SMALL EIGENVALUES AND THICK-THIN DECOMPOSITION IN NEGATIVE CURVATURE

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Abstract. Let M be a finite volume oriented Riemannian manifold of dimension $n \geq 3$ and curvature in $[-b^2, -1]$, with thick-thin decomposition $M = M_{\text{thick}} \cup M_{\text{thin}}$. Denote by $\lambda_k(M_{\text{thick}})$ the k-th eigenvalue for the Laplacian on M_{thick}, with Neumann boundary conditions. We show that $\lambda_k(M_{\text{thick}})/3 \leq \lambda_k(M)$ for all k for which $\lambda_k(M) < (n-2)^2/12$. If M is hyperbolic and of dimension 3 then $\lambda_k(M) \leq C \log(\text{vol}(M_{\text{thin}}) + 2)\lambda_k(M_{\text{thick}})$ for a fixed number $C > 0$ provided that $\lambda_k(M_{\text{thick}}) < 1/96$.

1. Introduction

The small part of the spectrum of the Laplace operator Δ acting on functions on a closed oriented hyperbolic surface S is quite well understood. Namely, if g denotes the genus of S then the $2g-3$-th eigenvalue $\lambda_{2g-3}(S)$ can be arbitrarily small [4], while $\lambda_{2g-2}(S) > \frac{1}{4}$ [18].

By the Gauss-Bonnet theorem, the volume of S equals $2\pi(2g-2)$, so these results relate the small part of the spectrum of S to its volume.

For $n \geq 3$, the small part of the spectrum of an oriented finite volume Riemannian manifold M of dimension n and sectional curvature $\kappa \in [-b^2, -1|$ for some $b \geq 1$ is less well understood. The manifold M admits a *thick-thin decomposition* $M = M_{\text{thick}} \cup M_{\text{thin}}$ which is determined as follows. There exists a constant $c = c(n) > 0$ such that $\varepsilon = b^{-1}c(n)$ is a *Margulis constant* for M, and M_{thick} is the set of all points in M of injectivity radius at least ε. Its complement M_{thin} is a disjoint union of so-called *Margulis tubes* and *cusps.* Here a Margulis tube is a tubular neighborhood of a closed geodesic of length at most 2ε, and a cusp is homeomorphic to the quotient of a horoball in the universal covering $\tilde{M}$ of M by a rank $n-1$ parabolic subgroup of the isometry group of $\tilde{M}$. After a small modification, we may assume that $M_{\text{thick}} = M - M_{\text{thin}}$ is a submanifold of M with smooth boundary [5].

Unlike in the case of surfaces, the submanifold M_{thick} is always connected, and this prevents the occurence of very small eigenvalues. More precisely, for every $n \geq 3$, Schoen [19] established the existence of a universal and

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explicit constant $\theta = \theta(n, b) > 0$ such that

$$\lambda_1(M) \geq \frac{\theta}{\mathrm{vol}(M)^2} \tag{1}$$

for *every* closed n-manifold M with curvature in $[-b^2, -1]$ (here in contrast to the work of Schoen, we normalize the metric on M so that the upper bound of the curvature is fixed). That this estimate extends without change to non-compact finite volume manifolds was established in [10, 9].

For hyperbolic manifolds of dimension $n = 3$, this bound is roughly sharp: White [20] proved that for fixed $r \geq 2, \delta > 0$ there exists a positive constant $a = a(r, \delta)$ such that

$$\lambda_1(M) \in [1/a\mathrm{vol}(M)^2, a/\mathrm{vol}(M)^2]$$

for any closed hyperbolic 3-manifold whose injectivity radius is bounded from below by δ and such that the rank of the fundamental group $\pi_1(M)$ of M is at most r. This rank is defined to be the minimal number of generators of $\pi_1(M)$, and it is bounded from above by a fixed multiple of the volume (Theorem 1.10 in [13]). A similar statement also holds true for random 3-dimensional hyperbolic mapping tori [1].

Since the injectivity radius at points in M_{thick} is at least ε, the submanifold M_{thick} of M is uniformly quasi-isometric to a finite graph G of uniformly bounded valence, with constant only depending on the dimension and the curvature bounds (see [2] for a detailed discussion of this fact in the case of hyperbolic manifolds). If $|G|$ denotes the number of vertices of G, then for $k \leq |G|$ the k-th eigenvalue $\lambda_k(M_{\mathrm{thick}})$ of M_{thick} with Neumann boundary conditions is uniformly comparable to the k-th eigenvalue $\lambda_k(G)$ of the graph Laplacian of G [17]. Here and in the sequel, we list eigenvalues as $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ with each eigenvalue repeated according to its multiplicity. Note that $|G|$ is proportional to the volume $\mathrm{vol}(M_{\mathrm{thick}})$ of M_{thick}, with multiplicative constants only depending on n and b.

Now for any graph G, there are precisely $|G|$ eigenvalues $0 = \lambda_0(G) < \lambda_1(G) \leq \cdots \leq \lambda_{|G|-1}(G)$, and (Lemma 1 of [8])

$$\sum_i \lambda_i(G) = |G|.$$

In particular, we have $\lambda_{|G|-1}(G) \geq 1$. As the volume of a finite volume oriented manifold M of curvature in $[-b^2, -1]$ is uniformly comparable to the volume of M_{thick}, this together with the result in [17] implies that there exists a constant $q = q(n, b) > 1$ so that $\lambda_{q\mathrm{vol}(M)}(M_{\mathrm{thick}}) \geq 1/q$.

To recover a relation between the spectrum of M and the volume of M which resembles the result known for hyperbolic surfaces, it is therfore desirable to relate the small part of the spectrum of M to the small part of the spectrum of M_{thick}, taken with Neumann boundary condition. The first purpose of this article is to establish such a relation. We show

Theorem 1. *For a suitable choice of a Margulis constant, we have*

$$\lambda_k(M) \geq \min\{\frac{1}{3}\lambda_k(M_{\text{thick}}), \frac{(n-2)^2}{12}\}$$

for every finite volume oriented Riemannian manifold M of dimension $n \geq 3$ and curvature $\kappa \in [-b^2, -1]$, and all $k \geq 0$.

The dependence of this estimate on the lower curvature bound $-b^2$ enters this result via the Margulis constant which depends on b. Note also that the constant $(n-2)^2/12$ is uniformly comparable to the lower bound $(n-1)^2/4$ for the bottom of the essential spectrum of a non-compact finite volume manifold of curvature $\kappa \in [-b^2, -1]$ (Corollary 3.2 of [14]). We expect that our methods can be used to extend Theorem 1 to geometrically finite manifolds of infinite volume and curvature in $[-b^2, -1]$. This could then be used to establish an improvement of Corollary 3.3 of [14] which contains the following statement as a special case: The number of eigenvalues contained in $(0, (n-2)^2/12)$ is at most $d^{\text{vol}(M)}$ for a fixed constant $d > 0$. However, we do not attempt to carry out such a generalization in this article.

As an application of Theorem 1, we recover the results of Schoen, of Dodziuk and of Randol and Dodziuk, and we relate the number of small eigenvalues to the volume as promised.

Corollary. *For all $n \geq 3, b \geq 1$ there exists a constant $\chi = \chi(n, b) > 0$ with the following property. Let M be a finite volume oriented Riemannian manifold of dimension n and curvature $\kappa \in [-b^2, -1]$; then*

(1) $\lambda_1(M) \geq \frac{\chi}{\text{vol}(M)^2}$.
(2) $\lambda_{\text{vol}(M)/\chi}(M) \geq \chi$.

Unlike in the work of Schoen, the constant $\chi(n, b)$ in the above Corollary is not explicit as it depends on a Margulis constant for Riemannian manifolds with curvature in $[-b^2, -1]$. However, for hyperbolic manifolds it can explicitly be estimated.

For hyperbolic 3-manifolds M we also obtain upper bounds for the small eigenvalues of M. We show

Theorem 2. *There exists a number $c > 0$ such that for every finite volume oriented hyperbolic 3-manifold M, we have*

$$\lambda_k(M) \leq c \log(\text{vol}(M_{\text{thin}}) + 2)\lambda_k(M_{\text{thick}})$$

for all $k \geq 1$ such that $\lambda_k(M_{\text{thick}}) < 1/96$.

The proof of Theorem 1 uses standard comparison results and a simple decomposition principle, and it is carried out in Section 2. Theorem 2 is shown in Section 3 with an explicit construction which is only valid for hyperbolic 3-manifolds.

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2. Bounding small eigenvalues from below

The goal of this section is to show Theorem 1.

Thus let M be a finite volume oriented Riemannian manifold of dimension $n \geq 3$ and sectional curvature $\kappa \in [-b^2, -1]$ for some $b \geq 1$. Then M admits a *thick-thin decomposition*

$$M = M_{\text{thin}} \cup M_{\text{thick}}.$$

For a number $\varepsilon = b^{-1}c(n) > 0$, a so-called *Margulis constant*, the thin part M_{thin} is the set of all points $x \in M$ with injectivity radius $\text{inj}(x) \leq \varepsilon$, and $M_{\text{thick}} = \{x \mid \text{inj}(x) \geq \varepsilon\}$. The set M_{thick} is a non-empty compact connected manifold with (perhaps non-smooth) boundary, and M_{thin} is a union of (at most) finitely many *Margulis tubes* and *cusps*.

A Margulis tube is a tubular neighborhood of a closed geodesic γ in M of length smaller than 2ε, and it is homeomorphic to $B^{n-1} \times S^1$ where B^{n-1} is the closed unit ball in $\mathbb{R}^{n-1}$. The geodesic γ is called the *core geodesic* of the tube.

