ASYMPTOTIC DIMENSION AND THE DISK GRAPH I

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Abstract. For an apsherical oriented 3-manifold M and a subsurface X of the boundary of M with empty or incompressible boundary we use surgery to identify a graph whose vertices are disks with boundary in X and which is quasi-isometrically embedded in the curve graph of X.

1. Introduction

Consider an oriented aspherical 3-manifold M and a subsurface X of the boundary of M. We require that the boundary of X either is empty, or it is incompressible in M. We also require that the Euler characteristic of X is negative. The prototypical example is a *handlebody* of genus $g \geq 2$, i.e. a compact three-dimensional manifold which can be realized as a closed regular neighborhood in $\mathbb{R}^3$ of an embedded bouquet of g circles. Its boundary is a closed oriented surface of genus g.

The *disk graph* $\mathcal{DG}(X)$ of (M, X) is the metric graph whose vertices are isotopy classes of properly embedded disks in M with boundary in X and where two such disks are connected by an edge of length one if they can be realized disjointly. Assigning to a disk its boundary then defines an embedding of the disk graph into the *curve graph* $\mathcal{CG}(X)$ of X.

The curve graph $\mathcal{CG}(X)$ is a locally infinite geodesic metric graph which is hyperbolic in the sense of Gromov [MM99]. The disk graph $\mathcal{DG}(X)$ is a *quasi-convex* subset of $\mathcal{CG}(X)$ [MM04]. This means that there exists a number $c > 1$ with the following property. For any two points $x, y \in \mathcal{DG}(X)$, there exists a path in $\mathcal{DG}(X)$ which connects x to y and which is contained in the c-neighborhood of a geodesic in $\mathcal{CG}(X)$ connecting x to y.

As $\mathcal{CG}(X)$ is a locally infinite, this does not imply that the inclusion $\mathcal{DG}(X) \to \mathcal{CG}(X)$ is a quasi-isometric embedding. Indeed, as was discovered by Masur and Schleimer [MS13], this is not the case. Namely, in the terminology of their paper, the disk graph of a handlebody of genus $g \geq 2$ has *holes* consisting of convex subsets of infinite diameter whose images in $\mathcal{CG}(X)$ have uniformly bounded diameter. Nevertheless, the main result of [MS13] shows that the disk graph of the handlebody is hyperbolic. Furthermore, somewhat indirectly, Masur and Schleimer describe how to "fill" the holes and, by adding edges to $\mathcal{DG}(X)$, to construct a graph whose vertices are disks and which is quasi-isometrically embedded in the curve graph.

The main purpose of this article is to define such a graph explicitly and to give a purely combinatorial proof that it embeds quasi-isometrically into the curve graph. This construction is used in [H16] to give an alternative proof of hyperbolicity of

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the disk graph and to determine its Gromov boundary. In [H17] we use the finer structure of the disk graph established along the way to show that its asymptotic dimension is finite.

A construction which is closer to the viewpoint of Masur and Schleimer is due to Ma. In the article [Ma14] one also finds an interpretation of some of our results using the viewpoint of of Masur and Schleimer.

To introduce the graph we are interested in, call a simple closed curve c on X *diskbusting* if c has an essential intersection with the boundary of every disk.

Define an *I-bundle generator* for X to be a diskbusting simple closed curve c on X with the following property. There is a compact surface F with a distinguished boundary component $\alpha \in \partial F$, and there is a homeomorphism of the orientable I-bundle $\mathcal{J}(F)$ over F into M which maps α to c and which maps the union of the horizontal boundary of F with the I-bundle over α onto the complement in X of a tubular neighborhood of the boundary of X.

Definition. The *super-conducting disk graph* is the graph $\mathcal{SDG}(X)$ whose vertices are isotopy classes of essential disks with boundary in X and where two vertices D_1, D_2 are connected by an edge of length one if and only if one of the following two possibilities holds.

(1) There is an essential simple closed curve on X which can be realized disjointly from both $\partial D_1, \partial D_2$.
(2) There is an I-bundle generator c for X which intersects both $\partial D_1, \partial D_2$ in precisely two points.

Since the distance in the curve graph $\mathcal{CG}(X)$ of X between two simple closed curves which intersect in two points does not exceed 3 [MM99], the natural vertex inclusion extends to a coarse 6-Lipschitz map $\mathcal{SDG}(X) \to \mathcal{CG}(X)$. We show

Theorem 1. *The natural vertex inclusion extends to a quasi-isometric embedding $\mathcal{SDG}(X) \to \mathcal{CG}(X)$.*

The constants for the quasi-isometric embeddings are bounded from above by an explicit quadratic polynomial in the Euler characteristic of X.

The requirement that the boundary of the surface X is incompressible in M is essential for Theorem 1. In the statement of the following result, we tacitly assume that the graph $\mathcal{DG}(X)$ is not trivial.

Theorem 2. *If X is a subsurface of the boundary of M of genus $g \geq 2$, with a single compressible boundary component, then the graph $\mathcal{DG}(X)$ is not a quasi-convex subset of the curve graph.*

Organization: In Section 2 we use surgery of disks to relate the distance in the superconducting disk graph to intersection numbers of boundary curves.

In Section 3 we give an effective estimate of the distance in the curve graph using train tracks. The results in this section are independent of the rest of the article.

Together with a construction of [MM04], this is used in Section 4 to show Theorem 1. In Section 5 we identify the Gromov boundary of $\mathcal{SDG}(X)$ in the case M is a handlebody of genus $g \geq 2$ and X is its boundary surface. The proof of Theorem 2 is contained in Section 6.

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2. Distance and intersection

In this section we consider an arbitrary oriented aspherical 3-manifold M together with a compact oriented subsurface X of the boundary of M. The surface X may have boundary ∂X, but any boundary component of X is supposed to be incompressible in M.

By a *disk* we always mean an embedded essential disk in M with boundary in X. As the boundary of X is not diskbounding by assumption, the boundary ∂D of such a disk is an essential curve in X. Two disks D_1, D_2 are in *normal position* if their boundary circles intersect in the minimal number of points and if every component of $D_1 \cap D_2$ is an embedded arc in $D_1 \cap D_2$ with endpoints in $\partial D_1 \cap \partial D_2$. In the sequel we always assume that disks are in normal position; this can be achieved by modifying one of the two disks with an isotopy.

Let D be any disk and let E be a disk which is not disjoint from D. A component α of $\partial E - D$ is called an *outer arc* of ∂E relative to D if there is a component E' of $E - D$ whose boundary is composed of α and an arc $\beta \subset D$. The interior of β is contained in the interior of D. We call such a disk E' an *outer component* of $E - D$. An outer component of $E - D$ intersects X in an outer arc α relative to D, and α intersects ∂D in opposite directions at its endpoints.

For every disk E which is not disjoint from D there are at least two distinct outer components E', E'' of $E - D$. There may also be components of $\partial E - D$ which leave and return to the same side of D but which are not outer arcs. An example of such a component is a subarc of ∂E which is contained in the boundary of a rectangle component of $E - D$ leaving and returning to the same side of D. The boundary of such a rectangle consists of two subarcs of ∂E with endpoints on ∂D which are homotopic relative to ∂D, and two arcs contained in D.

