T$  at  $\zeta(0)$  does not exceed  $\theta$ . This is just the statement in the second part of the lemma.  $\square$

As a consequence of Lemma 3.1, we obtain a sufficient condition for the existence of a biinfinite WP-geodesic which is forward and backward asymptotic to WP-rays with specific geometric properties.

**Corollary 3.2.** *For  $\epsilon > 0$  and  $\alpha > 0$  let  $k = k(\epsilon, \alpha) > 0$  be as in Lemma 3.1. Let  $\gamma_0 : [0, \tau] \rightarrow \mathcal{T}(S)$  be a WP-geodesic segment with  $\ell_{\epsilon\text{-thick}}(\gamma_0) \geq k(\epsilon, \alpha)$  and let  $\gamma_1, \gamma_2 : [0, \infty) \rightarrow \mathcal{T}(S)$  be infinite WP-rays issuing from  $\gamma_1(0) = \gamma_0(0), \gamma_2(0) = \gamma_0(\tau)$ . Assume that the angle at  $\gamma_0(0), \gamma_0(\tau)$  between the unit tangent vectors  $\gamma_0'(0), \gamma_1'(0)$  and between the unit tangent vectors  $-\gamma_0'(\tau), \gamma_2'(0)$  is at least  $\alpha$ . Assume moreover that there are measured geodesic laminations  $\mu_1, \mu_2$  which fill up  $S$  and such that the length of  $\mu_i$  is bounded along  $\gamma_i$  ( $i=1,2$ ). Then there is a unique biinfinite WP-geodesic  $\xi$  which is forward asymptotic to  $\gamma_1$  and backward asymptotic to  $\gamma_2$ .*

*Proof.* Let  $\gamma_0, \gamma_1, \gamma_2$  be as in the lemma. For  $t > 0$  consider the geodesic quadrangle  $Q_t$  with vertices  $\gamma_0(0), \gamma_0(\tau), \gamma_1(t), \gamma_2(t)$ . By Lemma 3.1, the side  $\xi_t$  of  $Q_t$  which connects  $\gamma_1(t)$  to  $\gamma_2(t)$  passes through a fixed compact neighborhood  $K$  of  $\gamma_0[0, \tau] \cap \mathcal{T}(S)_\epsilon$ . We parametrize  $\xi_t$  by arc length in such a way that  $\xi_t(0) \in K$ . Then the WP-distance between  $\gamma_0(0)$  and  $\xi_t(0)$  is uniformly bounded, independent of  $t$ .

By the CAT(0)-triangle comparison property, the Hausdorff distance with respect to the Weil-Petersson metric between the geodesic arc  $\gamma_1[0, t]$  and the subsegment of  $\xi_t$  connecting  $\xi_t(0)$  to  $\gamma_1(t)$  is uniformly bounded, independent of  $t$ . Similarly, the Hausdorff distance with respect to the Weil-Petersson metric between the geodesic arc  $\gamma_2[0, t]$  and the subsegment of  $\xi_t$  connecting  $\xi_t(0)$  to  $\gamma_2(t)$  is uniformly bounded.

The restriction to  $K$  of the unit tangent bundle for the Weil-Petersson metric is compact. Therefore there is a sequence  $t_i \rightarrow \infty$  such that the directions  $\xi'_{t_i}(0)$  of  $\xi_{t_i}$  at  $\xi_{t_i}(0)$  converge as  $i \rightarrow \infty$  to a direction  $v$ . Let  $\xi : (-r, T) \rightarrow \mathcal{T}(S)$  be the (maximal) WP-geodesic with initial velocity  $v$ . The WP-geodesics  $\xi_{t_i}$  converge uniformly on compact subsets of  $(-r, T)$  to  $\xi$ . In particular, if  $\xi$  is biinfinite (i.e. if  $r = T = \infty$ ) then  $\xi$  is indeed a geodesic which is forward asymptotic to  $\gamma_1$  and backward asymptotic to  $\gamma_2$ .

To see that  $\xi$  is indeed biinfinite, let  $\nu \in \mathcal{ML}$  be a measured geodesic lamination which fills up  $S$  and whose length is bounded along the geodesic ray  $\gamma_1$ . Such a measured geodesic lamination exists by assumption. Since  $\xi_t(0) \in K$  for all  $t$ , the  $\xi_t(0)$ -length of  $\nu$  is bounded independent of  $t > 0$ . Now for every  $t$  the forward endpoint  $\xi_t(\tau_t)$  of  $\xi_t$  equals  $\gamma_1(t)$  and hence by convexity of length functions along WP-geodesics, the length of  $\nu$  is bounded along  $\xi_t[0, \tau_t]$  by a universal constant, independent of  $t$ . Note also that  $\tau_t \rightarrow \infty$  ( $t \rightarrow \infty$ ). By continuity of the length pairing, we conclude that the length of  $\nu$  on  $\xi[0, T)$  is uniformly bounded. However, if  $T < \infty$  then there is a simple closed curve  $c$  on  $S$  so that the  $\xi(t)$ -length of  $c$  tends to zero as  $t \rightarrow T$  (see [W08] for more and for references). Since  $\nu$  fills up  $S$  we have  $i(c, \nu) > 0$  and therefore the length of  $\nu$  along  $\xi[0, T)$  tends to infinity as  $t \rightarrow T$  which is a contradiction. This argument also applies to the ray  $\xi(-r, 0]$  and yields that  $\xi$  is indeed a biinfinite WP-geodesic which is forward asymptotic to  $\gamma_1$  and backward asymptotic to  $\gamma_2$ .

To show that such a geodesic is unique, recall from comparison for CAT(0)-spaces that if there is a second such geodesic  $\xi'$  then  $\xi$  and  $\xi'$  bound a flat strip. Since the sectional curvature of the Weil-Petersson metric is negative, this is impossible. The corollary follows.  $\square$

#### 4. SHORT CURVES AND TWISTING

In Corollary 3.2 we established a sufficient condition for the existence of a biinfinite Weil-Petersson geodesic which is forward and backward asymptotic to two given Weil-Petersson geodesic rays. To apply this result, we have to find a sufficient condition for a Weil-Petersson geodesic to spend a definitive amount of time in the thick part of Teichmüller space.

We approach this problem by analyzing WP-geodesic segments of uniformly bounded length which enter deeply into the thin part of Teichmüller space. Our goal is a quantitative version of the following result of Wolpert [W03]: If  $\gamma : [0, r] \rightarrow \mathcal{T}(S)$  is any Weil-Petersson geodesic of uniformly bounded length and if  $\gamma$  enters the thin part of Teichmüller space then  $\gamma$  twists a definitive amount about one of the curves which becomes very short along  $\gamma$ . Or, put differently, if  $\gamma$  does not twist much about its short curves then  $\gamma$  can not be entirely contained in the thin part of Teichmüller space.

We continue to use the assumptions and notations from Sections 2-3. Following Masur and Minsky, we measure the amount of twisting about a curve  $\alpha \in \mathcal{C}(S)$  as follows (see in particular p. 919 of [MM00]). Fix a complete hyperbolic metric of finite volume  $h \in \mathcal{T}(S)$  and identify  $\alpha$  with the  $h$ -geodesic it defines. There is a locally isometric annular cover  $\tilde{A} \rightarrow S$  of  $(S, h)$  whose geodesic core curve  $\tilde{\alpha}$  projects isometrically onto  $\alpha$ . The hyperbolic annulus  $\tilde{A}$  admits a natural compactification to a closed annulus  $\hat{A}$  which is obtained as follows. The fundamental group  $\langle \alpha \rangle$  of  $\tilde{A}$  acts on the hyperbolic plane  $\mathbf{H}^2$  as a group of hyperbolic isometries fixing two points  $a \neq b$  in the ideal boundary  $\partial\mathbf{H}^2$  of  $\mathbf{H}^2$ . The quotient of  $\mathbf{H}^2 \cup (\partial\mathbf{H}^2 - \{a, b\})$  under the action of  $\langle \alpha \rangle$  is a compact annulus  $\hat{A}$  containing  $\tilde{A}$  as an open dense subset. Any geodesic in  $\tilde{A}$  for the hyperbolic metric which intersects the geodesic core curve  $\tilde{\alpha}$  transversely extends to a continuous path in  $\hat{A}$  connecting the two distinct boundary components of  $\hat{A}$ .

Let  $\mathcal{C}(\alpha)$  be the set of all simple paths in  $\hat{A}$  connecting the two distinct boundary components of  $\hat{A}$  modulo homotopies that fix the endpoints. Then  $\mathcal{C}(\alpha)$  is the set of vertices of a metric graph  $\mathcal{CG}(\alpha)$  whose edges are determined by requiring that two such homotopy classes of arcs  $\gamma, \gamma'$  are connected by an edge of length one if and only if  $\gamma, \gamma'$  have representatives with disjoint interior.

Following p. 920 of [MM00], define a projection  $\pi_\alpha$  of  $\mathcal{C}(S)$  into the family of all subsets of  $\mathcal{CG}(\alpha)$  as follows. Let  $\gamma \in \mathcal{C}(S)$  be represented by a simple closed  $h$ -geodesic. If  $\gamma$  does not intersect  $\alpha$  transversely then we define  $\pi_\alpha(\gamma) = \emptyset$ . Otherwise the preimage  $\tilde{\gamma}$  of  $\gamma$  in  $\tilde{A}$  has at least one component which extends continuously to an arc connecting the two distinct boundary components of  $\hat{A}$ . The set  $\pi_\alpha(\gamma)$  of all these components is a finite set of diameter at most 1 in  $\mathcal{CG}(\alpha)$ . This definition of  $\pi_\alpha$  is essentially independent of the choice of the hyperbolic metric  $h$  (see [MM00] for more details). If  $c$  is a *simple multi-curve*, i.e. a disjoint union of mutually not freely homotopic simple closed curves, then let  $\pi_\alpha(c)$  be the union of the projections of its components. As before, the diameter of  $\pi_\alpha(c)$  is at most one (Lemma 2.3 of [MM00]). The projection  $\pi_\alpha$  can be used to measure the relative twisting about  $\alpha$  of two simple closed curves which intersect  $\alpha$  transversely [MM00].

As in Section 2, let  $\chi_0 > 0$  be a Bers constant for  $S$  and let  $\Upsilon_{\mathcal{T}} : \mathcal{T}(S) \rightarrow \mathcal{CG}(S)$  be a map which associates to a hyperbolic metric  $x$  a simple closed curve of  $x$ -length at most  $\chi_0$ . We use the projections  $\pi_\alpha$  ( $\alpha \in \mathcal{C}(S)$ ) and Wolpert's description of the Weil-Petersson metric near its completion locus (see [W03, W07b]) to obtain information on the image of a Weil-Petersson geodesic  $\gamma$  under the map  $\Upsilon_{\mathcal{T}}$ . We are in particular interested in the twisting behavior of points in  $\Upsilon_{\mathcal{T}}(\gamma)$  about a simple closed curve which becomes very short along  $\gamma$ .

For this we first need a better control of the projections of WP-geodesics into the graph  $\mathcal{CG}(\alpha)$  for a simple closed curve  $\alpha \in \mathcal{C}(S)$ . We obtain such a control using the *pants graph*  $\mathcal{PG}(S)$  for  $S$ .

A pants decomposition  $P$  for  $S$  is changed to a pants decomposition  $P'$  by an *elementary move* if  $P'$  is obtained from  $P$  by replacing one of the pants curves  $\alpha$  of  $P$  by a curve which does not intersect  $P - \alpha$  and intersects  $\alpha$  in the minimal number of points (i.e. in precisely two points if the component of  $S - (P - \alpha)$  containing  $\alpha$  is a four-holed sphere, and in precisely one point if this component is a one-holed torus). The pants graph  $\mathcal{PG}(S)$  of  $S$  is the geodesic metric graph whose set of vertices is the set  $\mathcal{P}(S)$  of pants decompositions for  $S$  and where two such pants decompositions  $P, P'$  are connected by an edge of length one if and only if  $P'$  can be obtained from  $P$  by an elementary move.

As in Section 2, call a pants decomposition  $P$  for  $S$  a Bers decomposition for  $h \in \mathcal{T}(S)$  if the  $h$ -length of each of the components of  $P$  is bounded from above by  $\chi_0$ . Note that if  $\alpha \in \mathcal{C}(S)$  is any simple closed curve whose  $h$ -length is bigger than  $\chi_0$  then every Bers decomposition  $P$  for  $h$  contains a component which intersects  $\alpha$  transversely. In particular, the projection  $\pi_\alpha(P)$  is not empty, and its diameter  $\text{diam}(\pi_\alpha(P))$  is at most one.

Define a map

$$\Upsilon_{\mathcal{P}} : \mathcal{T}(S) \rightarrow \mathcal{PG}(S)$$

by associating to a hyperbolic metric  $x \in \mathcal{T}(S)$  a Bers decomposition for  $x$ . The following Lemma is a quantitative version of Lemma 2.2 for pants decompositions which enables us to control for a simple closed curve  $\alpha$  the projections into the graph  $\mathcal{CG}(\alpha)$  of Bers decompositions along Weil-Petersson geodesics. Compare also Section 3 of [B03].

**Lemma 4.1.** *There is a constant  $\chi_1 > \chi_0$  with the following property. Let  $\xi : [0, \sigma] \rightarrow \mathcal{T}(S)$  be a WP-geodesic segment of length  $\sigma \leq 1$  and let  $P_0, P_\sigma$  be Bers decompositions for  $\xi(0), \xi(\sigma)$ . Then  $P_0$  can be connected to  $P_\sigma$  by a path  $\rho$  in  $\mathcal{PG}(S)$  of length at most  $\chi_1$  with the following property. For every vertex  $P \in \mathcal{P}(S)$  passed through by  $\rho$  there is some  $t \in [0, \sigma]$  such that the  $\xi(t)$ -length of every component curve of  $P$  is smaller than  $\chi_1$ .*

*Proof.* By Lemma 3.12 of [W07a], there is a number  $a > 0$  with the following property. Let  $h \in \mathcal{T}(S)$  and let  $\alpha \in \mathcal{C}(S)$  be a simple closed curve of  $h$ -length  $\ell_h(\alpha) \leq 2\chi_0$ . Then the norm at  $h$  of the Weil-Petersson gradient of the length function  $x \rightarrow \ell_x(\alpha)$  is bounded from above by  $a$ . This implies the following. Let  $\tau \leq \chi_0/a$ , let  $\xi : [0, \tau] \rightarrow \mathcal{T}(S)$  be any WP-geodesic and let  $P$  be a Bers decomposition for  $\xi(0)$ . Then for every  $t \in [0, \tau]$  the  $\xi(t)$ -length of every component of  $P$  does not exceed  $2\chi_0$ .

Choose an integer  $\ell > a/\chi_0$ . Let  $\xi : [0, \sigma] \rightarrow \mathcal{T}(S)$  be a WP-geodesic of length  $\sigma \leq 1$  and let  $j \leq \ell$  be the smallest integer such that  $j/\ell \geq \sigma$ . Let  $P_0, P_\sigma$  be Bers decompositions for  $\xi(0), \xi(\sigma)$  and for each  $i \in \{1, \dots, j-1\}$  let  $P_i$  be a Bers decomposition for  $\xi(\frac{i}{\ell})$ . By the choice of the constant  $\ell$ , for each  $i$  the  $\xi(\frac{i}{\ell})$ -length of each component of  $P_i$  does not exceed  $2\chi_0$ . Since  $j \leq \ell$ , for the proof of the lemma it is enough to show the existence of a number  $\beta > 0$  with the following

property. For every  $x \in \mathcal{T}(S)$ , any pants decomposition  $Q_0$  of  $S$  with components of  $x$ -length at most  $2\chi_0$  can be connected to a given Bers decomposition  $Q_1$  for  $x$  by a path in  $\mathcal{PG}(S)$  of length at most  $\beta$  passing through vertices of the pants graph which are pants decompositions for  $S$  with components of  $x$ -length at most  $\beta$ .

Thus let  $x \in \mathcal{T}(S)$  and let  $Q_0, Q_1$  be such pants decompositions whose components are  $x$ -geodesics of length at most  $2\chi_0$  and  $\chi_0$ , respectively. Let  $c_1, \dots, c_k$  ( $0 \leq k \leq 3g - 3 + m$ ) be those components of  $Q_0$  which are also components of  $Q_1$ . Note that by the collar lemma, the set  $c_1, \dots, c_k$  contains every simple closed  $x$ -geodesic of sufficiently small length. Let  $\hat{S}$  be the completion of a connected component of the surface which we obtain by cutting  $S$  open along the curves  $c_1, \dots, c_k$  and which is different from a three-holed sphere.

The intersection  $Q_0 \cap \hat{S}$  of  $Q_0$  with the interior of  $\hat{S}$  is a pants decomposition for  $\hat{S}$ , and the same is true for the intersection  $Q_1 \cap \hat{S}$  of  $Q_1$  with the interior of  $\hat{S}$ . Thus  $(Q_0 \cup Q_1) \cap \hat{S}$  is an embedded connected piecewise geodesic graph in  $\hat{S}$  which decomposes  $\hat{S}$  into non-essential annuli, i.e. annuli whose core curves are homotopic to a boundary component of  $\hat{S}$  or to a puncture, and into topological discs with piecewise geodesic boundary. In particular, the injection of the graph into  $\hat{S}$  induces a surjection of fundamental groups. By the collar lemma and the length bound for the components of  $Q_0, Q_1$ , the number of intersections between  $Q_0$  and  $Q_1$  is bounded from above by a number only depending on  $\chi_0$  and the topological type of  $S$  and hence the same is true for the number of edges of the graph. The length of each edge does not exceed  $2\chi_0$ .

Up to the action of the mapping class group  $\text{Mod}(\hat{S})$  of  $\hat{S}$ , there are only finitely many pairs of pants decompositions of  $\hat{S}$  whose union is an embedded connected graph in  $\hat{S}$  with a uniformly bounded number of edges and whose complementary components are topological discs and non-essential annuli. By invariance under the action of  $\text{Mod}(\hat{S})$ , this means that there is a number  $m = m(\hat{S}) > 0$  only depending on the topological type of  $\hat{S}$ , there is a number  $n \leq m$  and there is a sequence of pants decompositions  $R_0 = Q_0 \cap \hat{S}, R_1, \dots, R_n = Q_1 \cap \hat{S}$  for  $\hat{S}$  with the following properties. For each  $i < n$ ,  $R_{i+1}$  can be obtained from  $R_i$  by an elementary move. Moreover, each simple closed curve on  $\hat{S}$  appearing as a pants curve of one of the pants decompositions  $R_i$  is freely homotopic to an edge path in the graph  $(Q_0 \cup Q_1) \cap \hat{S}$  of uniformly bounded length. Since the  $x$ -length of an edge of  $(Q_0 \cup Q_1) \cap \hat{S}$  is uniformly bounded, the  $x$ -length of each component curve of  $R_i$  is bounded from above by a constant which only depends on  $\hat{S}$ .

On the other hand, there are only finitely many topological types of subsurfaces of  $S$  which can arise as complementary components of a simple multi-curve on  $S$ . Thus a successive application of this construction to all components of  $S - (Q_0 \cap Q_1)$  shows that  $Q_0$  can be modified to  $Q_1$  in a uniformly bounded number of steps consisting of pants decompositions whose components have uniformly bounded  $x$ -length. The lemma is proven.  $\square$

Lemma 4.1 together with the results of [MM00] imply the following projection-diameter control.

