Examples for nonequivalence of symplectic capacities

Ursula Hamenstädt *
Mathematisches Institut der Unversität Bonn
Beringstraße 1, D-53115 Bonn, Germany
e-mail: ursula@math.uni-bonn.de

Abstract

We construct an open bounded star-shaped set $\Omega \subset \mathbb{R}^4$ whose cylindrical capacity is strictly bigger than its proper displacement energy. We also construct an open bounded set $\Omega_0 \subset \mathbb{R}^4$ whose proper displacement energy is stricly bigger than the displacement energy of its closure. [1]

1 Introduction

Consider the standard $2n-$dimensional euclidean space $\mathbb{R}^{2n}$ equipped with the euclidean symplectic form $\omega_0 = \sum_{i=1}^n dx_{2i-1} \wedge dx_{2i}$. In this paper we are interested in symplectic invariants of nonempty open subsets of $(\mathbb{R}^{2n}, \omega_0)$. One example of such an invariant is a *relative* or *nonintrinsic capacity* [MS] which associates to every open subset Ω of $\mathbb{R}^{2n}$ a number $c(\Omega) \in [0, \infty]$. This number $c(\Omega)$ measures the symplectic size of Ω in such a way that the following three properties hold.

A1 *Monotonicity:* $c(\Omega) \leq c(D)$ if there is a *global* symplectomorphism of $\mathbb{R}^{2n}$ which maps Ω into D.

A2 *Conformality:* $c(a\Omega) = a^2 c(\Omega)$ for all $a > 0$.

A3 *Nontriviality:* $c(B^{2n}(1)) = 1 = c(Z^{2n}(1))$ for the open normalized ball $B^{2n}(1)$ of radius $\sqrt{1/\pi}$ and the open symplectic cylinder $Z^{2n}(1) = B^2(1) \times \mathbb{R}^{2n-2}$ in the standard space $(\mathbb{R}^{2n}, \omega_0)$.

Here we use coordinates $(x_1, \dots, x_{2n})$ in $\mathbb{R}^{2n}$ and we write $B^{2n}(r) = \{x \in \mathbb{R}^{2n} \mid |x|^2 < r/\pi\}$ and $Z^{2n}(r) = B^2(r) \times \mathbb{R}^{2n-2} = \{x \in \mathbb{R}^{2n} \mid x_1^2 + x_2^2 < r/\pi\}$ for the ball and cylinder of capacity $r > 0$ in $\mathbb{R}^{2n}$.

*Research partially supported by SFB 256 and SFB 611.
[1] AMS subject classification: 53D05

The celebrated *non-squeezing lemma* of Gromov [G] shows that for $r > 1$ the ball $B^{2n}(r)$ does not admit a symplectic embedding into the cylinder $Z^{2n}(1)$. This implies that relative capacities do exist, and in fact there are many ways to define them. The resulting invariants do not coincide in general. We will consider the following four examples of such relative capacities.

The *Gromov width* assigns to an open set $\Omega \subset \mathbb{R}^{2n}$ the supremum $c_0(\Omega)$ of all numbers $r > 0$ such that there is a symplectic embedding of the ball $B^{2n}(r)$ into Ω. By monotonicity, the Gromov width is the smallest capacity which means that if c' is any relative capacity, then $c_0(\Omega) \leq c'(\Omega)$ for every open set $\Omega \subset \mathbb{R}^{2n}$.

Let $\mathcal{O}$ be the family of nonempty open bounded subsets of $\mathbb{R}^{2n}$. Our second example is the *cylindrical capacity* which associates to $\Omega \in \mathcal{O}$ the infimum $c_p(\Omega)$ of all numbers $r > 0$ for which there is a symplectomorphism of $\mathbb{R}^{2n}$ which maps Ω into the cylinder $Z^{2n}(r)$ [P]. If $\Omega \subset \mathbb{R}^{2n}$ is unbounded then we define $c_p(\Omega) = \sup\{c_p(\Omega') \mid \Omega' \in \mathcal{O}, \Omega' \subset \Omega\}$. By monotonicity, the cylindrical capacity is the biggest relative capacity which means that if c' is any relative capacity, then $c'(\Omega) \leq c_p(\Omega)$ for every $\Omega \in \mathcal{O}$.

Third the *displacement energy* is defined as follows. Recall that a compactly supported smooth time dependent function $H(t,x)$ on $[0,1] \times \mathbb{R}^{2n}$ induces a time-dependent Hamiltonian flow on $\mathbb{R}^{2n}$. Its time-one map φ is then a symplectomorphism of $\mathbb{R}^{2n}$. The group $\mathcal{D}$ of compactly supported symplectomorphisms obtained in this way is called the group of *compactly supported Hamiltonians* [HZ].

The *Hofer-norm* on the group $\mathcal{D}$ assigns to $\varphi \in \mathcal{D}$ the value

$$\|\varphi\| = \inf_H \Big(\sup_{t\in[0,1]} \big(\sup_{x\in\mathbb{R}^{2n}} H(t,x) - \inf_{x\in\mathbb{R}^{2n}} H(t,x)\big)\Big)$$

where H ranges over the set of all compactly supported time dependent functions whose Hamiltonian flows induce φ as their time-one map. The Hofer-norm $\| \; \|$ induces a bi-invariant distance function d on the group $\mathcal{D}$ by defining $d(\varphi,\psi) = \|\varphi \circ \psi^{-1}\|$, in particular we have $\|\varphi\| > 0$ for $\varphi \neq Id$ [HZ].

For a bounded set $A \subset \mathbb{R}^{2n}$ we define the *displacement energy* $d(A)$ to be the infimum of the Hofer norms $\|\varphi\|$ of all those $\varphi \in \mathcal{D}$ which *displace* A, i.e. for which we have $\varphi(A) \cap A = \emptyset$. If $A \subset \mathbb{R}^{2n}$ is unbounded then we define $d(A) = \sup\{d(A') \mid A' \subset A, A' \text{ bounded}\}$.

Since the cube $(0,1) \times (0,a) \subset \mathbb{R}^2$ of area $a > 0$ is displaced by the time-one map of the Hamiltonian flow induced by the time-independent function $H(t,x,y) = ax$, the displacement energy of the cylinder $(0,1) \times (0,a) \times \mathbb{R}^{2n-2} \subset \mathbb{R}^{2n}$ in $\mathbb{R}^{2n}$ is not bigger than its capacity $a > 0$. This implies in particular that $d(\Omega) \leq c_p(\Omega)$ for every open bounded subset of $\mathbb{R}^{2n}$. On the other hand, the displacement energy of an euclidean ball of capacity a is not smaller than a (this was first shown by Hofer; we refer to [HZ] and [LM] for proofs and references). Since moreover clearly $d(\Omega') \leq d(\Omega)$ if $\Omega' \subset \Omega$, the displacement energy is a relative capacity.

Following [HZ] we call two subsets A, B of $\mathbb{R}^{2n}$ *properly separated* if there is a symplectomorphism Ψ of $\mathbb{R}^{2n}$ such that $\overline{\Psi(A)} \subset \{x_1 < 0\}$ and $\overline{\Psi(B)} \subset$

$\{x_1 > 0\}$. Define the *proper displacement energy* $e(\Omega)$ of a set $\Omega \in \mathcal{O}$ to be the infimum of the Hofer-norms $\|\varphi\|$ of all those $\varphi \in \mathcal{D}$ for which $\varphi(\Omega)$ and Ω are properly separated. If $\Omega \subset \mathbb{R}^{2n}$ is unbounded we define $e(\Omega) = \sup\{e(\Omega') \mid \Omega' \subset \Omega, \Omega' \text{ bounded}\}$. As before, the proper displacement energy is a relative capacity. We have the inequalities $c_0(\Omega) \leq d(\Omega) \leq e(\Omega) \leq c_p(\Omega)$ for every set $\Omega \in \mathcal{O}$.

Even for star-shaped subsets of $\mathbb{R}^{2n}$ $(n \geq 2)$ our above capacities define different symplectic invariants. The earliest result known to me in this direction is due to Hermann [He]. He constructed for every $n \geq 2$ star-shaped Reinhardt-domains in $\mathbb{R}^{2n}$ with arbitrarily small volume and hence arbitrarily small Gromov width whose displacement energy is bounded from below by 1. For the estimate of the displacement energy he uses a remarkable result of Chekanov [C] who showed that the displacement energy of a closed Lagrangian submanifold of $\mathbb{R}^{2n}$ is positive.

The displacement energy of closed Lagrangian submanifolds is not the only obstruction for embeddings of a star-shaped set Ω into a cylinder of small capacity. We show.

Theorem A: *There is an open bounded starshaped subset Ω of $\mathbb{R}^4$ with $e(\Omega) < c_p(\Omega)$.*

We also compare the displacement energy and the proper displacement energy. We show.

Theorem B:

1. *Let $\Omega \subset \mathbb{R}^{2n}$ be open, bounded and connected. If $H^1(\overline{\Omega}, \mathbb{R}) = 0$ then $e(\Omega) = d(\overline{\Omega})$. If $\Omega \subset \mathbb{R}^{2n}$ is star-shaped then we have $d(\Omega) = e(\Omega)$.*
2. *There is an open bounded connected subset Ω_0 of $\mathbb{R}^4$ with smooth boundary and such that $d(\overline{\Omega_0}) < e(\Omega_0)$.*

A modification of our construction can be used to obtain for every $n \geq 2$ examples of open bounded subsets Ω of $\mathbb{R}^{2n}$ with the properties stated in Theorem A and in the second part of Theorem B.

The organization of this note is as follows. In Section 2 we compute various relative capacities for open bounded subsets of the plane $\mathbb{R}^2$. We show that the equality $d(\Omega) = c_p(\Omega)$ holds for every open connected set $\Omega \subset \mathbb{R}^2$ with smooth boundary, without further restrictions on the topology of $\overline{\Omega}$. However for a union Ω of open discs in the plane with smooth boundary and disconnected closure we have $d(\Omega) < c_p(\Omega)$.

In Section 3 we collect some results on symplectic embeddings and symplectic isotopies which are needed for the proof of Theorem A. The proof of Theorem A is completed in Section 4. Section 5 is devoted to the proof of Theorem B.

2 Capacities and relative capacities for subsets of $\mathbb{R}^2$

The main purpose of this section is to compute relative capacities for open subsets of $\mathbb{R}^2$. We begin with some general remarks on relative capacities for open bounded subsets of an arbitrary euclidean space $\mathbb{R}^{2n}$. For this we continue to use the notations from the introduction. In particular, we denote by $\mathcal{O}$ the family of all open bounded subsets of $\mathbb{R}^{2n}$. The cylindrical capacity assigns to $\Omega \in \mathcal{O}$ the infimum $c_p(\Omega)$ of all numbers $r > 0$ for which there is a symplectomorphism of $\mathbb{R}^{2n}$ which maps Ω into the cylinder $Z^{2n}(r)$.