Let T be such a Margulis tube, with core geodesic γ of length $\ell < 2\varepsilon$. We fix a parametrization of γ by arc length on the interval $[0, \ell)$. Let σ be the standard angular coordinate on the fibers of the unit normal bundle $N(\gamma)$ of γ in M obtained by parallel transport of the fibre over $\gamma(0)$ (this unit normal bundle is an S^{n-2}-bundle over γ), let s be the length parameter of γ and let $\rho \geq 0$ be the radial distance from γ. Via the normal exponential map, these functions define "coordinates" (i.e. a parametrization) (σ, s, ρ) for $T - \{\gamma\}$, defined on an open subset of $N(\gamma) \times (0, \infty)$ which will be specified below. In these "coordinates", the maps $\rho \to (\sigma, s, \rho)$ are unit speed geodesics with starting point on γ and initial velocity perpendicular to γ'.

There exists a continuous function

$$R : N(\gamma) \to (0, \infty), (\sigma, s) \to R(\sigma, s)$$

such that in these "coordinates", we have $T = \{\rho \leq R\}$. The metric on $T - \{\gamma\}$ is of the form $h(\rho) + d\rho^2$ where $h(\rho)$ is a family of smooth metrics on the hypersurfaces $\rho = \text{const}$.

Lemma 2.4 of [5] states that there exists a constant $\nu(n, b) > 0$ which can be computed as an explicit function of the constants $c(n), b, n$ such that

$$\min R \geq -\log \ell - \nu(n, b). \tag{2}$$

In particular, up to slightly adjusting the thick-thin decomposition and replacing M_{thick} by its union with all Margulis tubes with core geodesics of length ℓ so that $\log \ell \geq -3 - \nu(n, b)$, we may assume that for every component T of M_{thin}, the distance between the core geodesic and the boundary ∂T is at least three.

In general, the boundary of a Margulis tube need not be smooth. However, Theorem 2.14 of [5] shows that it can be perturbed to be smooth and of controlled geometry. We record this result for completeness.

Theorem 2.1. *Let $T \subset M$ be a Margulis tube, with core geodesic γ and boundary ∂T. Then there exists a smooth hypersurface $H \subset T - \gamma$ with the following properties.*

(1) *The angle θ between the tangent of the radial geodesic and the exterior normal to H is less than $\pi/2-\alpha$ for some $\alpha = \alpha(n, b) \in (0, \pi/2)$.*
(2) *The sectional curvatures of H with respect to the induced metric are bounded in absolute value by a constant depending only on n and b.*
(3) *H is homeomorphic to ∂T by pushing along radial arcs. The distance between $x \in H$ and its image $\bar{x} \in \partial T$ satisfies $d(x, \bar{x}) \leq bc(n)/50$.*

In the sequel we always assume that the boundary ∂T of a Margulis tube has the properties stated in Theorem 2.1. Then the injectivity radius of ∂T with respect to the induced metric is bounded from below by a positive constant only depending on the curvature bound and the dimension (Corollary 2.24 of [5]).

Our first goal is to obtain a better understanding of the volume element of a Margulis tube. To this end we record a variant of Lemma 1 of [10]. We begin with a simple comparison lemma.

Let $\tilde{M}$ be a simply connected complete Riemannian manifold of dimension n, with sectional curvature $\kappa \leq -1$. Let $\eta : \mathbb{R} \to \tilde{M}$ be a geodesic parametrized by arc length and let $Y_1(t), \ldots, Y_{n-1}(t)$ be orthonormal parallel vector fields along η, orthogonal to η'. Let moreover $J_1, \ldots, J_{n-1}$ be Jacobi fields along η, orthogonal to η', with the following initial conditions.

(1) $J_1(0) = Y_1(0)$, and $\frac{\nabla}{dt} J_1(0) = 0$.
(2) For $i = 2, \ldots, n-1$ we have $J_i(0) = 0$, and covariant derivatives $\frac{\nabla}{dt} J_i(t)|_{t=0} = Y_i(0)$.

Then $J(t) = (J_1(t), \ldots, J_{n-1}(t))$ can be viewed as an $(n-1, n-1)$-matrix with respect to the basis $Y_1(t), \ldots, Y_{n-1}(t)$. Define the matrix $A(t)$ by

$$\frac{\nabla}{dt} J(t) = A(t) J(t). \tag{3}$$

The matrix valued map $t \to A(t)$ satisfies the *Riccati equation*

$$A' + A^2 + R_{\eta'} = 0$$

where $R_{\eta'}$ is the curvature tensor of $\tilde{M}$ evaluated on η'. In particular, A is self-adjoint.

Using the notations from [12], for $t > 0$ we denote by $j(t)$ the determinant of the matrix $J(t)$.

Lemma 2.2. *For all $R > 0$ we have*

$$j(R) \geq (n-1) \int_0^R j(t) dt.$$

Proof. The proof follows from standard comparison. We use the above notations.

Let $\bar\eta:\mathbb{R}\to\mathbb{H}^n$ be a geodesic in the hyperbolic n-space $\mathbb{H}^n$. Let $\bar Y_1(t),\dots,\bar Y_{n-1}(t)$ be parallel vector fields along $\bar\eta$ such that for each t, $\bar Y_1(t),\dots,\bar Y_{n-1}(t)$ is an orthonormal basis of $\bar\eta'(t)^\perp$. Let $\bar J_i$ $(1\le i\le n-1)$ be the Jacobi fields along $\bar\eta$ defined by $\bar J_1(0)=\bar Y_1(0),\frac{\nabla}{dt}\bar J_1(0)=0$, and for $i\ge 2$ we require that $\bar J_i(0)=0$ and $\frac{\nabla}{dt}\bar J_i(0)=\bar Y_i(0)$. Write $\bar J(t)=(\bar J_1(t),\dots,\bar J_{n-1}(t))$ and view this as a matrix with respect to the basis $\bar Y_1(t),\dots,\bar Y_{n-1}(t)$ of $\bar\eta'(t)^\perp$.

The Jacobi fields $\bar J_i$ can explicitly be computed as follows. We have $\bar J_1(t)=\cosh(t)\bar Y_1(t)$, and $\bar J_i(t)=\sinh(t)\bar Y_i(t)$ for $i\ge 2$. In particular, if we denote by $\bar j(t)$ the determinant of $\bar J(t)$ then

$$\bar j(t)=\sinh^{n-2}(t)\cosh(t).$$

Thus $\bar j'(t)=(n-2)\sinh^{n-3}(t)\cosh^2(t)+\sinh^{n-1}(t)$ and hence

$$\bar j'(t)=((n-2)\coth(t)+\tanh(t))\bar j(t).$$

Using the notations from the lemma and the text preceding it, Theorem 3.2, Theorem 3.4 and Section 6.1 of [11] show that

$$j'(t)/j(t)\ge \bar j'(t)/\bar j(t) \tag{4}$$

for all $t>0$.

Write $b(t)=\frac{1}{n-1}\sinh^{n-1}(t)$; then $b(t)=\int_0^t\bar j(s)ds$ and

$$\frac{d}{dt}\log b(t)=\frac{b'(t)}{b(t)}=(n-1)\coth(t)>n-1 \tag{5}$$

for all t.

For $a(t)=\int_0^t j(s)ds$ we have $a'(t)=j(t)$. By the estimate (5), it now suffices to show that

$$\frac{d}{dt}\log a(t)=\frac{a'(t)}{a(t)}\ge\frac{d}{dt}\log b(t)$$

for all $t>0$.

Let $\delta>0$. It suffices to show that for all $t>0$ we have $\frac{d}{dt}\log a(t)\ge(1-\delta)\frac{d}{dt}\log b(t)$. To this end note that by comparison, the inequality holds true for all small t (see [11] for details). As this condition is closed, if the inequality does not hold for all t then there is a number $T_0>0$ so that the estimate holds true for $t\le T_0$, and it is violated for $T_0<t<T_0+\tau$ where $\tau>0$. Then we have $\frac{d}{dt}\log a(T_0)=(1-\delta)\frac{d}{dt}\log b(T_0)$,

Now

$$\frac{d^2}{dt^2}\log a=\frac{d}{dt}\frac{a'}{a}=\frac{a''a-(a')^2}{a^2}=\frac{a''}{a}-(\frac{a'}{a})^2=\frac{a'}{a}(\frac{a''}{a'}-\frac{a'}{a}).$$

Inequality (4) implies that $\frac{a''}{a'}(T_0)\ge\frac{b''}{b'}(T_0)$. As $\frac{a'}{a}(T_0)=(1-\delta)\frac{b'}{b}(T_0)$, we conclude that

$$\frac{d^2}{dt^2}\log a(T_0)>(1-\delta)\frac{d^2}{dt^2}\log b(T_0).$$

Using Taylor expansion, we deduce that $\frac{a'}{a}(s) \geq (1-\delta)\frac{b'}{b}(s)$ for all $s > T_0$ which are sufficiently close to T_0. This contradicts the choice of T_0. Since $\delta > 0$ was arbitrary, the lemma follows. □

A *cusp* T is an unbounded component of M_{thin}. It is homeomorphic to $N \times [0,\infty)$ where N is a closed manifold of dimension $n-1$. The manifold N is homeomorphic to the quotient of a horosphere in the universal covering $\tilde{M}$ of M by a parabolic subgroup of the isometry group of $\tilde{M}$. As before, the boundary of T need not be smooth, but everything said so far for boundaries of Margulis tubes is also valid for cusps (see the remark after Theorem 2.14 in [5]). In particular, Theorem 2.1 holds true for boundaries of cusps.

A version of Lemma 2.2 is also valid for cusps and follows with exactly the same argument. Namely, let T be a cusp, with boundary ∂T. Write $T = \partial T \times [0,\infty)$ where $\partial T \times \{s\}$ is the hypersurface of distance s to ∂T. Let $\eta : [0,\infty) \to T$ be a radial geodesic and let $J(t) = (J_1(t), \dots, J_{n-1}(t))$ be Jacobi fields with the following properties. The vectors $J_1(0), \dots, J_{n-1}(0)$ define an orthonormal basis of $\eta'(0)^\perp$, and $\|J(t)\| \to 0$ $(t \to \infty)$. Denote by $j(t)$ the determinant of $J(t)$, viewed as a matrix with respect to an orthonormal basis of $\eta'(t)^\perp$ defined by parallel vector fields along η.

