Small eigenvalues of geometrically finite manifolds

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Abstract

Let M be a complete geometrically finite manifold of bounded negative curvature, infinite volume and dimension at least 3. We give both a lower bound for the bottom of the spectrum of M and an upper bound for the number of the small eigenvalues of M. These bounds only depend on the dimension, curvature bounds and the volume of the one-neighborhood of the convex core. [1]

1 Introduction

Let M be a complete Riemannian manifold of dimension $n \geq 3$ with sectional curvature $-\kappa \leq K \leq -1$ for some $\kappa \geq 1$. Denote by X the universal covering of M; then there is a discrete torsion free subgroup Γ of the isometry group $\mathrm{Iso}(X)$ of X such that $M = X/\Gamma$. The group Γ acts by homeomorphisms on the *ideal boundary* ∂X of X. The *limit set* Λ of Γ is the smallest nonempty closed Γ-invariant subset of ∂X. The quotient under the action of Γ of the closed convex hull of Λ in X is a closed convex subset $C(M)$ of M, the so-called *convex core* of M.

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The manifold M is called *geometrically finite* if the volume of the one-neighborhood $C_1(M)$ of $C(M)$ in M is finite. This is in particular the case if the manifold M is of finite volume itself [B].

The Laplace-Beltrami operator Δ of M acts on the space of smooth functions with compact support and admits a unique extension to an unbounded self-adjoint operator on $L^2(M)$. The spectrum of $-\Delta$ is a closed subset $\sigma(M)$ of the half-line $[0,\infty)$ which is the disjoint union of the *essential spectrum* $\sigma_{\text{ess}}(M)$ and the *discrete spectrum* $\sigma_{\text{disc}}(M)$. The essential spectrum is a closed subset of $[0,\infty)$. The discrete spectrum consists of eigenvalues of finite multiplicity; they are isolated points in $\sigma(M)$.

In this note we are interested in the *small part* of the spectrum of M, i.e. the intersection of $\sigma(M)$ with the interval $[0,(n-1)^2/4)$. Notice that $(n-1)^2/4$ is just the bottom of the L^2-spectrum of hyperbolic n-space $\mathbf{H}^n$. Our first result is a positive lower bound for the bottom $\lambda_0(M)$ of the spectrum of M under the assumption that the volume of M is infinite. This bound only depends on the dimension n, the curvature bounds and the volume of $C_1(M)$ and generalizes an earlier result of Burger and Canary [BC] for geometrically finite hyperbolic manifolds, with a simpler proof.

We also show that the bottom of the essential spectrum of M is not smaller than $(n-1)^2/4$. Thus the small part of the spectrum consists of a collection of eigenvalues. We give an upper bound for the number of eigenvalues contained in a closed interval of the form $[0,(n-1)^2/4-\chi]$ for some $\chi>0$. This bound only depends on the dimension n, curvature bounds, the volume of $C_1(M)$ and the choice of χ and generalizes in a weaker form an estimate of Buser, Colbois and Dodziuk [BCD] for manifolds of finite volume and bounded negative curvature. Our results can be summarized as follows.

Theorem: *Let $n\geq 3,\kappa\geq 1$ and let M be a geometrically finite manifold of dimension n, infinite volume and sectional curvature contained in $[-\kappa,-1]$. Denote by $\text{vol}(C_1(M))$ the volume of the neighborhood of radius 1 of the convex core of M.*

1. *There is a constant $c_0=c_0(n,\kappa)>0$ such that*
$$\lambda_0(M)\geq\frac{c_0}{\text{vol}(C_1(M))^2}.$$
2. *The essential spectrum of M is contained in $[(n-1)^2/4,\infty)$.*

3. *For every $\chi > 0$ there is a constant $c_1 = c_1(n, \kappa, \chi) > 0$ such that the number of eigenvalues of M contained in the interval $(0, (n-1)^2/4 - \chi]$ is bounded from above by $c_1^{\mathrm{vol}(C_1(M))}$.*

In the case of geometrically finite subgroups Γ of the simple rank-one Lie groups $O_{\mathbb{F}}(n, 1)$ where $\mathbb{F}$ is one of the Archimedian fields $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ the bottom of the spectrum of the corresponding rank-one locally symmetric space M corresponds precisely to the *critical exponent* of Γ. In particular, the bottom of the spectrum of M is 0 if and only if this critical exponent equals $\delta_{\mathbb{F}}(n) = \dim(M) + \dim_{\mathbb{R}}\mathbb{F} - 2$. The proof of our theorem also yields the following result.