Let $E' \subset E$ be an outer component of $E - D$ whose boundary is composed of an outer arc α and a subarc $\beta = E' \cap D$ of D. The arc β decomposes the disk D into two half-disks P_1, P_2. The unions $Q_1 = E' \cup P_1$ and $Q_2 = E' \cup P_2$ are embedded disks in M which up to isotopy are disjoint and disjoint from D. For $i = 1, 2$ we say that the disk Q_i is obtained from D by *simple surgery* at the outer component E' of $E - D$ (see e.g. [S00] for this construction). Since D, E are in minimal position, the disks Q_1, Q_2 are essential.

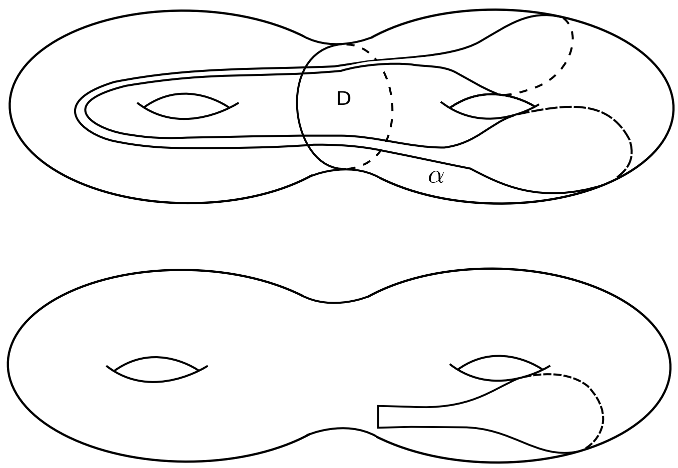
In the introduction we defined two graphs of disks with boundary in X. We called them *disk graph* $\mathcal{DG}(X)$ and *superconducting disk graph* $\mathcal{SDG}(X)$, respectively.

Each disk in M with boundary in X can be viewed as a vertex in the disk graph $\mathcal{DG}(X)$ and the superconducting disk graph $\mathcal{SDG}(X)$ of X. We will work with both graphs simultaneously. Denote by $d_{\mathcal{D}}$ (or $d_{\mathcal{S}}$) the distance in $\mathcal{DG}(X)$ (or in $\mathcal{SDG}(X)$). Note that for any two disks D, E we have

$$d_{\mathcal{S}}(D, E) \leq d_{\mathcal{D}}(D, E).$$

In the sequel we always assume that all curves and multicurves on X are essential. For two simple closed multicurves c, d on X let $\iota(c, d)$ be the geometric intersection

Figure A



number between c, d. The following lemma [MM04] implies that the graph $\mathcal{DG}(X)$ is connected. We provide the short proof for completeness.

Lemma 2.1. *Let $D, E \subset M$ be any two disks with boundary in X. Then D can be connected to a disk E' with boundary in X which is disjoint from E by at most $\log_2(\iota(\partial D, \partial E)/2)$ simple surgeries. In particular,*

$$d_{\mathcal{D}}(D, E) \leq \log_2(\iota(\partial D, \partial E)/2) + 1.$$

Proof. Let D, E be two disks in normal position, with boundary in X. Assume that D, E are not disjoint. Then there is an outer component E' of $E - D$. The endpoints of the outer arc $\partial E' \cap X$ decompose ∂D into two arcs β_1, β_2. Choose the arc with fewer intersections with ∂E, say the arc β_1. The disk D' obtained by simple surgery of D at this component which contains β_1 in its boundary is essential, with boundary in X. Moreover, D' is disjoint from D, i.e. we have $d_{\mathcal{D}}(D', D) = 1$, and

$$\iota(\partial E, \partial D') \leq \iota(\partial D, \partial E)/2. \tag{1}$$

The lemma now follows by induction on $\iota(\partial D, \partial E)$. □

Consider an oriented I-bundle $\mathcal{J}(F)$ over a compact (not necessarily oriented) surface F with (not necessarily connected) boundary ∂F. The boundary $\partial \mathcal{J}(F)$ decomposes into the *horizontal boundary* and the *vertical boundary*. The vertical boundary is the interior of the restriction of the I-bundle to ∂F and consists of a collection of pairwise disjoint open incompressible annuli. The horizontal boundary is the complement of the vertical boundary in $\partial \mathcal{J}(F)$.

For a given boundary component α of F, the union of the horizontal boundary of $\mathcal{J}(F)$ with the I-bundle over α is a compact connected orientable surface $F_\alpha \subset \partial \mathcal{J}(F)$. The boundary of F_α is empty if and only if the boundary of F is connected. If the boundary of F is not connected then F_α is properly contained in the boundary $\partial \mathcal{J}(F)$ of $\mathcal{J}(F)$. The complement $\partial \mathcal{J}(F) - F_\alpha$ is a union of incompressible annuli.

Definition 2.2. An *I-bundle generator* is an essential simple closed curve $\gamma \subset X$ with the following property. There is a compact surface F with non-empty boundary ∂F, there is a boundary component α of ∂F, and there is an orientation preserving embedding Ψ of the oriented I-bundle $\mathcal{J}(F)$ over F into M which maps α to γ and which maps F_α onto the complement in X of a tubular neighborhood of the boundary ∂X of X.

We call the surface F the *base* of the I-bundle generated by γ. Note that an I-bundle generator γ is *diskbusting*, i.e it intersects every disk in M with boundary in X.

If γ is a separating I-bundle generator in X with base surface F (i.e. $\gamma \subset X$ is a separating simple closed curve which also is an I-bundle generator) then F is orientable and the genus g of X is even. Moreover, the I-bundle $\mathcal{J}(F) = F \times [0,1]$ is trivial. The I-bundle over every essential arc in F with endpoints in ∂F is an embedded disk in M. If γ is a non-separating I-bundle generator in X then the base F of the I-bundle is non-orientable.

For each I-bundle $\mathcal{J}(F)$, there is an orientation reversing involution $\Phi : \mathcal{J}(F) \to \mathcal{J}(F)$ which acts as a reflection in the fiber. Up to isotopy, the I-bundle over any essential arc β on the surface F with endpoints in the same boundary component α is a Φ-invariant disk which intersects α in precisely two points.

If $D, E \subset M$ are disks in normal position then each component of $D - E$ is a disk. Furthermore, each component of $D \cap E$ is a properly embedded arc in D which decomposes D into two connected components. Therefore the graph dual to the cell decomposition of D whose two-cells are the components of $D - E$ is a tree. If $D - E$ only has two outer components then this tree is just a line segment. The following lemma analyzes the case that this holds true for both $D - E$ and $E - D$. For its formulation, we say that two simple closed curves c, d *fill* the surface X if c, d are contained in X and if there is no essential simple closed curve in X which is disjoint from $c \cup d$.

The proof of the following lemma uses a suggestion of a referee which lead to a considerable simplification of the argument.

