NEGATIVELY CURVED EINSTEIN METRICS ON GROMOV THURSTON MANIFOLDS

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Abstract. For every $n \geq 4$ we construct infinitely many mututally not homotopic closed manifolds of dimension n which admit a negatively curved Einstein metric but no locally symmetric metric.

1. Introduction

As a consequence of the solution to the geometrization conjecture by Perelmann, any closed manifold of dimension three which admits a negatively curved metric also admits a hyperbolic metric, and for surfaces, the corresponding statement is a classical consequence of the uniformization theorem. This statement is not true any more for closed negatively curved manifolds of dimension at least four.

Indeed, Gromov and Thurston [GT87] constructed for each dimension $n \geq 4$ and every $\epsilon > 0$ a closed manifold X of dimension n which admits a metric with curvature contained in the interval $[-1-\epsilon, -1+\epsilon]$ but which does *not* admit a hyperbolic metric. These manifolds are cyclic coverings of standard arithmetic hyperbolic manifolds, branched along a null-homologous totally geodesic submanifold of codimension two. In the sequel we call such branched coverings *Gromov–Thurston manifolds.*

The proof of non-existence of hyperbolic metrics on these manifolds is however indirect, that is, it it shown that among an infinite collection of candidate manifolds with pinched curvature, only finitely many admit hyperbolic metrics. Much later, Ontaneda [O20] gave a very general method for constructing closed Riemannian manifolds of any dimension $n \geq 4$ with arbitrarily pinched negative curvature. Some of the examples he found have in addition some non-zero Pontryagin numbers.

It is a natural question whether these manifolds admit distinguished metrics. This was partially answered affirmatively by Fine and Promoselli [FP20] who constructed negatively curved Einstein metrics on an infinite family of Gromov-Thurston manifolds in dimension four. These metrics do not have constant curvature and therefore by the work of Besson, Courtois and Gallot [BCG95], see also the survey [And10], these manifolds are not homotopy equivalent to hyperbolic manifolds.

The goal of this article is to extend this result to all dimensions. We show

Date: January 10, 2024.

AMS subject classification: 53C25, 53C21, 22E40.

Partially supported by the DFG Schwerpunktprogramm "Geometry at infinity".

Theorem 1. *For any $n \geq 4$ and any $\epsilon > 0$ there exist infinitely many pairwise non-homeomorphic smooth closed manifolds X of dimension n with the following properties.*

(1) X admits a Riemannian metric with sectional curvature contained in $[-1-\epsilon, -1+\epsilon]$.
(2) X admits an Einstein metric with negative sectional curvature.
(3) X is not *homeomorphic to any closed locally symmetric space.*

The examples in the Theorem are Gromov Thurston manifolds. For $n \geq 5$ they seem to be the first examples of negatively curved Einstein metrics on manifolds which are *not* locally symmetric.

Our construction builds on the ideas of Fine and Premoselli, however the examples we find in dimension four are different from the examples in [FP20].

The following question is motivated by the uniqueness of Einstein metrics on hyperbolic 4-manifolds [BCG95].

Question. Is it true that an Einstein metric on any closed hyperbolic manifold has constant curvature? Does every Gromov Thurston manifold admit an Einstein metric?

The answer to the first question is negative if the Einstein metrics are allowed to have conical singularities (see Remark 4.4).

1.1. **Sketch of proof.** In this subsection we outline the rough strategy for the proof of Theorem 1 and point out the difference to the proof of Fine–Premoselli.

We start by constructing a particular sequence of closed hyperbolic manifolds of which we take the branched cover. Namely, using subgroup separability in arithmetic hyperbolic lattices of simplest type, we construct for each $n \geq 4$ a sequence $(M_k)_{k\in\mathbb{N}}$ of closed hyperbolic manifolds that contain null-homologous closed totally geodesic codimension two submanifolds $\Sigma_k \subseteq M_k$ with at most two connected components and such that

$$\lim_{k\to\infty} \frac{\operatorname{diam}(\Sigma_k)}{R_k^\nu} = 0, \tag{1.1}$$

where R_k^ν is the normal injectivity radius of $\Sigma_k \subseteq M_k$ and, by a slight abuse of notation, $\operatorname{diam}(\Sigma_k)$ is the maximum of the diameters of the connected components of Σ_k.

As $\Sigma_k \subseteq M_k$ is homologous to zero, for any fixed integer $d \geq 2$ there exists a cyclic d-fold covering X_k of M_k, branched along Σ_k. Fine–Premoselli constructed an approximate Einstein metric $\bar{g}_k$ on X_k by gluing together a model Einstein metric and the hyperbolic metric, where the gluing takes place in the region of distance R_k away from Σ_k. We follow this strategy and choose as gluing parameter $R_k := \frac{1}{2}R_k^\nu$. From (1.1) we then deduce that

$$\int_{X_k} |\operatorname{Ric}(\bar{g}_k) + (n-1)\bar{g}_k|^2(x)\, d\operatorname{vol}_{\bar{g}_k}(x) \xrightarrow{k\to\infty} 0. \tag{1.2}$$

From this estimate, we obtain the Einstein metric from an application of the inverse function theorem, using a uniform a priori estimate for the so-called *Einstein operator*.

This leaves the question open whether the branched covering manifolds admit a locally symmetric metric. As our construction of the Einstein metrics uses a delicate volume estimate, to answer this question in the affirmative we can not rely on the indirect argument in [GT87]. Moreover the rigidity theorem in [BCG95] only holds in dimension four. Instead we show in Section 5 a result of independent interest whose precise version is Theorem 5.5. It states that given a pair (M, Σ) consisting of a closed hyperbolic n-manifold M $(n \geq 4)$ and a codimension two null-homologous embedded totally geodesic submanifold Σ of M of the form required for our construction, among the cyclic covers of M branched along Σ, at most one can be homeomorphic to a hyperbolic manifold.

1.2. **Structure of the article.** The article is organized as follows. In Section 2 we review the necessary background information. Namely, Section 2.1 introduces the Einstein operator. Section 2.2 explains how the De Giorgi–Nash–Moser estimates can be used to obtain C^0-estimates for the linearized Einstein operator. In Section 2.3 we recall the construction of the approximate Einstein metric on branched covers due to Fine–Premoselli. The algebraic results about arithmetic hyperbolic manifolds due to Bergeron and Bergeron–Haglund–Wise we use are contained in Section 2.4. These are then employed in Section 3 to construct the sequence of closed hyperbolic manifolds containing well-behaved codimension two submanifolds. In Section 4.1 we show that the linearized Einstein operator is invertible. The existence of negatively curved Einstein metrics is then proved in Section 4.2. Finally, in Section 5 we analyze Gromov Thurstion manifolds and, as an application, deduce that we can find such manifolds in any dimension to which our construction applies and which do not admit any locally symmetric metric.

Acknowledgement: U.H. thanks Alan Reid for pointing out the reference [BHW11], and Bena Tshishiku for pointing out the reference [CLW18].

2. Preliminaries

2.1. **The Einstein operator.** As mentioned in the introduction, we shall construct the Einstein metric by an application of the implicit function theorem for the so-called *Einstein operator* (see [Biq00, Section I.1.C], [And06, page 228] for more information). This operator is defined as follows.

Consider the operator $\Psi : g \to \mathrm{Ric}(g) + (n-1)g$ acting on smooth Riemannian metrics g on the manifold X, where Ric denotes the Ricci tensor. As the diffeomorphism group $\mathrm{Diff}(X)$ of the manifold X acts on metrics by pull-back and Ψ is equivariant for this action, the linearization of Ψ is *not* elliptic. To remedy this problem, for a given background metric $\bar{g}$ one defines the *Einstein operator* $\Phi_{\bar{g}}$ (in Bianchi gauge relative to $\bar{g}$) by

$$\Phi_{\bar{g}}(g) := \mathrm{Ric}(g) + (n-1)g + \frac{1}{2}\mathcal{L}_{(\beta_{\bar{g}}(g))^\sharp}(g), \tag{2.1}$$

where the musical isomorphism $\sharp$ is with respect to the metric g, and $\beta_{\bar{g}}$ is the *Bianchi operator* of $\bar{g}$ acting on symmetric $(0,2)$-tensors h by

$$\beta_{\bar{g}}(h) := \delta_{\bar{g}}(h) + \frac{1}{2}d\mathrm{tr}_{\bar{g}}(h) := -\sum_{i=1}^{n}(\nabla_{e_i}h)(\cdot, e_i) + \frac{1}{2}d\mathrm{tr}_{\bar{g}}(h). \tag{2.2}$$

The exact formula (2.1) is not important. What does matter is that, using the formula for the linearization of Ric ([Top06, Proposition 2.3.7]), one computes the linearization of $\Phi_{\bar{g}}$

at $\bar{g}$ to be

$$(D\Phi_{\bar{g}})_{\bar{g}}(h) = \frac{1}{2}\Delta_L h + (n-1)h. \tag{2.3}$$

Here Δ_L is the *Lichnerowicz Laplacian* acting on symmetric $(0,2)$-tensors h by

$$\Delta_L h = \nabla^*\nabla h + \mathring{\mathrm{Ric}}(h),$$

where $\nabla^*\nabla$ is the Connection Laplacian and $\mathring{\mathrm{Ric}}$ is the *Weitzenböck curvature operator* given by $\mathring{\mathrm{Ric}}(h)(x,y) = h(\mathrm{Ric}(x),y) + h(x,\mathrm{Ric}(y)) - 2\,\mathrm{tr}_g h(\cdot, R(\cdot,x)y)$ (see [Pet16, Section 9.3.2]).

Equation (2.3) shows that $(D\Phi_{\bar{g}})_{\bar{g}}$ is an elliptic operator. This opens up the possibility for an application of the implicit function theorem.

The main point is, as has been observed many times in the literature, that the Einstein operator can detect Einstein metrics. The following result can for example be found in [And06, Lemma 2.1].

Lemma 2.1 (Detecting Einstein metrics)**.** *Let $(X,\bar{g})$ be a complete Riemannian manifold, and let g be another metric on X so that*

$$\sup_{x\in X} |\beta_{\bar{g}}(g)|(x) < \infty \quad \textit{and} \quad \mathrm{Ric}(g) \le \lambda g \ \textit{ for some } \lambda < 0,$$

where $\beta_{\bar{g}}(\cdot)$ is the Bianchi operator of the background metric $\bar{g}$. Denote by $\Phi_{\bar{g}}$ the Einstein operator defined in (2.1). Then

$$\Phi_{\bar{g}}(g) = 0 \quad \textit{if and only if} \quad g \textit{ solves the system} \quad \begin{cases} \mathrm{Ric}(g) = -(n-1)g \\ \beta_{\bar{g}}(g) = 0 \end{cases}.$$

2.2. C^0**-.** To obtain C^0-estimates for the linearization of the Einstein operator, we use once again a standard tool, the De Giorgi–Nash–Moser estimates on manifolds in the following form.

Lemma 2.2 (C^0-estimate)**.** *For all $n \in \mathbb{N}$, $\alpha \in (0,1)$, $\Lambda \ge 0$, and $i_0 > 0$ there exist constants $\rho = \rho(n,\alpha,\Lambda,i_0) > 0$ and $C = C(n,\alpha,\Lambda,i_0) > 0$ with the following property. Let (X,g) be a Riemannian n-manifold satisfying*

$$|\sec(X,g)| \le \Lambda \quad \textit{and} \quad \mathrm{inj}(X,g) \ge i_0.$$

For $f \in C^0\big(\mathrm{Sym}^2(T^*X)\big)$ let $h \in C^2\big(\mathrm{Sym}^2(T^*X)\big)$ be a solution of

$$\frac{1}{2}\Delta_L h + (n-1)h = f.$$

Then it holds

$$|h|(x) \le C\Big(\|h\|_{L^2(B(x,\rho))} + \|f\|_{C^0(B(x,\rho))}\Big) \tag{2.4}$$

for all $x \in M$.

Here as customary, $\sec(X,g)$ denotes the sectional curvature of the metric g and $\mathrm{inj}(X,g)$ the injectivity radius, and $\mathrm{Sym}^2(T^*X)$ is the bundle of symmetric $(0,2)$ tensors on X.

In the proof we will make use of a result by Jost–Karcher [JK82, Satz 5.1] or Anderson [And90, Main Lemma 2.2] that, under the geometric assumptions on (X,g), around every point there exists harmonic charts of a priori size with good analytic control.

Proof. The desired C^0-bound will follow from the De Giorgi—Nash—Moser estimates. The problem is that De Giorgi–Nash–Moser estimates only hold for scalar equations, but not for systems. To remedy this, we show that $|h|$ satisfies an elliptic partial differential inequality.

We write $\Delta = \nabla^*\nabla = -\operatorname{tr}\nabla^2$ for the Connection Laplacian (on tensors and functions). Using $\frac{1}{2}\Delta(|h|^2) = \langle \Delta h, h\rangle - |\nabla h|^2$ and $f = \frac{1}{2}\Delta h + \frac{1}{2}\operatorname{Ric}(h) + (n-1)h$, we obtain

$$-\frac{1}{2}\Delta(|h|^2) = -2\langle f, h\rangle + \langle \operatorname{Ric}(h), h\rangle + 2(n-1)|h|^2 + |\nabla h|^2.$$

Note that $|\operatorname{Ric}(h)| \le C(n,\Lambda)|h|$ since $|\sec(M)| \le \Lambda$. Thus, together with the Cauchy-Schwarz inequality, the above equality implies

$$-\frac{1}{2}\Delta(|h|^2) + C(n,\Lambda)|h|^2 \ge -2|f||h| + |\nabla h|^2. \tag{2.5}$$

Suppose for the moment that $h \neq 0$ everywhere. Then $|h|$ is a nowhere vanishing C^2 function. Observe

$$|\nabla(|h|)| \le |\nabla h| \quad \text{and} \quad -\frac{1}{2}\Delta(|h|^2) = -|h|\Delta(|h|) + |\nabla(|h|)|^2.$$

Combining this with inequality (2.5) and dividing by $|h|$ shows

$$-\Delta(|h|) + C(n,\Lambda)|h| \ge -2|f|. \tag{2.6}$$

By [JK82, Satz 5.1] (also see [And90, Main Lemma 2.2], [And06, page 230] and [Biq00, Proposition I.3.2]) there exist $\rho = \rho(n,\alpha,\Lambda,i_0) > 0$ and $C = C(n,\alpha,\Lambda,i_0)$ with the following property. For all $x \in X$ there exists a *harmonic* chart $\varphi : B(x,2\rho) \subseteq X \to \mathbb{R}^n$ centered at x so that

$$e^{-Q}|v|_g \le |(D\varphi)(v)|_{\text{eucl.}} \le e^{Q}|v|_g \tag{2.7}$$

for all $v \in TB(x,2\rho)$, and for all $i,j = 1,\dots,n$

$$\|g^{\varphi}_{ij}\|_{C^{1,\alpha}} \le C, \tag{2.8}$$

where $Q > 0$ is a very small fixed constant, and $\|\cdot\|_{C^{2,\alpha}}$ is the usual Hölder norm of the coefficient functions in $\varphi(B(p,2\rho)) \subseteq \mathbb{R}^n$.