Let $\mathcal{D}$ be the group of compactly supported Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$. We denote by $\|\varphi\|$ the Hofer norm of $\varphi \in \mathcal{D}$. Recall from the introduction the definition of the displacement energy $d(\Omega)$ and the proper displacement energy $e(\Omega)$ of an open bounded set $\Omega \in \mathcal{O}$. The displacement energy and the proper displacement energy are relative capacities, and we have $d(\Omega) \leq d(\overline{\Omega}) \leq e(\Omega) \leq c_p(\Omega)$ for every $\Omega \in \mathcal{O}$.

The next lemma describes some first easy properties of the proper displacement energy and the cylindrical capacity.

Lemma 2.1: *Let Ω be an open and bounded subset of $\mathbb{R}^{2n}$ and let $K \supset \Omega$ be the union of the closure $\overline{\Omega}$ of Ω with the bounded components of $\mathbb{R}^{2n} - \overline{\Omega}$. Then we have.*

1. $e(\Omega) = \inf\{e(U) \mid U \in \mathcal{O}, U \supset K\}$, $c_p(\Omega) = \inf\{c_p(U) \mid U \in \mathcal{O}, U \supset K\}$.
2. *If $1 \leq m < n$ and if $\Omega = \Omega_1 \times \Omega_2$ for open bounded subsets Ω_1 of $\mathbb{R}^{2m}$, Ω_2 of $\mathbb{R}^{2n-2m}$ then*

$$e(\Omega) \leq \min\{e(\Omega_1), e(\Omega_2)\} \quad \text{and} \quad c_p(\Omega) \leq \min\{c_p(\Omega_1), c_p(\Omega_2)\}.$$

Proof: If $\varphi \in \mathcal{D}$ properly displaces Ω, then it properly displaces an open neighborhood of the closure $\overline{\Omega}$ of Ω, moreover the image under φ of a bounded component of $\mathbb{R}^{2n} - \overline{\Omega}$ is a bounded component of $\mathbb{R}^{2n} - \overline{\varphi(\Omega)}$.

Similarly, every symplectomorphism of $\mathbb{R}^{2n}$ which maps Ω into an open cylinder maps the union K of the closure of Ω with the bounded components of $\mathbb{R}^{2n} - \overline{\Omega}$ into the closed cylinder of the same capacity. From this 1) above is immediate.

To show 2) observe that every time dependent Hamiltonian function H_t on $\mathbb{R}^{2m}$ admits a natural extension to a function $\tilde{H}_t$ on $\mathbb{R}^{2n}$ which only depends on the first $2m$ coordinates. If the Hamiltonian flow on $\mathbb{R}^{2m}$ induced by H_t properly displaces an open bounded subset Ω_1 of $\mathbb{R}^{2m}$, then the Hamiltonian flow on $\mathbb{R}^{2n}$ induced by $\tilde{H}_t$ properly displaces every product of Ω_1 with an open bounded subset of $\mathbb{R}^{2n-2m}$. **q.e.d.**

Now we discuss briefly how a relative capacity gives rise to an *intrinsic* capacity. For this recall the definition of such an intrinsic capacity as introduced by Ekeland and Hofer [EH]. It associates to every symplectic manifold (M, ω) a number $c(M, \omega) \in [0, \infty]$ which measures the symplectic size of M and such that the following three properties hold.

A1′ *Monotonicity:* $c(M, \omega) \leq c(N, \tau)$ if there exists a symplectic embedding $\varphi\colon (M, \omega) \to (N, \tau)$.

A2′ *Conformality:* $c(M, \alpha\omega) = |\alpha| c(M, \omega)$ for all $\alpha \in \mathbb{R}, \alpha \neq 0$.

A3′ *Nontriviality:* $c(B^{2n}(1), \omega_0) = 1 = c(Z^{2n}(1), \omega_0)$

Notice that every intrinsic capacity naturally defines a relative capacity for open subsets of $\mathbb{R}^{2n}$. An example for an intrinsic capacity is the Gromov width. It assigns to (M, ω) the supremum $c_0(M, \omega)$ of all numbers $r > 0$ such that there is a symplectic embedding of the ball $B^{2n}(r)$ into (M, ω).

The Gromov width is in general strictly smaller than the displacement energy, even for open bounded star-shaped Reinhardt domains in $\mathbb{C}^n = \mathbb{R}^{2n}$ [He]. On the other hand, it is well known that for ellipsoids in $\mathbb{R}^{2n}$ all capacities and relative capacities coincide, and the same is true for convex Reinhardt domains in $\mathbb{C}^n$ [He]. Viterbo [V] showed that for general open bounded convex subsets Ω of $\mathbb{R}^{2n}$ we have $c_p(\Omega) \leq 4n^2 c_0(\Omega)$.

A relative capacity c for open subsets of $\mathbb{R}^{2n}$ induces an intrinsic symplectic capacity c' for symplectic manifolds (M, ω) by assigning to (M, ω) the infimum $c'(M, \omega)$ of all numbers $r \in (0, \infty]$ such that (M, ω) admits a symplectic embedding into an open subset Ω of $\mathbb{R}^{2n}$ with $c(\Omega) < r$. If (M, ω) does not admit a symplectic embedding into $\mathbb{R}^{2n}$ then we define $c'(M, \omega) = \infty$. We call c' the *capacity derived from the relative capacity c*.

Define the *outer capacity* c_1 to be the capacity derived from the cylindrical capacity. It assigns to a symplectic manifold (M, ω) the infimum $c_1(M, \omega)$ of all numbers $r > 0$ such that (M, ω) admits a symplectic embedding into the cylinder $Z^{2n}(r)$. If M does not admit a symplectic embedding into any cylinder in $\mathbb{R}^{2n}$, then $c_1(M, \omega) = \infty$. The capacity e_1 derived from the proper displacement energy e will be called the *strict displacement energy*.

To compute the above capacities and relative capacities for subsets of the plane $\mathbb{R}^2$ we recall the following result of Moser (see e.g. [HZ]).

Lemma 2.2: *Let D_1, D_2 be open bounded topological discs in $\mathbb{R}^2$ with smooth boundaries whose closures are contained in a common open topological disc D_0. If* $\text{area}(D_1) = \text{area}(D_2)$ *then there is an area preserving diffeomorphism of $\mathbb{R}^2$ which maps D_1 onto D_2 and equals the identity on $\mathbb{R}^2 - D_0$.*

Proof: We may assume that the boundary ∂D_0 of D_0 is smooth. Choose a diffeomorphism φ_0 of D_0 which equals the identity near ∂D_0, maps D_1 to D_2 and is area preserving on a small tubular neighborhood of the boundary ∂D_1 of D_1. Following Moser [HZ], since $\text{area}(D_1) = \text{area}(D_2)$ there is an area preserving diffeomorphism φ_1 of D_1 onto D_2 which coincides with φ_0 near the boundary of D_1. Similarly, there is an area preserving diffeomorphism φ_2 of $D_0 - \overline{D_1}$ onto $D_0 - \overline{D_2}$ which coincides with φ_0 near the boundary $\partial D_0 \cup \partial D_1$

of $D_0 - \overline{D_1}$. Then φ_1, φ_2 and the identity map on $\mathbb{R}^2 - \overline{D_0}$ can be combined to an area preserving diffeomorphism of $\mathbb{R}^2$ with the required properties. **q.e.d.**

Now we can show.

Lemma 2.3: *Let Ω be an open bounded subset of $\mathbb{R}^2$ with connected components Ω_i $(i \in I)$. Then we have.*

i) $c_0(\Omega) = \sup_i \ \text{area}(\Omega_i)$.

ii) $c_p(\Omega) = \text{area}(K)$ *where K is the union of $\overline{\Omega}$ with the bounded components of $\mathbb{R}^2 - \overline{\Omega}$.*

iii) $e(\Omega) = d(\overline{\Omega}) = \sup_j \ \text{area}(K_j)$ *where the sets K_j $(j \in J)$ are the connected components of the union of $\overline{\Omega}$ with the bounded components of $\mathbb{R}^2 - \overline{\Omega}$.*

Assume in addition that the area of the boundary of Ω vanishes. Then we have.

iv) $c_1(\Omega) = \text{area}(\Omega)$.

v) $e_1(\Omega) = c_0(\Omega)$.

Proof: Let Ω be an open bounded subset of $\mathbb{R}^2$. Then Ω has at most countably many connected components $\Omega_1, \Omega_2, \ldots$.

Write $a_0 = \sup_i \ \text{area}(\Omega_i)$. To show i) above observe that the image of an open ball in $\mathbb{R}^2$ under a symplectic embedding is connected and therefore $c_0(\Omega) \leq a_0$ by the definition of c_0. For the reverse inequality let $\epsilon > 0$ and choose a connected component Ω_i of Ω with $\text{area}(\Omega_i) > a_0 - \epsilon$. Since the area of Ω_i is the supremum of the areas of its compact subsets we can find a compact connected subset A of Ω_i with dense interior and smooth boundary and such that $\text{area}(A) > a_0 - 2\epsilon$.

Since A is a smooth compact connected two-dimensional manifold with boundary, its fundamental group is finitely generated. This means that we can find finitely many smooth pairwise disjoint arcs $\gamma_1, \ldots, \gamma_k$ in A such that $A - \cup_i \gamma_i$ is connected and simply connected. If we remove from A small open tubular neighborhoods of these arcs, then we obtain a compact subset C of A with smooth boundary and area at least $a_0 - 3\epsilon$ which is connected and simply connected (compare [HZ]). Then C is diffeomorphic to a closed disc. By Lemma 2.2 there is an area preserving diffeomorphism of the round disc $B^2(\text{area}(C))$ onto C. But this means that $c_0(\Omega) \geq c_0(C) = \text{area}(C) \geq a_0 - 3\epsilon$, and since $\epsilon > 0$ was arbitrary we conclude that $c_0(\Omega) = a_0$.

To compute $c_p(\Omega)$ let K be the union of the closure of Ω with the bounded components of $\mathbb{R}^2 - \overline{\Omega}$. Lemma 2.1 shows that $c_p(\Omega) = c_p(K)$, and by the definition of the proper outer capacity we have $c_p(K) \geq \text{area}(K) = a_1$.

Now the area of every compact set in $\mathbb{R}^2$ equals the infimum of the areas of its open neighborhoods and therefore we can find for every $\epsilon > 0$ a compact neighborhood A of K with the following properties.

1. $\text{area}(A) \leq \text{area}(K) + \epsilon$.

2. A has smooth boundary and only finitely many components.

3. $\mathbb{R}^2 - A$ is connected.

Let $A_1, \dots, A_k$ be the components of A and denote by κ_i the area of A_i. Since $\mathbb{R}^2 - A$ is connected, A is simply connected and therefore each component of A is diffeomorphic to a disc. A successive application of Lemma 2.2 shows that there is an area preserving diffeomorphism of $\mathbb{R}^2$ which maps each of the sets A_i into the rectangle

$$R_i = (0,1) \times (\sum_{j=1}^{i-1} \kappa_j + (i-1)\epsilon/k, \sum_{j=1}^{i} \kappa_j + i\epsilon/k).$$

But this means that A and hence Ω can be embedded into $(0,1) \times (0, a_1 + 2\epsilon)$ by an area preserving diffeomorphism of $\mathbb{R}^2$. Since $\epsilon > 0$ was arbitrary we conclude that $c_p(\Omega) = \text{area}(K) = a_1$.