Lemma 2.3. *Let $D, E \subset M$ be disks in normal position with boundary in the surface X. If $D - E$ and $E - D$ only have two outer components and if $\partial D, \partial E$ fill X then there exists an I-bundle $\mathcal{J}(F)$ over a compact surface F and an embedding $\Psi : \mathcal{J}(F) \to M$ with the following properties. There is a boundary component α of F such that $\Psi(\alpha)$ is an I-bundle generator in X, and D, E are the images under Ψ of I-bundles over embedded arcs δ, β in F with endpoints on α.*

Proof. Let D, E be two disks in normal position, with boundary in the surface X. Assume that $D - E$ and $E - D$ only have two outer components. Then each component of $D - E, E - D$ either is an outer component or a rectangle, i.e. a disk whose boundary consists of two components of $D \cap E$ and two arcs contained in $\partial D \subset X$ or $\partial E \subset X$, respectively. Assume that $\partial D, \partial E$ fill X. This means that $X - (\partial D \cup \partial E)$ is a union of disks and peripheral annuli about the boundary components of X.

Choose tubular neighborhoods $N(D), N(E)$ of D, E in M which are homeomorphic to an interval bundle over a disk and which intersect X in an embedded annulus. We may assume that the interiors $A(D), A(E)$ of these annuli are contained in the interior of X. Then $\partial N(D) - A(D)$, $\partial N(E) - A(E)$ is the union of two properly embedded disjoint disks in M isotopic to D, E. We may assume that $\partial N(D) - A(D)$ is in normal position with respect to $\partial N(E) - A(E)$ and that

$$S = \partial(N(D) \cup N(E)) - (A(D) \cup A(E))$$

is a compact surface with boundary which is properly embedded in M. Since M is assumed to be oriented, the boundary $\partial(N(D) \cup N(E))$ of $N(D) \cup N(E)$ has an induced orientation which restricts to an orientation of S.

Now note that $N(D) \cup N(E)$ has the structure of an interval bundle over a surface with the property that each intersection component of $D \cap E$ is a fibre of this bundle. Namely, for each outer component C of $D - E$ or $E - D$ choose two points in the interior of $\partial C \cap X$ so that the boundary of C can be viewed as a rectangle, with one side ρ the component of $D \cap E$ contained in the boundary of C. Foliate this rectangle in the standard way by intervals so that ρ is a leaf of this foliation. Similarly, each component of $D - E$ or $E - D$ which is not an outer components contains two components of $D \cap E$ in its boundary, and it can be foliated into intervals in such a way that the two components of $D \cap E$ in its boundary are leaves. This foliation of $D \cup E$ can naturally be extended to a foliation of $N(D) \cap N(E)$ by intervals. With the exception of a subarc of the boundary of an outer component, the leaves of this foliation intersect the boundary surface X only at their endpoints.

By assumption, $\partial D \cup \partial E$ decomposes X into a union of polygons, i.e. disks bounded by finitely many subarcs of $\partial D \cup \partial E$ and peripheral annuli. Such a polygon P is contained in the boundary of a component V of $M - (D \cup E)$. The intersection $\partial V \cap X$ has two connected components. One of these components is the polygon P, the other component P' either is a polygon component of $X - (\partial D \cup \partial E)$, or it contains a boundary component of X.

The complement of $P \cup P'$ in ∂V is a finite collection W of fibred rectangles. The base of such a rectangle is an edge in the boundary ∂P of P. The side of the rectangle opposite to the base is an arc in the boundary of P'. Since P is a topological disk, the same holds true for $\partial V - P'$. Now $\partial X \subset M$ is incompressible and therefore the subsurface P' of X is not boundary parallel. But this means that P' is a disk and hence ∂V is a two-sphere. Since M is aspherical, we conclude that V is a 3-ball.

As a consequence, each component V of $M - (D \cup E)$ which contains a polygonal component of $X - (\partial D \cup \partial E)$ in its boundary is a ball whose boundary consists of P, a finite union $\mathcal{R}$ of fibred rectangles with pase ∂P and a second polygonal component P' of $X - (\partial D \cup \partial E)$. The I-bundle structure on $\mathcal{R}$ naturally extends to an I-bundle structure on V. Therefore the union of $N(D) \cup N(E)$ with these components is an I-bundle whose boundary contains the complement of a small neighborhood of the boundary of X. The involution of the I-bundle preserves each component V of $M - (D \cup E)$ containing a polygon in $X - (\partial D \cup \partial E)$ in its boundary, and it exchanges the two components of $V \cap X$.

Now note that by construction, $\partial D, \partial E$ intersects the fixed point set of the involution only at two points, and these two points are contained in the interiors of the unique fibres of the bundle which are subarcs of the two outer components of $D - E, E - D$, respectively. As the intersection of this fixed point set with X is the generator γ of the I-bundle in the sense defined above, γ has all the properties stated in the Lemma. This completes the proof. $\square$

We use Lemma 2.3 to show

Proposition 2.4. *Let $D, E \subset M$ be essential disks with boundary in X. If there is an essential simple closed curve $\alpha \subset X$ which intersects $\partial D, \partial E$ in at most $k \geq 1$ points then $d_{\mathcal{S}}(D, E) \leq 2k + 4$.*

Proof. Let D, E be essential disks in normal position as in the proposition which are not disjoint.

Let α be an essential simple closed curve in X which intersects both ∂D and ∂E in at most $k \geq 1$ points. We may assume that these intersection points are disjoint from $\partial D \cap \partial E$.

Let $p \geq 2$ (or $q \geq 2$) be the number of outer components of $D - E$ (or of $E - D$). If $p = 2, q = 2$ then Lemma 2.3 shows that either $d_{\mathcal{S}}(D, E) \leq 1$ (in the case that $\partial D, \partial E$ do not fill up X) or $\partial D, \partial E$ intersect some I-bundle generator $\gamma \subset X$ in precisely two points, and we have $d_{\mathcal{S}}(D, E) = 1$.

Let $j \leq k, j' \leq k$ be the number of intersection points of D, E with α. As $d_{\mathcal{S}}(D, E) = 1$ if $j = j' = 0$ we may assume that $j + j' \geq 1$. Thus it suffices to show the following. If $\max\{p, q\} \geq 3$ then there is a simple surgery transforming the pair (D, E) to a pair (D', E') with the following properties.

1. (1) D' is disjoint from D, E' is disjoint from E.
2. (2) Either $D = D'$ or $E = E'$.
3. (3) The total number of intersections of α with $D' \cup E'$ is strictly smaller than $j + j'$.

To this end assume without loss of generality that $q \geq 3$. If $j/2 > j'/3$ then choose an outer component E_1 of $E - D$ with at most $j'/3$ intersections with α. This is possible because $E - D$ has at least three outer components. Let D_1 be the component of $D - E_1$ which intersects α in at most $j/2$ points. Replace D be the disk $D' = D_1 \cup E_1$ which is disjoint from D and has at most $j/2 + j'/3 < j$ intersections with α.

On the other hand, if $j/2 \leq j'/3$ then choose an outer component D_1 of $D - E$ with at most $j/2$ intersections with α. Let E_1 be the component of $E - D_1$ with at most $j'/2$ intersections with α and replace E by the disk $E' = E_1 \cup D_1$ which is disjoint from E and intersects α in at most $j/2 + j'/2 < j'$ points.

This is what we wanted to show. □

Remark: The arguments in this section use the fact that every simple surgery of a disk at an outer component of another disk yields an essential disk in M. They are not valid for surfaces $X \subset \partial M$ with compressible boundary.