Let $\bar{j}(t)$ be the corresponding function for Jacobi fields defined by a horoball in hyperbolic n-space. Explicit calcuation yields $\bar{j}(t) = e^{-(n-1)t}$ and hence $\bar{j}(T) = (n-1)\int_T^\infty \bar{j}(s)ds$ for all T. The comparison argument from the proof of Lemma 2.2 together with the results in [11] imply that the inequality $j(T) \geq \int_T^\infty j(s)ds$ holds true for cusps in a manifold M of curvature ≤ -1. In fact, this statement can also be obtained as a limiting case of Lemma 2.2 by reparametrizing the radial geodesic arc η of the Margulis tube, choosing $\eta(R)$ as a basepoint, renormalizing the Jacobi fields with rescaling and the Gram-Schmidt procedure and letting R tend to infinity.

As one consequence of the above discussion, by possibly replacing M_{thick} by a neighborhood of uniformly bounded radius we may assume that

$$\frac{3}{2}\text{vol}(M_{\text{thick}}) \geq \text{vol}(M). \tag{6}$$

Consider again a Margulis tube T in M. Recall that the boundary ∂T of T equals the set $\{\rho = R\}$ which can be viewed as a graph over the unit normal bundle $N(\gamma)$ of γ for some smooth function $R : N(\gamma) \to (0,\infty)$. Here we use Theorem 2.1 to assure that the function R is smooth. We equip ∂T with the volume element determined by this description (this is in general not the volume element of ∂T viewed as a smooth hypersurface in M). By this we mean that the volume element of the hypersurface ∂T at a point $(\sigma_0, s_0, R(\sigma_0, s_0))$ equals the radial projection of the volume element of the local hypersurface $\{(\sigma, s, R(\sigma_0, s_0))\}$ passing through $(\sigma_0, s_0, R(\sigma_0, s_0))$. This makes sense since by Theorem 2.1, radial geodesics intersect ∂T transversely. Or, equivalently, this volume element is chosen in such a way that the Jacobian at (σ, s) of the map $(\sigma, s) \to (\sigma, s, R(s,\sigma))$ equals the Jacobian of the normal exponential map $(\sigma, s) \to \exp(R(\sigma, s)(\sigma, s))$, and this is just the function j from Lemma 2.2. The same construction is also valid for a cusp T, and we equip ∂T with the corresponding volume element obtained by projection of the volume element on local hypersurfaces orthogonal to the radial geodesics.

Lemma 2.3. *Let $T \subset M_{\rm thin}$ be a Margulis tube or a cusp with boundary ∂T, and let f be a smooth function on T with $\int_T f^2 \geq \int_{\partial T} f^2$; then*

$$\int_T f^2 \leq \frac{4}{(n-2)^2}\int_T \|\nabla f\|^2.$$

Proof. We begin with the case that T is a Margulis tube. Let $\gamma : [0,\ell) \to M$ be a parametrization of the core geodesic of T by arc length. We use normal exponential coordinates and Lemma 2.2. Let ρ be the radial distance from γ and let $j(\sigma, s, \rho)$ be the Jacobian of the normal exponential map at the point with "coordinates" (σ, s, ρ). Then we have

$$\int_T f^2 = \int_{S^{n-2}} d\sigma \int_0^\ell ds \int_0^{R(\sigma,s)} f^2 j(\sigma,s,\rho)d\rho.$$

Define $a(\sigma,s,\rho) = \int_0^\rho j(\sigma,s,u)du$. By Lemma 2.2 and the definition of the volume element on ∂T, integration by parts along the radial rays from γ yields

$$\int_T f^2 \leq \frac{1}{n-1}\int_{\partial T} f^2 - 2\int_{S^{n-2}} d\sigma \int_0^\ell ds \int_0^{R(\sigma,s)} ff'a(\sigma,s,\rho)d\rho$$

where f' is the derivative of f in direction of the radial variable ρ.

By the assumption in the lemma, we have

$$\int_{\partial T} f^2 \leq \int_T f^2$$

and therefore taking absolute values and using Lemma 2.2 once more, we obtain

$$\begin{aligned}\frac{n-2}{n-1}\int_T f^2 &\leq \frac{2}{n-1}\int_{S^{n-2}} d\sigma \int_0^\ell ds \int_0^R |ff'|j(\sigma,s,\rho)d\rho \\ &= \frac{2}{n-1}\int_T |ff'| \leq \frac{2}{n-1}(\int_T f^2)^{1/2}(\int_T \|\nabla f\|^2)^{1/2}\end{aligned}$$

where the last inequality follows from Schwarz's inequality and from $|f'| \leq \|\nabla f\|$ (compare the proof of Lemma 1 of [10]). Dividing by $\frac{2}{n-1}(\int_T f^2)^{1/2}$ and squaring the resulting inequality yields the lemma in the case that T is a Margulis tube.

The argument for a cusp is almost identical. Thus let T be such a cusp, with boundary ∂T. Then ∂T is an closed manifold, and T is diffeomorphic to $\partial T \times [0,\infty)$. Let $\rho : T \to [0,\infty)$ be the radial distance from ∂T.

Let $f : T \to \mathbb{R}$ be a smooth square integrable function with $\int_T f^2 \geq \int_{\partial T} f^2$ and let $j : \partial T \times [0,\infty) \to (0,\infty)$ be the function which describes the volume form on T as discussed above. The version of Lemma 2.1 for cusps and integration by parts is used as in the proof for Margulis tubes. Since f is square integrable, as $R \to \infty$ we have $\int_{\rho=R} f^2 \to 0$ and hence the same calculation as before yields (here $d\mu$ is the volume element on ∂T used in the lemma)

$$\int_T f^2 = \int_{\partial T} d\mu \int_0^\infty f^2 j(x,\rho) d\rho = \lim_{R\to\infty} \int_{\partial T} d\mu \int_0^R f^2 j(x,\rho) d(\rho)$$
$$\leq \frac{1}{n-1}\Big(\int_{\partial T} f^2 + 2\int_{\partial T} d\mu \int_0^\infty f f' j(x,\rho) d\rho\Big).$$

As before, this yields the required estimate. Note that the change of sign in this formula stems from the fact that here ρ is the radial distance from the boundary, while for Margulis tubes, ρ denoted the radial distance from the core geodesic which seems more natural in that context. □

For a Margulis tube T or a cusp, let $\hat{T}$ be the set of all points $x \in T$ whose radial distance from the boundary ∂T is at least one. Thus if T is Margulis tube, determined by a closed geodesic and a function $R(\sigma, s) > 0$ on the normal bundle of that geodesic, then using "coordinates" (σ, s, ρ) as before, we have $\hat{T} = \{(\sigma, s, \rho) \in T \mid \rho \leq R(\sigma, s) - 1\}$. By our assumption on Margulis tubes, $T - \hat{T}$ is diffeomorphic to $\partial T \times [0,1)$. Write $\hat{M}_{\text{thick}} = M - \cup \hat{T}$ where the union is over all Margulis tubes and cusps.

For a smooth square integrable function f on M denote by

$$\mathcal{R}(f) = \int_M \|\nabla f\|^2 / \int_M f^2$$

the Rayleigh quotient of f.

Lemma 2.4. *Let $f : M \to \mathbb{R}$ be a smooth square integrable function with Rayleigh quotient $\mathcal{R}(f) < \frac{(n-2)^2}{12}$; then*

$$\int_{\hat{M}_{\text{thick}}} f^2 \geq \frac{1}{3} \int_M f^2.$$

Proof. By our assumption on the components of the thin part of M, the set

$$A = \hat{M}_{\text{thick}} - M_{\text{thick}}$$

is a union of *shells*, i.e. submanifolds of M with boundary which either are diffeomorphic to $S^{n-2} \times S^1 \times [0,1]$ (if the component is a Margulis tube) or to $N \times [0,1]$ (if the component is a cusp), where N is a quotient of a horosphere by a rank $n-1$ parabolic subgroup of the isometry group of the universal covering $\tilde{M}$ of M.

Let $f : M \to \mathbb{R}$ be a smooth function with $\mathcal{R}(f) < \frac{(n-2)^2}{12}$. We want to show that

$$\int_{\hat{M}_{\text{thick}}} f^2 \geq \frac{1}{3} \int_M f^2,$$

and to this end we assume to the contrary that $\int_{\hat{M}_{\text{thick}}} f^2 < \frac{1}{3} \int_M f^2$.

Recall that $M_{\rm thin}$ is a disjoint union of a finite number of Margulis tubes and cusps, say $M_{\rm thin} = \cup_{i=1}^k T_i$. For each of these tubes and cusps T_i, let r_i be the radial distance function to the boundary hypersurface, i.e. $r_i(x)$ is the length of the radial arc connecting the point $x \in T$ to ∂T. By reordering we may assume that there exists a number $p \leq k$ such that for all $i \leq p$, there is some $s_i \leq 1$ such that

$$\int_{\{r_i=s_i\}\cap T_i} f^2 \leq \int_{T_i\cap\{r_i\geq s_i\}} f^2$$

and that for $i > p$, such an s_i does not exist. Here we use the volume element on the hypersurfaces $\{r_i = s_i\}$ as in Lemma 2.3.