**Corollary 4.2.** *Let  $\xi : [0, R] \rightarrow \mathcal{T}(S)$  be any Weil-Petersson geodesic. Let  $\alpha \in \mathcal{C}(S)$  be a simple closed curve with  $\ell_{\xi(t)}(\alpha) \geq \chi_1$  for every  $t \in [0, R]$ , where  $\chi_1 > 0$  is as in Lemma 4.1. Let  $P, Q$  be Bers decompositions for  $\xi(0), \xi(R)$ . Then*

$$\text{diam}(\pi_\alpha(P) \cup \pi_\alpha(Q)) \leq 2\chi_1 R + 4\chi_1.$$

*Proof.* Let  $\xi : [0, R] \rightarrow \mathcal{T}(S)$  be a Weil-Petersson geodesic and let  $P_0, P_1$  be Bers decompositions for  $\xi(0), \xi(R)$ . Let moreover  $\alpha \in \mathcal{C}(S)$  be a simple closed curve with  $\ell_{\xi(t)}(\alpha) \geq \chi_1$  for every  $t \in [0, R]$ . By Lemma 4.1,  $P_0$  can be connected to  $P_1$  by a path in  $\mathcal{PG}(S)$  of length  $m \leq R\chi_1 + \chi_1$  which passes through vertices  $Q_0 = P_0, \dots, Q_m = P_1$  of the pants graph defined by pants decompositions with component curves of  $\xi(t)$ -length smaller than  $\chi_1$  for some  $t \in [0, R]$ . Since  $\ell_{\xi(t)}(\alpha) \geq \chi_1$  for all  $t$ , each of the pants decompositions  $Q_i$  intersects  $\alpha$  transversely.

Now by Lemma 2.3 of [MM00], for every pants decomposition  $P$  of  $S$  with an essential intersection with  $\alpha$ , the projection  $\pi_\alpha(P)$  of  $P$  to  $\mathcal{CG}(\alpha)$  is non-empty and of diameter at most 2. If for some  $i < m$  the pants decomposition  $Q_{i+1}$  is obtained from  $Q_i$  by an elementary move preserving at least one curve which intersects  $\alpha$  transversely, then the projection of this curve to  $\mathcal{CG}(\alpha)$  is contained in  $\pi_\alpha(Q_i) \cap \pi_\alpha(Q_{i+1})$  and therefore the diameter of  $\pi_\alpha(Q_i) \cup \pi_\alpha(Q_{i+1})$  is at most 4. Otherwise the elementary move which transforms  $Q_i$  to  $Q_{i+1}$  exchanges two simple closed curves  $\beta_i, \beta_{i+1}$ , and the connected component  $Y$  containing  $\beta_i$  of  $S - (Q_i - \beta_i)$  contains  $\alpha$ . Now  $Y$  is a four-holed sphere or a one-holed torus which is bounded by simple closed curves in  $Q_i \cap Q_{i+1}$ , and  $\alpha$  intersects both  $\beta_i$  and  $\beta_{i+1}$  transversely. However, in this case the diameter of  $\pi_\alpha(Q_i) \cup \pi_\alpha(Q_{i+1}) = \pi_\alpha(\beta_i) \cup \pi_\alpha(\beta_{i+1})$  is also bounded from above by 4 by another application of Lemma 2.3 of [MM00].

As a consequence and by induction, the diameter in  $\mathcal{CG}(\alpha)$  of the projection  $\pi_\alpha(P_0) \cup \pi_\alpha(P_1)$  is at most  $2(m+1) \leq 2\chi_1(R+1) + 2 \leq 2\chi_1(R+1) + 2\chi_1$  (note that  $\chi_1 > 1$  by construction). The corollary follows.  $\square$

Let again  $\chi_0 > 0$  be a Bers constant for  $S$  and let  $\chi_1 > \chi_0$  be as in Lemma 4.1. By the collar lemma and the fact that the distance in  $\mathcal{CG}(S)$  between any two simple closed curves  $\alpha, \beta \in \mathcal{C}(S)$  does not exceed  $i(\alpha, \beta) + 1$  (Lemma 2.1 of [MM99] and [Bw06]), there is a number  $p = p(\chi_0, \chi_1) > 0$  with the following property. Let  $x \in \mathcal{T}(S)$  and let  $\alpha$  be a simple closed curve on  $S$  whose  $x$ -length is at most  $\chi_0$ . If  $d_{\mathcal{C}}(\alpha, \beta) \geq p - 1$  then the  $x$ -length of  $\beta$  is bigger than  $\chi_1$ .

We use Corollary 4.2 and the results of Wolpert [W03] to control Weil-Petersson geodesics in the thin part of Teichmüller space.

**Proposition 4.3.** *For every  $R > 1, c > 0$  there is a number  $\epsilon = \epsilon(R, c) > 0$  with the following property. Let  $\zeta : [0, \sigma] \rightarrow \mathcal{T}(S)$  be a WP-geodesic segment of length  $\sigma \leq R$  and let  $P_0, P_\sigma$  be Bers decompositions for  $\zeta(0), \zeta(\sigma)$ . Let  $\alpha \in \mathcal{C}(S)$  be a curve whose distance in  $\mathcal{CG}(S)$  to any component of  $P_0, P_\sigma$  is at least  $p$ . Assume that  $\text{diam}(\pi_\beta(P_0) \cup \pi_\beta(P_\sigma)) \leq c$  for every simple closed curve  $\beta \in \mathcal{C}(S)$  with  $d_{\mathcal{C}}(\alpha, \beta) \leq 1$ . Then  $\ell_{\zeta(t)}(\alpha) \geq \epsilon$  for all  $t \in [0, \sigma]$ .*

*Proof.* The proof relies on Wolpert's description of Weil-Petersson geodesics near the completion locus of Teichmüller space as explained in [W03].

We argue by contradiction and we assume that the statement of the proposition does not hold. Then there are numbers  $R > 0, c > 0$  and there is a sequence  $\epsilon_i \rightarrow 0$ , a sequence of WP-geodesics  $\zeta_i : [0, \sigma_i] \rightarrow \mathcal{T}(S)$ , a sequence of numbers  $t_i \in (0, \sigma_i)$  and a sequence of simple closed curves  $\alpha_i \in \mathcal{C}(S)$  such that for every  $i$  the following holds true.

- (1)  $\sigma_i \leq R$ .
- (2) The distance in  $\mathcal{CG}(S)$  between  $\alpha_i$  and any component of some Bers decomposition  $P_i, P_{\sigma_i}$  of  $\zeta_i(0), \zeta_i(\sigma_i)$  is at least  $p$ .
- (3)  $\text{diam}(\pi_\beta(P_i) \cup \pi_\beta(P_{\sigma_i})) \leq c$  for every simple closed curve  $\beta \in \mathcal{C}(S)$  with  $d_{\mathcal{C}}(\alpha_i, \beta) \leq 1$ .
- (4) There is some  $t_i \in (0, \sigma_i)$  such that  $\ell_{\zeta_i(t_i)}(\alpha_i) \leq \epsilon_i$ .

By the choice of the constants  $p > 0$  and  $\chi_1 > 0$ , for each  $i$  we have  $\ell_{\zeta_i(0)}(\alpha_i) \geq \chi_1, \ell_{\zeta_i(\sigma_i)}(\alpha_i) \geq \chi_1$ . On the other hand, by Wolpert's gradient estimates for length functions (see Lemma 3.12 of [W07a]), for every  $\beta \in \mathcal{C}(S)$  the norm of the Weil-Petersson gradient of the length function  $x \rightarrow \ell_x(\beta)$  is uniformly bounded on  $\{x \mid \ell_x(\beta) \leq \chi_1\}$ . This implies that the Weil-Petersson distance between a point  $x \in \mathcal{T}(S)$  with  $\ell_x(\alpha_i) \geq \chi_1$  and a point  $y \in \mathcal{T}(S)$  with  $\ell_y(\alpha_i) \leq \chi_0/2$  is bounded from below by a universal constant  $a > 0$ . In particular, we have  $\sigma_i \geq 2a$  for all sufficiently large  $i$  and hence by passing to a subsequence we may assume that  $\sigma_i \rightarrow \sigma \in [2a, R]$ .

The mapping class group acts on the Weil-Petersson completion  $\overline{\mathcal{T}(S)}$  of Teichmüller space. The quotient  $\overline{\mathcal{T}(S)}/\text{Mod}(S)$  is just the Deligne-Mumford compactification of moduli space, in particular it is compact. Thus up to passing to another subsequence and up to the action of the mapping class group, we may assume that the initial points  $\zeta_i(0)$  of the geodesics  $\zeta_i$  converge to a point  $x_0 \in \overline{\mathcal{T}(S)}$  (compare the discussion in [W03]).

A point in  $\overline{\mathcal{T}(S)} - \mathcal{T}(S)$  is a surface with nodes, where a node is obtained by pinching a simple closed curve on  $S$  to a point. For a simple multi-curve  $c$  on  $S$  let  $\mathcal{T}(c) \subset (\overline{\mathcal{T}(S)} - \mathcal{T}(S))$  be the stratum of the completion locus for the Weil-Petersson metric which consists of all Riemann surfaces with nodes at the components of  $c$  (i.e. Riemann surfaces obtained from  $S$  by pinching the components of  $c$  to punctures). By Proposition 23 of [W03], up to passing to another subsequence and up to possibly a composition with Dehn multi-twists about the nodes of  $x_0$ , there exists a finite partition  $0 = t_0 < t_1 < \dots < t_k = \sigma$  of the interval  $[0, \sigma]$  and there are simple multi-curves  $c_0, c_1, \dots, c_k$  and points  $x_j \in \mathcal{T}(c_j)$  such that the following holds true.

For  $0 < j \leq k - 1$  consider the (possibly trivial) multi-curve  $\tau_j = c_j \cap c_{j+1}$ . If  $j < k - 1$  then  $\tau_j$  is a proper subset of both  $c_j$  and  $c_{j+1}$ , and  $\tau_0 = c_0 \cap c_1$  is a proper subset of  $c_1$ ,  $\tau_{k-1}$  is a proper subset of  $c_{k-1}$ . For each  $i$  and each  $1 \leq j \leq k - 1$  there is a Dehn multi-twist  $T_{(j,i)}$  about the components of  $c_j - \tau_j$  such that on the parameter interval  $[t_j, t_{j+1}]$  the arcs

$$(2) \quad T_{(j,i)} \circ \dots \circ T_{(1,i)} \zeta_i$$

converge as  $i \rightarrow \infty$  to the geodesic arc  $\xi_j$  in  $\overline{\mathcal{T}(S)}$  connecting  $x_j$  to  $x_{j+1}$  in the sense of parametrized unit-speed curves. The concatenation  $\xi : [0, \sigma] \rightarrow \overline{\mathcal{T}(S)}$  of the arcs  $\xi_j$  ( $0 \leq j \leq k-1$ ) is the piecewise Weil-Petersson geodesic  $\xi : [0, \sigma] \rightarrow \overline{\mathcal{T}(S)}$  connecting  $x_0$  to  $x_k$  and passing through  $x_1, \dots, x_{k-1}$  in this order.

For each  $i$  the  $\zeta_i(0)$ -length and the  $\zeta_i(\sigma_i)$ -length of the curve  $\alpha_i$  is bounded from below by  $\chi_1$ , and the minimum of the length of  $\alpha_i$  along the WP-geodesics  $\zeta_i$  tends to zero as  $i \rightarrow \infty$ . This implies that  $k \geq 2$  and that up to passing to a subsequence, there is a number  $j_2 \in \{1, \dots, k-1\}$  such that for every sufficiently large  $i$  we have

$$T_{(j_2-1, i)} \circ \dots \circ T_{(1, i)} \alpha_i = \alpha \in c_{j_2} - \tau_{j_2}.$$

By Proposition 23 of [W03], the sequence of Dehn multi-twists  $T_{(j_2, i)}$  about the components of  $c_{j_2} - \tau_{j_2}$  is unbounded as  $i \rightarrow \infty$ . In particular, up to passing to a subsequence there is a simple closed curve  $\beta \in c_{j_2} - \tau_{j_2}$  with the following property. Let  $T_\beta$  be the Dehn twist about  $\beta$ . Then there is a sequence  $r(i) \rightarrow \infty$  such that

$$T_{(j_2, i)} = T_\beta^{r(i)} \circ \hat{T}_{(j_2, i)}$$

where  $\hat{T}_{(j_2, i)}$  is a (possibly trivial) Dehn multi-twist about the components of  $c_{j_2} - \tau_{j_2} - \beta$ . Note that  $d_{\mathcal{C}}(\alpha, \beta) \leq 1$  and hence by invariance under the action of the mapping class group, for sufficiently large  $i$  the distance in  $\mathcal{CG}(S)$  between  $\alpha_i$  and

$$\beta_i = (T_{(j_2-1, i)} \circ \dots \circ T_{(1, i)})^{-1} \beta$$

is at most one. By property 2) above and the choice of  $p$ , this implies that the  $\zeta_i(0)$ -length of  $\beta_i$  and the  $\zeta_i(\sigma_i)$ -length of  $\beta_i$  is at least  $\chi_1$ . Moreover, the curve  $\beta_i$  becomes short along  $\zeta_i$ .

Assume first that  $\beta$  does not intersect any of the multi-curves  $c_j$  ( $0 \leq j \leq k$ ). Then  $\beta$  is invariant under each of the Dehn multi-twists  $T_{(j, i)}$  and we have  $\beta_i = \beta$  for all sufficiently large  $i$ . In particular, since the arcs  $T_{(j, i)} \circ \dots \circ T_{(1, i)} \zeta_i|_{[t_j, t_{j+1}]}$  converge as  $i \rightarrow \infty$  to the arc  $\xi_j$ , the  $\zeta_i(0)$ -length and the  $\zeta_i(\sigma_i)$ -length of  $\beta$  is bounded independent of  $i$ . Property 2) above and the choice of the number  $p$  yield that this length is at least  $\chi_1$ . Moreover, the curve  $\beta$  becomes short along the geodesics  $\zeta_i$  and we have  $d_{\mathcal{C}}(\beta, \alpha_i) \leq 1$  for all  $i$ .

Call a pants decomposition  $P$  of  $S$  a Bers decomposition for some  $x \in \overline{\mathcal{T}(S)}$  if the  $x$ -length of any component of  $P$  is at most  $\chi_0$  (this means in particular that  $P$  contains all nodes of  $x$ ). Length functions on  $\mathcal{T}(S)$  extend continuously to functions on  $\overline{\mathcal{T}(S)}$  with values in  $[0, \infty]$ . Thus by the collar lemma, there is a neighborhood  $U$  of  $x_0$  in  $\overline{\mathcal{T}(S)}$  such that the number of pants decompositions which are Bers decompositions for points in  $U$  is finite. Namely, if the simple closed curve  $\omega$  is a node of  $x_0$  then there is a neighborhood  $V$  of  $x_0$  in  $\overline{\mathcal{T}(S)}$  such that every Bers decomposition for every  $x \in V$  contains  $\omega$ . Consequently, by passing to another subsequence we may assume that the Bers decompositions  $P_i$  for  $\zeta_i(0)$  coincide for all  $i$ . We denote this common pants decomposition by  $P$ .

For each  $i$  let as before  $P_{\sigma_i}$  be a Bers decomposition for  $\zeta_i(\sigma_i)$ . We claim that

$$\text{diam}(\pi_\beta(P) \cup \pi_\beta(P_{\sigma_i})) \rightarrow \infty \ (i \rightarrow \infty).$$

Namely, the Dehn twist  $T_\beta$  about  $\beta$  commutes with each of the Dehn multi-twists  $T_{(j,i)}$ . Write  $\hat{T}_{(j,i)} = T_{(j,i)}$  for  $j \neq j_2$  and let as before  $\hat{T}_{(j,i)}$  be the (possibly trivial) Dehn multi-twist about the components of  $c_{j_2} - \tau_{j_2} - \beta$  such that  $T_{(j_2,i)} = T_\beta^{r(i)} \circ \hat{T}_{(j_2,i)}$  for some  $r(i) \in \mathbb{Z}$ . Then we have

$$T_{(k-1,i)} \circ \cdots \circ T_{(1,i)} = T_\beta^{r(i)} \circ (\hat{T}_{(k-1,i)} \circ \cdots \circ \hat{T}_{(1,i)})$$

for all  $i$ .

Now if  $\omega \in \mathcal{C}(S)$  is any simple closed curve and if  $T$  is any Dehn multi-twist about a simple closed curve  $\kappa \in \mathcal{C}(S)$  which is disjoint from  $\beta$  then  $\pi_\beta(\omega) = \pi_\beta(T\omega)$ . As a consequence, for each  $i$  we have

$$(3) \quad \pi_\beta(P_{\sigma_i}) = \pi_\beta(\hat{T}_{(k-1,i)} \circ \cdots \circ \hat{T}_{(1,i)}(P_{\sigma_i})) \subset \mathcal{CG}(\beta).$$

Proposition 23 of [W03] together with  $\text{Mod}(S)$ -invariance of the Weil-Petersson metric shows that the Weil-Petersson distance between the points

$$(T_{(k-1,i)} \circ \cdots \circ T_{(1,i)})^{-1}(x_k) = (\hat{T}_{(k-1,i)} \circ \cdots \circ \hat{T}_{(1,i)})^{-1}(T_\beta^{-r(i)}(x_k))$$

and the points  $\zeta_i(\sigma_i)$  converges to zero as  $i \rightarrow \infty$ . Using once more invariance of the Weil-Petersson metric under the mapping class group, we conclude that

$$d_{WP}(\hat{T}_{(k-1,i)} \circ \cdots \circ \hat{T}_{(1,i)}(\zeta_i(\sigma_i)), T_\beta^{-r(i)}(x_k)) \rightarrow 0 \quad (i \rightarrow \infty).$$

Now there are only finitely many Bers decompositions for all points in a small neighborhood of  $x_k$  and therefore by invariance under the action of the mapping class group and by (3) above, up to passing to a subsequence there is a Bers decomposition  $\hat{P}$  for  $x_k$  such that

$$\pi_\beta(P_{\sigma_i}) = \pi_\beta(T_\beta^{-r(i)}\hat{P})$$

for all sufficiently large  $i$ .

Since the diameter in  $\mathcal{CG}(\beta)$  of the projection  $\pi_\beta(P) \cup \pi_\beta(\hat{P})$  is finite and since  $|r(i)| \rightarrow \infty (i \rightarrow \infty)$  this implies that the diameters of the projections  $\pi_\beta(P) \cup \pi_\beta(P_{\sigma_i})$  tend to infinity with  $i$ . But  $P_i = P$  for all  $i$  and  $d_{\mathcal{C}}(\beta, \alpha_i) \leq 1$  and hence this contradicts the assumption 3) above.

For the proof of the proposition we are now left with analyzing the case that there is some  $j \in \{0, \dots, k\}$  such that  $\beta$  intersects the multi-curve  $c_j$ . Let  $j_1 \in \{0, \dots, j_2 - 1\}$  be the maximal number  $j \leq j_2 - 1$  such that  $i(c_j, \beta) \neq 0$ . If there is no such  $j$  then write  $j_1 = -1$ . Similarly, let  $j_3 \in \{j_2 + 1, \dots, k\}$  be the minimal number  $j \geq j_2 + 1$  such that  $i(c_j, \beta) \neq 0$ . If there is no such number then write  $j_3 = k + 1$ . By our assumption, we either have  $j_1 \geq 0$  or  $j_3 \leq k$ . For  $j_1 < j < j_3$ , the curve  $\beta$  is invariant under the Dehn multi-twist  $T_{(j,i)}$ . Define

$$\rho_i = T_{(j_1,i)} \circ \cdots \circ T_{(1,i)}\zeta_i.$$

If  $j_1 > -1$  then we have  $\ell_{\rho_i(t_{j_1})}(\beta) \rightarrow \infty (i \rightarrow \infty)$ ,  $\ell_{\rho_i(t_{j_1+1})}(\beta) \rightarrow 0 (i \rightarrow \infty)$  and therefore by convexity of length functions along Weil-Petersson geodesics, for sufficiently large  $i$  there is a unique number  $s_i \in (t_{j_1}, t_{j_2})$  such that  $\ell_{\rho_i(s_i)} = \chi_1$ . If  $j_1 = -1$  then using once more convexity and the assumption that the  $\zeta_i(0)$ -length of  $\beta_i$  is not smaller than  $\chi_1$  we also find a unique number  $s_i \in (t_{j_1}, t_{j_2})$  such that  $\ell_{\rho_i(s_i)} = \chi_1$ . Similarly we find for all sufficiently large  $i$  a unique number

$u_i \in (t_{j_2}, t_{j_3})$  such that  $\ell_{\rho_i(u_i)}(\beta) = \chi_1$ . By passing to a subsequence, we may assume that  $s_i \rightarrow s, u_i \rightarrow u$ . Using again Wolpert's gradient bound for length functions, we have  $s \in (t_{j_1}, t_{j_2})$  and  $u \in (t_{j_2}, t_{j_3})$ .