3. Distance in the curve graph

The purpose of this section is to establish an estimate for the distance in the curve graph of a compact oriented surface X of genus $g \geq 0$ with $m \geq 0$ boundary components and $3g-3+m \geq 2$. This estimate which will be essential for a geometric description of the superconducting disk graph.

The curve graph of a compact oriented surface X with boundary coincides with the curve graph obtained from X by replacing each boundary component by a puncture. As considering surfaces with punctures rather than bordered surfaces has advantages for our exposition, we consider in the remainder of this section an arbitrary closed oriented surface S from which a finite set of points have been deleted. This results in this section are independent from the rest of the paper.

The idea is to use *train tracks* on S. We refer to [PH92] for all basic notions and constructions regarding train tracks.

A train track η (which may just be a simple closed curve) is *carried* by a train track τ if there is a map $F : S \to S$ of class C^1 which is homotopic to the identity, with $F(\eta) \subset \tau$ and such that the restriction of the differential dF of F to the tangent line of η vanishes nowhere. Write $\eta \prec \tau$ if η is carried by τ. If $\eta \prec \tau$ then the image of η under a carrying map is a *subtrack* of τ which does not depend on the choice of the carrying map. Such a subtrack is a subgraph of τ which is itself a train track. Write $\eta < \tau$ if η is a subtrack of τ.

A train track τ is called *large* [MM99] if each complementary component of τ is either simply connected or a once punctured disk. A simple closed curve η carried by τ *fills* τ if the image of η under a carrying map is all of τ. A *diagonal extension* of a large train track τ is a train track ξ which can be obtained from τ by subdividing some complementary components which are not trigons or once punctured monogons.

A *trainpath* on τ is an immersion $\rho : [k, \ell] \to \tau$ which maps every interval $[m, m+1]$ diffeomorphically onto a branch of τ. We say that ρ is *periodic* if $\rho(k) = \rho(\ell)$ and if the inward pointing tangent of ρ at $\rho(k)$ equals the outward pointing tangent of ρ at $\rho(\ell)$. Any simple closed curve carried by a train track τ defines a periodic trainpath and a *transverse measure* on τ. The space of transverse measures on τ is a cone in a finite dimensional real vector space. Each of its extreme rays is spanned by a *vertex cycle* which is a simple closed curve carried by τ. A vertex cycle defines a periodic trainpath which passes through every branch at most twice, in opposite direction (Lemma 2.2 of [H06], see also [Mo03]).

Let η be a large train track. If $\eta \prec \tau$ then τ is large as well. In particular, if $\eta' < \eta$ is a large subtrack of η and if ξ is a diagonal extension of η', then a carrying map $F : \eta \to \tau$ induces a carrying map of η' onto a large subtrack τ' of τ, and it induces a carrying map of ξ onto a diagonal extension of τ'.

Definition 3.1. A pair $\eta \prec \tau$ of large train tracks is called *wide* if every simple closed curve which is carried by a diagonal extension of a large subtrack of η fills a diagonal extension of a large subtrack of τ.

We have

Lemma 3.2. *If $\sigma \prec \eta \prec \tau$ and if the pair $\eta \prec \tau$ is wide then $\sigma \prec \tau$ is wide.*

Proof. Let σ' be a large subtrack of σ and let ξ be a diagonal extension of σ'. Then the carrying map $\sigma \to \eta$ maps σ' onto a large subtrack η' of η, and it maps ξ to a diagonal extension ζ of η'. Similarly, η' is mapped to a large subtrack τ' of τ, and ζ is mapped to a diagonal extension ρ of τ'.

A simple closed curve α carried by ξ is carried by ζ. In particular, since $\eta \prec \tau$ is wide, α fills a large subtrack of ρ. From this the lemma follows. □

A *splitting and shifting sequence* is a finite sequence $(\tau_i)_{0 \le i \le n}$ of large train tracks so that for each i, τ_{i+1} can be obtained from τ_i by a sequence of *shifts* followed by a single *split*. We refer to p.119 of [PH92] and p.192 of [H06] for the definition of a split and a shift of a train track on S. We allow the split to be a *collision* (see p.119 of [PH92]), i.e. a split followed by the removal of the diagonal of the split. Such a collision reduces the number of branches of the train track. Note that τ_{i+1} is carried by τ_i for all i and the pair $\tau_{i+1} \prec \tau_i$ is *never* wide. Namely, the cone of transverse measures for τ_{i-1} maps via the carrying map onto the subcone of the cone of transverse measures on τ_i obtained by intersecting the latter cone with a half-space. This implies that there exists an extreme ray of the cone for τ_{i+1} which also is an extreme ray for the cone for τ_i, and such an extreme ray is spanned by a vertex cycle c for τ_{i+1} which maps to a vertex cycle of τ_i. However, vertex cycles do not fill large subtracks ([H06], see also [Mo03]).

For an essential simple closed curve c on S let $i(c) \in \{0, \dots, n\}$ be the largest number with the following property. There is a large subtrack η of $\tau_{i(c)}$ so that c is carried by a diagonal extension ξ of η and fills ξ. If no such number exists then put $i(c) = 0$.

The curve graph $\mathcal{CG}$ of S is the graph whose vertices are essential simple closed curves on S and where two such curves c, d are connected by an edge of length one if they can be realized disjointly. Define a projection $P : \mathcal{CG} \to (\tau_i)_{0 \leq i \leq n}$ by

$$P(c) = \tau_{i(c)}.$$

Extend the map P to the edges of $\mathcal{CG}$ by mapping an edge to the image of one of its endpoints.

Lemma 3.3. *Let c, d be disjoint simple closed curves on S. Assume that $P(c) = \tau_i$. If $\tau_i \prec \tau_j$ is wide then $P(d) = \tau_s$ for some $s \geq j$.*

Proof. Assume that $P(c) = \tau_i \prec \tau_j$ is wide. By the definition of the map P, there is a large subtrack η of τ_i so that c fills a diagonal extension ξ of η. By Lemma 4.4 of [MM99], since d is disjoint from c, d is carried by a diagonal extension ζ of ξ. Then ζ is a diagonal extension of η.

Since $\tau_i \prec \tau_j$ is wide, d fills a diagonal extension of a large subtrack of τ_j. This implies that $P(d) = \tau_s$ for some $s \geq j$. □

Define a distance function d_g on $(\tau_i)_{0 \leq i \leq n}$ as follows. For $i < j$, the *gap distance* $d_g(\tau_i, \tau_j)$ between τ_i and τ_j is the smallest number $k > 0$ so that there is a sequence $i_0 = i < i_1 < \cdots < i_k = j$ with the property that for each $p < k$, the pair $\tau_{i_{p+1}} \prec \tau_{i_p}$ is *not* wide. Note that this defines indeed a distance since for each ℓ the pair $\tau_{\ell+1} \prec \tau_\ell$ is not wide and hence $d_g(\tau_i, \tau_j) \leq j - i$. Moreover, the triangle inequality is immediate from Lemma 3.2.