We distinguish two cases. In the first case, $\sum_{i=1}^p \int_{T_i - A} f^2 \geq \frac{1}{3}\int_M f^2$. Lemma 2.3 then shows that

$$\int_M \|\nabla f\|^2 \geq \sum_{i=1}^p \int_{T_i\cap\{r_i\geq s_i\}} \|\nabla f\|^2$$
$$\geq \frac{(n-2)^2}{4}\sum_{i=1}^p \int_{T_i\cap\{r_i\geq s_i\}} f^2 \geq \frac{(n-2)^2}{12}\int_M f^2.$$

Thus we have $\mathcal{R}(f) \geq \frac{(n-2)^2}{12}$ which contradicts our assumption on f.

In the second case, we have $\sum_{i=1}^p \int_{T_i-A} f^2 < \frac{1}{3}\int_M f^2$. Then

$$\sum_{i=p+1}^k \int_{T_i-A} f^2 \geq \frac{1}{3}\int_M f^2. \tag{7}$$

But for each $i > p$, integration of the defining equation

$$\int_{T_i\cap\{r_i=s\}} f^2 \geq \int_{T_i\cap\{r_i\geq s\}} f^2 \tag{8}$$

over the shell $T_i \cap A = \{0 \leq r_i \leq 1\}$ yields

$$\int_0^1 ds \int_{T_i\cap\{r_i=s\}} f^2 = \int_{T_i\cap A} f^2 \geq \int_{T_i-A} f^2.$$

Summing over $i \geq p+1$ and using inequality (7), we obtain

$$\int_{\cup_{i=p+1}^k T_i\cap A} f^2 \geq \sum_{i=p+1}^k \int_{T_i-A} f^2 \geq \frac{1}{3}\int f^2.$$

As $A \subset \hat{M}_{\rm thick}$, this contradicts the assumption on f. The lemma follows. □

The next proposition completes the proof of Theorem 1 for a choice of a Margulis constant so that $\hat{M}_{\rm thick}$ as defined above is contained in the thick part of M for this constant.

Proposition 2.5. *For all $k \geq 0$ we have*

$$\lambda_k(M) \geq \min\{\frac{1}{3}\lambda_k(\hat{M}_{\text{thick}}), \frac{(n-2)^2}{12}\}$$

for any finite volume oriented Riemannian manifold M of dimension $n \geq 3$ and curvature $\kappa \in [-b^2 - 1]$.

Proof. Let M be a finite volume oriented Riemannian manifold of dimension $n \geq 3$ and curvature $\kappa \leq -1$. We use the previous notations.

If M is non-compact, then by Corollary 3.2 of [14], the bottom of the essential spectrum of M is not smaller than $(n-1)^2/4$. Thus the intersection of the spectrum of M with the interval $[0, \frac{(n-2)^2}{12}]$ consists of a finite number of eigenvalues.

Let $\mathcal{H}(M)$ be the Sobolev space of square integrable functions on M, with square integrable weak derivatives. Let $k > 0$ be such that $\lambda_k(M) < (n-2)^2/12$. Let $m \leq k-1$ be the largest number so that $\lambda_m(M) < \lambda_k(M)$. Note that as we count eigenvalues with multiplicities, we may have $m < k-1$. Choose a $k-m$-dimensional subspace $E_k \subset \mathcal{H}(M)$ of the eigenspace for the eigenvalue $\lambda_k(M)$ and define $E = V \oplus E_k$ where V is the direct sum of the eigenspaces for eigenvalues strictly smaller than $\lambda_k(M)$. In particular, E is a $k+1$-dimensional linear subspace of the Hilbert space $\mathcal{H}(M)$.

By construction, $\hat{M}_{\text{thick}}$ is a smooth manifold with smooth boundary. Denote by $\mathcal{H}(\hat{M}_{\text{thick}})$ the Sobolev space of square integrable functions on $\hat{M}_{\text{thick}}$ with square integrable weak derivatives. Here the weak derivative of a function f on $\hat{M}_{\text{thick}}$ is a vector field Y so that

$$\int_{\hat{M}_{\text{thick}}} \langle Y, X\rangle = -\int_{\hat{M}_{\text{thick}}} f\text{div}(X)$$

for all smooth vector fields X on $\hat{M}_{\text{thick}}$ with compact support in the interior of $\hat{M}_{\text{thick}}$. Green's formula implies that $\mathcal{H}(\hat{M}_{\text{thick}})$ contains all functions f on $\hat{M}_{\text{thick}}$ which are smooth up to and including the boundary.

As smooth functions are dense in $\mathcal{H}(M)$ and $\mathcal{H}(\hat{M}_{\text{thick}})$, there is a natural linear one-Lipschitz restriction map $\Pi : \mathcal{H}(M) \to \mathcal{H}(\hat{M}_{\text{thick}})$. We denote by W the image of the linear subspace E under Π.

We claim that $\dim(W) = k+1$. To this end assume otherwise. Then there is a normalized function $f \in E$ (i.e. $\int_M f^2 = 1$) whose restriction to $\hat{M}_{\text{thick}}$ vanishes. But by the definition of E, the Rayleigh quotient $\mathcal{R}(f)$ of f is at most $\lambda_k(M) < (n-2)^2/12$ and therefore Lemma 2.4 shows that $\int_{\hat{M}_{\text{thick}}} f^2 \geq \frac{1}{3}\int_M f^2$.

As $\dim(W) = k+1$ and as $\mathcal{H}(\hat{M}_{\text{thick}})$ is the function space for $\hat{M}_{\text{thick}}$ with Neumann boundary conditions (see p.14-17 in [6]), Rayleigh's theorem shows that there exists a normalized function $f \in E$ with $\mathcal{R}(\Pi(f)) = \mathcal{R}(f|\hat{M}_{\text{thick}}) \geq \lambda_k(\hat{M}_{\text{thick}})$. Furthermore, as $f \in E$, we have $\mathcal{R}(f) \leq \lambda_k(M)$. Note that f and $\Pi(f)$ are smooth.

By Lemma 2.4, we have $\int_{\hat{M}_{\rm thick}} f^2 \geq \frac{1}{3}\int_M f^2 = \frac{1}{3}$ and hence

$$\lambda_k(M) \geq \mathcal{R}(f) \geq \int_{\hat{M}_{\rm thick}} \|\nabla f\|^2 \geq \frac{1}{3}\mathcal{R}(f|_{\hat{M}_{\rm thick}}) \geq \frac{1}{3}\lambda_k(\hat{M}_{\rm thick}).$$

This is what we wanted to show. □

As an easy consequence, we obtain the estimate of Schoen [19].

Corollary 2.6. *For every $n \geq 3$ and every $b \geq 1$ there exists a number $\chi(n,b) > 0$ such that*

$$\lambda_1(M) \geq \frac{\chi(n,b)}{\mathrm{vol}(M)^2}$$

for every finite volume Riemannian manifold M of dimension n and curvature $\kappa \in [-b^2,-1]$.

Proof. Let $\varepsilon > 0$ be a Margulis constant for Riemannian manifolds M of dimension $n \geq 3$ and curvature in the interval $[-b^2,-1]$. As $M_{\rm thick} \neq \emptyset$, the manifold M contains an embedded ball of radius ε which is isometric to a ball of the same radius in a simply connected manifold of curvature contained in $[-b^2,-1]$. By comparison, the volume of such a ball is bounded from below by a universal constant $a(n,b)$ only depending on n and b and hence the volume of every Riemannian n-manifold of curvature in $[-b^2,-1]$ is bounded from below by $a(n,b)$. Thus Theorem 1 shows that $\lambda_1(M) \geq \min\{a(n,b)^2(n-2)^2/12\mathrm{vol}(M)^2, \lambda_1(M_{\rm thick})/3\}$.

The manifold with boundary $M_{\rm thick}$ is uniformly quasi-isometric to a finite connected graph G, and its first nontrivial eigenvalue $\lambda_1(M_{\rm thick})$ with Neumann boundary conditions is uniformly equivalent to the first nontrivial eigenvalue of the graph Laplacian [17] (see also Lemma 2.1 of [16] for an explicit formulation of this fact).

Now the first eigenvalue $\lambda_1(G)$ of a finite graph G satisfies $\lambda_1(G) \geq h^2(G)/2$ where $h(G)$ is the so-called *Cheeger constant* of the graph [8]. This Cheeger constant is defined to be

$$\min \frac{|E(U,U')|}{\min\{|U|,|U'|\}}$$

where U is a subset of the vertex set $\mathcal{V}(G)$ of G, $U' = \mathcal{V}(G) - U$ and where $E(U,U')$ is the set of all edges which connect a vertex in U to a vertex in U'. As G is connected, this Cheeger constant is at least $1/|\mathcal{V}(G)| \sim 1/\mathrm{vol}(M)$ which yields the corollary. □

3. Bounding small eigenvalues from above

In this section we restrict to the investigation of oriented hyperbolic 3-manifolds of finite volume. Our goal is to prove Theorem 2.

We begin with analyzing in more detail functions on Margulis tubes in such a hyperbolic 3-manifold M. For a sufficiently small Margulis constant $\varepsilon > 0$, the boundary ∂T of each such tube T is a flat torus whose injectivity radius for the induced metric roughly equals ε [5]. Furthermore, radial geodesics intersect ∂T orthogonally. Note that an analogous statement is not true in higher dimensions.

The *radius* $\mathrm{rad}(T)$ of the tube T is the distance of ∂T to the core geodesic γ. The following lemma is only valid in dimension three.

Lemma 3.1. *There is a number $q_1 = q_1(\varepsilon) > 0$ only depending on ε such that* $\mathrm{rad}(T) \geq \log \mathrm{vol}(\partial T) - q_1$.

Proof. Write $R = \mathrm{rad}(T)$. We may assume that $R > 1$. The tube T is isometric to the quotient of the tubular neighborhood $N(\tilde{\gamma}, R)$ of radius R of a geodesic $\tilde{\gamma}$ in $\mathbb{H}^3$ under an infinite cyclic group of loxodromic isometries. Up to conjugation, the generator ψ of this group is determined by its complex translation length $\chi \in \mathbb{C}$ with $\Re(\chi) = \ell < 2\varepsilon$. Here ℓ is the length of the core geodesic of T.