To compute $e(\Omega)$ recall from Lemma 2.1 that for every $\epsilon > 0$ there is an open neighborhood U of the union K of the closure of Ω with the bounded components of $\mathbb{R}^2 - \Omega$ such that $e(U) < e(\Omega) + \epsilon$. Denote by $a_2 > 0$ the supremum of the areas of a connected component of K. Then i) above applied to the set U yields $e(\Omega) + \epsilon > e(U) \geq c_0(U) \geq a_2$, and since $\epsilon > 0$ was arbitrary we deduce that $e(\Omega) \geq a_2$.

Now let $\epsilon \in (0, a_2/3)$ and let A be a compact neighborhood of K with smooth boundary such that the supremum of the areas of a connected component of A is not bigger than $a_2 + \epsilon$ and that $\mathbb{R}^2 - A$ is connected. Then A is simply connected and hence a finite union $A = \cup_{i=1}^{m} A_i$ of closed pairwise disjoint topological discs A_i. By a successive application of Lemma 2.2 there is an area preserving diffeomorphism of $\mathbb{R}^2$ which maps A into the disjoint union

$$R = \cup_{i=0}^{m-1} (0,1) \times (4ia_2, (4i+1)a_2 + 2\epsilon)$$

of m rectangles of area $a_2 + 2\epsilon$ which are separated by strips of width not smaller than $3a_2 - 2\epsilon > a_2 + 3\epsilon$. The collection of these rectangles is properly displaced by the time-one map of the Hamiltonian flow of the function $(x,y) \to (a_2 + 3\epsilon)x$. This function can be multiplied with a suitable cutoff-function to define a compactly supported Hamiltonian of Hofer-norm smaller than $a_2 + 4\epsilon$ which properly displaces R. This implies that $e(\Omega) < a_2 + 4\epsilon$, and since $\epsilon > 0$ was arbitrary we obtain that $e(\Omega) = a_2$.

The inequality $e(\Omega) \geq d(\overline{\Omega})$ is immediate from the definition of the displacement energy and the proper displacement energy. To show the reverse inequality let K be the union of $\overline{\Omega}$ with the bounded components of $\mathbb{R}^2 - \overline{\Omega}$. Then Lemma 2.1 shows that $d(\overline{\Omega}) = d(K)$. On the other hand, $e(\Omega)$ equals the supremum of the areas of the connected components of K, and this is not bigger than $d(K)$. Together we obtain that $e(\Omega) \leq d(K) = d(\overline{\Omega})$ which completes the proof of part iii) of our lemma.

We are left with showing iv) and v) of our lemma, and for this we assume from now on that the area of the boundary $\partial\Omega$ of Ω vanishes. Write $a_3 =$

area(Ω) = area($\overline{\Omega}$). We can find for every $\epsilon > 0$ a compact neighborhood A of $\overline{\Omega}$ with smooth boundary whose area is not bigger than $a_3 + \epsilon$. Then A and $\mathbb{R}^2 - A$ have only finitely many components.

Now if A_i is a component of A, then A_i is a smooth manifold with boundary and hence by Moser's result (compare Lemma 2.2) A_i can be embedded by an area preserving map into a rectangle whose area is arbitrarily close to the area of A_i. As before, we conclude from this that $c_1(\Omega) = a_3$.

To compute $e_1(\Omega)$ let $a_0 = c_0(\Omega)$ be the supremum of the areas of a connected component of Ω and let $\epsilon > 0$. By our assumption, the area of the closure of any component of Ω is not bigger than $c_0(\Omega) = a_0$. The above consideration shows that each of these components Ω_i can be embedded by an area preserving map into a rectangle of area smaller than area$(\Omega_i)(1+\epsilon)$. Since the area of Ω is finite this means that Ω admits an area preserving embedding into finitely many rectangles $R_1, \dots, R_k$ of area smaller than $a_0 + \epsilon$ each whose union is properly displaced by the time-one map of the flow of the function $(x, y) \to (a_0 + 2\epsilon)x$. From this we conclude that $e_1(\Omega) = c_0(\Omega)$. **q.e.d.**

Example: Define $\Omega = \big((0,3) \times (0,3) - [1,2] \times [1,2]\big) \cup (0,3) \times (3,4) \cup (0,1) \times (5,7)$. Lemma 2.3 shows that $c_0(\Omega) = 8$, $c_1(\Omega) = 13$, $c_p(\Omega) = 14$ and $e(\Omega) = 12$. In particular, for general open bounded subsets of $\mathbb{R}^2$ these capacities and relative capacities are all different.

3 Extensions of symplectic embeddings and isotopies

In the proof of Lemma 2.3 we used several times the fact that for every smooth area preserving map φ_0 which is defined on a neighborhood of a closed disc D in $\mathbb{R}^2$ we can find an area preserving diffeomorphism of $\mathbb{R}^2$ which coincides with φ_0 on some neighborhood of D. In this section we collect some existence results for extensions of symplectic embeddings in higher dimensions which are needed for the construction of our main examples.

Define a *proper symplectic embedding* of an open bounded subset Ω of $\mathbb{R}^{2n}$ into $\mathbb{R}^{2n}$ to be a symplectic embedding of a neighborhood of Ω in $\mathbb{R}^{2n}$ into $\mathbb{R}^{2n}$. We call the image of Ω under a proper symplectic embedding *properly equivalent* to Ω. With this notion, the annulus $\{0 < r < |x| < R\} \subset \mathbb{R}^2$ is not properly equivalent to a punctured disc, but any two open annuli $\{0 < r < |x| < R\}$ and $\{0 < r' < |x| < R'\}$ of the same area are properly equivalent. Notice however that there is no symplectomorphism of $\mathbb{R}^2$ which maps $\{0 < r < |x| < R\}$ to $\{0 < r' < |x| < R'\}$ if $r \neq r'$.

In general, if $\Omega \subset \mathbb{R}^{2n}$ is a bounded and open set and if $\varphi\colon \Omega \to \mathbb{R}^{2n}$ is a symplectic embedding then there may not exist a proper embedding of Ω which restricts to φ, even if Ω is homeomorphic to a ball. Once again, the difficulties can be seen already for subsets of the plane $\mathbb{R}^2$.

Namely, if we cut an annulus $\{0 < r < |x| < R\}$ in $\mathbb{R}^2$ along the half-line $\{(t, 0) \mid t > 0\}$ then the resulting set is open and homeomorphic to a ball and hence can be mapped to the ball $\{|x| < \sqrt{R^2 - r^2}\}$ of the same volume by an area preserving map. However this map does not admit an area preserving extension to a neighborhood of the closed annulus $\{0 < r \leq |x| \leq R\}$.

The following well known *neighborhood extension theorem* of Banyaga [B] (see also [MS] for a proof) gives a sufficient condition for the existence of a symplectomorphism of $\mathbb{R}^{2n}$ extending a given proper symplectic embedding of a suitable set $\Omega \in \mathcal{O}$.

Theorem 3.1: *Let $\Omega \in \mathcal{O}$ be an open bounded set such that $H^1(\overline{\Omega}, \mathbb{R}) = 0$. Then for every symplectic embedding φ of a neighborhood of $\overline{\Omega}$ into $\mathbb{R}^{2n}$ there is a symplectomorphism of $\mathbb{R}^{2n}$ which coincides with φ near $\overline{\Omega}$.*

Next we look at isotopies of proper symplectic embeddings of open bounded subsets $\Omega \in \mathcal{O}$ into $\mathbb{R}^{2n}$. We call a smooth 1-parameter family φ_t of symplectic embeddings of a neighborhood U of Ω into $\mathbb{R}^{2n}$ a *proper isotopy* of Ω if φ_0 is the inclusion.

An *extension* of a proper isotopy φ_t of Ω is defined to be a 1-parameter family $\tilde{\varphi}_t$ of symplectomorphisms of $\mathbb{R}^{2n}$ with compact support such that $\tilde{\varphi}_0 = Id$ and $\tilde{\varphi}_t|V = \varphi_t|V$ for some open neighborhood V of $\overline{\Omega}$ in the domain of definition for φ_t.

From the result of Banyaga we obtain (see [MS]):

Lemma 3.2: *If $H^1(\overline{\Omega}, \mathbb{R}) = 0$ then a proper isotopy of Ω admits an extension.*

Let again $\Omega \in \mathcal{O}$. A *strict symplectomorphism* of Ω is a symplectomorphism φ of Ω which equals the identity near the boundary of Ω. A *strict isotopy* of Ω is a 1-parameter family φ_t of symplectomorphisms of Ω which coincides with the identity near the boundary of Ω and such that $\varphi_0 = Id$.

With this notion we have.

Lemma 3.3: *Let $\Omega \subset \mathbb{R}^{2n}$ be open, bounded and star-shaped. Then any two strict symplectomorphisms of Ω are strictly isotopic.*

Proof: Let $\Omega \subset \mathbb{R}^{2n}$ be open, bounded and star-shaped with respect to the origin. We have to show that every strict symplectomorphism Ψ of Ω is strictly isotopic to the identity.

For this we use the arguments of Banyaga. Namely, since the support of Ψ is compact there is an isotopy Ψ_t of the identity with compact support in $a\Omega$ for some $a \geq 1$ and such that $\Psi_1 = \Psi$ (see [MS]). Define $\varphi_t(x) = \frac{1}{a}\Psi_t(ax)$. Then φ_t is a strict isotopy of Ω such that $\varphi_1(x) = \frac{1}{a}\Psi(ax)$. For $t \in [0, 1]$ write moreover $\zeta_t(x) = \frac{1}{a(1-t)+t}\Psi((a(1-t)+t)x)$; then $\zeta_0 = \varphi_1$ and $\zeta_1 = \Psi$ and therefore the composition of the isotopies φ_t and ζ_t define a strict isotopy of Ω as required. **q.e.d.**

A *proper embedding* of an open bounded set $C \in \mathcal{O}$ into Ω is a symplectic embedding of an open neighborhood U of C into Ω. Two proper embeddings

ψ_1, ψ_2 of C into Ω are *strictly isotopic* if there is a strict isotopy φ_t of Ω such that $\varphi_1\psi_1|C = \psi_2|C$. As a corollary of Lemma 3.3 we obtain.