The following is a consequence of Lemma 3.3. For its formulation, define a map P from a metric space X to a metric space Y to be *coarsely L-Lipschitz* for some $L > 1$ if $d(Px, Py) \leq Ld(x, y) + L$ for all $x, y \in X$.

Corollary 3.4. *The map $P : \mathcal{CG} \to ((\tau_i), d_g)$ is coarsely 2-Lipschitz.*

Define a map $\Upsilon : (\tau_i)_{0 \leq i \leq n} \to \mathcal{CG}$ by associating to the train track τ_i one of its vertex cycles. We have

Lemma 3.5. *The map $\Upsilon : ((\tau_i), d_g) \to \mathcal{CG}$ is coarsely 32-Lipschitz.*

Proof. It suffices to show the following. If $\tau \prec \eta$ is not wide then the distance in $\mathcal{CG}$ between a vertex cycle α of τ and a vertex cycle β of η is at most 22.

To this end note that if α is a simple closed curve which is carried by a large train track ξ then the image of α under a carrying map is a subtrack of ξ. If this subtrack is not large then α is disjoint from an essential simple closed curve α' which can be represented by an edge-path in ξ (possibly with corners) which passes through any branch of ξ at most twice. Since a vertex cycle of ξ passes through each branch of

ξ at most twice [Mo03, H06], this implies that α' intersects any vertex cycle of ξ in at most 4 points (Corollary 2.3 of [H06]). In particular, the distance in $\mathcal{CG}$ between α and any vertex cycle of ξ is at most 6 [MM99].

On the other hand, if τ is another large train track and if ξ is a diagonal extension of a large subtrack τ' of τ then a vertex cycle of ξ intersects a vertex cycle of τ' (which is also a vertex cycle of τ) in at most 4 points. Hence the distance in $\mathcal{CG}$ between a vertex cycle of τ' and a vertex cycle of ξ is at most 5, and the distance between a vertex cycle of ξ and any vertex cycle of τ is at most 10. Together we deduce that the distance in $\mathcal{CG}$ between α and any vertex cycle of τ does not exceed 16.

Now by definition, if $\tau \prec \eta$ is not wide then there is a curve α which is carried by a diagonal extension ξ of a large subtrack τ' of τ and such that the following holds true. A carrying map $\tau \to \eta$ induces a carrying map of ξ onto a diagonal extension ζ of a large subtrack of η. The train track ζ carries α and so that α does not fill a large subtrack of ζ. Since a carrying map $\xi \to \zeta$ maps a large subtrack of ξ onto a large subtrack of ζ, the curve α does not fill a large subtrack of ξ.

By the above discussion, the distance in $\mathcal{CG}$ between α and any vertex cycle of both τ and η is at most 16. This shows the lemma. □

Call a map Φ of a metric space (X,d) into a subset A of X a *coarse Lipschitz retraction* if there is a number $L>1$ with the following properties.

(1) $d(\Phi(x),\Phi(y)) \leq Ld(x,y)+L$.
(2) $d(x,\Phi(x)) \leq L$ whenever $x \in A$.

We are now ready to show

Corollary 3.6. *For any splitting and shifting sequence $(\tau_i)_{0\leq i\leq n}$ the map $\Upsilon \circ P$ is a coarse L-Lipschitz retraction of $\mathcal{CG}$ for a number $L>1$ not depending on (τ_i) or on the Euler characteristic of S.*

Proof. Let $d_{\mathcal{CG}}$ be the distance in the curve graph of S. By Corollary 3.4 and Lemma 3.5 it suffices to show that $d_{\mathcal{CG}}(\alpha, \Upsilon \circ P(\alpha)) \leq L$ for a universal constant $L>1$ and every vertex cycle α of a train track τ_i from the sequence.

To this end observe that since α is a vertex cycle of τ_i, α is carried by each of the train tracks τ_j for $j \leq i$, moreover α does not fill a diagonal extension of a large subtrack of τ_i. On the other hand, by definition of a wide pair, if $\tau_i \prec \tau_j$ is wide then α fills a large subtrack of τ_j. This means that $P(\alpha)=\tau_s$ for some $s \geq j$ so that the pair $\tau_i \prec \tau_{s+1}$ is *not* wide. The corollary now follows from Lemma 3.5. □

Remark: The above discussion immediately implies that the image under Υ of a splitting and shifting sequence of train tracks is an unparametrized quasi-geodesic in $\mathcal{CG}$ for a constant not depending on the Euler characteristic of S. A non-effective version of this result was earlier established in [MM04] (see also [H06]).

4. Quasi-geodesics in the superconducting disk graph

In this section we resume the discussion of an oriented 3-manifold M and a subsurface X of the boundary of M whose boundary is incompressible in M.

Recall the definition of the graph $\mathcal{SDG}(X)$. Our goal is to show that the natural map which associates to a disk its boundary defines a quasi-isometric embedding of $\mathcal{SDG}(X)$ into the curve graph $\mathcal{CG}(X)$ of X. To simplify the notation we identify in the sequel a disk in M with boundary in X with its boundary circle. Thus we view the vertex set of $\mathcal{SDG}(X)$ as a subset of the curve graph $\mathcal{CG}(X)$ of X.

The argument is based on the results in Section 2-3 and a construction from [MM04]. This construction uses a specific type of surgery sequences of disks which can be related to train tracks as follows.

Let $D, E \subset M$ be two disks in normal position, with boundary in X. Let E' be an outer component of $E - D$ and let D_1 be a disk obtained from D by simple surgery at E'.

Let α be the intersection of $\partial E'$ with X. Then up to isotopy, the boundary ∂D_1 of the disk D_1 contains α as an embedded subarc. Moreover, α is disjoint from E. In particular, given an outer component E'' of $E - D_1$, there is a distinguished choice for a disk D_2 obtained from D_1 by simple surgery at E''. The disk D_2 is determined by the requirement that α is *not* a subarc of ∂D_2. Then for an outer component of $E - D_2$ there is a distinguished choice for a disk D_3 obtained from D_2 by simple surgery at an outer component of $E - D_2$ etc. We call a surgery sequence (D_i) of this form a *nested* surgery path in direction of E. Note that the boundary of each disk D_i is composed of a single subarc of ∂D and a single subarc of ∂E.

The following result is due to Masur and Minsky (this is Lemma 4.2 of [MM04] which is based on Lemma 4.1 and the proof of Theorem 1.2 in that paper).

Proposition 4.1. *Let $D, E \subset X$ be any disks. Let $D = D_0, \dots, D_n$ be a nested surgery path in the direction of E which connects D to a disk D_n disjoint from E. Then for each $i \leq n$ there is a train track τ_i on X with a single switch such that the following holds true.*

1. *τ_i carries ∂E and ∂E fills up τ_i.*
2. *$\tau_{i+1} \prec \tau_i$.*
3. *The disk D_i intersects τ_i only at the switch.*

The train tracks τ_i in the proposition are constructed as follows.