Corollary: *Let Γ be a geometrically finite subgroup of $O_{\mathbb{F}}(n, 1)$ and denote by M the locally symmetric space defined by Γ.*

1. *The critical exponent of Γ equals $\delta_{\mathbb{F}}(n)$ if and only if Γ is a lattice.*

2. *The essential spectrum of M is contained in $[\delta_{\mathbb{F}}(n), \infty)$, and for every $\chi > 0$ the number of eigenvalues of M contained in $[0, \delta_{\mathbb{F}}(n) - \chi]$ is bounded from above by $c(\chi)^{\mathrm{vol}(C_1(M))}$.*

The first part of our corollary is well known for real hyperbolic manifolds (see [BC] and the references there) and, in a much stronger form, for general discrete subgroups of $O_{\mathbb{H}}(n, 1), O_{\mathbb{O}}(2, 1)$ (see [C]).

Our method is very general and can be applied to give a lower bound for the spectrum and the essential spectrum for the Laplacian acting on forms; however we do not address this question here. For a discussion of further related results we refer to [BC] and [BCD].

2 Tubes, cusps and ends

In this section we collect some preliminary results on the spectrum of tubes, cusps and ends in a geometrically finite Riemannian manifold M of dimension $n \geq 3$ and of bounded negative curvature $-\kappa \leq K \leq -1$. Recall that such a manifold M can be written in the form $M = X/\Gamma$ where Γ is a discrete torsion free subgroup of the isometry group $\mathrm{Iso}(X)$ of the universal covering X of M. The group Γ is called *elementary* if its action on the ideal boundary ∂X

of X has a finite orbit. An elementary torsion free discrete isometry group Γ either is an infinite cyclic group of loxodromic isometries or Γ is a parabolic group which fixes a point $\xi \in \partial X$ (see [B]).

We begin with estimating the bottom of the spectrum for the quotient of X under a torsion free elementary group Γ.

Lemma 2.1: *Let Γ be an infinite cyclic group of loxodromic isometries of X; then $\lambda_0(X/\Gamma) \geq (n-1)^2/4$.*

Proof: Since the bottom of the L^2-spectrum of $M = X/\Gamma$ for the operator $-\Delta$ equals the top of the positive spectrum of $-\Delta$ it is enough to find a positive $\Delta + (n-1)^2/4$-superharmonic function on M [S]. For this let $\tilde{\gamma}$ be the axis of the action of Γ on X; then $\tilde{\gamma}$ is a Γ-invariant geodesic in X, and Γ acts as a group of translations on $\tilde{\gamma}$. Denote by $\tau > 0$ the translation length on $\tilde{\gamma}$ of a generator Ψ of Γ; then for every $x \in X$ and every $k \in \mathbb{Z}$ we have $\text{dist}(x, \Psi^k x) \geq k\tau$.

Fix a point $x \in X$ and denote by $\rho_x(y)$ the distance of a point $y \in X$ to x. Write $\delta = (n-1)/2$; using Rauch's comparison theorem we have

$$\Delta(e^{-\delta\rho_x}) \leq e^{-\delta\rho_x}(-(n-1)\delta + \delta^2)$$

at least on $X - \{x\}$. Therefore the function $e^{-\delta\rho_x}$ is $\Delta + \delta^2$-superharmonic, moreover it is positive. On the other hand, by our above observation, for each $y \in X$ the series

$$\sum_k e^{-\delta\rho_{\Psi^k x}(y)}$$

converges and hence the assignment $y \to \sum_k e^{-\delta\rho_{\Psi^k x}(y)}$ defines a Γ-invariant positive $\Delta + \delta^2$-superharmonic function on X which projects to a positive $\Delta + \delta^2$-superharmonic function on $M = X/\Gamma$. This shows the lemma. $\square$

Next we look at the bottom of the spectrum of an elementary parabolic group. For this recall that for every complete Riemannian manifold N the bottom $\lambda_0(N)$ of the spectrum of N is the infimum of the *Rayleigh quotients* $\mathcal{R}(f) = \frac{\int |df|^2}{\int f^2}$ over all nontrivial smooth functions on N with compact support.

Lemma 2.2: *Let $\Gamma \subset \text{Iso}(X)$ be an elementary parabolic group; then $\lambda_0(X/\Gamma) \geq (n-1)^2/4$.*

Proof: Let $\Gamma \subset \text{Iso}(X)$ be an elementary parabolic group. Then Γ stabilizes a horosphere H in X, and $M = X/\Gamma$ is diffeomorphic to $H/\Gamma \times \mathbb{R}$ with a diffeomorphism which maps each line $\{z\} \times \mathbb{R}$ to a geodesic in M. If we denote by dx the volume element on H/Γ of the restriction of the Riemannian metric on M then the volume element ω on M can be represented in the form $\omega = dt \times hdx$ where $h : H/\Gamma \times \mathbb{R} \to (0, \infty)$ satisfies

$$\frac{\partial}{\partial t} h(x,t) \geq (n-1)h(x,t)$$

(see [HIH]).