The boundary $\partial N(\tilde{\gamma}, R)$ of $N(\tilde{\gamma}, R)$ is a flat two-sided infinite cylinder of circumference $2\pi \sinh R$. For a fixed identification of the fibre of the unit normal circle bundle of $\tilde{\gamma}$ over the point $\tilde{\gamma}(0)$ with the unit circle S^1, parallel transport along $\tilde{\gamma}$ and the normal exponential map determine global "coordinates" on $\partial N(\tilde{\gamma}, R)$. These "coordinates" consist in a diffeomorphism $S^1 \times \mathbb{R} \to \partial N(\tilde{\gamma}, R)$. The isometry ψ identifies the meridian $\exp(RS^1 \times \{0\})$ with the meridian $\exp(RS^1 \times \{\ell\})$ by an isometry which is given by rotation with angle $\Im\chi$ in these coordinates. In particular, the volume of the boundary torus ∂T of T equals

$$\mathrm{vol}(\partial T) = 2\pi\ell \sinh R \cosh R \tag{9}$$

and is independent of $\Im\chi$.

The second fundamental form of the torus ∂T is uniformly bounded, independent of $R > 1$. This implies that the length of the shortest geodesic on ∂T is contained in an interval $[\varepsilon, a\varepsilon]$ where $a > 0$ is a universal constant (compare Theorem 2.14 and the following discussion in [5]). As we are only interested in tubes of large radius, we may assume that the length $2\pi \sinh R$ of the meridian of ∂T is bigger than $a\varepsilon$.

Now the distance on $\partial N(\tilde{\gamma}, R)$ for the intrinsic path metric between the circles corresponding to the coordinates $s = 0, s = \ell$ equals $\ell \cosh R$. Here as before, s is the length parameter of γ. A shortest closed geodesic on the flat torus ∂T lifts to a straight line segment on $\partial N(\tilde{\gamma}, R)$. Since $2\pi \sinh R > a\varepsilon$, this line segment connects the circle $\{s = 0\}$ to the circle $\{s = k\ell\}$ for some integer $k \geq 1$. From this we conclude that

$$\ell \cosh R \leq a\varepsilon.$$

From the formula (9) for $\mathrm{vol}(\partial T)$, we deduce that $\sinh R \geq \frac{1}{2\pi a\varepsilon} \mathrm{vol}(\partial T)$ and hence

$$e^R \geq \frac{1}{\pi a\varepsilon} \mathrm{vol}(\partial T)$$

which is what we wanted to show. □

Remark 3.2. For hyperbolic manifolds of dimension $n \geq 4$, Proposition 2 of [3] states a reverse inequality: Namely, if U is any Margulis tube, and if $r(U)$ is the largest distance of a point in U to the boundary, then $\mathrm{Vol}(U) \geq d_n \sinh(\frac{1}{3}r(U))$ where $d_n > 0$ is a constant only depending on n. The case $\dim(M) = 3$ is very special as every point on the boundary of a Margulis tube has the same distance to the core geodesic. We refer to p.3 of [5] for more information.

In the statement of the next proposition, we use the fact that given a Margulis tube T of radius at least three, the volume of T is uniformly proportional to the volume of its boundary ∂T.

Proposition 3.3. *There exists a number $q_2 = q_2(\varepsilon) > 0$ with the following property. Let $T \subset M$ be a Margulis tube or a cusp with boundary ∂T. Let $f : \partial T \to \mathbb{R}$ be a function (not neccessarily of zero mean) whose Rayleigh quotient equals $d \geq 0$. Then there is an extension of f to a smooth function F on T with the following properties.*

(1) $\frac{1}{4}\int_{\partial T} f^2 \leq \int_T F^2 \leq \frac{1}{2}\int_{\partial T} f^2$.
(2) *The Rayleigh quotient of F is at most $dq_2 \log \mathrm{vol}(T)$.*
(3) *If $\int_{\partial T} f = 0$ then $\int_T F = 0$.*

Proof. Let γ be the core curve of the tube T. Use the coordinates (σ, s, ρ) on T, given by the angular coordinate σ on the unit normal circle over a point in γ, the length parameter s on γ and the distance ρ from γ. If $R > 0$ is the radius of T then the boundary ∂T of T is the surface $\rho = R$. This boundary is a flat torus whose injectivity radius equals ε up to a universal multiplicative constant.

By Lemma 3.1, the radius R of the tube satisfies

$$-\theta = \log \mathrm{vol}(\partial T) - q_1 - 1 - R \leq -1$$

where $q_1 > 0$ is a universal constant. By the inequality (2) in Section 2, we may assume that $R - \theta \geq 3$.

Let $f : \partial T = \{\rho = R\} \to \mathbb{R}$ be a smooth function with Rayleigh quotient $d \geq 0$. Decompose $f = f_0 + g$ where f_0 is a constant function and $\int_{\partial T} g = 0$. Extend f_0 to a constant function F_0 on T, and extend the function g to a function $G : T \to \mathbb{R}$ by

$$G(\sigma, s, \rho) = \begin{cases} g(\sigma, s, R) & \text{if } \theta + 1 \leq \rho \leq R; \\ (\rho - \theta) g(\sigma, s, R) & \text{if } \theta \leq \rho \leq \theta + 1; \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Then $F = F_0 + G$ is continuous, smooth away from the hypersurfaces $\rho = \theta + 1$ and $\rho = \theta$, with uniformly bounded derivative. Standard Sobolev theory then implies that F is contained in the Sobolev space of square integrable functions with square integrable weak derivative, and $F|\partial T = f$.

As for each $r \in (0, R)$ the radial projection $(\sigma, s, r) \to (\sigma, s, R)$ of the torus $\{\rho = r\}$ onto the boundary torus ∂T is a homothety, with dilation

$\sinh(R)\cosh(R)/\sinh(r)\cosh(r)$, we have $\int_T G = 0$. Furthermore, integration using the explicit description of the metric on T as a warped product metric (see Section 2 for details) yields that

$$\int_T (F_0+G)^2 \in [\frac{1}{4}\int_{\partial T}(f+g)^2, \frac{1}{2}\int_{\partial T}(f+g)^2]. \tag{11}$$

Namely, as $(F_0+G)^2(\sigma,s,\rho) \leq f^2(\sigma,s,R)$ for all ρ, the upper bound follows from

$$\int_T (F_0+G)^2 \leq \int_{\partial T} f^2 \int_\theta^R \frac{\sinh\rho\cosh\rho}{\sinh R\cosh R}d\rho = \frac{\sinh^2(R)-\sinh^2(\theta)}{2\sinh R\cosh R}\int_{\partial T} f^2.$$

To establish the lower bound, simply note that

$$\begin{aligned}\int_T (F_0+G)^2 &\geq \int_{\partial T} f^2 \int_{\theta+1}^R \frac{\sinh\rho\cosh\rho}{\sinh R\cosh R}d\rho \\ &= \frac{\sinh^2(R)-\sinh^2(\theta+1)}{2\sinh R\cosh R}\int_{\partial T} f^2.\end{aligned}$$

Up to adjusting the requirement on the radius of the Margulis tube, we may assume that $\sinh^2(\theta+1) \leq \frac{1}{4}\sinh(R)^2$ and $\frac{3}{4}\sinh(R) \geq \frac{1}{2}\cosh(R)$ which then results in the estimate stated in the first part of the proposition.

Now $\int_T \|\nabla(F_0+G)\|^2 = \int_T \|\nabla G\|^2$ and therefore for the second part of the statement in the proposition, it suffices to show that

$$\int_T \|\nabla G\|^2 \leq c_1(R-\theta)\int_{\partial T}\|\nabla g\|^2 \tag{12}$$

for a universal constant $c_1 > 0$. Namely, the discussion in the previous paragraph shows that the integral of the constant function one over the shell $\theta \leq \rho \leq R$ is bounded from below by $\frac{1}{4}\mathrm{vol}(\partial T)$. In particular, we have $\log\mathrm{vol}(T) \geq \log\mathrm{vol}(\partial T) - \log 4$ where $b > 0$ is a universal constant. On the other hand, by the choice of θ, we also have $R-\theta = \log\mathrm{vol}(\partial T) - q_1 - 1$.

To establish the estimat (12) recall from Chapter 2 of [6] that the first nonzero eigenvalue of ∂T is not smaller than $c_2/\mathrm{vol}(\partial T)^2$ where $c_2 > 0$ is a universal constant (here we use that the injectivity radius of ∂T is at least ε). In particular, by the definition of the number θ, this eigenvalue is not smaller than $c_3 e^{-2(R-\theta)}$ where $c_3 > 0$ is a universal constant. As a consequence, we have

$$\int_{\partial T}\|\nabla g\|^2 \geq c_3 e^{-2(R-\theta)}\int_{\partial T} g^2. \tag{13}$$

Now for $r \in [\theta+1, R]$, the radial projection of the torus $\{\rho = r\}$ onto ∂T scales the metric with a fixed constant. Moreover, the directional derivative of G in direction of the radial vector field vanishes. This implies that

$$\int_{\rho=r}\|\nabla G(\sigma,s,r)\|^2 = \int_{\partial T}\|\nabla G(\sigma,s,R)\|^2$$

for $r \in [\theta+1, R]$ and hence

$$\int_{\theta+1\leq\rho\leq R} \|\nabla G(s,t,\rho)\|^2 = (R-\theta-1)\int_{\partial T}\|\nabla g\|^2.$$

On the other hand, if $\theta < r < \theta+1$ then

$$\int_{\rho=r}\|\nabla G(s,t,\rho)\|^2 \leq \int_{\partial T}\|\nabla G(s,t,R)\|^2 + \int_{\rho=r} G(s,t,\rho)^2.$$

Here the second term in this inequality is the contribution of the derivative of G in direction of the radial vector field, and we use that this vector field is normal to the hypersurfaces $\{\rho = r\}$.