By the choice of  $\beta$  and the consideration in the first part of this proof, applied to the Weil-Petersson geodesics  $\rho_i$  and Bers decompositions  $Q_i^0$  for  $\rho_i(s_i), Q_i^1$  for  $\rho_i(u_i)$ , we conclude that the diameters of the projections

$$\pi_\beta(Q_i^0) \cup \pi_\beta(Q_i^1)$$

tend to infinity with  $i$ . By equivariance under the mapping class group, if

$$\beta_i = (T_{(j_1, i)} \circ \cdots \circ T_{(1, i)})^{-1} \beta$$

and if  $\hat{Q}_i^0, \hat{Q}_i^1$  are Bers decompositions for  $\zeta_i(s_i), \zeta_i(u_i)$ , then the diameter of

$$\pi_{\beta_i}(\hat{Q}_i^0) \cup \pi_{\beta_i}(\hat{Q}_i^1)$$

tends to infinity with  $i$  as well.

On the other hand, by convexity and invariance under the action of the mapping class group, the length of  $\beta_i$  is at least  $\chi_1$  on  $[0, s_i] \cup [u_i, \sigma_i]$ . Corollary 4.2, applied to the Weil-Petersson geodesics  $\zeta_i[0, s_i]$  and  $\zeta_i[u_i, \sigma_i]$  of length at most  $R$  and the curves  $\beta_i \in \mathcal{C}(S)$  then shows that

$$\max\{\text{diam}(\pi_{\beta_i}(P_i) \cup \pi_{\beta_i}(\hat{Q}_i^0)), \text{diam}(\pi_{\beta_i}(\hat{Q}_i^1) \cup \pi_{\beta_i}(P_{\sigma_i}))\} \leq 2\chi_1 R + 4\chi_1.$$

However, this means that  $\text{diam}(\pi_{\beta_i}(P_i) \cup \pi_{\beta_i}(P_{\sigma_i})) \rightarrow \infty$  ( $i \rightarrow \infty$ ) which contradicts assumption 3) above. The proposition is proven.  $\square$

## 5. LENGTH BOUNDS IN THE THICK PART OF TEICHMÜLLER SPACE

In this section we use the results from Section 4 to estimate the length of the intersection of a family of Weil-Petersson geodesics with the thick part of Teichmüller space. In particular, we show that for every  $\epsilon > 0$ , a Weil-Petersson geodesic  $\zeta : [0, \sigma] \rightarrow \mathcal{T}(S)$  which connects two points on a Teichmüller geodesic  $\gamma \subset \mathcal{T}(S)_\epsilon$  spends a fixed percentage of time in the thick part of Teichmüller space. The main two ingredients for the proof of this fact are Proposition 4.3 and the following consequence of hyperbolicity of the curve graph (which holds true for every hyperbolic geodesic metric space).

**Lemma 5.1.** *For every  $\kappa > 1$  there is a number  $b(\kappa) > 0$  and for every  $m > 0$  there are numbers  $\tau = \tau(\kappa, m) > 0, T = T(\kappa, m) > 0$  with the following properties. Let  $k > 0$ , let  $\rho : [0, k] \rightarrow \mathcal{CG}(S)$  be any geodesic in  $\mathcal{CG}(S)$  and let  $\omega : [0, r] \rightarrow \mathcal{CG}(S)$  be a one-Lipschitz curve of length  $r \leq \kappa k$  connecting  $\omega(0) = \rho(0)$  to  $\omega(r) = \rho(k)$ . Then there is a set  $A \subset \{0, \dots, k-1\}$  of cardinality at least  $\tau k$  and for every  $i \in A$  there are numbers  $r_i \in [0, r], s_i \in [r_i, r_i + T]$  with  $r_{i+1} \geq s_i$  and there is a number  $j_i \geq i + m$  so that*

$$d_{\mathcal{C}}(\omega(r_i), \rho(i)) \leq b(\kappa), d_{\mathcal{C}}(\omega(s_i), \rho(j_i)) \leq b(\kappa) \quad (i = 1, 2).$$

*Proof.* Let  $\rho : [0, k] \rightarrow \mathcal{CG}(S)$  be any geodesic in the curve graph. Let  $\Pi : \mathcal{CG}(S) \rightarrow \rho[0, k]$  be a shortest distance projection (i.e.  $\Pi$  associates to a point  $x \in \mathcal{CG}(S)$  a point  $\Pi(x) \in \rho[0, k]$  of minimal distance; note that such a point need not be unique, and the map  $\Pi$  need not be continuous). By hyperbolicity of  $\mathcal{CG}(S)$ , there are numbers  $a \geq 1, b > 0$  only depending on the hyperbolicity constant such that the projection  $\Pi$  satisfies the following contraction properties (see Section 2 of [MM99] for these properties in the case of the curve graph).

- (1) If  $d_{\mathcal{C}}(x, y) \leq 1$  then  $d_{\mathcal{C}}(\Pi(x), \Pi(y)) \leq a$ .
- (2) If  $d_{\mathcal{C}}(x, \Pi(x)) \geq a$  and  $d_{\mathcal{C}}(x, y) \leq bd_{\mathcal{C}}(x, \Pi(x))$  then  $d_{\mathcal{C}}(\Pi(x), \Pi(y)) \leq a$ .

As a consequence, the following holds true. For every  $p > a$  and for every 1-Lipschitz path  $\theta : [0, s] \rightarrow \mathcal{CG}(S)$  of length  $s > 0$  which does not intersect the  $p$ -neighborhood of  $\rho[0, k]$ , we have

$$(4) \quad \text{diam}(\Pi\theta[0, s]) \leq as/bp + a.$$

Namely, by property 2) above, the claim holds true if  $d_{\mathcal{C}}(\theta(0), \theta(\sigma)) < bp$  for all  $\sigma \in [0, s]$ . Otherwise let  $s_1 \geq bp$  be the smallest number such that  $d_{\mathcal{C}}(\theta(0), \theta(s_1)) = bp$ . Then  $d_{\mathcal{C}}(\Pi\theta(0), \Pi\theta(\sigma)) \leq a$  for every  $\sigma \in [0, s_1]$ . Moreover,  $d_{\mathcal{C}}(\theta(s_1), \Pi\theta(s_1)) \geq p$  and hence we can repeat this argument for the interval  $[s_1, s_2]$  where  $s_2 \geq s_1 + bp$  is the smallest number such that  $d_{\mathcal{C}}(\theta(s_1), \theta(s_2)) = bp$ . The estimate (4) now follows by induction.

Let  $\kappa > 1$  and let  $\omega : [0, r] \rightarrow \mathcal{CG}(S)$  be any 1-Lipschitz path of length  $r \leq \kappa k$  connecting  $\omega(0) = \rho(0)$  to  $\omega(r) = \rho(k)$ . Then property (1) above implies that the image of  $\omega[0, r]$  under the projection  $\Pi$  is  $a$ -dense in  $\rho[0, k]$ . This means that for every  $t \in [0, k]$  there is some  $s \in [0, r]$  such that  $d_{\mathcal{C}}(\Pi(\omega(s)), \rho(t)) \leq a$ . Moreover, we have  $\Pi(\omega(0)) = \rho(0)$  and  $\Pi(\omega(r)) = \rho(k)$ .

Identify  $\rho[0, k]$  with the line segment  $[0, k]$ . Choose inductively a sequence  $0 = q_0 < \dots < q_\ell < r \leq \kappa k$  by requiring that  $q_i$  is the smallest number such that  $d_{\mathcal{C}}(\Pi(\omega(q_i)), \rho(5ai)) \leq a$ . Note that  $k/5a \leq \ell \leq k/5a + 1$  and

$$\Pi(\omega(q_i)) + 3a \leq \Pi(\omega(q_{i+1})) \leq \Pi(\omega(q_i)) + 7a \text{ for all } i.$$

Define

$$A = \{i \leq \ell \mid q_{i+1} \leq q_i + 15\kappa a\}.$$

Since the length of  $\omega$  does not exceed  $\kappa k$  and  $\ell \geq k/5a$ , the cardinality of the subset  $A$  of  $\{1, \dots, \ell\}$  exceeds  $2\ell/3 > k/8a$ .

Now if  $i \in A$  then the arc  $\omega[q_i, q_{i+1}]$  of length at most  $15\kappa a$  is mapped by  $\Pi$  to a subset of  $[0, k]$  of diameter at least  $3a$ . Thus by the estimate (4), if  $\chi > 0$  is such that  $15\kappa a^2/b\chi + a = 3a$  then there is some  $j(i) \in [q_i, q_{i+1}]$  such that the distance between  $\omega(j(i))$  and  $\rho[0, k]$  is at most  $\chi$ . Since  $q_{i+1} \leq q_i + 15\kappa a$ , since  $\omega$  is a one-Lipschitz-curve and since the projection  $\Pi$  is distance minimizing, the distance between  $\omega(q_i)$  and  $\Pi\omega(q_i)$  is bounded from above by  $\chi + 15\kappa a$ , i.e. by a universal constant only depending on  $\kappa$ .

Let  $m > 1$  and let  $T_m : \mathbb{Z} \rightarrow \mathbb{Z}$  be the translation  $T_m(z) = z - m$ . Since the cardinality of  $A \subset \{0, \dots, \ell\}$  is not smaller than  $2\ell/3$ , the cardinality of the intersection  $A \cap T_m(A)$  is at least  $\ell/3 - m$ . The cardinality of a maximal subset

$Q \subset A \cap T_m(A)$  with the additional property that if  $z \in Q$  then  $q \notin Q$  for every  $q \in \{z+1, \dots, z+m\}$  is at least  $\ell/3m - m \geq k/15am - m$ .

Define  $\tau = \tau(\kappa, m) = 1/40\kappa ma$  and  $T = T(\kappa, m) = 30\kappa ma$ . Since the cardinality of  $Q$  is not smaller than  $k/15ma - m$  and the length of  $\omega$  does not exceed  $\kappa k$ , there is a subset  $\hat{Q} \subset Q$  of cardinality at least  $\tau k$  such that  $q_{i+m} - q_i \leq T$  for every  $i \in \hat{Q}$ .

Note that if  $i \in \hat{Q}$  then

$$\max\{d_{\mathcal{C}}(\omega(q_i), \rho(5ai)), d_{\mathcal{C}}(\omega(q_{i+m}), \rho(5a(i+m)))\} \leq \chi + 16\kappa a$$

and  $d_{\mathcal{C}}(\rho(5ai), \rho(5a(i+m))) \geq 5ma > m$  (recall that  $a \geq 1$ ). This shows the lemma with  $b(\kappa) = \chi + 15\kappa a + 5a$  and with  $r_i = q_i$  and  $s_i = q_{i+m}$  for  $i \in \hat{Q}$ .  $\square$

To use Lemma 5.1 to obtain a control on the intersection of a Weil-Petersson geodesic with the thick part of Teichmüller space we have to compare the distances  $d_{WP}$  and  $d_{\mathcal{T}}$  on  $\mathcal{T}(S)$ . Since we did not find a precise reference in the literature, we include such a (probably well known) comparison here.

**Lemma 5.2.** *There is a number  $L > 0$  such that for all  $x, y \in \mathcal{T}(S)$  the following holds true.*

- (1)  $d_{WP}(x, y) \leq Ld_{\mathcal{T}}(x, y) + L$ .
- (2)  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(x), \Upsilon_{\mathcal{T}}(y)) \leq Ld_{WP}(x, y) + L$ .

*Proof.* Let  $\gamma : [0, r] \rightarrow \mathcal{T}(S)$  be any Teichmüller geodesic. For the first part of the lemma, we have to show that  $d_{WP}(\gamma(0), \gamma(r)) \leq Lr + L$  for a universal constant  $L \geq 1$ .

For this let as before  $\chi_0 > 0$  be a Bers constant for  $S$ . Let  $\Upsilon_{\mathcal{P}} : \mathcal{T}(S) \rightarrow \mathcal{P}(S)$  be any map which associates to a hyperbolic metric  $x \in \mathcal{T}(S)$  a Bers decomposition for  $x$ . By a result of Brock [B03], the map  $\Upsilon_{\mathcal{P}}$  is a quasi-isometry with respect to the Weil-Petersson metric on  $\mathcal{T}(S)$  and the combinatorial metric  $d_{\mathcal{P}}$  on the pants graph  $\mathcal{PC}(S)$ : There is a number  $L_1 > 0$  such that

$$(5) \quad d_{WP}(x, y)/L_1 - L_1 \leq d_{\mathcal{P}}(\Upsilon_{\mathcal{P}}x, \Upsilon_{\mathcal{P}}y) \leq L_1d_{WP}(x, y) + L_1 \text{ for all } x, y \in \mathcal{T}(S).$$

Thus we only have to show that there is a constant  $L_1 > 0$  such that

$$(6) \quad d_{\mathcal{P}}(\Upsilon_{\mathcal{T}}x, \Upsilon_{\mathcal{T}}y) \leq L_1d_{\mathcal{T}}(x, y) + L_1 \text{ for all } x, y \in \mathcal{T}(S).$$

For this we proceed as in the proof of Lemma 4.1. Namely, by Lemma 3.1 of [W79], if  $P$  is a Bers decomposition for  $x \in \mathcal{T}(S)$  and if  $y \in \mathcal{T}(S)$  is such that  $d_{\mathcal{T}}(x, y) \leq 1$  then the  $y$ -length of every pants curve of  $x$  is bounded from above by  $e\chi_0$ . Thus by the collar theorem, for every Bers decomposition  $Q$  for  $y$  and any component  $\alpha$  of  $P$ ,  $\beta$  of  $Q$  the number of intersections between  $\alpha$  and  $\beta$  is uniformly bounded, independent of  $x, y$ .

Now up to the action of the mapping class group, there are only finitely many pairs of pants decompositions  $P, Q$  such that the number of intersections between  $P$  and  $Q$  is uniformly bounded (compare the proof of Lemma 4.1). By invariance, this means that the distance in  $\mathcal{PG}(S)$  between  $\Upsilon_{\mathcal{P}}(x), \Upsilon_{\mathcal{P}}(y)$  is bounded from above by a universal constant not depending on  $x, y$ . The inequality (6) now follows by

induction and the fact that any two points in  $\mathcal{T}(S)$  can be connected by a geodesic for the Teichmüller metric.

To show the second part of the lemma, let  $\Psi : \mathcal{P}(S) \rightarrow \mathcal{C}(S)$  be a map which associates to a pants decomposition  $P \in \mathcal{P}(S)$  one of its component curves. If  $P, P'$  are pants decompositions with  $d_{\mathcal{P}}(P, P') = 1$  then  $P'$  can be obtained from  $P$  by an elementary move and hence  $P \cap P' \neq \emptyset$  (recall that by assumption, the surface  $S$  is not a once punctured torus or a four-punctured sphere). This implies that the distance in  $\mathcal{CG}(S)$  between any component of  $P$  and any component of  $P'$  is at most 2. Since the pants graph  $\mathcal{PG}(S)$  is a geodesic metric graph, we have

$$d_{\mathcal{C}}(\Psi P_1, \Psi P_2) \leq 2d_{\mathcal{P}}(P_1, P_2) \text{ for all } P_1, P_2 \in \mathcal{P}(S).$$

Now the map  $\Upsilon_{\mathcal{T}} : \mathcal{T}(S) \rightarrow \mathcal{C}(S)$  which associates to a point  $x \in \mathcal{T}(S)$  a simple closed curve of  $x$ -length at most  $\chi_0$  can be chosen to coincide with  $\Psi \circ \Upsilon_{\mathcal{P}}$ . Together with inequality (5), the second part of the lemma follows.  $\square$

As before, let  $\chi_0 > 0$  be a Bers constant for  $S$ . By Lemma 2.2, the distance in  $\mathcal{CG}(S)$  between any two simple closed curves  $\alpha, \beta$  whose  $x$ -length is at most  $\chi_0$  for some  $x \in \mathcal{T}(S)$  is at most  $a(\chi_0) > 0$ . In particular, if  $\alpha \in \mathcal{C}(S)$  is of  $x$ -length at most  $\chi_0$  and if  $\beta \in \mathcal{C}(S)$  is such that  $d_{\mathcal{C}}(\alpha, \beta) \geq a(\chi_0) + 3$  then  $\beta$  jointly fills up  $S$  with any simple closed curve whose  $x$ -length is at most  $\chi_0$ .

For  $\alpha \in \mathcal{C}(S)$  recall from Section 4 the definition of the graph  $\mathcal{CG}(\alpha)$  and the definition of the projection  $\pi_{\alpha} : \mathcal{C}(S) \rightarrow \mathcal{CG}(\alpha)$ . The following lemma is a version of an observation in [MM00] and [R05]. We include a proof since we did not find the precise statement we need in the literature.

**Lemma 5.3.** *For every  $R > 0$  there are numbers  $\delta = \delta(R) > 0$ ,  $c = c(R) > 0$  such that for every Teichmüller geodesic arc  $\gamma : [0, r] \rightarrow \mathcal{T}(S)$  of length  $r \leq R$  such that  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(0), \Upsilon_{\mathcal{T}}\gamma(r)) \geq 2a(\chi_0) + 3$  the following is satisfied.*

- (1)  $\gamma[0, r] \subset \mathcal{T}(S)_{\delta}$ .
- (2)  $\text{diam}(\pi_{\beta}(\Upsilon_{\mathcal{T}}\gamma(0)) \cup \pi_{\beta}(\Upsilon_{\mathcal{T}}\gamma(r))) \leq c$  for all  $\beta \in \mathcal{C}(S)$ .

*Proof.* We first claim that for  $R > 0$  a Teichmüller geodesic  $\gamma : [0, r] \rightarrow \mathcal{T}(S)$  of length  $r \leq R$  such that  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(0), \Upsilon_{\mathcal{T}}\gamma(r)) \geq 2a(\chi_0) + 3$  is entirely contained in  $\mathcal{T}(S)_{\delta}$  where  $\delta = \chi_0 e^{-R}$ . Namely, by Lemma 3.1 of [W79], if  $\zeta : \mathbb{R} \rightarrow \mathcal{T}(S)$  is any Teichmüller geodesic and if  $s \in \mathbb{R}$ ,  $\alpha \in \mathcal{C}(S)$  are such that  $\ell_{\zeta(s)}(\alpha) \leq \delta$  then  $\ell_{\zeta(t)}(\alpha) \leq \chi_0$  for every  $t$  with  $|s - t| \leq \log \chi_0 - \log \delta$ . In particular, by Lemma 2.2, if  $|s - t_i| \leq \log \chi_0 - \log \delta$  for  $i = 1, 2$  then we have  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(\zeta(t_1)), \Upsilon_{\mathcal{T}}(\zeta(t_2))) \leq 2a(\chi_0)$ . This shows the first part of the lemma.

To show the second part, we argue by contradiction and we assume that the lemma does not hold. Then there is a number  $R > 0$  and there is a sequence of Teichmüller geodesic arcs  $\gamma_i : [0, r_i] \rightarrow \mathcal{T}(S)_{\delta}$  of length  $r_i \leq R$  (where  $\delta = \chi_0 e^{-R}$ ) and a sequence of simple closed curves  $\beta_i \in \mathcal{C}(S)$  such that

$$\text{diam}(\pi_{\beta_i}(\Upsilon_{\mathcal{T}}\gamma_i(0)) \cup \pi_{\beta_i}(\Upsilon_{\mathcal{T}}\gamma_i(r_i))) \geq i.$$

By equivariance under the action of the mapping class group and cocompactness of this action on  $\mathcal{T}(S)_\delta$ , we may assume that the geodesic arcs  $\gamma_i$  converge uniformly to a geodesic arc  $\gamma : [0, r] \rightarrow \mathcal{T}(S)_\delta$ .

