# PERIODIC ORBITS IN THE THIN PART OF STRATA 

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#### Abstract

Let $S$ be a closed oriented surface of genus $g \geq 0$ with $n \geq 0$ punctures and $3 g-3+n \geq 5$. Let $\mathcal{Q}$ be a connected component of a stratum in the moduli space $\mathcal{Q}(S)$ of area one meromorphic quadratic differentials on $S$ with $n$ simple poles at the punctures or in the moduli space $\mathcal{H}(S)$ of abelian differentials on $S$ if $n=0$. For a compact subset $K$ of $\mathcal{Q}(S)$ or of $\mathcal{H}(S)$, we show that the asymptotic growth rate of the number of periodic orbits for the Teichmüller flow $\Phi^{t}$ on $\mathcal{Q}$ which are entirely contained in $\mathcal{Q}-K$ is at least $h(\mathcal{Q})-1$ where $h(\mathcal{Q})>0$ is the complex dimension of $\mathbb{R}^{+} \mathcal{Q}$.


## 1. Introduction

For a closed oriented surface $S$ of genus $g \geq 0$, the moduli space $\mathcal{Q}(S)$ of area one meromorphic quadratic differentials with at most simple poles which are not squares of holomorphic one-forms decomposes into strata. Such a stratum is the subset of $\mathcal{Q}(S)$ of all quadratic differentials with the same number $n \geq 0$ of simple poles and the same number $\ell \geq 0$ of zeros of the same order $m_{i}(1 \leq i \leq \ell)$. Strata need not be connected, but they have only finitely many components [L08]. A connected component $\mathcal{Q}$ of a stratum is a real hypersurface in a complex algebraic orbifold of complex dimension

$$
h(\mathcal{Q})=2 g-2+\ell+n
$$

Similarly, for $g \geq 2$ the moduli space $\mathcal{H}(S)$ of area one abelian differentials on $S$ decomposes into strata. A stratum is the subset of $\mathcal{H}(S)$ of holomorphic one-forms with the same number $s \geq 0$ of zeros of the same order $k_{i}(1 \leq i \leq s)$. Again, strata need not be connected, but they have at most 3 components [KZ03]. A component $\mathcal{Q}$ of a stratum is a real hypersurface in a complex algebraic orbifold of complex dimension

$$
h(\mathcal{Q})=2 g-1+s
$$

The Teichmüller flow $\Phi^{t}$ acts on $\mathcal{Q}(S)$ and $\mathcal{H}(S)$, and this action preserves the strata. Each component of a stratum contains periodic orbits, and these orbits can be counted: Namely, for a subset $A$ of $\mathcal{Q}(S)$ or of $\mathcal{H}(S)$ and a number $R>0$, denote by $n_{A}(R)$ the number of period orbits in $A$ of length at most $R$. Then for any component $\mathcal{Q}$ of a stratum, we have [H13]

$$
n_{\mathcal{Q}}(R) \sim \frac{1}{h(\mathcal{Q}) R} e^{h(\mathcal{Q}) R} \quad(R \rightarrow \infty)
$$

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which means that the ratio of the numbers on both sides of $\sim$ tend to 1 as $R \rightarrow \infty$.
The mechanism behind this result is that the Teichmüller flow on components of strata is non-uniformly hyperbolic in a precise sense [H22]. However, components of strata are non-compact, and a major difficulty is the possibility that as $R \rightarrow \infty$, the number of periodic orbits of length at most $R$ which do not intersect some fixed compact set $K$ grows faster than the number of periodic orbits of length at most $R$ which intersect $K$. That this is not the case was made established by Eskin and Mirzakhani [EM11] and Eskin, Mirzakhani and Rafi [EMR19] who showed that for every $\epsilon>0$, there is a compact subset $K$ of $\mathcal{Q}$ with the property that the growth rate of the number of periodic orbits in $\mathcal{Q}$ which are entirely contained in $\mathcal{Q}-K$ is at most $h(\mathcal{Q})-1+\epsilon$.

The main goal of this article is to establish a converse of this result. We are interested in periodic orbits in components of strata which project into the thin part of moduli space, which ignores the possibility of periodic orbits in components of strata which are arbitrarily close to a component in the boundary of the stratum, obtained by colliding zeros or merging zeros and poles of the differentials.

Theorem 1. Let $\mathcal{Q}$ be a component of a stratum of area one meromorphic quadratic differentials with $n$ poles on a closed surface of genus $g \geq 0$ where $3 g-3+n \geq 5$ or of a stratum of area one abelian differentials on a surface of genus $g \geq 3$. Then for every compact set $K \subset \mathcal{Q}(S)$ we have

$$
\lim \inf _{R \rightarrow \infty} R e^{-(h(\mathcal{Q})-1) R} n_{\mathcal{Q}-K}(R)>0
$$

Note that the Teichmüller flow $\Phi^{t}$ on the space $\mathcal{Q}(1 ;-1)$ of area one meromorphic quadratic differentials with a single simple pole on a torus $T^{2}$ can be identified with the geodesic flow on the unit tangent bundle of the modular surface $S L(2, \mathbb{Z}) \backslash \mathbb{H}^{2}$. Thus in this case, there is a compact set $K$ which is intersected by every periodic orbit for $\Phi^{t}$. This shows that a constraint on the complexity of the stratum is necessary.

In analogy to finite volume locally symmetric manfolds of $\mathbb{Q}$-rank at least 2 , Theorem 1 can be viewed as a witness of higher rank for components of strata, with a small number of exceptions in low dimension. Indirectly, it draws on the fact that as is the case for locally symmetric manifolds of $\mathbb{Q}$-rank at least 2 , if $3 g-3+n \geq 5$ then the component $\mathcal{Q}$ (or its projectivization) admits a (partial) compactification which is built from components of strata of smaller complexity [MW17, BCGGM19].

Theorem 1 is not optimal. In forthcoming work which builds on the results in this article, we use a flexible symbolic coding of the Teichmüller flow to show that for every component $\mathcal{Q}$ of a stratum as in Theorem 1 and every compact set $K \subset \mathcal{Q}(S)$, the asymptotic growth rate of periodic orbits for $\Phi^{t}$ which are contained in $\mathcal{Q}-K$ is strictly larger than $h(\mathcal{Q})-1$. Periodic orbits entirely contained in a fixed compact set $K \subset \mathcal{Q}(S)$ are counted in [H10].

The main technical tool for the proof of Theorem 1 is the construction of combinatorial models for components $\mathcal{Q}$ of strata of area one abelian or quadratic
differentials. These models are adapted to the study of the dynamics of the Teichmüller flow and do not use cylinder decompositions. They are used to investigate pseudo-Anosov mapping classes which stabilize a component $\tilde{\mathcal{Q}}$ of the preimage of $\mathcal{Q}$ in the Teichmüller space of area one marked abelian or quadratic differentials. Unit cotangent lines of axes of these mapping classes acting on Teichmüller space project to periodic orbits in $\mathcal{Q}$, and each periodic orbit can be obtained in this way.

We use this construction to lift periodic orbits of the Teichmüller flow from the principal boundary of $\mathcal{Q}$ in the sense of [EMZ03] into $\mathcal{Q}-K$. The resulting periodic orbits fellow travel the orbits in the principal boundary used for their construction in a controlled way except for a subsegment whose length only depends on the compact set $K$.

The tools developed in this article can also be applied to construct orbits in a given stratum with an arbitrarily prescribed recursion behavior to compact subsets of moduli space. An example for this is given in the following statement. For its formulation, for a point $X$ in the moduli space $\mathcal{M}(S)$ of hyperbolic metrics on the surface $S$ denote by $\operatorname{syst}(X)$ be the systole of $X$, that is, the minimal length of a closed geodesic for the hyperbolic metric $X$.

Theorem 2. Let $\mathcal{Q}$ be a component of a stratum of area one meromorphic (or abelian) differentials on a surface of genus $g \geq 0$ with $n \geq 0$ simple poles. If $3 g-3+n \geq 5$ then there is a Teichmüller geodesic ray $\gamma:[0, \infty) \rightarrow \mathcal{M}(S)$ defined by a differential with uniquely ergodic vertical measured geodesic lamination and such that

$$
\lim \sup _{t \rightarrow \infty} \frac{1}{t} \log \operatorname{syst}(\gamma(t))<0
$$

The organization of the article is as follows. In Section 2 we collect some results from the literature in the form needed later on. In Section 3, we construct combinatorial models for components of strata. In Section 4 we use the classification of components of strata by Kontsevich and Zorich [KZ03] (for strata of abelian differentials) and Lanneau [L08] (for strata of quadratic differentials which are not squares of holomophic one-forms) to find for each component of a stratum with $3 g-3+n \geq 5$ such combinatorial models which encode the degeneration of differentials into suitably chosen components of the principal boundary. Section 5 translates information on dynamical properties of the Teichmüller flow on strata into the combinatorial setup. This is then used in Section 6 to prove Theorem 1.

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## 2. Train tracks and geodesic laminations

In this section we introduce some technical tools needed in the sequel. We begin with summarizing some constructions from [PH92, H09] which will be used throughout the paper. We then introduce a class of train tracks which will serve as
combinatorial models for components of strata in the later sections, and we discuss some of their properties.
2.1. Geodesic laminations. Let $S$ be an oriented surface of genus $g \geq 0$ with $n \geq 0$ marked points (punctures) and where $3 g-3+n \geq 2$. A geodesic lamination for a complete hyperbolic structure on $S$ of finite volume is a compact subset of $S$ which is foliated into simple geodesics. A geodesic lamination $\lambda$ is called minimal if each of its half-leaves is dense in $\lambda$. Thus a simple closed geodesic is a minimal geodesic lamination. A minimal geodesic lamination with more than one leaf has uncountably many leaves and is called minimal arational. Every geodesic lamination $\lambda$ consists of a disjoint union of finitely many minimal components and a finite number of isolated leaves. Each of the isolated leaves of $\lambda$ either is an isolated closed geodesic and hence a minimal component, or it spirals about one or two minimal components [CEG87].

A geodesic lamination $\lambda$ on $S$ is said to fill up $S$ if its complementary regions are all topological disks or once punctured monogons or once punctured bigons. Here a once puncture monogon is a once punctured disk with a single cusp at the boundary. A maximal geodesic lamination is a geodesic lamination whose complementary regions are all ideal triangles or once punctured monogons.

Definition 2.1. A geodesic lamination $\lambda$ is called large if $\lambda$ fills up $S$ and if moreover $\lambda$ can be approximated in the Hausdorff topology by simple closed geodesics.

Since every minimal geodesic lamination can be approximated in the Hausdorff topology by simple closed geodesics [CEG87], a minimal geodesic lamination which fills up $S$ is large. However, there are large geodesic laminations with finitely many leaves. We refer to [H09] for more detailed information.

The topological type of a large geodesic lamination $\nu$ is a tuple
$\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ where $1 \leq m_{1} \leq \cdots \leq m_{\ell}, \sum_{i} m_{i}=4 g-4+p_{1}, p_{1}+p_{2}=n$.
Here $\ell \geq 1$ is the number of complementary regions which are topological disks, and these disks are $m_{i}+2$-gons $(i \leq \ell)$. There are $p_{1}$ once punctured monogons and $p_{2}$ once punctured bigons. Let

$$
\mathcal{L L}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)
$$

be the space of all large geodesic laminations of type $\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ equipped with the restriction of the Hausdorff topology for compact subsets of $S$.

A measured geodesic lamination is a geodesic lamination $\lambda$ together with a translation invariant transverse measure. Such a measure assigns a positive weight to each compact arc in $S$ with endpoints in the complementary regions of $\lambda$ which intersects $\lambda$ nontrivially and transversely. The geodesic lamination $\lambda$ is called the support of the measured geodesic lamination; it consists of a disjoint union of minimal components. The space $\mathcal{M L}$ of all measured geodesic laminations on $S$ equipped with the weak*-topology is homeomorphic to $S^{6 g-7+2 n} \times(0, \infty)$. Its projectivization is the space $\mathcal{P} \mathcal{M} \mathcal{L}$ of all projective measured geodesic laminations.

The measured geodesic lamination $\mu \in \mathcal{M} \mathcal{L}$ fills up $S$ if its support fills up $S$. This support is then necessarily connected and hence minimal. Since a minimal geodesic lamination can be approximated in the Hausdorff topology by simple closed curves [CEG87], there exists a tuple $\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ such that the support of $\mu$ defines a point in the set $\mathcal{L} \mathcal{L}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$. The projectivization of a measured geodesic lamination which fills up $S$ is also said to fill up $S$.