Now let $f : M \to \mathbb{R}$ be any smooth function with compact support. We use an idea of Kean (see [BCD]). Namely, for $x \in H/\Gamma$ we have

$$(\int_{-\infty}^{\infty} |\frac{\partial f}{\partial t}|^2 h(x,t)dt)^{1/2} (\int_{-\infty}^{\infty} f^2 h(x,t)dt)^{1/2} \geq \int_{-\infty}^{\infty} |\frac{\partial f}{\partial t} f| h(x,t)dt$$
$$= \frac{1}{2}\int_{-\infty}^{\infty} |\frac{\partial}{\partial t}(f^2)| h(x,t)dt \geq \frac{1}{2}\int_{-\infty}^{\infty} f^2 \frac{\partial}{\partial t} h(x,t)dt \geq \frac{n-1}{2}\int_{-\infty}^{\infty} f^2 h(x,t)dt$$

and therefore

$$\int_{H/\Gamma}\int_{-\infty}^{\infty} |\frac{\partial f}{\partial t}|^2 h(x,t)dtdx \geq (\frac{n-1}{2})^2 \int_{H/\Gamma}\int_{-\infty}^{\infty} f^2 h(x,t)dtdx.$$

Thus the Rayleigh quotient of f is at least $(n-1)^2/4$. □

Now consider an arbitrary geometrically finite manifold M of curvature contained in $[-\kappa, -1]$. As in the introduction, let $C(M)$ be the convex core of M. Then $M - C(M)$ is an open subset of M which consists of finitely many connected components (see [B]). Recall that for each open subset Ω of a Riemannian manifold M the smallest Rayleigh quotient $\mu_1(\Omega)$ of Ω is defined to be the infimum of all Rayleigh quotients for all smooth functions f with compact support in Ω; in particular we have $\mu_1(\Omega) \geq \mu_1(M)$.

The next lemma is similar to Lemma 2.2. For its formulation, denote for a subset A of M and $r > 0$ by $B(A, r)$ the open r-neighborhood of A in M.

Lemma 2.3: *Let M be geometrically finite with curvature $-\kappa \leq K \leq -1$; then for every $r > 0$ we have $\mu_1(M - \overline{B(C(M), r)}) \geq (\tanh r)^2 (n-1)^2/4$.*

Proof: Since $B(C(M), r)$ is the r-neighborhood of a convex subset of M, its boundary is a smooth hypersurface $\partial B(C(M), r)$ in M. The second fundamental form of this boundary is positive definite. If we denote by N the outer normal of $\partial B(C(M), r)$ then the normal exponential map exp is a diffeomorphism of $\{tN \mid t \in (0,\infty)\}$ onto $M - \overline{B(C(M),r)}$. The pullback under exp of the volume element ω on $M - \overline{B(C(M),r)}$ can be written in the form $\omega = dt \times h(x,t)dx$ where dx is the volume element on $\partial B(C(M),r)$ and where $h : \partial B(C(M),r) \times [0,\infty) \to [1,\infty)$ is a smooth function. By comparison with a manifold of constant curvature -1, this function satisfies

$$\frac{\partial}{\partial t} h(x,t) \geq (n-1)h(x,t)\tanh(t+r)$$

for all $t \geq 0$. The lemma now follows from the arguments in the proof of Lemma 2.2. $\square$

Remark: Using the notations from the introduction, if Γ is a geometrically finite subgroup of the group $O_{\mathbb{F}}(n,1)$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ then using the standard computations for Jacobi fields in rank-one symmetric spaces we observe that the proofs of Lemma 2.1-2.3 remain valid if we replace the lower bound $(n-1)^2/4$ for the spectrum of our tubes, cusps and ends by the (sharp) bound $\delta_{\mathbb{F}}(n)$.

3 Proof of the theorem

We continue to look at a geometrically finite Riemannian manifold (M,g) of dimension $n \geq 3$ and sectional curvature $-\kappa \leq K \leq -1$. As before, we denote by X the universal covering of M and by Γ the fundamental group of M. For a number $\delta > 0$ let $M_{\mathrm{thin}(\delta)}$ be the set of all points in M at which the injectivity radius is smaller than δ. Following Margulis, there is a constant $\epsilon_0 > 0$ only depending on the dimension and the curvature bounds such that for every $\epsilon \leq \epsilon_0$ each connected component of $M_{\mathrm{thin}(\epsilon)}$ is isometric to an open connected subset of the quotient of X under an elementary torsion-free subgroup G of Γ. In the sequel we always denote by ϵ_0 this constant. For $\epsilon \in (0,\epsilon_0]$ define $M_{\mathrm{thick}(\epsilon)}$ to be the ϵ-*thick* part of M, i.e. the set of all points in M with injectivity radius at least ϵ. Since M is geometrically finite, the intersection $M_{\mathrm{thick}(\epsilon)} \cap C(M)$ is compact [B].