There is a neighborhood  $U$  of  $\gamma(0)$  such that there are only finitely many simple closed curves  $\alpha \in \mathcal{C}(S)$  whose  $x$ -length is at most  $\chi_0$  for some  $x \in U$ , and similarly we can find a neighborhood  $V$  of  $\gamma(r)$ . Thus by passing to a subsequence, we may assume that there are curves  $\alpha_0, \alpha_1 \in \mathcal{C}(S)$  such that  $\Upsilon_{\mathcal{T}}(\gamma_i(0)) = \alpha_0, \Upsilon_{\mathcal{T}}(\gamma_i(r_i)) = \alpha_1$  for all  $i$ .

Let  $P_0, P_1$  be pants decompositions of  $S$  containing  $\alpha_0, \alpha_1$  as components. Then  $P_0$  can be connected to  $P_1$  by a path  $\rho$  of finite length  $m \geq 0$  in the pants graph  $\mathcal{PC}(S)$  of  $S$ . The  $\gamma(0)$ -length of a component of a pants decomposition passed through by this path is bounded from above by some  $c > 0$ .

Now there are only finitely many simple closed curves  $\{\xi_1, \dots, \xi_k\}$  whose  $\gamma(0)$ -length is bounded from above by  $c$ . Let  $n = \max\{\text{diam}(\pi_{\xi_i}(\alpha_0) \cup \pi_{\xi_i}(\alpha_1)) \mid i\}$ . For every curve  $\xi \notin \{\xi_1, \dots, \xi_k\}$ , each of the vertices of the pants graph passed through by the path  $\rho$  is a pants decomposition which intersects  $\xi$  transversely. Therefore by Lemma 2.3 of [MM00] we have

$$\text{diam}(\pi_{\xi}(P_0) \cup \pi_{\xi}(P_1)) \leq 2m + 1$$

(see the proof of Lemma 4.1 for more details on this argument). In other words, for every  $\xi \in \mathcal{C}(S)$  the diameter of the projection  $\pi_{\xi}(P_0) \cup \pi_{\xi}(P_1)$  does not exceed  $\max\{n, 2m + 1\}$ . Since  $\pi_{\xi}(P_0) \cup \pi_{\xi}(P_1) \supset \pi_{\xi}(\alpha_0) \cup \pi_{\xi}(\alpha_1)$ , this is a contradiction and shows the second part of the lemma.  $\square$

We now use Proposition 4.3, and Lemma 5.1-5.3 to show that a WP-geodesic connecting two sufficiently far apart points on a Teichmüller geodesic  $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)_\epsilon$  whose image under  $\Upsilon_{\mathcal{T}}$  makes linear progress in  $\mathcal{CG}(S)$ .

**Proposition 5.4.** *For every  $\theta > 0$  there are numbers  $\delta = \delta(\theta) > 0$  and  $\eta = \eta(\theta) > 0$  with the following property. Let  $n \geq 1/\eta$  and let  $\gamma : [0, n] \rightarrow \mathcal{T}(S)$  be a Teichmüller geodesic such that*

$$d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(\gamma(0)), \Upsilon_{\mathcal{T}}(\gamma(n))) \geq \theta n.$$

*If  $\zeta : [0, \sigma] \rightarrow \mathcal{T}(S)$  is the Weil-Petersson geodesic connecting  $\zeta(0) = \gamma(0)$  to  $\zeta(\sigma) = \gamma(n)$  then  $\ell_{\delta\text{-thick}}(\zeta) \geq \eta n$ .*

*Proof.* Let  $\theta > 0$  and let  $\gamma : [0, n] \rightarrow \mathcal{T}(S)_\epsilon$  be any Teichmüller geodesic of length  $n \geq 2a(\chi_0) + 3/\theta$  such that

$$(7) \quad d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(\gamma(0)), \Upsilon_{\mathcal{T}}(\gamma(n))) \geq \theta n.$$

Here as before,  $a(\chi_0)$  is as in Lemma 2.2.

Lemma 5.2 yields that

$$(8) \quad \begin{aligned} d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(0), \Upsilon_{\mathcal{T}}\gamma(n))/L - 1 &\leq d_{WP}(\gamma(0), \gamma(n)) \\ &\leq 2Ln = Ld_{\mathcal{T}}(\gamma(0), \gamma(n)) \leq 2Ld_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(0), \Upsilon_{\mathcal{T}}\gamma(n))/\theta. \end{aligned}$$

Let  $\zeta : [0, \sigma] \rightarrow \mathcal{T}(S)$  be the WP-geodesic connecting  $\zeta(0) = \gamma(0)$  to  $\zeta(\sigma) = \gamma(n)$ . By the second part of Lemma 5.2, we have  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(\zeta(i)), \Upsilon_{\mathcal{T}}(\zeta(i+1))) \leq 2L$  for all  $i$ . Thus via connecting  $\Upsilon_{\mathcal{T}}(\zeta(i))$  to  $\Upsilon_{\mathcal{T}}(\zeta(i+1))$  by a geodesic segment in  $\mathcal{CG}(S)$  for each  $i$  and subsequent reparametrization we obtain a 1-Lipschitz curve  $\omega : [0, u] \rightarrow \mathcal{CG}(S)$  connecting  $\omega(0) = \Upsilon_{\mathcal{T}}(\gamma(0))$  to  $\omega(u) = \Upsilon_{\mathcal{T}}(\gamma(n))$  whose length  $u$  does not exceed  $2Ld_{WP}(\gamma(0), \gamma(n)) + 2L$ . Inequality (8) then implies that the length  $u$  of  $\omega$  is also bounded from above by

$$u \leq 4L^2 d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(0), \Upsilon_{\mathcal{T}}\gamma(n)) / \theta + 2L \leq \kappa d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(0), \Upsilon_{\mathcal{T}}\gamma(n))$$

where  $\kappa = 8L^2/\theta > 0$  only depends on  $\theta$  (note that  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(0), \Upsilon_{\mathcal{T}}\gamma(n)) \geq 4$ ). Moreover, every point on the curve  $\omega$  is of distance at most  $L$  from a point in  $\Upsilon_{\mathcal{T}}(\zeta[0, \sigma])$ .

Let  $k = d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(0), \Upsilon_{\mathcal{T}}\gamma(n)) \geq \max\{n/\theta, u/\kappa\}$ . By Theorem 2.4, there is a number  $L_0 > 1$  such that the curve  $t \rightarrow \Upsilon_{\mathcal{T}}(\gamma(t))$  is an unparametrized  $L_0$ -quasi-geodesic in the curve graph. Hence by hyperbolicity, the Hausdorff distance between its image and the image of a geodesic  $\rho : [0, k] \rightarrow \mathcal{CG}(S)$  with the same endpoints is bounded from above by a universal constant  $w_0 > 0$ .

Let  $p = p(\chi_0, \chi_1) \geq 1$  be as in Proposition 4.3. For  $\kappa = 4L^2\nu > 1$  let  $b(\kappa) > 0$  be as in Lemma 5.1 and write

$$B = b(\kappa) + w_0 + L.$$

Note that  $B > 0$  only depends on  $\theta$ . Apply Lemma 5.1 to the geodesic  $\rho$  in  $\mathcal{CG}(S)$  of length  $k$ , the one-Lipschitz arc  $\omega : [0, u] \rightarrow \mathcal{CG}(S)$  of length  $u \leq \kappa k$  connecting  $\omega(0) = \rho(0)$  to  $\omega(u) = \rho(k)$  and the constant

$$m = 2p + 4B + 2w_0 + 5a(\chi_0) + 8.$$

Note that  $m > 0$  only depend on  $\theta$ . Using the fact that every point on  $\omega[0, u]$  is of distance at most  $L$  from a point in  $\Upsilon_{\mathcal{T}}(\zeta[0, \sigma])$  and that the Hausdorff distance between  $\Upsilon_{\mathcal{T}}\gamma[0, n]$  and  $\rho[0, k]$  is at most  $w_0$ , we conclude that there are numbers  $\tau = \tau(\theta) > 0$  and  $T = T(\theta) > 0$  and there is a subset  $A_0$  of  $\{1, \dots, k\}$  of cardinality at least  $\tau k$  and for each  $i \in A_0$  there is some  $s_i \in [0, \sigma]$  and some  $t_i \in [0, n]$  such that the following holds true.

- (1)  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\zeta(s_i), \Upsilon_{\mathcal{T}}\gamma(t_i)) \leq B$ .
- (2) There are numbers  $e_i \leq T, r_i > 0$  such that  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\zeta(s_i + e_i), \Upsilon_{\mathcal{T}}\gamma(t_i + r_i)) \leq B$ .
- (3)  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(t_i), \Upsilon_{\mathcal{T}}\gamma(t_i + r_i)) \geq 2p + 4B + 5a(\chi_0) + 8$ .
- (4)  $s_{i+1} \geq s_i + e_i, t_{i+1} \geq t_i + r_i$ .

Since the length  $n$  of the Teichmüller geodesic arc  $\gamma$  does not exceed  $k/\theta$  and since the cardinality of  $A_0$  is at least  $\tau k$ , there is a subset  $A_1$  of  $A_0$  of cardinality at least  $\tau k/2$  such that  $r_i \leq 2/\theta\tau$  for every  $i \in A_1$ . Lemma 5.3 then yields the existence of a number  $c > 0$  only depending on  $\theta$  such that for every  $i \in A_1$  and for every  $\alpha \in \mathcal{C}(S)$  we have

$$(9) \quad \text{diam}(\pi_{\alpha}(\Upsilon_{\mathcal{T}}\gamma(t_i)), \pi_{\alpha}(\Upsilon_{\mathcal{T}}\gamma(t_i + r_i))) \leq c.$$

Now let  $\psi, \xi \in \mathcal{C}(S)$  and let  $s, t \in \mathbb{R}$  be such that

$$d_{\mathcal{C}}(\psi, \Upsilon_{\mathcal{T}}\gamma(t_i)) \leq B, d_{\mathcal{C}}(\xi, \Upsilon_{\mathcal{T}}\gamma(t_i + r_i)) \leq B.$$

If  $\alpha \in \mathcal{C}(S)$  is such that  $d_{\mathcal{C}}(\psi, \alpha) \geq B + 3, d_{\mathcal{C}}(\xi, \alpha) \geq B + 3$  then a geodesic in  $\mathcal{CG}(S)$  connecting  $\psi, \xi$  to  $\Upsilon_{\mathcal{T}}(\gamma(t_i)), \Upsilon_{\mathcal{T}}(\gamma(t_i + r_i))$  does not intersect the ball of radius 2 about  $\alpha$ . In particular, each of the vertices of  $\mathcal{CG}(S)$  passed through by these geodesics intersects  $\alpha$  transversely. Now by Lemma 2.3 of [MM00], if  $\beta_1, \beta_2 \in \mathcal{C}(S)$  are two curves of distance one in  $\mathcal{CG}(S)$  which intersect  $\alpha$  transversely then the diameter in  $\mathcal{CG}(\alpha)$  of the projection  $\pi_{\alpha}(\beta_1) \cup \pi_{\alpha}(\beta_2)$  is at most 2. An inductive application of this fact to a geodesic in  $\mathcal{CG}(S)$  connecting  $\psi, \xi$  to  $\Upsilon_{\mathcal{T}}(\gamma(t_i)), \Upsilon_{\mathcal{T}}(\gamma(t_i + r_i))$  shows that

$$\text{diam}(\pi_{\alpha}(\psi) \cup \pi_{\alpha}(\Upsilon_{\mathcal{T}}\gamma(t_i))) \leq B + 1, \text{diam}(\pi_{\alpha}(\xi) \cup \pi_{\alpha}(\Upsilon_{\mathcal{T}}\gamma(t_i + r_i))) \leq B + 1.$$

Together with the estimate (9) we obtain that

$$(10) \quad \text{diam}(\pi_{\alpha}(\psi), \pi_{\alpha}(\xi)) \leq c + 2B + 2.$$

For  $i \in A_1$  (where  $A_1 \subset \{1, \dots, k\}$  is as above) consider the WP-geodesic arc  $\zeta[s_i, s_i + e_i]$ . By the properties 1)-3) above, we have

$$d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\zeta(s_i), \Upsilon_{\mathcal{T}}\zeta(s_i + e_i)) \geq 2p + 2B + 5a(\chi_0) + 8.$$

Even though the map  $t \rightarrow \Upsilon_{\mathcal{T}}(\zeta(t))$  is not continuous, by the choice of the constant  $a(\chi_0)$  the set  $\Upsilon_{\mathcal{T}}\zeta[0, \sigma]$  is  $a(\chi_0)$ -dense in a curve connecting  $\Upsilon_{\mathcal{T}}(\zeta(0))$  to  $\Upsilon_{\mathcal{T}}(\zeta(\sigma))$ . Therefore there is a number  $t \in (s_i, s_i + e_i)$  so that

$$(11) \quad \begin{aligned} d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\zeta(t), \Upsilon_{\mathcal{T}}\zeta(s_i)) &\geq p + B + 2a(\chi_0) + 4, \\ d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\zeta(t), \Upsilon_{\mathcal{T}}\zeta(s_i + e_i)) &\geq p + B + 2a(\chi_0) + 4. \end{aligned}$$

Note that by the second part of Lemma 5.2, there is a number  $v_0 > 0$  such that  $\min\{|s_i - t|, |s_i + e_i - t|\} \geq v_0$ .

If  $P_i, Q_i$  are Bers decompositions for  $\zeta(s_i), \zeta(s_i + e_i)$ , then using once more the definition of  $a(\chi_0)$ , inequality (11) implies that the distance in  $\mathcal{C}(S)$  between any component of  $P_i, Q_i$  and every component  $\alpha$  of a Bers decomposition for  $\zeta(t)$  is at least  $p + B + 4$ . Thus the estimate (10), applied to components  $\psi, \xi$  of  $P_i, Q_i$  and to every  $\beta \in \mathcal{C}(S)$  with  $d_{\mathcal{C}}(\alpha, \beta) \leq 1$  (which is possible by the property 1) above) yields that

$$\text{diam}(\pi_{\beta}(P_i) \cup \pi_{\beta}(Q_i)) \leq c + 2B + 2.$$

Now  $\alpha$  is an arbitrary component of a Bers decomposition for  $\zeta(t)$  and the length  $e_i$  of the arc  $\zeta[s_i, s_i + e_i]$  does not exceed  $T$ . Therefore Proposition 4.3 shows the existence of a number  $\delta = \delta(\epsilon) > 0$  only depending on  $\theta$  such that  $\zeta(t) \in \mathcal{T}(S)_{2\delta}$ . Since by Wolpert's results the Weil-Petersson distance between  $\mathcal{T}(S)_{2\delta}$  and  $\mathcal{T}(S) - \mathcal{T}(S)_{\delta}$  is bounded from below by a universal constant (compare the beginning of the proof of Lemma 4.1 for a more precise statement), there is a number  $v \leq v_0$  such that  $\ell_{\delta\text{-thick}}(\zeta[s_i, s_i + e_i]) \geq v$ .

Now  $i \in A_1$  was arbitrary and the cardinality of  $A_1$  is not smaller than  $\tau k/2$ . Then the above consideration shows that  $\ell_{\delta\text{-thick}}(\zeta) \geq \tau k v/2 \geq \tau n v/2\theta$ . Since  $\tau, v$  only depend on  $\theta$ , the proposition follows.  $\square$

## 6. FELLOW TRAVELERS IN TEICHMÜLLER SPACE

In this section we complete the proof of the first part of Theorem 2 from the introduction. Our strategy is as follows. First we use the control of Weil-Petersson geodesics established in Sections 3-5 to associate to every Teichmüller geodesic ray  $\gamma_+ : [0, \infty) \rightarrow \mathcal{T}(S)_\epsilon$  a Weil-Petersson geodesic ray  $\eta(\gamma_+)$  whose ending measure is the class of the vertical projective measured geodesic lamination of the quadratic differential defining  $\gamma_+$ . We observe that  $\eta(\gamma_+)$  depends continuously on  $\gamma_+$ . Next we show that if  $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)_\epsilon$  is a biinfinite Teichmüller geodesic and if we write  $\gamma_+ = \gamma[0, \infty)$  and  $\gamma_-(t) = \gamma(-t)$  for  $t \in (-\infty, 0]$  then there is a biinfinite Weil-Petersson geodesic  $\eta(\gamma)$  which is forward asymptotic to  $\eta(\gamma_+)$  and backward asymptotic to  $\eta(\gamma_-)$ . Then we apply the argument from the proof of Proposition 2.6 to conclude that  $\eta(\gamma)$  is a uniform fellow-traveler of  $\gamma$ . Throughout the section, we use the assumptions and notations from Section 2-5.

Let  $\epsilon > 0$ . We begin with associating to a Teichmüller geodesic ray  $\gamma : [0, \infty) \rightarrow \mathcal{T}(S)_\epsilon$  a unique Weil-Petersson geodesic ray which admits the vertical projective measured geodesic lamination of  $\gamma$  as an ending measure. This ray is obtained as a limit of a sequence of WP-geodesics  $\zeta_n$  connecting  $\gamma(0)$  to  $\gamma(n)$  where  $n \rightarrow \infty$ . While up to passing to a subsequence we can always take a limit of the directions of the geodesics  $\zeta_n$  and obtain a ray  $\zeta$ , a priori this ray may be of bounded length even though the lengths of the rays  $\zeta_n$  tend to infinity with  $n$ . The next easy lemma deals with this issue. We formulate it in a form which will be used in Section 7 as well.

**Lemma 6.1.** *Let  $\gamma_i : \mathbb{R} \rightarrow \mathcal{T}(S)$  be a sequence of Teichmüller geodesics converging locally uniformly to a Teichmüller geodesic  $\gamma$ . Assume that the vertical measured geodesic lamination  $\nu$  of the quadratic differential which defines  $\gamma$  is uniquely ergodic. Let  $n_i \rightarrow \infty$  and for each  $i$  let  $\zeta_i : [0, \sigma_i] \rightarrow \mathcal{T}(S)$  be the WP-geodesic connecting  $\zeta_i(0) = \gamma_i(0)$  to  $\zeta_i(\sigma_i) = \gamma_i(n_i)$ . Then up to passing to a subsequence, the geodesics  $\zeta_i$  converge locally uniformly to an infinite WP-ray  $\zeta : [0, \infty) \rightarrow \mathcal{T}(S)$ . The length of  $\nu$  is bounded along  $\zeta$ .*

*Proof.* By Theorem 2.4, there is a number  $L_1 > 0$  such that the image under  $\Upsilon_{\mathcal{T}}$  of every Teichmüller geodesic  $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)$  in an unparametrized  $L_1$ -quasi-geodesic in  $\mathcal{CG}(S)$ . By hyperbolicity of the curve graph, this means in particular that there is a closed connected set  $J \subset \mathbb{R}$  and there is a geodesic  $\rho : J \rightarrow \mathcal{CG}(S)$  such that the Hausdorff distance between  $\rho(J)$  and  $\Upsilon_{\mathcal{T}}(\gamma(\mathbb{R}))$  is bounded from above by a universal constant  $p_1 > 0$ .

Let  $\gamma_i, \gamma : \mathbb{R} \rightarrow \mathcal{T}(S)$  be as in the lemma. Let  $\nu$  be the vertical measured geodesic lamination of  $\gamma$ , normalized in such a way that the  $\gamma(0)$ -length of  $\nu$  equals one. By assumption,  $\nu$  is uniquely ergodic. Its support, viewed as a point in the boundary  $\partial\mathcal{CG}(S)$  of the curve graph  $\mathcal{CG}(S)$ , is the endpoint of the unparametrized quasi-geodesic  $t \rightarrow \Upsilon_{\mathcal{T}}(\gamma(t))$  ( $t \in [0, \infty)$ ) in  $\mathcal{CG}(S)$  (compare the discussion in Section 2). If  $(\beta_i) \subset \mathcal{C}(S)$  is any sequence converging in  $\mathcal{CG}(S) \cup \partial\mathcal{CG}(S)$  to the support of  $\nu$  then by unique ergodicity of  $\nu$ , the projective measured geodesic laminations  $[\beta_i]$  supported in  $\beta_i$  converge in  $\mathcal{PML}$  to the projective measured geodesic lamination  $[\nu]$  which is the class of  $\nu$  (see Theorem 1.4 of [KI99] and Theorem 1.1 of [H06a]).