Corollary 3.4: *Let $\Omega \subset \mathbb{R}^{2n}$ be open, bounded and star-shaped, and let $B \subset \mathbb{R}^{2n}$ be open and bounded and such that $H^1(\overline{B}, \mathbb{R}) = 0$. Then any two proper embeddings ψ_1, ψ_2 of B into Ω are strictly isotopic.*

Proof: The case $n = 1$ follows immediately from Lemma 2.2, so assume that $n \geq 2$. Let $\Omega \subset \mathbb{R}^{2n}$ be open and star-shaped with respect to the origin. Then $\overline{\Omega}$ is simply connected and the same is true for $\mathbb{R}^{2n} - \Omega$. Let $B \subset \mathbb{R}^{2n}$ be open and bounded and such that $H^1(\overline{B}, \mathbb{R}) = 0$. Let $\psi_1, \psi_2 : B \to \Omega$ be proper embeddings. Choose an open neighborhood $U \supset \overline{B}$ of B such that ψ_i is defined on U $(i = 1, 2)$. Since $\mathbb{R}^{2n} - \Omega$ is simply connected and $H^1(\psi_i(\overline{B}), \mathbb{R}) = 0$ there is by Lemma 3.1 a symplectomorphism Ψ of $\mathbb{R}^{2n}$ whose restriction to a neighborhood of $\mathbb{R}^{2n} - \Omega$ equals the identity and whose restriction to a neighborhood of $\psi_1(\overline{B})$ which is contained in $\psi_1(U)$ coincides with $\psi_2 \circ \psi_1^{-1}$. Lemma 3.3 then shows that Ψ is isotopic to the identity with an isotopy which equals the identity on $\mathbb{R}^{2n} - \Omega$. **q.e.d.**

Corollary 3.5: *Let $\Omega \subset \mathbb{R}^{2n}$ be open, bounded and star-shaped and let $U \subset V \in \mathcal{O}$ be such that $\overline{U} \subset V$ and that $\overline{U}$ and $\overline{V}$ are simply connected. Let $\varphi : U \to \Omega$ be a proper embedding. If there is a proper embedding $\zeta : V \to \Omega$ then there is a proper embedding $\tilde{\zeta} : V \to \Omega$ whose restriction to U coincides with φ.*

Proof: Let $\varphi : U \to \Omega$ and $\zeta : V \to \Omega$ be proper embeddings. By Corollary 3.4 there is a symplectomorphism Ψ of Ω which equals the identity near the boundary and such that $\Psi \circ \varphi = \zeta|U$. Then $\Psi^{-1} \circ \zeta$ is a proper embedding of V whose restriction to U coincides with φ. **q.e.d.**

4 Cylindrical capacity and proper displacement energy

Using the assumptions and notations from the introduction and the beginning of Section 2, the goal of this section is to show.

Theorem 4.1: *There is an open bounded starshaped set $\Omega \subset \mathbb{R}^4$ such that $c_p(\Omega) > e(\Omega)$.*

For the proof of our theorem we will need the following simple lemma.

Lemma 4.2: *Let $h : \mathbb{R}^2 \to \mathbb{R}$ and $f : \mathbb{R}^{2n-2} \to \mathbb{R}$ be smooth functions with Hamiltonian flows φ_t, η_s. View h and f as functions on $\mathbb{R}^{2n}$ which only depend on the first two and last $2n-2$ coordinates respectively. Let ν_t be the Hamiltonian flow on $\mathbb{R}^{2n}$ of the function hf; then $\nu_t(x, z) = (\varphi_{tf(z)}(x), \eta_{th(x)}(z))$ for every $x \in \mathbb{R}^2$ and every $z \in \mathbb{R}^{2n-2}$.*

Proof: Let Z_h, Z_f be the Hamiltonian vector fields of h, f as functions on $\mathbb{R}^{2n}$ only depending on the first two and last $2n-2$ coordinates respectively. Then Z_h is a section of the 2-dimensional subbundle of $T\mathbb{R}^{2n}$ spanned by the basic vector fields $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$, and Z_f is a section of the $2n-2$-dimensional subbundle of $T\mathbb{R}^{2n}$ spanned by the basic vector fields $\frac{\partial}{\partial x_i}$ for $i \geq 3$. Moreover $fZ_h + hZ_f$ is the Hamiltonian vector field of the function hf. We denote by ν_t its Hamiltonian flow.

The functions h, f induce Hamiltonian flows φ_t, η_t on $\mathbb{R}^2, \mathbb{R}^{2n-2}$. Since h is constant along the orbits of φ_t and f is constant along the orbits of η_t, for every $x \in \mathbb{R}^2$ and every $z \in \mathbb{R}^{2n-2}$ we have $\nu_t(x,z) = (\varphi_{tf(z)}(x), \eta_{th(x)}(z))$ which shows the lemma. **q.e.d.**

Using our lemma we can now determine the cylindrical capacity of a special open bounded star-shaped set as follows.

Example 4.3: Consider $\mathbb{R}^4$ with the standard symplectic form ω_0. Define $Q_1 = \{(0,s,t,0) \mid -\frac{1}{2} \leq s \leq \frac{1}{2}, 0 \leq t \leq 2\}$. For a small number $\delta < 1/4$ let L be the convex cone in the (x_2,x_3)-plane with vertex at the origin whose boundary consists of the ray ℓ_1 through 0 and the point $(0,\delta,2,0)$ and the ray ℓ_2 through 0 and $(0,\frac{1}{2},2,0)$. Let $\tau > 2$ be the unique number with the property that the line $\{(0,s,\tau,0) \mid s \in \mathbb{R}\}$ intersects the cone L in a segment of length 1. Define $Q_2 = \{(0,x_2,x_3,0) \in L \mid x_3 \leq \tau\}$; then the boundary of Q_2 is a triangle with one vertex at the origin, a second vertex $z_1 \neq 0$ on the line ℓ_1 and the third vertex $z_2 \neq 0$ on the line ℓ_2. Let ℓ_3 be the line through z_2 which is parallel to ℓ_1. The lines ℓ_1, ℓ_3 bound a strip S which is foliated into line segments of length 1 which are parallel to the x_2-coordinate axis. Choose a large number $M > \tau + 2$ and define $Q_3 = \{(0,x_2,x_3,0) \in S \mid \tau \leq x_3 \leq M\}$.

By construction, the set $Q = Q_1 \cup Q_2 \cup Q_3$ is star-shaped with respect to 0, and for every $t > 0$ the line $\{x_3 = t, x_1 = x_4 = 0\}$ intersects Q in a connected segment of length at most 1.

Let $P_0 \subset \mathbb{R}^3 = \{x_1 = 0\}$ be the set which we obtain by rotating Q about the origin in the (x_3,x_4)-plane. The set $P = [-1/2,1/2] \times P_0$ is star-shaped with respect to the origin and it contains the cube $[-\frac{1}{2},\frac{1}{2}]^2 \times D$ where D is the disc of capacity $4\pi > 1$ in the (x_3,x_4)-plane. Thus the Gromov-width of P is not smaller than 1.

We claim that the cylindrical capacity of P equals 1. For this let $\epsilon > 0$ and choose a smooth function $\sigma : [0,\infty) \to [0,\infty)$ with the property that we have $Q \subset \{(0,s,t,0) \mid t \geq 0, \sigma(t) - 1/2 \leq s \leq \sigma(t) + 1/2 + \epsilon\}$. Such a function σ exists by the definition of Q, and we may assume that it vanishes identically on $[0,2]$. The Hamiltonian flow ζ_t of the smooth function $(x_3,x_4) \to \sigma(\sqrt{x_3^2+x_4^2})$ preserves the concentric circles about the origin. Define $h(x_1,x_2,x_3,x_4) = -x_1\sigma(\sqrt{x_3^2+x_4^2})$. By Lemma 4.2 and the fact that P is invariant under rotation about the origin in the (x_3,x_4)-plane we conclude that the image of P under the time-one map of the Hamiltonian flow of the function h equals the set $\hat{P} = \{(x_1,x_2,x_3,x_4) \mid (x_1, x_2 + \sigma(\sqrt{x_3^2+x_4^2}), x_3, x_4) \in P\}$ which is contained in the subset $[-1/2,1/2+\epsilon]^2 \times B^2(\pi M^2)$ of the cylinder

$[-1/2, 1/2+\epsilon]^2 \times \mathbb{R}^2 \subset \mathbb{R}^4$. Since $\epsilon > 0$ was arbitrary, the cylindrical capacity of P is not bigger than 1 and hence it coincides with the Gromov width of P.

Now we can complete the proof of Theorem 4.1. Let $Q \subset \{x_1 = 0, x_4 = 0\}$ be as in Example 4.3. Reflect Q along the line $\{x_3 = 0\}$ in the (x_2, x_3)-plane. We obtain a set $\tilde{Q}$ which is star-shaped with respect to the origin. Define $\tilde{P}_0 \subset \{x_1 = 0\}$ to be the set which be obtain by rotating $\tilde{Q}$ about the origin in the (x_3, x_4)-plane. Let $\tilde{P} = [-1/2, 1/2] \times \tilde{P}_0$. Then $\tilde{P}$ is star-shaped with respect to the origin and contains P as a proper subset.

We claim that $e(\tilde{P}) = 1$. To see this notice that for every $t > 0$ the intersection of $\tilde{Q}$ with the line $L_t = \{(0, s, t, 0) \mid s \in \mathbb{R}\}$ consists of at most 2 segments of length at most 1 each. Thus for every $\epsilon > 0$ we can find a smooth function f_ϵ on the half-plane $\{x_3 > 0\}$ in the plane $\{x_1 = x_4 = 0\}$ which satisfies $\sup_{z \in \tilde{Q}} f_\epsilon(z) - \inf_{z \in \tilde{Q}} f_\epsilon(z) \leq 1 + \epsilon$ and such that for every $t \geq 0$ its restriction to each of the at most two components of $L_t \cap \tilde{Q}$ equals a translation. We may choose f_ϵ in such a way that $f_\epsilon(x_2, x_3) = x_2$ for $0 < x_3 < 2$.

Extend the function f_ϵ to a function f on $\mathbb{R}^4$ which does not depend on the first coordinate and is invariant under rotation about the origin in the (x_3, x_4)-plane. The Hamiltonian vector field of the restriction of $-f$ to our set $\tilde{P}$ is of the form $\frac{\partial}{\partial x_1} + Z$ where the vector field Z is tangent to the concentric circles about the origin in the (x_3, x_4)-plane. Since $\tilde{P}$ is invariant under rotation about the origin in the (x_3, x_4)-plane we conclude that for every $s > 0$ the image of $\tilde{P}$ under the time-s map of the Hamiltonian flow of f equals the set $[-1/2+s, 1/2+s] \times \tilde{P}_0$. This means that the time-$(1+\epsilon)$ map of the Hamiltonian flow of f properly displaces $\tilde{P}$. Via multiplying f with a suitable cutoff-function we deduce that the proper displacement energy of $\tilde{P}$ is not bigger than $(1+\epsilon)^2$. Since $\epsilon > 0$ was arbitrary and since $c_0(\tilde{P}) \geq 1$ we have $e(\tilde{P}) = 1$.

We are left with showing that $c_p(\tilde{P}) > 1$. For this define $A \subset \mathbb{R}^2$ to be the closed annulus $\overline{B^2(\pi M^2) - B^2(\pi\tau^2)}$ of area $\pi(M^2 - \tau^2) \geq 4\pi$. By the discussion in Example 4.3 there is a small number $\rho > 0$ depending on the choice of δ in the construction of the set Q with the following properties.

1. For every $\epsilon \in (0, \rho)$ there is a symplectic embedding ψ_ϵ of a neighborhood of the star-shaped set $P \subset \tilde{P}$ into the cylinder $B^2(1+\epsilon) \times \mathbb{R}^2$ with the property that $\psi_\epsilon(P) \supset B^4(1-\epsilon) \cup B^2(1-\epsilon) \times A$ and $\psi_\epsilon(P \cap \{x_2 > 0\}) \supset B^2(\rho) \times B^2(\pi M^2)$.
2. There is a proper symplectic embedding of the standard ball $B^4(1/2 - \rho)$ into $\tilde{P} - P$ whose image $\tilde{B}$ is strictly isotopic in $\tilde{P} \cap \{x_2 < 0\} \supset \tilde{P} - P$ to a standard ball embedded in $B^4(1-\epsilon) \cap \{x_2 < 0\}$.