There is a universal constant $c_4 > 0$ such that for each $r \in [\theta, \theta+1]$ the volume of the level surface $\{\rho = r\}$ is not bigger than $c_4 e^{-2(R-\theta)}(\text{vol}(\partial T))$. Integration of this inequality over the interval $[\theta, \theta+1]$ yields $\int_{\theta\leq\rho\leq\theta+1} G^2 \leq c_4 e^{-2(R-\theta)}\int_{\partial T} g^2$. Hence by the estimate (13),

$$\int_{\theta\leq\rho\leq\theta+1} G^2 \leq \frac{c_4}{c_3}\int_{\partial T}\|\nabla g\|^2.$$

Together this implies

$$\int_T \|\nabla G\|^2 \leq (R-\theta+\frac{c_4}{c_3})\int_{\partial T}\|\nabla g\|^2. \tag{14}$$

The estimates (11,14) together with the choice of θ show part (1) and (2) of the proposition. The third part is immediate from the fact that by construction, if $\int_{\partial T} f = 0$ then $F = G$ and $\int_T G = 0$.

The case that T is a cusp is completely analogous but easier and will be omitted. □

As an immediate consequence of Proposition 3.3 we obtain

Corollary 3.4. *The first nonzero eigenvalue with Neumann boundary conditions of a three-dimensional hyperbolic Margulis tube or a cusp is at most* $q_3 \log \text{vol}(T)/\text{vol}(\partial T)^2$ *where* $q_3 > 0$ *is a universal constant.*

Proof. By the discussion in Chapter II, Section 2 of [6] and the fact that the injectivity radius of the flat torus ∂T is proportional to ε, there exists a universal number $a > 0$ and a function $f : \partial T \to \mathbb{R}$ with $\int_{\partial T} f = 0$ and

$$\int_{\partial T}\|\nabla f\|^2 \leq a\int_{\partial T} f^2/\text{vol}(\partial T)^2.$$

By Proposition 3.3, the function f can be extended to a function $F : T \to \mathbb{R}$ with $\int_T F = 0$ and Rayleigh quotient $\mathcal{R}(f) \leq q_2 a \log \text{vol}(T)/\text{vol}(\partial T)^2$. Now F is smooth up to and including the boundary, and therefore by Rayleigh's principle as explained in Chapter 1 Section 5 of [6], this implies that the first non-zero eigenvalue of T with Neumann boundary conditions is bounded from above by $q_2 a \log \text{vol}(T)/\text{vol}(\partial T)^2$ as claimed in the corollary. □

The strategy for the proof of Theorem 2 from the introduction consists in extending an eigenfunction f on $\hat{M}_{\rm thick}$ to a function on M using the construction in Proposition 3.3. But the control on square integrals and Rayleigh quotients established in Proposition 3.3 depends on a control of these data for the restriction of f to the boundary of $\hat{M}_{\rm thick}$, and there is no apparent relations to the global square norm and the global Rayleigh quotient of f. Note also that there is no useful Harnack inequality for the restriction of an eigenfunction on $\hat{M}_{\rm thick}$ to its boundary.

To overcome this difficulty we establish some a-priori control on the restriction of eigenfunctions for small eigenvalues to sufficiently large tubular neighborhoods of the boundary of $\hat{M}_{\rm thick}$- the price we have to pay is that we have to decrease the Margulis constant which appears in Theorem 2.

We begin with establishing a modified version of Lemma 2.3. We only formulate this lemma for hyperbolic 3-manifolds although it is valid for arbitrary finite volume manifolds of curvature contained in $[-b^2,-1]$, where integration over radial distance hypersurfaces has to be interpreted as in Section 2.

Define a *shell* in a Margulis tube or cusp T to be a subset of T of the form $\{s\leq\rho\leq t\}$ where ρ is up to an additive constant the *negative* of the radial distance from the boundary of T and $s<t$ (using the negative of the radial distance is for convenience here). The *height* of the shell equals $t-s$. We always write a shell N in the form $N=V\times[0,k]$ where k is the height of the shell, the manifold V is diffeomorphic to the boundary ∂T of the tube or cusp, the real parameter t is the radial distance from the boundary component $V\times\{0\}$, and the boundary component $V\times\{k\}$ is closer to the boundary ∂T of the tube or cusp. As before, we denote by $\mathcal{R}(f)$ the Rayleigh quotient of a function f.

Lemma 3.5. (1) *Let $k\geq 2$ and let $N=V\times[0,k]$ be a shell of height k in a Margulis tube or cusp T in a hyperbolic 3-manifold. Let $f:N\to\mathbb{R}$ be a function which is smooth up to and including the boundary. If $\int_{V\times[0,1]}f^2\geq 3\int_{V\times[k-1,k]}f^2$ then $\mathcal{R}(f)\geq\frac{1}{3}$.*

(2) *Let f be an eigenfunction on $\hat{M}_{\rm thick}$ with Neumann boundary conditions for an eigenvalue $\lambda<1$. Let $\tau>0$ be sufficiently small that the closed tubular neighborhood of radius τ of the boundary $\partial\hat{M}_{\rm thick}$ in $\hat{M}_{\rm thick}$ is a shell, diffeomorphic to $\partial\hat{M}_{\rm thick}\times[0,\tau]$. Then $\int_{\partial\hat{M}_{\rm thick}\times\{\tau\}}f^2\geq\int_{\partial\hat{M}_{\rm thick}\times\{0\}}f^2$.*

Proof. We begin with the proof of the first part of the lemma. Define

$$s=\min\{t\in[0,1]\mid\int_{V\times\{t\}}f^2\geq\frac{1}{3}\int_{V\times[0,1]}f^2\}\text{ and}$$
$$u=\max\{t\in[k-1,k]\mid\int_{V\times\{t\}}f^2\leq\int_{V\times[k-1,k]}f^2\}.$$

Then

$$\int_{V\times[0,s]} f^2 \leq \frac{1}{3}\int_{V\times[0,1]} f^2 \leq \frac{1}{3}\int_N f^2,$$

in particular, by the assumption on f in the first part of the lemma, we have

$$\int_{V\times[s,u]} f^2 \geq \int_N f^2 - \frac{1}{3}\int_{V\times[0,1]} f^2 - \int_{V\times[k-1,k]} f^2$$

$$\geq \int_N f^2 - \frac{2}{3}\int_{V\times[0,1]} f^2 \geq \frac{1}{3}\int_N f^2.$$

Let us compute the derivative of the function $a(t) = \int_{V\times\{t\}} f^2$. Recall that the radial projection $(x,t)\in V\times\{t\}\to(x,u)\in V\times\{u\}$ is a homothety which scales the Lebesgue measure of the flat torus $\rho=t$ by a factor $b(t,u)$, with $\frac{d}{dt}b(t,u)|_{u=t}=c(t)\geq 2$ (in fact, if T is a cusp then $c(t)\equiv 2$, and if T is a Margulis tube and if the radial distance of $V\times\{t\}$ from the core geodesic equals R, then $c(t)=(\cosh^2(R)+\sinh^2(R))/\sinh(R)\cosh(R)\geq 2$, compare the proof of Lemma 2.3). We obtain

$$\frac{d}{dt}\int_{V\times\{t\}} f^2 = c(t)\int_{V\times\{t\}} f^2 + 2\int_{V\times\{t\}} f\nu(f) \tag{15}$$

where ν is the outer normal field of the hypersurface $V\times\{t\}$.

Thus as $\int_{V\times\{u\}} f^2 \leq \int_{V\times[k-1,k]} f^2 \leq \int_{V\times\{s\}} f^2$, we conclude that

$$\int_{V\times\{u\}} f^2 - \int_{V\times\{s\}} f^2 = \int_{V\times[s,u]} c(t)f^2 + 2f\nu(f) \leq 0 \tag{16}$$

and hence

$$\int_{V\times[s,u]} c(t)f^2 \leq 2\left|\int_{V\times[s,u]} f\nu(f)\right|.$$

Using $c(t)\geq 2$ for all t and $\nu(f)^2\leq\|\nabla f\|^2$ and applying the Schwarz inequality as before, we deduce

$$\int_{V\times[s,u]} f^2 \leq \int_{V\times[s,u]} \|\nabla f\|^2 \leq \int_N \|\nabla f\|^2. \tag{17}$$

As $\int_{V\times[s,u]} f^2 \geq \frac{1}{3}\int_N f^2$, this provides the first statement in the lemma.

To show the second statement in the lemma, let f be an eigenfunction on $\hat{M}_{\rm thck}$ with Neumann boundary conditions and eigenvalue $\lambda<1$ and assume to the contrary that there is some $\tau>0$ so that the tubular neighborhood of radius τ about the boundary $\partial\hat{M}_{\rm thick}$ of $\hat{M}_{\rm thick}$ is a shell diffeomorphic to $\partial\hat{M}_{\rm thick}\times[0,\tau]$ and such that

$$\delta = \int_{\partial\hat{M}_{\rm thick}\times\{0\}} f^2 - \int_{\partial\hat{M}_{\rm thick}\times\{\tau\}} f^2 > 0. \tag{18}$$

Let $t_0 \in (0, \tau]$ be the smallest number for which the equation (18) holds true, with this number δ. We then have

$$\frac{d}{dt}\int_{\partial\hat{M}_{\text{thick}}\times\{t\}} f^2|_{t=t_0} \leq 0.$$

By formula (15), this implies that $\int_{\partial\hat{M}_{\text{thick}}\times\{t_0\}} f\nu(f) < 0$.