For an open subset Ω of M the Laplacian acting on smooth functions with compact support in Ω admits a unique self-adjoint extension Δ_Ω to an unbounded operator on $L^2(\Omega)$. The domain of $\Delta_\Omega^{1/2}$ is the usual Sobolev space $H^1(\Omega)$ which is the completion of the space of smooth functions with compact support in Ω with respect to the Sobolev norm $\|f\|^2 = \int f^2 + \int |df|^2$. Let $\sigma(\Omega)$ be the spectrum of $-\Delta_\Omega$.

We use the spectral theorem for $-\Delta_\Omega$ in the following form (see [D]). There is a finite measure μ on $\sigma(\Omega)\times\mathbb{N}$ and a unitary operator $U : L^2(\Omega)\to L^2(\sigma(\Omega)\times\mathbb{N}, d\mu)$ as follows. Define $h(s,n)=s$; then $f\in L^2(\Omega)$ is contained in the domain of $-\Delta_\Omega$ if and only if $hU(f)\in L^2(\sigma(\Omega)\times\mathbb{N},d\mu)$, and if this is the case we have $-U\Delta_\Omega U^{-1}(Uf)=hU(f)$. The spectral measure of such a function f is supported in an interval $[\lambda-\kappa,\lambda+\kappa]$ if and only if the function Uf is supported in $[\lambda-\kappa,\lambda+\kappa]\times\mathbb{N}$. Since $(u,q)\to\int g(du,dq)$ is the quadratic form of $-\Delta_\Omega^{1/2}$ this implies that for every $q\in H^1(\Omega)$ we have

$$|\int g(df,dq)-\lambda\int fq|\leq\kappa\int|fq|.$$

Using this inequality for $u=f$ we obtain in particular that the Raleigh quotient of f is contained in the interval $[\lambda-\kappa,\lambda+\kappa]$. Moreover, if f and q are contained in the domain of $-\Delta_\Omega$ and if their spectral measures are supported on disjoint subsets of $\sigma(\Omega)$ then $\int fq=\int g(df,dq)=0$.

Call a square integrable function f on M *normalized* if $\int f^2=1$. We have.

Lemma 3.1: *For every $\sigma>0$ there is a number $R_0=R_0(\sigma,\kappa,n)>0$, a number $\delta=\delta(\sigma,\kappa,n)>0$ and a number $\epsilon=\epsilon(\sigma,\kappa,n)<\epsilon_0$ with the following properties. Let $f\in H^1(M)$ be a normalized function with spectral measure supported in a subinterval of $[0,(n-1)^2/4-\sigma]$ of length at most δ; then there is a normalized function $Lf\in H^1(B(M_{\text{thick}(\epsilon)}\cap C(M),R_0))$ whose spectral measure is supported in $[0,(n-1)^2/4-\sigma/2]$ and such that*

$$\int(f-Lf)^2<1/4.$$

Proof: By the result of Margulis, for $\epsilon\leq\epsilon_0$ each connected component of $M_{\text{thin}(\epsilon)}$ is isometric to an open subset of the quotient of X under an elementary subgroup of the isometry group of X. Thus by domain monotonicity and Lemma 2.1, Lemma 2.2 the smallest Raleigh quotient $\mu_1(M_{\text{thin}(\epsilon_0)})$ of $M_{\text{thin}(\epsilon_0)}$ is not smaller than $(n-1)^2/4$. Lemma 2.3 also shows that $\mu_1(M-\overline{B(C(M),r)})\geq(\tanh r)^2(n-1)^2/4$.

Let $\sigma>0$ and define $\chi=\min\{1/16(12+3(n-1)^2),\sigma/2\}$ and $\nu=4\chi^3/(n+1)^2$. By Lemma 2.3 of [FH], applied to the closed set $\overline{B(C(M),r)}$ for a number $r>0$ which is sufficiently big that $(\tanh r)^2(n-1)^2/4\geq(n-1)^2/4-\sigma/2$ and to the closed set $M_{\text{thick}(\epsilon_0)}$, there is a number $\delta=\delta(\sigma)<\nu$ and a constant $R = R(\sigma) > 0$ with the following properties. Let $f \in H^1(M)$ be a normalized function whose spectral measure is supported in a closed subinterval of $[0,(n-1)^2/4-\sigma]$ of length at most δ; then

$$\int_{M-B(C(M),R)} f^2 + |df|^2 < \nu^2/4, \quad \int_{M-B(M_{\text{thick}(\epsilon_0)},R)} f^2 + |df|^2 < \nu^2/4.$$