Fix $x \in X$ and choose a harmonic chart $\varphi : B(x,2\rho) \to \mathbb{R}^n$ satisfying (2.7) and (2.8). In the local harmonic coordinates given by φ, the differential inequality (2.6) reads

$$g^{ij}_{\varphi}\partial_i\partial_j(|h|\circ\varphi^{-1}) + C(n,\Lambda)(|h|\circ\varphi^{-1}) \ge -2(|f|\circ\varphi^{-1}) \quad \text{in } \varphi\big(B(x_0,2\rho)\big) \subseteq \mathbb{R}^n.$$

Since the matrices (g^{ij}_{φ}) are uniformly elliptic by (2.7), the desired estimate (2.4) follows from the classical De Giorgi–Nash–Moser estimates (see [GT01, Theorem 8.17]) provided that $h \neq 0$ everywhere.

It remains to show that the assumption $h \neq 0$ can be dropped. Note that (2.4) is stable under C^2-convergence, that is, if (2.4) holds for a sequence of h_i and if $h_i \to h$ in the C^2-topology, then (2.4) also holds for h. Therefore, it suffices to construct a sequence h_i converging to h in the C^2-topology so that $h_i \neq 0$ everywhere.

An arbitrary $h \in C^2\big(\operatorname{Sym}^2(T^*X)\big)$ can be approximated in the C^2-topology by symmetric $(0,2)$-tensors h_i $(i \ge 1)$ which are transverse to the zero-section of $\operatorname{Sym}^2(T^*X)$. For reasons of dimension, such a section is disjoint from the zero-section, in other words, the tensors

h_i vanish nowhere. Therefore, the estimate (2.4) holds for all $h \in C^2\big(\mathrm{Sym}^2(T^*X)\big)$ and $x \in X$. □

2.3. **The approximate Einstein metric of Fine–Premoselli.** In this subsection we review the construction of the approximate Einstein metric on the branched cover of a hyperbolic manifold due to Fine–Premoselli. We refer the reader to [FP20, Section 3] for more information.

Let M be a closed hyperbolic manifold of dimension $n \geq 4$, and let $\Sigma \subseteq M$ be a closed null-homologous totally geodesic codimension two embedded submanifold. Fix an integer $d \geq 2$ and denote by $p : X \to M$ the cyclic d-fold cover branched along Σ. We refer to [FP20] and to Section 3 for an explicit construction of such branched covers.

Define $r : X \to \mathbb{R}$ by $r(x) = d_M(p(x), \Sigma)$ and $u = \cosh(r)$. Set

$$U_{\max} \coloneqq \cosh(R^\nu),$$

where R^ν is the normal injectivity radius of $\Sigma \subseteq M$. The construction of the approximate Einstein metric also depends on a choice of gluing parameter $U_{\mathrm{glue}} < \frac{1}{2}U_{\max}$. We will later choose $U_{\mathrm{glue}} = (U_{\max})^{1/2}$, though this is irrelevant at the moment.

The following proposition summarizes all the necessary information about the approximate Einstein metric that will be used later.

Proposition 2.3 (The approximate Einstein metric)**.** *There exists a smooth Riemannian metric $\bar{g}$ on the branched covering X with the following properties:*

(i) For all $m \in \mathbb{N}$ there exists a constant $C = C(m, n, d)$ such that

$$\|\mathrm{Ric}(\bar{g}) + (n-1)\bar{g}\|_{C^m(X,\bar{g})} \leq C U_{\mathrm{glue}}^{-(n-1)};$$

(ii) The tensor $\mathrm{Ric}(\bar{g}) + (n-1)\bar{g}$ is supported in the region $\left\{\frac{1}{2}U_{\mathrm{glue}} < u < U_{\mathrm{glue}}\right\}$;

(iii) There exists a constant $c = c(n, d) > 0$ such that $\sec(X, \bar{g}) \leq -c < 0$;

(iv) For all $U < U_{\max}$ the volume of the region $\left\{\frac{1}{2}U < u < U\right\}$ is bounded from above by

$$\mathrm{vol}_n\left(\left\{\frac{1}{2}U < u < U\right\}, \bar{g}\right) \leq C U^{n-1}\,\mathrm{vol}_{n-2}(\Sigma, g_{\mathrm{hyp}})$$

for a constant $C = C(n, d)$.

Points (i)-(iii) are contained in [FP20, Proposition 3.1]. Property (iii) requires that the gluing parameter U_{glue} is larger than a constant depending on n and d. As in [FP20], in our construction this will always be the case. Point (iv) follows from the explicit construction, which we are now going to explain.

Consider $\mathbb{H}^n$ and fix a totally geodesic copy $S \subseteq \mathbb{H}^n$ of $\mathbb{H}^{n-2}$. Then in exponential normal coordinates centered at S, the hyperbolic metric of $\mathbb{H}^n$ is given by

$$g_{\mathbb{H}^n} = dr^2 + \sinh^2(r)d\theta^2 + \cosh^2(r)g_S,$$

where g_S is the hyperbolic metric of S. Using the change of variables $u = \cosh(r)$, this becomes

$$g_{\mathbb{H}^n} = \frac{du^2}{u^2-1} + (u^2-1)d\theta^2 + u^2 g_S,$$

which is defined for $(u,\theta) \in (1,\infty) \times S^1$.

Fine–Premoselli consider metrics of the form

$$g = \frac{du^2}{V(u)} + V(u)d\theta^2 + u^2 g_S, \tag{2.9}$$

where V is a positive smooth function. The following is [FP20, Proposition 3.2].

Lemma 2.4. *The metric g defined in (2.9) solves $\mathrm{Ric}(g) + (n-1)g = 0$ if and only if V is of the form*

$$V(u) = u^2 - 1 + \frac{a}{u^{n-3}} \tag{2.10}$$

for some $a \in \mathbb{R}$.

Let g_a be the metric (2.9) for the function $V = V_a$ given by (2.10). Let u_a denote the largest zero of the function V_a. If $u_a > 0$, g_a is a smooth Riemannian metric for $u \in (u_a, \infty)$, but in general it will have a cone singularity along S at $u = u_a$ with cone angle depending on a. The following summarizes [FP20, Lemma 3.3].

Lemma 2.5. *There are explicit constants $a_{\max} = a_{\max}(n) > 0$ and $v = v(n) > 0$ such that the following hold.*

(i) We have $u_a > 0$ if and only if $a \in (-\infty, a_{\max}]$ and the map $a \mapsto u_a$ is a decreasing homeomorphism $(-\infty, a_{\max}] \to [v, \infty)$;
(ii) For each integer $d \geq 1$ there exists a unique $a = a(d) \in (-\infty, a_{\max})$ such that the cone angle of g_a along S at $u = u_a$ is $\frac{2\pi}{d}$.
(iii) The sequence $(a(d))_{d\in\mathbb{N}}$ is strictly increasing with $a(1) = 0$ and $a(d) \to a_{\max}$ as $d \to \infty$.

Therefore, the metric $g_{a(d)}$ defines a global smooth metric on the cyclic d-fold branched cover of $\mathbb{H}^n$ along S. Of course, this is also true in X, the cyclic d-fold branched cover of M along Σ, at least in a tubular neighbourhood of Σ.

The approximate Einstein metric $\bar{g}$ in Proposition 2.3 is then obtained by interpolating between $g_{a(d)}$ (defined in a tubular neighbourhood of Σ) and g_{hyp} (defined on $X \smallsetminus \Sigma$). Namely, $\bar{g}$ is as in (2.9) for a function V of the form

$$V = u^2 - 1 + \frac{a}{u^{n-3}}\chi(u),$$

where χ smooth cutoff function with $\chi = 1$ in $\{u \leq \frac{1}{2}U_{\mathrm{glue}}\}$ and $\chi = 0$ in $\{u \geq U_{\mathrm{glue}}\}$. We refer the reader to [FP20, Section 3.2] for further details.

The volume estimate in Proposition 2.3(iv) follows easily because $\bar{g}$ is of the form (2.9).

2.4. **Algebraic retraction and subgroup separability.** In this subsection we state the results about arithmetic hyperbolic manifolds and arithmetic groups from Bergeron–Haglund–Wise [BHW11] and Bergeron [Ber00] that are needed for the construction of good totally geodesic submanifolds of codimension two (see Proposition 3.1).

Consider a $\mathbb{Q}$- algebraic group $\mathbf{G}$ such that the group of its real points is the product, with finite intersection, of a compact group by the isometry group $\mathrm{O}^+(n,1)$ of $\mathbb{H}^n$ for some $n \geq 4$. We require that $\mathbf{G}$ comes by restriction of scalars from an orthogonal group over a totally real number field and that **G** is anisotropic over $\mathbb{Q}$. The arithmetic group Γ is the subgroup of **G** which is defined over the ring of integers in the totally real number field; it acts cocompactly on $\mathbb{H}^n$. The compact hyperbolic orbifold $\Gamma\backslash\mathbb{H}^n$ is called *standard*, and Γ is a called a standard arithmetic lattice or an arithmetic lattice of simplest type. A *sufficiently deep* congruence subgroup Γ' of Γ is known to be torsion free and hence acts freely on $\mathbb{H}^n$. Following [BHW11] we call the quotient $\Gamma'\backslash\mathbb{H}^n$ a *standard arithmetic hyperbolic manifold.*

A Γ-*hyperplane* in $\mathbb{H}^n$ is a totally geodesic hyperplane $H \subset \mathbb{H}^n$ with the property that $\mathrm{Stab}_\Gamma(H)$ acts cocompactly on H. If Γ is an arithmetic group, then it is well-known that there exists a Γ-hyperplane in $\mathbb{H}^n$ if and only if Γ is standard. Similarly, a Γ-*subspace* is a totally geodesic subspace Σ of $\mathbb{H}^n$ of arbitrary codimension so that $\mathrm{Stab}_\Gamma(\Sigma)$ acts cocompactly on Σ.

Definition 2.6. A subgroup Λ of a group Γ is called *separable* if for any $\gamma \in \Gamma \smallsetminus \Lambda$, there exists a finite index subgroup $\Gamma' \leqslant \Gamma$ such that $\Lambda \leqslant \Gamma'$ and $\gamma \notin \Gamma'$.

The following is a special case of a result of Bergeron (see [Ber00, Lemme principal] or [BHW11, Corollary 1.12]).

Theorem 2.7 (Subgroup Separability)**.** *Let* $M = \Gamma\backslash\mathbb{H}^n$ *be a standard arithmetic hyperbolic manifold and* Σ *a* Γ*-subspace. Then* $\mathrm{Stab}_\Gamma(\Sigma)$ *is separable in* Γ*.*

Note that if Γ is a group as in Theorem 2.7, if Σ is a Γ-subspace and if Γ' is a finite index subgroup of Γ, then $\mathrm{Stab}_{\Gamma'}(\Sigma) \leqslant \Gamma'$ is separable. Indeed, if $\gamma \in \Gamma' \smallsetminus \mathrm{Stab}_{\Gamma'}(\Sigma)$, then $\gamma \in \Gamma \smallsetminus \mathrm{Stab}_\Gamma(\Sigma)$. Thus if Γ'' is a finite index subgroup of Γ which contains $\mathrm{Stab}_\Gamma(\Sigma)$ but not γ, then $\Gamma'' \cap \Gamma'$ is a finite index subgroup of Γ' which contains $\mathrm{Stab}_{\Gamma'}(\Sigma)$ but not γ.

The following summarizes the results of [BHW11] that will be needed in the sequel.

Theorem 2.8 (Bergeron–Haglund–Wise)**.** *Let* $M = \Gamma\backslash\mathbb{H}^n$ *be a standard arithmetic hyperbolic manifold and* $H \subseteq \mathbb{H}^n$ *a* Γ*-hyperplane. Then there exists a subgroup of finite index* $\Gamma' \leqslant \Gamma$ *that retracts onto* $\mathrm{Stab}_{\Gamma'}(H)$*, that is, there is a group homomorphism*

$$\mathrm{retr} : \Gamma' \to \mathrm{Stab}_{\Gamma'}(H) \quad \textit{such that} \quad \mathrm{retr}|_{\mathrm{Stab}_{\Gamma'}(H)} = \mathrm{id}_{\mathrm{Stab}_{\Gamma'}(H)}.$$

Moreover, $\mathrm{Stab}_{\Gamma'}(H)\backslash H$ *is a standard arithmetic hyperbolic manifold, and the natural map*

$$\mathrm{Stab}_{\Gamma'}(H)\backslash H \longrightarrow \Gamma'\backslash\mathbb{H}^n$$

is an embedding whose image agrees with the projection of $H \subseteq \mathbb{H}^n$ *to* $\Gamma'\backslash\mathbb{H}^n$*.*

The first half of Theorem 2.8 is [BHW11, Theorem 1.2]. So it remains to explain why the second part easily follows from the results of [BHW11].

Proof. By [BHW11, Theorem 1.2] there exists a torsion free congruence subgroup $\Gamma' \leqslant \Gamma$ of finite index. Note that, for any Γ-hyperplane $H \subseteq \mathbb{H}^n$, the stabilizer $\mathrm{Stab}_{\Gamma'}(H)$ of H in Γ' is a congruence subgroup of the arithmetic group $\mathrm{Stab}_\Gamma(H)$.

Appealing to [BHW11, Theorem 1.4] we may assume, after possibly passing to a further finite index congruence subgroup, that $\Lambda = \mathrm{Stab}_{\Gamma'}(H)$ is a virtual retract of Γ', that is, there exists a finite index subgroup $Q \leqslant \Gamma'$ containing Λ and a homomorphism $Q \to \Lambda$ that is the identity when restricted to Λ.

Now $\Lambda\backslash H$ is a compact standard arithmetic manifold, and the natural map $\Lambda\backslash H \to Q\backslash\mathbb{H}^n$ induced by the inclusion $H \subseteq \mathbb{H}^n$ is an immersion. By Theorem 2.7 and the following remark, the subgroup Λ is separable in Q.

If the immersion $\Lambda\backslash H \to Q\backslash\mathbb{H}^n$ is not an embedding, then the hyperplane H is not precisely invariant under Q, that is, there exist $\gamma \in Q \smallsetminus \Lambda$ such that

$$\gamma(H) \cap H \neq \varnothing \quad \text{but} \quad \gamma(H) \neq H. \tag{2.11}$$

Then $\gamma(H)\cap H$ is the intersection of two totally geodesic hyperplanes and hence it is a totally geodesic submanifold of H of codimension one. As the action of Λ on H is cocompact, there exists a compact fundamental domain $D \subseteq H$ for the action of Λ on H. For any $\gamma \in Q \smallsetminus \Lambda$ satisfying (2.11) there exists, by precomposing with a suitable element from Λ, an element $\gamma' \in Q \smallsetminus \Lambda$ such that

$$\gamma'(H) \cap H \neq \varnothing \quad \text{and} \quad \gamma'(D) \cap D \neq 0. \tag{2.12}$$

Since the action of Q on $\mathbb{H}^n$ is discrete and D is compact, there exist only finitely many elements in $Q \smallsetminus \Lambda$ satisfying (2.12). Keeping in mind that Λ is separated in Q, we can find a finite index subgroup $Q' \leqslant Q$ which contains Λ but does not contain any of the elements satisfying (2.12), and hence also no element satisfying (2.11). Then the manifold $\Lambda\backslash H$ is embedded in $Q'\backslash\mathbb{H}^n$. Furthermore, the restriction of the retraction $Q \to \Lambda$ to Q' defines a retraction $Q' \to \Lambda$. This completes the proof. □

3. Good totally geodesic submanifolds of codimension two

The goal of this section is to prove the following proposition.

