GEOMETRY OF GRAPHS OF DISCS IN A HANDLEBODY:
SURGERY AND INTERSECTION

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Abstract. For a handlebody $H$ of genus $g \geq 2$ we use surgery to identify a graph whose vertices are discs and which is quasi-isometrically embedded in the curve graph of the boundary surface.

1. Introduction

The curve graph $CG$ of an oriented surface $S$ of genus $g \geq 0$ with $m \geq 0$ punctures and $3g - 3 + m \geq 2$ is the graph whose vertices are isotopy classes of essential (i.e. non-contractible and not homotopic into a puncture) simple closed curves on $S$. Two such curves are connected by an edge of length one if and only if they can be realized disjointly. The curve graph is a locally infinite hyperbolic geodesic metric space of infinite diameter [MM99].

The mapping class group $\text{Mod}(S)$ of all isotopy classes of orientation preserving homeomorphisms of $S$ acts on $CG$ as a group of simplicial isometries. This action is coarsely transitive, i.e. the quotient of $CG$ under this action is a finite graph. Curve graphs and their geometric properties turned out to be an important tool for the investigation of the geometry of $\text{Mod}(S)$ [MM00].

For mapping class groups of other manifolds, much less is known. Examples of manifolds with large and interesting mapping class groups are handlebodies. A handlebody of genus $g \geq 1$ is a compact three-dimensional manifold $H$ which can be realized as a closed regular neighborhood in $\mathbb{R}^3$ of an embedded bouquet of $g$ circles. Its boundary $\partial H$ is an oriented surface of genus $g$. The group $\text{Map}(H)$ of all isotopy classes of orientation preserving homeomorphisms of $H$ is called the handlebody group of $H$. The homomorphism which associates to an orientation preserving homeomorphism of $H$ its restriction to $\partial H$ induces an embedding of $\text{Map}(H)$ into $\text{Mod}(\partial H)$. We refer to [HH12] for more and for references.

For a number $L > 1$, a map $\Phi$ of a metric space $X$ into a metric space $Y$ is called an $L$-quasi-isometric embedding if

$$d(x, y)/L - L \leq d(\Phi x, \Phi y) \leq L d(x, y) + L \text{ for all } x, y \in X.$$  

It is called an $L$-quasi-isometry if in addition the $L$-neighborhood of $\Phi(X)$ is all of $Y$. The restriction homomorphism $\text{Map}(H) \to \text{Mod}(\partial H)$ is not a quasi-isometric embedding [HH12] and hence understanding the geometry of the mapping class group does not lead to an understanding of the geometry of $\text{Map}(H)$. In fact, the geometry of the handlebody group is quite different from the geometry of the mapping class group. Namely, the mapping class group is known to have quadratic Dehn

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function [Mo95] while the Dehn function of the handlebody group is exponential (in preparation).

To understand the difference between the geometry of these groups in a more systematic way one can try to develop tools for the handlebody groups which are similar to the tools used for the investigation of the mapping class group. In particular, there is an analog of the curve graph for a handlebody which is defined as follows.

An essential disc in $H$ is a properly embedded disc $(D, \partial D) \subset (H, \partial H)$ whose boundary $\partial D$ is an essential simple closed curve in $\partial H$.

**Definition 1.** The disc graph $\mathcal{DG}$ of $H$ is the graph whose vertices are isotopy classes of essential discs in $H$. Two such discs are connected by an edge of length one if and only if they can be realized disjointly.

Thus the disc graph can be viewed as the complete subgraph of the curve graph of $\partial H$ whose vertices are disbounding simple closed curves. The disc graph is a quasi-convex subgraph of the curve graph [MM04], and it is hyperbolic [MS13]. However, it is not quasi-isometrically embedded in the curve graph [MS13].

The goal of this note is to use surgery to establish an effective understanding of the relation between the geometry of the disc graph and the geometry of the curve graph. In a sequel to this paper [H11] we apply these results to obtain an efficient proof of hyperbolicity of the disc graph with only topological tools which moreover gives effective bounds on the hyperbolicity constants. Our methods equally apply to the investigation of sphere graphs in the connected sum of $g$ copies of $S^1 \times S^2$ and can be used for the investigation of the outer automorphism group of the free group.

To formulate our results we introduce another Map($H$)-graph whose vertices are discs. This graph is the analog of the free factor graph of a free group [KL09] and is defined as follows.

**Definition 2.** The electrified disc graph $\mathcal{EDG}$ is the graph whose vertices are isotopy classes of essential discs in $H$ and where two vertices $D_1, D_2$ are connected by an edge of length one if and only if there is an essential simple closed curve on $\partial H$ which can be realized disjointly from both $\partial D_1, \partial D_2$.

Since for any two disjoint essential simple closed curves $c, d$ on $\partial H$ there is a simple closed curve on $\partial H$ which can be realized disjointly from $c, d$ (e.g. one of the curves $c, d$), the electrified disc graph is obtained from the disc graph by adding some edges.

Call a simple closed curve $c$ on $\partial H$ dis bustling if $c$ has an essential intersection with the boundary of every disc.

Define an $I$-bundle generator for $H$ to be a dis bustling simple closed curve $c$ on $\partial H$ with the following property. There is a compact surface $F$ with connected boundary $\partial F$, and there is a homeomorphism of the orientable $I$-bundle $I(F)$ over $F$ onto $H$ which maps $\partial F$ to $c$. The curve $c$ is separating if and only the surface $F$ is orientable. The handlebody group preserves the set of $I$-bundle generators.

**Definition 3.** If $m = 0$ then the super-conducting disc graph is the graph $\mathcal{SDG}$ whose vertices are isotopy classes of essential discs in $H$ and where two vertices $D_1, D_2$ are connected by an edge of length one if and only if one of the following two possibilities holds.
There is a simple closed curve on $\partial H$ which can be realized disjointly from both $\partial D_1, \partial D_2$.

(2) There is an $I$-bundle generator $c$ for $H$ which intersects both $\partial D_1, \partial D_2$ in precisely two points.

In particular, the superconducting disc graph is obtained from the electrified disc graph by adding some edges.

Since the distance in the curve graph $CG$ of $\partial H$ between two simple closed curves which intersect in two points does not exceed 3 [MM99], the natural vertex inclusion extends to a coarse 6-Lipschitz map $SDG \to CG$. We show

**Theorem 1.** The natural vertex inclusion extends to a quasi-isometric embedding $SDG \to CG$.

The constants for the quasi-isometric embeddings are effectively computable and bounded from above by a cubic polynomial in the genus of $\partial H$.

A hyperbolic geodesic metric space admits a Gromov boundary. The Gromov boundary of the curve graph of $\partial H$ can be described as follows [K99, H06].

The space $L$ of geodesic laminations on $\partial H$ can be equipped with the coarse Hausdorff topology. In this topology, a sequence $(\lambda_i)$ of geodesic laminations converges to a geodesic lamination $\lambda$ if every accumulation point of $(\lambda_i)$ in the usual Hausdorff topology contains $\lambda$ as a sublamination. This topology is not $T_0$, but its restriction to the subset $\partial CG$ of all minimal geodesic laminations which fill up $\partial H$ (i.e. which have an essential intersection with every simple closed curve in $\partial H$) is Hausdorff. The Gromov boundary of the curve graph can naturally be identified with the subspace $\partial CG$ of $L$ equipped with the coarse Hausdorff topology [K99, H06].

Let

$$\partial H \subset \partial CG$$

be the closed subset of all points which are limits in the coarse Hausdorff topology of sequences of boundaries of discs in $H$. As a fairly easy consequence of Theorem 1 we obtain

**Corollary.** $\partial H$ with the coarse Hausdorff topology is the Gromov boundary of $SDG$.

To use the program developed by Masur and Minsky [MM99, MM00] for the investigation of the geometry of the handlebody groups one also has to understand disc graphs of handlebodies with spots, i.e. handlebodies with marked points on the boundary. Here as before, the disc graph of such a spotted handlebody is the complete subgraph of the curve graph of the marked boundary whose vertex set is the set of discbounding curves. The electrified disc graph is the graph whose vertices are discs which are connected by an edge of length one if either they are disjoint or if they are disjoint from a common essential simple closed curve.

For disc graphs of spotted handlebodies we obtain

**Theorem 2.** Let $H$ be a handlebody of genus $g \geq 2$ with $m \geq 1$ spots on the boundary.

(1) If $m = 1$ then the disc graph is not a quasi-convex subset of the curve graph.

(2) For $m \geq 2$ the map which associates to a disc its boundary is a 16-quasi-isometry $EDG \to CG$. 
In [H11] we show that the disc graph in handlebodies with two spots on the boundary is much more complicated and can effectively be used to understand cycles in the handlebody group which require exponential filling.

The organization of this paper is as follows. In Section 2 we use surgery of discs to relate the distance in the superconducting disc graph of a handlebody $H$ without spots to intersection numbers of boundary curves.

In Section 3 we give an effective estimate of the distance in the curve graph using train tracks. This together with a construction of [MM04] is used in Section 4 to show Theorem 1 and the corollary. The proof of Theorem 2 is contained in Section 5 and Section 6.

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2. Distance and intersection

In this section we consider a handlebody $H$ of genus $g \geq 2$ without spots on the boundary. We use surgery of discs to establish first estimates for the distance in the electrified disc graph $EDG$ and in the superconducting disc graph $SDG$ of $H$. We begin with introducing the basic surgery construction needed later on.

By a disc in the handlebody $H$ we always mean an essential disc in $H$. Two discs $D_1, D_2$ are in normal position if their boundary circles intersect in the minimal number of points and if every component of $D_1 \cap D_2$ is an embedded arc in $D_1 \cap D_2$ with endpoints in $\partial D_1 \cap \partial D_2$. In the sequel we always assume that discs are in normal position; this can be achieved by modifying one of the two discs with an isotopy.

Let $D$ be any disc and let $E$ be a disc which is not disjoint from $D$. A component $\alpha$ of $\partial E - D$ is called an outer arc of $\partial E$ relative to $D$ if there is a component $E'$ of $E - D$ whose boundary is composed of $\alpha$ and an arc $\beta \subset D$. The interior of $\beta$ is contained in the interior of $D$. We call such a disc $E'$ an outer component of $E - D$. An outer component of $E - D$ intersects $\partial H$ in an outer arc $\alpha$ relative to $D$, and $\alpha$ intersects $\partial D$ in opposite directions at its endpoints.

For every disc $E$ which is not disjoint from $D$ there are at least two distinct outer components $E', E''$ of $E - D$. There may also be components of $\partial E - D$ which leave and return to the same side of $D$ but which are not outer arcs. An example of such a component is a subarc of $\partial E$ which is contained in the boundary of a rectangle component of $E - D$ leaving and returning to the same side of $D$. The boundary of such a rectangle consists of two subarcs of $\partial E$ with endpoints on $\partial D$ which are homotopic relative to $\partial D$, and two arcs contained in $D$.