Assume to the contrary that $c_p(\tilde{P}) = 1$. Then there is for every $\epsilon \in (0, \rho)$ a proper symplectic embedding of $\tilde{P}$ into the cylinder $B^2(1+\epsilon) \times \mathbb{R}^2$. Since the closure of the disconnected set $P \cup \tilde{B} \subset \tilde{P}$ is simply connected and since the standard cylinder $B^2(1+\epsilon) \times \mathbb{R}^2$ is star-shaped with respect to the origin, we can apply Corollary 3.5 to proper embeddings of $P \cup \tilde{B}$ into $B^2(1+\epsilon) \times \mathbb{R}^2$. This means that there is a proper symplectic embedding Ψ of $\tilde{P}$ into $B^2(1+\epsilon) \times \mathbb{R}^2$ whose restriction to $P \subset \tilde{P}$ coincides with ψ_ϵ and which maps $\tilde{B}$ to a standard euclidean ball $\hat{B}$ which is contained in $B^2(1+\epsilon) \times \big(\mathbb{R}^2 - B^2(\pi M^2)\big)$ and can be obtained from $B^4(1/2-\rho)$ by a translation.

Let ω_1 be a standard volume form on the sphere S^2 whose total area is bigger than but arbitrarily close to $1+\epsilon$. Embed the disc $B^2(1+\epsilon)$ symplecticly into S^2. The image in S^2 of the annulus $B^2(1+\epsilon) - B^2(1-\epsilon)$ is contained in a closed round disc $D \subset S^2$ of area bigger than but arbitrarily close to 2ϵ. The complement of D in S^2 is the disc $B^2(1-\epsilon)$. The complement of the disc $B^2(\rho)$ in S^2 is area-preserving equivalent to a closed disc in $\mathbb{R}^2$.

Our embedding of $B^2(1+\epsilon)$ into (S^2, ω_1) extends to a symplectic embedding of $B^2(1+\epsilon) \times \mathbb{R}^2$ into $(S^2 \times \mathbb{R}^2, \omega = \omega_1 + \omega_0)$. Thus if $\tilde{P}$ admits a proper symplectic embedding into $B^2(1+\epsilon) \times \mathbb{R}^2$ then the standard linear embedding of $B^4(1/2-\rho)$ onto a ball in $S^2 \times (\mathbb{R}^2 - B^2(\pi M^2)) \subset S^2 \times \mathbb{R}^2$ is strictly isotopic in $S^2 \times (\mathbb{R}^2 - B^2(\pi M^2)) \cup D \times A \cup B^2(1+\epsilon) \times B^2(\pi\tau^2) \subset S^2 \times \mathbb{R}^2$ to the standard inclusion of $B^4(1/2-\rho)$ into $B^2(1+\epsilon) \times B^2(1/2-\rho)$.

However a suitable version of the *symplectic camel theorem* in dimension 4 [MDT] shows that for sufficiently small ϵ this is not possible. We formulate this version as a proposition which completes the proof of our Theorem 4.1.

Proposition 4.4: *Let $D \subset S^2$ be an open round disc of area $\frac{1}{4}$ in a standard sphere (S^2, ω_1) of area 1. Let $\omega = \omega_1 + \omega_2$ be a standard symplectic form on $S^2 \times \mathbb{R}^2$. For every $R \in (\frac{1}{4}, 1)$ a standard embedding of the ball $B^4(R)$ into $S^2 \times (\mathbb{R}^2 - B^2(3))$ is not properly isotopic in $S^2 \times (\mathbb{R}^2 - \overline{B^2(2)}) \cup D \times \partial B^2(2) \cup B^2(1) \times B^2(2) \subset S^2 \times \mathbb{R}^2$ to a standard embedding of $B^4(R)$ into $S^2 \times B^2(1)$.*

Proof: Using the notation from the proposition, let $R \in (\frac{1}{4}, 1)$ and let φ_0 be a standard embedding of the ball $B^4(R)$ of capacity R into $B^2(R) \times (\mathbb{R}^2 - B^2(3)) \subset S^2 \times (\mathbb{R}^2 - B^2(3))$. Let moreover φ_1 be a standard embedding of $B^4(R)$ into $B^2(1) \times B^2(1) \subset S^2 \times B^2(1)$. We argue by contradiction and we assume that φ_0 can be connected to φ_1 by a proper isotopy φ_t ($t \in [0,1]$) whose image is contained in the subset $S^2 \times (\mathbb{R}^2 - \overline{B^2(2)}) \cup D \times \partial B^2(2) \cup B^2(1) \times B^2(2)$ of the manifold $S^2 \times \mathbb{R}^2$.

We follow [MDT] and arrive at a contradiction in three steps.

Step 1

Let c be the boundary of the disc $B^2(2)$. Denote by ν the boundary circle of the disc $D \subset S^2$. Then $T = \nu \times c$ is a Lagrangian torus embedded in $S^2 \times \mathbb{R}^2$. For a fixed point y on c, the circle $\nu \times \{y\}$ bounds the embedded disc $D \times \{y\} \subset S^2 \times \{y\}$. We call such a disc a *standard flat disc.* It defines a homotopy class of maps of pairs from a closed unit disc $(D_0, \partial D_0) \subset \mathbb{R}^2$ into $(S^2 \times \mathbb{R}^2, T)$.

Let $\mathcal{J}$ be the space of all smooth almost complex structures J on $S^2 \times \mathbb{R}^2$ which *calibrate* the symplectic form ω (i.e. such that $g(v,w) = \omega(v, Jw)$ defines a Riemannian metric on $S^2 \times \mathbb{R}^2$). In the sequel we mean by a pseudoholomorphic disc a disc which is holomorphic with respect to some structure $J \in \mathcal{J}$. For $J \in \mathcal{J}$ define a *J-filling* of the torus T to be a 1-parameter family of disjoint, J-holomorphic discs which are homotopic as maps of pairs to the standard flat disc, whose boundaries foliate T and whose union $F(J)$ is diffeomorphic to $D \times S^1$ and does not intersect $(S^2 - D) \times c$. The set $F(J)$ then necessarily disconnects $\Omega = S^2 \times \mathbb{R}^2 - (S^2 - D) \times c$.

For $t \in [0,1]$ let $J_t \in \mathcal{J}$ be an almost complex structure depending continuously on t. We require that $J_0 = J_1$ is the standard complex structure and that the restriction of J_t to $\varphi_t B^4(R)$ coincides with $(\varphi_t)_* J_0$ (where by abuse of notation we denote by J_0 the natural complex structure on $\mathbb{R}^4$ and on $S^2 \times \mathbb{R}^2$). Such structures exist since the space $\mathcal{J}$ is contractible.

Assume that for every $t \in [0,1]$ there is a unique J_t-filling $F(J_t)$ of T depending continuously on t in the Hausdorff topology for closed subsets of $S^2 \times \mathbb{R}^2$. Since each filling $F(J_t)$ disconnects Ω, the set

$$X = \{(t,x) \mid x \in F(J_t), 0 \le t \le 1\}$$

disconnects $[0,1] \times \Omega$. Now J_0 and J_1 are standard and the standard filling of T by flat discs separates $\varphi_0 B^4(R)$ from $\varphi_1 B^4(R)$. Thus the points $(0, \varphi_0(0))$ and $(1, \varphi_1(0))$ are contained in different components of $[0,1] \times \Omega - X$. Therefore the path $(t, \varphi_t(0))$ must intersect X. In other words, for some t, there is a J_t-holomorphic disc C through $\varphi_t(0)$ with boundary on T and which is contained in the homotopy class of the standard flat disc. The connected component of $\varphi_t^{-1} C \cap B^4(R)$ containing 0 is a holomorphic disc with respect to the standard integrable complex structure and hence it is a minimal surface with boundary on the boundary of $B^4(R)$. This implies that the area of this disc is not smaller than R [G]. On the other hand, the area of every J_t-holomorphic disc with boundary on T which is homotopic to the standard flat disc equals the area $\frac{1}{4}$ of the standard flat disc (recall that the boundary torus T is Lagrangian). Since $\frac{1}{4} < R$ by assumption this is a contradiction (compare [MDT] p.178).

By the above it is now enough to construct for every $t \in [0,1]$ an almost complex structure $J_t \in \mathcal{J}$ whose restriction to $\varphi_t B^4(R)$ coincides with $(\varphi_t)_* J_0$ and such that for every $t \in [0,1]$ the torus T admits a unique J_t-filling depending continuously on $t \in [0,1]$. For this we follow again [MDT].

Step 2

Let H be an oriented hypersurface in an almost complex 4-manifold (N, J). There is a unique two-dimensional subbundle ξ of the tangent bundle of H which is invariant under J. We call H *J-convex* if for one (and hence any) one-form α on H whose kernel equals ξ and which defines together with the restriction of J to ξ the orientation of H and for every $0 \neq v \in \xi$ we have $d\alpha(v, Jv) > 0$.

Let $H \subset S^2 \times \mathbb{R}^2$ be a smooth hypersurface which contains the Lagrangian torus T and bounds an open domain $U_H \subset S^2 \times \mathbb{R}^2$ which contains the open disc bundle $D \times c - T$. We equip H with the orientation induced by the outer normal of U_H. Denote by $\mathcal{J}_H \subset \mathcal{J}$ the set of all almost complex structures $J \in \mathcal{J}$ for which H is J-convex and which coincide with the standard complex structure J_0 near T. If the set $\mathcal{J}_H$ is not empty then we can find some $\tilde{J} \in \mathcal{J}_H$ which coincides with J_0 on a neighborhood of the disc bundle $D \times c$. Then the standard flat discs define a $\tilde{J}$-filling of T.

Let $J_t \subset \mathcal{J}_H$ ($t \in [1,2]$) be a differentiable curve (there is some subtlety here about the differentiable structure of $\mathcal{J}_H$ which will be ignored in the sequel, compare [MDT]). Assume that for every $t \in [1,2]$ there is a holomorphic disc D_t of J_t with boundary on T depending continuously on t and such that D_1 is a standard flat disc. Since H is J_t-convex, Lemma 2.4 of [MD2] and Proposition 3.2 in [MDT] show that each of the discs D_t meets the hypersurface H transversely at its boundary, and its interior is contained in U_H.

By assumption, each of the structures J_t coincides with the standard complex structure near the torus T. Thus there is an open neighborhood V of T in $S^2 \times \mathbb{R}^2$ and a J_t- antiholomorphic involution in T on V (see [MDT]). Using this involution we can double our domain U_H near T [MDT] and use intersection theory for pseudoholomorphic spheres in almost complex manifolds to conclude that each of our discs D_t is embedded. Moreover any two different such discs for the same structure $J \in \mathcal{J}_H$ do not intersect [MD1].