Proposition 3.1 (Codimension two submanifolds). *For each $n \geq 4$ and any standard arithmetic hyperbolic manifold M, there is a sequence of finite covers $(M_k)_{k\in\mathbb{N}}$ of M containing closed embedded totally geodesic submanifolds $\Sigma_k \subset H_k \subset M_k$ with the following properties:*

(i) The manifolds Σ_k are all isometric, and they are of codimension 2.
(ii) Σ_k is null-homologous in the embedded connected hypersurface $H_k \subset M_k$;
(iii) Σ_k has at most two connected components;
(iv) We have

$$\lim_{k\to\infty} \frac{\operatorname{diam}(\Sigma_k)}{R_k^\nu} = 0,$$

where R_k^ν is the normal injectivity radius of $\Sigma_k \subseteq M_k$ and, by abuse of notation, $\operatorname{diam}(\Sigma_k)$ is the maximum of the diameters of the connected components of Σ_k.

Before we come to the proof of Proposition 3.1, we first show how it can be used to control the L^2-norm of the approximate Einstein metrics.

Namely, let $(M_k)_{k\in\mathbb{N}}$ and $\Sigma_k \subseteq M_k$ be as in Proposition 3.1. Fix an integer $d \geq 2$, and denote by X_k the cyclic d-fold cover of M_k branched along Σ_k. Set

$$U_{k;\max} := \cosh(R_k^\nu) \quad \text{and} \quad U_{k;\text{glue}} := \left(U_{k;\max}\right)^{\frac{1}{2}}. \tag{3.1}$$

Let $\bar{g}_k$ be the approximate Einstein metric on X_k given by Proposition 2.3. Then we easily obtain the following from Proposition 2.3 and Proposition 3.1.

Corollary 3.2 (Small L^2-norm). *For the L^2-norm of* $\operatorname{Ric}(\bar{g}_k) + (n-1)\bar{g}_k$ *we have*

$$\int_{X_k} |\operatorname{Ric}(\bar{g}_k) + (n-1)\bar{g}_k|^2(x)\, d\operatorname{vol}_{\bar{g}_k}(x) \xrightarrow{k\to\infty} 0.$$

This is the key estimate that will enable us to use a fairly simple perturbation argument for the construction of Einstein metrics.

Proof of Corollary 3.2. By Proposition 2.3(i),(ii) the tensor $\operatorname{Ric}(\bar{g}_k)+(n-1)\bar{g}_k$ is supported in the region $\left\{\frac{1}{2}U_{k;\text{glue}} < u < U_{k;\text{glue}}\right\}$ and it is uniformly bounded from above by

$$\|\operatorname{Ric}(\bar{g}_k) + (n-1)\bar{g}_k\|_{C^0(X_k,\bar{g}_k)} \le C U_{k;\text{glue}}^{-(n-1)} \tag{3.2}$$

It follows from Proposition 3.1(iii),(iv) and the definition (3.1) of $U_{k;\text{glue}}$ that for all $\varepsilon > 0$ and $k \ge k_0(\varepsilon)$ large enough we have

$$\operatorname{vol}_{n-2}(\Sigma_k, g_{\text{hyp}}) \le C\exp\big((n-3)\operatorname{diam}(\Sigma_k)\big) \le C\exp\left(\frac{1}{2}\varepsilon R_k^{\nu}\right) \le C U_{k;\text{glue}}^{\varepsilon}.$$

Together with the volume bound in Proposition 2.3(iv) this implies

$$\operatorname{vol}_n\left(\left\{\frac{1}{2}U_{k;\text{glue}} < u < U_{k;\text{glue}}\right\}, \bar{g}\right) \le C U_{k;\text{glue}}^{n-1}\operatorname{vol}_{n-2}(\Sigma, g_{\text{hyp}}) \le C U_{k;\text{glue}}^{n-1+\varepsilon}. \tag{3.3}$$

Combining (3.2) and (3.3) implies the desired estimate given that we choose $\varepsilon < n-1$. □

We now come to the proof of Proposition 3.1.

Proof of Proposition 3.1. The proof of Proposition 3.1 is divided into three steps.

Step 1. Let M be a standard arithmetic hyperbolic manifold. Because of Theorem 2.8 and [BHW11, Theorem 1.4], after passing to a finite cover, we may assume that $M = \Gamma\backslash\mathbb{H}^n$ where Γ is a torsion free cocompact lattice and that there exists a Γ-hyperplane $\tilde{H} \subseteq \mathbb{H}^n$ with the following properties:

(1) $\operatorname{Stab}_\Gamma(\tilde{H})\backslash\tilde{H}$ is a standard arithmetic hyperbolic manifold which is embedded in $\Gamma\backslash\mathbb{H}^n$;
(2) There is a retraction $\operatorname{retr}: \Gamma \to \operatorname{Stab}_\Gamma(\tilde{H})$;
(3) Any geometrically finite subgroup of Γ is a virtual retract.

Since $\operatorname{Stab}_\Gamma(\tilde{H})\backslash\tilde{H}$ is a standard arithmetic hyperbolic manifold, there exists a $\operatorname{Stab}_\Gamma(\tilde{H})$-hyperplane $\tilde{\Sigma} \subseteq \tilde{H}$ and a subgroup $Q \leqslant \operatorname{Stab}_\Gamma(\tilde{H})$ of finite index such that

$$\Sigma = \operatorname{Stab}_Q(\tilde{\Sigma})\backslash\tilde{\Sigma}$$

is a standard arithmetic hyperbolic manifold embedded in $Q\backslash H$.

The preimage $\Gamma' \coloneqq \operatorname{retr}^{-1}(Q)$ of $Q \leqslant \operatorname{Stab}_\Gamma(\tilde{H})$ under the retraction $\operatorname{retr}: \Gamma \to \operatorname{Stab}_\Gamma(\tilde{H})$ is a finite index subgroup of Γ. Note that $\operatorname{Stab}_{\Gamma'}(\tilde{H}) = \Gamma' \cap \operatorname{Stab}_\Gamma(\tilde{H}) = Q$, and so the retraction of Γ restricts to a retraction $\operatorname{retr}: \Gamma' \to \operatorname{Stab}_{\Gamma'}(\tilde{H})$. Moreover, any geometrically finite subgroup of Γ' is still a virtual retract of Γ'.

Therefore, we have obtained a finite index subgroup $\Gamma' \leqslant \Gamma$ such that $\operatorname{Stab}_{\Gamma'}(\tilde{\Sigma})\backslash\tilde{\Sigma}$, $\operatorname{Stab}_{\Gamma'}(\tilde{H})\backslash\tilde{H}$ and $\Gamma'\backslash\mathbb{H}^n$ are all standard arithmetic hyperbolic manifolds that are smoothly embedded in each other and so that (2),(3) above still hold for Γ'.

For ease of notation we will from now on replace Γ' by Γ in our notations and put $M = \Gamma\backslash\mathbb{H}^n$. Furthermore we put

$$\Gamma_\Sigma := \mathrm{Stab}_\Gamma(\tilde{\Sigma}), \quad \Sigma := \Gamma_\Sigma\backslash\tilde{\Sigma}, \quad \Gamma_H := \mathrm{Stab}_\Gamma(\tilde{H}), \quad H := \Gamma_H\backslash\tilde{H}.$$

This notation is slightly different than in Section 2.4, where we used H to denote a hyperplane in $\mathbb{H}^n$. We hope that this leads to no confusion.

Step 2. Let $R > 0$ be arbitrary. We will now show that one can pass to a finite-sheeted cover $M_R \to M$ that keeps Σ fixed but so that the normal injectivity radius radius of $\Sigma \subseteq M_R$ is at least R. To achieve this we will exploit the subgroup separability from Theorem 2.7.

Proof of Step 2. We first make the following observation: If the normal injectivity radius of $\Sigma \subseteq M$ is at less than R, then there exists $\gamma \in \Gamma$ such that

$$d_{\mathbb{H}^n}(\gamma \cdot \tilde{x}_0, \tilde{x}_0) \le 2\big(R + \mathrm{diam}(\Sigma)\big) \quad \text{and} \quad \gamma \notin \Gamma_\Sigma, \tag{3.4}$$

where $\tilde{x}_0 \in \tilde{\Sigma}$ is some basepoint. Indeed, if the normal injectivity radius of $\Sigma \subseteq M$ is less than R, then there exists a geodesic $\sigma : [0,1] \to M$ of length at most $2R$ such that

$$\sigma(i) \in \Sigma \quad \text{and} \quad \sigma'(i) \perp T_{\sigma(i)}\Sigma \quad \text{for } i = 0,1.$$

Let $x_0 \in \Sigma$ be the projection of the chosen basepoint $\tilde{x}_0 \in \tilde{\Sigma}$. Choose a distance minimizing geodesic τ_i in Σ from x_0 to $\sigma(i)$. Then the concatenation $\tau_0 \cdot \sigma \cdot \tau_1^{-1}$ is a loop based at x_0 of length at most $2(R + \mathrm{diam}(\Sigma))$. Clearly, this loop is *not* homotopic to a loop in Σ. This proves the existence of an element $\gamma \in \pi_1(M, x_0) \cong \Gamma$ satisfying (3.4).

Note that, for $R > 0$ fixed, there are only finitely many elements $\gamma \in \Gamma$ satisfying the conditions in (3.4). Therefore, by Theorem 2.7, there is a finite index subgroup Γ' of Γ containing Γ_Σ such that $\gamma \notin \Gamma'$ for all $\gamma \in \Gamma$ satisfying (3.4).

Since Γ' is a subgroup of Γ of finite index, we know that $\mathrm{Stab}_{\Gamma'}(\tilde{H})$ is a subgroup of finite index in $\mathrm{Stab}_\Gamma(\tilde{H})$ and hence it is a cocompact torsion free lattice. In particular, $\mathrm{Stab}_{\Gamma'}(\tilde{H})$ is a geometrically finite subgroup of Γ. Thus from property (3) in Step 1 we obtain a finite index subgroup $Q \leqslant \Gamma$ such that there is a retraction $\mathrm{retr}' : Q \to \mathrm{Stab}_{\Gamma'}(\tilde{H})$. As $\mathrm{Stab}_{\Gamma'}(\tilde{H}) = \mathrm{Stab}_{\Gamma'\cap Q}(\tilde{H})$, we obtain a retraction $\mathrm{retr}' : \Gamma' \cap Q \to \mathrm{Stab}_{\Gamma'\cap Q}(\tilde{H})$ by restriction. Moreover, $\Gamma_\Sigma \leqslant \mathrm{Stab}_{\Gamma'}(\tilde{H})$ implies $\Gamma_\Sigma \leqslant \Gamma' \cap Q$. The finite cover $M' \to M$ associated to $\Gamma' \cap Q$ has the desired properties. For ease of notation, we will in the following simply write Γ' for $\Gamma' \cap Q$. This completes the proof of Step 2. ■

The construction in Step 2 yields a finite cover M' of M and connected totally geodesic submanifolds $\Sigma \subseteq H' \subseteq M' = \Gamma'\backslash\mathbb{H}^n$ such that the normal injectivity radius of $\Sigma \subseteq M'$ is larger than an arbitrary multiple of $\mathrm{diam}(\Sigma)$. If Σ is null-homologous in H' (and hence in M') we are done. So we assume that Σ is *not* null-homologous in H'.

Step 3. Finally we show that there is a two-sheeted cover $\hat{M} \to M'$ such that the preimage $\hat{\Sigma} \subseteq \hat{M}$ of Σ is null-homologous in $\hat{M}$ and the preimage $\hat{H}$ of H' in $\hat{M}$ is connected.

Proof of Step 3. In accordance with Fine–Premoselli [FP20, Definition 2.2] we say that $\Sigma \subseteq H'$ *separates* H' if $H' \setminus \Sigma$ is disconnected. It is well-known that this can be detected algebraically. Namely, Σ determines a class $[\Sigma] \in H_{n-2}(H';\mathbb{Z}/2\mathbb{Z})$ and hence, by Poincaré duality, also a homomorphism

$$\rho : \pi_1(H') \to \mathbb{Z}/2\mathbb{Z}$$

that counts the number of intersections mod 2 of a generic loop with Σ. Then Σ separates H' if and only if ρ is trivial (see for example [FP20, proof of Lemma 2.3]). Therefore, if ρ is non-trivial, then the two-sheeted cover $\hat{H} \to H'$ associated to $\ker(\rho)$ is connected, and the preimage $\hat{\Sigma} \subseteq \hat{H}$ of Σ will separate $\hat{H}$. In particular, $\hat{\Sigma}$ is null-homologous in $\hat{H}$. Note that as $\pi_1(\Sigma)$ is contained in the kernel of ρ, the manifold $\hat{\Sigma}$ has precisely two connected components, each of which is isometric to Σ.

Precomposing with the retraction $\mathrm{retr}' : \Gamma' \to \mathrm{Stab}_{\Gamma'}(\tilde{H})$ we can extend ρ to a homomorphism defined on Γ'. Let $\hat{\Gamma} \leqslant \Gamma'$ be the kernel of this homomorphism. Then $\hat{M} = \hat{\Gamma}\backslash\mathbb{H}^n$ contains the two-sheeted cover $\hat{H} \to H'$ of H as an embedded totally geodesic submanifold. Since $\hat{\Sigma}$ is null-homologous in $\hat{H}$, it is also null-homologous in $\hat{M}$. Furthermore, as $\hat{\Gamma} \leqslant \Gamma'$, the normal injectivity radius of $\hat{\Sigma}$ is at least R. This completes the proof of Step 3. ■

Recall that if Σ is not homologous to zero then $\hat{\Sigma}$ has precisely two connected components, isometric to Σ, and the normal injectivity radius can *not* become smaller. Therefore, this completes the proof of Proposition 3.1. □

4. Construction of the Einstein metric

Let M_k be a sequence of closed hyperbolic n-manifolds with a totally geodesic codimension two submanifold Σ_k as in Proposition 3.1, X_k the cyclic d-fold covering of M_k branched along Σ_k, and $\bar{g}_k$ the approximate Einstein metric on X_k given by Proposition 2.3.

The goal of this section is to prove Theorem 1, that is, to show that X_k admits a negatively curved Einstein metric. By Lemma 2.1 it suffices to show that the Einstein operator $\Phi_k = \Phi_{\bar{g}_k}$ defined in (2.1) has a zero sufficiently close to the zero section. We will achieve this by an application of the Inverse Function Theorem.