Let $E' \subset E$ be an outer component of $E - D$ whose boundary is composed of an outer arc $\alpha$ and a subarc $\beta = E' \cap D$ of $D$. The arc $\beta$ decomposes the disc $D$ into two half-discs $P_1, P_2$. The unions $Q_1 = E' \cup P_1$ and $Q_2 = E' \cup P_2$ are embedded discs in $H$ which up to isotopy are disjoint and disjoint from $D$. For $i = 1, 2$ we say that the disc $Q_i$ is obtained from $D$ by simple surgery at the outer component $E'$ of $E - D$ (see e.g. [S00] for this construction). Since $D, E$ are in minimal position, the discs $Q_1, Q_2$ are essential.
Each disc in $H$ can be viewed as a vertex in the disc graph $\mathcal{DG}$, the electrified disc graph $\mathcal{EDG}$ and the superconducting disc graph $\mathcal{SDG}$. We will work with all three graphs simultaneously. Denote by $d_D$ (or $d_E$ or $d_S$) the distance in $\mathcal{DG}$ (or in $\mathcal{EDG}$ or in $\mathcal{SDG}$). Note that for any two discs $D, E$ we have
$$d_S(D,E) \leq d_E(D,E) \leq d_D(D,E).$$

In the sequel we always assume that all curves and multicurves on $\partial H$ are essential. For two simple closed multicurves $c,d$ on $\partial H$ let $\iota(c,d)$ be the geometric intersection number between $c,d$. The following lemma [MM04] implies that the graph $\mathcal{DG}$ is connected. We provide the short proof for completeness.

**Lemma 2.1.** Let $D,E \subset H$ be any two discs. Then $D$ can be connected to a disc $E'$ which is disjoint from $E$ by at most $\iota(\partial D, \partial E)/2$ simple surgeries. In particular,
$$d_D(D,E) \leq \iota(\partial D, \partial E)/2 + 1.$$ 

**Proof.** Let $D, E$ be two discs in normal position. Assume that $D, E$ are not disjoint. Then there is an outer component of $E - D$. A disc $D'$ obtained by simple surgery of $D$ at this component is essential in $\partial H$. Moreover, $D'$ is disjoint from $D$, i.e. we have $d_D(D', D) = 1$, and
$$\iota(\partial E, \partial D') \leq \iota(\partial D, \partial E) - 2.$$ 

The lemma now follows by induction on $\iota(\partial D, \partial E)$. \hfill \Box

**Lemma 2.2.** Let $D, E \subset H$ be discs. If there is an essential simple closed curve $\alpha \subset \partial H$ which intersects $\partial E$ in at most one point and which intersects $\partial D$ in at most $k \geq 1$ points then $d_E(D, E) \leq \log_2 k + 3.$

**Proof.** Let $D, E \subset H$ be any two discs. If $D, E$ are disjoint then there is nothing to show, so assume that $\partial D \cap \partial E \neq \emptyset$. Let $\alpha \subset \partial H$ be a simple closed curve which intersects $\partial E$ in at most one point and which intersects $\partial D$ in $k \geq 0$ points. If $\alpha$ is disjoint from both $D, E$ then $d_E(D, E) \leq 1$ by definition of the electrified disc
A simple closed multicurve $\gamma$ in $\partial H$ is called discbusting if $\gamma$ intersects every disc.

**Definition 2.3.** An I-bundle generator in $\partial H$ is a discbusting simple closed curve $\gamma \subset \partial H$ with the following property. There is an oriented I-bundle $\mathcal{J}(F)$ over a compact surface $F$ with connected boundary $\partial F$, and there is an orientation preserving homeomorphism of $\mathcal{J}(F)$ onto $H$ which maps $\partial F$ to $\gamma$.

We call the surface $F$ the base of the I-bundle generated by $\gamma$.

If $\gamma$ is a separating I-bundle generator, with base surface $F$, then $F$ is orientable and we have $g = 2n$ for some $n \geq 1$. Moreover, the I-bundle $\mathcal{J}(F) = F \times [0,1]$ is trivial. The I-bundle over every essential arc in $F$ with endpoints in $\partial F$ is an embedded disc in $H$. If $\gamma$ is non-separating then the base $F$ of the I-bundle is non-orientable.

There is an orientation reversing involution $\Phi : H \to H$ whose fixed point set intersects $\partial H$ precisely in $\gamma$. This involution acts as a reflection in the fiber. The union of an essential embedded arc $\alpha$ in $F$ with endpoints on $\partial F$ with its image under $\Phi$ is the boundary of a disc in $H$ (there is a small abuse of notation here—since the fixed point set of $\Phi$ intersects $\partial H$ in a subset of the fibre over $\partial F$). This disc is just the I-bundle over $\alpha$.

If $D, E \subset H$ are discs in normal position then each component of $D - E$ is a disc and therefore the graph dual to the cell decomposition of $D$ whose two-cells are the components of $D - E$ is a tree. If $D - E$ only has two outer components then this tree is just a line segment. The following lemma analyzes the case that this holds true for both $D - E$ and $E - D$.

**Lemma 2.4.** Let $D, E \subset H$ be discs in normal position. If $D - E$ and $E - D$ only have two outer components then one of the following two possibilities is satisfied.

1. $d_\xi(D, E) \leq 4$. 

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\[ \gamma \subset \partial H \text{ intersects } I_\alpha \cup \gamma \]

\[ \text{union of an essential embedded arc } \]

\[ \text{a compact surface } F \]

\[ \text{embedded disc in } \]

\[ \text{Definition 2.3.} \]

\[ \text{disc}. \]

\[ \text{We have } \]

\[ \partial H \text{ since the fixed point set of } \Phi \text{ intersects } J_\beta \]

\[ \text{J}. \]

\[ \text{Thus by perhaps exchanging } \]

\[ \text{D}_\beta \text{ and we have } \]

\[ \text{graph. Thus by perhaps exchanging } D \text{ and } E \text{ we may assume that } k \geq 1. \text{ Via a small homotopy we may moreover assume that } \alpha \text{ is disjoint from } D \cap E. \]

\[ \text{We modify } D \text{ as follows. There are at least two outer components of } E - D. \text{ Since } \alpha \text{ intersects } \partial E \text{ in at most one point, one of these components, say the component } E', \text{ is disjoint from } \alpha. \text{ The boundary of } E' \text{ decomposes } D \text{ into two subdiscs } P_1, P_2. \text{ Assume without loss of generality that } P_1 \text{ contains fewer intersection points with } \alpha \text{ than } P_2. \text{ Then } P_1 \text{ intersects } \alpha \text{ in at most } k/2 \text{ points. The disc } D' = P_1 \cup E' \text{ has at most } k/2 \text{ intersection points with } \alpha, \text{ and up to isotopy, it is disjoint from } D. \text{ In particular, we have } d_\xi(D, D') = 1. \]

\[ \text{Repeat this construction with } D', E. \text{ After at most } \log_2 k + 1 \text{ such steps we obtain a disc } D_1 \text{ which either is disjoint from } E \text{ or is disjoint from } \alpha. \text{ The distance between } D \text{ and } D_1 \text{ in the graph } \mathcal{EDG} \text{ is at most } \log_2 k + 1. \]

\[ \text{If } D_1 \text{ and } E \text{ are disjoint then } d_\xi(D_1, E) \leq 1 \text{ and } d_\xi(D, E) \leq \log_2 k + 2 \text{ and we are done. Otherwise apply the above construction to } D_1, E \text{ but with the roles of } D_1 \text{ and } E \text{ exchanged. We obtain a disc } E_1 \text{ which is disjoint from both } E \text{ and } \alpha, \text{ in particular it satisfies } d_\xi(E_1, E) = 1. \text{ The discs } D_1, E_1 \text{ are both disjoint from } \alpha \text{ and therefore } d_\xi(D_1, E_1) \leq 1 \text{ by the definition of the electrified disc graph. Together this shows that } d_\xi(D, E) \leq \log_2 k + 1 + d_\xi(D_1, E) \leq \log_2 k + 3. \]

\[ \square \]
(2) $D, E$ intersect some $I$-bundle generator $\gamma$ in $\partial H$ in precisely two points.

Proof. Let $D, E$ be two discs in normal position. Assume that $D − E$ and $E − D$ only have two outer components. Then each component of $D − E, E − D$ either is an outer component or a rectangle, i.e. a disc whose boundary consists of two components of $D \cap E$ and two arcs contained in $\partial D \subset \partial H$ or $\partial E \subset \partial H$, respectively. Since $d_{\mathcal{E}}(D, E) = 1$ if there is an essential simple closed curve in $\partial H$ which is disjoint from $\partial D \cup \partial E$, we may assume without loss of generality that $\partial D \cup \partial E$ fills up $\partial H$. This means that $\partial H − (\partial D \cup \partial E)$ is a union of discs and peripheral annuli.

Choose tubular neighborhoods $N(D), N(E)$ of $D, E$ in $H$ which are homeomorphic to an interval bundle over a disc and which intersect $\partial H$ in an embedded annulus. We may assume that the interiors $A(D), A(E)$ of these annuli are contained in the interior of $\partial H$. Then $\partial N(D) − A(D), \partial N(E) − A(E)$ is the union of two properly embedded disjoint discs in $H$ isotopic to $D, E$. We may assume that $\partial N(D) − A(D)$ is in normal position with respect to $\partial N(E) − A(E)$ and that

$$S = \partial(N(D) \cup N(E)) − (A(D) \cup A(E))$$

is a compact surface with boundary which is properly embedded in $H$. Since $H$ is assumed to be oriented, the boundary $\partial(N(D) \cup N(E))$ of $N(D) \cup N(E)$ has an induced orientation which restricts to an orientation of $S$.

Let $(P, \sigma)$ be a pair consisting of an outer component $P$ of $D − E$ and an orientation $\sigma$ of $D$ which induces an orientation of $P$. The orientation $\sigma$ together with the orientation of $H$ determines an orientation of the normal bundle of $D$ and hence $\sigma$ determines a side of $D$ in $N(D)$, say the right side. We claim that the component $Q$ of the surface $S$ containing the copy of $P$ in $\partial N(D)$ to the right of $D$ is a disc which contains precisely one other pair $(P', \sigma')$ of this form, i.e. $P'$ is an outer component of $D − E$ or of $E − D$, and $\sigma'$ an orientation of $P'$.

Namely, by construction, each component of $\partial N(D) − (A(D) \cup N(E))$ either corresponds to an outer component of $D − E$ and the choice of a side, or it corresponds to a rectangle component of $D − E$ and a choice of a side. A component corresponding to a rectangle is glued at each of its two sides which are contained in the interior of $H$ to a component of $\partial N(E) − (A(E) \cup N(D))$. In other words, up to homotopy, the component $Q$ of $S$ can be written as a chain of oriented discs beginning with $P$ and alternating between components of $D − E$ and $E − D$ equipped with one of the two possible orientations. Since $Q$ is embedded in $H$ and contains $P$, this chain cannot be a cycle and hence it has to terminate at an oriented outer component of $D − E$ or $E − D$ which is distinct from $(P, \sigma)$.

To summarize, each pair $(P, \sigma)$ consisting of an outer component $P$ of $D − E$ or $E − D$ and an orientation $\sigma$ of $D$ or $E$ determines a unique component of the oriented surface $S$. This component is a properly embedded disc in $H$ which is disjoint from $D \cup E$. Each such disc corresponds to precisely two such pairs $(P, \sigma)$, so there is a total of four such discs. Denote these discs by $Q_1, \ldots, Q_4$. If one of these discs is essential, say if this holds true for the disc $Q_i$, then $Q_i$ is an essential disc disjoint from both $D, E$ with boundary in $X$ and hence $d_{\mathcal{E}}(D, Q_i) ≤ 1, d_{\mathcal{E}}(E, Q_i) ≤ 1$ and we are done.