There is a continuous symmetric pairing $\iota: \mathcal{M} \mathcal{L} \times \mathcal{M L} \rightarrow[0, \infty)$, the so-called intersection form, which extends the geometric intersection number between simple closed curves.
2.2. Train tracks. A train track on $S$ is an embedded 1-complex $\tau \subset S$ whose edges (called branches) are smooth arcs with well-defined tangent vectors at the endpoints. At any vertex (called a switch) the incident edges are mutually tangent. Through each switch there is a path of class $C^{1}$ which is embedded in $\tau$ and contains the switch in its interior. A simple closed curve component of $\tau$ contains a unique bivalent switch, and all other switches are at least trivalent. The complementary regions of the train track have negative Euler characteristic, which means that they are different from disks with 0,1 or 2 cusps at the boundary and different from annuli and once-punctured disks with no cusps at the boundary. We always identify train tracks which are isotopic. Throughout we use the book [PH92] as the main reference for train tracks. All train tracks will be marked, that is, we think of a train track $\tau$ as a (coarsely well defined) point in the marking graph of the subsurface of $S$ filled by $\tau$. This subsurface is a small neighborhood of the union of $\tau$ with all complementary components of $\tau$ which are topological disks or once punctured topological disks.

A train track is called generic if all switches are at most trivalent. For each switch $v$ of a generic train track $\tau$ which is not contained in a simple closed curve component, there is a unique half-branch $b$ of $\tau$ which is incident on $v$ and which is large at $v$. This means that every germ of an arc of class $C^{1}$ on $\tau$ which passes through $v$ also passes through the interior of $b$. A half-branch which is not large is called small. A branch $b$ of $\tau$ is called large (or small) if each of its two halfbranches is large (or small). A branch which is neither large nor small is called mixed.

Remark 2.2. As in [H09], all train tracks are assumed to be generic. Unfortunately this leads to a small inconsistency of our terminology with the terminology found in the literature.

A trainpath on a train track $\tau$ is a $C^{1}$-immersion $\rho:[k, \ell] \rightarrow \tau$ such that for every $i<\ell-k$ the restriction of $\rho$ to $[k+i, k+i+1]$ is a homeomorphism onto a branch of $\tau$. More generally, we call a $C^{1}$-immersion $\rho:[a, b] \rightarrow \tau$ a generalized trainpath. A trainpath $\rho:[k, \ell] \rightarrow \tau$ is closed if $\rho(k)=\rho(\ell)$ and if either the image of $\rho$ is a closed curve component of $\tau$ or if precisely one of the half-branches $\rho[k, k+1 / 2], \rho[\ell-1 / 2, \ell]$ is large.

A generic train track $\tau$ is orientable if there is a consistent orientation of the branches of $\tau$ such that at any switch $s$ of $\tau$, the orientation of the large half-branch
incident on $s$ extends to the orientation of the two small half-branches incident on $s$. If $C$ is a complementary polygon of an oriented train track then the number of sides of $C$ is even. In particular, a train track which contains a once punctured monogon component is not orientable (see p. 31 of [PH92] for a more detailed discussion).

A train track or a geodesic lamination $\eta$ is carried by a train track $\tau$ if there is a map $F: S \rightarrow S$ of class $C^{1}$ which is homotopic to the identity and maps $\eta$ into $\tau$ in such a way that the restriction of the differential of $F$ to the tangent space of $\eta$ vanishes nowhere; note that this makes sense since a train track has a tangent line everywhere. We call the restriction of $F$ to $\eta$ a carrying map for $\eta$. Write $\eta \prec \tau$ if the train track $\eta$ is carried by the train track $\tau$. Then every geodesic lamination $\nu$ which is carried by $\eta$ is also carried by $\tau$.

A train track fills up $S$ if its complementary components are topological disks or once punctured monogons or once punctured bigons. Note that such a train track $\tau$ is connected. Let $\ell \geq 1$ be the number of those complementary components of $\tau$ which are topological disks. Each of these disks is an $m_{i}+2$-gon for some $m_{i} \geq 1(i=1, \ldots, \ell)$. The topological type of $\tau$ is defined to be the ordered tuple $\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ where $1 \leq m_{1} \leq \cdots \leq m_{\ell}$ and $p_{1}$ (or $p_{2}$ ) is the number of once punctured monogons (or once punctured bigons); then $\sum_{i} m_{i}=4 g-4+p_{1}$ and $p_{1}+p_{2}=n$. If $\tau$ is orientable then $p_{1}=0$ and $m_{i}$ is even for all $i$. A train track of topological type $\left(1, \ldots, 1 ;-p_{1}, 0\right)$ is called maximal. The complementary components of a maximal train track are all trigons, that is, topological disks with three cusps at the boundary, or once punctured monogons.

A transverse measure on a generic train track $\tau$ is a nonnegative weight function $\mu$ on the branches of $\tau$ satisfying the switch condition: for every trivalent switch $s$ of $\tau$, the sum of the weights of the two small half-branches incident on $s$ equals the weight of the large half-branch. Particular such transverse measures are the counting measures of simple multicurves $c$ carried by $\tau$. Such a measure associates to a branch $b$ the number of the preimages of an interior point of $b$ under the carrying map. The weight of every branch with respect to this measure is integral. In particular, the ratio of weights of any two branches is rational, and we call a transverse measure with this property rational. The set of rational measures is invariant under scaling, and it is dense is the cone of all transverse measures on $\tau$.

A subtrack $\sigma$ of a train track $\tau$ is a subset of $\tau$ which is itself a train track. Then $\sigma$ is obtained from $\tau$ by removing some of the branches, and we write $\sigma<\tau$. A vertex cycle for $\tau$ is defined to be an embedded subtrack of $\tau$ which either is a simple closed curve or a dumbbell, that is, it consists of two loops with one cusp which are connected by an embedded segment joining the cusps (that this definition is equivalent to the definition defined in other works can for example be found in [Mo03], see also [H06]). An orientable train track does not contain dumbbells. Each vertex cycle supports a single transverse measure up to scale.

The following is well known and will be used several times in the sequel. We refer to [Mo03] for a comprehensive discussion.

Lemma 2.3. Let $\mathcal{V}(\tau)$ be the space of all transverse measures on $\tau$.
(1) $\mathcal{V}(\tau)$ has the structure of a cone over a compact convex polyhedron in a finite dimensional vector space.
(2) The vertices of the polyhedron are up to scaling the measures supported on the vertex cycles.
(3) There exists a natural homeomorphism of $\mathcal{V}(\tau)$, equipped with the euclidean topology, onto the closed subspace of $\mathcal{M} \mathcal{L}$ of all measured geodesic laminations carried by $\tau$.

The train track is called recurrent if it admits a transverse measure which is positive on every branch. We call such a transverse measure $\mu$ positive, and we write $\mu>0$ (see [PH92] for more details).

If $b$ is a small branch of $\tau$ which is incident on two distinct switches of $\tau$ then the graph $\sigma$ obtained from $\tau$ by removing $b$ is a subtrack of $\tau$. We then call $\tau$ a simple extension of $\sigma$. Note that formally to obtain the subtrack $\sigma$ from $\tau-b$ we may have to delete the switches on which the branch $b$ is incident.

Lemma 2.4. (1) A simple extension $\tau$ of a recurrent non-orientable connected train track $\sigma$ is recurrent. Moreover,

$$
\operatorname{dim} \mathcal{V}(\tau)=\operatorname{dim} \mathcal{V}(\sigma)+1
$$

(2) An orientable simple extension $\tau$ of a recurrent orientable connected train track $\sigma$ is recurrent. Moreover,

$$
\operatorname{dim} \mathcal{V}(\tau)=\operatorname{dim} \mathcal{V}(\sigma)+1
$$

Proof. If $\tau$ is a simple extension of a connected train track $\sigma$ then $\sigma$ can be obtained from $\tau$ by the removal of a small branch $b$ which is incident on two distinct switches $s_{1}, s_{2}$. Then $s_{i}$ is an interior point of a branch $b_{i}$ of $\sigma(i=1,2)$.

If $\sigma$ is moreover non-orientable and recurrent then there is a trainpath $\rho_{0}$ : $[0, t] \rightarrow \tau-b$ which begins at $s_{1}$, ends at $s_{2}$ and such that the half-branch $\rho_{0}[0,1 / 2]$ is small at $s_{1}=\rho_{0}(0)$ and that the half-branch $\rho_{0}[t-1 / 2, t]$ is small at $s_{2}=\rho_{0}(t)$. Extend $\rho_{0}$ to a closed trainpath $\rho$ on $\tau-b$ which begins and ends at $s_{1}$. This is possible since $\sigma$ is non-orientable, connected and recurrent. There is a closed trainpath $\rho^{\prime}:[0, u] \rightarrow \tau$ which can be obtained from $\rho$ by replacing the trainpath $\rho_{0}$ by the branch $b$ traveled through from $s_{1}$ to $s_{2}$. The counting measure of $\rho^{\prime}$ on $\tau$ satisfies the switch condition and hence it defines a transverse measure on $\tau$ which is positive on $b$. On the other hand, every transverse measure on $\sigma$ defines a transverse measure on $\tau$. Thus since $\sigma$ is recurrent and since the sum of two transverse measures on $\tau$ is again a transverse measure, the train track $\tau$ is recurrent as well. Moreover, we have $\operatorname{dim} \mathcal{V}(\tau) \geq \operatorname{dim} \mathcal{V}(\sigma)+1$.

Let $k$ be the number of branches of $\tau$. Label the branches of $\tau$ with the numbers $\{1, \ldots, k\}$ so that the number $k$ is assigned to $b$. Let $e_{1}, \ldots, e_{k}$ be the standard basis of $\mathbb{R}^{k}$ and define a linear map $A: \mathbb{R}^{k} \rightarrow \mathbb{R}^{k}$ by $A\left(e_{i}\right)=e_{i}$ for $i \leq k-1$ and $A\left(e_{k}\right)=\sum_{i} \nu(i) e_{i}$ where $\nu$ is the weight function on $\{1, \ldots, k-1\}$ defined by the trainpath $\rho_{0}$. The map $A$ is a surjection onto a linear subspace of $\mathbb{R}^{k}$ of codimension one, moreover $A$ preserves the linear subspace $V$ of $\mathbb{R}^{k}$ defined by the switch conditions for $\tau$. In particular, the corank of $A(V)$ in $V$ is at most one.

But $A(V)$ is contained in the space of solutions of the switch conditions on $\sigma$ and consequently its corank in $V$ is at least one. Thus equality holds.

To summarize, we obtain that indeed, $\operatorname{dim} \mathcal{V}(\tau)=\operatorname{dim} \mathcal{V}(\sigma)+1$ which completes the proof of the first part of the lemma. The second part follows in exactly the same way, and its proof will be omitted.

As a consequence we obtain
Corollary 2.5. (1) $\operatorname{dim} \mathcal{V}(\tau)=2 g-2+\ell+p_{1}+p_{2}$ for every non-orientable recurrent train track $\tau$ of topological type ( $m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}$ ).
(2) $\operatorname{dim} \mathcal{V}(\tau)=2 g-1+\ell+p_{2}$ for every orientable recurrent train track $\tau$ of topological type $\left(m_{1}, \ldots, m_{\ell} ; 0, p_{2}\right)$.

Proof. The complementary components of a non-orientable recurrent train track $\tau$ of topological type $\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ can be subdivided in $4 g-4+p_{1}-\ell$ steps into trigons by successively adding small branches. The once punctured bigon components can be subdivided into a trigon and a once punctured monogon. A repeated application of the first part of Lemma 2.4 shows that the resulting train track $\eta$ is maximal and recurrent. Since for every maximal recurrent train track $\eta$ on a surface with $n=p_{1}+p_{2}$ punctures we have $\operatorname{dim} \mathcal{V}(\eta)=6 g-6+2 n$ (see [PH92]), the first part of the corollary follows from the formula in the first part of Lemma 2.4.

