HYPERBOLICITY OF THE GRAPH OF NON-SEPARATING MULTICURVES

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Abstract. A non-separating multicurve on a surface S of genus $g \geq 2$ with $m \geq 0$ punctures is a multicurve c so that $S - c$ is connected. For $k \geq 1$ define the graph $\mathcal{NC}(S,k)$ of non-separating k-multicurves to be the graph whose vertices are non-separating multicurves with k components and where two such multicurves are connected by an edge of length one if they can be realized disjointly and differ by a single component. We show that if $k < g/2+1$ then $\mathcal{NC}(S,k)$ is hyperbolic.

1. Introduction

The *curve graph* $\mathcal{CG}$ of an oriented surface S of genus $g \geq 0$ with $m \geq 0$ punctures and $3g - 3 + m \geq 2$ is the graph whose vertices are isotopy classes of essential (i.e. non-contractible and not homotopic into a puncture) simple closed curves on S. Two such curves are connected by an edge of length one if and only if they can be realized disjointly. The curve graph is a locally infinite δ-hyperbolic geodesic metric space of infinite diameter [MM99] for a number $\delta > 0$ not depending on the surface [A12, B12, CRS13, HPW13].

The *mapping class group* $\mathrm{Mod}(S)$ of all isotopy classes of orientation preserving homeomorphisms of S acts on $\mathcal{CG}$ as a group of simplicial isometries. This action is *coarsely transitive*, i.e. the quotient of $\mathcal{CG}$ under this action is a finite graph. Curve graphs and their geometric properties turned out to be an important tool for the investigation of the geometry of $\mathrm{Mod}(S)$ [MM00].

If the genus g of S is positive then for each $k \leq g$ we can define another $\mathrm{Mod}(S)$-graph $\mathcal{NC}(S,k)$ as follows. Vertices of $\mathcal{NC}(S,k)$ are *non-separating k-multicurves*, i.e. multicurves ν consisting of k components such that $S-\nu$ is connected. Two such multicurves are connected by an edge of length one if they can be realized disjointly and differ by a single component. The mapping class group of S acts coarsely transitively as a group of simplicial isometries on the graph of non-separating k-multicurves. In fact, the action is transitive on vertices. Note that $\mathcal{NC}(S,1)$ is just the complete subgraph of $\mathcal{CG}$ whose vertex set consists of all non-separating simple closed curves in S.

The goal of this note is to show

Theorem. *For $g \geq 2$ and $k < g/2 + 1$ the graph $\mathcal{NC}(S,k)$ of non-separating k-multicurves is hyperbolic.*

For the proof of the theorem, we adopt a strategy from [H13]. Namely, we begin with showing that for $g \geq 2$ the graph $\mathcal{NC}(S,1)$ is hyperbolic. This is easy if S has

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at most one puncture, in fact in this case the inclusion map $\mathcal{NC}(S,1) \to \mathcal{CG}$ is a quasi-isometry (see Section 3). If S has at least two punctures then this inclusion is not a quasi-isometry any more. In this case we apply a tool from [H13]. This tool is also used in Section 4 to successively add components to the multicurve until the number $k < g/2 + 1$ of components is reached.

We summarize the results from [H13] which we need in Section 2. At the end of this note we give an example indicated to us by Tarik Aougab and Saul Schleimer which shows that the strict bound $g/2 + 1$ for the number of components of the multicurve in the Theorem is sharp.

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2. Hyperbolic extensions of hyperbolic graphs

In this section we consider any (not necessarily locally finite) metric graph $(\mathcal{G}, d)$ (i.e. edges have length one). Let $\mathcal{C}$ be any finite, countable or empty index set. For a given family $\mathcal{H} = \{H_c \mid c \in \mathcal{C}\}$ of complete connected subgraphs of $\mathcal{G}$ define the *$\mathcal{H}$-electrification* of $\mathcal{G}$ to be the metric graph $(\mathcal{EG}, d_{\mathcal{E}})$ which is obtained from $\mathcal{G}$ by adding vertices and edges as follows.

For each $c \in \mathcal{C}$ there is a unique vertex $v_c \in \mathcal{EG} - \mathcal{G}$. This vertex is connected with each of the vertices of H_c by a single edge of length one, and it is not connected with any other vertex.

Definition 2.1. For a number $r > 0$ the family $\mathcal{H}$ is called *r-bounded* if for $c \neq d \in \mathcal{C}$ the intersection $H_c \cap H_d$ has diameter at most r where the diameter is taken with respect to the intrinsic path metric on H_c and H_d.

A family which is r-bounded for some $r > 0$ is simply called *bounded*.

In the sequel all parametrized paths γ in $\mathcal{G}$ or $\mathcal{EG}$ are supposed to be *simplicial*. This means that they are defined on a closed connected subset of the reals whose finite endpoints (if any) are integers. We require that the image of every integer is a vertex, and that the restriction to an integral interval $[k, k+1]$ either is homeomorphism onto an edge, or it is constant. In particular, simplicial paths are continuous.

Call a simplicial path γ in $\mathcal{EG}$ *efficient* if for every $c \in \mathcal{C}$ we have $\gamma(k) = v_c$ for at most one integer k. Note that if γ is an efficient simplicial path in $\mathcal{EG}$ which passes through $\gamma(k) = v_c$ for some $c \in \mathcal{C}$ then $\gamma(k-1) \in H_c, \gamma(k+1) \in H_c$. This is true because the vertex $v_c \in \mathcal{EG}$ is only connected with vertices in H_c by an edge.

For a number $L > 1$, an *L-quasi-geodesic* in $\mathcal{EG}$ is a path $\gamma : [a, b] \to \mathcal{EG}$ such that for all $a \leq s < t \leq b$ we have

$$|t - s|/L - L \leq d(\gamma(s), \gamma(t)) \leq L|t - s| + L.$$

In slight deviation from this standard definition, throughout we require in the sequel that all quasi-geodesics are simplicial, in particular, they are continuous. We will often but not always state this explicitly.

Definition 2.2. The family $\mathcal{H}$ has the *bounded penetration property* if it is r-bounded for some $r > 0$ and if for every $L > 0$ there is a number $p(L) > 2r$ with the following property. Let γ be an efficient simplicial L-quasi-geodesic in $\mathcal{EG}$, let $c \in \mathcal{C}$ and let $k \in \mathbb{Z}$ be such that $\gamma(k) = v_c$. If the distance in H_c between $\gamma(k-1)$ and $\gamma(k+1)$ is at least $p(L)$ then every efficient simplicial L-quasi-geodesic γ' in $\mathcal{EG}$ with the same endpoints as γ passes through v_c. Moreover, if $k' \in \mathbb{Z}$ is such that $\gamma'(k') = v_c$ them the distance in H_c between $\gamma(k-1), \gamma'(k'-1)$ and between $\gamma(k+1), \gamma'(k'+1)$ is at most $p(L)$.

Let $\mathcal{H}$ be as in Definition 2.2. Define an *enlargement* $\hat{\gamma}$ of an efficient simplicial L-quasi-geodesic $\gamma : [0, n] \to \mathcal{EG}$ with endpoints $\gamma(0), \gamma(n) \in \mathcal{G}$ as follows. Let $0 < k_1 < \cdots < k_s < n$ be those points such that $\gamma(k_i) = v_{c_i}$ for some $c_i \in \mathcal{C}$. Then $\gamma(k_i - 1), \gamma(k_i + 1) \in H_{c_i}$. For each $i \leq s$ replace $\gamma[k_i - 1, k_i + 1]$ by a simplicial geodesic in the graph H_{c_i} with the same endpoints. Note that since we require that the endpoints of γ are vertices in $\mathcal{G}$, an enlargement of γ is a path with the same endpoints.