Otherwise define a cycle to be a subset $C$ of $\{Q_1, \ldots, Q_4\}$ of minimal cardinality so that the following holds true. Let $Q_i \in C$ and assume that $Q_i$ contains a pair $(B, \zeta)$ consisting of an outer component $B$ of $D − E$ (or of $E − D$) and an orientation $\zeta$ of $D$ (or $E$). If $Q_j$ is the disc containing the pair $(B, \zeta')$ where $\zeta'$ is the orientation
of $D$ (or $E$) distinct from $\zeta$ then $Q_j \in C$. Note that two distinct cycles are disjoint. The length of the cycle is the number of its components.

For each cycle $C$ we construct a properly embedded annulus $(A(C), \partial A(C)) \subset H$ as follows. Remove from each of the discs $Q_i$ in the cycle the subdiscs which correspond to outer components of $D - E, E - D$ and glue the discs along the boundary arcs of these outer components.

To be more precise, let $B$ be an outer component of $D - E$ corresponding to a subdisc of $Q_i$ and let $\beta = B \cap E$. Then the complement of $B$ in $Q_i$ (with a small abuse of notation) contains the arc $\beta$ in its boundary, and the orientation of $Q_i$ defines an orientation of $\beta$. There is a second disc $Q_j$ in the cycle which contains $B$ and which induces on $\beta$ the opposite orientation ($Q_j$ is not necessarily distinct from $Q_i$). Glue $Q_i - B$ to $Q_j - B$ along $\beta$ and note that the resulting surface is oriented. Doing this with each of the outer components of $D - E$ and $E - D$ contained in the cycle yields a properly embedded annulus $A(C) \subset H$ as claimed.

If there is a cycle $C$ of odd length then for one of the two discs $D, E$, say the disc $D$, the cycle contains precisely one outer component (with both orientations). Since the discs $Q_i$ are disjoint from $D \cup E$, this means that a boundary curve $\gamma$ of $A(C)$ intersects the disc $D$ in precisely one point, and it intersects the disc $E$ in at most two points. In particular, $\gamma$ is an essential curve in $\partial H$. Lemma 2.2 now shows that $d_\varepsilon(D, E) \leq 4$.

Similarly, if there is a cycle $C$ of length two then there are two possibilities. The first case is that the cycle contains both an outer component of $D - E$ and an outer component of $E - D$. Then a boundary curve $\gamma$ of $A(C)$ intersects each of the discs $D, E$ in precisely one point. In particular, $\gamma$ is an essential curve in $\partial H$ and $d_\varepsilon(D, E) \leq 3$.

If $C$ contains both outer components of say the disc $D$ then a boundary curve $\gamma$ of $A(C)$ intersects $\gamma$ of $A(C)$ in precisely two points, and it is disjoint from $E$. Let $D'$ be a disc obtained from $D$ by a simple surgery at an outer components of $E - D$. Then either $D'$ is disjoint from both $D, E$ (which is the case if $D'$ is composed of an outer component of $E - D$ and an outer component of $D - E$) or $D'$ intersects $\gamma$ in precisely one point and is disjoint from $D$. As before, we conclude from Lemma 2.2 that $d_\varepsilon(D, E) \leq 4$.

We are left with the case that there is a single cycle $C$ of length four. Let $\gamma_1, \gamma_2$ be the two boundary curves of $A(C)$. Then $\gamma_1, \gamma_2$ are simple closed curves in $X$ which are freely homotopic in the handlebody $H$. We claim that $\gamma_1, \gamma_2$ are freely homotopic in $\partial H$. Namely, assume that the discs $Q_i$ are numbered in such a way that $Q_1$ and $Q_{i+1}$ share one outer component of $D - E$ or $E - D$. Glue the discs $Q_1, \ldots, Q_4$ successively to a single disc $Q$ with the surgery procedure described above (namely, if $P$ is the outer component of $D - E$ or $E - D$ contained in both $Q_1, Q_2$ then remove $P$ from $Q_1, Q_2$ and glue $Q_1$ to $Q_2$ along the resulting boundary arc to form a disc $Q_3$, glue $Q_1$ to $Q_3$ to form a disc $Q_5$, and glue $Q_2$ to $Q_4$ to obtain the disc $Q$). Since by assumption none of the discs $Q_i$ is essential, the disc $Q$ is contractible. In particular, $Q$ is homotopic with fixed boundary to an embedded disc in $\partial H$. Now the annulus $A(C)$ is obtained from $Q$ by identifying two disjoint boundary arcs and hence $A(C)$ is homotopic into $\partial H$. Assume from now on that $A(C) \subset \partial H$. 

By construction, each of the simple closed curves \( \partial D, \partial E \) intersects \( A(C) \) in precisely two arcs connecting the two boundary components of \( A(C) \). These are exactly the boundary arcs of the outer components of \( D - E, E - D \).

The intersection arcs \( \partial D \cap A(C), \partial E \cap A(C) \) decompose \( A(C) \) into four rectangles. The annulus \( A(C) \) has a natural structure of an \( I \)-bundle over one of its boundary circles, with \( \partial D \cap A(C), \partial E \cap A(C) \) as a union of fibres.

Let \( \zeta \subset \partial D \) be a component of \( \partial D - A(C) \). Then \( \zeta \) is a union of boundary components of rectangles embedded in \( D \). Two opposite sides of such a rectangle \( R \) are contained in \( \partial D \). There is a unique side \( \rho \) of \( R \) which is contained in \( \zeta \), and the side opposite to \( \rho \) is contained in the component \( \zeta' \) of \( \partial D - A(C) \) disjoint from \( \zeta \). This decomposition of \( D \) into rectangles determines for \( D \) the structure of an \( I \)-bundle over \( \zeta \). Each component of \( D \cap E \) is a fibre of this \( I \)-bundle, and the two components of \( \partial D \cap A(C) \) are fibres as well. Similarly, \( E \) is an \( I \)-bundle over each component \( \xi \) of \( \partial E - A(C) \).

Since \( \partial D \cup \partial E \) fills up \( \partial H \), each component of \( \partial H - (\partial D \cup \partial E) \) is a polygon, i.e. a disc bounded by finitely many subarcs of \( \partial D, \partial E \). Such a polygon \( P \) is contained in the boundary of a component \( V \) of \( H - (D \cup E) \). The boundary \( \partial V \) of \( V \) has two connected components contained in \( \partial H \). One of these components is the polygon \( P \), the other component \( P' \) either is a polygon component of \( \partial H - (\partial D \cup \partial E) \), or it contains a boundary component of \( \partial H \).

The complement of \( P \cup P' \) in \( \partial V \) is a finite collection \( W \) of fibred rectangles. The base of such a rectangle is an edge of the boundary \( \partial P \) of the polygon \( P \). Using again the fact that \( \partial D \cup \partial E \) fills up \( \partial H \), if \( P' \) is a polygon in \( \partial H \) then \( V \) is a 3-ball.

As a consequence, each component \( V \) of \( H - (D \cup E) \) is a ball whose boundary consists of \( P \), a finite union \( R \) of fibred rectangles with base \( \partial P \) and a second polygonal component \( P'' \) of \( X - (\partial D \cup \partial E) \). The \( I \)-bundle structure on \( R \) naturally extends to an \( I \)-bundle structure on \( V \). Therefore \( H \) is an \( I \)-bundle. The involution of the \( I \)-bundle which exchanges the endpoints of the interval \( I \) preserves each component \( V \) of \( H - (D \cup E) \), and it exchanges the two components of \( V \cap \partial H \). This completes the proof of the lemma. \( \square \)

We use Lemma 2.4 to improve Lemma 2.2 as follows.

**Proposition 2.5.** Let \( D, E \subset H \) be essential discs. If there is an essential simple closed curve \( \alpha \subset \partial H \) which intersects \( \partial D, \partial E \) in at most \( k \geq 1 \) points then \( d_s(D, E) \leq 2k + 4 \).

**Proof.** Let \( D, E \) be essential discs in normal position as in the proposition which are not disjoint.

Let \( \alpha \) be an essential simple closed curve in \( \partial H \) which intersects both \( \partial D \) and \( \partial E \) in at most \( k \geq 1 \) points. We may assume that these intersection points are disjoint from \( \partial D \cap \partial E \).

Let \( p \geq 2 \) (or \( q \geq 2 \)) be the number of outer components of \( D - E \) (or of \( E - D \)). If \( p = 2, q = 2 \) then Lemma 2.4 shows that either \( d_s(D, E) \leq 4 \) or \( \partial D, \partial E \) intersect some \( I \)-bundle generator \( \gamma \) in precisely two points, and we have \( d_s(D, E) = 1 \).

Let \( j \leq k, j' \leq k \) be the number of intersection points of \( D, E \) with \( \alpha \). If \( \min\{j, j'\} \leq 1 \) then \( d_s(D, E) \leq \log_2 k + 3 \) by Lemma 2.2. Thus it suffices to show the following. If \( \max\{p, q\} \geq 3 \) and \( \min\{j, j'\} \geq 2 \) then there is a simple surgery transforming the pair \( (D, E) \) to a pair \( (D', E') \) with the following properties.

(1) \( D' \) is disjoint from \( D, E' \) is disjoint from \( E \).
(2) Either $D = D'$ or $E = E'$.
(3) The total number of intersections of $\alpha$ with $D' \cup E'$ is strictly smaller than $j + j'$.

To this end assume without loss of generality that $q \geq 3$. If $j/2 > j'/3$ then choose an outer component $E_1$ of $E - D$ with at most $j'/3$ intersections with $\alpha$. This is possible because $E - D$ has at least three outer components. Let $D_1$ be a component of $D - E_1$ which intersects $\alpha$ in at most $j/2$ points. Then $D_1 \cup E_1$ is a disc which is disjoint from $D$ and has at most $j/2 + j'/3 < j$ intersections with $\alpha$.

On the other hand, if $j/2 \leq j'/3$ then choose an outer component $D_1$ of $D - E$ with at most $j/2$ intersections with $\alpha$. Let $E_1$ be a component of $E - D_1$ with at most $j'/2$ intersections with $\alpha$ and replace $E$ by the disc $E_1 \cup D_1$ which is disjoint from $E$ and intersects $\alpha$ in at most $j/2 + j'/2 < j'$ points.

This is what we wanted to show. \qed

For easy reference we note

**Corollary 2.6.** Let $H$ be a handlebody of genus $g \geq 2$ without spots. Let $D, E \subseteq H$ be discs and assume that there is a simple closed curve $\gamma$ on $\partial H$ which intersects both $\partial D, \partial E$ in at most $k \geq 1$ points; then $d_S(D, E) \leq 2k + 4$.

**Remark:** The arguments in this section use the fact that every simple surgery of a disc at an outer component of another disc yields an essential disc in $H$. They are not valid for handlebodies with spots.

### 3. Distance in the curve graph

The purpose of this section is to establish an estimate for the distance in the curve graph of the boundary of the handlebody $H$ which will be essential for a geometric description of the superconducting disc graph. The results in this section are valid for an arbitrary oriented surface $S$ of genus $g \geq 0$ with $m \geq 0$ punctures and $3g - 3 + m \geq 2$.

The idea is to use train tracks on $S$. We refer to [PH92] for all basic notions and constructions regarding train tracks.