Now Gromov's compactness theorem is valid for pseudoholomorphic discs with Lagrangian boundary condition [O]. The area of each pseudoholomorphic disc with boundary on T and in the homotopy class of the standard flat disc coincides with the area $\frac{1}{4}$ of the standard flat disc. Moreover since our torus T is rational [P] and $\frac{1}{4}$ is the generator of the subgroup of $\mathbb{R}$ induced by evaluation of ω_0 on $\pi_2(\mathbb{R}^4, T)$, $\frac{1}{4}$ is the minimal area of any pseudoholomorphic disc whose boundary is contained in T. This implies that bubbling off of holomorphic spheres and holomorphic discs can not occur. Therefore we can use standard Fredholm theory for the Cauchy Riemann operator [MDT] to compute for a dense set of points in $\mathcal{J}_H$ the parameter space of pseudoholomorphic discs with boundary on T and which are homotopic to the standard flat disc. As a consequence [MDT], for every $J \in \mathcal{J}_H$ which can be connected to our fixed structure $\tilde{J}$ by a differentiable curve in $\mathcal{J}_H$ there is a unique J-filling $F(J)$ of T which depends continuously on $J \in \mathcal{J}_H$ in the Hausdorff topology for closed subsets of $S^2 \times \mathbb{R}^2$ (here uniqueness means uniqueness of the image and hence we divide the family of all holomorphic discs by the group of biholomorphic automorphisms of the unit disc in $\mathbb{C}$).

Together with Step 1 above we conclude that our proposition follows if we can construct a hypersurface H in $S^2 \times \mathbb{R}^2$ with the following properties.

1. H contains T and bounds an open set U_H containing the open disc bundle $D \times c - T$.
2. U_H contains a neighborhood of $\cup_{t \in [0,1]} \varphi_t B^4(R)$.
3. $\mathcal{J}_H \neq \emptyset$.

Namely, for such a hypersurface H we can choose a fixed almost complex structure $J \in \mathcal{J}_H$ whose restriction to the disc bundle D coincides with the standard structure. For each $t \in [0,1]$ we modify J near $\varphi_t B^4(R)$ in such a way that the modified structure J_t is unchanged near the hypersurface H and coincides with $(\varphi_t)_* J_0$ on $\varphi_t B^4(R)$. We can do this in such a way that J_t depends differentiably on t. For each of the structures J_t there is then a unique J_t-filling of T depending continuously on t.

By our assumption, the closure of the set $\cup_{t\in[0,1]}\varphi_t B^4(R)$ is contained in $S^2 \times (\mathbb{R}^2 - \overline{B^2(2)}) \cup D \times \partial B^2(2) \cup B^2(1) \times B^2(2)$ and hence it is enough to find a hypersurface H in the symplectic manifold

$$N = S^2 \times (\mathbb{R}^2 - \overline{B^2(2)}) \cup B^2(1) \times \partial B^2(2) \cup \mathbb{R}^2 \times B^2(2)$$

with properties 1-3. In the third step of our proof we construct such a hypersurface.

Step 3

Let $A \subset \mathbb{R}^2$ be a closed circular annulus containing the circle c in its interior. We assume that A is small enough so that the closure of the set $\cup_{t\in[0,1]}\varphi_t B^4(R)$ intersects $S^2 \times A$ in $B^2(\frac{1}{4}) \times A$. We also require that there is some $a > 0$ such that $S^2 \times A \subset S^2 \times \mathbb{R}^2$ is symplectomorphic to the quotient of the bundle $S^2 \times [-a, a] \times \mathbb{R}$ under a translation τ in the plane $\mathbb{R}^2$ in such a way that the torus T is the quotient of the standard circle bundle $\partial B^2(r) \times \{0\} \times \mathbb{R}$. The standard complex structure on $S^2 \times \mathbb{R}^2$ is invariant under the translation τ and projects to an integrable complex structure on a neighborhood of $S^2 \times A$ in $S^2 \times \mathbb{R}^2$ which calibrates ω.

For small $\sigma \in (0, a)$ write $\ell_\sigma = \{(0, 0, -\sigma, s) \mid s \in \mathbb{R}\} \subset \mathbb{R}^4$. The circle bundle $\partial B^2(\frac{1}{4}) \times \{0\} \times \mathbb{R}$ is contained in the boundary ∂U_σ of a tubular neighborhood U_σ of some radius $r(\sigma) > \frac{1}{4}$ about the line ℓ_σ. The hypersurface ∂U_σ with its orientation as the boundary of U_σ is convex with respect to the euclidean metric.

Let X_σ be the gradient of the function $z \to \frac{1}{2}\,\mathrm{dist}(z, \ell_\sigma)^2$. Then X_σ is perpendicular to the hypersurface ∂U_σ. If we denote by $\iota_{X_\sigma}\omega_0$ the 1-form $\omega_0(X_\sigma, \cdot)$ then $d(\iota_{X_\sigma}\omega_0) = dx_1 \wedge dx_2$. This implies that ∂U_σ is convex with respect to J_0 at every point in ∂U_σ at which X_σ is not contained in the span of the standard vector fields $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$. In other words, ∂U_σ is J_0-convex away from the circle bundle $\partial B^2(r(\sigma)) \times \{(-\sigma, s) \mid s \in \mathbb{R}\}$.

Let $E \subset \mathbb{R}^4 - \ell_\sigma$ be any smooth embedded hypersurface which divides $\mathbb{R}^4$ into two components and which is everywhere transverse to the vector field X_σ. Then the kernel of the 1-form $\iota_{X_\sigma}\omega_0$ intersects the tangent bundle of E in a two-dimensional subbundle ξ. The restriction of ω_0 to ξ is non-degenerate and hence there is a unique almost complex structure $\tilde{J}_\sigma$ on ξ which calibrates $\omega_0|\xi$. This structure $\tilde{J}_\sigma$ extends to an almost complex structure J_σ near E which calibrates ω_0.

Denote by U_E the connected component of $\mathbb{R}^4 - E$ which contains the line ℓ_σ. We equip E with the orientation as the boundary of U_E. In the case that E equals the boundary of the tubular neighborhood U_σ, the almost complex structure J_σ coincides on $E = \partial U_\sigma$ with the restriction of the integrable complex structure J_0. Moreover, if E is contained in the open half-space $\{x_3 > -\sigma\}$ then E is J_σ-convex.

Now choose a hypersurface E which is contained in $\{0 \le x_3 \le a/2\}$ and which bounds a noncompact convex set V containing $\{x_3 > a/2\}$. We assume that E is invariant under translations along the lines parallel to ℓ_σ. This implies that E projects to a smooth hypersurface $\tilde{E}$ in $\mathbb{R}^2 \times A$.

Assume that the intersection of E with the hyperplane $\{x_3 = 0\}$ equals the line $\ell_0 = \{(0, 0, 0, s) \mid s \in \mathbb{R}\}$ and that for sufficiently small $\sigma > 0$ the vector field X_σ is everywhere transverse to E. Then for small $\sigma > 0$ the vector field X_σ is everywhere transverse to the boundary of the set $U_\sigma \cup V \cup \{x_3 < -\sigma/2\} = W$. Since E is contained in $\{0 \leq x_3 \leq a/2\}$, the boundary ∂W of W projects to a hypersurface in $\mathbb{R}^2 \times A$ which bounds an open set $\tilde{W}$. The set $\tilde{W}$ in turn projects to a set $\hat{W} \subset N$. By our explicit construction we may assume that $\hat{W}$ contains the closure of the set $\cup_{t \in [0,1]} \varphi_t B^4(R)$. Moreover after a slight perturbation we may assume that the boundary ∂W of W is smooth.

The vector field X_σ is transverse to ∂W. Thus X_σ defines an almost complex structure J_σ on a neighborhood of ∂W which coincides with the standard structure near the torus T and such that ∂W is J_σ-convex. Since X_σ is invariant under the translation τ we may assume that the same is true for the almost complex structure J_σ. Therefore this almost complex structure projects to an almost complex structure on a neighborhood of $\partial\tilde{W}$ which we denote again by J_σ. The hypersurface $\partial\tilde{W}$ is J_σ-convex.

By construction, there is a circle γ in $\mathbb{R}^2 - \overline{B^2(2)}$ such that the intersection of $\partial\tilde{W}$ with the set $(\mathbb{R}^2 - B^2(1/2)) \times (\mathbb{R}^2 - B^2(2))$ is contained in the hypersurface $\mathbb{R}^2 \times \gamma$. But this just means that $\partial\tilde{W}$ projects to a smooth hypersurface H in the set N.

Denote by α the restriction of the 1-form $\iota_{X_\sigma}\omega_0$ to the hyperplane $Q = \{x_3 = -\sigma/2\}$. Since the vector field X_σ is invariant under rotation about the origin in the (x_1, x_2)-plane and the translations along the lines parallel to ℓ_σ we have $\alpha = \varphi(r)d\theta + \rho(r)dx_4$ where (r, θ) are polar coordinates about 0 in the (x_1, x_2)-plane. Then $d\alpha = \varphi'(r)dr \wedge d\theta + \rho'(r)dr \wedge dx_4$. The restriction of $d\alpha$ to the kernel of α vanishes nowhere, and this is equivalent to the inequality $\varphi'(r)\rho(r) - \rho'(r)\varphi(r) > 0$. Moreover we have $\varphi > 0, \rho > 0$.

Let $\tilde{\varphi}, \tilde{\rho}$ be functions on $(0, \infty)$ which coincide with φ, ρ on $(0, 1/2]$ and with the following additional properties.

1. $\tilde{\varphi}'\tilde{\rho} - \tilde{\rho}'\tilde{\varphi} > 0$.
2. $\tilde{\varphi}$ and $\tilde{\rho}$ do not have a common zero.
3. $\tilde{\varphi}(1) = 0$.

Such functions $\tilde{\varphi}, \tilde{\rho}$ can easily be constructed. We replace the one-form α on Q by the one-form $\tilde{\alpha} = \tilde{\varphi}(r)d\theta + \tilde{\rho}(r)dx_4$. Then $\tilde{\alpha}$ is a contact form on Q which coincides with the contact form α on $B^2(1/2) \times \mathbb{R}^2 \cap Q$. Moreover, this contact form projects to a contact form on the hypersurface $\mathbb{R}^2 \times \gamma$ which we denote again by $\tilde{\alpha}$. The kernel of $\tilde{\alpha}$ on $\partial B^2(1) \times \gamma$ is spanned by the vector fields $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$ and hence this kernel projects to a plane-bundle on the projection of $(\overline{B^2(1)} - B^2(1/2)) \times \gamma$ to N which we obtain by mapping each circle $\partial B^2(1) \times \{y\}$ to a point.

In other words, there is a modification $\tilde{J}_\sigma$ of the almost complex structure J_σ which coincides with the integrable complex structure near the torus T and which projects to an almost complex structure on a neighborhood of H in N.

This structure is the restriction of a smooth almost complex structure $\hat{J}_\sigma$ on N which calibrates ω. The oriented hypersurface H is the boundary of an open set U containing $\cup_{t\in[0,1]}\varphi_t B^4(R)$, and it is $\hat{J}_\sigma$-convex.