For a number $k > 0$ define a subset Z of the metric graph $\mathcal{G}$ to be *k-quasi-convex* if any geodesic with both endpoints in Z is contained in the k-neighborhood of Z. In particular, up to perhaps increasing the number k, any two points in Z can be connected in Z by a (not necessarily continuous) path which is a k-quasi-geodesic in $\mathcal{G}$.

In Section 5 of [H13] the following is shown.

Theorem 2.3. *Let $\mathcal{G}$ be a metric graph and let $\mathcal{H} = \{H_c \mid c \in \mathcal{C}\}$ be a bounded family of complete connected subgraphs of $\mathcal{G}$. Assume that the following conditions are satisfied.*

(1) *There is a number $\delta > 0$ such that each of the graphs H_c is δ-hyperbolic.*
(2) *The $\mathcal{H}$-electrification $\mathcal{EG}$ of $\mathcal{G}$ is hyperbolic.*
(3) *$\mathcal{H}$ has the bounded penetration property.*

Then $\mathcal{G}$ is hyperbolic. There is a number $L > 1$ such that enlargements of geodesics in $\mathcal{EG}$ are L-quasi-geodesics in $\mathcal{G}$. The subgraphs H_c are uniformly quasi-convex.

In fact, although this was not stated explicitly, one obtains that the graph $\mathcal{G}$ is δ'-hyperbolic for a number $\delta' > 0$ only depending on the hyperbolicity constant for $\mathcal{EG}$, the common hyperbolicity constant δ for the subgraphs H_c and the constants which enter in the bounded penetration property.

3. Hyperbolicity of the graph of non-separating curves

In this section we consider an arbitrary surface S of genus $g \geq 2$ with $m \geq 0$ punctures. Let $\mathcal{CG}$ be the curve graph of S and let $\mathcal{NC}(S, 1)$ be the complete subgraph of $\mathcal{CG}$ whose vertex set consists of non-separating curves. The goal of this section is to show

Proposition 3.1. *The graph $\mathcal{NC}(S, 1)$ is hyperbolic.*

Example: If S is a surface of genus $g = 1$ then any two disjoint non-separating simple closed curves in S are homotopic after closing the punctures and the graph $\mathcal{NC}(S, 1)$ is not connected.

Define a properly embedded connected incompressible subsurface X of S to be *thick* if the genus of X equals g. This is equivalent to stating that each of the boundary circles of X is separating in S and that moreover there is no non-separating simple closed curve in S which is contained in $S - X$. Observe that the only thick subsurface of a surface S with at most one puncture is S itself.

If $X \subset S$ is thick then each component of $S - X$ is a bordered punctured sphere with connected boundary. If we collapse each boundary circle of X to a puncture then we can view X as a surface of finite type whose genus equals the genus of S. In particular, we can look at thick subsurfaces of X. However, thick subsurfaces of X are precisely the thick subsurfaces of S which are contained in X.

For a thick subsurface X of S and for $p \geq 1$ define a graph $\mathcal{A}(X, p)$ as follows. Vertices of $\mathcal{A}(X, p)$ are non-separating simple closed curves in X. Two such vertices c, d are connected by an edge of length one if either they are disjoint or if they are both contained in a proper thick subsurface Y of X of Euler characteristic $\chi(X)+p$. Note that if $p \geq -\chi(X) - 2g + 2$ then $\mathcal{A}(X, p) = \mathcal{NC}(X, 1)$.

Recall that for a number $L \geq 1$ two geodesic metric spaces Y, Z are *L-quasi-isometric* if there is a map $F : Y \to Z$ so that

$$d(x, y)/L - L \leq d(Fx, Fy) \leq Ld(x, y) + L \,\forall x, y \in Y$$

and that for all $z \in Z$ there is some $y \in Y$ with $d(Fy, z) \leq L$. In general, quasi-isometries are not continuous. A map $F : Y \to Z$ is called *coarsely L-Lipschitz* if $d(Fx, Fy) \leq Ld(x, y) + L$ for all $x, y \in Y$.

Let $\mathcal{CG}(X)$ be the curve graph of X.

Lemma 3.2. *For every thick subsurface X of S the vertex inclusion extends to a 2-quasi-isometry $\mathcal{A}(X, 1) \to \mathcal{CG}(X)$.*

Proof. Since two simple closed curves which are contained in a proper thick subsurface Y of X are disjoint from a boundary circle of Y which is essential in X, the vertex inclusion extends to a coarsely 2-Lipschitz map $\mathcal{A}(X, 1) \to \mathcal{CG}(X)$. Thus it suffices to show that the distance in $\mathcal{A}(X, 1)$ between any two non-separating simple closed curves does not exceed twice their distance in $\mathcal{CG}(X)$.

To this end let $\gamma : [0, n] \to \mathcal{CG}(X)$ be a simplicial geodesic connecting two non-separating simple closed curves $\gamma(0), \gamma(n)$. We construct first a simplicial geodesic $\tilde{\gamma}$ in $\mathcal{CG}(X)$ with the same endpoints such that for each i, the curve $\tilde{\gamma}(i)$ either is non-separating or it decomposes X into a thick subsurface of Euler characteristic $\chi(X) + 1$ and a three-holed sphere. Call such a simple closed curve (with either of these two properties) *admissible* in the sequel.

For the construction of $\tilde{\gamma}$ replace first each of the vertices $\gamma(2i)$ with even parameter $0 < 2i < n$ by an admissible curve. Namely, if $\gamma(2i)$ is not admissible then $\gamma(2i)$ decomposes X into two surfaces X_1, X_2 which are different from three holed spheres.

If $\gamma(2i - 1), \gamma(2i + 1)$ are contained in distinct components of $X - \gamma(2i)$ then they are disjoint and hence they are connected in $\mathcal{CG}(X)$ by an edge. This implies that we can shorten γ with fixed endpoints. Since γ is length minimizing this is impossible.

Thus $\gamma(2i - 1), \gamma(2i + 1)$ are contained in the same component of $X - \gamma(2i)$, say in X_1. Then $X_2 = X - X_1$ either has positive genus and hence contains a non-separating curve, or it is a sphere with at least four holes and contains an admissible separating curve. Thus there is an admissible curve $\tilde{\gamma}(2i) \subset X_2$, and this curve is disjoint from $\gamma(2i-1) \cup \gamma(2i+1)$. Replace $\gamma(2i)$ by $\tilde{\gamma}(2i)$. This process leaves the points $\gamma(2i + 1)$ with odd parameter unchanged.

In a second step, replace with the same construction each of the points $\gamma(2i+1)$ with odd parameter by an admissible curve. Let $\tilde{\gamma}:[0,n]\to\mathcal{CG}(X)$ be the resulting simplicial geodesic. The image of every vertex is admissible.

The geodesic $\tilde{\gamma}$ is now modified as follows. Replace each edge $\tilde{\gamma}[i,i+1]$ connecting two separating admissible simple closed curves $\tilde{\gamma}(i),\tilde{\gamma}(i+1)$ by an edge path in $\mathcal{CG}(X)$ of length 2 with the same endpoints so that the middle vertex is a non-separating simple closed curve. This is possible because if c_1,c_2 are two disjoint separating admissible curves then $c_1\cup c_2$ is disjoint from some non-separating simple closed curve in X. The length of the resulting path $\hat{\gamma}$ is at most twice the length of γ.