Since each connected component of the ϵ_0-thin part of M is isometric to a connected subset of the quotient of X under an elementary group, there are numbers $\epsilon_1 \in (0,\epsilon_0]$, $\epsilon \in (0,\epsilon_1]$ only depending on the dimension, the curvature bounds and the radius R (which only depends on σ) such that $B(M_{\text{thick}(\epsilon_0)},R) \subset M_{\text{thick}(\epsilon_1)}$ and $B(M_{\text{thick}(\epsilon_1)},R) \subset M_{\text{thick}(\epsilon)}$. But this just means that $B(C(M),R) \cap B(M_{\text{thick}(\epsilon_0)},R) \subset B(C(M),R) \cap M_{\text{thick}(\epsilon_1)} \subset B(C(M) \cap M_{\text{thick}(\epsilon)},R)$; thus our function f as above satisfies

$$\int_{M-B(C(M)\cap M_{\text{thick}(\epsilon)},R)} f^2 + |df|^2 < \nu^2/2.$$

Choose a smooth function $\tilde{u} : \mathbb{R} \to [0,1]$ which equals 1 on $(-\infty,0]$, vanishes on $[2,\infty)$ and whose gradient is pointwise bounded in norm by 1. Define $u(x) = \tilde{u}(\text{dist}(x, C(M) \cap M_{\text{thick}(\epsilon)}) - R)$; then u is supported in the $R+2$-neighborhood $\Omega = B(C(M) \cap M_{\text{thick}(\epsilon)}, R+2)$ of $C(M) \cap M_{\text{thick}(\epsilon)}$ and its gradient is pointwise bounded in norm by 1. The function uf is contained in $H^1(\Omega)$ and it satisfies

$$\begin{aligned}|\int f^2 - (uf)^2| + \int (f-uf)^2 \leq & 2\int_{M-B(C(M)\cap M_{\text{thick}(\epsilon)},R)} f^2 < \nu^2 \quad \text{and} \\ & \int |df - d(uf)|^2 + |\int |df|^2 - |d(uf)|^2| \\ \leq 2 \int_{M-B(C(M)\cap M_{\text{thick}(\epsilon)},R)} & |df|^2 + f^2 + |2ufg(df,du)| < 3\nu^2.\end{aligned}$$

Let λ be the midpoint of an interval of length at most $\delta \leq \nu$ which contains the support of the spectral measure for f. The function uf lies in the domain of the operator Δ_Ω and therefore uf admits an L^2-orthogonal decomposition of the form $uf = \alpha + \varphi + \beta$ where the spectral measure of α is supported in $[0, \lambda - \chi^2]$, the spectral measure of φ is supported in $[\lambda - \chi^2, \lambda + \chi]$ and the spectral measure of β is supported in $[\lambda + \chi, \infty)$. Denote by $\|\psi\|$ the L^2-norm of a square integrable function ψ on Ω or M

and denote similary by $\|\xi\|$ the L^2-norm of a square integrable one-form ξ on Ω or M. Define $Lf = \alpha + \varphi/\|\alpha + \varphi\|$. For the proof of the lemma it is now enough to show that $\|\beta\| < 1/8$. Namely, if this is the case then we have $|\|\alpha+\varphi\| - 1| \leq \|\alpha+\varphi - f\| < 1/4$ and hence $\|Lf - \alpha - \varphi\| < 1/4$ and $\|f - Lf\| \leq \|f - \alpha - \varphi\| + \|\alpha + \varphi - Lf\| < 1/2$.

For this estimate of $\|\beta\|$ we proceed as in [FH]. Observe first that we have

$$\|\alpha\|^2 = \int \alpha(uf) = \int \alpha f + \int \alpha(uf - f) \leq \int \alpha f + \nu\|\alpha\|.$$

Now f is normalized and the Raleigh quotient of α is not bigger than $\lambda - \chi^2 < (n-1)^2/4$ and consequently we obtain that

$$\begin{aligned}(\lambda - \chi^2)\|\alpha\|^2 \geq \|d\alpha\|^2 &= \int g(d\alpha, d(uf)) \geq \int g(d\alpha, df) - \sqrt{3}\nu\|d\alpha\| \\ &\geq \lambda \int \alpha f - \nu(\|\alpha\| + \sqrt{3\lambda}\|\alpha\|) \geq \lambda\|\alpha\|^2 - \nu\|\alpha\|(\lambda + 1 + \sqrt{3\lambda}).\end{aligned}$$

Since $\lambda + 1 + \sqrt{3\lambda} \leq (n+1)^2/4$ we conclude that $\|\alpha\| \leq \nu(n+1)^2/4\chi^2 = \chi$ by our choice of ν.

On the other hand, the square norm of β can be estimated as follows. By construction we have

$$\begin{aligned}(1 + 3\nu^2)(\lambda + \nu) &\geq (1 + 3\nu^2)\|df\|^2 \geq \|d(uf)\|^2 \\ &= \|d\alpha\|^2 + \|d\varphi\|^2 + \|d\beta\|^2 \geq (\lambda - \chi^2)\|\varphi\|^2 + (\lambda + \chi)\|\beta\|^2.\end{aligned}$$

Since $\|\varphi\|^2 + \|\beta\|^2 = \|uf\|^2 - \|\alpha\|^2 \geq 1 - \nu^2 - \|\alpha\|^2 \geq 1 - 2\chi^2$ we obtain from this that

$$(1 + 3\nu^2)(\lambda + \nu) \geq (1 - 2\chi^2)(\lambda - \chi^2) + \chi\|\beta\|^2$$

and hence $\chi\|\beta\|^2 \leq \nu + 3\nu^2(\lambda+\nu) + \chi^2 + 2\chi^2(\lambda - \chi^2)$ and $\|\beta\|^2 \leq \chi(3+3\lambda)$. By our choice of χ this means that $\|\beta\| < 1/8$ from which the lemma follows. ☐

Recall that if $\Omega \subset M$ is an open set with compact closure then the spectrum of Ω consists of a countable number of eigenvalues going to ∞. In the sequel we always count our eigenvalues with multiplicities. We have.