Together this means that H and $\hat{J}_\sigma$ satisfy the properties 1-3 above. This finishes the proof of our proposition. **q.e.d.**

5 Displacement and proper displacement

This section is devoted to the proof of Theorem B from the introduction. We continue to use the assumptions and notations from Section 2-4.

Recall that the *displacement energy* $d(\Omega)$ of an open bounded set $\Omega \subset \mathbb{R}^{2n}$ equals the infimum of the Hofer norms of all symplectomorphisms Ψ of $\mathbb{R}^{2n}$ such that $\Psi\Omega \cap \Omega = \emptyset$. For every $\Omega \in \mathcal{O}$ the displacement energy $d(\overline{\Omega})$ of the closure $\overline{\Omega}$ of Ω is not smaller than the proper displacement energy $e(\Omega)$ of Ω. By Lemma 2.3, in the case $n = 1$ equality $d(\overline{\Omega}) = e(\Omega)$ always holds.

We begin the proof of our theorem with the following easy corollary of the neighborhood extension theorem of Banyaga [B] (see Theorem 3.1).

Lemma 5.1: *Let $\Omega \in \mathcal{O}$ be such that $H^1(\overline{\Omega}, \mathbb{R}) = 0$; then $d(\overline{\Omega}) = e(\Omega)$.*

Proof: Since we always have $e(\Omega) \geq d(\overline{\Omega})$ we have to show the reverse inequality under the assumption that $H^1(\overline{\Omega}, \mathbb{R}) = 0$. For this we only have to consider the case $n \geq 2$.

Let $\epsilon > 0$ and let $\Psi \in \mathcal{D}$ be a compactly supported Hamiltonian symplectomorphism of $\mathbb{R}^{2n}$ of Hofer norm smaller than $d(\overline{\Omega})+\epsilon$ and such that $\Psi(\overline{\Omega})\cap\overline{\Omega} = \emptyset$. Then there is an open neighborhood U of $\overline{\Omega}$ such that $\Psi(U) \cap U = \emptyset$.

Assume without loss of generality that $U \subset \{x_1 < 0\}$. Let e_1 be the first basis vector of the standard basis of $\mathbb{R}^{2n}$, choose some $\mu < \inf\{x_1(z) \mid z \in U\}$ and define $W = U \cup (U - \mu e_1)$. Then W contains two copies of $\overline{\Omega}$ in its interior which are separated by the hyperplane $\{x_1 = 0\}$. Moreover the set $\Omega \cup \Psi\Omega$ admits a natural proper symplectic embedding into W whose restriction to Ω is just the inclusion.

Since $H^1(\overline{\Omega} \cup \overline{\Psi\Omega}, \mathbb{R}) = 0$, by the Banyaga extension theorem this proper symplectic embedding of $\Omega \cup \Psi\Omega$ into W can be extended to a symplectomorphism η of $\mathbb{R}^{2n}$. Then $\eta \circ \Psi \circ \eta^{-1}$ is a symplectomorphism of Hofer-norm smaller than $d(\overline{\Omega}) + \epsilon$ which properly displaces $\Omega = \eta(\Omega)$. This shows that $e(\Omega) \leq d(\overline{\Omega}) + \epsilon$, and since $\epsilon > 0$ was arbitrary the lemma follows. **q.e.d.**

The following example shows the second part of Theorem B from the introduction.

Example 5.2: Let $\epsilon \in (0, 1/24)$ be a small number and define $Q_1 = [0, 1]^2 - (\epsilon, 1 - \epsilon)^2 \subset \mathbb{R}^2$ and $P_1 = Q_1 \times [-10, 10]^2 \subset \mathbb{R}^4$. For $\delta > 0$ write

$$Q_\delta = [-1, 0] \times [-20, 20] - (-4\epsilon, -\epsilon) \times (-10 - \delta, 10 + \delta) \quad \text{and}$$
$$P_\delta = P_1 \cup ([2\epsilon, 1 - 2\epsilon] \times Q_\delta \times [-10, 10]) \cup [0, 1] \times [0, \epsilon] \times [-20, 20]^2 \subset \mathbb{R}^4.$$

Let $e_2 = (0,1,0,0) \in \mathbb{R}^4$; by construction we have $P_\delta + (1+2\epsilon)e_2 \subset \mathbb{R}^4 - P_\delta$ and therefore P_δ can be displaced by the time-one map of the Hamiltonian flow of the function $(x_1,x_2,x_3,x_4) \to (1+2\epsilon)x_1$. This implies that $d(P_\delta) \le 1+2\epsilon$.

Now let

$$D = \{\frac{1}{4}, \frac{3}{4}\} \times [-4\epsilon, -\epsilon] \times [-10-\delta, 10+\delta] \times \{-5, 5\}$$

be the union of four Lagrangian discs with boundary on P_δ and define $\hat{P}_\delta = P_\delta \cup D$.

We claim that for sufficiently small $\delta > 0$ the displacement energy of $\hat{P}_\delta$ is not smaller than $3/2 > 1+2\epsilon$. To see this we use a result of Chekanov [C]. He showed that for circles γ_1, γ_2 in the plane $\mathbb{R}^2$ the displacement energy of the Lagrangian split-torus $\gamma_1 \times \gamma_2 \subset \mathbb{R}^4$ equals the minimum of the areas of the discs enclosed by one of the curves γ_i. Thus it is enough to find such a Lagrangian split-torus which is contained in our set $\hat{P}_\delta$ and whose displacement energy is bigger than $3/2$.

Denote by $\Pi_0 : \mathbb{R}^4 \to \mathbb{R}^2 = \{x_3 = x_4 = 0\}$ the canonical projection. For a point $x \in \hat{P}_\delta$ consider the set $A(x) = \{\Pi_0 x\} \times \mathbb{R}^2 \cap \hat{P}_\delta$. If x is contained in $\hat{P}_\delta \cap \{x_2 > \epsilon\}$ then $A(x)$ equals the square $[-10,10]^2$ and therefore it contains the boundary ∂R_δ of the rectangle $R_\delta = [-10,10] \times [-5-\delta, 5]$ of area $200+20\delta$.

If $x = (x_1,x_2,x_3,x_4)$ is contained in $\hat{P}_\delta - P_\delta$ then $x_1 \in \{\frac{1}{4}, \frac{3}{4}\}, x_2 \in (-4\epsilon, -\epsilon)$ and

$$A(x) = [-10-\delta, 10+\delta] \times \{-5,5\} \cup ([-20,20] - (-10-\delta, 10+\delta)) \times [-10,10]$$

contains the boundary $\partial\tilde{R}_\delta$ of the square $\tilde{R}_\delta = [-10-\delta, 10+\delta] \times [-5,5]$ of area $200+20\delta$. Similarly, if $(x_1,x_2,x_3,x_4) \in P_\delta$ is such that $x_2 \in [-1,-4\epsilon] \cup [-\epsilon, 0]$ then once again, $A(x)$ contains $\partial\tilde{R}_\delta$. If $x_2 \in [0,\epsilon]$ then $A(x)$ contains both the boundary of R_δ and of $\tilde{R}_\delta$.

By our choice of $R_\delta, \tilde{R}_\delta$ there is a linear area preserving map L_δ of $\mathbb{R}^2$ which maps the rectangle $\tilde{R}_\delta$ onto the rectangle R_δ. Its distance to the identity in the usual norm on $SL(2,\mathbb{R})$ tends to 0 with δ. In particular, if we multiply the generator of the corresponding one-parameter group in $SL(2,\mathbb{R})$ by a suitable cutoff-function then we conclude that we can deform R_δ into $\tilde{R}_\delta$ by the time-one map of the Hamiltonian flow of a function g_δ with support in $[-20,20]^2$ whose L^∞-norm tends to 0 as $\delta \to 0$. We view g_δ as a function on $\mathbb{R}^4$ which only depends on x_3, x_4.

Let $\sigma : \mathbb{R} \to [0,1]$ be a smooth function with support in $[0,\infty)$ which equals 1 on $[\epsilon,\infty)$, and let Ψ be the time-one map of the flow of the function $\sigma(x_2)g_\delta$. By Lemma 4.2, for sufficiently small $\delta > 0$ and every $x_1 \in (\frac{1}{4}, \frac{3}{4}), x_2 \in [0,\epsilon]$ the image of the set $(x_1,x_2) \times \partial\tilde{R}_\delta$ under Ψ is contained in $\hat{P}_\delta$. But this just means that for sufficiently small δ the set $\hat{P}_\delta$ contains the image under Ψ of the Lagrangian split-torus which is the product of the curve $\partial\tilde{R}_\delta$ with a Jordan curve γ in the plane $\{x_3 = x_4 = 0\}$ bounding a disc of area not smaller than $1+(1-4\epsilon)(1-8\epsilon) > 3/2$. The displacement energy of this torus is bigger than $3/2$ and therefore we have $d(\hat{P}_\delta) > 3/2 > d(P_\delta)$.

Now let $\delta > 0$ be sufficiently small that our above estimate for the displacement energy of $\hat{P}_\delta$ holds. For $\rho \in (0, 1/4)$ there is a symplectomorphism $\Phi \in \mathcal{D}$ of Hofer norm smaller than $e(P_\delta)+\rho$ and a symplectomorphism η of $\mathbb{R}^4$ such that $\eta\overline{P_\delta} \subset \{x_1 < 0\}$ and $\Phi\eta\overline{P_\delta} \subset \{x_1 > 0\}$. We may assume that $\eta\hat{P}_\delta \subset \{x_1 < 0\}$.

Let V be an open neighborhood of P_δ which is mapped by $\Phi\eta$ to $\{x_1 \in [\nu, 1/\nu]\}$ for some $\nu > 0$. Choose an open neighborhood U of $¶_\delta$ whose closure $\overline{U}$ is contained in V. We may assume that $(D - U) \cap \overline{V}$ consists of four Lagrangian annuli with smooth boundary.

By the neighborhood extension theorem of Banyaga, applied to the union with $\{x_1 \leq 0\}$ of a compact simply connected neighborhood of $\Phi\eta V$ in $\{x_1 > 0\}$, we can find a symplectomorphism ξ of $\mathbb{R}^4$ which equals the identity on $\{x_1 \leq 0\}$ and maps $\Phi\eta V$ to a translate of V by the vector $(2\nu, 0, 0, 0) = Z$. Then $\xi\Phi\xi^{-1}$ is a symplectomorphism of Hofer norm smaller than $e(P_\delta) + \rho$ which maps $\eta V \supset \eta(P_\delta)$ to $V + Z$.

Now the set D is a union of four Lagrangian discs and therefore it has vanishing proper displacement energy. This means that we then can find a symplectomorphism of $\mathbb{R}^4$ with arbitrarily small Hofer norm which fixes U pointwise and maps $\xi\Phi\eta D - Z$ to D. Then the set $\eta\hat{P}_\delta$ is mapped to $\{x_1 > 0\}$ by the composition of our map $\xi\Phi\xi^{-1}$ of Hofer-norm smaller than $e(P_\delta)+\rho$ and a symplectomorphism whose Hofer norm is arbitrarily small. Since the Hofer norm of the composition of two symplectomorphisms is not bigger than the sum of the Hofer norms of each component [HZ] and since $\rho > 0$ was arbitrary we conclude that the proper displacement energy of $\hat{P}_\delta$ is not bigger than the proper displacement energy of P_δ. In particular, we have $e(P_\delta) \geq d(\hat{P}_\delta) > d(P_\delta)$.