A train track $\eta$ (which may just be a simple closed curve) is carried by a train track $\tau$ if there is a map $F : S \to S$ of class $C^1$ which is homotopic to the identity, with $F(\eta) \subset \tau$ and such that the restriction of the differential $dF$ of $F$ to the tangent line of $\eta$ vanishes nowhere. Write $\eta \prec \tau$ if $\eta$ is carried by $\tau$. If $\eta \prec \tau$ then the image of $\eta$ under a carrying map is a subtrack of $\tau$ which does not depend on the choice of the carrying map. Such a subtrack is a subgraph of $\tau$ which is itself a train track. Write $\eta < \tau$ if $\eta$ is a subtrack of $\tau$.

A train track $\tau$ is called large [MM99] if each complementary component of $\tau$ is either simply connected or a once punctured disc. A simple closed curve $\eta$ carried by $\tau$ fills $\tau$ if the image of $\eta$ under a carrying map is all of $\tau$. A diagonal extension of a large train track $\tau$ is a train track $\xi$ which can be obtained from $\tau$ by subdividing some complementary components which are not trigons or once punctured monogons.

A trainpath on $\tau$ is an immersion $\rho : [k, \ell] \to \tau$ which maps every interval $[m, m + 1]$ diffeomorphically onto a branch of $\tau$. We say that $\rho$ is periodic if $\rho(k) = \rho(\ell)$ and if the inward pointing tangent of $\rho$ at $\rho(k)$ equals the outward pointing tangent of $\rho$ at $\rho(\ell)$. Any simple closed curve carried by a train track $\tau$ defines a periodic trainpath and a transverse measure on $\tau$. The space of transverse
Lemma 3.3. Let $\eta$ be a large train track. If $\eta \prec \tau$ then $\tau$ is large as well. In particular, if $\eta' \prec \eta$ is a large subtrack of $\eta$ and if $\xi$ is a diagonal extension of $\eta'$, then a carrying map $F : \eta \to \tau$ induces a carrying map of $\eta'$ onto a large subtrack $\tau'$ of $\tau$, and it induces a carrying map of $\xi$ onto a diagonal extension of $\tau'$.

Definition 3.1. A pair $\eta \prec \tau$ of large train tracks is called wide if every simple closed curve which is carried by a diagonal extension of a large subtrack of $\eta$ fills a diagonal extension of a large subtrack of $\tau$.

We have

Lemma 3.2. If $\sigma \prec \eta \prec \tau$ and if the pair $\eta \prec \tau$ is wide then $\sigma \prec \tau$ is wide.

Proof. Let $\sigma'$ be a large subtrack of $\sigma$ and let $\xi$ be a diagonal extension of $\sigma'$. Then the carrying map $\sigma \to \eta$ maps $\sigma'$ onto a large subtrack $\eta'$ of $\eta$, and it maps $\xi$ to a diagonal extension $\zeta$ of $\eta'$. Similarly, $\eta'$ is mapped to a large subtrack $\tau'$ of $\tau$, and $\zeta$ is mapped to a diagonal extension $\rho$ of $\tau'$.

A simple closed curve $\alpha$ carried by $\xi$ is carried by $\zeta$. In particular, since $\eta \prec \tau$ is wide, $\alpha$ fills a large subtrack of $\rho$. From this the lemma follows. □

A splitting and shifting sequence is a finite sequence $(\tau_i)_{0 \leq i \leq n}$ of train tracks so that for each $i$, $\tau_{i+1}$ can be obtained from $\tau_i$ by a sequence of shifts followed by a single split. We allow the split to be a collision, i.e. to reduce the number of edges. Note that $\tau_{i+1}$ is carried by $\tau_i$ for all $i$ and the pair $\tau_{i+1} \prec \tau_i$ is not wide.

Recall from the introduction the definition of the curve graph $CG$ of $S$. For an essential simple closed curve $c$ on $S$ let $i(c) \in \{0, \ldots, n\}$ be the largest number with the following property. There is a large subtrack $\eta$ of $\tau_{i(c)}$ so that $c$ is carried by a diagonal extension $\xi$ of $\eta$ and fills $\xi$. If no such number exists then put $i(c) = 0$.

Define a projection $P : CG \to (\tau_i)_{0 \leq i \leq n}$ by

$$P(c) = \tau_{i(c)}.$$  

Extend the map $P$ to the edges of $CG$ by mapping an edge to the image of one of its endpoints.

Lemma 3.3. Let $c, d$ be disjoint simple closed curves on $S$. Assume that $P(c) = \tau_i$. If $\tau_i \prec \tau_j$ is wide then $P(d) = \tau_s$ for some $s \geq j$.

Proof. Assume that $P(c) = \tau_i \prec \tau_j$ is wide. By the definition of the map $P$, there is a large subtrack $\eta$ of $\tau_j$ so that $c$ fills a diagonal extension $\xi$ of $\eta$. By Lemma 4.4 of [MM99], since $d$ is disjoint from $c$, $d$ is carried by a diagonal extension $\zeta$ of $\xi$. Then $\zeta$ is a diagonal extension of $\eta$.

Since $\tau_i \prec \tau_j$ is wide, $d$ fills a diagonal extension of a large subtrack of $\tau_j$. This implies that $P(d) = \tau_s$ for some $s \geq j$. □

Define a distance function $d_g$ on $(\tau_i)_{0 \leq i \leq n}$ as follows. For $i < j$, the gap distance $d_g(\tau_i, \tau_j)$ between $\tau_i$ and $\tau_j$ is the smallest number $k > 0$ so that there is a sequence

$$i_0 = i < i_1 < \cdots < i_k = j$$

with the property that for each $p < k$, the pair $\tau_{i_{p+1}} \prec \tau_{i_p}$ is not wide. Note that this defines indeed a distance since for each $\ell$
Corollary 3.4. The map \( \tau \) is not wide and hence \( d_\rho(\tau_i, \tau_j) \leq j - i \). Moreover, the triangle inequality is immediate from Lemma 3.2.

The following is a consequence of Lemma 3.3. For its formulation, define a map \( P \) from a metric space \( X \) to a metric space \( Y \) to be coarsely \( L \)-Lipschitz for some \( L > 1 \) if \( d(Px, Py) \leq Ld(x, y) + L \) for all \( x, y \in X \).

**Corollary 3.4.** The map \( P : CG \to ((\tau_i), d_\rho) \) is coarsely 2-Lipschitz.

Define a map \( \Upsilon : (\tau_i)_{0 \leq i \leq n} \to CG \) by associating to the train track \( \tau_i \) one of its vertex cycles. We have

**Lemma 3.5.** The map \( \Upsilon : (\tau_i), d_\rho \to CG \) is coarsely 22-Lipschitz.

**Proof.** It suffices to show the following. If \( \tau \prec \eta \) is not wide then the distance in \( CG \) between a vertex cycle \( \alpha \) of \( \tau \) and a vertex cycle \( \beta \) of \( \eta \) is at most 22.

To this end note that if \( \alpha \) is a simple closed curve which is carried by a large train track \( \xi \) then the image of \( \alpha \) under a carrying map is a subtrack of \( \xi \). If this subtrack is not large then \( \alpha \) is disjoint from an essential simple closed curve \( \alpha' \) which can be represented by an edge-path in \( \xi \) (possibly with corners) which passes through any branch of \( \xi \) at most twice. Since a vertex cycle of \( \xi \) passes through each branch of \( \xi \) at most twice, this implies that \( \alpha' \) intersects a vertex cycle of \( \xi \) in at most 4 points (see [H06] for details). In particular, the distance in \( CG \) between \( \alpha \) and a vertex cycle of \( \xi \) is at most 6 [MM99].

On the other hand, if \( \tau \) is another large train track and if \( \xi \) is a diagonal extension of a large subtrack \( \tau' \) of \( \tau \) then a vertex cycle of \( \xi \) intersects a vertex cycle of \( \tau \) in at most 4 points. Hence the distance in \( CG \) between a vertex cycle of \( \tau \) and a vertex cycle of \( \xi \) is at most 5. Together we deduce that the distance in \( CG \) between \( \alpha \) and a vertex cycle of \( \xi \) does not exceed 11.

Now by definition, if \( \tau \prec \eta \) is not wide then there is a curve \( \alpha \) which is carried by a diagonal extension \( \xi \) of a large subtrack \( \tau' \) of \( \tau \) and such that the following holds true. A carrying map \( \tau \to \eta \) induces a carrying map of \( \xi \) onto a diagonal extension \( \zeta \) of a large subtrack of \( \eta \). The train track \( \zeta \) carries \( \alpha \) and so that \( \alpha \) does not fill a large subtrack of \( \zeta \). Since a carrying map \( \xi \to \zeta \) maps a large subtrack of \( \xi \) onto a large subtrack of \( \zeta \), the curve \( \alpha \) does not fill a large subtrack of \( \zeta \).

By the above discussion, the distance in \( CG \) between \( \alpha \) and any vertex cycle of both \( \tau \) and \( \eta \) is at most 11. This shows the lemma.

Call a map \( \Phi \) of a metric space \( (X, d) \) into a subset \( A \) of \( X \) a coarse Lipschitz retraction if there is a number \( L > 1 \) with the following properties.

1. \( d(\Phi(x), \Phi(y)) \leq Ld(x, y) + L \).
2. \( d(x, \Phi(x)) \leq L \) whenever \( x \in A \).

We are now ready to show

**Corollary 3.6.** For any splitting and shifting sequence \( (\tau_i)_{0 \leq i \leq n} \) the map \( \Upsilon \circ P \) is a coarse \( L \)-Lipschitz retraction of \( CG \) for a number \( L > 1 \) not depending on \( (\tau_i) \) or on the Euler characteristic of \( S \).

**Proof.** Let \( d_{CG} \) be the distance in the curve graph of \( S \). By Corollary 3.4 and Lemma 3.5 it suffices to show that \( d_{CG}(\alpha, \Upsilon \circ P(\alpha)) \leq L \) for a universal constant \( L > 1 \) and every vertex cycle \( \alpha \) of a train track \( \tau_i \) from the sequence.

To this end observe that since \( \alpha \) is a vertex cycle of \( \tau_i \), \( \alpha \) is carried by each of the train tracks \( \tau_j \) for \( j \leq i \), moreover \( \alpha \) does not fill a diagonal extension of a large
subtrack of \( \tau_i \). On the other hand, by definition of a wide pair, if \( \tau_i \prec \tau_j \) is wide then \( \alpha \) fills a large subtrack of \( \tau_j \). This means that \( P(\alpha) = \tau_s \) for some \( s \geq j \) so that the pair \( \tau_i \prec \tau_{s+1} \) is not wide. The corollary now follows from Lemma 3.5. \( \square \)

**Remark:** The above discussion immediately implies that the image under \( T \) of a splitting and shifting sequence of train tracks is an unparametrized quasi-geodesic in \( CG \) for a constant not depending on the Euler characteristic of \( S \). A non-effective version of this result was earlier established in [MM04] (see also [H06]).

4. **Quasi-geodesics in the superconducting disc graph**

Recall the definition of the graph \( SDG \). Our goal is to show that the natural map which associates to a disc its boundary defines a quasi-isometric embedding of \( SDG \) into the curve graph \( CG \) of \( \partial H \). To simplify the notation we identify in the sequel a disc in \( H \) with its boundary circle. Thus we view the vertex set of \( SDG \) as a subset of the curve graph \( CG \).