The path $\hat{\gamma}$ can be viewed as a path in $\mathcal{A}(X,1)$ by simply erasing all vertices which are separating admissible simple closed curves. Namely, each such vertex v is the boundary circle of a thick subsurface Y of X of Euler characteristic $\chi(X)+1$. The two adjacent vertices are non-separating simple closed curves contained in Y. Thus by the definition of $\mathcal{A}(X,1)$, these curves are connected in $\mathcal{A}(X,1)$ by an edge. This shows that the endpoints of γ are connected in $\mathcal{A}(X,1)$ by a path whose length does not exceed twice the distance in $\mathcal{CG}(X)$ between the endpoints. □

Since a surface with at most one puncture does not admit any proper thick subsurface we obtain as an immediate corollary

Corollary 3.3. *If S has at most one puncture then the inclusion $\mathcal{NC}(S,1)\to\mathcal{CG}$ is a 1-quasi-isometry.*

Write $\mathcal{A}(p)=\mathcal{A}(S,p)$. Our goal is to use Lemma 3.2 and induction on p to show that $\mathcal{A}(p)$ is hyperbolic for all p. Since $\mathcal{A}(p)=\mathcal{NC}(S,1)$ for $p\geq-\chi(S)-2g+2$, this then shows Proposition 3.1.

Let now $p-1\geq 1$ and let X be a thick subsurface of S such that $\chi(X)=\chi(S)+p-1$. Let H_X be the complete subgraph of $\mathcal{A}(p)$ whose vertex set consists of all non-separating simple closed curves which are contained in X. Let $\mathcal{H}=\{H_X\mid X\}$. Our goal is to apply Theorem 2.3. to the graph $\mathcal{A}(p)$ and its $\mathcal{H}$-electrification. The next easy observation is the basic setup for the induction step.

Lemma 3.4. *$\mathcal{A}(p-1)$ is 2-quasi-isometric to the $\mathcal{H}$-electrification of $\mathcal{A}(p)$.*

Proof. Let $\mathcal{E}$ be the $\mathcal{H}$-electrification of $\mathcal{A}(p)$. Let c,d be any two simple closed curves which are connected in $\mathcal{A}(p-1)$ by an edge. Then either c,d are disjoint and hence connected in $\mathcal{A}(p)$ by an edge, or c,d are contained in a thick subsurface X of S of Euler characteristic $\chi(X)=\chi(S)+p-1$. Thus c,d are vertices in H_X and hence the distance between c,d in $\mathcal{E}$ is at most two. This shows that the vertex inclusion $\mathcal{A}(p-1)\to\mathcal{E}$ is two-Lipschitz.

That this is in fact a 2-quasi-isometry follows from the observation that $\mathcal{A}(p)$ is obtained from $\mathcal{A}(p-1)$ by deleting some edges. Moreover, the endpoints of an embedded simplicial path in $\mathcal{E}$ of length 2 whose midpoint is a special vertex not contained in $\mathcal{A}(p)$ are non-separating simple closed curves which are contained in a thick subsurface X of S of Euler characteristic $\chi(S)+p-1$ and henc they are connected by an edge in $\mathcal{A}(p-1)$. □

Our goal is now to check that the family $\mathcal{H}=\{H_X\mid X\}$ has the properties stated in Theorem 2.3. The following lemma together with Lemma 3.2 implies that the graphs H_X are δ-hyperbolic for a universal constant $\delta>0$.

Lemma 3.5. *H_X is isometric to $\mathcal{A}(X,1)$.*

Proof. Let X be a thick subsurface of S of Euler characteristic $\chi(S)+p-1$. If c_1,c_2 are two non-separating simple closed curves contained in X then c_1,c_2 are connected in $\mathcal{A}(p)$ by an edge if either c_1,c_2 are disjoint or if c_1,c_2 are contained in a thick subsurface X_0 of S of Euler characteristic $\chi(X_0)=\chi(S)+p=\chi(X)+1$.

Now the thick subsurface X_0 can be chosen to be contained in X. Namely, any thick subsurface of S of Euler characteristic $\chi(S)+\ell$ ($\ell\geq 0$) can be described as the complement in S of a small neighborhood of an embedded forest in S (i.e. an embedded possibly disconnected graph with no cycles) with ℓ edges whose vertices are the punctures of S.

Assume for the moment that there is a forest G defining X which is the union of a connected component $\hat G$ and isolated points. Then $\hat G$ has precisely p vertices and $p-1$ edges where $p-1=\chi(X)-\chi(S)$. Since the graph G_0 defining X_0 has p edges, there is at least one puncture x of S which is not contained in $\hat G$ and which is the endpoint of an edge e of G_0. If up to homotopy with fixed endpoints, e intersects $\hat G$ at most at the second endpoint then the complement of a small neighborhood of $\hat G\cup e$ is a thick subsurface Y of X of Euler characteristic $\chi(S)+p$ which contains $c_1\cup c_2$. This is what we wanted to show.

If e intersects $\hat G$ in an interior point which can not be removed with a homotopy of e with fixed endpoints, let e_0 be the subarc of e with endpoints x and the first intersection point with $\hat G$. Concatenation of e_0 with a subarc of an edge of $\hat G$ and modification of the resulting arc with a small homotopy with fixed endpoints yields an embedded are $\hat e$ in S whose interior is disjoint from the interior of $\hat G$ so that $\hat G\cup\hat e$ defines a thick subsurface Y of X of Euler characteristic $\chi(S)+p$ disjoint from $c_1\cup c_2$.

The general case is treated in the same way. Namely, there is at least one edge e of G_0 which connects two distinct connected components of G, and the argument above can be applied to the edge e.

As a consequence, c_1,c_2 are connected by an edge in $\mathcal{A}(X,1)$ which is what we wanted to show. ☐

Lemma 3.6. *The family of subgraphs H_X of $\mathcal{A}(p)$ where X runs through the thick subsurfaces of S of Euler characteristic $\chi(S)+p-1$ is bounded.*

Proof. Let X,Y be two thick subsurfaces of S of Euler characteristic $\chi(S)+p-1$. If $X\neq Y$ then up to homotopy, $X\cap Y$ is a (possibly disconnected) subsurface of X whose Euler characteristic is strictly bigger than the Euler characteristic of X. In particular, the diameter in the curve graph of X of the set of simple closed curves contained in $X\cap Y$ is uniformly bounded. Thus the lemma follows from Lemma 3.2 and Lemma 3.5. ☐

The proof of the bounded penetration property is more involved. To this end recall from [MM00] that for every proper connected subsurface X of S there is a *subsurface projection* π_X of $\mathcal{CG}$ into the subsets of the arc and curve graph of X. This projection associates to a simple closed curve c in S which is not disjoint from X the intersection components $\pi_X(c)$ of c with X, viewed as a subset of the arc and curve graph of X. The diameter of the image is at most one. If c is disjoint from X then this projection is empty. The arc and curve graph of X is 2-quasi-isometric to the curve graph of X (see [MM00]).