Since  $\gamma_i \rightarrow \gamma$  uniformly on compact sets, as  $i \rightarrow \infty$  longer and longer subarcs of the uniform unparametrized quasi-geodesic  $\Upsilon_{\mathcal{T}} \circ \gamma$  are uniformly fellow-traveled by subarcs of the uniform unparametrized quasi-geodesics  $\Upsilon_{\mathcal{T}} \circ \gamma_i$ . Thus for a sequence  $n_i \rightarrow \infty$ , the curves  $c_i = \Upsilon_{\mathcal{T}}(\gamma_i(n_i))$  converge in  $\mathcal{CG}(S) \cup \partial\mathcal{CG}(S)$  to the support of  $\nu$  (compare the more detailed argument in the proof of Proposition 3.4 of [H08]). As a consequence, we have  $[c_i] \rightarrow [\nu]$  in  $\mathcal{PML}$ .

For  $i > 0$  let  $\zeta_i$  be the WP-geodesic connecting  $\zeta_i(0) = \gamma_i(0)$  to  $\zeta_i(\sigma_i) = \gamma(n_i)$ . After passing to a subsequence we may assume that the directions  $v_i = \zeta_i'(0)$  of the geodesics  $\zeta_i$  converge as  $i \rightarrow \infty$  to a direction  $v$  with footpoint  $\gamma(0)$ . Let  $\zeta : [0, T) \rightarrow \mathcal{T}(S)$  be the WP-ray with direction  $v$ . Then  $\zeta_i \rightarrow \zeta$  uniformly on compact subsets of  $[0, T)$ .

We claim that the length of  $\nu$  is bounded along  $\zeta$ . Namely, for each  $i$  let  $\mu_i$  be the measured geodesic lamination of  $\gamma_i(0)$ -length one which we obtain from  $\Upsilon_{\mathcal{T}}(\gamma_i(n_i))$  by multiplication with a positive constant. Since  $\gamma_i(0) \rightarrow \gamma(0)$  ( $i \rightarrow \infty$ ) there is a number  $\epsilon > 0$  such that  $\gamma_i(0) \in \mathcal{T}(S)_{\epsilon}$  for all  $i$ . This means that the shortest length of a simple closed curve for the metric  $\gamma_i(0)$  is not smaller than  $\epsilon$ . Therefore  $\mu_i$  is obtained from the counting measure for  $\Upsilon_{\mathcal{T}}(\gamma_i(n_i))$  by multiplication with a constant which does not exceed  $1/\epsilon$ . By definition of the map  $\Upsilon_{\mathcal{T}}$ , the  $\gamma_i(n_i)$ -length of  $\mu_i$  is not bigger than  $\chi_0/\epsilon$ . Thus by convexity, the length of  $\mu_i$  along  $\zeta_i$  is uniformly bounded, independent of  $i$ .

By the above consideration and continuity of the length pairing, as  $i \rightarrow \infty$  the measured geodesic laminations  $\mu_i$  converge weakly to the measured geodesic lamination  $\nu$ . Since  $\zeta_i \rightarrow \zeta$  locally uniformly, continuity of the length pairing implies that the length of  $\nu$  is uniformly bounded along  $\zeta$  (compare the more detailed argument in the proof of Proposition 2.6).

Since  $\nu$  fills up  $S$ , the WP-ray  $\zeta$  is infinite. Namely, otherwise there is a simple closed curve  $c$  on  $S$  so that the  $\zeta(t)$ -length of  $c$  tends to zero as  $t \rightarrow T$ . But  $i(c, \nu) > 0$  and consequently in this case the length of  $\nu$  is unbounded along  $\zeta$  which is a contradiction. The lemma is proven.  $\square$

The following angle control is the second and last preparatory step toward the construction of a WP-ray associated to a Teichmüller geodesic  $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)_{\epsilon}$ .

**Lemma 6.2.** *For every  $\epsilon > 0$  there are numbers  $\alpha = \alpha(\epsilon) > 0$  and  $R_0 = R_0(\epsilon) > 0$  with the following property. Let  $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)_{\epsilon}$  be a Teichmüller geodesic. Let  $R_1, R_2 \geq R_0$  and let  $\xi_1, \xi_2$  be the Weil-Petersson geodesics which connect  $\gamma(0) = \xi_1(0) = \xi_2(0)$  to  $\gamma(-R_1), \gamma(R_2)$ . Then the angle  $\angle_{\gamma(0)}(\xi_1'(0), \xi_2'(0))$  at  $\gamma(0)$  between the geodesics  $\xi_1, \xi_2$  is at least  $\alpha$ .*

*Proof.* We argue by contradiction and we assume that there is some  $\epsilon > 0$  for which the statement of the lemma does not hold. Then there is a sequence of Teichmüller geodesics  $\gamma_i : \mathbb{R} \rightarrow \mathcal{T}(S)_{\epsilon}$  and a sequence of numbers  $R_i, T_i \rightarrow \infty$  so that the angle between the WP-geodesics  $\zeta_i, \xi_i$  connecting  $\gamma_i(0)$  to  $\gamma_i(-R_i), \gamma_i(T_i)$  tends to zero as  $i \rightarrow \infty$ .

By invariance under the action of the mapping class group and cocompactness of the action of  $\text{Mod}(S)$  on  $\mathcal{T}(S)_\epsilon$ , up to passing to a subsequence we may assume that the Teichmüller geodesics  $\gamma_i$  converge as  $i \rightarrow \infty$  to a Teichmüller geodesic  $\gamma$ . By Lemma 6.1, by passing to another subsequence we may assume that the WP-geodesics  $\zeta_i, \xi_i$  converge as  $i \rightarrow \infty$  to infinite Weil-Petersson rays  $\zeta, \xi : [0, \infty) \rightarrow \mathcal{T}(S)$ .

Let  $q_h, q_v$  be the horizontal and vertical measured geodesic lamination, respectively, of the area one quadratic differential  $q$  which is the unit cotangent vector of  $\gamma$  at  $\gamma(0)$ . By Lemma 6.1, the length of  $q_h$  is bounded along  $\zeta$  and the length of  $q_v$  is bounded along  $\xi$ .

On the other hand, by assumption, the angle at  $\gamma_i(0)$  between  $\zeta_i, \xi_i$  converges to zero and therefore we have  $\zeta = \xi$ . As a consequence, the lengths of both  $q_h, q_v$  are bounded along  $\zeta$ . But the measured geodesic laminations  $q_h, q_v$  jointly fill up  $S$  (i.e. we have  $i(\mu, q_h) + i(\mu, q_v) > 0$  for every measured geodesic lamination  $\mu$  on  $S$ ) and hence the function on  $\mathcal{T}(S)$  which associates to a point  $x \in \mathcal{T}(S)$  the value  $\ell_x(q_h) + \ell_x(q_v)$  is proper (see Theorem 1.2 of [K92]). Since  $\zeta$  is an infinite ray, this is a contradiction. The lemma follows.  $\square$

Recall that two infinite Weil-Petersson rays  $\gamma, \zeta : [0, \infty) \rightarrow \mathcal{T}(S)$  are *asymptotic* if the function

$$t \rightarrow d_{WP}(\gamma(t), \zeta(t))$$

is bounded. Lemma 5.4, Lemma 6.1 and Lemma 6.2 is used to show.

- Lemma 6.3.** (1) *Let  $\epsilon > 0$  and let  $\gamma_+ : [0, \infty) \rightarrow \mathcal{T}(S)_\epsilon$  be a Teichmüller geodesic ray. For  $n > 0$  let  $\zeta_n$  be the WP-geodesic connecting  $\zeta_n(0) = \gamma(0)$  to  $\zeta_n(\tau_n) = \gamma(n)$ . Then the geodesics  $\zeta_n$  converge as  $n \rightarrow \infty$  to an infinite WP-ray  $\eta(\gamma_+)$  depending continuously on  $\gamma_+$ .*
- (2) *Let  $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)_\epsilon$  be a biinfinite Teichmüller geodesic with forward and backward rays  $\gamma_+ = \gamma[0, \infty)$ ,  $\gamma_- = \gamma(-\infty, 0]$ . Then up to parametrization, there is a unique biinfinite WP-geodesic  $\eta(\gamma)$  which is forward asymptotic to  $\eta(\gamma_+)$  and backward asymptotic to  $\eta(\gamma_-)$ .*

*Proof.* Let  $\epsilon > 0$  and let  $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)_\epsilon$  be a biinfinite Teichmüller geodesic. We show first that as  $n \rightarrow \infty$ , the WP-geodesics  $\zeta_n$  connecting  $\gamma(0) = \zeta_n(0)$  to  $\gamma(n) = \zeta_n(\tau_n)$  converge as  $n \rightarrow \infty$  uniformly on compact sets to an infinite WP-ray  $\zeta$ . By Lemma 6.1, for this it is enough to show that the directions  $\zeta'_n(0)$  of the geodesics  $\zeta_n$  form a Cauchy sequence in the space of directions at the point  $\gamma(0)$ . In other words, we have to show that for every  $\theta > 0$  there is a number  $n(\theta) > 0$  such that for  $n \geq n(\theta)$  the angle between  $\zeta'_n(0)$  and  $\zeta'_{n(\theta)}(0)$  does not exceed  $\theta$ .

To see that this is the case, let  $\alpha = \alpha(\epsilon) > 0$  be as in Lemma 6.2. By the main theorem in [H08] (see also [MM99, R05]), there is a number  $\theta = \theta(\epsilon) > 0$  such that  $d_{\mathcal{T}}(\Upsilon_{\mathcal{T}}\gamma(s), \Upsilon_{\mathcal{T}}\gamma(t)) \geq \theta|s - t|$  whenever  $|s - t| \geq 1/\theta$ . Let  $\delta = \delta(\theta) = \delta(\epsilon) > 0$  be as in Lemma 5.4. For  $\theta > 0$  let  $m = m(\delta, \alpha, \theta/2) > 0$  be as in the second part of Lemma 3.1. By Proposition 5.4 (and using the notations from the proposition) there is a number  $n_0 = n_0(\theta) > 0$  with the following property. Let  $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)_\epsilon$  be a Teichmüller geodesic and for  $n \geq n/\eta$  we have  $\ell_{\delta\text{-thick}}(\zeta_n) \geq m = m(\delta, \alpha, \theta/2)$ .

Let  $R_0 = R_0(\epsilon) > 0$  be as in Lemma 6.2 and let  $n_1 = n_1(\theta) = \max\{R_0, m/\eta\}$ . By Lemma 6.2, for  $n \geq 2n_1$  the angle at  $\gamma(n_1)$  of the WP-geodesic triangle with vertices  $\gamma(0), \gamma(n_1), \gamma(n)$  is at least  $\alpha$ . Thus by the second part of Lemma 3.1 and the choice of  $n_1 \geq m/\eta$ , for  $n \geq 2n_1$  the angle at  $\gamma(0)$  between the directions  $\zeta'_n(0), \zeta'_{n_1}(0)$  of the geodesics  $\zeta_n, \zeta_{n_1}$  does not exceed  $\theta/2$ . In particular, for all  $n, N > 2n_1$  the angle at  $\gamma(0)$  between the directions  $\zeta'_n(0), \zeta'_N(0)$  does not exceed  $\theta$ . Thus indeed the geodesics  $\zeta_n$  converge as  $n \rightarrow \infty$  locally uniformly to an infinite WP-ray  $\eta(\gamma_+)$ .

The above argument also shows that the WP-geodesic  $\eta(\gamma_+)$  depends continuously on  $\gamma$ . Namely, for every  $\theta > 0$  and every Teichmüller geodesic  $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)$ , for every  $n \geq n_1$  the angle between the direction of the Weil-Petersson ray  $\eta(\gamma_+)$  and the direction of the geodesic  $\zeta_n$  connecting  $\gamma(0)$  to  $\gamma(n)$  is at most  $\theta$ . Since finite-lengths WP-geodesic segments with endpoints in  $\mathcal{T}(S)$  depend smoothly on their endpoints, continuous dependence of  $\eta(\gamma_+)$  on the Teichmüller geodesic  $\gamma$  immediately follows. This shows the first part of the lemma.

To show the second part of the lemma, for  $\delta = \delta(\epsilon) > 0, \alpha = \alpha(\epsilon) > 0$  as above let  $k = k(\delta, \alpha) > 0$  be as in the first part of Lemma 3.1. By Lemma 5.4 and Lemma 3.1 the following holds true. Let  $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)_\epsilon$  be a Teichmüller geodesic and let  $\zeta_t : [0, \sigma] \rightarrow \mathcal{T}(S)$  be the WP-geodesic connecting  $\gamma(0)$  to  $\gamma(t)$  for some  $t \geq k/\eta$ . Then  $\ell_{\delta\text{-thick}}(\zeta) \geq k$ .

Now let  $T_i \rightarrow \infty, U_i \rightarrow \infty$ . Let again  $R_0 = R_0(\epsilon) > 0$  be as in Lemma 6.2. For sufficiently large  $i > 0$  such that  $T_i \geq k/\eta + 2R_0, U_i \geq R_0$  let  $\zeta_i, \xi_i$  be the WP-geodesics connecting  $\gamma(0)$  to  $\gamma(T_i), \gamma(-U_i)$ . Let moreover  $\hat{\zeta}_i$  be the WP-geodesic connecting  $\gamma(k/\eta + R_0)$  to  $\gamma(T_i)$ . By the first part of this lemma and Cat(0)-comparison, the geodesics  $\zeta_i, \hat{\zeta}_i$  converge as  $i \rightarrow \infty$  to infinite asymptotic WP-rays  $\zeta, \hat{\zeta}$ , and the geodesics  $\xi_i$  converge as  $i \rightarrow \infty$  to an infinite WP-ray  $\xi$ .

By the choice of  $R_0$ , for each sufficiently large  $i$  the angle at  $\gamma(0), \gamma(k/\eta + R_0)$  of the WP-geodesic quadrangle with vertices  $\gamma(0), \gamma(k/\eta + R_0), \gamma(T_i), \gamma(-U_i)$  is at least  $\alpha$ . Thus by the choice of  $\eta$  and Corollary 3.2, there is a unique biinfinite WP-geodesic which is forward asymptotic to  $\hat{\zeta}$  and backward asymptotic to  $\xi$ . Since  $\hat{\zeta}$  is asymptotic to  $\zeta$ , this completes the proof of the lemma.  $\square$

Now we are ready to show the first part of Theorem 2 and of Corollary 3 from the introduction. As in the introduction, we always denote by  $J, J'$  a closed connected subset of  $\mathbb{R}$ .

**Proposition 6.4.** (1) *For every  $\epsilon > 0$  there is a number  $R = R(\epsilon) > 0$  with the following property. For every Teichmüller geodesic  $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$  there is a Weil-Petersson geodesic  $\xi : J' \rightarrow \mathcal{T}(S)$  with  $d_H(\gamma(J), \xi(J')) \leq R$ .*

(2) *Let  $K \subset \mathcal{Q}(S)$  be any compact set which is invariant under the Teichmüller geodesic flow  $\Phi_T^t$ . Then there is a conjugacy  $\Lambda : K \rightarrow \mathcal{Q}_{WP}(S)$  of the restriction of  $\Phi_T^t$  to  $K$  into the geodesic flow  $\Phi_{WP}^t$  for the Weil-Petersson metric.*

*Proof.* The argument is similar to the proof of Proposition 2.6. Namely, let  $J \subset \mathbb{R}$  be a closed connected set containing 0 and let  $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$  be a Teichmüller geodesic. We say that the projective measured geodesic lamination  $[\beta]$  defined by a simple closed curve  $\beta \in \mathcal{C}(S)$  is *realized* at some  $t \in J$  if the  $\gamma(t)$ -length of  $\beta$  does not exceed  $\chi_0$ . If  $J$  contains  $[0, \infty)$  then we say that the projectivization  $[q_v]$  of the vertical measured geodesic lamination defined by the unit cotangent vector  $q$  of  $\gamma$  at  $\gamma(0)$  is realized at the right endpoint of  $J$ , and similarly for a left infinite endpoint.

Let  $\Gamma$  be the set of all triples  $(\gamma : J \rightarrow \mathcal{T}(S)_\epsilon, \lambda_+, \lambda_-)$  with the following properties.

- (1)  $J \subset \mathbb{R}$  is a closed connected set containing 0.
- (2)  $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$  is a Teichmüller geodesic.
- (3)  $\lambda_+, \lambda_-$  are measured geodesic laminations of  $\gamma(0)$ -length one, and the projective measured geodesic lamination  $[\lambda_+]$  is realized at the right endpoint of  $J$ , the projective measured geodesic lamination  $[\lambda_-]$  is realized at the left endpoint of  $J$ .

We equip  $\Gamma$  with the product topology, using the weak\*-topology on  $\mathcal{ML}$  for the second and the third component of the triple and the compact-open topology for the arc  $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$ . Note that this topology is metrizable. Moreover,  $\Gamma$  is invariant under the natural action of the mapping class group.

We claim that the action of  $\text{Mod}(S)$  on  $\Gamma$  is cocompact. Since  $\text{Mod}(S)$  acts cocompactly on  $\mathcal{T}(S)_\epsilon$ , for this it is enough to show that the following holds true. If  $\gamma_i : J_i \rightarrow \mathcal{T}(S)_\epsilon$  ( $i > 0$ ) is any sequence of Teichmüller geodesics which converge locally uniformly to a Teichmüller geodesic  $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$ , if the projective measured geodesic lamination  $[\lambda_i]$  is realized at the right endpoint of  $J_i$  and if  $[\lambda_i] \rightarrow [\lambda]$  in  $\mathcal{PML}$  ( $i \rightarrow \infty$ ) then  $[\lambda]$  is realized at the right endpoint of  $J$ . However, that this holds true was shown in the proof of Proposition 3.4 of [H08] (compare also the argument in the proof of Proposition 2.6).

To each triple  $(\gamma : J \rightarrow \mathcal{T}(S)_\epsilon, \lambda_+, \lambda_-) \in \Gamma$  associate a Weil-Petersson geodesic  $\rho(\gamma, \lambda_+, \lambda_-)$  as follows. Assume first that  $J = [-a, b]$  is bounded. Then there is up to parametrization a unique WP-geodesic  $\xi$  connecting  $\gamma(-a)$  to  $\gamma(b)$ . The restriction to  $\xi$  of the function which associates to  $x \in \mathcal{T}(S)$  the sum  $\ell_x(\lambda_+) + \ell_x(\lambda_-)$  is strictly convex and non-constant (unless  $a = b = 0$ ) and hence it assumes a unique minimum along  $\xi$  [W07a]. Let  $\rho(\gamma, \lambda_+, \lambda_-)$  be the parametrization of  $\xi$  so that this minimum is assumed at  $\rho(\gamma, \lambda_+, \lambda_-)(0)$ .

If  $J$  is one-sided unbounded, say if  $J = [-a, \infty)$  for some  $a \geq 0$ , then there is a unique infinite WP-geodesic ray  $\xi$  issuing from  $\gamma(-a)$  which is asymptotic to  $\eta(\gamma_+)$  where  $\eta(\gamma_+)$  is as in Lemma 6.3. The function  $x \rightarrow \ell_x(\lambda_+) + \ell_x(\lambda_-)$  assumes a unique minimum along  $\xi$ . We define  $\rho(\gamma, \lambda_+, \lambda_-)$  to be the parametrization of  $\xi$  for which this minimum is assumed at  $\rho(\gamma, \lambda_+, \lambda_-)(0)$ .

If  $J$  is two-sided infinite then let  $\rho(\gamma, \lambda_+, \lambda_-)$  be the parametrization of the geodesic  $\eta(\gamma)$  as in the second part of Lemma 6.3 so that the minimum of the function  $x \rightarrow \ell_x(\lambda_+) + \ell_x(\lambda_-)$  along  $\eta(\gamma)$  is assumed at  $\rho(\gamma, \lambda_+, \lambda_-)(0)$ .