Recall from (2.3) that the linearization of the Einstein operator at the background metric $\bar{g}_k$ is given by

$$\mathcal{L} = (D\Phi_k)_{\bar{g}_k} = \frac{1}{2}\Delta_L + (n-1)\,\mathrm{id}\,.$$

We will first show in Section 4.1 that $\mathcal{L}$ is an invertible linear operator between suitable Banach spaces. Section 4.2 then contains the proof of Theorem 1.

4.1. Invertibility of the linearized Einstein operator.

It is a classic result of Koiso [Koi78, Section 3] (also see [Bes08, Lemma 12.71]) that for a closed Einstein manifold with negative sectional curvature, the operator $\mathcal{L}$ has a uniform L^2-spectral gap (only depending on the negative upper curvature bound). By an adaptation of Koiso's argument, the same is also true for the approximate Einstein metrics $\bar{g}_k$.

Lemma 4.1 (L^2-spectral gap)**.** *There exists a constant $\lambda = \lambda(n,d) > 0$ such that for all sufficiently large k we have*

$$\lambda \int_{X_k} |h|^2 \, d\,\mathrm{vol}_{\bar{g}_k} \le \int_{X_k} \langle \mathcal{L}h, h\rangle d\,\mathrm{vol}_{\bar{g}_k}$$

for all $h \in C^2\big(\operatorname{Sym}^2(T^*X_k)\big)$.

For a detailed proof we refer the reader to [FP20, Proposition 4.3], which is a bit more general than what we need here.

Fix a Hölder parameter $\alpha \in (0,1)$. We equip $C^{0,\alpha}\big(\operatorname{Sym}^2(T^*X_k)\big)$ with the hybrid norm

$$\|f\|_0 := \max\Big\{\|f\|_{C^{0,\alpha}(X_k,\bar{g}_k)}, \|f\|_{L^2(X_k,\bar{g}_k)}\Big\}. \tag{4.1}$$

Similarly, we equip $C^{2,\alpha}\big(\operatorname{Sym}^2(T^*X_k)\big)$ with the hybrid norm

$$\|h\|_2 := \max\Big\{\|h\|_{C^{2,\alpha}(X_k,\bar{g}_k)}, \|h\|_{H^2(X_k,\bar{g}_k)}\Big\}, \tag{4.2}$$

where $\|\cdot\|_{H^2(X_k,\bar{g}_k)}$ is the Sobolev norm

$$\|h\|_{H^2(X_k,\bar{g}_k)} := \Big(\int_{X_k} |h|^2 + |\nabla h|^2 + |\Delta h|^2 \, d\operatorname{vol}_{\bar{g}_k}\Big)^{\frac{1}{2}}.$$

Here the Hölder norm of a tensor is defined by the Hölder norm of the coefficients of the tensor in a harmonic chart defined on balls of a priori size (for a detailed account we refer to [HJ22, Proof of Proposition 2.5]).

Using the C^0-estimate from Lemma 2.2 with the L^2-estimate from Lemma 4.1, it is now straightforward to show that $\mathcal{L}$ is invertible (with universal constants).

Proposition 4.2 ($\mathcal{L}$ is uniformly invertible)**.** *There exists a constant* $C = C(\alpha, n, d)$ *with the following property. For all* k *sufficiently large, the linearized Einstein operator*

$$\mathcal{L} : \Big(C^{2,\alpha}\big(\operatorname{Sym}^2(T^*X_k)\big), \|\cdot\|_2\Big) \longrightarrow \Big(C^{0,\alpha}\big(\operatorname{Sym}^2(T^*X_k)\big), \|\cdot\|_0\Big)$$

is invertible, and

$$\|\mathcal{L}\|_{\mathrm{op}}, \|\mathcal{L}^{-1}\|_{\mathrm{op}} \le C,$$

where $\|\cdot\|_0$ *resp.* $\|\cdot\|_2$ *is the norm defined in (4.1) resp. (4.2).*

Proof. It is clear that $\|\mathcal{L}\|_{\mathrm{op}}$ is bounded by a universal constant. It will suffice to prove the a priori estimate $\|h\|_2 \le C\|\mathcal{L}h\|_0$ for all $h \in C^{2,\alpha}\big(\operatorname{Sym}^2(T^*X_k)\big)$. Indeed, given the a priori estimate, standard arguments show that $\mathcal{L}$ is surjective; consequently $\mathcal{L}$ is invertible and $\|\mathcal{L}^{-1}\|_{\mathrm{op}} \le C$ due to the a priori estimate.

Clearly, $\|h\|_{L^2(X_k)} \le C\|\mathcal{L}h\|_{L^2(X_k)}$ because $\mathcal{L}$ has a uniform L^2-spectral gap (Lemma 4.1). Since $\mathcal{L} = \frac{1}{2}\Delta_L + (n-1)\,\mathrm{id}$, this L^2-estimate implies

$$\|h\|_{H^2(X_k)} \le C\|\mathcal{L}h\|_{L^2(X_k)}. \tag{4.3}$$

Moreover, the well-known Schauder estimates (see [HJ22, Proposition 2.5]) state

$$\|h\|_{C^{2,\alpha}(X_k)} \le C\Big(\|\mathcal{L}h\|_{C^{0,\alpha}(X_k)} + \|h\|_{C^0(X_k)}\Big). \tag{4.4}$$

The C^0-estimate (2.4) together with (4.3) yields

$$\|h\|_{C^0(X_k)} \le C\Big(\|h\|_{L^2(X_k)} + \|\mathcal{L}h\|_{C^0(X_k)}\Big) \le C\Big(\|\mathcal{L}h\|_{L^2(X_k)} + \|\mathcal{L}h\|_{C^0(X_k)}\Big). \tag{4.5}$$

Keeping in mind the definitions (4.1) and (4.2) of the norms $\|\cdot\|_0$ and $\|\cdot\|_2$, the desired a priori estimate $\|h\|_2 \le C\|\mathcal{L}h\|_0$ follows by combining (4.3), (4.4) and (4.5). □

4.2. **Proof of the Main Theorem.** The goal of this subsection is to present the proof of our main result.

Theorem 4.3 (Existence of Einstein metrics)**.** *For all sufficiently large k there exists a metric $\hat{g}_k$ on X_k such that*

$$\mathrm{Ric}(\hat{g}_k) + (n-1)\hat{g}_k = 0 \quad \textit{and} \quad \sec(X_k, \hat{g}_k) \le -c(n,d) < 0.$$

Moreover,

$$\|\hat{g}_k - \bar{g}_k\|_{C^{2,\alpha}(X_k,\bar{g}_k)} \xrightarrow{k\to\infty} 0.$$

Before we come to the proof we point out that, for k sufficiently large, the Einstein metric $\hat{g}_k$ can *not* be locally symmetric. Indeed, $\sec(\Sigma_k, \bar{g}_k) = -u_{a(d)}^{-2} < -1$ by (2.9) and Lemma 2.5. Thus $\sec(\Sigma_k, \hat{g}_k) < -1$ for all k sufficiently large, and so $\hat{g}_k$ can *not* be (real) hyperbolic. Moreover, by construction, the metric $\bar{g}_k$ is hyperbolic outside of a tubular neighborhood of Σ_k. Hence, outside of a tubular neighborhood of Σ_k, $\sec(X_k, \hat{g}_k)$ is very close to -1, and so $\hat{g}_k$ can *not* be complex- or quaternionic hyperbolic nor the Cayley plane.

In fact, in Section 5 we will show that for a slightly restricted choice of the hyperbolic manifolds M_k, at most one of the cyclic branched coverings X_k can *not* admit any locally symmetric metric.

Proof. We equip $C^{k,\alpha}\big(\mathrm{Sym}^2(T^*X_k)\big)$ with the norm $\|\cdot\|_k$ defined in (4.1) and (4.2) ($k = 0, 2$); $B(h, r)$ shall denote the balls with respect to these norms.

Any element in $B(\bar{g}_k, 1/2) \subseteq C^{2,\alpha}\big(\mathrm{Sym}^2(T^*X_k)\big)$ is a positive definite $(0,2)$-tensor, that is, a Riemannian metric on X_k. Let $\Phi_k = \Phi_{\bar{g}_k}$ be the Einstein operator defined in (2.1), which we consider as an operator

$$\Phi_k : B(\bar{g}_k, 1/2) \subseteq C^{2,\alpha}\big(\mathrm{Sym}^2(T^*X_k)\big) \to C^{0,\alpha}\big(\mathrm{Sym}^2(T^*X_k)\big).$$

Denote by $\mathcal{L} = (D\Phi_k)_{\bar{g}_k}$ the linearization of Φ_k at the background metric $\bar{g}_k$. By Proposition 4.2 there exists a universal constant $C_0 = C_0(\alpha, n, d)$ such that, for all k sufficiently large, $\mathcal{L}$ is invertible with $\|\mathcal{L}\|_{\mathrm{op}}, \|\mathcal{L}^{-1}\|_{\mathrm{op}} \le C_0$. Moreover, by possibly enlarging C_0, it is clear that the map $g \mapsto (D\Phi_k)_g$ is C_0-Lipschitz. Therefore, applying (a quantitative version of) the Inverse Function Theorem implies that there exist constants $\varepsilon_0 = \varepsilon_0(\alpha, n, d) > 0$ and $C = C(\alpha, n, d)$ with the following property: For each $f \in C^{0,\alpha}\big(\mathrm{Sym}^2(T^*X_k)\big)$ with $\|f - \Phi_k(\bar{g}_k)\|_0 \le \varepsilon_0$ there exists a metric $g_f \in C^{2,\alpha}\big(\mathrm{Sym}^2(T^*X_k)\big)$ such that

$$\Phi_k(g_f) = f \quad \text{and} \quad \|g_f - \bar{g}_k\|_2 \le C\|f - \Phi_k(\bar{g}_k)\|_0.$$

Note that $\Phi_k(\bar{g}_k) = \mathrm{Ric}(\bar{g}_k) + (n-1)\bar{g}_k$. Hence it follows from Proposition 2.3(i) and Corollary 3.2 that $\|\Phi_k(\bar{g}_k)\|_0 \to 0$ as $k \to \infty$. In particular, for all k sufficiently large, $f = 0$ satisfies $\|f - \Phi_k(\bar{g}_k)\| \le \varepsilon_0$. Therefore, there exists a metric $\hat{g}_k$ on X_k such that

$$\Phi_k(\hat{g}_k) = 0 \quad \text{and} \quad \|\hat{g}_k - \bar{g}_k\|_2 \xrightarrow{k\to\infty} 0.$$

In particular, as $\sec(X_k, \bar{g}_k) \le -c(n,d) < 0$ by Proposition 2.3(iii), also $\sec(X_k, \hat{g}_k) < 0$ for all k sufficiently large. Therefore, $\Phi_k(\hat{g}_k) = 0$ implies $\mathrm{Ric}(\hat{g}_k) + (n-1)\hat{g}_k = 0$ due to Lemma 2.1. This completes the proof. □

For the formulation of the next remark, note that there is a natural action of the cyclic group C_d of order d on the d-fold branched cover X_k.

Remark 4.4. For all k sufficiently large, the Einstein metric $\hat{g}_k$ on X_k given by Theorem 4.3 is C_d-invariant. In particular, for all k sufficiently large (depending on d), the hyperbolic manifolds M_k admit negatively curved Einstein metrics with a conical singularity and cone angle $\frac{2\pi}{d}$ along the codimension two submanifold $\Sigma_k \subseteq M_k$.

Proof. In the proof of Theorem 4.3, the Einstein metric $\hat{g}_k$ was the zero of the Einstein operator Φ_k obtained from an application of the Inverse Function Theorem. Since the Inverse Function Theorem can be proved using the Banach Fixed Point Theorem, $\hat{g}_k$ is of the form $\bar{g}_k + \hat{h}_k$, where $\hat{h}_k$ is a fixed point of the operator

$$\Psi_k : C^{2,\alpha}\big(\mathrm{Sym}^2(T^*X_k)\big) \to C^{2,\alpha}\big(\mathrm{Sym}^2(T^*X_k)\big),\ h \mapsto h - \mathcal{L}^{-1}\big(\Phi_k(\bar{g}_k + h)\big).$$

Using the definition (2.1) of the Einstein operator, one can easily check that if a Riemannian metric g on X_k is φ-invariant for some $\varphi \in \mathrm{Isom}(X_k, \bar{g}_k)$, then also $\Phi_k(g)$ is φ-invariant. As the fixed point $\hat{h}_k$ is given by the limit $\lim_{i\to\infty} \Psi_k^i(0)$, this shows that $\hat{h}_k$, and hence $\hat{g}_k$, is $\mathrm{Isom}(X_k, \bar{g}_k)$-invariant. However, it is apparent from the construction of $\bar{g}_k$ explained in Section 2.3 that $\bar{g}_k$ is C_d-invariant. Therefore, also $\hat{g}_k$ is C_d-invariant. □

5. Einstein manifolds not homeomorphic to locally symmetric spaces

By Theorem 4.3 there exist negatively curved Einstein metrics on some branched covers X of certain hyperbolic manifolds M. The construction is valid for all covering degrees smaller than a number depending on M. As M varies, this maximal covering degree can be arbitrarily large. The goal of this section is to show that for any dimension $n \ge 4$, we find infinitely many such branched coverings which are not homeomorphic to a locally symmetric manifold.

We start with the following basic observation.

Proposition 5.1. *Let M be a closed hyperbolic n-manifold and $\Sigma \subseteq M$ a closed null-homologous totally geodesic submanifold of codimension two. Then the cyclic d-fold covering X of M branched along Σ is* not *homeomorphic to any locally symmetric manifold, except possibly hyperbolic manifolds.*

Proof. Arguing by contradiction, we assume that X is homeomorphic to a locally symmetric manifold N that is *not* hyperbolic. We first observe that this locally symmetric manifold has to be of real rank one. Namely, since X is aspherical, a locally symmetric metric on a manifold homeomorphic to X is of non-positive curvature. By a theorem of Wolf (Theorem 4.2 of [W62]), a cocompact lattice in a semisimple Lie group of real rank r contains a subgroup isomorphic to $\mathbb{Z}^r$. However, as X carries a negatively curved metric [GT87], by

Preissmann's theorem [Pr42, Théorème 10] (also see [dC92, Theorem 3.2 in Chapter 12]) any abelian subgroup of $\pi_1(X)$ is infinite cyclic.

It remains to show that X is *not* homeomorphic to any complex-, quaternionic- or Cayley-hyperbolic manifold. If X is homotopy equivalent to a complex hyperbolic manifold N, then there is a degree $d \geq 2$ map $\Pi : N \to M$. Since M has constant negative curvature, the map Π is homotopic to a harmonic map. But by a theorem of Sampson [Sa86], any harmonic map from a compact Kähler manifold into a real hyperbolic manifold is trivial in homology of dimension larger than two, which contradicts the fact that the degree of the map Π is positive (unless $n = 2$ and N is also real hyperbolic).