Recall that every vertex of any of the graphs $\mathcal{A}(p)$ ($p \geq 1$) is a non-separating simple closed curve in S. By definition of a thick subsurface of S, for any such curve c and every thick subsurface Z of S we have $\pi_Z(c) \neq \emptyset$. This fact will be used throughout the remainder of this section.

We need the following result from [MM00] (in the version formulated in Lemma 6.5 of [H13]).

Proposition 3.7. *For every number $L > 1$ there is a number $\xi(L) > 0$ with the following property. Let Y be a proper connected subsurface of S and let γ be a simplicial path in $\mathcal{CG}$ which is an L-quasi-geodesic. If $\pi_Y(v) \neq \emptyset$ for every vertex v on γ then*

$$\mathrm{diam}\pi_Y(\gamma) < \xi(L).$$

If $\gamma : [0, n] \to \mathcal{A}(S, 1)$ is any geodesic then for all j, the curves $\gamma(j)$ and $\gamma(j + 1)$ either are disjoint and hence connected in $\mathcal{CG}$ by an edge, or they are contained in a common thick subsurface Y of S of Euler characteristic $\chi(S) + 1$. In the second case replace the edge $\gamma[j, j + 1]$ by an edge path in $\mathcal{CG}$ of length two connecting the same endpoints which passes through an essential simple closed curve in the complement of Y. We call $\tilde{\gamma}$ a *canonical modification* of γ. By Lemma 3.2 and its proof, $\tilde{\gamma}$ is a simplicial path in $\mathcal{CG}$ which is a 2-quasi-geodesic.

We now define a family of geodesics in $\mathcal{A}(S, 1)$ which serve as substitutes for the tight geodesics as introduced in [MM00]. Namely, for numbers $\kappa > 0, p \geq 1$ define a simplicial path $\zeta : [0, k] \to \mathcal{A}(S, 1)$ to be *(κ,p)-good* if the following holds true. Let $X \subset S$ be any thick subsurface of Euler characteristic $\chi(X) \geq \chi(S) + p$; then there is a number $u = u(X) \in [0, k)$ with the following property.

(1) For every $j \leq u$, $\mathrm{diam}(\pi_X(\zeta(0) \cup \zeta(j))) \leq \kappa$.
(2) For every $j > u$, $\mathrm{diam}(\pi_X(\zeta(j) \cup \zeta(k))) \leq \kappa$.

Thus in a good simplicial path, big subsurface projections into thick subsurfaces can be explicitly localized.

We use Proposition 3.7 to show

Lemma 3.8. *There is a number $\kappa_1 > 0$ such that any two vertices in $\mathcal{A}(S, 1)$ can be connected by a $(\kappa_1, 1)$-good geodesic.*

Proof. Let c_1, c_2 be non-separating simple closed curves and let $\gamma : [0, k] \to \mathcal{A}(S, 1)$ be a simplicial geodesic connecting c_1 to c_2, with canonical modification $\tilde{\gamma} : [0, \tilde{k}] \to \mathcal{CG}$.

Let $\ell_1 < \cdots < \ell_s$ be such that for each i the curves $\tilde{\gamma}(\ell_i), \tilde{\gamma}(\ell_i + 2)$ are both separating and such that the subsurface of S filled by $\tilde{\gamma}(\ell_i) \cup \tilde{\gamma}(\ell_i + 2)$ (i.e. the smallest subsurface of S which contains $\tilde{\gamma}(\ell_i) \cup \tilde{\gamma}(\ell_i + 2)$) is a holed sphere whose complement Z in S is thick (we may have $s = 0$, i.e. there may not be such a pair of vertices). Then $\tilde{\gamma}(\ell_i + 1)$ is a non-separating simple closed curve contained in Z. Let $\tilde{\gamma}_1(\ell_i + 1)$ be a non-separating simple closed curve contained in Z which is contained in the one-neighborhood of the subsurface projection $\pi_Z(c_2)$ of c_2. That the subsurface projection $\pi_Z(c_2)$ is not empty follows since c_2 is non-separating and hence can not be contained in $S - Z$.

Replace $\tilde{\gamma}(\ell_i + 1)$ by $\tilde{\gamma}_1(\ell_i + 1)$. The simplicial path $\tilde{\gamma}_1$ constructed in this way is a canoncial modification of a geodesic γ_1 in $\mathcal{A}(S, 1)$ connecting c_1 to c_2. We claim that γ_1 is a $(\xi(2), 1)$-good geodesic in $\mathcal{A}(S, 1)$ where $\xi(2) > 0$ is as in Proposition 3.7.

Namely, if Z is an arbitrary thick subsurface of S then since γ_1 is a geodesic in $\mathcal{A}(S,1)$. there are at most two parameters $k, k+\iota$ (here $\iota = 0$ or $\iota = 2$) such that $\tilde\gamma_1(k), \tilde\gamma_1(k+\iota)$ is disjoint from Z. Since $\tilde\gamma_1$ is a 2-quasi-geodesic in $\mathcal{CG}$, if there is at most one such point (which is in particular the case if the Euler characteristic of Z equals $\chi(S)+1$, i.e. if there is a unique essential curve disjoint from Z) then the properties (1),(2) for Z with $\kappa = \xi(2)$ are immediate from Lemma 3.2 and Proposition 3.7. Otherwise the property follows from the construction of γ_1 and the fact that for subsurfaces $X \subset Y \subset S$ and any simple closed curve c we have $\pi_X(c) = \pi_X(\pi_Y(c))$ (with a small abuse of notation). □

We use Proposition 3.7 and Lemma 3.8 to define a *level p hierarchy path* in $\mathcal{A}(p)$ connecting two non-separating simple closed curves c_1, c_2 as follows. The starting point is a $(\kappa_1, 1)$-good geodesic $\gamma : [0,k] \to \mathcal{A}(S,1)$. For any j so that the curves $\gamma(j), \gamma(j+1)$ are not disjoint there is a thick subsurface Y_j of Euler characteristic $\chi(Y_j) = \chi(S)+1$ so that $\gamma(j), \gamma(j+1) \subset Y_j$. Replace the edge $\gamma[j, j+1]$ by a simplicial $(\kappa_1, 1)$-good geodesic in $\mathcal{A}(Y_j, 1)$ with the same endpoints. The resulting path is an edge-path in the subgraph $\mathcal{A}(2)$ of $\mathcal{A}(1)$. Proceed inductively and construct in p such steps a simplicial path in $\mathcal{A}(p) \subset \mathcal{A}(1)$ connecting c_1 to c_2 which we call a *level p hierarchy path*.

Lemma 3.9. *For every $p \geq 1$ there is a number $\kappa_p > 0$ such that a level p hierarchy path in $\mathcal{A}(p)$ is (κ_p, p)-good.*

Proof. We proceed by induction on p. The case $p = 1$ follows from the definition of a hierarchy path and Lemma 3.8. Thus assume that the lemma holds true for all $p - 1 \geq 1$.

Let $\gamma : [0,n] \to \mathcal{A}(p)$ be a level p hierarchy path. The construction of γ is as follows. There is a level $p-1$ hierarchy path $\zeta : [0,s] \to \mathcal{A}(p-1)$, and there are numbers $0 \leq \tau_1 < \cdots < \tau_q < s$ such that for each i, the edge $\zeta[\tau_i, \tau_i + 1]$ connects two non-separating simple closed curves which are contained in a thick subsurface Z_i of S of Euler characteristic $\chi(S) + p - 1$. For $\ell \notin \{\tau_1, \dots, \tau_q\}$, the simple closed curves $\zeta(\ell), \zeta(\ell+1)$ are disjoint. The hierarchy path γ is obtained from ζ by replacing each of the edges $\zeta[\tau_i, \tau_i + 1]$ by a $(\kappa_1, 1)$-good geodesic in $\mathcal{A}(Z_i, 1)$ with the same endpoints.