Now by Lemma 6.3 and continuity and convexity of length functions, the assignment which associates to  $(\gamma : J \rightarrow \mathcal{T}(S)_\epsilon, \lambda_+, \lambda_-) \in \Gamma$  the point  $\rho(\gamma, \lambda_+, \lambda_-)(0)$  is continuous, moreover it is equivariant under the action of the mapping class group. Since  $\text{Mod}(S)$  acts cocompactly on  $\Gamma$ , this means that for every  $(\gamma : J \rightarrow \mathcal{T}(S)_\epsilon, \lambda_+, \lambda_-) \in \Gamma$  the Teichmüller distance between  $\gamma(0)$  and  $\rho(\gamma, \lambda_+, \lambda_-)(0)$  is uniformly bounded. The first part of the proposition now follows as in the proof of Proposition 2.6.

For the second part of the proposition, let  $K \subset \mathcal{Q}(S)$  be a compact set which is invariant under the Teichmüller geodesic flow  $\Phi_{\mathcal{T}}^t$ . Let  $\tilde{K}$  be the preimage of  $K$  in the bundle  $\mathcal{Q}^1(S)$  of all area one quadratic differentials on  $\mathcal{T}(S)$ . Let  $P : \mathcal{Q}^1(S) \rightarrow \mathcal{T}(S)$  be the canonical projection. Then there is a number  $\epsilon > 0$  such that the projection  $P(\tilde{K})$  is contained in  $\mathcal{T}(S)_\epsilon$ . For  $q \in \tilde{K}$  there is a WP-geodesic  $\tilde{\gamma}$  whose Hausdorff distance to the Teichmüller geodesic  $t \rightarrow P\Phi_{\mathcal{T}}^t q$  is uniformly bounded. If  $q_v, q_h$  are the vertical and horizontal measured geodesic lamination of  $q$ , respectively, then the function  $x \rightarrow \ell_x(q_h) + \ell_x(q_v)$  on  $\mathcal{T}(S)$  is proper, and its restriction to  $\tilde{\gamma}$  is strictly convex. Define  $\tilde{\Lambda}(q)$  to be the unit tangent of  $\tilde{\gamma}$  at the unique point on  $\tilde{\gamma}$  where this function assumes its minimum. It follows from the proof of the first part of this proposition that the map  $\tilde{\Lambda}$  is continuous. Since the length of  $q_v$  is strictly decreasing along  $\tilde{\gamma}$  and the length of  $q_h$  is strictly increasing, the restriction of  $\tilde{\Lambda}$  to the flow line  $t \rightarrow \Phi_{\mathcal{T}}^t q$  for the Teichmüller flow is a homeomorphism onto the flow line  $t \rightarrow \Phi_{WP}^t(\tilde{\Lambda}q)$ . Now the map  $\tilde{\Lambda}$  is clearly equivariant under the action of the mapping class group and hence it projects to a map  $\Lambda : K \rightarrow \mathcal{Q}_{WP}(S)$  which defines a conjugacy of the Teichmüller flow on  $K$  into the geodesic flow for the Weil-Petersson metric.  $\square$

## 7. INVARIANT MEASURES

This section is devoted to the proof of Theorem 1 from the introduction. Our strategy is to construct for every  $\Phi_{\mathcal{T}}^t$ -invariant ergodic Borel probability measure  $\mu$  on  $\mathcal{Q}(S)$  a measurable conjugacy  $\Lambda$  of the restriction of  $\Phi_{\mathcal{T}}^t$  to a suitably chosen invariant Borel subset of full  $\mu$ -mass into the geodesic flow of the Weil-Petersson metric. For this we use an extension of the arguments in Section 6. A good geometric control of this conjugacy allows to conclude that there is a  $\Phi_{\mathcal{T}}^t$ -invariant Borel probability measure in the measure class of  $\Lambda_*(\mu)$ .

We continue to use the assumptions and notations from Sections 2-6. Let again  $\Upsilon_{\mathcal{T}} : \mathcal{T}(S) \rightarrow \mathcal{CG}(S)$  be a map which associates to a point  $x \in \mathcal{T}(S)$  a simple closed curve of  $x$ -length at most  $\chi_0$  where  $\chi_0 > 0$  is a Bers constant for  $S$ . Our first goal is to study the distances  $d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}\gamma(0), \Upsilon_{\mathcal{T}}\gamma(t))$  as  $t \rightarrow \infty$  for a Teichmüller geodesic  $\gamma$  whose cotangent line projects to an orbit of the Teichmüller geodesic flow on  $\mathcal{Q}(S)$  which is typical for an invariant ergodic Borel probability measure on  $\mathcal{Q}(S)$ .

For this we face the problem that the map  $\Upsilon_{\mathcal{T}}$  may not be measurable, and the function  $(x, y) \rightarrow d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(x), \Upsilon_{\mathcal{T}}(y))$  on  $\mathcal{T}(S) \times \mathcal{T}(S)$  may not be measurable either. We resolve this problem by using a construction from [H08].

Namely, choose a smooth function  $\sigma : [0, \infty) \rightarrow [0, 1]$  with  $\sigma[0, \chi_0] \equiv 1$  and  $\sigma[2\chi_0, \infty) \equiv 0$ . For every  $h \in \mathcal{T}(S)$  we obtain a finite Borel measure  $\mu_h$  on  $\mathcal{C}(S)$  by defining

$$\mu_h = \sum_{\beta} \sigma(\ell_h(\beta)) \delta_{\beta}$$

where  $\delta_{\beta}$  denotes the Dirac mass at  $\beta$ . By the collar lemma, the number of simple closed geodesics on  $(S, h)$  of length at most  $2\chi_0$  is bounded from above independent of  $h$ . Thus the total mass of  $\mu_h$  is bounded from above and below by a universal positive constant, and by Lemma 2.2, the diameter of the support of  $\mu_h$  in  $\mathcal{CG}(S)$  is uniformly bounded as well.

Define a symmetric non-negative function  $\rho$  on  $\mathcal{T}(S) \times \mathcal{T}(S)$  by

$$\rho(h, h') = \int_{\mathcal{C}(S) \times \mathcal{C}(S)} d_{\mathcal{C}}(\cdot, \cdot) d\mu_h \times d\mu_{h'} / \mu_h(\mathcal{C}(S)) \mu_{h'}(\mathcal{C}(S)).$$

Lemma 3.3 of [H08] shows that the function  $\rho$  is continuous and invariant under the diagonal action of  $\text{Mod}(S)$ . Moreover, there is a universal constant  $a_0 > 0$  such that

$$(12) \quad \rho(h, h') - a_0 \leq d_{\mathcal{C}}(\Upsilon_{\mathcal{T}}(h), \Upsilon_{\mathcal{T}}(h')) \leq \rho(h, h') + a_0 \text{ for all } h, h' \in \mathcal{T}(S).$$

For  $q \in \mathcal{Q}(S)$  and for  $t \geq 0$  we can now define a number  $r(q, t) \geq 0$  as follows. A lift  $\tilde{q}$  of  $q$  to  $\mathcal{Q}^1(S)$  determines a Teichmüller geodesic  $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)$  with initial unit cotangent  $\tilde{q}$ . By invariance of the function  $\rho$  under the diagonal action of the mapping class group, the number  $r(q, t) = \rho(\gamma(0), \gamma(t))$  does not depend on the choice of  $\tilde{q}$ . The function  $r : \mathcal{Q}(S) \times \mathbb{R} \rightarrow [0, \infty)$  is continuous. We have.

**Lemma 7.1.** *Let  $\mu$  be a  $\Phi_{\mathcal{T}}^t$ -invariant ergodic Borel probability measure on  $\mathcal{Q}(S)$ . Then there is a number  $b > 0$  such that*

$$\lim_{t \rightarrow \infty} \frac{1}{t} r(q, t) = b$$

for  $\mu$ -almost every  $q \in \mathcal{Q}(S)$ .

*Proof.* By the triangle inequality for  $d_{\mathcal{C}}$  and the estimate (12) above, for  $q \in \mathcal{Q}(S)$  and  $s, t > 0$  we have

$$r(q, s+t) \leq r(q, s) + r(\Phi_{\mathcal{T}}^s q, t) + 2a_0$$

where  $a_0 > 0$  is as in (12). Thus by the subadditive ergodic theorem [Kr85], for  $\mu$ -almost every  $q \in \mathcal{Q}(S)$  the limit

$$\lim_{t \rightarrow \infty} \frac{1}{t} r(q, t)$$

exists and does not depend on  $q$ . (Here we apply the subadditive ergodic theorem to the function  $(q, t) \rightarrow r(q, t) + 2a_0$ ).

We have to show that this limit is positive almost everywhere. For this recall from Theorem 2.4 that there is a number  $L_1 > 0$  such that the image under  $\Upsilon_{\mathcal{T}}$  of any Teichmüller geodesic is an unparametrized  $L_1$ -quasi-geodesic in  $\mathcal{CG}(S)$ . By the Poincaré recurrence theorem, the  $\Phi_{\mathcal{T}}^t$ -orbit of  $\mu$ -almost every  $q \in \mathcal{Q}(S)$  is recurrent under the Teichmüller flow. Therefore the vertical measured geodesic lamination of  $\mu$ -almost every  $q \in \mathcal{Q}(S)$  is uniquely ergodic [M82]. As a consequence, if  $\gamma : \mathbb{R} \rightarrow$

$\mathcal{T}(S)$  is a geodesic whose unit cotangent line is a lift of an orbit of  $\Phi_{\mathcal{T}}^t$  which is typical for  $\mu$  then the map  $t \rightarrow \Upsilon_{\mathcal{T}}(\gamma(t))$  is an unparametrized  $L_1$ -quasi-geodesic of infinite length [K199, H06a]. This means that  $r(q, t) \rightarrow \infty$  ( $t \rightarrow \infty$ ) for  $\mu$ -almost every  $q$ . Moreover, by Lemma 2.4 of [H08] and the estimate (12) above, there is a number  $c > 0$  such that

$$(13) \quad r(q, t + s) \geq r(q, s) + r(q, t) - c \text{ for all } q \in \mathcal{Q}(S), s, t > 0,$$

Now the function  $r : \mathcal{Q}(S) \times \mathbb{R} \rightarrow [0, \infty)$  is continuous and the measure  $\mu$  is Borel regular. Hence there is a compact subset  $A$  of  $\mathcal{Q}(S)$  with  $\mu(A) > 3/4$  and a number  $t_0 > 0$  such that  $r(q, t_0) \geq 4c$  for all  $q \in A$  where  $c > 0$  is as in (13) above. For the continuous non-negative function  $\psi(q) = r(q, t_0)$  on  $\mathcal{Q}(S)$  we then have  $\int \psi d\mu \geq 3c$ . The Birkhoff ergodic theorem implies that

$$\frac{1}{t_0} \int_0^{t_0} \left( \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{i=0}^{m-1} \psi(\Phi_{\mathcal{T}}^{it_0} \Phi^s q) \right) ds \geq 3c$$

for  $\mu$ -almost every  $q$ . The estimate (13) shows that  $r(q, mt_0) \geq \sum_{i=0}^{m-1} \psi(\Phi^{it_0} q) - mc$  and therefore

$$\liminf_{m \rightarrow \infty} \frac{1}{mt_0} r(q, mt_0) \geq 2c/t_0$$

for  $\mu$ -almost every  $q$ . This completes the proof of the lemma.  $\square$

The next observation is analogous to Lemma 6.2. For easier comparison, we use the same notations.

**Lemma 7.2.** *Let  $\mu$  be a  $\Phi_{\mathcal{T}}^t$ -invariant ergodic Borel probability measure on  $\mathcal{Q}(S)$ . Then there is a compact set  $K \subset \mathcal{Q}(S)$  with  $\mu(K) \geq 1/2$  and there is a number  $\alpha = \alpha(K) > 0$  and a number  $R_0 = R_0(K) > 0$  with the following property. Let  $q \in K$  and let  $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)$  be a Teichmüller geodesic whose initial unit cotangent vector is a lift of  $q$ . Let  $R_1, R_2 \geq R_0$  and let  $\xi_1, \xi_2$  be the Weil-Petersson geodesics which connect  $\gamma(0) = \xi_1(0) = \xi_2(0)$  to  $\gamma(-R_1), \gamma(R_2)$ . Then the angle  $\angle_{\gamma(0)}(\xi_1'(0), \xi_2'(0))$  at  $\gamma(0)$  between the geodesics  $\xi_1, \xi_2$  is at least  $\alpha$ .*

*Proof.* Let  $\mathcal{F} : \mathcal{Q}(S) \rightarrow \mathcal{Q}(S)$  be the flip  $q \rightarrow -q$ . The image under  $\mathcal{F}$  of every  $\Phi_{\mathcal{T}}^t$ -invariant Borel probability measure  $\mu$  is a  $\Phi_{\mathcal{T}}^t$ -invariant Borel probability measure. In particular, by the Poincaré recurrence theorem, there is a  $\Phi_{\mathcal{T}}^t$ -invariant Borel set  $A \subset \mathcal{Q}(S)$  with  $\mu(A) = 1$  such that every  $q \in A$  is forward and backward recurrent, i.e. its forward and backward orbit under  $\Phi_{\mathcal{T}}^t$  intersects a fixed compact neighborhood  $U$  of  $q$  for arbitrarily large times. Then for every  $q \in A$  the vertical and the horizontal measured geodesic lamination of  $q$  is uniquely ergodic [M82]. Since  $\mu$  is Borel regular, there is a compact set  $K \subset A$  with  $\mu(K) > 1/2$ . We claim that for this set  $K$  there are numbers  $\alpha > 0, R_0 > 0$  as stated in the lemma.

For this we proceed as in the proof of Lemma 6.2. Namely, we argue by contradiction and we assume that the statement of the lemma does not hold. Let  $\tilde{K} \subset \mathcal{Q}^1(S)$  be a compact set which projects onto  $K$ . Then there is a sequence of Teichmüller geodesics  $\gamma_i : \mathbb{R} \rightarrow \mathcal{T}(S)$  with unit cotangent vector  $\gamma_i'(0) \in \tilde{K}$  and a sequence of numbers  $R_i, T_i \rightarrow \infty$  so that the angle between the WP-geodesics  $\zeta_i, \xi_i$  connecting  $\gamma_i(0)$  to  $\gamma_i(-R_i), \gamma_i(T_i)$  tends to zero as  $i \rightarrow \infty$ .

By compactness of  $\tilde{K}$ , up to passing to a subsequence we may assume that the Teichmüller geodesics  $\gamma_i$  converge as  $i \rightarrow \infty$  to a Teichmüller geodesic  $\gamma$  with unit cotangent  $q \in \tilde{K}$  at  $\gamma(0)$ . By Lemma 6.1, by passing to another subsequence we may assume that the WP-geodesics  $\zeta_i, \xi_i$  converge as  $i \rightarrow \infty$  to infinite Weil-Petersson rays  $\zeta, \xi : [0, \infty) \rightarrow \mathcal{T}(S)$ .

Let  $q_h, q_v$  be the horizontal and vertical measured geodesic lamination, respectively, of the area one quadratic differential  $q \in \tilde{K}$ . Since  $\tilde{K} \subset \tilde{A}$ , the measured geodesic laminations  $q_h, q_v$  are uniquely ergodic. By Lemma 6.1, the length of  $q_h$  is bounded along  $\zeta$  and the length of  $q_v$  is bounded along  $\xi$ .

On the other hand, by assumption, the angle at  $\gamma_i(0)$  between  $\zeta_i, \xi_i$  converges to zero and therefore we have  $\zeta = \xi$ . As a consequence, the lengths of both  $q_h, q_v$  are bounded along  $\zeta$ . But the measured geodesic laminations  $q_h, q_v$  jointly fill up  $S$  (i.e. we have  $i(\mu, q_h) + i(\mu, q_v) > 0$  for every measured geodesic lamination  $\mu$  on  $S$ ) and hence the function on  $\mathcal{T}(S)$  which associates to a point  $x \in \mathcal{T}(S)$  the value  $\ell_x(q_h) + \ell_x(q_v)$  is proper (see Theorem 1.2 of [K92]). Since  $\zeta$  is an infinite WP-ray, this is impossible. The lemma follows.  $\square$

For a  $\Phi_{\mathcal{T}}^t$ -invariant Borel subset  $A$  of  $\mathcal{Q}(S)$  define a *measurable conjugacy* of the restriction of  $\Phi_{\mathcal{T}}^t$  to  $A$  into the geodesic flow of the Weil-Petersson metric to be an injective measurable map  $\Lambda : A \rightarrow \mathcal{Q}_{WP}(S)$  such that there is a measurable function  $\psi : A \times \mathbb{R} \rightarrow \mathbb{R}$  with the following properties.

- (1)  $\psi(x, 0) = 0$  for all  $x \in A$ .
- (2) For each fixed  $x \in A$  the function  $\psi(x, \cdot) : s \rightarrow \psi(x, s)$  is a homeomorphism.
- (3)  $\Lambda(\Phi_{\mathcal{T}}^t x) = \Phi_{WP}^{\psi(x, t)} \Lambda(x)$  for all  $x \in A, t \in \mathbb{R}$ .

We use the above considerations to construct for a  $\Phi_{\mathcal{T}}^t$ -invariant ergodic Borel probability measure  $\mu$  on  $\mathcal{Q}(S)$  a measurable conjugacy of the restriction of the Teichmüller geodesic flow to an invariant Borel set of full  $\mu$ -measure into the geodesic flow of the Weil-Petersson metric. The construction uses the ideas from Section 6. Denote as before by  $P : \mathcal{Q}^1(S) \rightarrow \mathcal{T}(S)$  the canonical projection.

**Lemma 7.3.** *Let  $\mu$  be a  $\Phi_{\mathcal{T}}^t$ -invariant ergodic Borel probability measure on  $\mathcal{Q}(S)$ . Then there is a  $\Phi_{\mathcal{T}}^t$ -invariant Borel set  $A \subset \mathcal{Q}(S)$  with  $\mu(A) = 1$  and there is a measurable conjugacy  $\Lambda : A \rightarrow \mathcal{Q}_{WP}(S)$  of the restriction of  $\Phi_{\mathcal{T}}^t$  to  $A$  into the geodesic flow of the Weil-Petersson metric. Moreover, there is a number  $c = c(\mu) > 0$  with the following property. Let  $\tilde{\Lambda} : \mathcal{Q}^1(S) \rightarrow \mathcal{Q}_{WP}^1(S)$  be a lift of  $\Lambda$ , defined on the preimage  $\tilde{A}$  of  $A$ . Then for every  $\tilde{q} \in \tilde{A}$  there are sequences  $t_i$  ( $i \in \mathbb{Z}$ ) with  $t_i \rightarrow \pm\infty$  ( $i \rightarrow \pm\infty$ ) and such that  $d_{\mathcal{T}}(P\Phi_{\mathcal{T}}^{t_i}\tilde{q}, P\tilde{\Lambda}(\Phi_{\mathcal{T}}^{t_i}\tilde{q})) \leq c$ .*

*Proof.* Let  $\mu$  be a  $\Phi_{\mathcal{T}}^t$ -invariant ergodic Borel probability measure on  $\mathcal{Q}(S)$  and let  $A_0$  be the set of all points in  $\mathcal{Q}(S)$  for which the Birkhoff ergodic theorem holds true in forward and backward direction (i.e. for the measures  $\mu$  and  $\mu \circ \mathcal{F}$  as  $t \rightarrow \infty$  where  $\mathcal{F} : q \rightarrow -q$  is the flip). Then  $A_0$  is a  $\Phi_{\mathcal{T}}^t$ -invariant Borel subset of  $\mathcal{Q}(S)$  of full measure.

Let  $r : \mathcal{Q}(S) \times \mathbb{R} \rightarrow \mathbb{R}$  be as in Lemma 7.1. By Lemma 7.1, there is a number  $\beta > 0$  such that

$$\lim_{t \rightarrow \infty} \frac{1}{t} r(q, t) \geq 2\beta, \lim_{t \rightarrow \infty} \frac{1}{t} r(-q, t) \geq 2\beta$$

for  $\mu$ -almost every  $q \in \mathcal{Q}(S)$ . Let  $\mathcal{F} : \mathcal{Q}(S) \rightarrow \mathcal{Q}(S)$  be the flip  $v \rightarrow \mathcal{F}(v) = -v$  and let  $A \subset A_0$  be a  $\Phi_{\mathcal{T}}^t$ -invariant Borel set of full measure so that this holds true for every  $q \in A \cup \mathcal{F}(A)$ .