Let $\alpha = \partial D, \beta = \partial E$. Assume that the curves α, β are smooth (for a smooth structure on X) and fill up X. This means that the complementary components of $\alpha \cup \beta$ are all polygons or once holed polygons where in our setting, a hole is a boundary component of X. Let P be a complementary polygon which has at least 6 sides. Such a polygon exists since the Euler characteristic of X is negative. Its edges are subsegments of α and β. Let I be a boundary edge of P contained in α. Collapse $\alpha - I$ to a single point with a homotopy F of X. This can be done in such a way that the restriction of F to β is nonsingular everywhere. The resulting graph has a single vertex. Collapsing the bigons in the graph to single arcs yields a train track τ with a single switch [MM04].

Let $b \subset \beta$ be an outer arc for $E - D$ and let $a \subset \alpha - I$ be the subarc of α which is bounded by the endpoints of b and which does not intersect the interval I. Then $a \cup b$ is the boundary of a disk D_1 obtained from D by nested surgery at b. The new train track τ_1 obtained from the above construction is obtained from $\beta \cup a$ by collapsing the arc a to a single point (we refer to [MM04] for details).

In the formulation of the following result, $\chi(X)$ denotes the Euler characteristic of the surface X.

Theorem 4.2. *There is an explicit quadratic polynomial p such that the vertex inclusion defines a $p(|\chi(X)|)$-quasi-isometric embedding $\mathcal{SDG}(X) \to \mathcal{CG}(X)$. In particular, $\mathcal{SDG}(X)$ is a hyperbolic geodesic metric graph.*

Proof. As before, let $d_{\mathcal{S}}$ be the distance in $\mathcal{SDG}(X)$ and let $d_{\mathcal{CG}}$ be the distance in $\mathcal{CG}(X)$. We have to show the existence of a quadratic polynomial p with the following property. If D, E are any disks then

$$d_{\mathcal{S}}(D,E) \leq p(|\chi(X)|)d_{\mathcal{CG}}(\partial D, \partial E).$$

By Proposition 4.1, there is a nested surgery path $D = D_0, \dots, D_n$ connecting the disk $D_0 = D$ to a disk D_n which is disjoint from E, and there is a sequence $(\tau_i)_{0\leq i\leq n}$ of one-switch train tracks on X such that $\tau_{i+1} \prec \tau_i$ for all $i < n$ and that D_i intersects τ_i only at the switch.

By Theorem 2.3.1 of [PH92], there is a splitting and shifting sequence $\tau_0 = \eta_0 \succ \dots \succ \eta_s = \tau_n$ connecting τ_0 to τ_n and a sequence $0 = u_0 < \dots < u_n = s$ so that $\eta_{u_q} = \tau_q$ for $0 \leq q \leq n$. Since the disk D_i intersects τ_i only at the switch, the boundary ∂D_i of D_i intersects a vertex cycle of τ_i in at most two points and hence the distance in the curve graph $\mathcal{CG}(X)$ between ∂D_i and a vertex cycle of τ_i is at most three. Now the disks D_i and D_{i+1} are disjoint and consequently the distance in $\mathcal{CG}(X)$ between a vertex cycle of τ_i and a vertex cycle of τ_{i+1} is at most 7.

Corollary 3.6 implies that the map Υ which associates to the train track η_u one of its vertex cycles is a quasi-isometric embedding of the splitting and shifting sequence (η_u), equipped with the gap distance d_g, into the curve graph of X. The discussion in the previous paragraph implies that this statement also holds true for the restriction of Υ to the subsequence (τ_i) of (η_u), equipped with the restriction of the gap distance. Thus by the definition of the gap distance, it suffices to show the existence of a universal number $b > 0$ with the following property. Let $k < i$ be such that the pair $\tau_i \prec \tau_k$ is *not* wide; then $d_{\mathcal{S}}(D_i, D_k) \leq b|\chi(X)|^2$.

Since $\tau_i \prec \tau_k$ is not wide there is a large subtrack τ_i' of τ_i, a diagonal extension ζ_i of τ_i' and a simple closed curve α carried by ζ_i with the following property. Let τ_k' be the image of τ_i' under a carrying map $\tau_i \to \tau_k$ and let ζ_k be the diagonal extension of τ_k' which is the image of ζ_i under a carrying map induced by a carrying map $\tau_i' \to \tau_k'$. Then α does not fill a large subtrack of ζ_k.

Since ζ_i is a diagonal extension of the large subtrack τ_i' of τ_i and since D_i intersects τ_i only at the switch, the intersection number between ∂D_i and ζ_i is bounded from above by a constant $\kappa \geq 2$ which does not exceed a constant multiple of the Euler characteristic of X.

For each $p \in [k, i]$, the image of τ_i' under a carrying map $\tau_i \to \tau_p$ is a large subtrack τ_p' of τ_p, and there is a diagonal extension ζ_p of τ_p' which carries α. We may assume that $\zeta_u \prec \zeta_p$ for $u \geq p$. The disk D_p intersects ζ_p in at most κ points.

For $p \in [k, i]$ let $\beta_p \prec \zeta_p$ be the subtrack of ζ_p filled by α. Then β_p is connected and not large. The union Y_p of a thickening of β_p with the components of $X - \beta_p$ which are simply connected is a proper connected subsurface of X for all p. The boundary of Y_p can be realized as a union of simple closed curves which are embedded in β_p (but with cusps). The carrying map $\beta_{p+1} \to \beta_p$ maps Y_{p+1} into Y_p. In particular, either the boundary of Y_{p+1} coincides up to homotopy with the boundary of Y_p or Y_{p+1} is a *proper* subsurface of Y_p. In the latter case, the Euler characteristic of Y_p is strictly smaller than the Euler characteristic of Y_{p+1}. In other words, the subsurfaces Y_p are nested, and hence their number is bounded from above by a universal constant $h > 0$ depending linearly on the Euler characteristic of S.

Since ∂D_p intersects ζ_p in at most κ points, the number of intersections between ∂D_p and ∂Y_p is bounded from above by $2\kappa > 0$. As a consequence, there are h essential simple closed curves $c_1, \dots, c_h$ in X so that for every $p \in [k, i]$ there is some $r(p) \in \{1, \dots, h\}$ with

$$\iota(\partial D_p, c_{r(p)}) \leq 2\kappa. \tag{2}$$

Each of the curves c_j is a fixed boundary component of one of the subsurfaces Y_p.

By reordering, assume that $r(i) = 1$. Let v_1 be the minimum of all numbers $p \in [k, i]$ such that $r(v_1) = 1$. Proposition 2.4 shows that

$$d_{\mathcal{S}}(D_i, D_{v_1}) \leq 4\kappa + 4. \tag{3}$$

On the other hand, we have $d_{\mathcal{S}}(D_{v_1}, D_{v_1 - 1}) = 1$. Again by reordering, assume that $r(v_1 - 1) = 2$ and repeat this construction with the disks $D_{v_1 - 1}, \dots, D_k$ and the curve c_2. In $a \leq h$ steps we construct in this way a decreasing sequence $i \geq v_1 > \dots > v_a = k$ such that $d_{\mathcal{S}}(D_{v_u}, D_{v_{u-1}}) \leq 4\kappa + 5$ for all $u \leq a$. From (2, 3) we conclude that

$$d_{\mathcal{S}}(D_i, D_k) \leq h(4\kappa + 5).$$

Together with the explicit bounds for κ and h, this yields the theorem. □

5. Gromov boundary in the case of handlebodies

A hyperbolic geodesic metric space Y admits a *Gromov boundary*. This boundary is a topological space on which the isometry group of Y acts as a group of homeomorphisms. In this section we explicitly determine the Gromov boundary of the superconducting disk graph $\mathcal{SDG} = \mathcal{SDG}(\partial H)$ for a handlebody H of genus $g \geq 2$. Recall that the boundary ∂H of H is a closed oriented surface of genus g.