By induction hypothesis, ζ is $(\kappa_{p-1}, p-1)$-good for a number $\kappa_{p-1} > 1$ not depending on ζ. Thus for any thick subsurface Z of S of Euler characteristic $\chi(Z) \geq \chi(S) + p$ there is a number $u \in [0,s]$ so that for all $j \leq u$ we have $\operatorname{diam}(\pi_Z(\zeta(0) \cup \zeta(j))) \leq \kappa_{p-1}$ and similarly for $j \geq u+1$.

Let now $i > 0$ be such that $\tau_i < u$. There is a subarc ρ of γ which is a $(\kappa_1, 1)$-good geodesic in $\mathcal{A}(Z_i, 1)$ connecting $\zeta(\tau_i)$ to $\zeta(\tau_i + 1)$. By the definition of a $(\kappa_1, 1)$-good geodesic in $\mathcal{A}(Z_i, 1)$, since

$$\begin{aligned}\operatorname{diam}(\pi_Z(\zeta(\tau_i) \cup \zeta(\tau_i+1))) &\leq \operatorname{diam}(\pi_Z(\zeta(0) \cup \zeta(\tau_i))) + \operatorname{diam}(\pi_Z(\zeta(0) \cup \zeta(\tau_i+1))) \\ &\leq 2\kappa_{p-1},\end{aligned}$$

for each vertex $\rho(t)$ on the geodesic ρ the diameter of the subsurface projection $\pi_Z(\zeta(\tau_i) \cup \rho(t))$ does not exceed $2\kappa_{p-1} + \kappa_1$. Then for each t we have

$$\operatorname{diam}(\pi_Z(\zeta(0) \cup \rho(t))) \leq 3\kappa_{p-1} + \kappa_1 = \kappa_p.$$

This argument is also valid for $\tau_i > u$.

Finally if $\tau_i = u$ then we can apply the same reasoning as before to the κ_1-good geodesic ρ and obtain the statement of the lemma. □

Proof of Proposition 3.1: By Lemma 3.3, if S has at most one puncture then the inclusion $\mathcal{NC}(S,1)\to\mathcal{CG}$ is a quasi-isometry.

If the number of punctures is at least two then we show by induction on p the following.

a) The graph $\mathcal{A}(p)$ is hyperbolic.
b) Level p hierarchy paths are uniform quasi-geodesics in $\mathcal{A}(p)$.
c) For every $L>1$ there is a number $\xi(L,p)>0$ with the following property. Let γ be a simplicial path in $\mathcal{A}(p)$ which is an L-quasi-geodesic, and let $\tilde\gamma$ be the canonical modification of γ. If Y is a thick subsurface of S of Euler characteristic $\chi(Y)\geq\chi(S)+p$, and if $\pi_Y(v)\neq\emptyset$ for every vertex v on $\tilde\gamma$ then $\mathrm{diam}\pi_Y(\gamma)<\xi(L,p)$.

The case $p=1$ follows from Lemma 3.2, Proposition 3.7 and the definition of a canonical modification of a simplicial path in $\mathcal{A}(S,1)$. Assume that the claim holds true for $p-1\geq 1$.

For a thick subsurface X of Euler characteristic $\chi(X)=\chi(S)+p-1$ let as before H_X be the complete subgraph of $\mathcal{A}(p)$ whose vertex set consists of all non-separating simple closed curves contained in X, and let $\mathcal{H}=\{H_X\mid X\}$. By Lemma 3.4, $\mathcal{A}(p-1)$ is 2-quasi-isometric to the $\mathcal{H}$-electrification of $\mathcal{A}(p)$. Moreover by construction, level p hierarchy paths are enlargements of level $p-1$ hierarchy paths. Therefore by the induction hypothesis, to establish properties a),b) above for p it suffices to show that the family $\mathcal{H}$ is bounded and satisfies the assumptions (1),(3) in the statement of Theorem 2.3.

Lemma 3.6 shows that the family $\mathcal{H}=\{H_X\mid X\}$ is bounded.

By Lemma 3.5, H_X is isometric to $\mathcal{A}(X,1)$ and hence by Lemma 3.2, H_X is δ-hyperbolic for a number $\delta>0$ not depending on X. The bounded penetration property for $\mathcal{H}$ follows from property c) above, applied to thick subsurfaces of Euler characteristic $\chi(S)+p$ and quasi-geodesics in $\mathcal{A}(p-1)$ (compare [H13]). Thus by Theorem 2.3 and the induction hypothesis, $\mathcal{A}(p)$ is hyperbolic, and level p hierarchy paths are uniform quasi-geodesics in $\mathcal{A}(p)$.

We are left with verifying property c) above for $\mathcal{A}(p)$. By Lemma 3.9, this property holds true for level p hierarchy paths with the number $\kappa_p>0$ replacing $\xi(L,p)$. The argument in the proof of Lemma 6.5 of [H13] then yields this property for an arbitrary L-quasi-geodesic in $\mathcal{A}(p)$ for a suitable number $\xi(L,p)>0$.

Namely, by hyperbolicity, for every $L>1$ there is a number $n(L)>1$ so that for every L-quasi-geodesic $\eta:[0,k]\to\mathcal{A}(p)$ of finite length, the Hausdorff distance between the image of η and the image of a level p hierarchy path γ with the same endpoints does not exceed $n(L)$.

Let $Y\subset S$ be a thick subsurface of Euler characteristic $\chi(Y)\geq\chi(S)+p$. Assume that

$$\mathrm{diam}(\pi_Y(\eta(0)\cup\eta(k)))\geq 2\kappa_p+L(4n(L)+10). \tag{1}$$

By the properties of level p hierarchy paths, if $\tilde\gamma$ denotes the canonical modification of γ then there is some $u\in\mathbb{Z}$ so that $\tilde\gamma(u)\in A$ where $A\subset\mathcal{CG}$ is the set of all curves which are disjoint from Y.

By the choice of $n(L)$, the quasi-geodesic η passes through the $n(L)$-neighborhood of Y. By this we mean that there is a vertex x on η and a simplicial path in $\mathcal{A}(p)$ of length at most $n(L)$ which connects x to a non-separating simple closed curve contained in Y.

Let $s_0+1 \leq t_0-1$ be the smallest and biggest number, respectively, so that $\eta(s_0+1), \eta(t_0-1)$ are contained in the $n(L)$-neighborhood of Y in the sense defined in the previous paragraph. The distance in $\mathcal{A}(p)$ between $\eta(s_0)$ and $\eta(t_0)$ does not exceed $2n(L)+3$.