Let  $\tilde{A}$  be the preimage of  $A$  in  $\mathcal{Q}^1(S)$ . We show first that for every  $\tilde{q} \in \tilde{A}$  the following is satisfied. Let  $\tilde{q}_v$  be the vertical measured geodesic lamination of  $\tilde{q}$ . Let  $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)$  be the Teichmüller geodesic with initial velocity  $\tilde{q}$ . For  $n > 0$  let  $\zeta_{\tilde{q}, n}$  be the Weil-Petersson geodesic connecting  $\zeta_{\tilde{q}, n}(0) = \gamma(0)$  to  $\zeta_{\tilde{q}, n}(\sigma_n) = \gamma(\sigma_n)$ . Then the geodesics  $\zeta_{\tilde{q}, n}$  converge as  $n \rightarrow \infty$  to a WP-ray  $\zeta_{\tilde{q}}$  with ending measure  $[\tilde{q}_v]$ , the class of  $\tilde{q}_v$ . Moreover, the initial directions  $\zeta_{\tilde{q}}'(0)$  of the geodesics  $\zeta_{\tilde{q}}$  depend measurably on  $\tilde{q}$ .

For the proof of this claim we proceed as in Section 6. Namely, let  $a_0 > 0$  be as in inequality (12). For  $k > 0$  let  $B(k) \subset A$  be the set of all points  $z \in A$  such that  $r(q, m) \geq m\beta + a_0$  for every integer  $m \geq k$ . Since the function  $r$  is continuous, for each  $k$  the set  $B(k)$  is a Borel subset of  $A$ , and  $\cup_k B(k) = A$ . If  $\tilde{q} \in \mathcal{Q}^1(S)$  projects into  $B(k)$  then  $d_C(\Upsilon_{\mathcal{T}}(P\tilde{q}), \Upsilon_{\mathcal{T}}(P\Phi_{\mathcal{T}}^m \tilde{q})) \geq \beta m$  for every integer  $m \geq k$ .

Let  $K \subset \mathcal{Q}(S)$  and let  $\alpha = \alpha(K) > 0, R_0 = R_0(K) > 0$  be as in Lemma 7.2. By the Birkhoff ergodic theorem, for every  $q \in A$  the orbit of  $q$  under the Teichmüller flow satisfies

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \chi_K(\Phi_{\mathcal{T}}^t q) dt = \mu(K) \geq 1/2$$

where  $\chi_K$  denotes the characteristic function of  $K$ .

Let  $\delta = \delta(\beta) > 0$  and let  $\eta = \eta(\beta) > 0$  be as in Proposition 5.4. Let  $\theta > 0$  and let  $m = m(\delta, \alpha, \theta/2) > 0$  be as in the second part of Lemma 3.1. Let  $k > 0$  and let  $N = \max\{k, m/\eta\}$ . For an integer  $p > N$  let  $B(k, p) \subset B(k)$  be the set of all points  $q \in B(k)$  such that there is some  $t \in (p-1, p]$  with  $\Phi_{\mathcal{T}}^t q \in K$ . By construction,  $B(k, p)$  is a Borel subset of  $B(k)$ , and we have  $\cup_{p \geq N} B(k, p) = B(k)$ .

Let  $q \in B(k, p)$  for some  $p \geq N$  and let  $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)$  be a Teichmüller geodesic whose unit cotangent vector  $\tilde{q}$  at  $\gamma(0)$  is a lift of  $q$ . By the definition of the set  $B(k, p)$ , there is some  $t \in (p-1, p]$  such that the unit cotangent  $\Phi_{\mathcal{T}}^t \tilde{q}$  of  $\gamma$  at  $t$  is contained in the preimage  $\tilde{K} \subset \mathcal{Q}^1(S)$  of  $K$ . By Proposition 5.4 and the definition of  $B(k)$ , we have  $\ell_{\delta\text{-thick}}(\zeta_{\tilde{q}, t}) \geq m$ . By the choice of  $t$  and  $R_0$ , for  $T > p + R_0$  the angle at  $\gamma(t)$  of the Weil-Petersson geodesic triangle with vertices  $\gamma(0), \gamma(t), \gamma(T)$  is at least  $\alpha$ . Hence Lemma 3.1 implies that for all  $n, T > p + R_0$  the angle at  $\gamma(0)$  between the geodesics  $\zeta_{\tilde{q}, n}$  and  $\zeta_{\tilde{q}, T}$  is smaller than  $\theta$ . Since  $\theta > 0$  was arbitrary, together with Lemma 6.1 convergence of the geodesics  $\zeta_{\tilde{q}, n}$  to a geodesic ray  $\zeta_{\tilde{q}}$  with ending measure  $[q_v]$  follows.

Let  $\tilde{B}(k)$  be the preimage of  $B(k)$  in  $\mathcal{Q}^1(S)$ . By continuous dependence of the initial velocities of the geodesics  $\zeta_{\tilde{q}, n}$  on  $\tilde{q} \in \mathcal{Q}^1(S)$  and on  $n > 0$ , by uniformity of the above angle estimate over all points  $q \in B(k, p)$  and by the fact that  $\theta > 0$  was arbitrary, the initial cotangent vector  $\zeta_{\tilde{q}}'(0)$  of  $\zeta_{\tilde{q}}$  depends measurably on  $\tilde{q} \in$

$\tilde{B}(k)$ . But  $\cup_k \tilde{B}(k) = \tilde{A}$  and hence it is immediate from the construction that the assignment  $\tilde{q} \in \tilde{A} \rightarrow \zeta_{\tilde{q}}'(0)$  is measurable. Moreover, this map is equivariant under the action of the mapping class group. This completes the proof of the above claim. Note also that by construction, if  $s \in \mathbb{R}$  is arbitrary then for every  $\tilde{q} \in \tilde{A}$  the WP-geodesic ray  $\zeta_{\Phi^{-s}\tilde{q}}$  is asymptotic to  $\zeta_{\tilde{q}}$ . Moreover, all of the above holds true for all  $\tilde{q} \in \mathcal{F}(\tilde{A})$ .

Next we show that for every  $\tilde{q} \in \tilde{A}$  there is a unique Weil-Petersson geodesic  $\xi(\tilde{q})$  which is forward asymptotic to  $\zeta_{\tilde{q}}$  and backward asymptotic to  $\zeta_{-\tilde{q}}$ . By the remark in the previous paragraph, it is enough to show this property for points  $\tilde{q}$  which are contained in the preimage  $\tilde{K}$  of  $K$  in  $\mathcal{Q}^1(S)$ . Let again  $\delta = \delta(\beta) > 0$  be as in Proposition 5.4 and let  $\alpha = \alpha(K) > 0, R_0 = R_0(K) > 0$  be as in Lemma 7.2. Let  $k = k(\delta, \alpha) > 0$  be as in the first part of Lemma 3.1. By Proposition 5.4 and the choice of the set  $A$  there is a number  $R_1 > R_0 + k$  such that  $\Phi_{\mathcal{T}}^{R_1}(\tilde{q}) \in \tilde{K}$  and that moreover  $\ell_{\delta\text{-thick}}(\zeta_{\tilde{q}, R_1}) \geq k$ . By the choice of  $K$ , for all  $s > R_0$  the angle at  $P\tilde{q}$  between the Weil-Petersson geodesics connecting  $P\tilde{q}$  to  $P\Phi_{\mathcal{T}}^{-s}\tilde{q}, P\Phi_{\mathcal{T}}^{R_1}\tilde{q}$  is at least  $\alpha$ . Similarly, the angle at  $P\Phi_{\mathcal{T}}^{R_1}\tilde{q}$  between the Weil-Petersson geodesics connecting  $P\Phi_{\mathcal{T}}^{R_1}\tilde{q}$  to  $P\tilde{q}, P\Phi_{\mathcal{T}}^{s+R_1}\tilde{q}$  is at least  $\alpha$ . Corollary 3.2 now shows that there is a unique Weil-Petersson geodesic  $\xi(\tilde{q})$  which is forward asymptotic to  $\zeta_{\tilde{q}}$  and backward asymptotic to  $\zeta_{-\tilde{q}}$ . This geodesic depends measurably on  $\tilde{q}$ . Moreover, its ending measures are the classes  $[\tilde{q}_v], [\tilde{q}_h]$  of the vertical and the horizontal measured geodesic laminations of  $\tilde{q}$ .

We complete the proof of the lemma using the arguments in the proof of Proposition 6.4. Namely, let  $\tilde{q} \in \tilde{A}$  and let  $\tilde{q}_v, \tilde{q}_h$  be the vertical and the horizontal measured geodesic lamination of  $\tilde{q}$ , respectively. The function  $x \rightarrow \ell_x(\tilde{q}_v) + \ell_x(\tilde{q}_h)$  is strictly convex along the Weil-Petersson geodesic  $\xi(\tilde{q})$  and assumes a unique minimum. Let  $\tilde{\Lambda}(\tilde{q})$  be the unit cotangent vector of  $\xi(\tilde{q})$  at the unique minimum for this function along  $\xi(\tilde{q})$ . As in the proof of Proposition 6.4, the assignment  $t \rightarrow \tilde{\Lambda}(\Phi_{\mathcal{T}}^t \tilde{q})$  is a homeomorphism of the orbit of the Teichmüller geodesic flow through  $\tilde{q}$  onto the orbit of the Weil-Petersson geodesic flow through  $\tilde{\Lambda}(\tilde{q})$ . Since the Teichmüller geodesic  $t \rightarrow P\Phi_{\mathcal{T}}^t \tilde{q}$  is uniquely determined by the projective classes of the horizontal and vertical measured geodesic laminations, respectively, and these projective measured geodesic laminations are the ending measures of the WP-geodesic determined by  $\tilde{\Lambda}(\tilde{q})$ , the map  $\tilde{q} \rightarrow \tilde{\Lambda}(\tilde{q})$  is injective. Moreover, the assignment  $\tilde{q} \rightarrow \tilde{\Lambda}(\tilde{q})$  is measurable and equivariant under the action of the mapping class group. Thus this map projects to a measurable map  $\Lambda : A \rightarrow \mathcal{Q}_{WP}(S)$  which defines a measurable conjugacy of the Teichmüller geodesic flow on  $A$  into the Weil-Petersson geodesic flow.

Since the map  $\Lambda : A \rightarrow \mathcal{Q}_{WP}(S)$  is measurable, there is a compact set  $K_0 \subset A$  with  $\mu(K_0) > 1/2$  such that the restriction of  $\Lambda$  to  $K_0$  is continuous. Let  $\tilde{K}_0$  be the preimage of  $K$  in  $\mathcal{Q}^1(S)$  and let  $\tilde{\Lambda}$  be the lift of  $\Lambda$  constructed above. Then the restriction of  $\tilde{\Lambda}$  to  $\tilde{K}_0$  is continuous and equivariant under the action of the mapping class group. As a consequence, there is some  $c > 0$  such that  $d_{\mathcal{T}}(P\tilde{q}, P\tilde{\Lambda}(\tilde{q})) \leq c$  for every  $\tilde{q} \in \tilde{K}_0$ . Now if  $q \in A$  is arbitrary and if  $\tilde{q}$  is a lift of  $q$ , then by ergodicity and the choice of  $A$ , the orbit of  $\tilde{q}$  under the Teichmüller geodesic flow intersects  $\tilde{K}_0$  for arbitrarily large positive and negative times. Now if  $t_i \subset \mathbb{R}$  is any sequence

such that  $\Phi_{\mathcal{T}}^{t_i}(\tilde{q}) \in \tilde{K}_0$  then  $d_{\mathcal{T}}(P\Phi_{\mathcal{T}}^{t_i}(\tilde{q}), P(\tilde{\Lambda}\Phi_{\mathcal{T}}^{t_i}(\tilde{q}))) \leq c$ . This completes the proof of the lemma.  $\square$

Finally we are able to complete the proof of Theorem 1 from the introduction. Denote by  $h(\mu) \geq 0$  the entropy of a  $\Phi_{\mathcal{T}}^t$ -invariant Borel probability measure  $\mu$  on  $\mathcal{Q}(S)$  (or of a  $\Phi_{\text{WP}}^t$ -invariant Borel probability measure  $\mu$  on  $\mathcal{Q}_{\text{WP}}(S)$ ).

**Proposition 7.4.** *There is a continuous injective map*

$$\Theta : \mathcal{M}_{\mathcal{T}}(\mathcal{Q}(S)) \rightarrow \mathcal{M}_{\text{WP}}(\mathcal{Q}_{\text{WP}}(S)).$$

Moreover, there is a constant  $c > 0$  such that

$$h(\Theta(\mu)) \geq ch(\mu)$$

for all  $\mu \in \mathcal{M}_{\mathcal{T}}(\mathcal{Q}(S))$ .

*Proof.* Let  $\mu$  be a  $\Phi_{\mathcal{T}}^t$ -invariant ergodic Borel probability measure on  $\mathcal{Q}(S)$ . By Lemma 7.3, there is a Borel subset  $A \subset \mathcal{Q}(S)$  and there is a measurable conjugacy  $\Lambda$  of  $A$  into the Weil-Petersson geodesic flow.

We claim that there is a  $\Phi_{\text{WP}}^t$ -invariant Borel probability measure on  $\mathcal{Q}_{\text{WP}}(S)$  whose measure class equals  $\Lambda_*(\mu)$ . Namely, since the Teichmüller space  $\mathcal{T}(S)$  is contractible and the Teichmüller metric on  $\mathcal{T}(S)$  is complete, the action of  $\Phi_{\mathcal{T}}^t$  on  $\mathcal{Q}^1(S)$  is proper. This implies that the space of geodesics  $\mathcal{G}(S)$  for the Teichmüller metric is a locally compact  $\text{Mod}(S)$ -space (which is naturally homeomorphic to a  $\text{Mod}(S)$ -invariant open subset of  $\mathcal{PML} \times \mathcal{PML} - \Delta$ , see [K92]).

The measure  $\mu$  induces a locally finite  $\Phi_{\mathcal{T}}^t$ -invariant  $\text{Mod}(S)$ -invariant measure  $\tilde{\mu}$  on  $\mathcal{Q}^1(S)$ . Via desintegration, the measure  $\tilde{\mu}$  projects to a locally finite  $\text{Mod}(S)$ -invariant measure  $\hat{\mu}$  on  $\mathcal{G}(S)$ . Since  $\mu$  is ergodic under the Teichmüller flow, the measure  $\hat{\mu}$  is ergodic under the action of  $\text{Mod}(S)$ . By Lemma 7.3, there is a  $\text{Mod}(S)$ -equivariant injective Borel map  $\hat{\Lambda}$  of a  $\text{Mod}(S)$ -invariant Borel subset  $B$  of  $\mathcal{G}(S)$  of full  $\hat{\mu}$ -mass onto a  $\text{Mod}(S)$ -invariant Borel subset  $\hat{\Lambda}(B)$  of the space  $\mathcal{G}_{\text{WP}}(S)$  of biinfinite complete geodesics for the Weil-Petersson metric. The product of the push-forward  $\hat{\Lambda}_*(\hat{\mu})$  of the measure  $\hat{\mu}$  with the standard Lebesgue measure on the flow lines of the Weil-Petersson geodesic flow defines a  $\Phi_{\text{WP}}^t$ -invariant  $\text{Mod}(S)$ -invariant locally finite Borel measure on the bundle  $\mathcal{Q}_{\text{WP}}^1(S)$  of quadratic differentials of norm one for the Weil-Petersson metric. This measure induces a locally finite Borel measure  $\mu_0$  on  $\mathcal{Q}_{\text{WP}}(S)$ . The measure  $\mu_0$  is ergodic under the action of the Weil-Petersson geodesic flow since the measure  $\hat{\mu}$  and hence  $\hat{\Lambda}_*(\hat{\mu})$  is ergodic under the action of  $\text{Mod}(S)$ .

We show next that the measure  $\mu_0$  on  $\mathcal{Q}_{\text{WP}}(S)$  is *finite*. To see that this is the case, let  $A \subset \mathcal{Q}(S)$  be an invariant Borel set of full  $\mu$ -mass on which the measurable conjugacy  $\Lambda$  constructed in Lemma 7.3 is defined. Let  $\psi : A \times \mathbb{R} \rightarrow \mathbb{R}$  be the measurable function defined by the conjugacy. The function  $\psi$  satisfies the *cocycle identity*

$$(14) \quad \psi(x, s+t) = \psi(x, s) + \psi(\Phi_{\mathcal{T}}^s x, t) \quad (x \in A, s, t \in \mathbb{R}).$$

Moreover, its restriction to  $A \times [0, \infty)$  is non-negative, and we have  $\psi(x, s) \leq \psi(x, t)$  for  $s \leq t$ .

Let  $L > 0$  be as in Lemma 5.2, let  $q \in A$  be a typical point for  $\mu$  and let  $\tilde{q}$  be a lift of  $q$  to  $\mathcal{Q}^1(S)$ . By Lemma 7.3 there is a number  $c > 0$  and there is a sequence  $t_i \rightarrow \infty$  such that  $d_{\mathcal{T}}(P\Phi_{\mathcal{T}}^{t_i}\tilde{q}, P\tilde{\Lambda}(\Phi_{\mathcal{T}}^{t_i}\tilde{q})) \leq c$ . Now  $\tilde{\Lambda}(\Phi_{\mathcal{T}}^{t_i}\tilde{q}) = \Phi_{WP}^{\psi(q, t_i)}\tilde{\Lambda}(\tilde{q})$  and hence since geodesics for the Weil-Petersson metric are globally length minimizing, Lemma 5.2 shows that there is a constant  $C > 0$  depending on  $q$  such that  $\psi(q, t_i) \leq Lt_i + C$  for all  $i$ . As a consequence, we have

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \psi(q, t) \leq L.$$

Since  $q \in A$  was an arbitrary point, the Birkhoff ergodic theorem together with the cocycle identity (14) implies that the (non-negative) function  $x \rightarrow \psi(x, 1)$  is integrable with respect to  $\mu$ , and we have  $\int \psi(x, 1) d\mu \leq L$ .

Using the cocycle identity (14) and the Birkhoff ergodic theorem once more, we conclude that for  $\mu$ -almost every  $q \in A$  the limit

$$\lim_{t \rightarrow \infty} \frac{1}{t} \psi(q, t)$$

exists and equals  $\int \psi(q, 1) d\mu \leq L$ .

We claim that the total mass of the measure  $\mu_0$  on  $\mathcal{Q}_{WP}(S)$  does not exceed  $L$ . For this assume for a moment that for every  $q \in A \subset \mathcal{Q}(S)$  the function  $t \rightarrow \psi(q, t)$  is continuously differentiable, with derivative  $f(q)$  at  $t = 0$  depending measurably on  $q$ . By invariance of  $\mu$  under the Teichmüller flow we have

$$\int f d\mu = \int \left( \int_0^1 f(\Phi_{\mathcal{T}}^t q) dt \right) d\mu(q) = \int \psi(q, 1) d\mu \leq L.$$

On the other hand, by construction, for  $\mu$ -almost every  $q$  the Radon-Nikodym derivative of the measure  $\mu_0$  with respect to the measure  $\Lambda_*(\mu)$  equals  $f(q)$  and therefore  $\mu_0(\mathcal{Q}_{WP}(S)) = \int f d\mu \leq L$  as claimed.

The claim for the general case follows in the same way. Namely, let  $\sigma : \mathbb{R} \rightarrow [0, 1]$  be a smooth function supported in  $[-1, 1]$  with  $\int \sigma = 1$ . For  $q \in A$  define  $\Psi(q) = \Phi_{WP}^{s(q)}(\Lambda(q))$  where  $s(q) = \int \psi(q, t) \sigma(t) dt$ . Then  $\Psi$  defines a measurable conjugacy of  $A$  into the geodesic flow of the Weil-Petersson metric which induces the measure  $\mu_0$  as before. If  $\rho : A \times \mathbb{R} \rightarrow \mathbb{R}$  is the cocycle defined by this conjugacy, then by the Birkhoff ergodic theorem we have  $\int \rho(q, 1) d\mu = \int \psi(q, 1) d\mu \leq L$ . Moreover, by construction, for every  $q \in A$  the function  $t \rightarrow \rho(q, t)$  is continuously differentiable, with measurable differential. As a consequence, indeed  $\mu_0(\mathcal{Q}_{WP}(S)) \leq L$  holds true.