By a celebrated result of Novikov [Nov65, Theorem 1], the rational Pontryagin classes are a homeomorphism invariant. In particular, the Pontryagin numbers are a homeomorphism invariant. By a result of Lafont–Roy [LR07, Theorem B] all Pontryagin numbers of X vanish, while it is a well-known consequence of the Hirzebruch proportionality principle [Hir56, Satz 2 and Equation (2)] that closed quaternionic- or Cayley-hyperbolic manifolds have some non-zero Pontryagin numbers (see [LR07, Corollary 3]). Therefore, X can also *not* be homeomorphic to a quaternionic- or Cayley-hyperbolic manifold. ☐

As a consequence of Proposition 5.1 and the work of Besson, Courtois and Gallot [BCG95] we obtain.

Corollary 5.2. *If* $\dim(X) = 4$ *and* X *admits an Einstein metric as constructed in Theorem 4.3 then* X *is not homeomorphic to a locally symmetric manifold.*

Proof. Using the notations from Proposition 5.1, if X is homeomorphic to a locally symmetric manifold M then M is real hyperbolic. As X admits an Einstein metric g, it is a consequence of [BCG95, Théorème 9.6] (also see [And10, Corollary 4.6]) that X is diffeomorphic to M and g is of constant curvature. However, the curvature of the Einstein metric g on X is not constant, from which the corollary follows. ☐

In the remainder of this section, which is independent of the rest of the article, we show that for any $n \geq 4$ there are infinitely many arithmetic hyperbolic manifolds M of dimension n which admit branched covers to which our construction of Einstein metrics applies, but such that at most one of these branched covers can be homeomorphic to a hyperbolic manifold. Together with Theorem 5.5, this completes the proof of Theorem 1.

We begin with collecting some more specific information on the standard arithmetic hyperbolic manifolds used in our construction. Let k be a totally real number field of degree d over $\mathbb{Q}$ equipped with a fixed embedding into $\mathbb{R}$ which we refer to as the identity embedding. Let V be an $(n+1)$-dimensional vector space over k equipped with a quadratic form q (with associated symmetric matrix Q) defined over k which has signature $(n,1)$ at the identity embedding, and signature $(n+1,0)$ at the remaining embeddings. Such a quadratic form is called *admissible*. We require in the sequel that q is *anisotropic* over $\mathbb{Q}$. This means that $q = 0$ has no rational solution.

Let R_k be the ring of integers of the number field k and let $\mathrm{O}(q, R_k)$ be the group of automorphisms of the quadratic form q which are defined over R_k, that is,

$$\mathrm{O}(q, R_k) := \{X \in \mathrm{GL}_{n+1}(R_k) \mid X^t Q X = Q\}.$$

A subgroup Γ of the isometry group $\mathrm{O}^+(n,1)$ of the hyperbolic space $\mathbb{H}^n$ is called an *arithmetic group of simplest type* if Γ is commensurable with a conjugate of an arithmetic group $\mathrm{O}(q,R_k)$. As the quadratic form q is admissible and anisotropic over $\mathbb{Q}$, an arithmetic group of simplest type Γ is a cocompact lattice in $\mathrm{O}^+(n,1)$. Thus $\Gamma\backslash\mathbb{H}^n$ is a compact hyperbolic orbifold with singularities corresponding to the fixed points of Γ. We refer to [Em23, Example 6.30] for more information.

Example 5.3. The quadratic form

$$q(x) = -\sqrt{2}x_0^2 + x_1^2 + \cdots + x_n^2$$

on $\mathbb{R}^{n+1}$ is defined over the quadratic extension $\mathbb{Q}(\sqrt{2})$ of $\mathbb{Q}$. Evaluation on the non-identity embedding $\mathbb{Q}(\sqrt{2}) \to \mathbb{R}$ given by $\sqrt{2} \to -\sqrt{2}$ shows that q is admissible, moreover it is anisotropic over $\mathbb{Q}$. The upper paraboloid $\{x \in \mathbb{R}^{n+1} \,|\, q(x) = -1 \text{ and } x_0 > 0\}$ is a model for $\mathbb{H}^n$.

The ring of integers of the number field $\mathbb{Q}(\sqrt{2})$ is the ring $\mathbb{Z}[\sqrt{2}]$ and hence

$$\mathrm{O}(q,\mathbb{Z}[\sqrt{2}]) = \mathrm{O}(q) \cap \mathrm{GL}_{n+1}\left(\mathbb{Z}[\sqrt{2}]\right)$$

is a cocompact lattice in $\mathrm{O}^+(n,1)$.

Standard theory of quadratic forms (see [La73]) provides an equivalence over k of the quadratic form q to an admissible diagonal quadratic form. Thus we may assume without loss of generality that

$$q(x) = -a_0x_0^2 + a_1x_1^2 + \cdots + a_nx_n^2$$

with $a_i \in k, a_i > 0$. Put $\Gamma = \mathrm{O}(q,R_k)$.

Let $\iota \in \mathrm{Isom}(\mathbb{H}^n)$ be the geometric involution that acts via reflection in the x_1-variable, that is,

$$\iota(x_0,x_1,x_2,\ldots,x_n) = (x_0,-x_1,x_2,\ldots,x_n).$$

Then $\tilde{H} := \mathrm{Fix}(\iota) = \{x \in \mathbb{H}^n \,|\, x_1 = 0\}$ is a hyperplane. The quadratic form

$$q_0(x) = -a_0x_0^2 + a_2x_2^2 + \cdots + a_nx_n^2$$

on the linear subspace $V_0 = \{x_1 = 0\}$ of V defined over k is admissible and anisotropic over $\mathbb{Q}$. Under the obvious identifications we then have $\mathrm{Stab}_\Gamma(\tilde{H}) = \mathrm{O}^+(n-1,1) \cap \mathrm{O}(q,R_k)$, so that, by the same reason as above, the quotient $\mathrm{Stab}_\Gamma(\tilde{H})\backslash\tilde{H}$ is compact. This means that, in the terminology of Section 2.4, $\tilde{H}$ is a Γ-hyperplane. Furthermore, we have $\iota\Gamma\iota = \Gamma$.

Consider the sequence $\Gamma_m \triangleleft \Gamma$ of congruence subgroups defined as the kernel of the natural homomorphism

$$\Gamma \to \mathrm{GL}_{n+1}\left(R_k\right) \to \mathrm{GL}_{n+1}\left(R_k/\mathcal{O}_m\right)$$

where $\mathcal{O}_m$ is a sequence of mutually distinct prime ideals in R_k. For sufficiently large m the group Γ_m is sufficiently deep and hence torsion free. The quotient manifold $N_m = \Gamma_m\backslash\mathbb{H}^n$ is a standard arithmetic hyperbolic manifold. Moreover, by construction, N_m is oriented.

As kernels are normal subgroups, one easily checks $\iota\Gamma_m\iota^{-1} = \Gamma_m$. It follows that ι descends to an isometric involution of $N_m = \Gamma_m\backslash\mathbb{H}^n$, again denoted by ι. The fixed point set of this involution is a (possibly disconnected) totally geodesic submanifold of codimension one.

Fix a component H of this submanifold. We may assume that H is the projection to N_m of the Γ_m-hyperplane $\tilde{H}$. Following the construction in Section 3, we know that $\mathrm{Stab}_{\Gamma_m}(\tilde{H})$ is a virtual retract of Γ_m. Let $\Gamma'_m \leqslant \Gamma_m$ be a finite index subgroup containing the fundamental group $\mathrm{Stab}_{\Gamma_m}(\tilde{H})$ of H which retracts onto $\mathrm{Stab}_{\Gamma_m}(\tilde{H})$.

Lemma 5.4. *There exists a ι-invariant finite index subgroup $\Gamma^0_m \leqslant \Gamma'_m$ which contains $\mathrm{Stab}_{\Gamma_m}(\tilde{H})$.*

In particular, by restricting the retraction $\mathrm{ret}: \Gamma'_m \to \mathrm{Stab}_{\Gamma_m}(\tilde{H})$ to Γ^0_m, we see that Γ^0_m also retracts onto $\mathrm{Stab}_{\Gamma^0_m}(\tilde{H}) = \mathrm{Stab}_{\Gamma_m}(\tilde{H})$.

Proof. As $\Gamma'_m \leqslant \Gamma_m$ has finite index and Γ_m is ι-invariant, $\Gamma^0_m := \Gamma'_m \cap \iota\Gamma'_m\iota^{-1}$ is a ι-invariant finite index subgroup of Γ_m. Moreover, by inspecting the action of the differential, one can check that $\iota^{-1}\mathrm{Stab}_{\Gamma_m}(\tilde{H})\iota \subseteq \mathrm{Stab}_{\Gamma_m}(\tilde{H}) \subseteq \Gamma'_m$. This then implies $\mathrm{Stab}_{\Gamma_m}(\tilde{H}) \leqslant \Gamma^0_m$. ☐

The group Γ^0_m is invariant under conjugation by ι, and this action of ι on Γ^0_m is nontrivial. Thus ι acts as an isometric involution on $M_m = \Gamma^0_m \backslash \mathbb{H}^n$. Its fixed point set is a disjoint union of totally geodesic embedded hyperplanes containing the quotient H of $\tilde{H}$ under the action of $\mathrm{Stab}_{\Gamma_m}(\tilde{H})$.

By the construction in Section 4, by perhaps passing to a two-sheeted covering $\hat{M}_m$ of M_m, we may assume that the preimage $\hat{H}$ of H in $\hat{M}_m$ contains a totally geodesic embedded hyperplane $\hat{\Sigma}_m$ which is homologous to zero and consists of at most two connected components. The involution ι may not lift to $\hat{M}_m$, but it lifts to the covering $\tilde{M}_m$ of $\hat{M}_m$ of degree at most two with fundamental group $\pi_1(\hat{M}_m) \cap \iota\pi_1(\hat{M}_m)\iota^{-1}$. Note that as the hyperplane H in M_m is contained in the fixed point set of the involution ι, if $\pi_1(\hat{M}_m) < \pi_1(M_m)$ is not invariant under conjugation by ι, then the preimage of $\hat{H}$ in $\tilde{M}_m$ consists of two components of $\hat{H}$, each of which contains a totally geodesic null homologous hyperplane as required in the construction in the beginning of this article.

Using this construction, Theorem 1 is an immediate consequence of Theorem 4.3, Proposition 5.1 and the following main result of this section.

Theorem 5.5. *Let M be an oriented closed hyperbolic manifold of dimension $n \geq 4$ and let $H \subset M$ to a totally geodesic embedded hyperplane. Assume that H is contained in the fixed point set of an orientation reversing isometric involution ι and that H contains a (possibly disconnected) embedded totally geodesic hyperplane Σ which is homologous to zero in H. Then for at most one $d \geq 2$, the cyclic d-fold covering of M branched along Σ can be homeomorphic to a hyperbolic manifold.*

Remark 5.6. We believe that for $n \geq 4$, no nontrivial branched cover of a closed hyperbolic n-manifold admits a hyperbolic metric. This question is closely related to Mostow rigidity for hyperbolic cone manifolds in dimension at least four which seems to be a fairly uncharted territory (see however [KS12] for a related result). Note that due to the solution of the geometrization conjecture, in general branched covers of closed hyperbolic 3-manifolds do admit hyperbolic metrics.

The proof of Theorem 5.5 is inspired by [GT87, Remark 3.6], though it does not directly follow from it.

From now on we denote by M a closed hyperbolic manifold as in Theorem 5.5, containing the embedded hypersurface H. Let N be the compact hyperbolic manifold with totally geodesic boundary ∂N which is obtained by cutting M open along H, that is, N is the metric completion of $M \setminus H$. If H is non-separating, then N is connected, otherwise N has two connected components. The boundary ∂N of N is totally geodesic and consists of two copies of H containing one copy of Σ each. Thus N is isometric to the convex core of an infinite volume convex cocompact hyperbolic manifold $\hat{N}$. The main tool towards Theorem 5.5 is the following result.

Proposition 5.7. *If the cyclic d-fold branched cover X of M admits a hyperbolic metric, then there exists a hyperbolic cone manifold $N^{\#}$ satisfying the following properties:*

(i) *$N^{\#}$ is homotopy equivalent to N.*
(ii) *The boundary $\partial N^{\#}$ of $N^{\#}$ is path isometric to ∂N.*
(iii) *The singular locus of $N^{\#}$ consists of the two copies of Σ in $\partial N^{\#}$. The cone angle at each of these copies equals $2\pi/d$.*

Remark 5.8. As the dimension of N is at least four, by a result of Kerckhoff and Storm [KS12, Theorem 2.5] there is no continuous deformation of the convex cocompact hyperbolic manifold $\hat{N}$ within the space of convex cocompact hyperbolic manifolds. Using stability of convex cocompact representations, this is equivalent to stating the following.

Let Γ be the fundamental group of N, viewed as the fundamental group of the unique complete hyperbolic manifold $\hat{N}$ which contains N as its convex core. Let $\rho : \Gamma \to O^+(n,1)$ be a discrete faithful homomorphism defining $\hat{N}$ (which is unique up to conjugation). Then ρ does not admit any nontrivial deformations (where nontrivial means that it is not a deformation by conjugation).

This does not reduce Theorem 5.5 to Proposition 5.7 as the result of Kerckhoff and Storm does not rule out that there are isolated faithful convex cocompact representations of Γ which are not conjugate to ρ. We shall address this question in forthcoming work.

The remainder of this article is devoted to the proof of Theorem 5.5. It is divided into three subsections. We always consider a degree d branched covering X of M, and we assume that X admits a hyperbolic metric.

By Mostow rigidity, any homotopy self-equivalence σ of X is homotopic to a unique isometry $\sigma^{\#}$ of X. Furthermore, by uniqueness, the map

$$\mathrm{Homeo}(X) \to \mathrm{Isom}(X),\ \sigma \mapsto \sigma^{\#}$$

which associates to a homeomorphism the unique isometry homotopic to it is a group homomorphism.

In the first subsection we construct an involution j of X. The fixed point set of j is an explicit hypersurface in X which is homeomorphic to a manifold constructed from the hypersurface H. We show that the fixed point set of the isometric involution $j^{\#}$ of X homotopic to j is a totally geodesic hypersurface diffeomorphic to the fixed point set of j.

This is used in the second subsection to show that there is a fundamental domain for the action of the isometric realization of the cyclic deck group of X which can be cut open along a totally geodesic hyperplane with boundary to yield the cone manifold whose existence is predicted in Proposition 5.7. In the final subsection the proof of Theorem 5.5 is completed.

5.1. **Fixed point sets of isometries.** Let M be as in the statement of Theorem 5.5, containing the hypersurface $H \supset \Sigma$. By assumption, Σ bounds a submanifold $H_0 \subset H$. Put $H_1 = H \setminus H_0$.

Fix an integer $d \geq 2$ and let X be the d-fold covering of M branched along the totally geodesic submanifold $\Sigma \subset H \subset M$. This covering can be realized as follows. Let M_{cut} be the obtained from M by cutting along H_0, that is, M_{cut} is the metric completion of $M - H_0$. Thus M_{cut} is a compact (topological) manifold whose boundary consists of two copies H_0^- and H_0^+ of H_0 intersecting in Σ. The manifold X is obtained by gluing d copies $M_{\text{cut}}^1, \ldots, M_{\text{cut}}^d$ of M_{cut} along the boundary, so that the copy of H_0^+ in M_{cut}^i is glued to the copy of H_0^- in M_{cut}^{i+1} (where the superscripts i are taken $\bmod d$).