To show the second part of the corollary, let $\tau$ be an orientable recurrent train track of type $\left(m_{1}, \ldots, m_{\ell} ; 0, p_{2}\right)$. Then $m_{i}$ is even for all $i$. Add a branch $b_{0}$ to $\tau$ which cuts some complementary component of $\tau$ into a trigon and a second polygon with an odd number of sides. The resulting train track $\eta_{0}$ is not recurrent since a trainpath on $\eta_{0}$ can pass through $b_{0}$ at most once. However, we can add to $\eta_{0}$ another small branch $b_{1}$ which cuts some complementary component of $\eta_{0}$ with at least 4 sides into a trigon and a second polygon such that the resulting train track $\eta$ is non-orientable and recurrent. The inward pointing tangent of $b_{1}$ is chosen in such a way that there is a trainpath traveling through both $b_{0}$ and $b_{1}$. The counting measure of any simple closed curve which is carried by $\eta$ gives equal weight to the branches $b_{0}$ and $b_{1}$. But this just means that $\operatorname{dim} \mathcal{V}(\eta)=\operatorname{dim} \mathcal{V}(\tau)+1$ (see the proof of Lemma 2.4 for a detailed argument). By the first part of the corollary, we have $\operatorname{dim} \mathcal{V}(\eta)=2 g-2+\ell+p_{2}+2$ and consequently $\operatorname{dim} \mathcal{V}(\tau)=2 g-1+\ell+p_{2}$ as claimed.

Definition 2.6. A train track $\tau$ of topological type $\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ which carries a minimal large geodesic lamination $\nu \in \mathcal{L} \mathcal{L}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ is called fully recurrent.

Remark 2.7. It is not hard to construct train tracks which are recurrent but not fully recurrent. Since this fact is not important for what follows we do not give explicit examples here.

If a train track $\eta$ is carried by a train track $\tau$, then the identity of $S$ induces a map from the set of complementary components of $\tau$ into the set of complementary components of $\eta$. Thus up to homotopy, the complementary components of $\tau$ are
obtained from the complementary components of $\eta$ by subdivision. In particular, the number of complementary components of $\tau$ is not smaller than the number of complementary components of $\eta$, and if $\nu \in \mathcal{L} \mathcal{L}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ is carried by a train track $\tau$ of topological type $\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$, then a carrying map $\nu \rightarrow \tau$ is surjective.

Note that by definition, a fully recurrent train track is connected and fills up $S$. Since a minimal geodesic lamination supports a transverse measure, a fully recurrent train track $\tau$ is recurrent.

There are two simple ways to modify a fully recurrent train track $\tau$ to another fully recurrent train track. Namely, if $b$ is a mixed branch of $\tau$ then we can shift $\tau$ along $b$ to a new train track $\tau^{\prime}$. This new train track carries $\tau$ and hence it is fully recurrent since it carries every geodesic lamination which is carried by $\tau$ [PH92, H09].

Similarly, if $e$ is a large branch of $\tau$ then we can perform a right or left split of $\tau$ at $e$ as shown in Figure A. The new small branch in the split track is called the

Figure A

diagonal of the split. A (right or left) split $\tau^{\prime}$ of a train track $\tau$ is carried by $\tau$. If $\tau$ is of topological type $\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$, if $\nu \in \mathcal{L} \mathcal{L}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ is carried by $\tau$ and if $e$ is a large branch of $\tau$, then there is a unique choice of a right or left split of $\tau$ at $e$ such that the split track $\eta$ carries $\nu$. In particular, $\eta$ is fully recurrent. Note however that there may be a split of $\tau$ at $e$ such that the split track is not fully recurrent any more (see Section 2 of [H09] for details).

To each train track $\tau$ which fills up $S$ one can associate a dual bigon track $\tau^{*}$ (Section 3.4 of [PH92]). There is a bijection between the complementary components of $\tau$ and those complementary components of $\tau^{*}$ which are not bigons, i.e. disks with two cusps at the boundary. This bijection maps a complementary component $C$ of $\tau$ which is an $n$-gon for some $n \geq 3$ to an $n$-gon component of $\tau^{*}$ contained in $C$, and it maps a once punctured monogon or bigon $C$ to a once punctured monogon or bigon contained in $C$. If $\tau$ is orientable then the orientation of $S$ and an orientation of $\tau$ induce an orientation on $\tau^{*}$, that is, $\tau^{*}$ is orientable.

There is a notion of carrying for bigon tracks which is analogous to the notion of carrying for train tracks. Measured geodesic laminations which are carried by the bigon track $\tau^{*}$ can be described as follows. A tangential measure on a train track $\tau$ of type $\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ assigns to a branch $b$ of $\tau$ a weight $\mu(b) \geq 0$ such that for every complementary $k$-gon of $\tau$ or once punctured bigon with consecutive sides
$c_{1}, \ldots, c_{k}$ and total mass $\mu\left(c_{i}\right)$ (counted with multiplicities) the following holds true.
(1) $\mu\left(c_{i}\right) \leq \mu\left(c_{i-1}\right)+\mu\left(c_{i+1}\right)$.
(2) $\sum_{i=j}^{k+j-1}(-1)^{i-j} \mu\left(c_{i}\right) \geq 0, j=1, \ldots, k$.

The complementary once punctured monogons define no constraint on tangential measures. Our definition of tangential measure on $\tau$ is stronger than the definition given on p. 22 of [PH92] and corresponds to the notion of a metric as defined on p. 184 of [P88]. We do not use this terminology here since we find it misleading.

The space of all tangential measures on $\tau$ has the structure of a convex cone in a finite dimensional real vector space. By Lemma 2.1 of [P88], every tangential measure on $\tau$ determines a simplex of measured geodesic laminations which hit $\tau$ efficiently. The dimension of this simplex equals the number of complementary components of $\tau$ with an even number of sides. The supports of these measured geodesic laminations are carried by the bigon track $\tau^{*}$, and every measured geodesic lamination which is carried by $\tau^{*}$ can be obtained in this way. The train track $\tau$ is called transversely recurrent if it admits a tangential measure which is positive on every branch.

In general, a measured geodesic lamination $\nu$ which hits $\tau$ efficiently does not determine uniquely a tangential measure on $\tau$ either. Namely, let $s$ be a switch of $\tau$ and let $a, b, c$ be the half-branches of $\tau$ incident on $s$ and such that the half-branch $a$ is large. If $\beta$ is a tangential measure on $\tau$ and if $\nu$ is a measured geodesic lamination in the simplex determined by $\beta$ then it may be possible to drag the switch $s$ across some of the leaves of $\nu$ and modify the tangential measure $\beta$ on $\tau$ to a tangential measure $\mu \neq \beta$. Then $\beta-\mu$ is a multiple of a vector of the form $\delta_{a}-\delta_{b}-\delta_{c}$ where $\delta_{w}$ denotes the function on the branches of $\tau$ defined by $\delta_{w}(w)=1$ and $\delta_{w}(a)=0$ for $a \neq w$.
Definition 2.8. Let $\tau$ be a train track of topological type $\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$.
(1) $\tau$ is called fully transversely recurrent if its dual bigon track $\tau^{*}$ carries a minimal large geodesic lamination $\nu \in \mathcal{L} \mathcal{L}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$.
(2) $\tau$ is called large if $\tau$ is fully recurrent and fully transversely recurrent.

For a large train track $\tau$ let $\mathcal{V}^{*}(\tau) \subset \mathcal{M} \mathcal{L}$ be the set of all measured geodesic laminations whose support is carried by $\tau^{*}$. Each of these measured geodesic laminations corresponds to a family of tangential measures on $\tau$. With this identification, the pairing

$$
\begin{equation*}
(\nu, \mu) \in \mathcal{V}(\tau) \times \mathcal{V}^{*}(\tau) \rightarrow \sum_{b} \nu(b) \mu(b) \tag{1}
\end{equation*}
$$

is just the restriction of the intersection form on measured lamination space (Section 3.4 of [PH92]). Moreover, $\mathcal{V}^{*}(\tau)$ is naturally homeomorphic to a convex cone in a real vector space. The dimension of this cone coincides with the dimension of $\mathcal{V}(\tau)$.

From now on we denote by $\mathcal{L} \mathcal{T}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ the set of all isotopy classes of large train tracks on $S$ of type $\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$.

## 3. Combinatorial models for components of strata

The goal of this section is to relate large train tracks to components of strata of abelian or quadratic differentials.

For a closed oriented surface $S_{g, n}$ of genus $g \geq 0$ with $n \geq 0$ marked points (punctures) let $\tilde{\mathcal{Q}}\left(S_{g, n}\right)$ be the bundle of marked area one holomorphic quadratic differentials with either a simple pole or a regular point at each of the marked points and no other pole over the Teichmüller space $\mathcal{T}\left(S_{g, n}\right)$ of marked complex structures on $S_{g, n}$.

Fix a complete hyperbolic metric on $S_{g, n}$ of finite area. A quadratic differential $q \in \tilde{\mathcal{Q}}\left(S_{g, n}\right)$ is determined by a pair $\left(\lambda^{+}, \lambda^{-}\right)$of measured geodesic laminations which bind $S$, which means that we have $\iota\left(\lambda^{+}, \mu\right)+\iota\left(\lambda^{-}, \mu\right)>0$ for every measured geodesic lamination $\mu$. The vertical measured geodesic lamination $\lambda^{+}$for $q$ corresponds to the equivalence class of the vertical measured foliation of $q$. The horizontal measured geodesic lamination $\lambda^{-}$for $q$ corresponds to the equivalence class of the horizontal measured foliation of $q$. These foliations are the pull-back of the foliation of $\mathbb{C}$ into straight lines parallel to the imaginary or real axis, respectively, by a system of charts on the complement of the singular points of $q$ for which $q$ takes the form $d z^{2}$ (or $d z$ if $z$ is a holomorphic one-form).

For $p_{1} \leq n, p_{2}=n-p_{1}$ and $\ell \geq 1$, an $\ell$-tuple $\left(m_{1}, \ldots, m_{\ell}\right)$ of positive integers $1 \leq$ $m_{1} \leq \cdots \leq m_{\ell}$ with $\sum_{i} m_{i}=4 g-4+p_{1}$ defines a stratum $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ in $\tilde{\mathcal{Q}}\left(S_{g, n}\right)$. This stratum consists of all marked quadratic differentials with $p_{1}$ simple poles, $p_{2}$ regular marked points and $\ell$ zeros of order $m_{1}, \ldots, m_{\ell}$. We require that these differentials are not squares of holomorphic one-forms. The stratum is a complex manifold of dimension

$$
\begin{equation*}
h=2 g-2+\ell+p_{1}+p_{2} . \tag{2}
\end{equation*}
$$

In general, such a stratum is not connected, but most strata have only finitely many connected components [CS21, H21]. These components are permuted by the mapping class group $\operatorname{Mod}\left(S_{g, n}\right)$ of $S_{g, n}$.

The closure in $\tilde{\mathcal{Q}}\left(S_{g, n}\right)$ of a stratum is a union of components of strata. As strata are invariant under the action of the mapping class group $\operatorname{Mod}\left(S_{g, n}\right)$ of $S_{g, n}$, they project to strata in the moduli space $\mathcal{Q}\left(S_{g, n}\right)=\tilde{\mathcal{Q}}\left(S_{g, n}\right) / \operatorname{Mod}\left(S_{g, n}\right)$ of quadratic differentials on $S_{g, m}$. Denote by $\mathcal{Q}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ the projection of the stratum $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$. The strata in moduli space need not be connected, but their connected components have been identified by Lanneau [L08]. A stratum in $\mathcal{Q}\left(S_{g, n}\right)$ has at most two connected components. The number of components of the stratum $\mathcal{Q}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ equals the number of components of $\mathcal{Q}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, 0\right)$.

Similarly, let $\tilde{\mathcal{H}}\left(S_{g, n}\right)$ be the bundle of marked holomorphic one-forms over Te ichmüller space $\mathcal{T}\left(S_{g, n}\right)$ of $S_{g, n}$. Each of the marked points of $S_{g, n}$ is required to be a regular marked point for the differential. In particular, the bundle is non-empty only if $g \geq 1$. For an $\ell$-tuple $k_{1} \leq \cdots \leq k_{\ell}$ of positive integers with $\sum_{i} k_{i}=2 g-2$, the stratum $\tilde{\mathcal{H}}\left(k_{1}, \ldots, k_{\ell} ; n\right)$ of marked holomorphic one-forms on $S$ with $\ell$ zeros
of order $k_{i}(i=1, \ldots, \ell)$ and $n$ regular marked points is a complex manifold of dimension

$$
\begin{equation*}
h=2 g-1+\ell+n \tag{3}
\end{equation*}
$$

It projects to a stratum $\mathcal{H}\left(k_{1}, \ldots, k_{\ell} ; n\right)$ in the moduli space $\mathcal{H}\left(S_{g, n}\right)$ of area one holomorphic one-forms on $S_{g, n}$. Strata of holomorphic one-forms in moduli space need not be connected, but the number of connected components of a stratum is at most three [KZ03].