On the other hand, as f is an eigenfunction for the eigenvalue λ, with Neumann boundary conditions, Green's formula yields that

$$\int_{\partial\hat{M}_{\text{thick}}\times[0,t_0]} f\Delta f = -\lambda\int_{\partial\hat{M}_{\text{thick}}\times[0,t_0]} f^2$$
$$= \int_{\partial\hat{M}_{\text{thick}}\times\{t_0\}} f\nu(f) - \int_{\partial\hat{M}_{\text{thick}}\times[0,t_0]} \|\nabla f\|^2.$$

Thus $\int_{\partial\hat{M}_{\text{thick}}\times[0,t_0]} \|\nabla f\|^2 \leq \lambda\int_{\partial\hat{M}_{\text{thick}}\times[0,t_0]} f^2$. As $\lambda < 1$ by assumption, this violates the first inequality in the formula (17). Note that this inequality applies since by our setup, the estimate in inequality (16) holds true for $u = t_0$ and $s = 0$. □

Corollary 3.6. *Let M be a hyperbolic 3-manifold and let $\hat{M}_{\text{thick}}$ be the tubular neighborhood of radius* 96 *about M_{thick}. Let f be an eigenfunction for Δ on $\hat{M}_{\text{thick}}$ with respect to the eigenvalue $\lambda < 1/96$, with Neumann boundary conditions. Let N be the tubular neighborhood of radius one about $\partial\hat{M}_{\text{thick}}$ in $\hat{M}_{\text{thick}}$; then*

$$\int_N f^2 \leq \frac{1}{32}\int_{\hat{M}_{\text{thick}}} f^2.$$

Proof. Let f be an eigenfunction on $\hat{M}_{\text{thick}}$ with respect to the eigenvalue $\lambda < 1/96$. Parametrize $\hat{M}_{\text{thick}} - M_{\text{thick}} = W$ as $W = V \times [0, 96]$ where $\partial\hat{M}_{\text{thick}} = V \times \{0\}$. With this notation, we have $N = V \times [0, 1]$.

We distinguish two cases. In the first case, there exists some $k \in [1, 95]$ so that $\int_{V\times[0,1]} f^2 \geq 3\int_{V\times[k,k+1]} f^2$. By the first part of Lemma 3.5, we have $\mathcal{R}(f|_{V\times[0,k+1]}) \geq 1/3$. This implies that

$$\int_{\hat{M}_{\text{thick}}} \|\nabla f\|^2 \geq \int_{V\times[0,k+1]} \|\nabla f\|^2 \geq \frac{1}{3}\int_{V\times[0,k+1]} f^2.$$

But f is an eigenfunction for Δ with respect to the eigenvalue $\lambda < 1/96$, with Neumann boundary conditions. Thus $\mathcal{R}(f) \leq 1/96$, and

$$\int_{\hat{M}_{\text{thick}}} \|\nabla f\|^2 \leq \frac{1}{96}\int_{\hat{M}_{\text{thick}}} f^2.$$

These two estimates together yield

$$\int_{V\times[0,1]} f^2 \leq \int_{V\times[0,k+1]} f^2 \leq \frac{1}{32}\int_{\hat{M}_{\text{thick}}} f^2$$

as claimed.

In the second case, we have $\int_{V\times[0,1]} f^2 \leq 3\int_{V\times[k,k+1]} f^2$ for all $k\leq 95$. But this implies as before that

$$\int_{V\times[0,1]} f^2 \leq \frac{1}{32}\int_{V\times[0,96]} f^2 \leq \frac{1}{32}\int_{\hat{M}_{\rm thick}} f^2$$

as claimed. □

As another consequence, we obtain Theorem 2 from the introduction.

Proposition 3.7. *There is a constant $q_4>0$ with the following property. Let M be a finite volume oriented hyperbolic 3-manifold; then for a suitable choice of a Margulis constant, we have*

$$\lambda_k(M)\leq q_4(\log {\rm vol}(M_{\rm thin})+2)\lambda_k(M_{\rm thick})$$

for all k so that $\lambda_k(M_{\rm thick})<1/96$.

Proof. Let M be a finite volume oriented hyperbolic 3-manifold. Denote by $\hat{M}_{\rm thick}$ the tubular neighborhood of radius 96 of the thick part $M_{\rm thick}$ of M for some choice of Margulis constant.

Let $k\geq 1$ be such that $\lambda_k(\hat{M}_{\rm thick})=\lambda_k<1/96$ and let $f:\hat{M}_{\rm thick}\to\mathbb{R}$ be an eigenfunction for the eigenvalue λ_k, with Neumann boundary conditions. Then f is a smooth function on $\hat{M}_{\rm thick}$ which solves the Laplace equation

$$\Delta(f)+\lambda_k f=0.$$

Our goal is to extend the function f on $\hat{M}_{\rm thick}$ to a function F on M which is contained in the Sobolev space $\mathcal{H}(M)$ of square integrable functions on M, with square integrable weak derivative, in such a way that the Rayleigh quotient of F is controlled by the Rayleigh quotient of f and hence by λ_k.

The tubular neighborhood N of $\partial\hat{M}_{\rm thick}$ in $\hat{M}_{\rm thick}$ of radius 1 is diffeomorphic to $\partial\hat{M}_{\rm thick}\times[0,1]$, where the real parameter is the distance ρ from the boundary $\partial\hat{M}_{\rm thick}$ of $\hat{M}_{\rm thick}$. The metric on N is a warped product metric. By Corollary 3.6, we have

$$\int_N f^2 \leq \frac{1}{32}\int_{\hat{M}_{\rm thick}} f^2. \tag{19}$$

Let $m>0$ be such that $\int_N \|\nabla f\|^2 - m\lambda_k\int_N f^2=0$. Then we can find a number $\delta\in[0,1]$ so that

$$\int_{\partial\hat{M}_{\rm thick}\times\{\delta\}} \|\nabla f\|^2 \leq m\lambda_k\int_{\partial\hat{M}_{\rm thick}\times\{\delta\}} f^2. \tag{20}$$

Assume for the moment that

$$\int_{\partial\hat{M}_{\rm thick}\times\{\delta\}} f^2 \leq \int_N f^2 = \frac{1}{m\lambda_k}\int_N \|\nabla f\|^2. \tag{21}$$

Let $V=\hat{M}_{\rm thick}-\partial\hat{M}_{\rm thick}\times[0,\delta]$. Then V is a smooth submanifold of $\hat{M}_{\rm thick}$ with boundary $\partial V=\partial\hat{M}_{\rm thick}\times\{\delta\}$. Moreover, $M-V$ is a union of Margulis tubes and cusps as before. Extend the restriction of f to ∂V as in

Proposition 3.3. This yields a function $F : M \to (0, \infty)$ which is continuous, smooth away from ∂V, with smooth restriction to ∂V. Furthermore, F is square integrable, and its derivative (which exists in the strong sense away from the compact hypersurface ∂V) is pointwise bounded. Thus F is contained in the Sobolev space $\mathcal{H}(M)$ of square integrable functions with square integrable weak derivative.

The Rayleigh quotient of F can be estimated as follows. Recall the definition of the number $m > 0$. Let $\mathcal{T} = (M - \hat{M}_{\text{thick}}) \cup N \supset M - V$. By the first part of Proposition 3.3 and the assumption (21), we have

$$\int_{\mathcal{T}} F^2 = \int_{\mathcal{T}\cap V} F^2 + \int_{\mathcal{T}-V} F^2 \le \int_N f^2 + \frac{1}{2}\int_{\partial\hat{M}_{\text{thick}}\times\{\delta\}} f^2 \le \frac{3}{2}\int_N f^2.$$

Using the second part of Proposition 3.3 and the estimates (20) and decomposing integrals as before, we have

$$\begin{aligned}\int_{\mathcal{T}} \|\nabla F\|^2 &\le \int_{\mathcal{T}\cap V} \|\nabla f\|^2 + m\lambda_k q_2(\log \operatorname{vol}(\mathcal{T}))\int_{\mathcal{T}-V} F^2 \\ &\le m\lambda_k(1 + \frac{1}{2}q_2 \log \operatorname{vol}(\mathcal{T}))\int_N f^2 \end{aligned} \tag{22}$$

where the last inequality uses the definition of the constant $m > 0$.

We compute

$$\begin{aligned}\int_M \|\nabla F\|^2 &= \int_{M-\mathcal{T}} \|\nabla F\|^2 + \int_{\mathcal{T}} \|\nabla F\|^2 \\ &\le \int_{\hat{M}_{\text{thick}}} \|\nabla f\|^2 + \int_{\mathcal{T}} \|\nabla F\|^2 \le \lambda_k\big(\int_{\hat{M}_{\text{thick}}} f^2 + m(1 + \frac{1}{2}q_2 \log \operatorname{vol}(\mathcal{T}))\int_N f^2\big)\end{aligned} \tag{23}$$

where the last inequality follows from (22).

Recall that the Rayleigh quotient of the function f on $\hat{M}_{\text{thick}}$ equals λ_k. Thus as $\int_N \|\nabla f\|^2 = m\lambda_k \int_N f^2$, we have $\int_N f^2 \le \frac{1}{m}\int_{\hat{M}_{\text{thick}}} f^2$. Inserting into the estimate (23) implies that

$$\int_M \|\nabla F\|^2 \le \lambda_k(2 + \frac{1}{2}q_2 \log \operatorname{vol}(\mathcal{T}))\int_{\hat{M}_{\text{thick}}} f^2.$$

To complete the control on the Rayleigh quotient of F, we compare the square norm of F to the square norm of f. For convenience, extend the eigenfunction f on $\hat{M}_{\text{thick}}$ to all of M by 0. As $F = f$ on $V = \hat{M}_{\text{thick}} - \partial\hat{M}_{\text{thick}} \times [0, \delta]$, using inequality (19) we estimate

$$\begin{aligned}\frac{31}{32}\int_{\hat{M}_{\text{thick}}} f^2 &\le \int_V f^2 \le \int_V f^2 + \int_{M-V} F^2 = \int_M F^2 \\ &\le \int_V f^2 + \frac{1}{2}\int_{\partial\hat{M}_{\text{thick}}\times\{\delta\}} f^2 \le \frac{65}{64}\int_M f^2.\end{aligned} \tag{24}$$

Consequently, assuming without loss of generality that $q_2 \geq 2$, we deduce that the Rayleigh quotient of F is bounded from above by

$$q_2(\log \mathrm{vol}(M_{\mathrm{thin}}) + 2)\lambda_k(\hat{M}_{\mathrm{thick}}).$$

This is the property of the extension function F we were aiming at.