Let $\mathcal{L}$ be the space of all *geodesic laminations* on ∂H (for some fixed hyperbolic metric) equipped with the *coarse Hausdorff topology*. In this topology, a sequence (μ_i) converges to a lamination μ if every accumulation point of (μ_i) in the usual Hausdorff topology contains μ as a sublamination. Note that the coarse Hausdorff topology on $\mathcal{L}$ is not T_0, but its restriction to the subspace $\partial\mathcal{CG} \subset \mathcal{L}$ of all minimal geodesic laminations which fill up ∂H (i.e. which intersect every simple closed geodesic transversely) is Hausdorff. The space $\partial\mathcal{CG}$ equipped with the coarse Hausdorff topology can naturally be identified with the Gromov boundary of the curve graph $\mathcal{CG}$ of ∂H [K99, H06].

Let $\mathrm{Map}(H)$ be the *handlebody group*, which is defined to be the subgroup of the mapping class group $\mathrm{Mod}(\partial H)$ of the boundary surface consisting of all isotopy classes of diffeomorphisms which extend to diffeomorphisms of H. The group $\mathrm{Map}(H)$ acts on the graph $\mathcal{SDG}$ as a group of simplicial automorphisms.

The handlebody group $\mathrm{Map}(H)$ also acts on $\partial\mathcal{CG}$ as a group of transformations preserving the closed subset

$$\partial\mathcal{H} \subset \partial\mathcal{CG}$$

of all geodesic laminations which are limits in the coarse Hausdorff topology of boundaries of disks in H. It acts on the Gromov boundary $\partial\mathcal{SDG}$ of $\mathcal{SDG}$ as well.

Lemma 5.1. *The Gromov boundary of $\mathcal{SDG}$ is a closed* $\mathrm{Map}(H)$*-invariant subset of $\partial\mathcal{H}$.*

Proof. Since by Theorem 4.2 the vertex inclusion $\mathcal{SDG} \to \mathcal{CG}$ defines a quasi-isometric embedding, the Gromov boundary of $\mathcal{SDG}$ is the subset of the Gromov boundary of $\mathcal{CG}$ of all endpoints of quasi-geodesic rays in $\mathcal{CG}$ which are contained in $\mathcal{SDG}$.

By the main result of [H06] (see [K99] for an earlier account of a similar statement), a simplicial quasi-geodesic ray $\gamma : [0,\infty) \to \mathcal{CG}$ defines the endpoint lamination $\nu \in \partial\mathcal{CG}$ if and only if the curves $\gamma(i)$ converge as $i \to \infty$ in the coarse Hausdorff topology to ν. As a consequence, the Gromov boundary of $\mathcal{SDG}$ is a subset of $\partial\mathcal{H}$, and this subset is clearly $\mathrm{Map}(H)$-invariant.

We are left with showing that the Gromov boundary of $\mathcal{SDG}$ is a closed subset of $\partial\mathcal{CG}$. To this end note that by Theorem 4.2, there is a number $p > 1$ such that for every $L > 1$, any L-quasi-geodesic in $\mathcal{SDG}$ is an Lp-quasi-geodesic in $\mathcal{CG}$. Moreover, for a suitable choice of p, any vertex in $\mathcal{SDG}$ can be connected to any point in the Gromov boundary of $\mathcal{SDG}$ by a p-quasi-geodesic.

Now let (ν_i) be a sequence in the Gromov boundary of $\mathcal{SDG}$ which converges in $\partial\mathcal{CG}$ to a lamination ν. Let ∂D be the boundary of a disk and let $\gamma : [0,\infty) \to \mathcal{CG}$ be a quasi-geodesic ray issuing from $\gamma(0) = \partial D$ with endpoint ν. By hyperbolicity of $\mathcal{CG}$ and by the discussion in the previous paragraph, there is a number $R > 0$ and for every $k \geq 0$ there is some $i(k) > 0$ such that a p-quasi-geodesic in $\mathcal{SDG} \subset \mathcal{CG}$ connecting $\gamma(0)$ to $\nu_{i(k)}$ passes through the R-neighborhood of $\gamma(k)$ in $\mathcal{CG}$. Since $k > 0$ was arbitrary, this implies that the entire quasi-geodesic ray γ is contained in the R-neighborhood of the subset $\mathcal{SDG}$ of $\mathcal{CG}$. Using once more hyperbolicity, we conclude that there is a quasi-geodesic ray in $\mathcal{CG}$ connecting $\gamma(0)$ to ν which is entirely contained in $\mathcal{SDG}$. But this just means that ν is contained in the Gromov boundary of $\mathcal{SDG}$. □

By naturality, the action of the handlebody group $\mathrm{Map}(H)$ on the Gromov boundary $\partial\mathcal{SDG}$ of $\mathcal{SDG}$ is compatible with the action of the mapping class group on the Gromov boundary of the curve graph. From Lemma 5.1 and the following observation (which is essentially contained in Theorem 1.2 of [M86]), we conclude that $\partial\mathcal{H}$ is indeed the Gromov boundary of $\mathcal{SDG}$.

Lemma 5.2. *The action of the handlebody group* $\mathrm{Map}(H)$ *on* $\partial\mathcal{H}$ *is minimal.*

Proof. Let (∂D_i) be a sequence of boundaries of disks D_i converging in the coarse Hausdorff topology to a geodesic lamination $\mu \in \partial\mathcal{H}$. For each i let E_i be a disk which is disjoint from D_i. Since the space of geodesic laminations equipped with the usual Hausdorff topology is compact, up to passing to a subsequence the sequence (∂E_i) converges in the Hausdorff topology to a geodesic lamination ν which does not intersect μ (we refer to [K99, H06] for details of this argument). Now μ is minimal and fills up ∂H and therefore the lamination ν contains μ as a sublamination. This just means that (∂E_i) converges in the coarse Hausdorff topology to μ.

Since the genus of H is at least two, for every separating disk in H we can find a disjoint non-separating disk. Thus the discussion in the previous paragraph shows that every $\mu \in \partial\mathcal{H}$ is a limit in the coarse Hausdorff topology of a sequence of non-separating disks. However, the handlebody group acts transitively on non-separating disks. Minimality of the action of $\mathrm{Map}(H)$ on $\partial\mathcal{H}$ follows. □

As an immediate consequence of Lemma 5.1 and Lemma 5.2 we obtain

Corollary 5.3. *$\partial\mathcal{H}$ is the Gromov boundary of $\mathcal{SDG}$.*

6. Disk graphs for surfaces with compressible boundary

In this final section we show that the results from Sections 2-4 are not valid for graphs of disks in an oriented aspherical 3-manifold M with boundary in a compact subsurface X of the boundary of M of genus $g \geq 2$, with connected compressible boundary. As before, disks are required to be essential in M, and their boundaries are required to be essential curves in X, in particular their boundaries are not allowed to be homotopic to a boundary component of X. We continue to use the terminology from Section 2.