Let $\iota = \iota_M : M \to M$ be the isometric involution whose fixed point set contains $H \subseteq \text{Fix}(\iota)$. Since locally near H, ι_M acts as a reflection in H, it exchanges the two components of $U \setminus H$ where U is a tubular neighborhood of H in M. Thus ι_M acts as an involution on M_{cut} which exchanges H_0^+ and H_0^- and fixes $W = \text{Fix}(\iota_M) \setminus H_0 \supseteq H_1$.

As a consequence, ι_M induces an involution ι of X with the property that $\iota(M_{\text{cut}}^i) = M_{\text{cut}}^{d+2-i}$ and so that the restrictions $\iota : M_{\text{cut}}^i \to M_{\text{cut}}^{d+2-i}$ are identified with $\iota : M_{\text{cut}} \to M_{\text{cut}}$ (superscripts are again taken $\bmod d$). If the degree d of the covering is odd, then the fixed point set of this involution is the union of the copy W^1 of W in M_{cut}^1 and the set $M_{\text{cut}}^{(d-1)/2} \cap M_{\text{cut}}^{(d+1)/2}$, which is naturally homeomorphic to H_0 and is glued to $H_1^1 \subset W^1$ along Σ. If d is even then the fixed point set of ι consists of the two copies of W in M_{cut}^1 and $M_{\text{cut}}^{d/2+1}$ glued along Σ.

Let ζ be a generator of the cyclic deck group of $X \to M$. It cyclically permutes the copies $M_{\text{cut}}^1, \ldots, M_{\text{cut}}^d$ of M_{cut} in X. If the degree d is even define $j = \zeta \circ \iota$ (read from right to left), and for odd degree define $j = \iota$.

Fact 5.9.
- If d is even, then the fixed point set of j in X is the union $H_0^{d,1} \cup H_0^{d,1+d/2}$ of the copies of $H_0 \subset H$ in M_{cut}^1 and $M_{\text{cut}}^{1+d/2}$, and the copies $H_0^{d,1}$ and $H_0^{d,1+d/2}$ of H_0 are glued along Σ (see Figure 1).
- If d is odd then the fixed point set of j in X is the union $W^1 \cup H_0$ of the copy of $W \supseteq H_1$ in M_{cut}^1 and $M_{\text{cut}}^{(d-1)/2} \cap M_{\text{cut}}^{(d+1)/2}$. In particular, this fixed point set is homeomorphic to $\text{Fix}(\iota_M)$.

The fixed point set of each of the involutions $\zeta^i \circ j \circ \zeta^{-i}$ $(i = 0, \ldots, d-1)$ is the embedded submanifold $\zeta^i(\text{Fix}(j))$ of X. Their union cuts X up into the d copies of M_{cut} if d is even, and into $2d$ copies of N if d is odd. We call any diffeomorphism of X contained in the finite group of diffeomorphisms of X generated by j and ζ an *admissible* diffeomorphism of X.

The following result seems be known to the experts, and it was claimed in [GT87] for the generator ζ of the deck group of $X \to M$ (except for Remark 3.4, this is not used in

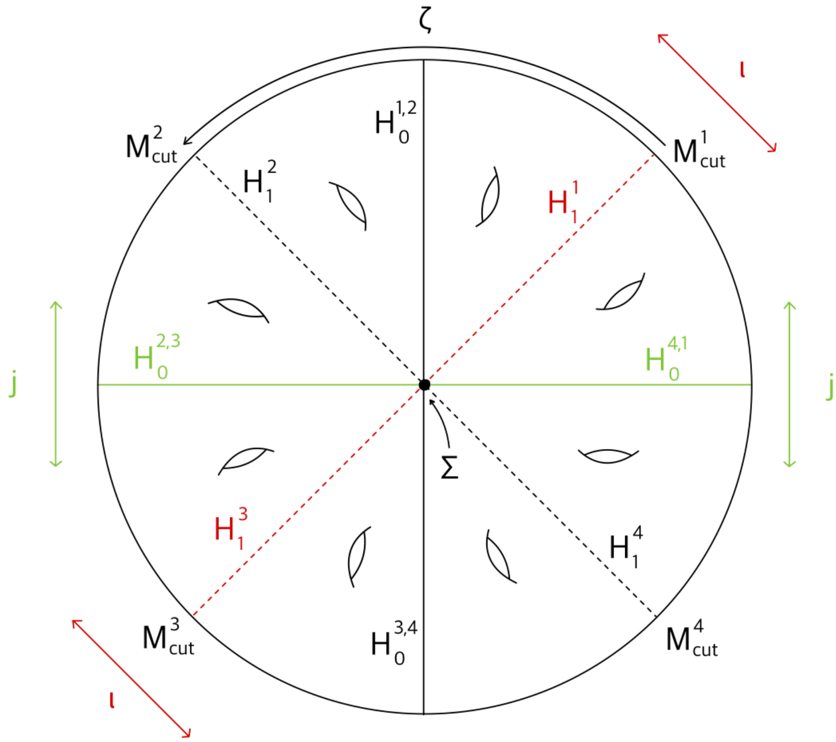

FIGURE 1. The cyclic 4-fold branched cover. The involutions ι and j act via reflection along the colored submanifolds and ζ via rotation around Σ.

[GT87]). In view of the fact that in the presence of fixed point sets of positive dimension, a finite group of diffeomorphisms of a hyperbolic manifold of dimension $n \geq 3$ need not be conjugate to its isometric realization (see the main result of [CLW18] for the case of finite groups of homeomorphisms and the included remark about the case of diffeomorphisms), we present a proof.

Proposition 5.10. *Let ϕ be an admissible diffeomorphism of X and let $\phi^{\#}$ be the isometry of X homotopic to ϕ. Then the fixed point set* $\mathrm{Fix}(\phi^{\#}) \subset X$ *of $\phi^{\#}$ is (abstractly) diffeomorphic to* $\mathrm{Fix}(\phi)$. *Moreover,* $\mathrm{Fix}(\phi^{\#})$ *and* $\mathrm{Fix}(\phi)$ *are freely homotopic inside X.*

Proof. Let ϕ be any nontrivial admissible diffeomorphism of X. Then either ϕ is an orientation reversing involution whose fixed point set is a disjoint union of submanifolds of codimension one contained in the preimage of $\mathrm{Fix}(\iota_M)$, or it is a power of ζ with fixed point set Σ. In particular, the fixed point set of ϕ is a (possibly disconnected) orientable hyperbolic manifold of dimension $n-2$ or $n-1$ containing Σ.

We first observe that $\phi^{\#}$ does have fixed points. Indeed, otherwise $X \to \langle \phi^{\#} \rangle \backslash X$ would be a finite-sheeted covering map between manifolds. As X is a closed hyperbolic manifold, it is aspherical and $\pi_1(X)$ has trivial center. But then [HJ24, Lemma 2.2] implies that $\phi^\#$ can *not* be homotopic to a map of finite order and non-empty fixed point set, contradicting the fact that $\phi^\#$ is homotopic to ϕ.

Since a component $Z^\#$ of $\mathrm{Fix}(\phi^\#)$ is a totally geodesic submanifold of X containing the unique closed geodesic in X freely homotopic to an element of $\pi_1(Z^\#)$, no two distinct components of $\mathrm{Fix}(\phi^\#)$ can be freely homotopic. As a consequence, it suffices to show that for every component Z of $\mathrm{Fix}(\phi)$ (or $Z^\#$ of $\mathrm{Fix}(\phi^\#)$) there exists a component $Z^\#$ of $\mathrm{Fix}(\phi^\#)$ (or Z of $\mathrm{Fix}(\phi)$) which is diffeomorphic and freely homotopic to Z (or $Z^\#$).

Thus let Z be a component of $\mathrm{Fix}(\phi)$ and choose a basepoint $x \in Z$. Let ϕ_* be the automorphism of $\pi_1(X,x)$ induced by ϕ. We divide the proof into six steps.

Claim 1. *We have* $\mathrm{Fix}(\phi_*) = \pi_1(Z,x)$*.*

Proof of Claim 1. The hyperbolic metric on M lifts to a hyperbolic cone metric h on X which is smooth away from Σ and with cone angle $2d\pi$ along Σ. Thus h is locally a CAT(-1)-metric. Therefore every homotopy class $\alpha \in \pi_1(X,x)$ has a unique geodesic representative for the metric h which is a geodesic loop based at x. The map ϕ is an isometry for h fixing Z pointwise.

Note that the inclusion $\pi_1(Z,x) \leqslant \mathrm{Fix}(\phi_*)$ trivially holds. To prove equality, we argue by contradiction and assume that there exists a class $[\gamma] \in \mathrm{Fix}(\phi_*) \smallsetminus \pi_1(Z,x)$. This class is represented by a unique geodesic loop γ based at x that does *not* entirely lie in Z. As ϕ is an isometry, $\phi(\gamma)$ is the geodesic representative of $\phi_*([\gamma]) = [\gamma]$, and hence $\phi(\gamma) = \gamma$ by uniqueness. As Z is a connected component of $\mathrm{Fix}(\phi)$, we get $\gamma \subseteq Z$, contradicting $[\gamma] \notin \pi_1(Z,x)$. ∎

The argument in the proof of Claim 1 also applies to the map $\phi^\#$ as an isometry for the hyperbolic metric on X and shows that if $Z^\#$ is any component of the fixed point set $\mathrm{Fix}(\phi^\#)$ of $\phi^\#$, which is a totally geodesic submanifold of X, and if $y \in Z^\#$, then $\pi_1(Z^\#, y) = \mathrm{Fix}(\phi^\#_*) \subset \pi_1(X,y)$.

Since $\phi^\#$ has fixed points, by changing the hyperbolic metric with an isotopy, that is, replacing it by its pullback by a diffeomorphism of X isotopic to the identity, we may assume that $x \in \mathrm{Fix}(\phi^\#)$. Then $\phi^\#$ induces an automorphism $\phi^\#_*$ of $\pi_1(X,x)$.

Claim 2: *Let* $[\gamma] \in \pi_1(Z,x)$ *and let* $\gamma^\#$ *be the closed geodesic for the hyperbolic metric which is freely homotopic to* γ*; then* $\gamma^\# \subset \mathrm{Fix}(\phi^\#)$*.*

Proof of Claim 2: Since ϕ and $\phi^\#$ are homotopic, there exists an element $\alpha \in \pi_1(X,x)$ such that $\phi^\#_* = \alpha\phi_*\alpha^{-1}$. As $[\gamma] \in \pi_1(Z,x) \subset \mathrm{Fix}(\phi_*)$ we know that $\phi^\#_*([\gamma])$ is conjugate to $[\gamma]$. In other words, $\phi^\#_*$ preserves the conjugacy class of $[\gamma]$.

Let $\gamma^\#$ be the unique oriented closed geodesic for the hyperbolic metric on X in the free homotopy class of $[\gamma]$. Since $\phi^\#$ preserves the conjugacy class of $[\gamma]$ and is an isometry, it preserves $\gamma^\#$ as an oriented unparameterized circle.

We argue by contradiction and we assume that $\gamma^\# \not\subset \mathrm{Fix}(\phi^\#)$. If there are no fixed points of $\phi^\#$ on $\gamma^\#$ then as $\phi^\#$ preserves the hyperbolic norm of the tangent of $\gamma^\#$, it acts on the immersed circle $\gamma^\# \subset X$ as a nontrivial rotation. Then $\gamma^\#$ admits a lift $\tilde\gamma^\#$ to the universal covering $\mathbb{H}^n$ of X so that a lift $\widetilde{\phi^\#}$ of $\phi^\#$ acts on $\tilde\gamma^\#$ as a nontrivial translation. But any isometry of $\mathbb{H}^n$ which preserves a geodesic and acts on it as a non-trivial translation is loxodromic and hence fixed point free. This violates the fact that $\phi^\#$ and hence $\widetilde{\phi^\#}$ have fixed points.

As a conclusion, the restriction of $\phi^\#$ to $\gamma^\#$ has fixed points. Let $y \in \gamma^\#$ be such a fixed point. Since $\phi^\#$ preserves $\gamma^\#$ as a set, the differential $d_y\phi^\#$ of $\phi^\#$ at y maps the (oriented) tangent v of $\gamma^\#$ at x to $\pm v$. If $d_y\phi^\#(v) = v$ then $\gamma^\# \subset \mathrm{Fix}(\phi^\#)$ since an isometry maps geodesics parameterized by arc length to geodesics parameterized by arc length, and geodesics are determined by their tangent at a single point. This contradicts the assumption $\gamma^\# \notin \mathrm{Fix}(\phi^\#)$.

Therefore $d_y\phi^\#(v) = -v$ and $\phi^\#$ reverses the orientation of $\gamma^\#$. Then $[\gamma^\#]$ is conjugate to its inverse in $\pi_1(X,x)$. This is equivalent to stating that there exists an element of $\pi_1(X,x)$ acting as the deck group of X on $\mathbb{H}^n$ which exchanges the endpoints in the ideal boundary $\partial\mathbb{H}^n$ of $\mathbb{H}^n$ of a lift of $\gamma^\#$, contradicting the fact that any isometry with this property has a fixed point. Together this completes the proof of Claim 2. ∎

Claim 3: *Up to changing the hyperbolic metric with an isotopy, we have* $\phi_* = \phi_*^\#$*, in particular* $\mathrm{Fix}(\phi_*^\#) = \mathrm{Fix}(\phi_*)$.

Proof of Claim 3: Let $\gamma \subset Z$ be a (nontrivial) closed geodesic for the hyperbolic cone metric h on X. Note that such a geodesic exists since the dimension of each component of $\mathrm{Fix}(\phi)$ is at least two and Z is totally geodesic for h. Let $x \in \gamma$ and let as before $\gamma^\#$ be the closed geodesic for the hyperbolic metric on X which is freely homotopic to γ.

Choose a point $x^\# \in \gamma^\#$ and an embedded arc $a : [0,1] \to X$, smooth up to and including the endpoints, which connects x to $x^\#$ and such that $a \circ \gamma \circ a^{-1}$ (read from right to left) is homotopic to $\gamma^\#$ in $\pi_1(X, x^\#)$. Let N be a tubular neighborhood of a. There exists a smooth isotopy $[0,1] \times X \to X$ of X which is the identity outside of N and pushes the point x along a. Let Λ be the endpoint map of this isotopy. Then Λ maps γ to a based loop at $x^\#$ which is homotopic to $\gamma^\#$ and hence up to replacing the hyperbolic metric by its pull-back under Λ, we may assume that $x \in \gamma^\#$ and that the homotopy classes of γ and $\gamma^\#$ in $\pi_1(X,x)$ coincide.