To summarize, for an eigenfunction f on $\hat{M}_{\mathrm{thick}}$ with Neumann boundary conditions and eigenvalue $\lambda_k < 1/96$, we constructed an extension $F : M \to \mathbb{R}$ with controlled Rayleigh quotient as predicted in the proposition provided that the function f fulfills assumption (21).

Our next goal is to reduce the general case where we can not assure that the inequality (21) is fulfilled to the case where it holds true.

To this end recall from the second part of Lemma 3.5 that

$$\int_{\partial \hat{M}_{\mathrm{thick}} \times \{0\}} f^2 \leq \int_N f^2.$$

Let $\beta : [0, \delta] \to \mathbb{R}$ be such that the function $F : \partial \hat{M}_{\mathrm{thick}} \times [0, \delta] = W \to (0, \infty)$ defined by $F(x, s) = e^{\beta(s)} f(x, \delta)$ satisfies

$$\int_{\partial \hat{M}_{\mathrm{thick}} \times \{s\}} F^2 = \int_{\partial \hat{M}_{\mathrm{thick}} \times \{s\}} f^2 \text{ for all } s \in [0, \delta].$$

Clearly we have $\int_W F^2 = \int_W f^2$.

Let ν be the vector field on W which equals the exterior normal field of the hypersurfaces $\hat{M}_{\mathrm{thick}} \times \{s\}$, defining the orientation of the interval. We claim that

$$\int_W \nu(F)^2 \leq \int_W \nu(f)^2.$$

To show the claim we evoke formula (15) from the proof of Lemma 3.5 which gives

$$\frac{d}{dt} \int_{\partial \hat{M}_{\mathrm{thick}} \times \{t\}} f^2 = c(t) \int_{\partial \hat{M}_{\mathrm{thick}} \times \{t\}} f^2 + 2 \int_{\partial \hat{M}_{\mathrm{thick}} \times \{t\}} f\nu(f).$$

By the definition of the function F, for all $t \in [0, \delta]$ we therefore have

$$\int_{\partial \hat{M}_{\mathrm{thick}} \times \{t\}} f\nu(f) = \int_{\partial \hat{M}_{\mathrm{thick}} \times \{t\}} F\nu(F).$$

But $\nu(F)(x, t) = \beta'(t)F(x, t)$ and hence

$$\int_{\partial \hat{M}_{\mathrm{thick}} \times \{t\}} F\nu(F) = \beta'(t) \int_{\partial \hat{M}_{\mathrm{thick}} \times \{t\}} F^2,$$

furthermore

$$\begin{aligned} \int_{\partial \hat{M}_{\mathrm{thick}} \times \{t\}} \nu(F)^2 &= |\beta'(t)|^2 \int_{\partial \hat{M}_{\mathrm{thick}} \times \{t\}} F^2 \\ &= \Big(\int_{\partial \hat{M}_{\mathrm{thick}} \times \{t\}} F\nu(F)\Big)^2 \Big/ \int_{\partial \hat{M}_{\mathrm{thick}} \times \{t\}} F^2. \end{aligned} \tag{25}$$

Observe that the last line in equation (25) coincides with the corresponding expression for the function f. By the Schwarz' inequality,

$$|\int_{\partial\hat{M}_{\rm thick}\times\{t\}} f\nu(f)| \leq (\int_{\partial\hat{M}_{\rm thick}\times\{t\}} f^2)^{1/2}(\int_{\partial\hat{M}_{\rm thick}\times\{t\}} \nu(f)^2)^{1/2}$$

and hence $\int_W \nu(F)^2 \leq \int_W \nu(f)^2$ as claimed.

Following the discussion in the proof of Proposition 3.3, for $s,t \in [0,\delta]$ the radial projection of the hypersurface $\partial\hat{M}_{\rm thick}\times\{s\}$ onto the hypersurface $\partial\hat{M}_{\rm thick}\times\{t\}$ is bilipschitz, with uniformly bounded bilipschitz constant not depending on M. This implies that there is a universal constant $c > 0$ such that for each $s \in [0,\delta]$, the Rayleigh quotient of the restriction of F to $\partial\hat{M}_{\rm thick} \times \{s\}$ is not bigger than c times the Rayleigh quotient of the restriction of F to $\partial\hat{M}_{\rm thick} \times \{\delta\}$. Together with the estimate on normal derivatives and the choice of δ, we obtain

$$\int_{\partial\hat{M}_{\rm thick}\times[0,\delta]} \|\nabla F\|^2 \leq cm\lambda_k \int_{\partial\hat{M}_{\rm thick}\times[0,\delta]} f^2 + \int_{\partial\hat{M}_{\rm thick}\times[0,\delta]} \|\nabla f\|^2.$$

Use Proposition 3.3 to extend the restriction of the function F to $\hat{M}_{\rm thick}\times \{0\}$ to the complement of $\hat{M}_{\rm thick}$ in M (which consists of a collection of Margulis tubes and cusps). It now follows from the beginning of this proof that the resulting function has all properties predicted in the proposition.

To provide more details of this estimate, let again $\mathcal{T} = M - \hat{M}_{\rm thick} \cup N$. Then $f = F$ on $M - \mathcal{T}$, and by Proposition 3.3 and inequality (19),

$$\int_{\mathcal{T}} F^2 \leq \frac{3}{2}\int_N f^2 \leq \frac{3}{64}\int_{\hat{M}_{\rm thick}} f^2. \tag{26}$$

This implies as in the estimate (24) that

$$\int_M f^2 \leq \int_M F^2 \leq \frac{67}{64}\int_M f^2. \tag{27}$$

On the other hand, we have

$$\int_{\mathcal{T}} \|\nabla F\|^2 \leq \lambda_k q_2 m(\log {\rm vol}(\mathcal{T}) + c)\int_N f^2$$

which yields the required estimate on Rayleigh quotients as above.

Now let us assume that for each $j < k$ we constructed from an eigenfunction f_i on $\hat{M}_{\rm thick}$ with eigenvalue $\lambda_i(M_{\rm thick})$ the function F_i. Let E_{k-1} be the linear span of the functions F_i for $i < k$. Assume that the function F_k is normalized so that $\int F_k^2 = 1$.

The function F_k may not be orthogonal to E_{k-1} for the L^2-inner product. Denote by H the L^2-orthogonal projection of F_k to E_{k-1}. We claim that $\int_M F_k H \leq 6/7$.

To this end let h be the (finite) linear combination of eigenfunctions on $\hat{M}_{\rm thick}$ with Neumann boundary condition which give rise to H. Then $\int f_k h = 0$ (since $h \in E_{k-1}$), $\int_M H^2 \leq 1$ (since $\int_M F_k^2 = 1$) and $\int_M f_k^2 \leq \frac{32}{31}$

by (24,27). Furthermore, by the construction of F_k and using the inequality $(a-b)^2 \leq 2a^2 + 2b^2$ and the estimate (26), we have

$$\int_M (F_k - f_k)^2 \leq 2\int_N f_k^2 + 2\int_{\mathcal{T}} F_k^2 \leq 5\int_N f_k^2 \leq \frac{5}{32}\frac{32}{31} = \frac{5}{31}$$

and consequently

(28)

$$\int_M F_k H = \int_M (F_k - f_k + f_k)H = \int_M (F_k - f_k)H + \int_M f_k(H-h) + f_k h$$
$$\leq (\int_M (F_k - f_k)^2)^{1/2} + \sqrt{\frac{32}{31}}(\int_M (H-h)^2)^{1/2} \leq 6/7$$

(where the last inequality follows from the rough estimate $\sqrt{\frac{5}{31}} + \sqrt{\frac{32}{31}}\sqrt{\frac{5}{31}} \leq 26/31 \leq 6/7$).

Now H is the orthogonal projection of F_k to E_{k-1} and therefore we have $\int_M H(F_k - H) = 0$. Together with the estimate (28), we find

$$\int_M (F_k - H)^2 = \int_M F_k(F_k - H) = 1 - \int_M F_k H \geq \frac{1}{7}.$$

By the above construction, there exists a universal constant $c > 0$ so that

$$\int_M \|\nabla F_k\|^2 \leq \lambda_k(2 + c \log \mathrm{vol}(M_{\mathrm{thin}})) \int_{\hat{M}_{\mathrm{thick}}} F_k^2$$

and the same holds true for H, with λ_k replaced by λ_{k-1}. But this implies that the Rayleigh quotient of $F_k - H$ is bounded from above by a fixed multiple of $\lambda_k(M_{\mathrm{thick}})(1+\log \mathrm{vol}(M_{\mathrm{thin}}))$. As k with $\lambda_k < 1/96$ was arbitrary, the proposition now follows from Rayleigh's principle (see [6] for details). □

Remark 3.8. For a compact hyperbolic manifold M of dimension $n \geq 4$, Burger and Schroeder [3] proved the upper bound

$$\lambda_1(M) \leq \frac{\alpha_n + \beta_n \log \mathrm{vol}(M)}{\mathrm{diam}(M)}$$

with constants α_n, β_n depending only on n. As hyperbolic 3-manifolds with cusps can be Dehn-filled, with fixed volume bound, this result is in general false for hyperbolic 3-manifolds.

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