The argument is based on the results in Section 2-3 and a construction from [MM04]. This construction uses a specific type of surgery sequences of discs which can be related to train tracks as follows.

Let \( H \) be a handlebody of genus \( g \geq 2 \) without spot. Let \( D, E \subset H \) be two discs in normal position. Let \( E' \) be an outer component of \( E-D \) and let \( D_1 \) be a disc obtained from \( D \) by simple surgery at \( E' \).

Let \( \alpha \) be the intersection of \( \partial E' \) with \( \partial H \). Then up to isotopy, the boundary \( \partial D_1 \) of the disc \( D_1 \) contains \( \alpha \) as an embedded subarc. Moreover, \( \alpha \) is disjoint from \( E \). In particular, given an outer component \( E'' \) of \( E-D_1 \), there is a distinguished choice for a disc \( D_2 \) obtained from \( D_1 \) by simple surgery at \( E'' \). The disc \( D_2 \) is determined by the requirement that \( \alpha \) is not a subarc of \( \partial D_2 \). Then for an outer component of \( E-D_2 \) there is a distinguished choice for a disc \( D_3 \) obtained from \( D_2 \) by simple surgery at an outer component of \( E-D_2 \) etc. We call a surgery sequence \((D_i)\) of this form a *nested* surgery path in direction of \( E \). Note that the boundary of each disc \( D_i \) is composed of a single subarc of \( \partial D \) and a single subarc of \( \partial E \).

The following result is due to Masur and Minsky (this is Lemma 4.2 of [MM04] which is based on Lemma 4.1 and the proof of Theorem 1.2 in that paper).

**Proposition 4.1.** Let \( D, E \subset \partial H \) be any discs. Let \( D = D_0, \ldots, D_n \) be a nested surgery path in direction of \( E \) which connects \( D \) to a disc \( D_n \) disjoint from \( E \). Then for each \( i \leq n \) there is a train track \( \tau_i \) on \( \partial H \) with a single switch such that the following holds true.

1. \( \tau_i \) carries \( \partial E \) and \( \partial E \) fills up \( \tau_i \).
2. \( \tau_{i+1} \prec \tau_i \).
3. The disc \( D_i \) intersects \( \tau_i \) only at the switch.

The train tracks \( \tau_i \) in the proposition are constructed as follows.

Let \( \alpha = \partial D, \beta = \partial E \). Assume that the curves \( \alpha, \beta \) are smooth (for a smooth structure on \( \partial H \)) and fill up \( \partial H \). This means that the complementary components of \( \alpha \cup \beta \) are all polygons or once holed polygons where in our setting, a hole is a boundary component of \( \partial H \). Let \( P \) be a complementary polygon which has at least 6 sides. Such a polygon exists since the Euler characteristic of \( \partial H \) is negative. Its edges are subsegments of \( \alpha \) and \( \beta \). Let \( I \) be a boundary edge of \( P \) contained in \( \alpha \). Collapse \( \alpha - I \) to a single point with a homotopy \( F \) of \( \partial H \). This can be done in such a way that the restriction of \( F \) to \( \beta \) is nonsingular everywhere. The resulting
graph has a single vertex. Collapsing the bigons in the graph to single arcs yields a train track $\tau$ with a single switch [MM04].

Let $b \subset \beta$ be an outer arc for $E - D$ and let $a \subset \alpha - I$ be the subarc of $\alpha$ which is bounded by the endpoints of $b$ and which does not intersect the interval $I$. Then $a \cup b$ is the boundary of a disc $D_1$ obtained from $D$ by nested surgery at $b$. The new train track $\tau_1$ obtained from the above construction is obtained from $\beta \cup a$ by collapsing the arc $a$ to a single point (we refer to [MM04] for details).

In the remainder of this section we identify a disc in $H$ with its boundary circle on $\partial H$. Thus we view the vertex set of the superconducting disc graph $SDG$ as a subset of the vertex set of the curve graph $CG$ of $\partial H$. We are now ready to show

**Theorem 4.2.** There is an effectively computable number $L > 1$ such that the vertex inclusion defines an L- quasi-isometric embedding $SDG \rightarrow CG$.

**Proof.** As before, let $d_S$ be the distance in $SDG$ and let $d_{CG}$ be the distance in $CG$. We have to show the existence of a number $c > 0$ with the following property. If $D, E$ are any discs then

$$d_S(D, E) \leq cd_{CG}(\partial D, \partial E).$$

By Proposition 4.1, there is a nested surgery path $D = D_0, \ldots, D_n$ connecting the disc $D_0 = D$ to a disc $D_n$ which is disjoint from $E$, and there is a sequence $(\tau_i)_{0 \leq i \leq n}$ of one-switch train tracks on $\partial H$ such that $\tau_{i+1} \prec \tau_i$ for all $i < n$ and that $D_i$ intersects $\tau_i$ only at the switch.

By Theorem 2.3.1 of [PH92], there is a splitting and shifting sequence $\tau_0 = \eta_0 \prec \cdots \prec \eta_s = \tau_n$ connecting $\tau_0$ to $\tau_n$ and a sequence $0 = u_0 \prec \cdots \prec u_n = s$ so that $\eta_{u_q} = \tau_q$ for $0 \leq q \leq n$. Since the disc $D_i$ intersects $\tau_i$ only at the switch, the boundary $\partial D_i$ of $D_i$ intersects a vertex cycle of $\tau_i$ in at most two points and hence the distance in the curve graph $CG$ between $\partial D_i$ and a vertex cycle of $\tau_i$ is at most three. Now the discs $D_i$ and $D_{i+1}$ are disjoint and consequently the distance in $CG$ between a vertex cycle of $\tau_i$ and a vertex cycle of $\tau_{i+1}$ is at most 7. Thus by Corollary 3.6 and the definition of the no-gap distance, it suffices to show the existence of an effectively computable number $b > 0$ with the following property. Let $k < i$ be such that the pair $\tau_i \prec \tau_k$ is not wide; then $d_S(D_i, D_k) \leq b$.

Since $\tau_i \prec \tau_k$ is not wide there is a large subtrack $\tau'_i$ of $\tau_i$, a diagonal extension $\zeta_i$ of $\tau'_i$ and a simple closed curve $\alpha$ carried by $\zeta_i$ with the following property. Let $\tau''_k$ be the image of $\tau'_i$ under a carrying map $\tau_i \rightarrow \tau_k$ and let $\zeta_k$ be the diagonal extension of $\tau''_k$ which is the image of $\zeta_i$ under a carrying map induced by a carrying map $\tau'_i \rightarrow \tau''_k$. Then $\alpha$ does not fill a large subtrack of $\zeta_k$.

Since $\zeta_i$ is a diagonal extension of the large subtrack $\tau'_i$ of $\tau_i$ and since $D_i$ intersects $\tau_i$ only at the switch, the intersection number between $\partial D_i$ and $\zeta_i$ is bounded from above by a constant $\kappa \geq 2$ which does not exceed a constant multiple of the Euler characteristic of $S$.

For each $p \in [k, i]$, the image of $\tau'_i$ under a carrying map $\tau_i \rightarrow \tau_p$ is a large subtrack $\tau'_p$ of $\tau_p$, and there is a diagonal extension $\zeta_p$ of $\tau'_p$ which carries $\alpha$. We may assume that $\zeta_p \prec \zeta_p$ for $u \geq p$. The disc $D_p$ intersects $\zeta_p$ in at most $\kappa$ points.

For each $p \in [k, i]$ let $\beta_p \prec \zeta_p$ be the subtrack of $\zeta_p$ filled by $\alpha$. Then $\beta_p$ is connected and not large. The union $Y_p$ of a thickening of $\beta_p$ with the components of $\partial H - \beta_p$ which are either simply connected or once holed discs is a proper connected subsurface of $\partial H$ for all $p$. The boundary of $Y_p$ can be realized as a union of simple closed curves which are embedded in $\beta_p$ (but with cusps). The
carrying map $\beta_{p+1} \to \beta_p$ maps $Y_{p+1}$ into $Y_p$. In particular, either the boundary of $Y_{p+1}$ coincides up to homotopy with the boundary of $Y_p$ or $Y_{p+1}$ is a \emph{proper} subsurface of $Y_p$. This means that there is a boundary circle of $Y_{p+1}$ which is essential in $Y_p$ and hence in $\partial H$. In other words, the subsurfaces $Y_p$ are nested, and hence their number is bounded from above by a universal constant $h > 0$ depending linearly on the Euler characteristic of $S$.

Since $\partial D_p$ intersects $\zeta_p$ in at most $\kappa$ points, the number of intersections between $\partial D_p$ and $\partial Y_p$ is bounded from above by a universal constant $\chi > 0$ ($\kappa$ times the maximal number of branches of a train track on $\partial H$ will do, and this number is quadratic in the Euler characteristic of $S$). As a consequence, there are $h$ essential simple closed curves $c_1, \ldots, c_h$ in $\partial H$ so that for every $p \in [k, i]$ there is some $r(p) \in \{1, \ldots, h\}$ with

$$\iota(\partial D_p, c_{r(p)}) \leq \chi.$$ 

Each of the curves $c_j$ is a fixed boundary component of one of the subsurfaces $Y_p$.

By reordering, assume that $r(i) = 1$. Let $v_1$ be the minimum of all numbers $p \in [k, i]$ such that $r(v_1) = 1$. Proposition 2.5 shows that

$$d_S(D_i, D_{v_1}) \leq 2\chi + 8.$$ 

On the other hand, we have $d_S(D_{v_1}, D_{v_1-1}) = 1$. Again by reordering, assume that $r(v_1 - 1) = 2$ and repeat this construction with the discs $D_{v_1-1}, \ldots, D_k$ and the curve $c_2$. In $a \leq h$ steps we construct in this way a decreasing sequence $i \geq v_1 > \cdots > v_a = k$ such that $d_S(D_{v_u}, D_{v_{u-1}}) \leq 2\chi + 9$ for all $u \leq a$. This implies that

$$d_S(D_i, D_k) \leq h(2\chi + 9)$$

which is what we wanted to show. 

By a result of Bell and Fujiwara [BF08], the asymptotic dimension of the curve graph of a subsurface $Y$ of $\partial H$ is bounded from above independent of $Y$. Thus as an immediate consequence of Theorem 4.2 we observe

\textbf{Corollary 4.3.} The graph $SDG$ is hyperbolic, and its asymptotic dimension is finite.

As a corollary, we obtain the statement of the theorem from the introduction.

\textbf{Corollary 4.4.} Let $H$ be a handlebody of genus $g \geq 2$ without spots. Then the vertex inclusion $SDG \to CG$ is a quasi-isometric embedding.

The remainder of this section is devoted to the proof of Corollary 1 from the introduction. To this end recall that a hyperbolic geodesic metric space $Y$ admits a \emph{Gromov boundary}. This boundary is a topological space on which the isometry group of $Y$ acts as a group of homeomorphisms.