We are left with constructing a symplectomorphism of arbitrarily small Hofer norm which fixes U pointwise and maps $\xi\Phi\eta D - Z$ to D. For this observe that by our construction the set $\xi\Phi\eta\hat{P}_\delta - Z$ is the image of $\hat{P}_\delta$ under a Hamiltonian isotopy which fixes U pointwise. In other words, we may assume that there is a smooth time-dependent function h_t which vanishes identically on U and such that $\xi\Phi\eta\hat{P}_\delta - Z$ is the image of $\hat{P}_\delta$ under the time-one map of the Hamiltonian flow ν_t of h_t. In particular, for each $t \geq 0$ the image of D under ν_t is a Lagrangian submanifold of $\mathbb{R}^4$ which coincides near its boundary with D.

Every Lagrangian submanifold of $\mathbb{R}^4$ which is C^1-close to a Lagrangian plane $L = \{x_1 = c_1, x_4 = c_2\}$ $(c_1, c_2 \in \mathbb{R})$ is the graph of a closed 1-form on L via the identification of the cotangent bundle of L with $\mathbb{R}^4$ [MS]. By the definition of P_δ and D and smoothness of ν_t this means that there is a number $k > 0$ depending on ν_t with the following properties.

1. $|\nu_t^{-1} \circ \nu_{t+1/k}x - x| < \frac{1}{8}$ for every $x \in D$.
2. For every $m < k$ there is a smooth function f^m on D with support in the interior of D and such that $\nu_{(m-1)/k}^{-1} \circ \nu_{m/k}D$ equals the graph of df^m.

For each m we can extend f^m to a smooth function on $\mathbb{R}^4$ which vanishes identically on U and whose restriction to the $\frac{1}{8}$-neighborhood of each component of D only depends on the coordinates x_2, x_3. Then $\nu_{(m-1)/k}^{-1} \circ \nu_{m/k}|D$ coincides with the time-one map of the Hamiltonian flow μ_t^m of the function f^m.

For $\epsilon > 0$ there is a number $\sigma > 0$ such that the Hofer norm of each of the symplectomorphism μ_σ^m is smaller than $\epsilon/2k$. The set $\mu_{1-\sigma}^m D$ intersects $\mu_1^m D$ only in the critical points of f^m. Let ρ be a smooth function on $\mathbb{R}^4$ which vanishes on $\mu_1^m D$ and is constant 1 outside an arbitrarily small neighborhood of $\mu_1^m D$. The restriction to $\cup_{0 \leq s \leq 1-\sigma} \mu_s^m D$ of the function ρf^m coincides outside an arbitrarily small neighborhood of the critical points of f^m on D with f^m. If we denote by $\tilde{\mu}$ the time-one map of its Hamiltonian flow then the image of $\nu_{m/k}^{-1} \circ \nu_{(m+1)/k} D$ under the map $\tilde{\mu}^{-1} \mu_{-\sigma}^m \tilde{\mu}$ is contained in an arbitrarily small neighborhood of D. This then implies that for every given $\epsilon > 0$ there is a symplectomorphism κ_m of $\mathbb{R}^4$ of Hofer norm smaller than ϵ/k which fixes P_δ pointwise and maps $\nu_{(m-1)/k}^{-1} \circ \nu_{m/k} D$ to D.

For $m \leq k$ define $\kappa'_m = \nu_{(m-1)/k} \circ \kappa_m \circ \nu_{(m-1)/k}^{-1}$. The Hofer norm of κ'_m is not bigger than ϵ/k. By construction we have $D = \kappa'_1 \circ \cdots \circ \kappa'_k(\nu_1 D)$. Since the Hofer norm of the composition $\kappa'_1 \circ \cdots \circ \kappa'_k$ is not bigger than the sum of the Hofer norms of the maps κ'_i we conclude that $\nu_1 D$ can be mapped to D by a symplectomorphism of Hofer norm smaller than ϵ which fixes P_δ pointwise. This concludes the proof of our inequality $e(P_\delta) \geq d(\hat{P}_\delta)$.

Our argument is also valid for the computation of the displacement energy and proper displacement energy of the interior of the compact set P_δ, with the same conclusion. Thus we obtain an example as stated in the second part of Theorem B from the introduction. Notice that by construction the first real cohomology group of our set P_δ equals $\mathbb{R}^2$.

We conclude our paper with an investigation of the relation between the displacement energy of a set $\Omega \in \mathcal{O}$ and the displacement energy of its closure $\overline{\Omega}$. We first give an easy example which shows that the equality $d(\Omega) = d(\overline{\Omega})$ does not even hold for open bounded topological balls with smooth boundary.

Example 5.3: Let $\epsilon \in (0, 1/2)$ and define

$$Q_\epsilon = \overline{B^2(4)} \times [0,1]^2 \cup (\overline{B^2(8)} - B^2(4)) \times [1-\epsilon, 2-\epsilon] \times [0,1].$$

Then Q_ϵ is a closed topological ball with piecewise smooth boundary whose interior we denote by U_ϵ. By construction, the sets U_ϵ and $U_\epsilon + (0,0,1,0)$ are disjoint and hence U_ϵ can be displaced by the time-one map of the Hamiltonian flow of the function $f(x_1, x_2, x_3, x_4) = -x_4$. This implies that $d(U_\epsilon) \leq 1$. Since U_ϵ contains the open cylinder $B^2(4) \times (0,1)^2$ of displacement energy 1 we conclude that $d(U_\epsilon) = 1$.

On the other hand, the closure Q_ϵ of U_ϵ contains the split Lagrangian torus $T^2 = \partial B^2(4) \times \partial([0, 2-\epsilon] \times [0,1])$ of displacement energy $2 - \epsilon$ and therefore we have $d(Q_\epsilon) > d(U_\epsilon)$. Via replacing the squares in our construction by discs with smooth boundary we can also find an example of a topological ball Ω with smooth boundary $\partial\Omega$ and such that $d(\overline{\Omega}) > d(\Omega)$.

Recall that a (not necessarily smooth) hypersurface H in $\mathbb{R}^{2n}$ is of *contact type* if there is a conformal vector field ξ defined near the hypersurface H (i.e.

such that the Lie-derivative of ω_0 with respect to ξ coincides with ω_0) which is transverse to H in the sense that the flow lines of ξ intersect H transversely. Let φ_t be the local flow of ξ and assume that there is some $\epsilon > 0$ such that φ_t is defined near H for all $t \in (-\epsilon, \epsilon)$. If H is the boundary of a bounded open set $\Omega \in \mathcal{O}$ then for $t \in (0, \epsilon)$ the set $\varphi_t H$ is the boundary of a neighborhood of $\overline{\Omega}$, and for $t \in (-\epsilon, 0)$ the set $\varphi_t H$ is contained in Ω. The hypersurface H is called of *restricted contact type* if the conformal vector field ξ can be defined on all of $\mathbb{R}^{2n}$. For example, if $\Omega \in \mathcal{O}$ is starshaped with respect to 0 and if the lines through 0 intersect the boundary $\partial\Omega$ transversely then $\partial\Omega$ is of restricted contact type.

Our last lemma shows that the difficulty encountered in our example 5.3 does not occur for open sets with boundary of restricted contact type.

Lemma 5.4: *Let Ω be an open bounded set in $\mathbb{R}^{2n}$ $(n \geq 2)$. If the boundary of Ω is of restricted contact type then $d(\Omega) = d(\overline{\Omega})$.*

Proof: Let $\Omega \in \mathcal{O}$ be an open bounded subset of $\mathbb{R}^{2n}$ whose boundary is of restricted contact type.

Let ξ be a conformal vector field on $\mathbb{R}^{2n}$ which intersects the boundary of Ω transversely. Assume that there is a neighborhood U of Ω and a number $\epsilon > 0$ such that the local flow φ_t of ξ is defined on $(-2\epsilon, 2\epsilon) \times U$. Then the image of Ω under the time-ϵ map of the flow φ_t of ξ is a neighborhood of $\overline{\Omega}$. Since $\varphi_t^* \omega_0 = e^t \omega_0$ for all t and wherever this is defined, by conformality the displacement energy of $\varphi_\epsilon \Omega$ is not bigger than $e^\epsilon d(\Omega)$. But this means that $d(\overline{\Omega}) \leq e^\epsilon d(\Omega)$, and since $\epsilon > 0$ was arbitrary we conclude that $d(\overline{\Omega}) = d(\Omega)$. **q.e.d.**

6 References

[B] A. Banyaga, *Sur la structure du groupe des diffeomorphismes qui préservent une forme symplectique*, Comm. Math. Helv. **53** (1978), 174-227.

[C] Y. Chekanov, *Lagrangian intersections, symplectic energy, and area of holomorphic curves*, Duke Math. J. 95 (1998), 213-226.

[EH] I. Ekeland, H. Hofer, *Symplectic topology and Hamiltonian dynamics*, Math. Zeit. 20 (1990), 355-378.

[E] Y. Eliashberg, *Filling by holomorphic discs and its applications*, London Math. Society Lecture Notes, Series 151 (1991), 45-67.

[G] M. Gromov, *Pseudo-holomorphic curves in symplectic manifolds*, Invent. Math. 82 (1985), 307-347.

[He] D. Hermann, *Non-equivalence of symplectic capacities for open sets with restricted contact type boundary*, Preprint, Orsay 1998.

[Ho] H. Hofer, *On the topological properties of symplectic maps*, Proc. R. Soc. Edinb., Sect. A, 115 (1990), 25-38.

[HZ] H. Hofer, E. Zehnder, *Symplectic invariants and Hamiltonian dynamics*, Birkhäuser, Basel 1994.

[LM] F. Lalonde, D. McDuff, *The geometry of symplectic energy*, Ann. Math. **141** (1995), 349-371.

[MD1] D. McDuff, *The local behaviour of holomorphic curves in almost complex 4-manifolds,* J. Diff. Geom. 34 (1991), 143-164.

[MD2] D. McDuff, *Symplectic manifolds with contact type boundaries*, Invent. Math 103 (1991), 651-671.

[MS] D. McDuff, D. Salamon, *Introduction to symplectic topology*, Oxford Univ. Press, Oxford (1995).

[MDT] D. McDuff, L. Traynor, *The 4-dimensional symplectic camel and related results*, in Symplectic geometry (ed. D. Salamon), London Math. Society Lecture Notes 192, 169-182, Cambridge University Press 1993.

[O] Y.-G. Oh, *Removal of boundary singularities of pseudo-holomophic curves with Lagrangian boundary conditions,* Comm. Pure Appl. Math. 45 (1992), 121-139.

[P] L. Polterovich, *The geometry of the group of symplectic diffeomorphisms*, Birkhäuser, Basel 2001.

[V] C. Viterbo, *Metric and isoperimetric problems in symplectic geometry*, J. AMS. 13 (2000), 411-431.