Recall that $\phi_*^\# = \alpha\phi_*\alpha^{-1}$ for some $\alpha \in \pi_1(X,x)$ (the element α may have changed in the course of this proof). Since

$$[\gamma] = [\gamma^\#] = \alpha[\gamma]\alpha^{-1}$$

we know that α centralizes the homotopy class $[\gamma]$ of γ. As $\pi_1(X,x)$ is the fundamental group of a hyperbolic manifold, the centralizer of $[\gamma]$ equals the infinite cyclic group generated by $[\gamma]$. In particular, $\phi_*(\alpha) = \alpha$ since $[\gamma] \in \mathrm{Fix}(\phi_*)$, moreover $\phi_*^\#$ preserves the fixed point set $\pi_1(Z,x)$ of ϕ_*.

The component Z of the fixed point set of ϕ is a closed oriented hyperbolic manifold of dimension at least two. Thus any nontrivial inner automorphism of $\pi_1(Z,x)$ has infinite order. Now $\phi^\#$ is an isometry of finite order and hence the order of $\phi_*^\#$ is finite as well. But $\phi_*^\#(\beta) = \alpha\beta\alpha^{-1}$ for all $\beta \in \pi_1(Z,x)$ and consequently $\alpha = e$ and $\phi_* = \phi_*^\#$. This completes the proof of Claim 3. ∎

We showed so far that for every component Z of $\mathrm{Fix}(\phi)$ there exists a component $Z^\#$ of $\mathrm{Fix}(\phi^\#)$ whose fundamental group is isomorphic to the fundamental group of Z. Each component Z of $\mathrm{Fix}(\phi)$ and corresponding component $Z^\#$ of $\mathrm{Fix}(\phi^\#)$ is naturally equipped with a hyperbolic metric. Its dimension equals the cohomological dimension of its fundamental group. Thus if the dimension of Z is at least three, then by Mostow rigidity, the manifolds Z and $Z^\#$ are isometric and freely homotopic inside X. If the dimension of Z equals two then the manifolds Z and $Z^\#$ are diffeomorphic as the diffeomorphism type of a closed surface is determined by its fundamental group. Furthermore, Z and $Z^\#$ are freely homotopic inside X.

It remains to show that there is no component of $\mathrm{Fix}(\phi^\#)$ which is not freely homotopic to a component of $\mathrm{Fix}(\phi)$. This is carried out in the rest of this proof.

Claim 4: $X \setminus \mathrm{Fix}(\phi)$ *is aspherical.*

Proof of Claim 4: As X is a closed hyperbolic manifold by assumption, its universal covering $\tilde{X}$ is diffeomorphic to $\mathbb{R}^n$. Furthermore, $\mathrm{Fix}(\phi) \subset X$ is an embedded closed totally geodesic submanifold for the CAT(-1)-hyperbolic cone metric h on X whose codimension either equals one or two. The preimage Y of $X \setminus \mathrm{Fix}(\phi)$ in $\tilde{X}$ is the complement in $\tilde{X}$ of a countable union of properly embedded submanifolds diffeomorphic either to $\mathbb{R}^{n-1}$ or to $\mathbb{R}^{n-2}$.

If the codimension of these subspaces equals one then Y is a disjoint union of countably many contractible spaces. If the codimension of these subspaces equals two then Y is homotopy equivalent to the wedge of countably many circles, each corresponding to a loop encircling one of the codimension two complementary subspaces. Hence Y is aspherical. Since Y is a covering of $X \setminus \mathrm{Fix}(\phi)$, the space $X \setminus \mathrm{Fix}(\phi)$ is aspherical as well. ∎

Claim 5: $X \setminus \mathrm{Fix}(\phi)$ *has the homotopy type of a finite CW-complex, and its fundamental group is center free.*

Proof of Claim 5: There are two cases possible for the map ϕ. In the first case, $\mathrm{Fix}(\phi)$ is a finute disjoint union of compact codimension one submanifolds in X, and in the second case, we have $\mathrm{Fix}(\phi) = \Sigma$. In both cases, $X \setminus \mathrm{Fix}(\phi)$ is homotopy equivalent to a compact manifold with boundary, which can be chosen to be the complement of a small open tubular neighborhood of $\mathrm{Fix}(\phi)$. Hence $X \setminus \mathrm{Fix}(\phi)$ has the homotopy type of a finite CW-complex.

To see that $\pi_1(X \setminus \mathrm{Fix}(\phi))$ is center free, note that if $\mathrm{Fix}(\phi)$ is a disjoint union of hyperplanes, then cutting X open along the corresponding components of $\mathrm{Fix}(\phi^\#)$ yields a (possibly disconnected) compact hyperbolic manifold N with totally geodesic boundary whose fundamental group is a torsion free hyperbolic group and hence center free.

If $\mathrm{Fix}(\phi) = \Sigma$ then putting $G = \pi_1(X \setminus \mathrm{Fix}(\phi))$, the homomorphism $\rho : G \to \pi_1(X)$ induced by the inclusion $X \setminus \mathrm{Fix}(\phi) \to X$ is surjective. Thus as $\pi_1(X)$ is torsion free and center free, an element in the center of G is contained in the kernel of the homomorphism ρ and hence it is contained in the center of $\ker(\rho)$. But the kernel of ρ is the fundamental group of the preimage Y of $X \setminus \Sigma$ in the universal covering $\mathbb{H}^n$ of X. As Y has the homotopy type of a countable wedge of circles, this fundamental group is an infinitely generated free group and hence center free. ∎

The following claim completes the proof of the proposition.

Claim 6: *There can not be any component of* $\mathrm{Fix}(\phi^{\#})$ *that is not freely homotopic to a component of* $\mathrm{Fix}(\phi)$.

Proof of Claim 6: We showed so far that there exists a union Q of components of $\mathrm{Fix}(\phi^{\#})$ which is abstractly diffeomorphic to $\mathrm{Fix}(\phi)$ and freely homotopic to $\mathrm{Fix}(\phi)$ in X. The manifolds $X \smallsetminus \mathrm{Fix}(\phi)$ and $X \smallsetminus Q$ have isomorphic fundamental groups, and by Claim 4 and its analog for $X \smallsetminus Q$, they are aspherical. As a consequence, $X \smallsetminus \mathrm{Fix}(\phi)$ and $X \smallsetminus Q$ are homotopy equivalent.

Let $\Lambda : X \smallsetminus \mathrm{Fix}(\phi) \to X \smallsetminus Q$ be a homotopy equivalence, with homotopy inverse Λ^{-1}. The map ϕ acting on $X \smallsetminus \mathrm{Fix}(\phi)$ is homotopic to the map $\hat{\phi} = \Lambda^{-1} \circ \phi^{\#} \circ \Lambda$, read from right to left. Thus via an identification of $\pi_1(X \smallsetminus \mathrm{Fix}(\phi))$ with $\pi_1(X \smallsetminus Q)$ via the homotopy equivalence Λ, the maps ϕ and $\phi^{\#}$ induce the same outer automorphisms of $\pi_1(X \smallsetminus \mathrm{Fix}(\phi))$. Furthermore, by construction, ϕ and $\phi^{\#}$ have the same order, say $m \geq 2$.

We now follow [HJ24, Lemma 2.2]. The finite order diffeomorphism ϕ restricts to a fixed point free finite order diffeomorphism on $X \smallsetminus \mathrm{Fix}(\phi)$. Let $\bar{X} = \langle \phi \rangle \backslash (X \smallsetminus \mathrm{Fix}(\phi))$ be the quotient of X under the free action of ϕ. There exists an exact sequence

$$1 \to \pi_1(X \smallsetminus \mathrm{Fix}(\phi)) \to \pi_1(\bar{X}) \to \mathbb{Z}/m\mathbb{Z} \to 1.$$

Since ϕ and $\phi^{\#}$ induce the same outer automorphism of $\pi_1(X \smallsetminus \mathrm{Fix}(\phi))$, this sequence splits if the map $\phi^{\#}$ acting on $X \smallsetminus Q$ has a fixed point. However, as $\pi_1(X \smallsetminus \mathrm{Fix}(\phi))$ is center free by Claim 5, if the sequence splits then $\mathbb{Z}/m\mathbb{Z}$ is a subgroup of $\pi_1(\bar{X})$, which is impossible as $X \smallsetminus \mathrm{Fix}(\phi)$ and hence $\bar{X}$ has the homotopy type of a finite CW complex by Claim 5. We refer to [HJ24, Lemma 2.2] for more information on this line of argument. As a conclusion, the action of $\phi^{\#}$ on $\pi_1(X \smallsetminus Q)$ is fixed point free which completes the proof of Claim 6. ■ □

Remark. The above proof is valid for all covers X of a hyperbolic manifold M, branched along a totally geodesic nullhomologous submanifold Σ of codimension two. It shows that if X admits a hyperbolic metric, then the fixed point set of an isometry of X homotopic to an element of the deck group of $X \to M$ is diffeomorphic to the branch locus Σ, thus confirming [GT87].

5.2. **The proof of Proposition 5.7.** In this subsection we assume as before that X admits a hyperbolic metric. Let j be the involution of X described in Fact 5.9, and ζ be the generator of the deck group of $X \to M$ which cyclically permutes the copies $M^1_{\mathrm{cut}}, \dots, M^d_{\mathrm{cut}}$ of M_{cut} in X.

From now on we always denote by F the component of $\mathrm{Fix}(j)$ containing Σ and by $F^{\#}$ the homotopic component of $\mathrm{Fix}(j^{\#})$ whose existence was shown in Proposition 5.10. By Proposition 5.10 and Mostow rigidity for closed hyperbolic manifolds of dimension $n-1 \geq 3$, there exists an isometry $\psi : F \to F^{\#}$ which maps Σ to the fixed point set $\Sigma^{\#}$ of $\zeta^{\#}$. Furthermore, with a homotopy we may identify Σ and $\Sigma^{\#}$ in X. For each $i = 0, \dots, d-1$, the map $(\zeta^{\#})^i \circ \psi \circ \zeta^{-i}$ maps $\zeta^i(F)$ isometrically onto $(\zeta^{\#})^i(F^{\#})$.

After possibly changing the hyperbolic metric of X with an isotopy, we may assume that for each connected component Σ_0 of Σ we have $\Sigma_0 \cap \Sigma_0^{\#} \neq \emptyset$, where $\Sigma_0^{\#} = \psi_0(\Sigma_0)$. So, for each component, we can fix a basepoint $x_0 \in \Sigma_0 \cap \Sigma_0^\#$, and we may assume without loss of generality that $\psi_0(x_0) = x_0$. We call such a basepoint *preferred.* Due to Proposition 5.10, we may also assume that

$$\pi_1(\Sigma_0, x_0) = \pi_1(\Sigma_0^\#, x_0) \quad \text{and} \quad \pi_1(F, x_0) = \pi_1(F^\#, x_0).$$

In the sequel, the fundamental group $\pi_1(X, x_0)$ will always be represented with respect to a fixed choice x_0 of preferred basepoint.

Although by Proposition 5.10, the cyclic group generated by $\zeta^\#$ acts freely on $X \setminus \Sigma^\#$ and the manifold $F^\#$ is homotopic to F, this does not necessarily imply that $\zeta^\#(F^\#) \cap F^\# = \Sigma^\#$. The following lemma takes care of this issue.

Lemma 5.11. *(1) The differential of $\zeta^\#$ acts on the normal bundle of $\Sigma^\#$ by a rotation with angle $2\pi/d$.*

(2) We have $F^\# \cap \zeta^\#(F^\#) = \Sigma^\#$.

Proof. We begin with the proof of the second part of the lemma. We may assume that $\pi_1(F, x_0) = \pi_1(F^\#, x_0)$ and $\pi_1(\zeta(F), x_0) = \pi_1(\zeta^\#(F^\#), x_0)$. Thus for a choice of lift $\tilde{x}_0$ of x_0 to the universal covering $\mathbb{H}^n$, limit sets of these groups in the ideal boundary $\partial\mathbb{H}^n$ and of their conjugates, acting as subgroups of the deck group, coincide.

Let $\widetilde{F^\#} \subset \mathbb{H}^n$ and $\widetilde{\zeta^\# F^\#} \subset \mathbb{H}^n$ be the (unique) lifts of $F^\#$ and $\zeta^\#(F^\#)$, respectively, which pass through $\tilde{x}_0$. Each component of $F^\# \cap \zeta^\#(F^\#)$, which is a totally geodesic embedded hyperplane in $F^\#$, lifts to precisely one $\pi_1(F^\#, x_0)$-orbit of intersections of $\widetilde{F^\#}$ with $\pi_1(X, x_0)(\widetilde{\zeta^\# F^\#})$ (using the deck group action) and hence to a $\pi_1(F^\#, x_0)$-orbit of intersections of the boundary sphere of $\widetilde{F^\#}$ with the boundaries of the hyperplanes in the orbit of $\widetilde{\zeta^\# F^\#}$. These boundary spheres are precisely the limit sets of the conjugates of the group $\pi_1(\zeta^\#(F^\#), x_0) = \pi_1(\zeta(F), x_0)$ in $\partial\mathbb{H}^n$. Since $F \cap \zeta(F) = \Sigma$, the number of $\pi_1(F^\#, x_0)$-orbits of such intersection spheres is at most the number of components of $\Sigma = \Sigma^\#$. Thus we have $F^\# \cap \zeta^\#(F^\#) = \Sigma^\#$ which completes the proof of the second part of the lemma.

Let $\Sigma_0^\#$ be a component of $\Sigma^\#$. This is a totally geodesic submanifold of X of codimension two contained in the fixed point set of $\zeta^\#$. Since $\zeta^\#$ is a non-trivial orientation preserving isometry of X of order d, its differential acts on the normal bundle of $\Sigma_0^\#$ as a rotation with rotation angle $2\pi k/d$ where k is a generator of the cyclic group of order d. We have to show that $k = 1$.

Consider again lifts $\tilde{\zeta}, \tilde{\zeta}^\#$ of $\zeta, \zeta^\#$ to the universal covering $\mathbb{H}^n$ of X, chosen so that they fix pointwise the same component $\widetilde{\Sigma_0^\#}$ of the universal covering of $\Sigma_0^\#$, which is a totally geodesic subspace of $\mathbb{H}^n$ of codimension two. The differential of $\tilde{\zeta}^\#$ acts on the normal bundle of $\widetilde{\Sigma^\#}$ as a rotation with rotation angle $2\pi k/p$.

The choice of basepoint $\tilde{x}_0 \in \widetilde{\Sigma_0^\#}$ determines an identification of the unit tangent sphere $T^1_{\tilde{x}_0}\mathbb{H}^n$ of $\mathbb{H}^n$ at $\tilde{x}_0$ with the ideal boundary $\partial\mathbb{H}^n = S^{n-1}$ of $\mathbb{H}^n$ by associating to a unit tangent vector v the equivalence class of the geodesic ray with initial velocity v. The limit set in $\partial\mathbb{H}^n = S^{n-1}$ of the stabilizer of $\widetilde{F^\#}$ in the deck group $\pi_1(X, x_0)$ equals the boundary

$\partial\widetilde{F^\#}$ of $\widetilde{F^\#}$, which is an equator sphere of codimension one in $T_{\tilde{x}_0}\mathbb{H}^n = \partial\mathbb{H}^n$. It contains the ideal boundary of $\widetilde{\Sigma_0^\#}$ as an equator sphere. We also know that $\zeta^\#(\partial\widetilde{F^\#})$ coincides with the limit set $\zeta(\partial\tilde{F})$ of the group $\pi_1(\zeta(F), x_0)$ acting on $\mathbb{H}^n$.