Let $L$ be the space of all geodesic laminations on $\partial H$ (for some fixed hyperbolic metric) equipped with the coarse Hausdorff topology. In this topology, a sequence $(\mu_i)$ converges to a lamination $\mu$ if every accumulation point of $(\mu_i)$ in the usual Hausdorff topology contains $\mu$ as a sublamination. Note that the coarse Hausdorff topology on $L$ is not $T_0$, but its restriction to the subspace $\partial CG \subset L$ of all minimal geodesic laminations which fill up $\partial H$ (i.e. which intersect every simple closed geodesic transversely) is Hausdorff. The space $\partial CG$ equipped with the coarse Hausdorff topology can naturally be identified with the Gromov boundary of $CG$ [K99, H06].
Let \( \partial H \subset \partial CG \) be the closed subset of all geodesic laminations which are limits in the coarse Hausdorff topology of boundaries of discs in \( H \). The handlebody group Map(\( H \)) acts on \( \partial CG \) as a group of transformations preserving the subset \( \partial H \). The Gromov boundary of \( SDG \) can now fairly easily be determined from Corollary 4.4. To this end we first observe

**Lemma 4.5.** The Gromov boundary of \( SDG \) is a closed Map(\( H \))-invariant subset of \( \partial H \).

**Proof.** Since the vertex inclusion \( SDG \to CG \) defines a quasi-isometric embedding, the Gromov boundary of \( SDG \) is the subset of the Gromov boundary of \( CG \) of all endpoints of quasi-geodesic rays in \( CG \) which are contained in \( SDG \).

By the main result of [H06] (see [K99] for an earlier account of a similar statement), a simplicial quasi-geodesic ray \( \gamma : [0, \infty) \to CG \) defines the endpoint lamination \( \nu \in \partial CG \) if and only if the curves \( \gamma(i) \) converge as \( i \to \infty \) in the coarse Hausdorff topology to \( \nu \). As a consequence, the Gromov boundary of \( SDG \) is a subset of \( \partial H \), and this subset is clearly Map(\( H \))-invariant.

We are left with showing that the Gromov boundary of \( SDG \) is a closed subset of \( \partial CG \). To this end note that by Corollary 4.4, there is a number \( p > 1 \) such that for every \( L > 1 \), any \( L \)-quasi-geodesic in \( SDG \) is an \( L^p \)-quasi-geodesic in \( CG \). Moreover, for a suitable choice of \( p \), any vertex in \( SDG \) can be connected to any point in the Gromov boundary of \( SDG \) by a \( p \)-quasi-geodesic.

Now let \( (\partial D_i) \) be a sequence of boundaries of discs \( D_i \) converging in the coarse Hausdorff topology to a geodesic lamination \( \mu \in \partial H \). For each \( i \) let \( E_i \) be a disc which is disjoint from \( D_i \). Since the space of geodesic laminations equipped with the usual Hausdorff topology is compact, up to passing to a subsequence the sequence \( (\partial E_i) \) converges in the Hausdorff topology to a geodesic lamination \( \nu \) which does not intersect \( \mu \) (we refer to [K99, H06] for details of this argument). Now \( \mu \) is minimal.
and fills up \( \partial H \) and therefore the lamination \( \nu \) contains \( \mu \) as a sublamination. This just means that \( (\partial E_i) \) converges in the coarse Hausdorff topology to \( \mu \).

Since the genus of \( H \) is at least two, for every separating disc in \( H \) we can find a disjoint non-separating disc. Thus the discussion in the previous paragraph shows that every \( \mu \in \partial H \) is a limit in the coarse Hausdorff topology of a sequence of non-separating discs. However, the handlebody group acts transitively on non-separating discs. Minimality of the action of \( \text{Map}(H) \) on \( \partial SDG \) follows. \( \square \)

As an immediate consequence of Lemma 4.5 and Lemma 4.6 we obtain

**Corollary 4.7.** \( \partial H \) is the Gromov boundary of \( SDG \).

5. Discs for a handlebody with a single spot

A handlebody with spots is a handlebody \( H \) of genus \( g \geq 1 \) with \( m \geq 1 \) marked points, called spots, on the boundary. We view the boundary \( \partial H \) of such a handlebody as a surface with \( m \) punctures.

A disc in a handlebody with spots is a disc \( D \) in \( H \) whose boundary \( \partial D \) is disjoint from the spots. We require that \( \partial D \) is an essential simple closed curve in \( \partial H \), i.e. it is neither contractible nor homotopic into a spot. Two such discs are isotopic if there is an isotopy between them which is disjoint from the spots. We use the terminology introduced in Section 2 also for handlebodies with spots.

For the remainder of this section we consider a handlebody \( H \) of genus \( g \geq 2 \) with a single spot \( p \). Our goal is to prove the first part of Theorem 2 from the introduction.

Let \( H_0 \) be the handlebody obtained from \( H \) by forgetting the spot and let

\[
\Phi : H \to H_0
\]

be the natural forgetful map. Let \( CG \) be the curve graph of \( \partial H \) and let \( CG(\partial H_0) \) be the curve graph of \( \partial H_0 \).

**Lemma 5.1.** The map \( \Phi \) induces a simplicial surjection

\[
\Pi : CG \to CG(\partial H_0)
\]

which maps discbounding curves to discbounding curves.

**Proof.** Since \( H \) has a single spot, the image under the map \( \Phi \) of an essential simple closed curve \( \gamma \) on \( \partial H \) is an essential simple closed curve \( \Phi(\gamma) \) on \( \partial H_0 \). The curve \( \gamma \) is discbounding if and only if this is the case for \( \Phi(\gamma) \). Moreover, if \( \gamma, \delta \) are disjoint then this holds true for \( \Phi(\gamma), \Phi(\delta) \) as well. This immediately implies the lemma. \( \square \)

Theorem 7.1 of [KLS09] shows that for every simple closed curve \( c \) in \( \partial H_0 \) the preimage \( \Pi^{-1}(c) \) of \( c \) in the curve graph \( CG \) of \( \partial H \) is a tree \( T_c \) which can be described as follows.

Let \( H^2 \) be the universal covering of \( \partial H_0 \) and let \( S(c) \) be the preimage of \( c \) in \( H^2 \). Let \( T \) be the tree whose vertex set is the set of components of \( H^2 - S(c) \), and where two components are connected by an edge if their closures intersect. View the spot \( p \) as the basepoint for the fundamental group of \( \partial H_0 \). We assume that \( p \) does not lie on \( c \). Then \( \pi_1(\partial H_0, p) \) acts transitively on \( T \) as a group of simplicial isometries. The tree \( T_c \) is \( \pi_1(\partial H_0, p) \)-equivariantly isomorphic to \( T \) (see Section 7 of [KLS09] for details).
Lemma 5.2. For each $\gamma \in \mathcal{G}$ there is an isometric section $\Lambda : \mathcal{G}(H_0) \to \mathcal{G}$ for the projection $\Pi : \mathcal{G} \to \mathcal{G}(H_0)$ passing through $\gamma$.

Proof. Fix a hyperbolic metric on $\partial H_0$. Every simple closed curve on $\partial H_0$ can be represented by a unique simple closed geodesic. The union of these geodetics has area zero and hence there is a point $x \in \partial H_0$ which is not contained in any such geodesic. View $x$ as the marked point in $\partial H$.

Define a map $\Lambda : \mathcal{G}(\partial H_0) \to \mathcal{G}$ as follows. To a vertex of $\mathcal{G}(\partial H_0)$, i.e. a simple closed curve on $\partial H_0$, associate the isotopy class $\Lambda(\gamma)$ of $\gamma$ viewed as a curve in $\partial H = (\partial H_0, x)$. Clearly $\Pi \circ \Lambda$ is the identity on vertices of $\mathcal{G}(\partial H_0)$. If $\alpha, \beta$ are disjoint simple closed curves on $\partial H_0$ then their geodesic representatives are disjoint as well, and $\Lambda(\alpha), \Lambda(\beta)$ are disjoint simple closed curves in $\partial H$. Thus $\Lambda$ extends to a simplicial section. Since $\Pi$ is distance non-increasing, $\Lambda$ is an isometric embedding of $\mathcal{G}(\partial H_0)$ into $\mathcal{G}$.

Consider the Birman exact sequence
\begin{equation}
0 \to \pi_1(\partial H_0, x) \to \text{Mod}(\partial H) \to \text{Mod}(\partial H_0) \to 0.
\end{equation}
The action of $\pi_1(\partial H_0, x)$ on the curve graph of $\partial H$ is fibre preserving. Indeed, the restriction of this action to a fixed fibre is just the action of $\pi_1(\partial H_0, x)$ on the geometric realization of the fibre as described in the text preceding this proof. In particular, this action is transitive on vertices in the fibres (see [KLS09] for details). Equivalently, for every simple closed curve $\alpha$ on $\partial H$ and any curve $\beta \in \Pi^{-1}(\Pi(\alpha))$ there is a mapping class $\psi \in \pi_1(\partial H_0, x) < \text{Mod}(\partial H)$ so that $\psi(\alpha) = \beta$.

Since the map $\psi$ acts as a fibre preserving simplicial isometry on the curve graph of $\partial H$, the composition $\psi \circ \Lambda$ is a new isometric section for the map $\Pi$. This implies that for every vertex of $\mathcal{G}$ there is an isometric section for the projection $\Pi$ whose image contains that point. This is what we wanted to show. $\square$

Remark: 1) The proof of Lemma 5.2 contains some additional information. Namely, two simple closed curves $\gamma, \delta$ on $\partial H$ are contained in the image of an isometric section $\mathcal{G}(\partial H_0) \to \mathcal{G}$ as constructed by Harer [KLS09] and used in the proof of Lemma 5.2 if the spot is not contained in a component of $\partial H - (\gamma \cup \delta)$ which is a punctured bigon, i.e. a component whose boundary consists of a single subarc of $\gamma$ and a single subarc of $\delta$.

2) In [MjS12] (see also [H05]), the authors define a metric graph bundle to consist of graphs $V, B$ and a surjective simplicial map $p : V \to B$ with the following properties.

(1) For each $b \in \mathcal{V}(B)$, $F_b = p^{-1}(b)$ is a connected subgraph of $V$, and the inclusion maps $i : \mathcal{V}(F_b) \to V$ are uniformly metrically proper for the path metric $d_b$ induced on $F_b$ as measured by a non-decreasing surjective function $f : N \to N$.

(2) For any two adjacent vertices $b_1, b_2$ of $\mathcal{V}(B)$, each vertex $x_1$ of $F_{b_1}$ is connected by an edge with a vertex in $F_{b_2}$.

If the fibres are trees then we speak about a metric tree bundle.

By Lemma 5.2, the graph projection $\Pi : \mathcal{G} \to \mathcal{G}(H_0)$ satisfies the second property in the definition of a metric tree bundle. However, the first property
states that for every $b \in B$ the distance in $V$ between any two points $x, y \in F_b$ is bounded from below by $f(d_b(x,y))$, and this property is violated for the map $\Pi$. As an example, let $c_1, c_2$ be two disjoint non-separating simple closed curves in $\partial H_0$, let $x \in \partial H_0 - (c_1 \cup c_2)$ and let $\zeta$ be a loop in $\pi_1(\partial H_0 - c_1, x)$ which fills $\partial H_0 - c_1$. Then the point pushing map $\beta$ defined by $\zeta$ via the Birman exact sequence (2) acts as a hyperbolic isometry and hence with positive translation length on the fibre over $c_2$ although the distance in $CG(H)$ between any two points on the orbit is at most two.