We continue to use the assumptions and notations from Section 2. For a marked large train track $\tau \in \mathcal{L T}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ let

$$
\mathcal{Q}(\tau) \subset \tilde{\mathcal{Q}}\left(S_{g, n}\right)
$$

be the set of all marked quadratic differentials whose horizontal measured geodesic lamination is contained in $\mathcal{V}(\tau)$ via the identification of $\mathcal{V}(\tau)$ with a (not necessarily open) cone in $\mathcal{M} \mathcal{L}$ and whose vertical measured geodesic lamination is carried by the dual bigon track $\tau^{*}$ of $\tau$. Since $\tau$ and $\tau^{*}$ both carry a minimal large geodesic lamination, and such a lamination supports a transverse measure and fills $S=S_{g, n}$, for a large train track $\tau$ on $S=S_{g, n}$ the set $\mathcal{Q}(\tau)$ is not empty. Recall that no geodesic lamination can be carried by both $\tau$ and $\tau^{*}$.

Given two measure laminations ( $\mu, \nu$ ) which bind $S$, it is in general not easy to determine the stratum of the quadratic or abelian differential $z$ determined by $(\mu, \nu)$ due to possibility of horizontal or vertical saddle connections. Such a saddle connection is a geodesic segment for the singular euclidean metric defined by $z$ which connects two singular points of $z$ (here we exclude a regular marked point) and does not contain a singular point in its interior. The next lemma shows that train tracks can to used to this end.

Lemma 3.1. (1) Let $\tau \in \mathcal{L} \mathcal{T}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ be non-orientable and let $q \in \mathcal{Q}(\tau)$. If the support of the horizontal measured geodesic lamination of $q$ is contained in $\mathcal{L} \mathcal{L}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ then $q \in \tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$.
(2) Let $\tau \in \mathcal{L} \mathcal{T}\left(m_{1}, \ldots, m_{\ell} ; 0, p_{2}\right)$ be orientable and let $q \in \mathcal{Q}(\tau)$. If the support of the horizontal measured geodesic lamination of $q$ is contained in $\mathcal{L} \mathcal{L}\left(m_{1}, \ldots, m_{\ell} ; 0, p_{2}\right)$ then $q \in \tilde{\mathcal{H}}\left(m_{1} / 2, \ldots, m_{\ell} / 2 ; p_{2}\right)$.

Proof. A marked quadratic differential $z \in \tilde{\mathcal{Q}}\left(S_{g, n}\right)$ defines a singular euclidean metric on $S_{g, n}$. A singular point for $z$ is a zero or a pole or a marked regular point. A separatrix is a maximal geodesic segment or ray which begins at a singular point and does not contain a singular point in its interior.

The complex structure on $S_{g, n}$ underlying $z$ determines a complete finite area hyperbolic metric $h$ on $S_{g, n}$ with cusps at the $p_{1}$ marked points appearing in the definition. Let $\xi$ be the support of the horizontal measured geodesic lamination of the quadratic differential $z$, realized in the hyperbolic metric $h$. By [L83], the geodesic lamination $\xi$ can be obtained from the horizontal foliation of $z$ by cutting $S_{g, n}$ open along each horizontal separatrix and straightening the remaining leaves so that they become geodesics for $h$. In particular, up to homotopy, a horizontal saddle connection $s$ of $z$ is contained in the interior of a complementary component $C$ of $\xi$ which is uniquely determined by $s$.

Let $\tau \in \mathcal{L} \mathcal{T}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ be non-orientable. Let $q \in \mathcal{Q}(\tau)$ and assume that the $\operatorname{support} \operatorname{supp}(\mu)$ of the horizontal measured geodesic lamination $\mu \in \mathcal{V}(\tau)$ of $q$ is contained in $\mathcal{L} \mathcal{L}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$. Then $\operatorname{supp}(\mu)$ is non-orientable since otherwise $\tau$ inherits an orientation from $\operatorname{supp}(\mu)$ via a carrying map $\operatorname{supp}(\mu) \rightarrow \tau$. Since $\operatorname{supp}(\mu) \in \mathcal{L} \mathcal{L}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$, the orders of the zeros of the quadratic differential $q$ are obtained from the orders $m_{1}, \ldots, m_{\ell}$ by subdivision. Moreover, $q \in$ $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ if and only if this subdivision is trivial, which is equivalent to stating that $q$ does not have any horizontal saddle connection.

To show that this is indeed the case note first that the closure in $\tilde{\mathcal{Q}}\left(S_{g, n}\right)$ of the stratum $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ is the union of $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ with components of strata obtained by colliding some singular points. Thus it suffices to find a sequence $q_{j}$ of marked quadratic differentials which are contained in the closure of $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ and such that $q_{j} \rightarrow q$.

For the construction of such a sequence, let $\beta_{j} \in \mathcal{V}^{*}(\tau)$ be a sequence of rational points, that is, measured geodesic laminations supported on simple closed multicurves, so that $\beta_{j}$ converges as $j \rightarrow \infty$ to the vertical measured lamination of $q$. Such a sequence exists since rational points are dense in $\mathcal{V}^{*}(\tau)$. As $\mu$ is minimal and fills $S_{g, n}$, for all $j$ the pair ( $\mu, \beta_{j}$ ) binds $S_{g, n}$ (since the only measured laminations on $S_{g, n}$ whose intersection with $\mu$ vanish have the same support as $\mu$ ) and hence defines a quadratic differential $q_{j} \in \tilde{\mathcal{Q}}\left(S_{g, n}\right)$ with $q_{j} \rightarrow q$. Our goal is to show that $q_{j}$ is contained in the closure of $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ and hence in $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ by the choice of $\mu$.

Consider for the moment a quadratic differential $u \in \mathcal{Q}(\tau)$ with horizontal measured geodesic lamination $\mu$ which admits a horizontal saddle connection $\alpha$ connecting the zeros $x_{1}, x_{2}$. The weight deposited on $\alpha$ by the transverse measure of the vertical measured geodesic lamination of $u$ is positive. By Remark 2.7 and the discussion in the beginning of this proof, there exists a homotopy of $S_{g, n}$ which maps $\mu$ onto $\tau$ and which maps $x_{1}, x_{2}$ into a (uniquely determined) complementary component $C$ of $\tau$. The component $C$ has at least four sides, and if $D$ is the complementary component of $\mu$ which corresponds to $C$, then there exists a pair of non-adjacent sides $a, b$ of $D$ corresponding to a pair of non-adjacent sides of $C$ such that the transverse measure of the vertical measured geodesic lamination of $u$ gives positive mass to geodesic lines whose intersection components with $D$ have endpoints on the sides $a, b$.

We use this observation as follows. For each $j$, the simple closed multicurve $\beta_{j}$ can be homotoped to a collection of closed trainpaths on the dual bigon track $\tau^{*}$ of $\tau$. These paths intersect $\tau$ transversely in interior points of branches. If $C$ is any component of $S_{g, n}-\tau$, then any component of the intersection of $\beta_{j}$ with $C$ has its endpoints on consecutive sides of $C$ (see Section 3.4 of [PH92] for details on this fact).

The geodesic lamination $\operatorname{supp}(\mu)$ lifts to a geodesic lamination $\hat{\mu}$ in the hyperbolic plane $\mathbb{H}^{2}$ which is the universal covering of $S_{g, n}$, equipped with a complete finite volume hyperbolic metric. on $\mathbb{H}^{2}$. The lift of $D$ is a $\pi_{1}\left(S_{g, n}\right)$-invariant collection of ideal polygons in $\mathbb{H}^{2}$. Trainpaths on $\tau^{*}$ lift to uniform quasi-geodesic in
$\mathbb{H}^{2}$ which uniformly fellow-travel their geodesic representatives. Thus if $\hat{D} \subset \mathbb{H}^{2}$ is a component of the preimage of $D$, then the geodesic representatives of the lifts of the trainpaths corresponding to a component of the multi-curve $\beta_{j}$ do not contain any subarc crossing through $\hat{D}$, with endpoints on non-adjacent sides of $\hat{D}$. Together with the discussion in the previous paragraph, we conclude that $q_{j}$ does not have a horizontal saddle connection and hence it is contained in the closure of $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ which is what we wanted to show.

This yields the first part of the lemma, and the second part follows with precisely the same argument.

We use Lemma 3.1 to show
Proposition 3.2. (1) Let $\tau \in \mathcal{L} \mathcal{T}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ be a large non-orientable train track. Then there is a component $\tilde{\mathcal{Q}}$ of $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ such that $\mathcal{Q}(\tau)$ is the closure in $\tilde{\mathcal{Q}}\left(S_{g, n}\right)$ of an open path connected subset of $\tilde{\mathcal{Q}}$.
(2) For every large orientable train track $\tau \in \mathcal{L T}\left(m_{1}, \ldots, m_{\ell} ; 0, n\right)$ there is a component $\tilde{\mathcal{Q}}$ of $\tilde{\mathcal{H}}\left(m_{1} / 2, \ldots, m_{\ell} / 2, n\right)$ such that $\mathcal{Q}(\tau)$ is the closure in $\tilde{\mathcal{H}}\left(S_{g, n}\right)$ of an open path connected subset of $\tilde{\mathcal{Q}}$.

Proof. In the proof of the proposition, we do not distinguish between the orientable and the non-orientable case.

Let $\tau \in \mathcal{L} \mathcal{L}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ and let $\mu \in \mathcal{V}(\tau)$ be such that the support $\operatorname{supp}(\mu)$ of $\mu$ is contained in $\mathcal{L} \mathcal{L}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$. Notice that such a point is contained in the interior of $\mathcal{V}(\tau)$. If $\beta \in \mathcal{V}^{*}(\tau)$ is arbitrary then the measured geodesic laminations $\mu, \beta$ bind $S_{g, n}$ (since the support of $\beta$ is different from the support of $\mu$ and $\operatorname{supp}(\mu)$ fills up $\left.S_{g, n}\right)$. Hence if we put $\hat{\beta}=\beta / \iota(\mu$, beta) then the pair $(\mu, \mid h a t \beta)$ defines a point $q(\mu, \hat{\beta}) \in \mathcal{Q}(\tau)$. By Lemma 3.1, we have $q(\mu, \hat{\beta}) \in$ $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$.

Recall that $\mathcal{V}^{*}(\tau)$ is homeomorphic to a cone over a closed cell whose dimension equals half of the dimension of $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$. Let $V$ be the interior of this cell. By continuity and invariance of domain, we conclude that the set $\{q(\mu, \beta) \mid \beta \in V\}$ is an open subset of the strong stable manifold of in $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ defined by $\mu$. In period coordinates, such a strong stable manifold consists of quadratic differentials in $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ with the same real part.

As measured geodesic laminations which are minimal and of the same topological type as $\tau$ are dense in the set of all measured laminations in such a strong stable manifold (see [KMS86] for a comprehensive discussion of this fact), we conclude that measured geodesic laminations with this property are dense in $\mathcal{V}^{*}(\tau)$.

Choose a measured geodesic lamination $\nu \in \mathcal{V}^{*}(\tau)$ whose support $\operatorname{supp}(\nu)$ is contained in $\mathcal{L} \mathcal{L}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$. Using exactly the same reasoning as above, we deduce that for each $\alpha \in \mathcal{V}(\tau)$, the pair $(\hat{\alpha}, \nu)$ defines a quadratic differential $q(\hat{\alpha}, \nu) \in \tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ where $\hat{\alpha}=\alpha / \iota(\alpha, \nu)$. Furthermore, the measured
laminations whose supports are minimal and of the same topological type as $\tau$ are dense in $\mathcal{V}(\tau)$.

As a consequence, there exists a dense subset of $Q(\tau)$ which is contained in $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$. As $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ is locally closed in $\tilde{\mathcal{Q}}\left(S_{g, n}\right)$, the intersection $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right) \cap \mathcal{Q}(\tau)$ is open and dense in $\mathcal{Q}(\tau)$. Thus to complete the proof of the proposition, it suffices to show that the set of all pairs $(\alpha, \beta) \in$ $\mathcal{V}(\tau) \times \mathcal{V}^{*}(\tau)$ which gives rise to a differential $q(\alpha, \hat{\beta}) \in \tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ is path connected. Since area renormalization is continuous, to this end it suffices to construct paths of differentials whose areas may be different from one.