Now recall that ζ acts as an isometry with respect to the CAT(-1) hyperbolic cone metric h on X, which is quasi-isometric to the hyperbolic metric, and it acts as a cyclic permutation on the totally geodesic submanifolds $\zeta^i(F)$. Thus viewing $\partial\mathbb{H}^n$ as the ideal boundary of the universal covering of X, equipped with the hyperbolic cone metric h, we obtain that there is a component of $\partial\mathbb{H}^n \setminus (\partial\tilde{F} \cup \zeta(\partial\tilde{F}))$ which does not intersect any of the spheres $\zeta^i(\partial\tilde{F})$. By identifying $\partial\widetilde{F^\#}$ with the unit tangent sphere of $\widetilde{F^\#}$ at $\tilde{x}_0$, which is an equator sphere in $T^1_{\tilde{x}_0}\mathbb{H}^n$, and $\zeta^\#(\partial\widetilde{F^\#})$ with the unit tangent space of $\zeta^\#(F^\#)$ at $\tilde{x}_0$, we deduce that there is a component of $T^1_{\tilde{x}_0}\mathbb{H}^n \setminus (T^1_{\tilde{x}_0}F^\# \cup d\zeta^\#(T^1_{\tilde{x}_0}F^\#))$ not intersecting any of the spheres $d\zeta^\#(T^1_{\tilde{x}_0}F^\#)$ if and only if the differential of $\zeta^\#$ acts on the normal bundle of $\Sigma_0^\#$ as a rotation with rotation angle $2\pi/d$. This completes the proof of the lemma. □

Remark. The proof of the first part of Lemma 5.11 relies on the analysis of limit sets of stabilizers of preimages of the totally geodesic hyperplane $H \subset M$. It remains valid even if H is not fixed by an isometric involution.

With these preliminary results at hand, we can now prove Proposition 5.7.

Proof of Proposition 5.7. By construction, the subspace $F \cup \zeta(F)$ of X separates X. If d is even, then by the definition of the map j, the complement $X - (F \cup \zeta(F))$ contains two connected components whose closures are homeomorphic to M_{cut}. If d is odd then it contains one connected component whose closure is homeomorphic to M_{cut}. In both cases, let Z be the closure of such a component. Its boundary consists of two copies of H_0 glued along Σ.

By Lemma 5.11, there exists a corresponding component $M^\#_{\text{cut}}$ of $X - (F^\# \cup \zeta^\#(F^\#))$. The boundary of its closure $Z^\#$ is connected and consists of two copies of H_0 meeting along Σ with an angle $2\pi/d$. Identifying Σ and $\Sigma^\#$ as before and choosing a basepoint $x \in \Sigma$, we claim that $\pi_1(Z, x) = \pi_1(Z^\#, x)$.

Namely, by Proposition 5.10, it holds that $\pi_1(\partial Z, x) = \pi_1(\partial Z^\#, x)$. As ∂Z is a separating hypersurface in X homotopic to $\partial Z^\#$, by the theorem of Seifert-van Kampen, we know that

$$\pi_1(X, x) = \pi_1(Z, x) *_{\pi_1(\partial Z, x)} \pi_1(X \setminus Z, x) = \pi_1(Z^\#, x) *_{\pi_1(\partial Z^\#, x)} \pi_1(X \setminus Z^\#, x).$$

It then follows from the normal form for amalgamated products [LS01, p.186] that $\pi_1(Z^\#, x)$ is isomorphic to either $\pi_1(Z, x)$ or to $\pi_1(X \setminus Z, x)$.

If $d = 2$ then $\pi_1(Z, x)$ is isomorphic to $\pi_1(X \setminus Z, x)$ and the claim is clear. If $d \geq 3$ then note that $\zeta_* = \zeta_*^\#$ maps $\pi_1(Z, x)$ to a proper subgroup of $\pi_1(X \setminus Z, x)$, and it maps $\pi_1(X \setminus Z, x)$ to a proper supergroup of $\pi_1(Z, x)$. Furthermore, it maps $\pi_1(Z^\#, x)$ to a proper subgroup of $\pi_1(X \setminus Z^\#, x)$, and it maps $\pi_1(X \setminus Z^\#, x)$ to a proper supergroup of $\pi_1(Z^\#, x)$. Thus we have $\pi_1(Z, x) = \pi_1(Z^\#, x)$ as claimed.

Note that if d is odd, then the component $Z^\#$ contains a totally geodesic hypersurface isometric to H_1 which intersects the boundary of $Z^\#$ along Σ. There exists an isometric involution of $M^{\#}_{\text{cut}}$ which exchanges the two copies of H_0 in its boundary and hence H_1 meets the boundary of $M^{\#}_{\text{cut}}$ with an angle of π/d. We refer to Fact 5.9 for more information. Cutting $Z^{\#}$ open along this hypersurface then yields a cone manifold with the properties stated in Proposition 5.7.

We do not give a proof of the proposition in the case that the covering degree d is even as for the completion of the proof of Theorem 5.5, we only need the above information on the submanifold $Z^{\#}$ of X. □

5.3. **Proof of Theorem 5.5.** We showed so far that the existence of a hyperbolic metric on the d-fold covering X of M branched along Σ gives rise to a convex cocompact hyperbolic manifold N_d with two boundary components, each of which is path isometric to the hypersurface H. The manifold is singular along $\Sigma \subset H$, with cone angle (or bending angle) π/d, and it is homotopy equivalent to the hyperbolic manifold N with totally geodesic boundary obtained by cutting M open along H. Note that N is connected if and only if the hypersurface H is non-separating.

Construct a new manifold W by gluing $2d$ copies N_d^i $(i = 1, \dots, 2d)$ of N_d along the boundary as follows. Let $\partial N_d^{\pm}$ be the two distinct boundary components of N, and let $(\partial N_d^i)^{\pm}$ be the corresponding boundary components of N_d^i. Each of these components contains a copy of Σ which decomposes the component into two connected components $(H^i_{0,d})^{\pm}, (H^i_{1,d})^{\pm}$. For each odd $i \le 2d$ identify $(H^i_{0,d})^{\pm}$ with $(H^{i+1}_{0,d})^{\pm}$, and for even $i \le 2d$ identify $(H^i_{1,d})^{\pm}$ with $(H^{i+1}_{1,d})^{\pm}$. As the cone angle of $\partial N_d^{\pm}$ along Σ equals π/d, the hyperbolic metrics on the bordered manifolds N_d^i induce a smooth hyperbolic metric on W.

Lemma 5.12. *The manifold W is a two-sheeted unbranched covering of X.*

In fact, if the hypersurface H in M is non-separating, then the same holds true for the component F containing Σ of the fixed point set of the lift ι to X of the involution ι_M specified in Fact 5.9. Then W is the two-sheeted covering of X so that the preimage of F consists of two components which separate W. If $H \subset M$ is separating, then N consists of two connected components, and W consists of two copies of X.

Proof. The case that H is separating is clear from the above remark. Thus assume that H is non-separating. Then the same holds true for the hypersurface F. Cut X open along F. The resulting manifold Q is connected and has two boundary components $\partial Q^-, \partial Q^+$ which are homeomorphic and path isometric to H. The metric is singular along the two copies of the totally geodesic submanifold $\Sigma^{\#}$ in the two boundary components of Q.

Glue a second copy $\hat{Q}$ of Q to Q along the boundary in such a way that the boundary component $\partial\hat{Q}^-$ is glued to the boundary component ∂Q^+, and the boundary component $\partial\hat{Q}^+$ is glued to the boundary component ∂Q^-. The resulting manifold is equipped with a smooth hyperbolic metric, and admits an obvious two sheeted unbranched covering onto X. The lemma follows. □

We are now in a position to present the proof of Theorem 5.5.

Proof of Theorem 5.5. We divide the proof into two claims.

Claim 1: *Among the branched coverings of M of even degree $d \in 2\mathbb{N}$, at most one can be homeomorphic to a hyperbolic manifold.*

Proof of Claim 1: We argue by contradiction and we assume that there are distinct multiples of $d_1 \neq d_2 \in 2\mathbb{N}$ such that the cyclic d_i-fold branched cover $X^{(d_i)}$ admits a smooth hyperbolic metric for $i = 1, 2$. Then, for each $i = 1, 2$, Proposition 5.7 (see the end of the proof for an explicit statement) implies that there exists a hyperbolic cone manifold $M_{\text{cut}}^{2\pi/d_i}$ with totally geodesic boundary $\partial M_{\text{cut}}^{2\pi/d_i}$ homeomorphic and path isometric to ∂M_{cut}, with singular set isometric to Σ, cone angle $2\pi/d_i$ along Σ, and $\pi_1(M_{\text{cut}}^{2\pi/d_i}) = \pi_1(M_{\text{cut}})$.

Note that $\frac{d_1}{2}\frac{2\pi}{d_1} + \frac{d_2}{2}\frac{2\pi}{d_2} = 2\pi$. Therefore, we can glue $d_1/2$ copies of $M_{\text{cut}}^{2\pi/d_1}$ and $d_2/2$ copies of $M_{\text{cut}}^{2\pi/d_2}$ in cyclic order along the components of $\partial M_{\text{cut}}^{2\pi/d_i} \setminus \Sigma$ to a smooth hyperbolic manifold Y (see Figure 2). An application of the Seifert–van Kampen theorem shows that the fundamental group of Y is isomorphic to the fundamental group of the $(d_1+d_2)/2$-fold cyclic cover X of M branched along Σ. In particular, this fundamental group admits a finite group of automorphisms generated by an element ζ_* of order $(d_1+d_2)/2$ and an involution j_* corresponding to the automorphisms induced by the homeomorphisms ζ and j of X (notations are as before). By the proof of Proposition 5.7, for each $i = 0, \ldots, (d_1+d_2)/2 - 1$, the fixed point group of $\zeta_*^i \circ j_* \circ \zeta_*^{-i}$ is the fundamental group of an embedded codimension one submanifold F_i that, by construction of the hyperbolic metric on Y, is already totally geodesic (see Figure 2). Moreover, for some i the totally geodesic submanifolds F_i and F_{i+1} intersect with angle $2\pi/d_1$, while for other i they intersect with angle $2\pi/d_2$ (see Figure 2).

By Mostow rigidity, there exist isometries $\zeta^\#, j^\#$ of the hyperbolic manifold Y of order $(d_1+d_2)/2$ and 2, respectively, that induce the outer automorphism given by ζ_* and j_*. By Lemma 5.11, the fixed point set of $\zeta^\#$ is a codimension two totally geodesic submanifold $\Sigma^\#$ freely homotopic to Σ, and thus $\Sigma^\# = \Sigma$ since Σ is already totally geodesic in Y. Similarly, by Proposition 5.7, the fixed point set $(\zeta^\#)^i(F^\#)$ of the involution $(\zeta^\#)^i \circ j^\# \circ (\zeta^\#)^{-i}$ is a totally geodesic hyperplane freely homotopic to the manifold F_i satisfying $\pi_1(F_i) = \text{Fix}(\zeta_*^i \circ j_* \circ \zeta_*^{-i})$, and thus $(\zeta^\#)^i(F^\#) = F_i$ since F_i is already hyperbolic. However, as $\zeta^\#$ acts by rotation with a fixed angle in the normal bundle of Σ, the intersection angle of $(\zeta^\#)^i(F^\#)$ and $(\zeta^\#)^{i+1}(F^\#)$ is the same for all i. But this contradicts the fact that, by construction, the intersection angle of F_i with F_{i+1} varies between $2\pi/d_1$ and $2\pi/d_2$ (see Figure 2), completing the proof of the claim. ∎

Claim 2: *No branched covering of M of odd degree $d \geq 3$ can be homeomorphic to a hyperbolic manifold.*

Proof of Claim 2: Assume that there exists a covering X of M branched along Σ of odd degree d which admits a hyperbolic metric. By Proposition 5.7, there exists a hyperbolic cone manifold N_d homotopy equivalent to the manifold N with cone angle π/d along the copies of Σ in each boundary component of N_d. Glue d copies of N_d to the manifold $N = M \setminus H$ along the boundary as described in Lemma 5.12. Note that this is possible because d is odd. The resulting manifold is homotopy equivalent to a double unbranched

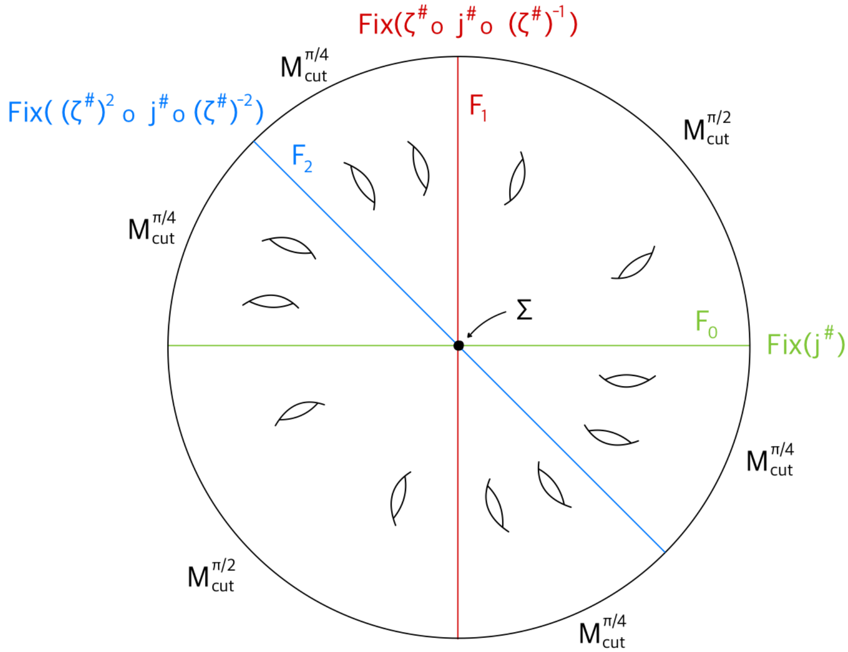

FIGURE 2. The case $d_1 = 4$ and $d_2 = 8$. The colored submanifolds F_i satisfying $\pi_1(F_i) = \mathrm{Fix}(\zeta_*^i \circ j_* \circ \zeta_*^{-i})$ are already totally geodesic and intersect at Σ with angles $\pi/2$ or $\pi/4$.

covering of the branched covering X of M of degree $\frac{d+1}{2}$ as in Lemma 5.12, and it is equipped with a smooth hyperbolic metric.

On the other hand, as $W \to X$ is a two-sheeted unbranched covering, the hyperbolic metric on X lifts to a hyperbolic metric on W. However, it follows precisely as in the proof of Claim 1 that this leads to a contradiction. ■

□

Remark 5.13. The proof of Theorem 5.5 for branched covers of hyperbolic manifolds of dimension $n \geq 4$ depends in a crucial way on the validity of Mostow rigidity for closed hyperbolic manifolds of dimension $n-1$ and hence is not valid for $n = 3$.

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