Thus we can not use Theorem 4.2 to deduce that the superconducting disc graph of $H$ is quasi-isometrically embedded in the curve graph $CG$ of $\partial H$. In fact we have

**Proposition 5.3.** The disc graph of $H$ is not a quasi-convex subset of the curve graph of $\partial H$.

**Proof.** The curve graph of $\partial H$ is hyperbolic, and pseudo-Anosov elements of the mapping class group of $\partial H$ act as hyperbolic isometries on the curve graph. If $\gamma \in \pi_1(\partial H_0, x)$ is filling, i.e. if $\gamma$ decomposes $\partial H_0$ into discs, then the image $\Psi(\gamma)$ of $\gamma$ in $\text{Mod}(\partial H)$ via the Birman exact sequence (2) is pseudo-Anosov [Kr81, KLS09].

Let $\varphi$ be a diffeomorphism of $\partial H_0$ which fixes the basepoint $x$ and which defines a pseudo-Anosov element of $\text{Mod}(H_0)$. We require that a quasi-axis for the action of $\varphi$ on $CG(H_0)$ passes uniformly near the boundary of a disc and that moreover for any discbounding simple closed curve $\zeta$, the distance in $CG(H_0)$ between $\varphi^k \zeta$ and the quasi-convex subset of discbounding curves tends to infinity as $k \to \infty$.

Such a pseudo-Anosov element can be found as follows. Each pseudo-Anosov element fixes two filling projective measured laminations, and the set of pairs of such fixed points is dense in $\mathcal{PML} \times \mathcal{PML}$. The closure in $\mathcal{PML}$ of the set of discbounding simple closed curves is nowhere dense in $\mathcal{PML}$, and a pseudo-Anosov element $\varphi$ whose pair of fixed points is contained in the complement will do.

Since $\varphi$ fixes the point $x$, it acts on the fundamental group $\pi_1(\partial H_0, x)$ of $\partial H_0$, moreover it can be viewed as an element of $\text{Mod}(\partial H)$. We denote this element of $\text{Mod}(\partial H)$ again by $\varphi$.

Let $\beta \subset \partial H_0 - \{x\}$ be a discbounding curve near a quasi-axis of $\varphi$. Via the isometric section $\lambda : CG(\partial H_0) \to CG$ defined by $x$ as in Lemma 5.2, we can view $\beta$ as a discbounding curve in $\partial H$. Let $\gamma \in \pi_1(\partial H_0, x)$ be a filling curve. For each $k > 0$ let $\beta_k$ be the image of $\beta$ under point-pushing by the curve $\varphi^k(\gamma)$. Then $\beta_k$ is discbounding, moreover we have

$$\beta_k = (\varphi^k \circ \Psi(\gamma) \circ \varphi^{-k})(\beta).$$

A quasi-axis in $CG$ of the pseudo-Anosov element $\varphi^k \circ \Psi(\gamma) \circ \varphi^{-k}$ is the image under $\varphi^k$ of a quasi-axis of $\Psi(\gamma)$. By hyperbolicity of $CG$, via perhaps replacing $\gamma$ by a multiple we may assume that a geodesic in $CG$ connecting $\beta$ to $\beta_k$ is close in the Hausdorff topology to the composition of three arcs. The first arc connects $\beta$ to a quasi-axis $\zeta$ of $\varphi^k \circ \Psi(\gamma) \circ \varphi^{-k}$, the second arc travels along $\zeta$ and the third arc connects $\zeta$ to $\beta_k$. However, by the choice of $\varphi$, for suitable choices of $k$ and suitable multiplicities of $\gamma$, such a curve is arbitrarily far in $CG$ from the set of discbounding curves.

\[\square\]

6. Handlebodies with at least two spots

In this section we show the second part of Theorem 2 from the introduction. We continue to use the notations from Section 2-4.
As mentioned at the end of Section 2, the fact that surgery of discs in handlebodies with spots may produce peripheral discs causes substantial difficulty. Indeed, we showed in Section 5 that for handlebodies \( H \) with a single spot and genus \( g \geq 2 \), boundaries of discs do not form a quasi-convex subset of the curve graph of \( \partial H \).

To understand disc graphs in handlebodies with at least two spots we first establish a weaker analog of Lemma 2.1 and then analyze in detail discs which become peripheral after a single surgery.

As a warm-up, observe that for handlebodies \( H \) with a single spot, Lemma 2.1 is valid. Namely, for any two discs \( D, E \) with boundary in \( \partial H \) which are not disjoint and for any outer component \( E' \) of \( E - D \), at least one of the discs obtained from \( D \) by surgery at \( E' \) is not peripheral. Thus the proof of Lemma 2.1 carries over without modification. Denote as before by \( d_D \) and \( d_E \) the distance in the disc graph and the electrified disc graph, respectively. We obtain

**Lemma 6.1.** If \( \partial H \) contains a single spot then for any discs \( D, E \) in \( H \) we have

\[
d_D(D, E) \leq \iota(\partial D, \partial E)/2 + 1.
\]

For convenience of notation in the proof of the following lemma, we define the intersection between a peripheral curve \( \alpha \) in \( \partial H \) and any other curve \( \beta \) in \( \partial H \) as

\[
\iota(\alpha, \beta) = 0.
\]

**Lemma 6.2.** Let \( H \) be a handlebody with \( n \geq 2 \) spots. Then for any two discs \( D, E \) in \( H \) we have

\[
d_E(D, E) \leq \iota(\partial D, \partial E)/2 + 1.
\]

**Proof.** As in the proof of Lemma 2.1, we proceed by induction on \( \iota(\partial D, \partial E) \). The case \( \iota(\partial D, \partial E) = 0 \) is immediate from the definitions. Thus assume that the claim of the lemma holds true whenever \( \iota(\partial D, \partial E) \leq k - 1 \) for some \( k \geq 1 \).

Let \( \alpha = \partial D, \beta = \partial E \subset \partial H \) be discbounding simple closed curves with \( \iota(\alpha, \beta) = k \). If there is an essential simple closed curve \( \gamma \subset \partial H \) disjoint from \( \alpha \cup \beta \) then \( d_E(D, E) \leq 1 \) by the definition of the electrified disc graph and there is nothing to show. Thus assume that \( \alpha \cup \beta \) decomposes \( \partial H \) into discs and one-holed discs. Then a component of \( \partial H - (\alpha \cup \beta) \) is a polygon or one-holed polygon with sides alternating between subarcs of \( \alpha \) and subarcs of \( \beta \). A complementary polygon has at least four sides. A *punctured bigon* is a complementary component which is a one-holed disc bounded by a single subarc of \( \alpha \) and a single subarc of \( \beta \).

Let \( E' \) be an outer component of \( E - D \) (see Section 2 for the terminology). Surgery of \( D \) at \( E' \) yields two discs \( B_1, B_2 \) in \( H \). The boundaries of these discs are simple closed curves \( \alpha_1, \alpha_2 \) in \( \partial H \) which are disjoint from \( \alpha \), moreover \( \iota(\alpha_i, \beta) \leq k - 2 = \iota(\alpha, \beta) - 2 \). If at least one of the discs \( B_1, B_2 \) is essential, say if this holds true for \( B_1 \), then \( B_1 \) is a disc with \( d_E(D, B_1) = 1 \) and \( \iota(\partial B_1, \partial E) \leq k - 2 \). Thus the induction hypothesis can be applied to \( B_1 \) and \( E \) and shows the lemma.

If both discs \( B_1, B_2 \) are non-essential then \( \alpha = \partial D \) bounds a twice punctured disc \( D_0 \) which is embedded in \( \partial H \). The curve \( \beta = \partial E \) decomposes \( D_0 \) into two once punctured discs \( A_1, A_2 \) and a set of rectangles.

The once punctured disc \( A_i \) (\( i = 1, 2 \)) is bounded by a subarc \( \alpha_i \) of \( \alpha \) and a subarc \( \beta_i \) of \( \beta \). Let \( p_i \) be the spot contained in \( A_i \). The arc \( \alpha_i \) is an outer arc for the disc \( E \). For \( i = 1, 2 \) surger \( E \) at the outer arc \( \alpha_i \) to a disc whose boundary \( \beta_i \) is obtained from \( \beta \) by replacing the arc \( \beta_i \) with the arc \( \alpha_i \). Then \( \iota(\alpha, \beta_i) \leq \iota(\alpha, \beta) - 2 \) and hence if either \( \beta_1 \) or \( \beta_2 \) is essential then the claim follows as before from the induction hypothesis.
Thus we are left with the case that both curves $\beta_1, \beta_2$ are peripheral. Then $\beta$ bounds a disc $E_0 \subset \partial H$ punctured at $p_1, p_2$. The intersection $D_0 \cap E_0$ is a union of the once punctured bigons $A_1, A_2$ and a collection of rectangles. More precisely, if $R = E_0 - (A_1 \cup A_2)$ then the intersections of $R$ with $\partial D_0$ decompose $R$ into a chain of rectangles $R_1, \ldots, R_s$ such that for even $i$, the rectangle $R_i$ is contained in the twice punctured disc $D_0$, and for odd $i$ the rectangle $R_i$ is contained in $\partial H - D_0$.

The number $s$ of rectangles is odd, and $\iota(\partial D, \partial E) = \iota(\alpha, \beta) = 2s + 2$.

Assume for the moment that $s = 1$, i.e. that the rectangle $R$ does not intersect $D_0$. Then $R$ is homotopic relative to $\alpha$ to an embedded arc $\rho$ in $\partial H - D_0$. Thus $\partial H$ contains an essential simple closed curve $\gamma$ which is disjoint from $D_0 \cup R$, and $\gamma$ is disjoint from both $D_0, E_0$ and hence from $D$ and $E$. This shows that $d_\text{E}(D, E) \leq 1$ which is what we wanted to show.

If $s \geq 3$ then assume without loss of generality that the arc $a_1 \subset A_1$ is contained in the boundary of the rectangle $R_1$. Homotope the rectangle $R_2$ relative to $\alpha$ to a rectangle $\hat{R}_1$ such that the side $\rho$ of $\hat{R}_1$ which is opposite to $a_1$ is a subarc of $a_2 \subset \alpha$. Attach to $\rho$ a disc $G \subset A_2$ punctured at $p_2$. The union $A_1 \cup \hat{R}_1 \cup G$ is a disc $B_0 \subset \partial H$ punctured at $p_1$ and $p_2$. The boundary of $B_0$ bounds a properly embedded disc $B$ in $H$. By the case $s = 1$ discussed above, we have $d_\text{E}(D, B) \leq 1$.

On the other hand, $\iota(\partial B, \beta) \leq 2s - 2 = \iota(\alpha, \beta) - 4$ and hence the claim follows as before from the induction hypothesis. \hfill $\Box$

We next establish a version of Proposition 2.5 for discs $D, E$ in handlebodies with at least two spots on the boundary which become peripheral after closing one of the spots. To ease terminology we say that a disc $D \subset H$ encloses two spots $p_1, p_2$ in $\partial H$ if $D$ is homotopic with fixed boundary to a disc $D_0 \subset \partial H$ punctured at the points $p_1, p_2$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figureB}
\caption{Figure B}
\end{figure}
Lemma 6.3. Let $D, E$ be two discs in $H$ which enclose the same spots $p_1 \neq p_2 \in X$. If there is a simple closed curve $\gamma \subset \partial H$ which intersects $\partial D, \partial E$ in at most $k \geq 1$ points then $d_\mathcal{C}(D, E) \leq 2k + 12$.