Thus let $(\alpha, \beta),\left(\alpha^{\prime}, \beta^{\prime}\right)$ be two pairs with $q(\alpha, \beta) \in \tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ where $q(\alpha, \beta)$ denotes the differential defined by $\alpha, \beta$, of area $\iota(\alpha, \beta)$. Now binding $S_{g, n}$ is an open condition for pairs of measured laminations, and $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right) \cap$ $\mathcal{Q}(\tau)$ is open in $\mathcal{Q}(\tau)$ by the above discussion. Since moreover the set of all measured laminations whose support is of type $\mathcal{L} \mathcal{L}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ is dense in $\mathcal{V}(\tau)$, there exist laminations $\mu, \mu^{\prime} \in \mathcal{V}(\tau)$ with the following properties.
(1) $\operatorname{supp}(\mu), \operatorname{supp}\left(\mu^{\prime}\right)$ are of type $\mathcal{L} \mathcal{L}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$.
(2) There is a path $c, c^{\prime}:[0,1] \rightarrow \mathcal{V}(\tau)$ connecting $\alpha$ to $\mu, \alpha^{\prime}$ to $\mu^{\prime}$ so that for every $t \in[0,1]$, the pair $(c(t), \beta)$ and $\left(c^{\prime}(t), \beta^{\prime}\right)$ binds $S_{g, n}$, and the paths $t \rightarrow q(c(t), \beta)$ and $t \rightarrow q\left(c^{\prime}(t), \beta^{\prime}\right)$ are contained in $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ up to area renormalization.

By the beginning of this proof, for every measured lamination $\xi \in \mathcal{V}^{*}(\tau)$, the pair $(\mu, \xi)$ determines a quadratic differential $q(\mu, \xi) \in \tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ (up to area renormalization), and the same holds true for the pair $\left(\mu^{\prime}, \xi\right)$. Choose a measured lamination $\nu \in \mathcal{V}^{*}(\tau)$ whose support is contained in $\mathcal{L} \mathcal{L}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$. Using the discussion in the previous paragraphs, the differential $q(\mu, \beta)$ can be connected to $q(\mu, \nu)$ by a path in $Q(\tau)$ which is contained in $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right) \cap$ $\mathcal{Q}(\tau)$, and there also is such a path connecting $q\left(\mu^{\prime}, \beta^{\prime}\right)$ and $q\left(\mu^{\prime}, \nu\right)$. Another application of this argument shows that the differential $q\left(\mu^{\prime}, \nu\right)$ can be connected to $q(\mu, \nu)$ by a path in $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right) \cap \mathcal{Q}(\tau)$.

Together we find that the differentials $q(\alpha, \beta)$ and $q\left(\alpha^{\prime}, \beta^{\prime}\right)$ can both be connected to $q(\mu, \nu)$ by a path in $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right) \cap Q(\tau)$ (up to area renormalization). As the pairs $(\alpha, \beta),\left(\alpha^{\prime}, \beta^{\prime}\right) \in \mathcal{V}(\tau) \times \mathcal{V}^{*}(\tau)$ were arbitrarily chosen with the property that they determine quadratic differentials in $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$, the proposition follows.

The next proposition is a converse to Proposition 3.2 and shows that train tracks can be used to define coordinates on components of strata.

Proposition 3.3. (1) For every $q \in \tilde{\mathcal{H}}\left(k_{1}, \ldots, k_{s} ; n\right)$ there is an orientable train track $\tau \in \mathcal{L} \mathcal{T}\left(2 k_{1}, \ldots, 2 k_{s} ; 0, n\right)$ so that $q$ is an interior point of $\mathcal{Q}(\tau)$.
(2) For every $q \in \tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ there is a non-orientable train track $\tau \in \mathcal{L T}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ so that $q$ is an interior point of $\mathcal{Q}(\tau)$.

Furthermore, if $q$ contains a horizontal cylinder then $\tau$ can be chosen in such a way that the core curve of this cylinder is embedded in $\tau$.

Proof. Let $q \in \tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ and let $\Sigma=\left\{u_{1}, \ldots, u_{s}\right\}\left(s=\ell+p_{1}+p_{2}\right)$ be the singular set of $q$, that is, the union of the zeros and poles and marked regular points.

Recall that $q$ defines a singular euclidean metric $h$ on $S_{g, n}$ as well as two measured foliations, the horizontal and the vertical measured foliation. If $x$ is a singular point of the metric $h$, then $x$ is a cone point of cone angle $k \pi$ for some positive integer $k \neq 2$. There are precisely $k$ horizontal and precisely $k$ vertical separatrices which begin at $x$. The zeros of the differential correspond to cone points with cone angle $k \pi$ for some $k \geq 3$.

Choose a number $\epsilon>0$ which is smaller than $1 / 8$-th of the distance in the metric $h$ between any two singular points. Let $u_{i} \in \Sigma$ be a singular point of cone angle $k \pi$ for some $k \geq 1$. There exists a neighborhood $V_{i}$ of $u_{i}$ with the following properties. The boundary $\partial V_{i}$ of $V_{i}$ is a polygon with $2 k$ sides. The sides are alternating between vertical arcs of fixed length $\sigma<\epsilon / 10$ and horizontal arcs. The midpoint of a vertical arc is a point of distance $\epsilon$ on a horizontal separatrix through $u_{i}$. Note that the polygon is uniquely determined by these requirements.

Out of the polygons $V_{i}(i \leq s)$ we construct a train track $\eta_{i}$ with stops whose switches are the midpoints of the vertical sides of the polygon $\partial V_{i}$. Thus each switch is a point of distance $\epsilon$ to the singular point $u_{i}$ on a horizontal separatrix $\zeta_{i}$.

Two different switches on separatrices $\zeta_{i}^{1}, \zeta_{i}^{2}$ starting at $x_{i}$ are connected by a branch in $\eta_{i}$ if the angle at $x_{i}$ between $\zeta_{i}^{1}, \zeta_{i}^{2}$ equals $\pi$, or, equivalently, if there is a path in $\partial V_{i}$ connecting $\zeta_{i}^{1}, \zeta_{i}^{2}$ which travels through precisely one horizontal side of $\partial V_{i}$. These branches are constructed in such a way that all the vertical sides of the polygons $\partial V_{i}$ are replaced by a cusp. Furthermore, we require that all branches are contained in $V_{i}$ and do not intersect $u_{i}$. Figure B shows this construction.


Figure B

The construction can be done in such a way that $\eta_{i}$ is transverse to the vertical measured foliation of $q$ - more precisely, by adjusting the constant $\sigma$ we can assume that the tangent of $\eta_{i}$ is arbitrarily close to the horizontal direction and that each branch of $\eta_{i}$ is an arc of arbitrarily small geodesic curvature for the euclidean metric $h$. There is a complementary component $C_{i}$ of $V_{i}-\eta_{i}$ which is a polygon with $2 k$ cusps. Its closure is contained in $V_{i}$ and meets $\partial V_{i}$ only at the cusps. It contains the singular point $u_{i}$. The cusps of the component are the vertices of $\eta_{i}$. The component $C_{i}$ is a once punctured monogon if $u_{i}$ is a pole, that is, if $k=1$, a once punctured bigon if $u_{i}$ is a regular marked point, or an $m_{i}+2$-gon if $u_{i}$ is a zero of order $m_{i}$.

Let $\hat{\eta}$ be the union of the train tracks with stops $\eta_{i}$; this union consists of $\ell+p_{1}+p_{2}$ connected components, and it has $\sum_{i}\left(m_{i}+2\right)+p_{1}+2 p_{2}$ vertices. The graph $\hat{\eta}$ is transverse to the vertical foliation of $q$. By construction, $\hat{\eta}$ also is transverse to the straight line foliation on $S$ defined by any direction for the singular euclidean metric on $S$ which is sufficiently close to the vertical direction.

A generalized bigon track is a graph with all properties of a train track except that we allow the existence of complementary bigons, and we allow complementary annuli. Out of the train track with stops $\hat{\eta}$ we construct a generalized bigon track $\eta$ on $S$ by inductively replacing a stop by a switch and adding additional branches as follows.

For $i \leq s$ let $\beta$ be a vertical side of the polygon $\partial V_{i}$. Then for small enough $d>0, \beta$ is a side of an euclidean (right angled) rectangle $R_{d}$ in $S$ of width $d$ which intersects the polygonal disk $V_{i}$ precisely in $\beta$. We require that the interior of $R_{d}$ is disjoint from the disks $V_{j}$. The area of $R_{d}$ equals $\sigma d$. Thus by consideration of area, there exists a smallest number $d_{0}>0$ such that the vertical side $\beta^{\prime}$ of $R_{d_{0}}$ distinct from $\beta$ intersects one of the polygonal disks $V_{j}$. Then $\beta^{\prime} \cap \partial V_{j}$ is a (possibly degenerate) subarc of a vertical side $\xi$ of $\partial V_{j}$.

There are two possibilities. In the first case, $\beta^{\prime}=\xi$. Then $V_{i} \cup R_{d_{0}} \cup V_{j}$ contains a horizontal saddle connection joining $u_{i}$ to $u_{j}$. Connect the cusp of the component $\eta_{i}$ of $\hat{\eta}$ contained in $V_{i}$ to the cusp of the component of $\hat{\eta}$ contained in $V_{j}$ by the subsegment of the saddle connection which is contained in $R_{d_{0}}$. This construction yields a new generalized bigon track $\hat{\eta}^{\prime}$ with stops, and the number of stops of $\hat{\eta}^{\prime}$ equals the number of stops of $\hat{\eta}$ minus two.

The second possibility is that $\beta^{\prime} \cap \xi$ is a proper subarc of $\xi$ (perhaps degenerate to a single point). Then precisely one of the endpoints of $\xi$ is contained in $\beta^{\prime}$; denote this point by $z$. The point $z$ is a vertex of the polygon $V_{j}$ and hence it is contained in a horizontal side of the polygon $V_{j}$ and determines a branch $b$ of the train track with stops $\hat{\eta}$. Connect the midpoint $y$ of the side $\beta$ of $R_{d_{0}}$ (which is a stop of $\eta_{i}$ ) to an interior point of the branch $b$ with an arc $\nu$ in such a way that the union $\eta_{i} \cup \nu \cup \eta_{j}$ is a generalized bigon track $\hat{\eta}^{\prime}$ with stops, and that there is an arc of class $C^{1}$ contained in $\hat{\eta}^{\prime}$ connecting the stop $y$ of $\eta_{i}$ to the endpoint of the branch $b$ distinct from $z$. In $\hat{\eta}^{\prime}$, the midpoint $y$ of the vertical side $\beta$ of $V_{i}$ (which was a stop in $\eta_{i}$ ) is a trivalent switch. The generalized bigon track $\hat{\eta}^{\prime}$ has fewer stops than $\hat{\eta}$.

Doing this construction with each of the stops of $\hat{\eta}$ replaces $\hat{\eta}$ by a generalized bigon track $\eta$. This can be done in such a way that each branch of $\eta$ is a smooth arc whose tangent line is everywhere close to the horizontal subbundle of the tangent bundle of $S_{g, n}-\Sigma$.

We show next that a complementary component of $\eta$ which does not contain a singular point of $q$ either is a bigon or an annulus with no singular point on the boundary. For this it suffices to show that the Euler characteristic of each complementary region which does not contain any marked point vanishes. This Euler characteristic is computed by giving each cusp in its boundary the value -1 (see [PH92] for this computation). Thus a disk with 3 cusps at the boundary has Euler characteristic -1 . The sum of the Euler characteristics of the complementary regions of $\eta$ containing a zero of $q$, a puncture or a marked regular point equals the Euler characteristic of $S_{g, n}$. By the Gauss Bonnet theorem, there are no complementary monogons of $\eta$ without marked point in the interior. Namely, up to adjusting the constant $\sigma$, the total geodesic curvature of any branch of $\eta$ can be chosen to be arbitrarily small, while the total number of branches is bounded from above by a constant independent of $\sigma$. Thus the Euler characteristic of every complementary component of $\eta$ is non-positive and hence the Euler characteristic of every complementary component not containing a singular point of $q$ or a marked regular point has to vanish. Hence each such component either is a bigon or an annulus. An annulus component corresponds to a horizontal cylinder of $q$.

To construct a train track out of $\eta$ we begin with collapsing successively the complementary bigons of $\eta$. Namely, the set of all directions for the flat metric defined by $q$ which are tangent to some saddle connection is countable and hence we can find arbitrarily near the vertical direction a direction which is not tangent to any saddle connection. By construction of $\eta$, we may assume that this direction is transverse to $\eta$. For simplicity of exposition we will call this direction vertical in the sequel. We use the singular foliation defined by this direction as follows.

Let $B$ be a complementary component of $\eta$ which is a bigon. The boundary $\partial B$ of this bigon consists of two $\operatorname{arcs} a_{1}, a_{2}$ which are nearly horizontal and which meet tangentially at their endpoints. The vertical foliation is transverse to these sides, and non-singular in the bigon.