A level p hierarchy path connecting $\eta(0)$ to $\eta(s_0)$ is contained in the $n(L)$-neighborhood of $\eta[0,s_0]$ and hence its canonical modification does not pass through A. Similarly, a canonical modification of a level p hierarchy path connecting $\eta(t_0)$ to $\eta(k)$ does not pass through A. By the assumption (1), and the properties of hierarchy paths, this implies that

$$\operatorname{diam}(\pi_Y(\eta(s_0)\cup\eta(t_0))) \geq L(4n(L)+10).$$

The distance in $\mathcal{A}(p)$ between $\eta(s_0), \eta(t_0)$ is at most $2n(L)+3$, and hence since η is an L-quasi-geodesic, the length of the segment $\eta[s_0,t_0]$ is at most $L(2n(L)+4)$. Then the length of a canonical modification $\tilde\eta[s,t]$ of $\eta[s_0,t_0]$ is at most $L(4n(L)+8)$. Now if c,d are disjoint simple closed curves which intersect Y then the diameter of $\pi_Y(c\cup d)$ is at most one. Thus if $\tilde\eta(\ell)$ intersects Y for all ℓ then

$$\operatorname{diam}(\pi_Y(\tilde\eta(s)\cup\tilde\eta(t))) = \operatorname{diam}(\pi_Y(\eta(s_0)\cup\eta(t_0))) \leq L(4n(L)+8)$$

which is a contradiction.

This completes the induction step and proves Proposition 3.1. □

The arguments in [H13] can now be used without modification to identify the Gromov boundary of $\mathcal{NC}(S,1)$. To this end let $\mathcal{L}$ be the set of all geodesic laminations on S equipped with the coarse Hausdorff topology. In this topology, a sequence (ν_i) converges to ν if any limit in the usual Hausdorff topology of a convergent subsequence contains ν as a sublamination.

For each thick subsurface X of S let $\mathcal{L}(X)\subset\mathcal{L}$ be the set of all minimal geodesic laminations which fill up X, equipped with the coarse Hausdorff topology. We have

Corollary 3.10. *The Gromov boundary of $\mathcal{NC}(S,1)$ equals $\cup_X\mathcal{L}(X)$ equipped with the coarse Hausdorff topology.*

4. Proof of the theorem

In this section we consider an oriented surface S of genus $g\geq 2$ with $m\geq 0$ punctures. In the introduction we defined for $n\geq 1$ the graph $\mathcal{NC}(S,n)$ of non-separating multicurves in S with n components. Our goal is to show

Theorem 4.1. *For $n<g/2+1$ the graph $\mathcal{NC}(S,n)$ is hyperbolic.*

The case $n=1$ is just Proposition 3.1. For $2\leq n<g/2+1$ we use induction on n similiar to the arguments in the proof of Proposition 3.1. There are no new tools needed, however all the constructions have to be adjusted to the situation at hand.

We begin with describing an electrification of the graph $\mathcal{NC}(S,n)$. First, for a non-separating $(n-1)$-multicurve $\nu\in\mathcal{NC}(S,n-1)$ let H_ν be the complete subgraph of $\mathcal{NC}(S,n)$ whose vertex set consists of all non-separating n-multi-curves containing ν. We have

Lemma 4.2. *There is a natural graph isomorphism $H_\nu\to\mathcal{NC}(S-\nu,1)$.*

Proof. If $\beta\in H_\nu$ is any non-separating n-multicurve containing ν then $\beta-\nu$ is a non-separating simple closed curve in $S-\nu$. If $\beta,\beta'\in H_\nu$ are connected by an edge then the non-separating simple closed curves $\beta - \nu$ and $\beta' - \nu$ are disjoint and hence they are connected in $\mathcal{NC}(S - \nu, 1)$ by an edge.

Vice versa, the union with ν of any non-separating simple closed curve c in $S - \nu$ is a non-separating multicurve in H_ν. If $c' \subset S - \nu$ is non-separating and disjoint from c then $\nu \cup c$ and $\nu \cup c'$ are connected by an edge in H_ν.

This shows the lemma. □

Let $\mathcal{H} = \{H_\nu \mid \nu \in \mathcal{NC}(S, n-1)\}$.

Lemma 4.3. *$\mathcal{NC}(S, n-1)$ is quasi-isometric to the $\mathcal{H}$-electrification of $\mathcal{NC}(S, n)$.*

Proof. Let $\mathcal{E}$ be the $\mathcal{H}$-electrification of $\mathcal{NC}(S, n)$. Define a vertex embedding $\Lambda : \mathcal{NC}(S, n-1) \to \mathcal{E}$ by associating to a non-separating $(n-1)$-multicurve c any non-separating n-multicurve $\Lambda(c)$ containing c. We claim that Λ is coarsely 8-Lipschitz.

To see this let c_0, c_1 be connected by an edge in $\mathcal{NC}(S, n-1)$. Then c_1 is obtained from c_0 by removing a component a from c_0 and replacing it by a component b disjoint from c_0.

The union $c_0 \cup b$ is a multicurve with n components. If this multicurve is non-separating then we can view it as a vertex $x \in \mathcal{E}$. Since $\Lambda(c_0)$ is a non-separating n-multicurve containing c_0, the distance in $\mathcal{E}$ between $\Lambda(c_0)$ and x equals at most 2. Similarly, the distance in $\mathcal{E}$ between $\Lambda(c_1)$ and x is at most 2 and hence the distance in $\mathcal{E}$ between $\Lambda(c_0)$ and $\Lambda(c_1)$ is at most 4.

If $c_0 \cup b$ is not non-separating then $a \cup b$ is a bounding pair in $S - (c_0 - a) = S - (c_1 - b)$. Choose a non-separating simple closed curve $\omega \in S - (c_0 - a)$ which is disjoint from $a \cup b$ so that both $c_0 \cup \omega$ and $c_1 \cup \omega$ are non-separating. Such a curve exists since the genus $g - n + 2$ of $S - (c_0 \cup a)$ is at least three. Apply the argument from the previous paragraph to the non-separating $(n-1)$-multicurves $c_0, (c_0 - a) \cup \omega$ and to $(c_0 - a) \cup \omega, c_1$. We conclude that the distance in $\mathcal{E}$ between $\Lambda(c_0)$ and $\Lambda(c_1)$ is at most 8.

On the other hand, a map which associates to a vertex $x \in \mathcal{NC}(S, n) \subset \mathcal{E}$ an $(n-1)$-multicurve contained in x is a coarsely Lipschitz coarse inverse of Λ. Thus indeed Λ is a quasi-isometry. □

By Lemma 4.2, for each vertex $\nu \in \mathcal{NC}(S, n-1)$ the complete connected subgraph H_ν of $\mathcal{NC}(S, n)$ is quasi-isometric to the hyperbolic graph $\mathcal{NC}(S - \nu, 1)$. However, by the results in Section 3, the graph $\mathcal{NC}(S - \nu, 1)$ is not quasi-isometric to the curve graph of $S - \nu$. Thus controlling distances in these subgraphs via subsurface projection is not immediate. Moreover, for a subsurface X of S of genus $g - n + 1$ there are in general many different non-separating $(n-1)$-multicurves disjoint from X.

To resolve this problem we use exactly the strategy from Section 3. Namely, we introduce intermediate graphs $\mathcal{NC}(S, n, p)$ $(p \geq 1)$ which are defined as follows. Vertices of $\mathcal{NC}(S, n, p)$ are non-separating n-multicurves. Two such multicurves ν_0, ν_1 are connected by an edge of length one if $\hat{\nu} = \nu_0 \cap \nu_1$ is an $(n-1)$-multicurve and if the non-separating simple closed curves $a = \nu_0 - \hat{\nu}$ and $b = \nu_1 - \hat{\nu}$ are connected by an edge in the graph $\mathcal{A}(S - \hat{\nu}, p)$.