Corollary 3.2:

1. *The essential spectrum of M is contained in* $[(n-1)^2/4, \infty)$.

2. *There is a constant $\chi > 0$ with the following property. For $\sigma > 0$ let $R_0 = R_0(\sigma, \kappa, n) > 0$ be as in Lemma 3.1 and let $q \geq 0$ be the number of eigenvalues of $\Omega = B(M_{\text{thick}(\epsilon)} \cap C(M), R_0)$ which are contained in the interval $(0, (n-1)^2/4 - \sigma/2]$; then the number of eigenvalues of M contained in the interval $(0, (n-1)^2/4 - \sigma]$ is not bigger than χ^q.*

Proof: Let $\sigma > 0$ and let $R_0 = R_0(n, \kappa, \sigma) > 0$ be as in Lemma 3.1. Since the closure of $\Omega = B(M_{\text{thick}(\epsilon)} \cap C(M), R_0)$ is compact, its spectrum consists of a countable number of eigenvalues which increase to ∞. Let E be the sum of the eigenspaces for $-\Delta_\Omega$ with respect to those eigenvalues which are not bigger than $(n-1)^2/4 - \sigma/2$. Then E is a finite-dimensional linear subspace of $H^1(\Omega)$, and its orthogonal complement $E^\perp$ in $H^1(\Omega)$ is the closed vector subspace which admits a Hilbert basis consisting of the complete set of eigenfunctions with respect to all eigenvalues contained in $((n-1)^2/4 - \sigma/2, \infty)$.

Let $\delta = \delta(\sigma) > 0$ be as in Lemma 3.1 and let $f_1, \dots, f_\ell \subset H^1(M)$ be a collection of mutually L^2-orthogonal normalized functions whose spectral measures are supported in a subinterval of $[0, (n-1)^2/4 - \sigma]$ of length at most δ. By Lemma 3.1 there is a collection $g_1, \dots, g_\ell \subset E$ of normalized functions with $\int (f_i - g_i)^2 < 1/4$. Then the functions g_i are contained in the unit sphere S in the finite-dimensional vector space E equipped with the L^2-inner product, and their pairwise euclidean distance is at least $1/4$. In other words, the balls in S of radius $1/8$ which are centered at the points $g_1, \dots, g_\ell$ are pairwise disjoint. But this implies that there is a universal constant $\chi > 0$ such that the cardinality of the set $\{g_1, \dots, g_\ell\}$ and hence the cardinality of the set $\{f_1, \dots, f_\ell\}$ is bounded from above by $\chi^{\dim(E)}$.

Thus the dimension of the vector-subspace of $H^1(M)$ which is spanned by all functions whose spectral measures are supported in a subinterval of $[0, (n-1)^2/4 - \sigma]$ of length at most δ is bounded from above by $\chi^{\dim(E)}$. This means that the essential spectrum of M does not intersect the interval $[0, (n-1)^2/4 - \sigma]$, and the dimension of the sum of the eigenspaces of $-\Delta$ for eigenvalues contained in $[0, (n-1)^2/4 - \sigma]$ is bounded from above by $\chi^{\dim(E)}$. Since $\sigma > 0$ was arbitrary the corollary follows. □

Corollary 3.2 shows the second part of the theorem from the introduction and easily implies the third.

Corollary 3.3: *For every $\sigma > 0$ there is a constant $c_1 = c_1(\sigma, n, \kappa) > 0$ such that the number of eigenvalues of M which are contained in the interval*

$[0,(n-1)^2/4-\sigma]$ *is not bigger than* $c_1^{\mathrm{vol}(C_1(M))}$.

Proof: Let $\sigma > 0$ and let $\epsilon = \epsilon(\sigma) > 0$ and $R_0 = R_0(\sigma) > 0$ be as in Lemma 3.1. Notice that ϵ and R_0 only depend on σ and the curvature bounds. Denote by Ω the open R_0-neighborhood of $M_{\mathrm{thick}(\epsilon)} \cap C(M)$; then the closure of Ω is compact. We claim that there is a constant $\zeta > 0$ only depending on R_0, the dimension and the curvature bounds such that the volume of Ω is bounded from above by $\zeta \mathrm{vol}(C_1(M))$.