Let $x \in a_{1}$; we claim that $x$ is the starting point of a vertical segment whose interior is contained in the interior of $B$ and whose endpoint $F(x)$ is contained in the second side $a_{2}$ of $\partial B$. Namely, by transversality and compactness, $x$ is the starting point of a vertical arc $\gamma$ whose interior is contained in the interior of $B$ and whose second endpoint $y$ is contained in a side of $\partial B$. If $y$ is contained in the same side $a_{1}$ of $\partial B$ as $\gamma(0)$ then $y$ bounds together with the subarc of $a_{1}$ connecting $x$ to $y$ an euclidean disk whose boundary consists of two smooth arcs with small curvature which meet at the endpoints with an angle close to $\pi / 2$. However, this violates the Gauss Bonnet theorem. Thus indeed, $B$ is foliated by vertical arcs with one endpoint on $a_{1}$ and the second endpoint on $a_{2}$.

Now observe that although the boundary of $B$ may not be embedded in $S_{g, n}$ (we only know that the interior of $B$ is embedded), the two endpoints of any vertical arc as above are distinct since there is no vertical saddle connection and hence
no vertical closed geodesic by assumption. This means that we can collapse these vertical arcs to points and collapse in this way the bigon $B$ to a single arc. Let $\hat{\theta}$ be the generalized bigon track obtained in this way.

There is a map $F_{0}: S_{g, n} \rightarrow S_{g, n}$ of class $C^{1}$ which is homotopic to the identity, which equals the identity in a small neighborhood of the bigon $B$ and which maps $\eta$ to $\hat{\theta}$ by collapsing the vertical arcs crossing through $B$. As the sides of $B$ are nearly horizontal, the differential of the restriction of the collapsing map $F_{0}$ to each horizontal arc vanishes nowhere.

Using once more the fact that vertical trajectories do not contain loops, we can repeat this process with any other bigon. In finitely many such steps we construct a generalized bigon track $\theta$ and a map $F: S_{g, n} \rightarrow S_{g, n}$ with the following properties.
(1) $\theta$ does not have any complementary bigon components.
(2) $F$ is homotopic to the identity and of class $C^{1}$.
(3) $F(\eta)=\theta$.
(4) The differential of the restriction of $F$ to the horizontal foliation of $q$ vanishes nowhere, and it maps the intersection of the horizontal foliation of $q$ with the bigon complementary components of $\eta$ to smoothly immersed arcs in $\theta$.

The generalized bigon track $\theta$ may not be a train track as it may have complementary components which are annuli. However, the above construction can also be used to collapse annuli to circles. To this end let $A$ be a complementary annulus of $\theta$. By construction of $\theta, A$ is contained in a horizontal cylinder $C$ for $q$, and its closure does not contain a singular point of $q$. Furthermore, its boundary curves are transverse to the vertical foliation.

Let $a_{1}, a_{2}$ be the two boundary curves of $A$. For a point $x \in a_{1}$, there is a unique subarc $v(x)$ of a vertical trajectory starting at $x$ which is entirely contained in $A$ and connects $x$ to a point $\psi(x)$ contained in the boundary of $A$. Using once more the Gauss Bonnet theorem, we conclude that in fact $\psi(x) \in a_{2}$. As there are no vertical cylinders and the closure of $A$ does not contain singular points, we have $\psi(x) \neq x$. Furthermore, the arc $v(x)$ depends smoothly on $x$.

By the discussion in the previous paragraph, for each $x \in a_{1}$ we can collapse the arc $v(x)$ to a point. The result is a new generalized bigon track which carries the horizontal measured geodesic lamination of $q$ and is such that the number of complementary components which are annuli is strictly smaller than the number of annuli components of $\theta$. Repeating this construction with all the finitely many annuli components of $S_{g, n}-\theta$, we construct in this way from $\theta$ a train track $\tau$. There is a map $F: S_{g, n} \rightarrow S_{g, n}$ of class $C^{1}$ with the following property.
(1) $F(\eta)=\tau$.
(2) $F$ equals the identity near the singular points of $q$.
(3) The restriction of the differential of $F$ to the tangent bundle of the horizontal foliation of $q$ vanishes nowhere.

As a consequence, the train track $\tau$ carries the horizontal measured foliation of $q$. Furthermore, the vertical measured geodesic lamination of $q$ hits $\tau$ efficiently (see [PH92]) and hence it is carried by $\tau^{*}$. Each complementary component of $\tau$ contains precisely one singular point of $q$, and the component is a $k+2$-gon if and only if the singular point is a zero of order $k$. This yields that $\tau$ is of topological type $\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$.

We are left with showing that $\tau$ is large. Now by construction, $\tau$ carries the horizontal geodesic lamination of $e^{i s} q$ provided that $s$ is sufficiently close to 0 . But the set of directions for the singular euclidean metric defined by $q$ so that the horizontal foliation in this direction is minimal and of the type predicted by the number and multiplicities of the zeros of $q$ is dense [KMS86]. This implies that $\tau$ carries a geodesic lamination which is minimal, large and of the same topological type as $q$. Similarly, for $s$ sufficiently close to zero, the vertical measured geodesic lamination of $e^{i s} q$ hits $\tau$ sufficiently. Thus as before, $\mathcal{V}^{*}(\tau)$ carries a minimal large geodesic lamination of the same topological type as $\tau$. In other words, $\tau$ has all the properties required in the proposition.

Now if $q$ is an abelian differential, then the horizontal and vertical foliations of $q$ are orientable. As the initial generalized bigon track $\hat{\eta}$ is constructed from the horizontal foliation of $q$, it inherits an orientation from the horizontal foliation of $q$. The collapsing construction uses the orientable vertical foliation, and it is straightforward that this construction respects orientations as well. Then the resulting train track $\tau$ is orientable.

We summarize the discussion in this section as follows.
Let $\mathcal{Q}$ be a component of the stratum $\mathcal{Q}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)$ of $\mathcal{Q}\left(S_{g, n}\right)$ (or of the stratum $\mathcal{H}\left(m_{\tilde{\mathcal{L}}} / 2, \ldots, m_{\ell} / 2 ; p\right)$ of $\left.\mathcal{H}\left(S_{g, p}\right)\right)$ and let $\tilde{\mathcal{Q}}$ be the preimage of $\mathcal{Q}$ in $\tilde{\mathcal{Q}}\left(S_{g, n}\right)$ (or in $\left.\tilde{\mathcal{H}}\left(S_{g, n}\right)\right)$. Then there is a collection

$$
\mathcal{L T}(\tilde{\mathcal{Q}}) \subset \mathcal{L T}\left(m_{1}, \ldots, m_{\ell} ;-p_{1}, p_{2}\right)
$$

of large marked train tracks $\tau$ of the same topological type as $\mathcal{Q}$ such that for every $\tau \in \mathcal{L} \mathcal{T}(\tilde{\mathcal{Q}})$, the set $\mathcal{Q}(\tau)$ contains an open path connected subset of $\tilde{\mathcal{Q}}$.

The set $\mathcal{L T}(\tilde{\mathcal{Q}})$ is invariant under the action of the mapping class group. Its quotient $\mathcal{L} \mathcal{T}(\mathcal{Q})$ under this action is finite and is called the set of combinatorial models for $\mathcal{Q}$. The set

$$
\cup_{\tau \in \mathcal{L} \mathcal{T}(\tilde{\mathcal{Q}})} \mathcal{Q}(\tau)
$$

is closed, $\operatorname{Mod}(S)$-invariant and contains $\tilde{\mathcal{Q}}$ as an open dense subset, that is, it coincides with the closure of $\tilde{\mathcal{Q}}$ in $\tilde{\mathcal{Q}}\left(S_{g, n}\right)$.

Lemma 3.4. Let $\mathcal{Q}$ be a component of a stratum, with preimage $\tilde{\mathcal{Q}}$ in $\tilde{\mathcal{Q}}\left(S_{g, n}\right)$, let $\tau \in \mathcal{L T}(\tilde{\mathcal{Q}})$ and let $\eta$ be a large train track of the same topological type as $\tau$ which is carried by $\tau$. Then $\eta \in \mathcal{L} \mathcal{T}(\tilde{\mathcal{Q}})$.

Proof. A point in $\mathcal{Q}(\tau)$ is defined by a pair $(\lambda, \nu)$ where $\lambda \in \mathcal{V}(\tau)$ and where $\nu \in \mathcal{V}^{*}(\tau)$. If we choose $\lambda$ in such a way that its $\operatorname{support} \operatorname{supp}(\lambda)$ is of the same topological type as $\tau$ and such that $\lambda$ is carried by the train track $\eta$, then $(\lambda, \nu)$
defines a differential in $\mathcal{Q}(\eta) \cap \mathcal{Q}(\tau)$. It then follows from Proposition 3.2 that $\eta \in \mathcal{L} \mathcal{T}(\mathcal{Q})$.

As a fairly immediate consequence of the above discussion and Section 3 of [H09], we obtain a method to construct large train tracks of a given topological type. Namely, for a fixed choice of a complete hyperbolic metric on $S$ of finite volume and numbers $a>0, \epsilon>0$ there is a notion of $a$-long train track which $\epsilon$-follows a large geodesic lamination $\lambda$. By definition, this means the following. Fix a complete finite volume hyperbolic metric on $S_{g, n}$. The straightening of a train track $\tau$ is obtained from $\tau$ by replacing each branch $b$ by a geodesic segment which is homotopic with fixed endpoints to $b$. We require that the length of each of the straightened edges is at least $a$, that their tangent lines are contained in the $\epsilon$-neighborhood of the projectivized tangent bundle of $\lambda$ and that moreover the straightening of every trainpath on $\tau$ is a piecewise geodesic whose exterior angles at the breakpoints are not bigger than $\epsilon$.

Lemma 3.2 of [H09] shows that for every geodesic lamination $\lambda$ on $S_{g, n}$ and every $\epsilon>0$ there is an $a$-long generic transversely recurrent train track $\tau$ which carries $\lambda$ and $\epsilon$-follows $\lambda$.
Corollary 3.5. Let $\tau \in \mathcal{L} \mathcal{T}(\tilde{\mathcal{Q}})$ and let $\lambda$ be a minimal large geodesic lamination of the same topological type as $\tau$ which is carried by $\tau$. Then for sufficiently small $\epsilon>0$, an a-long train track $\eta$ which $\epsilon$-follows $\lambda$ is contained in $\mathcal{L T}(\tilde{\mathcal{Q}})$.

Proof. By construction, if $\lambda$ is large, then for sufficiently small $\epsilon$ and sufficiently large $a>0$, an $a$-long train track $\eta$ which $\epsilon$-follows $\lambda$ is of the same topological type as $\lambda$. Furthermore, $\eta$ carries a minimal large geodesic lamination of the same topological type as $\eta$ and hence $\eta$ is fully recurrent and transversely recurrent.

If $\lambda$ is carried by a large train track $\tau$ then for sufficiently small $\epsilon>0$ and sufficiently large $a>0, \eta$ is carried by $\tau$ (see Section 3 of [H09]). Then a large geodesic lamination which is carried by $\tau^{*}$ is carried by $\eta^{*}$ and hence $\eta$ is large as claimed.

## 4. Components of strata: The principal boundary

In Section 3, a combinatorial model for every component $\mathcal{Q}$ of a stratum in $\mathcal{Q}\left(S_{g, n}\right)$ or in $\mathcal{H}\left(S_{g, n}\right)$ was constructed. The purpose of this section is to refine this construction and obtain models for specific types of degenerations of the stratum.

We begin with introducing the degenerations we are interested in. The framework for these degenerations is in the spirit of the "you see what you get" partial compactification of strata introduced in [MW17]. Although we will not make use of this work, we reproduce Definition 2.2 of [MW17]. In its formulation, $\Sigma_{i}$ is the singular set of the quadratic differential $q_{i}$ on the Riemann surface $X_{i}$.
Definition 4.1. Say that $\left(X_{j}, q_{j}, \Sigma_{j}\right)$ converges to ( $X, q, \Sigma$ ) if there are decreasing neighborhoods $U_{j} \subset X$ with $\cap U_{j}=\Sigma$ such that the following holds. There are maps $g_{j}: X-U_{j} \rightarrow X_{j}$ that are diffeomorphisms onto their range, such that
(1) $g_{j}^{*}\left(q_{j}\right)$ converges to $q$ in the compact open topology on $X-\Sigma$.
(2) The injectivity radius at points not in the image of $g_{j}$ goes to zero uniformly in $j$.

With this definition, we allow to erase zero area components of a limiting surface with nodes.