The strategy is now to deduce hyperbolicity of $\mathcal{NC}(S, n, 1)$ from hyperbolicity of $\mathcal{NC}(S, n-1)$, and for $p \geq 2$ to deduce hyperbolicity of $\mathcal{NC}(S, n, p)$ from hyperbolicity of $\mathcal{NC}(S, n, p-1)$.

For a non-separating $(n-1)$-multicurve $\nu \in \mathcal{NC}(S, n-1)$ let $H_\nu(1)$ be the complete subgraph of $\mathcal{NC}(S, n, 1)$ whose vertex set consists of all non-separating

n-multicurves containing ν. Define moreover

$$\mathcal{H}(1)=\{H_\nu(1)\mid \nu\}.$$

The following is immediate from the reasoning in Lemma 4.2 and Lemma 4.3.

Lemma 4.4. (1) *There is a natural graph isomorphism $H_\nu(1)\to\mathcal{A}(S-\nu,1)$.*
(2) *$\mathcal{NC}(S,n-1)$ is quasi-isometric to the $\mathcal{H}(1)$-electrification of $\mathcal{NC}(S,n,1)$.*

Our goal is to apply Theorem 2.3 to the family $\mathcal{H}(1)$ of subgraphs of $\mathcal{NC}(S,n,1)$ to deduce hyperbolicity of $\mathcal{NC}(S,n,1)$ from hyperbolicity of $\mathcal{NC}(S,n-1)$. To this end we have to check that the assumptions in the theorem are satisfied.

For $\nu\neq\zeta\in\mathcal{NC}(S,n-1)$, the vertex set of the intersection $H_\nu(1)\cap H_\zeta(1)$ is the set of all non-separating n-multicurves which contain both ν and ζ and hence it consists of at most one point. Thus $\mathcal{H}(1)$ is bounded.

By the first part of Lemma 4.4 and by Lemma 3.2, for every $\nu\in\mathcal{NC}(S,n-1)$ the graph $H_\nu(1)$ is δ-hyperbolic for a number $\delta>0$ which does not depend on ν.

The final step is the verification of the bounded penetration property, which is more involved.

Let $\mathcal{E}$ be the $\mathcal{H}(1)$-electrification of $\mathcal{NC}(S,n,1)$. Let $\beta:[0,k]\to\mathcal{E}$ be an efficient simplicial quasi-geodesic. If the integer $i<k$ is such that $\beta(i),\beta(i{+}1)\in\mathcal{NC}(S,n,1)$ then $\beta(i)$ and $\beta(i+1)$ are n-multicurves, and $\nu=\beta(i)\cap\beta(i+1)$ is an $(n-1)$-multicurve such that $\beta(i),\beta(i+1)\in H_\nu(1)$. If $\beta(i)=v_\nu$ is a special vertex defined by an $(n-1)$-multicurve ν then $\beta(i-1),\beta(i+1)\in H_\nu(1)$.

As in Section 2, an enlargement of β is a path $\hat\beta:[0,m]\to\mathcal{NC}(S,n,1)$ defined as follows. For each i such that $\beta(i)=v_\nu$ for some $\nu\in\mathcal{NC}(S,n-1)$, replace the arc $\beta[i-1,i+1]$ by a path of the form $j\to\nu\cup\zeta(j)$ where ζ is a simplicial geodesic in $\mathcal{A}(S-\nu,1)$ connecting $\beta(i-1)-\nu$ to $\beta(i+1)-\nu$.

By induction, we now assume that for every $L>1$ there is number $\kappa'(L)>0$ with the following property. Let $X\subset S$ be a connected subsurface of genus $g/2<h\leq g-n+1<g$. If $\alpha:[0,\ell]\to\mathcal{NC}(S,n-1)$ is an L-quasi-geodesic with the property that $\pi_X(\alpha(i))\neq\emptyset$ for all i then the diameter of $\pi_X(\cup_i\alpha(i))$ in the curve graph of X does not exceed $\kappa'(L)$. Note that the case $n=2$ holds true by the results in Section 3.

Lemma 4.3 then implies that for every $L>1$ there is a number $\kappa(L)>3L$ with the following property. Let $\beta:[0,k]\to\mathcal{E}$ be an efficient L-quasi-geodesic. If $X\subset S$ is a connected subsurface of genus $g/2<h\leq g-n+1$ so that the diameter of $\pi_X(\beta(0)\cup\beta(k))$ in the curve graph of X is at least $\kappa(L)$, then there is an $(n-1)$-multicurve $\nu\in\mathcal{NC}(S,n-1)$ disjoint from X, and there is some $i<k$ so that $\beta(i)\in H_\nu(1)\subset\mathcal{NC}(S,n,1)$.

As in Section 3, an enlargement $\hat\beta$ of β admits a *canonical modification* $\tilde\beta$ as follows. If $\beta(i)=\nu\cup a,\beta(i+1)=\nu\cup b$ are such that the simple closed curves a,b are not disjoint but contained in a thick subsurface X of $S-\nu$ of Euler characteristic $\chi(X)=\chi(S)+1=\chi(S-\nu)+1$, then replace the edge $\beta[i,i+1]$ in $\mathcal{NC}(S,n,1)$ by an edge-path $\zeta[j-1,j+1]$ of length two in the space of (not necessarily non-separating) n-multicurves so that $\zeta(j-1)=\beta(i)=\nu\cup a,\zeta(j+1)=\beta(i+1)=\nu\cup b$ and that $\zeta(j)=\nu\cup c$ for a (perhaps separating) simple closed curve $c\subset S-\nu$ which is disjoint from a,b and X.

The following can now be derived from the results in Section 2, Lemma 4.2 and Lemma 4.3.

Proposition 4.5. (1) *The graph $\mathcal{NC}(S,n,1)$ is hyperbolic.*

(2) *For every $L>1$ there are numbers $L'>1$, $\kappa(L)>0$ with the following property. Let $\hat\beta:[0,k]\to\mathcal{NC}(S,n,1)$ be an enlargement of an efficient L-quasi-geodesic in $\mathcal{E}$.*

(a) *$\hat\beta$ is an L'-quasi-geodesic in $\mathcal{NC}(S,n,1)$.*

(b) *Let $X\subset S$ be a connected subsurface of genus $h\in(g/2,g-n+1]$ and Euler characterisitic $\chi(X)=\chi(S)+1$. If $\mathrm{diam}(\pi_X(\beta(0)\cup\beta(k)))\geq\kappa(L)$ then a canonical modification of $\hat\beta$ passes through the complement of X.*

Proof. By the results in Section 2, by Lemma 4.2 and Lemma 4.3, to show hyperbolicity of $\mathcal{NC}(S,n,1)$ we only have to show the bounded penetration property for efficient quasi-geodesics in $\mathcal{E}$ and the family of subgraphs $\mathcal{H}(1)$.

To this end let $\beta:[0,k]\to\mathcal{E}$ be an efficient L-quasi-geodesic in $\mathcal{E}$ and let $\nu\in\mathcal{NC}(S,n-1)$. Assume that

$$\mathrm{diam}(\pi_{S-\nu}(\beta(0)\cup\beta(k)))\geq 3\kappa(L).$$

Then by induction hypothesis, β passes through $H_\nu(1)$.

The diameter of $H_\nu(1)$ in $\mathcal{E}$ equals two. Thus if $0\leq i\leq j\leq k$ are the first and last points, respectively, of the intersection of β with $H_\nu(1)$, then the length $j-i$ of $\beta[i,j]$ is at most $3L<\kappa(L)$.