To see this recall that $B(C(M),1/2)$ is the $1/2$-neighborhood of a convex subset of M and therefore the second fundamental form of the boundary $\partial B(C(M),1/2)$ of $B(C(M),1/2)$ at a point $x \in \partial B(C(M),1/2)$ is not bigger than the second fundamental form of a ball of radius $1/2$ in M which meets $\partial B(C(M),1/2)$ tangentially at x. Standard comparison arguments then imply that the second fundamental form of $\partial B(C(M),1/2)$ is globally uniformly bounded. Comparison with a manifold of constant curvature κ then shows that the volume of $B(C(M),R_0)$ is bounded from above by a fixed multiple of $\mathrm{vol}(C_1(M))$.

As a consequence of this and Corollary 3.2 we only have to show that the number of eigenvalues of $-\Delta_\Omega$ contained in the interval $[0,(n-1)^2/4]$ is bounded from above by a constant multiple of $\mathrm{vol}(\Omega)$.

For this recall from Lemma 3.1 and its proof that there there is a number $\rho > 0$ only depending on R_0, ϵ and the curvature bounds such that for every $z \in \overline{\Omega}$ the injectivity radius of M at z is at least ρ. Choose a collection of points $\{p_1,\dots,p_\ell\} \subset \Omega$ of maximal cardinality whose pairwise distances are at least ρ. Then the closed balls of radius ρ centered at the points p_i cover Ω, and the balls of radius $\rho/2$ centered at the points p_i are pairwise disjoint. Since the injectivity radius at each of the points p_i is at least ρ, the volume of the balls of radius $\rho/2$ about the points p_i is bounded from below by a universal constant only depending on the curvature bounds. Thus the cardinality of our collection is bounded from above by a constant multiple of $\mathrm{vol}(\Omega)$.

For $i \geq 1$ define $U_i = \{z \in \Omega \mid \mathrm{dist}(z,p_i) \leq \mathrm{dist}(z,p_j) \,\mathrm{for}\, j \neq i\}$. The sets U_i form a partition of Ω, and each of the sets U_i is contained in the ball of radius ρ about p_i. Now the smallest nonzero eigenvalue with Neumann boundary conditions of a ball of radius ρ in a simply connected manifold of curvature contained in $[-\kappa,-1]$ is not smaller than $(n-1)^2/4$ (compare [BCD]) and therefore by domain monotonicity, the second eigenvalue for each of the sets U_i with mixed boundary conditions (i.e. Neumann boundary conditions on $\partial U_i \cap \Omega$ and Dirichlet boundary conditions on $\partial\Omega$) is not smaller than $(n-1)^2/4$ as well. On the other hand, the number of small eigenvalues on Ω is bounded from above by the total number of small eigenvalues with Neumann boundary conditions for any partition of Ω and therefore this number is bounded from above by a constant multiple of the volume of $C_1(M)$. Together with Corollary 3.2, this yields the corollary. ☐

We are left with showing the first part of the theorem from the introduction. For this denote by $\lambda_0 = \lambda_0(M)$ the bottom of the spectrum for $-\Delta$ on M. By Corollary 3.2, if $\lambda_0 < (n-1)^2/4$ then λ_0 is an eigenvalue of $-\Delta$, and there is up to a constant a unique solution f of the equation $\Delta + \lambda_0(M) = 0$. This function f is square integrable and positive. For the purpose of our theorem we then may assume that such an eigenfunction f exists. For $\sigma = (n-1)^2/8$ denote by $R_0 = R_0(n, \kappa, \sigma)$ the constant as in Lemma 3.1. Let $\epsilon_1 > 0$ be sufficiently small that $B(M_{\text{thick}(\epsilon_0)}, R_0) \subset M_{\text{thick}(\epsilon_1)}$; notice once more that ϵ_1 can be chosen to depend only on the dimension and the curvature bounds. We have.

Lemma 3.4: *There is a number $\beta > 0$ only depending on the dimension and the curvature bounds of M with the following property. Let f be a positive normalized eigenfunction on M with respect to the eigenvalue $\lambda_0 < (n-1)^2/8$; then $f(x) \leq \beta\sqrt{\lambda_0}$ for all $x \in M_{\text{thick}(\epsilon_1)} \cap \partial B(C(M), R_0))$.*

Proof: By the infinitesimal Harnack inequality of Cheng and Yau [CY], applied to our positive normalized eigenfunction f with eigenvalue $\lambda_0 \in (0, (n-1)^2/8]$, there is a number $c_1 > 0$ only depending on the curvature bounds such that the function f satisfies $|d \log f| \leq c_1$ on $M_{\text{thick}(\epsilon_1)}$. In particular, if for a point $x \in M_{\text{thick}(\epsilon_1)} \cap \partial B(C(M), R_0)$ and a number $\alpha > 0$ we have $f(x) \geq \alpha\sqrt{\lambda_0}$ then there is a ball $B \subset \partial B(C(M), R_0)$ about x with the following property. Let N be the exterior normal of $B(C(M), R_0)$ and let exp be the normal exponential map; then $\int_{\exp(B\times(0,\infty))} f^2 \geq c_2\alpha^2\lambda_0$ where $c_2 > 0$ is a universal constant.