There are two specific types of degenerating sequences which will be used in the sequel.

## 1) The shrinking half-pillowcase:

Let $q$ be a quadratic differential on $S_{g, n}$. Choose a singular or marked regular point $x$ on $S_{g, n}$ and cut $S_{g, n}$ open along a geodesic segment $\alpha$ issuing from $x$ of length $s>0$. If $x$ is a singular point of $q$ then we require that $x$ is the only singular point contained in $\alpha$, and if $x$ is a regular point then we require that $\alpha$ does not contain any singular point. The cut open surface has a geodesic circle as boundary. Glue a foliated cylinder $C$ to this circle whose opposite boundary is divided into two arcs of the same length which are identified to form half of a pillowcase as shown in Figure C. This does not change the genus of $S_{g, n}$, but it adds two punctures


Figure C
to $S_{g, n}$, and it increases the cone angle of each of the endpoints of $\alpha$ by $\pi$. Note that for the fixed point $x$, the half-pillowcase is described by 4 real parameters: The direction and the length of its cutting arc $\alpha$, the height of the cylinder and the position of one of the simple poles on the top of the half-pillowcase, which is determined by the choice of a point on the (oriented) cutting arc $\alpha$.

We call a sequence of quadratic differentials containing a half-pillowcase whose circumferences and heights tend to zero and which degenerate in the sense of Definition 4.1 to the surface with the half-pillowcase removed a shrinking half-pillowcase. The areas of the half-pillowcases, that is, the products of their circumferences and heights, tend to zero.

## 2) The shrinking cylinder:

Let again $q$ be a quadratic differential defining a flat metric on $S_{g, n}$. Given two singular or marked regular points $x_{1} \neq x_{2}$ on $S_{g, n}$, cut the surface $S_{g, n}$ open along two geodesic arcs of the same length and the same direction starting at $x_{i}$ with no singular point in the interior, and glue the two boundary circles to the two boundary circles of a flat cylinder. The result is a singular flat metric on the surface $S_{g+1, n}$. The core curve of the attached cylinder in $S_{g+1, n}$ is non-separating, and the direction of the geodesics in its core equals the direction of the cutting arcs used in the construction. We leave it to the reader to check that this construction depends again on 4 real parameters (this fact is not needed in the sequel, see also [EMZ03]).

There is a modification of this construction as follows. Cut the surface $S_{g, n}$ open along a single geodesic arc $\alpha$ issuing from a singular or marked regular point $x$ with at most one singular point on the boundary and no interior singular point. Identify the endpoints of $\alpha$; the resulting surface has two geodesic boundary circles of the same length and the same direction. Glue the boundary of a flat cylinder to these two boundary circles. As before, the result of this construction is a singular flat metric on the surface $S_{g+1, n}$.

We call a sequence of flat surfaces containing a nonseparating cylinder which degenerate to the surface with nodes by shrinking the width and the heights of the cylinder to zero a shrinking cylinder. Note that the constructions discussed above change the area of the singular flat metric, which can be corrected with the usual area renormalization.

Our goal is to construct combinatorial models for quadratic or abelian differentials which capture these two types of degenerations to differentials on surfaces with nodes. These models will then be used to construct periodic orbits of the Teichmüller flow in the thin part of moduli space. Assume from now on that $3 g-3+n \geq 5$. This rules out spheres with at most 7 punctures, tori with at most 4 punctures and a surface of genus 2 with at most 1 puncture.
Definition 4.2. An essential simple closed curve $c$ on $S_{g, n}$ is called elementary if either
a) $c$ is non-separating or
b) $n \geq 2$ and $c$ decomposes $S_{g, n}$ into a surface $S_{g, n-1}$ and a twice punctured disk.

An elementary pair is a pair $\left(c_{1}, c_{2}\right)$ consisting of disjoint elementary curves $c_{1}, c_{2}$ on $S$. If both $c_{1}$ and $c_{2}$ are non-separating then we require that $S_{g, n}-\left(c_{1} \cup c_{2}\right)$ is connected.

Since $3 g-3+n \geq 5$ by assumption, the complement $S_{g, n}-\left(c_{1} \cup c_{2}\right)$ of an elementary pair in $S=S_{g, n}$ contains a (unique) component which is not a three holed sphere. In the sequel we tacitly identify a complementary component of a curve $c$ in $S$ (or any curve system) with its metric completion, that is, we view $S-c$ as a surface with boundary.
Definition 4.3. A primitive vertex cycle for a large train track $\tau$ is a simple closed curve $c$ embedded in $\tau$ which consists of a large branch and a small branch.

If $c$ is a primitive vertex cycle in $\tau$ then there are two half-branches incident on the two switches of $\tau$ in $c$ which are not contained in $c$. Since $\tau$ is large by assumption, these two half-branches lie on the two different sides of $c$ in an annulus neighborhood of $c$ in $S$. Namely, otherwise there is a complementary component of $\tau$ containing a simple closed curve which is neither contractible nor homotopic into a puncture.

We call a primitive vertex cycle of a large train track $\tau$ clean if its underlying simple closed curve $c$ is elementary and if moreover a branch $b$ which is incident on a switch in $c$ and which is not contained in $c$ satisfies one of the two following conditions.
(1) $b$ is a small branch.
(2) $n \geq 2, c$ is separating and $b$ is contained in the twice punctured disk component of $S-c$.

As there are two types of elementary curves, there are two types of clean vertex cycles. To relate these types to the degeneration of quadratic differentials, note that removing a clean vertex cycle $c$ from $\tau$ as well as all the branches of $\tau$ adjacent to $c$ and branches in a 3-holed sphere component of $S-c$ yields a train track $\tau^{\prime}$ on the complementary component $S_{0}$ of $S-c$ which is not a three holed sphere.

## Type I: The shrinking half-pillowcase.

If $n \geq 2$ and if $c$ is a separating clean vertex cycle of $\tau$, then there is a complementary component $C$ for the train track $\tau^{\prime}$ on $S_{0}$ which is an annulus whose core curve is homotopic to $c$. There is a component $\gamma$ of $\partial C$ embedded in $\tau^{\prime}$, and $\gamma$ contains at least one cusp of $\tau^{\prime}$ (since otherwise $\tau$ has a complementary component which is a bigon).

Let $S_{0}^{\prime}$ be the surface obtained from $S_{0}$ by replacing the boundary circle of $S_{0}$ by a marked point (puncture). The genus of $S_{0}^{\prime}$ coincides with the genus of $S$, and the number of marked points has decreased by one. If the component $\gamma$ of $\partial C$ contains at most two cusps, then $\tau^{\prime}$ is a large train track on $S_{0}^{\prime}$, where the puncture replacing the curve $c$ may be a marked regular point. By Section 3, $\tau^{\prime}$ determines a component of a stratum of differentials on $S^{\prime}=S_{0}^{\prime}$. Passing from $S^{\prime}$ back to $S$ corresponds to a shrinking half-pillowcase obtained by cutting $S^{\prime}$ open along a geodesic segment $\alpha$ for the flat metric of a differential on $S^{\prime}$ which does not contain any singular point. If $\gamma$ contains at least three cusps then remove from $S_{0}^{\prime}$ the marked point enclosed by $\gamma$ and denote the resulting surface by $S^{\prime}$. In both cases, $\tau^{\prime}$ is a large train track on the surface $S^{\prime}$.

Type II: The shrinking cylinder.
If $c$ is a non-separating clean vertex cycle of $\tau$ then both branches incident on $c$ are small. The genus of the surface with boundary $S_{0}$ obtained by cutting $S$ open along $c$ equals $g-1$. Let $S_{0}^{\prime}$ be obtained from $S_{0}$ by replacing the two boundary circles (which are copies of $c$ ) by a marked point. As before, $\tau^{\prime}$ defines a large train track on the surface $S^{\prime}$ which either coincides with $S_{0}^{\prime}$ or is obtained from $S_{0}^{\prime}$ by
removing one or both of the special marked points. These choices depend on the number of cusps of the complementary components of $c$ in $\tau^{\prime}$.

Let as before $\mathcal{Q}$ be a component of a stratum of quadratic or abelian differentials on $S$. Call a train track $\tau$ in special form for $\mathcal{Q}$ if $\tau \in \mathcal{L T}(\mathcal{Q})$ and if there is an elementary pair $\left(c_{1}, c_{2}\right)$ for $S$ with the following additional property.
$(*) \tau$ contains each of the curves $c_{1}, c_{2}$ as a clean vertex cycle, and the graph obtained by removal of $c_{1}, c_{2}$ and their adjacent branches as well as all branches contained in a 3 -hold sphere component of $S-\left(c_{1} \cup c_{2}\right)$ is a large train track on a subsurface of $S$

The rest of this section is devoted to the construction of train tracks in special form for all components of strata of abelian or quadratic differentials on $S_{g, n}$ without marked regular points provided that $3 g-3+n \geq 5$.

For simplicity, write $\mathcal{Q}\left(m_{1}, \ldots, m_{\ell} ;-n\right)$ instead of $\mathcal{Q}\left(m_{1}, \ldots, m_{\ell} ;-n, 0\right)$, and write $\mathcal{H}\left(k_{1}, \ldots, k_{s}\right)$ instead of $\mathcal{H}\left(k_{1}, \ldots, k_{s} ; 0\right)$. Motivated by the strategy in [KZ03] and [L08], the idea is to start with explicit train tracks for components of strata on surfaces with small complexity and uses these train tracks as building blocks for the construction of train tracks for all strata. The first type of modification consists in subdividing complementary components as follows.

Let $C$ be a complementary component of a train track $\eta$ which is a disk with $k \geq 4$ cusps on its boundary $\partial C$. Then $C$ can be subdivided into two components by adding a small branch which connects two non-adjacent sides of the component. The resulting train track $\tau$ is a simple extension of $\eta$ as defined in Section 2. If $\eta$ is orientable and if the number of cusps of $\partial C$ at least six, then this subdivision can be done in such a way that the components have an even number of cusps and that $\tau$ is orientable as well. By Proposition 2.4, if $\tau, \eta$ are either both orientable or both non-orientable then $\tau$ is large if and only if this holds true for $\eta$. In the sequel we always choose subdivisions of complementary components of orientable train tracks in such a way that the resulting train track is orientable.

Following [L08], strata of quadratic differentials with at least three simple poles are connected. We use this fact to observe

Lemma 4.4. Components of strata of differentials on the two-sphere $S^{2}$ with $n \geq 7$ simple poles or on the two-torus $T^{2}$ with $n \geq 4$ simple poles admit a train track in special form.

Proof. Figure D shows large train tracks with at least two clean vertex cycles $c_{1}, c_{2}$ on $S_{0, n}$ for $n=6,7$ and a single complementary component which is not a oncepunctured monogon. Such a train track belong to the stratum of meromorphic differentials on $S=S^{2}$ with a single zero and 6 or 7 simple poles since such strata are connected [L08].

The train track $\tau$ for the stratum of differentials on $S_{0,6}$ with 6 poles is not in special form. Namely, removal of the vertex cycles $c_{1}, c_{2}$ as well as all adjacent
branches and all branches in the three-holed sphere components of $S_{0,6}-\left(c_{1} \cup c_{2}\right)$ from $\tau$ does not result in a large train track for a subsurface of $S_{0,6}$.

In contrast, the train track $\tau$ for the stratum of differentials on $S_{0,7}$ with a single zero and 7 poles is in special form: The train track obtained from $\tau$ by removal of the vertex cycles $c_{1}, c_{2}$, all adjacent branches and all branches in the three-holed sphere components of $S_{0,7}-\left(c_{1} \cup c_{2}\right)$ is a large train track for the stratum of differentials on $S^{2}$ with 4 simple poles.

To construct train tracks in special form for strata of differentials on $S^{2}$ with a single zero and at least 8 simple poles just attach more copies of a circle enclosing two punctures and containing a once puncture monogon to one of the two train tracks shown in Figure D. Train tracks in special form for arbitrary strata of dif-

ferentials on $S^{2}$ with at least 7 simple poles are obtained from the train tracks for strata with a single zero by subdivision of complementary components.

Figure E shows large train tracks containing at least two clean vertex cycles for the stratum of differentials on the torus $T^{2}$ with a single zero and 3 or 4 simple poles. As before, the train track for the stratum with 4 simple poles is in special form. To construct train tracks in special form for a stratum of differentials on a torus with a single zero and at least 5 simple poles, attach more copies of a circle enclosing two punctures and containing a once punctured monogon. As before, train tracks in special form for arbitrary strata of differentials on the torus with at least 4 simple poles are constructed from these train tracks by subdivision of complementary components.