We claim that  $h(\Theta(\mu)) \geq h(\mu)/L$  for every measure  $\mu \in \mathcal{M}(\mathcal{Q}(S))$ . For this we use Rudolph's theorem (see Section 11.4 in [CFS82]) and Abramov's formula. Namely, let  $\epsilon > 0$ . Then there is a *special representation* of the flow  $\Phi_{\mathcal{T}}^t$  on  $(A, \mu)$  given by a Lebesgue space  $(M, \nu)$ , a measure preserving automorphism  $T : (M, \nu) \rightarrow (M, \nu)$  and a roof function  $\rho : M \rightarrow \{p, 1\}$  where  $p \in (1 - \epsilon, 1)$ . The flow  $\Phi_{\mathcal{T}}^t$  on  $A$  is just the vertical flow on the space  $\{(x, t) \in M \times \mathbb{R} \mid 0 \leq t < \rho(x)\} / \sim$  where  $(x, \rho(x)) \sim (Tx, 0)$  for all  $x$ . The measure  $\mu$  is the product of  $\nu$  with the Lebesgue measure on  $\mathbb{R}$ . Let  $h_1 \geq 0$  be the entropy of the  $T$ -invariant measure  $\nu$ . By Abramov's formula, the entropy  $h(\mu)$  of  $\mu$  equals  $h_1 / \int \rho d\mu_1$ . Since  $\rho$  assumes values in  $[1 - \epsilon, 1]$  we have  $\int \rho d\mu_1 \in [1 - \epsilon, 1]$ . In particular, the entropy  $h(\mu)$  is within  $h_1 \epsilon$  of the entropy of  $\mu_1$ .

The space  $(M, \nu)$  can be thought of a section for the flow  $\Phi_T^t$ . In particular, the conjugacy  $\Lambda$  determines a first return time  $\hat{\rho}$  to the set  $\Lambda(M) \subset \mathcal{Q}_{WP}(S)$  for the Weil-Petersson flow. Using again Abramov's formula, the entropy of  $\Theta(\mu)$  equals  $h_1 / \int \hat{\rho} d\mu_1$ . Now for each  $t \in [0, 1 - \epsilon]$  the set  $\Phi_T^t M$  is a section for  $\Phi_T^t$  as well. Since  $t \rightarrow \psi(x, t)$  is increasing,

But

$$\int \hat{\rho} d\mu_1 \leq \frac{1}{1 - \epsilon} \int_M \int_0^{1 - \epsilon} \psi(\Phi_T^t x, 1) dt d\hat{\mu} \leq \frac{1}{1 - \epsilon} \int \psi(x, 1) d\nu \leq \frac{1}{(1 - \epsilon)^2} L$$

(for the last estimate, note that since  $\mu$  is a probability measure, the total mass of  $\nu$  is contained in the interval  $[1, \frac{1}{(1 - \epsilon)^2}]$ , and as before,  $L > 0$  is as in Lemma 5.2). Since  $\epsilon > 0$  was arbitrary, we have  $h(\Theta(\mu)) \geq h(\mu)/L$  as claimed.

We show next that the map  $\Theta$  is injective. Namely, by construction, if  $\mu$  is an ergodic  $\Phi_T^t$ -invariant Borel probability measure on  $\mathcal{Q}(S)$  and if  $\tilde{q} \in \mathcal{Q}^1(S)$  is the lift of a typical point for  $\mu$  with vertical and horizontal measured geodesic laminations  $\tilde{q}_v, \tilde{q}_h$ , respectively, then there is a lift of a typical point for  $\Theta(\mu)$  which determines a biinfinite Weil-Petersson geodesic with forward and backward ending measures  $[\tilde{q}_v], [\tilde{q}_h]$ . Since the  $\Phi_T^t$ -orbit of  $\tilde{q}$  is determined by the pair  $([\tilde{q}_v], [\tilde{q}_h])$ , this means that the pairs of all ending measures of all geodesics whose initial cotangents are typical for  $\Theta(\mu)$  determine  $\Theta(\mu)$  and hence  $\mu$ . In other words, the map  $\Theta$  is injective.

Finally we show that the map  $\Theta$  is continuous with respect to the weak\*-topology. By convexity, for this it is enough to show the following. Let  $(\mu_i) \subset \mathcal{M}(\mathcal{Q}(S))$  be a sequence of ergodic measures converging to an ergodic measure  $\mu$ . Then  $\Theta(\mu_i) \rightarrow \Theta(\mu)$ .

To see that this is the case, note first that the locally finite  $\text{Mod}(S)$ -invariant measures  $\hat{\mu}_i$  on  $\mathcal{G}(S)$  which are the disintegrations of the lifts  $\tilde{\mu}_i$  of the measures  $\mu_i$  to  $\mathcal{Q}^1(S)$  converge weakly to the locally finite  $\text{Mod}(S)$ -invariant measure  $\hat{\mu}$ . Let  $K \subset \mathcal{G}(S)$  be any compact set consisting of typical points for  $\hat{\mu}$  in the above sense. In particular,  $K$  is contained in the support of  $\hat{\mu}$ . Let  $U_j$  be an open relative compact neighborhood of  $K$  with  $U_j \supset U_{j+1}$  and  $\cap_j U_j = K$ . For each  $j$  we have  $\liminf_{i \rightarrow \infty} \hat{\mu}_i(U_j) \geq \hat{\mu}(K)$ . Since this is valid for each  $j$  and since the measures  $\hat{\mu}_j$  are Borel regular, for every  $j$  we can find a number  $i(j) > 0$  with  $i(j+1) > i(j)$  and a compact subset  $K_j \subset U_j$  consisting of typical points for  $\hat{\mu}_{i(j)}$  and such that  $\hat{\mu}_{i(j)}(K_j) \geq \hat{\mu}(K)$ . Since  $\hat{\mu}_{i(j)} \rightarrow \hat{\mu}$  weakly and  $K$  is contained in the support of  $\hat{\mu}$ , the sets  $K_{i(j)}$  converge in the Hausdorff topology to  $K$ . In particular, for every  $\gamma \in K$  there is a sequence  $\gamma_j \in K_{i(j)}$  with  $\gamma_j \rightarrow \gamma$ .

As in the beginning of this proof, let  $\hat{\Lambda}_i$  be the maps defined on  $\text{Mod}(S)$ -invariant Borel subsets of  $\mathcal{G}(S)$  of full  $\hat{\mu}_i$ -mass into the space  $\mathcal{G}_{WP}(S)$  of geodesics for the Weil-Petersson metric and let  $\hat{\Lambda}$  be defined for  $\hat{\mu}$ . We may assume that  $\hat{\Lambda}_i$  is defined on  $K_i$  for all  $i$  and that  $\hat{\Lambda}$  is defined on  $K$ . Since  $\hat{\Lambda}$  is measurable, by eventually decreasing the size of  $K$  we may assume that the restriction of  $\hat{\Lambda}$  to  $K$  is continuous. By the definition of the weak\*-topology for Borel probability measures on  $\mathcal{Q}_{WP}(S)$  it now suffices to show the following. If  $\gamma_j \in K_j$  and if  $\gamma_j \rightarrow \gamma \in K$  then  $\hat{\Lambda}_j(\gamma_j) \rightarrow \hat{\Lambda}(\gamma)$ .

To see this we proceed as in the proof of Lemma 7.3. Namely, let  $\tilde{\Lambda} : \tilde{A} \subset \mathcal{Q}^1(S) \rightarrow \mathcal{Q}_{WP}(S)$  be the measurable conjugacy which induces  $\hat{\Lambda}$ . Since the restriction of  $\hat{\Lambda}$  to  $K$  is continuous, the restriction of  $\tilde{\Lambda}$  to the set  $B$  of all unit cotangent vectors of all geodesics in  $K$  is continuous as well.

Let  $\pi : \mathcal{Q}^1(S) \rightarrow \mathcal{PML}$  be the map which associates to a quadratic differential its vertical projective measured geodesic lamination. The map  $\pi$  is continuous and equivariant with respect to the action of the mapping class group. We claim that there is a compact subset  $V$  of  $B$  of positive measure, a number  $\alpha < \pi$  and an open neighborhood  $U$  of the diagonal in  $\mathcal{PML} \times \mathcal{PML}$  with the following property. If  $z \in V$  and  $\nu \in \mathcal{PML}$  are such that  $(\pi(z), \nu) \in U$  and if  $\xi$  is a recurrent WP-ray issuing from  $\xi(0) = P\tilde{\Lambda}(z)$  with ending measure  $\nu$  then the angle between the unit cotangent of  $\xi$  at  $\xi(0)$  and  $\tilde{\Lambda}(z)$  is at most  $\alpha$ . Here as before,  $P : \mathcal{Q}^1(S) \rightarrow \mathcal{T}(S)$  is the canonical projection.

To see this, assume otherwise. Then there is a sequence  $(z_i) \subset V$  and there is a sequence  $(\xi_i)$  of recurrent WP-rays issuing from  $P\tilde{\Lambda}(z_i) = \xi_i(0)$  with ending measures  $\nu_i$  such that the pair  $(\pi(z_i), \nu_i) \in \mathcal{PML} \times \mathcal{PML}$  converges to the diagonal and such that the angle between the unit cotangent  $\xi'_i(0)$  of  $\xi_i$  and  $-\tilde{\Lambda}(z_i)$  tends to zero. By compactness, up to passing to a subsequence we may assume that  $z_i \rightarrow z \in V$  and hence that  $\nu_i \rightarrow \pi(z)$ . Let  $\zeta : \mathbb{R} \rightarrow \mathcal{T}(S)$  be the Weil-Petersson geodesic with initial cotangent  $\zeta'(0) = \tilde{\Lambda}(z)$ . Since the restriction of  $\tilde{\Lambda}$  to  $V$  is continuous, the WP-rays  $\xi_i$  converge locally uniformly to the ray  $t \rightarrow \zeta(-t)$ . By continuity and convexity of the length pairing, since  $\nu_i \rightarrow \pi(z)$  the length of a measured geodesic lamination  $z_v$  in the class  $\pi(z)$  is bounded along the ray  $t \rightarrow \zeta(-t)$ . On the other hand, if  $z_h$  is the horizontal measured geodesic lamination of  $z$  then the length of  $z_h$  is bounded along  $t \rightarrow \zeta(-t)$  as well and hence we conclude that the function  $t \rightarrow \ell_{\zeta(-t)}(z_v) + \ell_{\zeta(-t)}(z_h)$  ( $t \in [0, \infty)$ ) is bounded. However, as before, since  $z_h, z_v$  jointly fill up  $S$  the function  $h \rightarrow \ell_h(z_v) + \ell_h(z_h)$  on  $\mathcal{T}(S)$  is proper and hence we arrive at a contradiction. This shows the claim.

Let again  $\gamma \in K$  be a geodesic which is typical for the measure  $\hat{\mu}$ . Let  $\tilde{q}$  be the unit cotangent of  $\gamma$  at  $\gamma(0)$  and let  $\zeta$  be the Weil-Petersson geodesic with initial unit cotangent  $\tilde{\Lambda}(\tilde{q})$ . It follows from Proposition 5.4 and Lemma 7.1 that there are numbers  $\delta > 0, \theta > 0$  depending on  $\mu$  such that  $\ell_{\delta\text{-thick}}(\zeta[0, T]) \geq \theta T$  for all sufficiently large  $T > 0$ . On the other hand, by the Birkhoff ergodic theorem, the projection to  $\mathcal{Q}(S)$  of the cotangent line of  $\gamma$  passes through the projection of the set  $V$  for arbitrarily large times. This then implies as before that for  $k = k(\delta, \alpha)$  there are numbers  $t_2 > t_1$  with  $t_2 - t_1 \geq k/\theta$  and such that the unit cotangents of  $\gamma(t_1), \gamma(t_2)$  are contained in a translate of  $V$  under the mapping class group.

As a consequence of the angle estimate established above and of the first part of Lemma 3.1, every biinfinite recurrent Weil-Petersson geodesic whose forward and backward ending measure is sufficiently close in  $\mathcal{PML}$  to the forward and backward ending measure of  $\zeta$  passes through a fixed compact subset  $C$  of  $\mathcal{T}(S)$ . Now for the sequence  $\gamma_j \rightarrow \gamma$  as above and for sufficiently large  $j$  the ending measures of the geodesics  $\hat{\Lambda}_j(\gamma_j)$  are close to the ending measures of  $\zeta = \hat{\Lambda}(\gamma)$ . For sufficiently large  $j$  the geodesics  $\hat{\Lambda}_j(\gamma_j)$  intersect the compact set  $C$ . Thus up to passing to a subsequence, the geodesics  $\hat{\Lambda}_j(\gamma_j)$  converge to a geodesic  $\rho$ . Now by continuity of

the length pairing, the length of the vertical measured geodesic lamination  $\tilde{q}_v$  of  $\tilde{q}$  is bounded along  $\rho[0, \infty)$ . Since the geodesic  $\Lambda(\gamma)$  is recurrent, this implies that  $\rho$  is asymptotic to  $\zeta$  (compare [BMM08a]). Similarly, the ray  $t \rightarrow \rho(-t)$  is asymptotic to  $t \rightarrow \zeta(-t)$ . But this just means that  $\rho = \zeta$  up to parametrization and hence  $\hat{\gamma}_j \rightarrow \hat{\gamma}$  ( $j \rightarrow \infty$ ). This completes the proof of the proposition.  $\square$

**Remark:** It is not too difficult to see that the image of the map  $\Theta$  is dense in the space of all invariant measures for the Weil-Petersson metric equipped with the weak\*-topology. However, we do not know whether or not  $\Theta$  is surjective.

There is a  $\Phi_{\mathcal{T}}^t$  invariant Borel probability measure on  $\mathcal{Q}(S)$  in the Lebesgue measure class whose entropy equals  $6g - 6 + 2m$  [M82]. Thus Proposition 7.4 immediately implies.

**Corollary 7.5.** *For  $c > 0$  as in Theorem 1, there is a  $\Phi_{WP}^t$ -invariant Borel probability measure on  $\mathcal{Q}(S)_{WP}$  with full support and entropy not smaller than  $6g - g + 2m$ ,*

## REFERENCES

- [Bo86] F. Bonahon, *Les bouts des variétés hyperboliques de dimension 3*, Ann. Math.124 (1986), 71–158.
- [Bw06] B. Bowditch, *Intersection numbers and the hyperbolicity of the curve complex*, J. reine angew. Math. 598 (2006), 105–129.
- [BH99] M. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*, Springer Grundlehren 319, Springer, Berlin 1999.
- [B03] J. Brock, *The Weil-Petersson metric and volumes of 3-dimensional hyperbolic convex cores*, J. Amer. Math. Soc. 16 (2003), 495–535.
- [BMM08a] J. Brock, H. Masur, Y. Minsky, *Asymptotics of Weil-Petersson geodesics I: ending laminations, recurrence, and flows*, arXiv:0802.1370.
- [BMM08b] J. Brock, H. Masur, Y. Minsky, *Asymptotics of Weil-Petersson geodesic II: bounded geometry and unbounded entropy*, to appear.
- [B92] P. Buser, *Geometry and spectra of compact Riemann surfaces*, Birkhäuser, Boston 1992.
- [C93] R. Canary, *Ends of hyperbolic 3-manifolds*, J. Amer. Math. Soc. 6 (1993), 1–35.
- [CR07] Y.E. Choi, K. Rafi, *Comparison between Teichmüller and Lipschitz metrics*, J. London Math. Soc. 76 (2007), 739–756.
- [CRS06] Y.E. Choi, K. Rafi, C. Series, *Lines of minima and Teichmüller geodesics*, arXiv:math.GT/0605135, to appear in Geom. Funct. Anal.
- [CFS82] I.P. Cornfeld, S.V. Fomin, Y.G. Sinai, *Ergodic theory*, Springer Grundlehren 245, Springer, New York 1982.
- [DW03] G. Daskolopoulos, R. Wentworth, *Classification of Weil-Petersson isometries*, Amer. J. Math. 125 (2003), 941–975.
- [GH90] E. Ghys, P. de la Harpe, *Sur les groupes hyperboliques d’après Mikhael Gromov*, Birkhäuser, Basel 1990.
- [H06a] U. Hamenstädt, *Train tracks and the Gromov boundary of the complex of curves*, in “Spaces of Kleinian groups” (Y. Minsky, M. Sakuma, C. Series, eds.), London Math. Soc. Lec. Notes 329 (2006), 187–207.
- [H06b] U. Hamenstädt, *Geometry of the mapping class groups II: (Quasi-)geodesics*, arXiv:math.GR/0511349, revised version, May 2006.
- [H07a] U. Hamenstädt, *The geometry of the curve complex and of Teichmüller space*, in “Handbook of Teichmüller theory”, (A. Papadopoulos, ed.), European Math. Soc. 2007, 447–467.
- [H07b] U. Hamenstädt, *Invariant measures for the Teichmüller flow*, preprint 2007.

- [H08] U. Hamenstädt, *Stability of quasi-geodesics in Teichmüller space*, preprint 2008.
- [K92] S. Kerckhoff, *Lines of minima in Teichmüller space*, Duke Math. J. 65 (1992), 187–213.
- [Kl99] E. Klarreich, *The boundary at infinity of the curve complex and the relative Teichmüller space*, unpublished manuscript, 1999.
- [Kr85] U. Krengel, *Ergodic theorems*, Walter de Gruyter, 1985.
- [M82] H. Masur, *Interval exchange transformations and measured foliations*, Ann. Math. 115 (1982), 169–201.
- [MM99] H. Masur, Y. Minsky, *Geometry of the complex of curves I: Hyperbolicity*, Invent. Math. 138 (1999), 103–149.
- [MM00] H. Masur, Y. Minsky, *Geometry of the complex of curves II: Hierarchical structure*, Geom. Funct. Anal. 10 (2000), 902–974.
- [Mi96] Y. Minsky, *Extremal length estimates and product regions in Teichmüller space*, Duke Math. J. 83 (1996), 249–286.
- [Mo03] L. Mosher, *Stable Teichmüller quasigeodesics and ending laminations*, Geom. Top. 7 (2003), 33–90.
- [PT07] A. Papadopoulos, G. Theret, *On Teichmüller’s metric and Thurston’s asymptotic metric on Teichmüller space*, in “Handbook of Teichmüller theory”, Vol. 1 (A. Papadopoulos, Ed.), European Math. Soc. 2007, 111–204.
- [PWW07] M. Pollicott, H. Weiss, S. Wolpert, *Topological dynamics of the Weil-Petersson geodesic flow*, arXiv:0711.3221.
- [R05] K. Rafi, *A characterization of short curves of a Teichmüller geodesic*, Geom. Top. 9 (2005), 179–202.
- [R07a] K. Rafi, *Thick-thin decomposition of quadratic differentials*, Math. Res. Lett. 14 (2007), 333–341.
- [R07b] K. Rafi, *A combinatorial model for the Teichmüller metric*, Geom. Funct. Anal. 17 (2007), 936–959.
- [W79] S. Wolpert, *The length spectra as moduli for compact Riemann surfaces*, Ann. Math. 109 (1979), 323–351.
- [W03] S. Wolpert, *Geometry of the Weil-Petersson completion of Teichmüller space*, in “Surveys in Differential Geometry VIII: Papers in honor of Calabi, Lawson, Siu and Uhlenbeck”, Intl. Press, Cambridge, MA 2003, 357–393.
- [W07a] S. Wolpert, *Behavior of geodesic-length functions on Teichmüller space*, J. Diff. Geom. 79 (2008), 277–343.
- [W07b] S. Wolpert, *Extension of the Weil-Petersson connection*, arXiv:0709.2513.
- [W08] S. Wolpert, *The Weil-Petersson metric geometry*, arXiv:0801.0175.

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