Let X_0 be obtained from X by capping off the boundary ∂X (i.e. identify the boundary ∂X with a single point). Note that X_0 can be viewed as a subsurface of the boundary of a submanifold M_0 of M with boundary. There exists a natural map $\Phi : X \to X_0$.

Let $\mathcal{CG}(X)$ be the curve graph of X and let $\mathcal{CG}(X_0)$ be the curve graph of X_0. The following simple and well known fact is the essential feature that distinguishes the case of a single compressible boundary component from the case of more than one compressible boundary components.

Lemma 6.1. *The map* Φ *induces a simplicial surjection*

$$\Pi : \mathcal{CG}(X) \to \mathcal{CG}(X_0)$$

which maps diskbounding curves to diskbounding curves.

Proof. Since ∂X has connected compressible boundary, the image under the map Φ of an essential simple closed curve γ on X is an essential simple closed curve $\Phi(\gamma)$ on X_0. The curve γ is diskbounding if and only if this is the case for $\Phi(\gamma)$. Moreover, if γ, δ are disjoint then this holds true for $\Phi(\gamma), \Phi(\delta)$ as well. This immediately implies the lemma. □

The special property of a boundary surface with connected compressible boundary which enters Lemma 6.1 is also reflected in the fact that we can use surgery of disks to construct paths in the disk graph which reduce distances. Namely, for any two disks D, E with boundary in X which are not disjoint and for any outer component E' of $E - D$, at least one of the disks obtained from D by surgery at E' is not peripheral. Thus if we denote as before by $d_{\mathcal{D}}$ the distance in the disk graph then the proof of Lemma 2.1 yields the following

Lemma 6.2. *If* X *has connected compressible boundary then for any disks* D, E *in* M *we have*

$$d_{\mathcal{D}}(D, E) \leq \iota(\partial D, \partial E)/2 + 1.$$

We complete the article with the proof of Theorem 2.

Proposition 6.3. *The graph of disks with boundary in* X *is not a quasi-convex subset of the curve graph of* X*.*

Proof. The curve graph $\mathcal{CG}(X)$ of X is hyperbolic, and pseudo-Anosov elements of the mapping class group of X act as hyperbolic isometries.

Let p be the image of the boundary of X under the map $\Phi : X \to X_0$. View the point p as a basepoint for the fundamental group of X_0. Denote by Γ the quotient of the mapping class group of X by its center, which is just the mapping class group of a surface of genus g with one marked point. In the *Birman exact sequence*

$$0 \to \pi_1(X_0, p) \xrightarrow[\Psi]{\Gamma} \mathrm{Mod}(X_0) \to 0, \tag{4}$$

an element $\gamma \in \pi_1(X_0, p)$ is mapped to a so-called *point-pushing map* $\Psi(\gamma) \in \Gamma$. If $\gamma \in \pi_1(X_0, p)$ is *filling*, i.e. if γ decomposes X_0 into disks, then the image $\Psi(\gamma)$ of γ in Γ via the Birman exact sequence is pseudo-Anosov [Kr81, KLS09]. Choose such a filling curve γ and assume that a quasi-axis ζ for $\Psi(\gamma)$, viewed as a hyperbolic isometry of $\mathcal{C}calG(X)$, passes uniformly near the boundary of a disk.

Let φ be a diffeomorphism of X_0 which fixes the basepoint p and which defines a pseudo-Anosov element of $\mathrm{Mod}(X_0)$. We require that a quasi-axis for the action of φ on $\mathcal{CG}(X_0)$ passes uniformly near the boundary of a disk and that moreover for any diskbounding simple closed curve β, the distance in $\mathcal{CG}(X_0)$ between $\varphi^k\beta$ and the quasi-convex subset of diskbounding curves tends to infinity as $k \to \infty$. Note that the results from Section 4 apply to the graph of disks in the 3-manifold M_0 with boundary in X_0.

Such a pseudo-Anosov element can be found as follows. Each pseudo-Anosov element fixes two projective measured laminations which fill up X_0. This means that the complementary components of the lamination are all simply connected. The set of pairs of such fixed points is dense in $\mathcal{PML} \times \mathcal{PML}$ (here $\mathcal{PML}$ denote the Thurston sphere of projective measured geodesic laminations on X_0). The closure in $\mathcal{PML}$ of the set of diskbounding simple closed curves is nowhere dense in $\mathcal{PML}$ (see [M86] for details and note that by standard 3-dimensional topology, the image of the fundamental group of X_0 in the fundamental group of M_0 can not be trivial as the genus of X_0 is at least two). Any pseudo-Anosov element φ whose pair of fixed points is contained in the complement will do. We also assume that the fixed point sets of φ and $\Psi(\gamma)$ on $\mathcal{PML}$ are disjoint.

Since φ fixes the point p, it acts on the fundamental group $\pi_1(X_0, p)$ of X_0, moreover it can be viewed as an element of Γ. We denote this element of Γ again by φ.

The idea is now to conjugate the point-pushing map $\Psi(\gamma)$ by high powers φ^k of φ. The resulting mapping class is pseudo-Anosov, and $\varphi^k\zeta$ is a quasi-axis for its action on $\mathcal{CG}(X_0)$, with all constants uniform in k. As φ acts with north-south dynamics on $\mathcal{PML}$ and on the Gromov boundary of the curve graph of X_0, by hyperbolicity of $\mathcal{CG}(X_0)$, for large enough k the quasi-axis $\varphi^k\zeta$ for the action of $\varphi^k \circ \Psi(\gamma) \circ \varphi^{-k}$ is arbitrarily far from the quasi-convex disk set.

Now let $\beta \subset X_0 - \{p\}$ be a diskbounding curve near a quasi-axis of φ. Since β avoids p we can view β as a diskbounding curve in X. For each $k > 0$ let β_k be the image of β under point-pushing along the curve $\varphi^k(\gamma)$. Then β_k is diskbounding, moreover we have

$$\beta_k = (\varphi^k \circ \Psi(\gamma) \circ \varphi^{-k})(\beta).$$

By hyperbolicity of $\mathcal{CG}(X)$, via perhaps replacing γ by a multiple (and hence replacing the point pushing map $\Psi(\gamma)$ by some power) we may assume that a geodesic in $\mathcal{CG}(X)$ connecting β to β_k is close in the Hausdorff topology to the composition of three arcs. The first arc connects β to the quasi-axis $\varphi^k(\zeta)$ of $\varphi^k \circ \Psi(\gamma) \circ \varphi^{-k}$, the second arc travels along $\varphi^k(\zeta)$, and the third arc connects $\varphi^k(\zeta)$ to β_k.

However, by the choice of φ, for suitable choices of k and suitable multiplicities of γ, such a geodesic contains points which are arbitrarly far in $\mathcal{CG}(X)$ from the set of diskbounding curves. □

Remark 6.4. The proof of Proposition 6.3 replies implicitly on the fact that the point pushing subgroup of the mapping class group $\mathrm{Mod}(X)$ of surface of genus $g \geq 2$ with a single puncture is exponentially distorted in $\mathrm{Mod}(X)$.

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