We now use the arguments in the proof of Lemma 2.2 and Lemma 2.3. Namely, let $h(y,t)$ be the function on $\partial B(C(M), R_0) \times [0,\infty)$ which determines the pullback under exp of the volume element of M with respect to the product measure on $\partial B(C(M), R_0) \times [0,\infty)$ as in Lemma 2.2. Then $h(y,0) = 1$ for all $y \in \partial B(C(M), R_0)$ and moreover $\frac{\partial}{\partial t} h(y,t) \geq (n-1)h(y,t)/\sqrt{2}$ for every $y \in B$ and $t \geq 0$. For $y \in \partial B(C(M), R_0)$ we conclude as in the proof of Lemma 2.2 that

$$\begin{aligned}(\int_0^\infty |\frac{\partial f}{\partial t}|^2 h(y,t)dt)^{1/2}(\int_0^\infty f^2 h(y,t)dt)^{1/2} &\geq \int_0^\infty |\frac{\partial f}{\partial t} f| h(y,t)dt \\ = \frac{1}{2}\int_0^\infty |\frac{\partial}{\partial t}(f^2)| h(y,t)dt &\geq -\frac{1}{2}\int_0^\infty \frac{\partial}{\partial t}(f^2) h(y,t)dt \\ = \frac{1}{2}f^2(y) + \frac{1}{2}\int_0^\infty f^2 \frac{\partial}{\partial t} h(y,t)dt &\geq \frac{1}{2}f^2(y) + \frac{n-1}{2\sqrt{2}}\int_0^\infty f^2 h(y,t)dt.\end{aligned}$$

Integration over B then shows that $\int_{\exp(\mathrm{B}\times(0,\infty))} |\frac{\partial f}{\partial t}|^2 \geq \frac{(n-1)^2}{8} c_2\alpha^2\lambda_0$ which implies that $\alpha^2 \leq \frac{8}{c_2(n-1)^2}$. This shows the lemma. □

Now we are ready for the proof of the first part of our theorem. Recall that it is enough to show the existence of a constant $\nu > 0$ only depending on the curvature bounds such that with $R_0 = R_0(n, \kappa, (n-1)^2/8) > 0$ as in Lemma 3.1 we have $\lambda_0 \geq \nu/(\mathrm{vol}B(C(M), R_0))^2$. For this we follow [DR] and argue by contradiction. Assume that $\lambda_0 = \lambda_0(M) < \alpha/\mathrm{vol}\,(B(C(M), R_0))^2$ for a small constant $\alpha > 0$. Denote by $K_1, \dots, K_m$ the connected components of $M_{\mathrm{thick}(\epsilon_1)} \cap B(C(M), R_0)$. Then for every $j \in \{1, \dots, m\}$ any two points in K_j can be connected by a chain $B_1, \dots, B_k$ of overlapping balls of radius $\epsilon_1/2$ where $k \leq c_1\mathrm{vol}(K_j)$ for a universal constant $c_1 > 0$ and for which each B_i intersects atmost a fixed number ℓ of other B_j's. Now if f is an eigenfunction on M with respect to the eigenvalue λ_0, then by the arguments of [DR] the oscillation of f on each of the sets K_j is not bigger than $c_2(\alpha/\mathrm{vol}(B(C(M), R_0)))^{1/2}$ where $c_2 > 0$ is a universal constant. On the other hand, each of the components K_j intersects $\partial B(C(M), R_0)$ (compare [BC]) and by Lemma 3.4 the value of f at such a boundary point does not exceed $\beta\sqrt{\lambda_0}$ where $\beta > 0$ is a universal constant. As as consequence, we have $|f| \leq c_3(\alpha/\mathrm{vol}(B(C(M), R_0)))^{1/2}$ on $\cup_{i=1}^m K_i$ where $c_3 > 0$ is another universal constant and hence $\int_{\cup K_j} f^2 \leq c_3^2\alpha$. For sufficiently small α this value is strictly smaller than 1/2. However Lemma 3.1 implies that $\int_{\cup K_j} f^2 \geq 3/4$ which is a contradiction.

Remark: The constant $(n-1)^2/4$ for our spectral bounds appears in our above argument only in the form of a lower bound for the spectrum of the tubes, cusps and ends of our geometrically finite manifold. Thus using the remark at the end of Section 2, in the case of a geometrically finite locally symmetric space we can use the sharp bound $\delta_{\mathbb{F}}(n)$ in our proof and obtain the same statements, with $(n-1)^2/4$ replaced by $\delta_{\mathbb{F}}(n)$. This then yields the corollary from the introduction.

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