An application of the induction hypothesis to $\beta[0,i]$ and to $\beta[j,k]$ shows that

$$\mathrm{diam}(\pi_X(\beta(0)\cup\beta(i)))\leq\kappa(L) \text{ and } \mathrm{diam}(\pi_X(\beta(j)\cup\beta(k)))\leq\kappa(L)$$

and therefore $\mathrm{diam}(\pi_X(\beta(i)\cup\beta(j)))\geq\kappa(L)>3L$.

Hence $\beta[i,j]$ passes through the special vertex v_ν. Moreover, the points $\beta(i),\beta(j)$ are of the form $\nu\cup a(i),\nu\cup a(j)$ for non-separating simple closed curves $a(i),a(j)\subset S-\nu$ such that $\mathrm{diam}(\pi_{S-\nu}(\beta(0)\cup a(i)))\leq\kappa(L)$ and $\mathrm{diam}(\pi_X(\beta(k)\cup a(j)))\leq\kappa(L)$. This immediately implies the bounded penetration property for the regions $H_\nu(1)$ and completes the proof of hyperbolicity of $\mathcal{NC}(S,n,1)$.

The other statements are also immediate from the induction hypothesis and Theorem 2.3. ☐

Next we show that hyperbolicity of $\mathcal{NC}(S,n,p-1)$ implies hyperbolicity of $\mathcal{NC}(S,n,p)$. To this end we proceed as in Section 3. The proofs are completely analogous to the proofs in Section 3.

Once more, our goal is to apply Theorem 2.3. For this denote for a connected subsurface X of S of genus $g-n+1$ and Euler characteristic $\chi(X)=\chi(S)+p-1$ by B_X the complete subgraph of $\mathcal{NC}(S,n,p-1)$ whose vertices consist of all non-separating n-multicurves ν which are disjoint from the boundary of X. By assumption on the genus of X, every non-separating n-multicurve ν intersects X and therefore a vertex $\nu\in B_X$ has at least one component which is contained in X.

In the next lemma, the assumption $n-1<g/2$ is used in an essential way.

Lemma 4.6. *There is a number $L_0>1$ not depending on X such that B_X is L_0-quasi-isometric to $\mathcal{A}(X,1)$.*

Proof. Let $\nu_0\subset S-X$ be a non-separating $(n-1)$-multicurve. Then for every simple closed non-separating curve $b\subset X$, the union $\nu_0\cup b$ is a vertex in $\mathcal{NC}(S,n,p)$.

Moreover, by Lemma 3.5, two such pairs $\nu_0 \cup a, \nu_0 \cup b$ are connected by an edge in $\mathcal{NC}(S,n,p)$ if and only if a,b are connected by an edge in $\mathcal{A}(X,1)$.

Now let $\zeta \in B_X$ be arbitrary. Then the number k of components of ζ contained in X is non-zero. Let $c_1,\dots,c_k$ be these components. Then $c_1 \cup \dots \cup c_k$ is non-separating k-multicurve contained in X.

Choose a non-separating n-multicurve $\zeta' \supset c_1 \cup \dots \cup c_k$ contained in X. Such a multicurve exists since $n < h$. By the definition of B_X, the distance in B_X between ζ and ζ' equals $n-k$. Moreover, the distance in B_X between ζ' and and a non-separating n-multicurve containing ν_0 equals $n-1$. But this just means that the subspace of B_X of non-separating n-multicurves containing ν_0 is coarsely dense. Consequently, associating to a vertex $\zeta \in B_X$ a component of ζ contained in X defines a Lipschitz map $B_X \to \mathcal{A}(X,1)$.

That this map coarsely does not decrease distances follows from the fact that any vertex $\zeta \in B_X$ contains at least one component which is contained in X, and adjacent vertices contain components which either are disjoint or contained in a common thick subsurface of X of Euler characteristic $\chi(X)+1$. □

Define a family of subgraphs

$$\mathcal{B}(p) = \{B_X \mid X\}$$

of $\mathcal{NC}(S,n,p)$ where X runs through the subsurfaces of S of genus $g-n+1$ and Euler characteristic $\chi(X) = \chi(S)+p-1$. By Lemma 4.6 and Lemma 3.2, each of the graphs B_X is quasi-isometric to the curve graph of X, in particular it is hyperbolic.

We claim that the family $\mathcal{B}(p)$ is bounded. To this end let X,Y be two subsurfaces of S of the same genus $g-n+1$ and the same Euler characteristic $\chi(S)+p-1$. Let $\nu \in B_X \cap B_Y$; then ν is a non-separating n-multicurve disjoint from the boundaries of both X,Y. Since the genus of X,Y equals $g-n+1$, at least one component of ν is contained in $X \cap Y$. However, since $X \neq Y$, $X \cap Y$ is a proper subsurface of X. By Lemma 4.6 and Lemma 3.2, B_X is quasi-isometric to the curve graph of X and hence the diameter of $B_X \cap B_Y$ is uniformly bounded.

Lemma 4.6 shows that each of the graphs B_X is δ-hyperbolic for a number $\delta > 0$ not depending on X. Moreover, by the definition of the graphs $\mathcal{NC}(S,n,p)$, the graph $\mathcal{NC}(S,n,p-1)$ is quasi-isometric to the $\mathcal{B}$-electrification of $\mathcal{NC}(S,n,p-1)$. Thus for an application of Theorem 2.3, we are left with showing the bounded penetration property.

However, this property follows by the induction assumption on subsurface projection and Lemma 4.6.

Example: The following example was observed by Tarik Aougab and Saul Schleimer and shows that the bound $n < g/2+1$ in Theorem 1 is sharp.

Namely, let S be a closed surface of genus 4 and let d be a separating simple closed curve which decomposes S into two surfaces X_1, X_2 of genus 2 with one boundary component. Let φ_i be a pseudo-Anosov element in the mapping class group of X_i and let a_i be a non-separating simple closed curve in X_i $(i=1,2)$. Let moreover c be a non-separating simple closed curve which is disjoint from a_i and intersects d in two points so that a_1, a_2, c defines a non-separating 3-multicurve ν.

For all $k, \ell \in \mathbb{Z}$ the pair $(\varphi_1^k, \varphi_2^\ell)$ defines a reducible mapping class for S, moreover it is easy to see that the distance in $\mathcal{NC}(S,3)$ between ν and $(\varphi_1^k, \varphi_2^\ell)\nu$ is comparable to $k+\ell$. The reason is that the subsurface projection of any non-separating 3-multicurve in S into each of the subsurfaces X_1, X_2 does not vanish. But this just means that $\mathcal{NC}(S,3)$ contains a quasi-isometrically embedded $\mathbb{R}^2$.

Define a subsurface Y of S to be *n-heavy* if the genus of Y is at least $g-n+1$. Let $\mathcal{L}(Y)$ be the set of all minimal geodesic laminations which fill up Y. Similarly to Corollary 3.10 we have

Corollary 4.7. *The Gromov boundary of $\mathcal{NC}(S,n)$ equals $\cup_Y \mathcal{L}(Y)$ equipped with the coarse Hausdorff topology where Y passes through the n-heavy subsurfaces of S.*

Remark: The main result in this note can also be obtained with the tools developed in [MS13]. To the best of our knowledge, these tools do not have any obvious advantage over the tools we used.

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