Proof. Let $D, E$ be as in the lemma, with boundaries $\partial D = \alpha, \partial E = \beta$. Then $\alpha, \beta$ bound discs $D_0, E_0 \subset \partial H$ punctured at the points $p_1, p_2$. Up to isotopy, the intersection $D_0 \cap E_0$ is a union of two once punctured bigons $A_1, A_2$ and a disjoint union of rectangles. The punctured bigon $A_i$ ($i = 1, 2$) is bounded by a subarc of $\alpha$ and a subarc of $\beta$. Assume that $A_i$ contains the spot $p_i$ ($i = 1, 2$).

Let $\gamma \subset \partial H$ be a simple closed curve which intersects both $\alpha, \beta$ in at most $k$ points. If $\gamma$ is not disjoint from $\alpha$ then $\gamma \cap D_0$ is a union of at most $k/2$ pairwise disjoint arcs. By modifying $\gamma$ with an isotopy we may assume that these arcs are disjoint from the punctured discs $A_1, A_2$. For $i = 1, 2$ choose a compact arc $c_i \subset D_0$ which connects the punctured bigon $A_i$ to $\gamma$ and whose interior is disjoint from $\gamma$.

Let moreover $\gamma_0$ be one of the two subarcs of $\gamma$ which connect the two endpoints of $c_1, c_2$ on $\gamma$. The concatenation $c_1 \circ \gamma_0 \circ c_2^{-1}$ (read from left to right) of $c_1, \gamma_0, c_2^{-1}$ is an embedded arc in $\partial H$ connecting $A_1$ to $A_2$.

Let $C_0 \subset \partial H$ be an embedded rectangle with two opposite sides contained in the interior of $\partial A_1 \cap D_0, \partial A_2 \cap D_0$ which is a thickening of the arc $c_1 \circ \gamma_0 \circ c_2^{-1}$. The union of $C_0$ with $A_1 \cup A_2$ is a disc $B_0$ punctured at $p_1, p_2$ whose boundary $\partial B_0$ intersects $\gamma$ in at most two points, one intersection point near each of the endpoints $c_1, c_2$, and it intersects $\partial D_0 = \alpha$ in at most $2k$ points.

Let $B \subset H$ be a properly embedded disc with boundary $\partial B = \partial B_0$. The disc $B$ encloses the spots $p_1, p_2$. By Lemma 6.2, we have $d_\mathcal{C}(D, B) \leq k + 1$. Thus via replacing $D$ by $B$ we may assume that $\gamma$ intersects $D$ in at most two points. Repeating this construction with the disc $E$ implies that it suffices to show the following. If the simple closed curve $\gamma \subset \partial H$ intersects each of the curves $\partial D, \partial E$ in at most two points then $d_\mathcal{C}(D, E) \leq 10$. Note that since $\partial D, \partial E$ are separating simple closed curves, the curve $\gamma$ intersects $\partial D_0 = \partial D, \partial E_0 = \partial E$ in either two or zero points.

In the case that $\gamma$ is disjoint from both $D_0, E_0$ we are done, so assume (via possibly exchanging $D_0$ and $E_0$) that $\gamma$ intersects $\partial E_0$ in precisely two points. If $\gamma$ is not disjoint from $D_0$ then we may assume that $\gamma \cap D_0$ is disjoint from $E_0$. Moreover, in this case we may assume that each of the two curves obtained from $\gamma$ by replacing $\gamma \cap D_0$ by a subarc of $\alpha = \partial D$ with the same endpoints is not peripheral. Namely, otherwise $\gamma$ bounds a disc $B$ in $H$ and the claim follows from Lemma 6.2 applied to $D, B$ and to $B, E$.

Let $R$ be the component of $E_0 - D_0$ containing $\gamma \cap E_0$. Let $E_1$ be the component of $E_0 - R$ which contains the once punctured disc $A_1$. Note that $E_1$ is disjoint from $\gamma$. The intersection $E_1 \cap D_0$ is a finite union of disjoint rectangles. Let $R \subset E_1 \cap D_0$ be the component which is closest to the once punctured disc $A_2$ in $D_0$. This means that there is an embedded arc $c_3 \subset D_0$ with one endpoint in $R$ and the second endpoint in $A_2$ which intersects $E_1$ only at one endpoint. Let $E_2 \subset E_1$ be the union of the component of $E_1 - R$ containing $A_1$ with $R$. Note that $E_2$ is disjoint from $\gamma$. The union of the once punctured disc $E_2$, a thickening of the arc $c_3$ and the once punctured disc $A_2$ is a twice punctured disc $V$ embedded in $\partial H$. Let $D_1 \subset H$ be a properly embedded disc with boundary $\partial D_1 = \partial V$. Then $D_1$ encloses the spots $p_1, p_2$. Moreover, if the rectangle component $R$ of $E_0 - D_0$ containing the
intersection of \( E_0 \) with \( \gamma \) is the component of \( E_0 - D_0 \) which contains \( \partial A_2 \cap \partial D_0 \) in its boundary then \( \iota(\partial D_1, \partial E) = 4 \).

We distinguish three cases.

Case 1: \( \gamma \cap D_0 \neq \emptyset \) and \( \hat{R} \) is contained in the component of \( D_0 - \gamma \) containing \( A_1 \), or, equivalently, \( \gamma \cap D_0 \) lies between \( \hat{R} \) and \( A_2 \).

Then the punctured disc \( V \) and hence \( D_1 \) is disjoint from the essential simple closed curve \( \gamma' \) which is obtained from \( \gamma \) by replacing the \( \gamma \cap D_0 \) by the subarc of \( \alpha \) with the same endpoints which contains \( A_2 \cap \alpha \). Since both \( D \) and \( D_1 \) are disjoint from \( \gamma' \), we have \( d_E(D, D_1) = 1 \). Moreover, \( D_1 \) intersects \( \gamma \) in precisely two points. Replace \( D \) by \( D_1 \).

Case 2: \( \gamma \cap D_0 = \emptyset \).

Then \( D_1 \) is disjoint from \( \gamma \), and \( d_E(D, D_1) = 1 \). Replace \( D \) by \( D_1 \).

Case 3: \( \gamma \cap D_0 \neq \emptyset \) and \( \hat{R} \) is contained in the component of \( D_0 - \gamma \) containing \( A_2 \).

Then \( D_1 \) is disjoint from \( \gamma \). Thus the pairs of discs \( D, D_1 \) and \( D_1, E \) satisfy the hypothesis in Case 2. Therefore this case follows from two applications of Case 2, applied to \( D, D_1 \) and \( D_1, E \) provided that we can show that under the assumption of Case 2 above, we have \( d_E(D, E) \leq 5 \).

We now continue to investigate Case 1 and Case 2. Using the above notations for Case 1 and Case 2, let \( Q \) be the component of \( D_0 - \hat{R} \) containing \( A_2 \). The components of \( V \cap E_0 \) which are different from once punctured discs are up to isotopy contained in \( Q \cap E_0 \). Since the subdisc \( E_1 \subset E_0 - R \) is disjoint from \( Q \) this implies that the rectangle component \( R_1 \) of \( E_0 - V \) which contains \( \gamma \cap E_0 \) has one side on the boundary of the disc component of \( V \cap E_0 \) which is punctured at \( p_1 \).

Reapply the above construction with the discs \( D_1 \) and \( E \), but with the roles of the punctures \( p_1, p_2 \) exchanged. Let \( V' \) be the twice punctured disc obtained from this construction. Since the component of \( E_0 - V \) containing the intersection with \( \gamma \) is the rectangle adjacent to the bigon component of \( V \cap E_0 \) punctured at \( p_1 \), the remark before Case 1 above shows that \( \partial V' \cap \partial E_0 \) consists of four points. In particular, if \( D_2 \subset H \) is the properly embedded disc with boundary \( \partial D_2 = \partial V' \) then \( d_E(D_2, E) \leq 3 \) by Lemma 6.2.

An application of the above analysis to \( D_1 \) and \( E \) yields the following. If either \( \gamma \) is disjoint from \( D_1 \) or if \( \gamma \) is not disjoint from \( D_2 \) then \( d_E(D_1, D_2) = 1 \), moreover \( d_E(D_2, E) \leq 3 \) and hence \( d_E(D_1, E) \leq 4 \).

To summarize, we have.

i) In Case 2 above, \( d_E(D, E) \leq 5 \).

ii) In Case 1, either the pairs of discs \( D_1, E \) also satisfy Case 1 and then \( d_E(D, E) \leq 5 \), or there is a disc \( D' \) which is disjoint from \( \gamma \).

iii) In Case 3, there is a disc \( D' \) which is disjoint from \( \gamma \).

As a consequence, either \( d_E(D, E) \leq 5 \) or there is a disc \( D' \) with \( d_E(D', D) \leq 5 \) and \( d_E(D', E) \leq 5 \) which is what we wanted to show. \( \square \)

We use Lemma 6.2 and Lemma 6.3 to show the second part of Theorem 2.

**Proposition 6.4.** Let \( H \) be a handlebody with \( n \geq 2 \) spots. Then the map \( \mathcal{E} \mathcal{D} \mathcal{G} \to \mathcal{C} \mathcal{G} \) which associates to a disc its boundary is a 16-quasi-isometry.

**Proof.** Let \( \gamma \subset \partial H \) be any simple closed curve. Let \( p_1, p_2 \) be two spots of \( \partial H \). Then there is an embedded arc \( \alpha \subset \partial H \) connecting \( p_1 \) to \( p_2 \) which intersects \( \gamma \) in
at most one point. A thickening of $\alpha$ is a disc bounding simple closed curve in $X$ which intersects $\gamma$ in at most two points. Thus any simple closed curve in $\partial H$ is at distance two in $CG$ from a disc bounding simple closed curve. Moreover, by Lemma 6.2, for any disc $D$ in $H$ there is a disc $D'$ which encloses the spots $p_1, p_2$ and such that $d_E(D, D') \leq 2$.

As a consequence, it suffices to show the following. If the discs $D, E$ both enclose the spots $p_1, p_2$ in $\partial H$ then

$$d_E(D, E) \leq 16d_{CG}(\partial D, \partial E)$$

where $d_{CG}$ denotes the distance in the curve graph of $\partial H$.

Let $(\gamma_i)_{0 \leq i \leq \ell}$ be a geodesic in $CG$ connecting $\partial D = \gamma_0$ to $\partial E = \gamma_\ell$. The curve $\gamma_i$ is disjoint from $\gamma_{i+1}$.

Since for each $i < \ell$ the simple closed curves $\gamma_i, \gamma_{i+1}$ are disjoint, there is a simple arc in $\partial H$ connecting $p_1$ to $p_2$ which intersects each of the curves $\gamma_i$ and $\gamma_{i+1}$ in at most one point. A thickening of such an arc is a curve $\beta_{i,i+1}$ which bounds a disc $B_{i,i+1}$ enclosing $p_1$ and $p_2$. The curve $\beta_{i,i+1}$ intersects both $\gamma_i$ and $\gamma_{i+1}$ in at most two points.

By Lemma 6.3,

$$d_E(B_{i-1,i}, B_{i,i+1}) \leq 16 \forall i.$$ 

This means that $D$ can be connected to $E$ by a path in $E\mathcal{DG}$ whose length does not exceed $16d_{CG}(\partial D, \partial E)$. This shows the proposition. \hfill $\square$

References


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