Remark 4.5. Lemma 4.4 shows the existence of train tracks in special form for strata of differentials on $S_{0, n}$ or $S_{1, n}$ where $3 g-3+n \geq 4$, which is slightly better than what appears in the statement of Theorem 1.

Call a component of a stratum of abelian or quadratic differentials hyperelliptic if it consists of differentials on hyperelliptic surfaces which are invariant under the hyperelliptic involution. Lemma 4.4 is used to show

Lemma 4.6. Let $\mathcal{Q}$ be a hyperelliptic component of a stratum of quadratic or abelian differentials on a surface of genus at least 3. Then there is a train track $\tau$ in special form for $\mathcal{Q}$.

Proof. Let $\mathcal{Q}$ be a hyperelliptic component of a stratum of quadratic differentials on a surface $S$ of genus $g \geq 3$ with $n \geq 0$ simple poles. Such a hyperelliptic component is obtained by pull-back of a stratum $\hat{\mathcal{Q}}$ of quadratic differentials on the punctured sphere $S_{0, m}$ with a double branched covering map.

By the main result of [L04] (see also Theorem 1.2 of [L08]), the component $\hat{\mathcal{Q}}$ consists of differentials with $n \geq 8$ simple poles, and the cover is ramified at all or at all but one of the poles.

Let $\eta$ be a train track in special form for $\hat{\mathcal{Q}}$ as constructed in Lemma 4.4, with two clean vertex cycles $c_{1}, c_{2}$. The vertex cycles $c_{1}, c_{2}$ cut from the punctured sphere two twice punctured disks $P_{1}, P_{2}$.

Choose the branched covering in such a way that it is ramified at each of the punctures in $P_{1}, P_{2}$. The preimage of $\eta$ under this covering is an embedded graph $\hat{\eta}$ in the surface $S$. The preimage of the vertex cycle $c_{i}$ consists of two embedded simple closed curves which bound an embedded annulus $A_{i}$. The annulus $A_{i}$ contains the preimage in $\hat{\eta}$ of the two punctures in the 3 -holed sphere component of $S_{0, m}-c_{i}$ as shown in the middle part of Figure F. It is subdivided into two bigons with an interior marked point by the preimage of $\eta$. The marked point is a preimage of one of the ramification points. Remove these marked points and the branches in the
interior of the annulus $A_{i}$ and identify the two boundary circles of $A_{i}$ as shown in Figure F so that they form a single simple closed curve $v_{i}(i=1,2)$.


Collapse each remaining bigon in $\hat{\eta}$ containing a single preimage of a ramification point to a single arc as described above. The resulting graph $\tau$ is a large train track on $S$ which is contained in $\mathcal{L T}(\mathcal{Q})$. The train track $\tau$ contains the curves $v_{i}$ as primitive vertex cycles. By construction, the curves $v_{i}$ are non-separating and do not form a bounding pair and hence they define an elementary curve system. Moreover, since $c_{i}$ is a clean primitive vertex cycle for $\eta$, the primitive vertex cycle $v_{i}$ is clean for $\tau$. As the train track $\eta$ for $\hat{\mathcal{Q}}$ was in special form, naturality of the construction implies that $\tau$ is in special form for $\mathcal{Q}$.

The same reasoning also applies for hyperelliptic components of abelian differentials. Namely, in this case the branched cover defining the component is ramified at each of the simple poles on the two-sphere and thus it is ramified at at least 8 simple poles. The above argument then shows that there is a train track in special form for the component.

To treat non-hyperelliptic components we construct from a large train track $\eta$ of topological type $\left(m_{1}, \ldots, m_{\ell} ;-n\right)$ on a surface of genus $g \geq 0$ with $n$ punctures a train track $\tau$ of type $\left(m_{1}, \ldots, m_{\ell}+4 ;-n\right)$ on a surface of genus $g+1$ by attaching a handle as follows.

The train track $\eta$ has a complementary polygon $P$ with $m_{\ell}+2$ sides. Attach two $\operatorname{arcs} b_{1}, b_{2}$ of class $C^{1}$ to the interior of two branches of $\eta$ which are contained in two different sides of the polygon $P$ in such a way that $b_{1}, b_{2}$ are disjoint and embedded in $P$. Attach a simple closed curve $c_{i} \subset P$ of class $C^{1}$ to the arc $b_{i}$ which meets $b_{i}$ only at its free endpoint and is tangent to $b_{i}(i=1,2)$. We require that the curves $c_{1}, c_{2}$ are disjoint and bound disjoint embedded disks $D_{1}, D_{2}$ in the interior of $P$.

Remove the interiors of the disks $D_{1}, D_{2}$ from $P$. The boundary of the resulting surface consists of the curves $c_{1}, c_{2}$. Glue $c_{1}$ to $c_{2}$ with a diffeomorphism which reverses the boundary orientation of $D_{i}$. The result is a surface of genus $g+1$ with $n$ punctures which carries a train track $\tau$ of topological type $\left(m_{1}, \ldots, m_{\ell}+4 ;-n\right)$. It contains the image of the curves $c_{i}$ under a glueing map as a clean vertex cycle. Note that if $\eta$ is orientable, then for a suitable choice of the $\operatorname{arcs} b_{1}, b_{2}$, the train track $\tau$ is orientable as well. In the sequel we always assume that the construction preserves orientability if applicable. We then call $\tau$ the train track obtained from $\eta$ by attaching a handle.

Lemma 4.7. The train track $\tau$ obtained from $\eta$ by attaching a handle is large. Moreover, it is orientable if and only if this holds true for $\eta$.

Proof. By construction, the train track $\tau$ is orientable if and only if this holds true for $\eta$. Moreover, $\eta$ can be viewed as a subtrack of $\tau$.

Now $\eta$ is a large train track and hence it carries a minimal large geodesic lamination of type $\left(m_{1}, \ldots, m_{\ell} ;-n\right)$. This geodesic lamination defines a minimal geodesic lamination $\lambda_{0}$ on $\tau$. The train track $\tau$ contains a primitive vertex cycle $c_{0}$ which is disjoint from $\lambda_{0}$ and which is the image of the curves $c_{1}, c_{2}$ under the glueing process. The union $\lambda_{0} \cup c_{0}$ is a geodesic lamination carried by $\tau$. This lamination is not large, but it is a sublamination of a large geodesic lamination which is the union of $\lambda_{0} \cup c_{0}$ with two isolated leaves which pass through the two branches of $\tau$ connecting $c_{0}$ to the subtrack $\eta$ and which spiral from one side about $\lambda_{0}$, from the other side about $c_{0}$. Thus $\tau$ carries a large geodesic lamination. The same argument also shows that the dual bigon track $\tau^{*}$ carries a large geodesic lamination. In other words, $\tau$ is large.

For the construction of train tracks in special form for all components of strata we use the classifiction of components due to Kontsevich and Zorich [KZ03] (for abelian differentials) and Lanneau [L08] (for quadratic differentials).

Proposition 4.8. (1) For every $g \geq 4$ the stratum $\mathcal{H}(2 g-2)$ has three connected components. One of these components is hyperelliptic, the other two are distinguished by the parity of the spin structure they define.
(2) The stratum $\mathcal{H}(4)$ has two components. One of the components is hyperelliptic, the other consists of abelian differentials defining an odd spin structure.
(3) $\mathcal{H}(2)$ is connected.
(4) For every $g \neq 3,4$ and every $n \geq 0$ the stratum $\mathcal{Q}(4 g-4+n ;-n)$ is connected.
(5) The strata $\mathcal{Q}(12 ; 0)$ and $\mathcal{Q}(9 ;-1)$ have two connected components, and $\mathcal{Q}(4 ; 0)=\emptyset$.

We are now ready to show
Proposition 4.9. Let $\mathcal{Q}$ be a non-hyperelliptic component of a stratum of quadratic or abelian differentials on $S$ where $3 g-3+n \geq 5$. Then there is a train track $\tau$ in special form for $\mathcal{Q}$.

Proof. We divide the proof of the proposition into four steps. The case $g=0$ and $g=1$ is covered by Lemma 4.4.

Step 1: Strata of quadratic differentials with at least three poles.
By the classification of Lanneau [L08], every stratum in moduli space consisting of meromorphic quadratic differentials with at least three poles is connected. Thus for $n \geq 3$ and any $g \geq 2$, we can construct a large train track in special form for the stratum $\mathcal{Q}(4 g-4+n ;-n)$ by attaching handles to a train track on the torus as described in Lemma 4.7. Although for $n=3$, this train track is not in special
form, it contains 2 clean vertex cycles, and the train obtained from it by attaching a handle is in special form. These train tracks can be subdivided to train tracks in special form for any stratum of quadratic differentials with at least three poles.

Step 2: Strata of abelian differentials with a single zero.
The moduli space $\mathcal{H}(2)$ of abelian differentials with a single zero on a surface of genus 2 is connected. It consists of differentials which define an even spin structure Figure G below shows a large train track $\eta \in \mathcal{L} \mathcal{T}(\mathcal{H}(2))$. It contains a single clean vertex cycle. Attaching a handle to $\eta$ result in a train track in special form on a surface of genus 3 .

Figure G


For $g=3$, the stratum $\mathcal{H}(4)$ consists of two components. One of these components is hyperelliptic. The two components are distinguished by the parity of the spin structure they define [KZ03]. The parity of the spin structure for the hyperelliptic component is even. By Lemma 4.6, it suffices to show that attaching a handle to the train track $\eta$ in Figure G results in a train track in special form for the component with odd spin structure. To this end we compute from a large train track $\tau \in \mathcal{L T}(\mathcal{Q})$ the parity of the spin structure of the component $\mathcal{Q}$.

The parity of the spin structure defined by an abelian differential $\omega$ can be calculated as follows (see p. 643 of [KZ03]). For a smooth simple closed curve $\alpha$ on $S$ not passing through a zero of $\omega$ define $\operatorname{ind}_{\alpha} \in \mathbb{Z}$ to be the total change of angle between the tangent of $\alpha$ and the vector tangent to the vertical foliation of $\omega$. Let $\left\{\alpha_{i}, \beta_{i} \mid i=1, \ldots, g\right\}$ be any system of $2 g$ smooth simple closed curves which define a symplectic basis for $H_{1}(S, \mathbb{Z})$ with the above property. Then

$$
\operatorname{Arf}(\omega)=\sum_{i=1}^{g}\left(i n d_{\alpha_{i}}+1\right)\left(i n d_{\beta_{i}}+1\right)(\bmod 2)
$$

This formula enables us to calculate the parity using a train track. Namely, a large orientable train track $\tau$ of type $(4 g-4 ; 0)$ has a single complementary component $C$ which is a $4 g$-gon. Let $\alpha$ be a smooth simple closed curve on $S$ which intersects $\tau$ transversely in finitely many points contained in the interior of some branches of $\tau$. Define the index $r_{\tau}(\alpha) \in \mathbb{Z} / 2 \mathbb{Z}$ of $\alpha$ as follows.

Choose a numbering of the sides of the complementary region of $\tau$ in counterclockwise order. Choose also an orientation of $\alpha$. A transverse intersection point

$p \in \alpha \cap \tau$ is contained in precisely two sides $s_{1}, s_{2}$ of $C$. Write $r(p)=s_{2}-s_{1}+$ $1(\bmod 2)=s_{1}-s_{2}+1(\bmod 2)$ and define

$$
r_{\tau}(\alpha)=\sum_{p} r(p) \in \mathbb{Z} / 2 \mathbb{Z}
$$

Note that if $\alpha^{\prime}$ is isotopic to $\alpha$ with an isotopy which moves some subarc of $\alpha$ across a switch then this number is unchanged, and the same holds true if $\alpha$ is fixed and $\tau$ is modified by a split.

Choose smooth simple closed curves $\left\{\alpha_{i}, \beta_{i} \mid i=1, \ldots, g\right\}$ which define a symplectic basis of $H_{1}(S, \mathbb{Z})$. Assume that each of the curves $\alpha_{i}$ intersects $\tau$ in finitely many points which are contained in the interior of some branch of $\tau$. Define

$$
\varphi(\tau)=\sum_{i=1}^{g}\left(r_{\tau}\left(\alpha_{i}\right)+1\right)\left(r_{\tau}\left(\beta_{i}\right)+1\right) \in \mathbb{Z} / 2 \mathbb{Z}
$$

and call this number the parity of the spin structure of $\tau$.
Recall that $\eta$ is a large train track with a single complementary component $C$. If the train track $\tau$ is obtained from $\eta$ by attaching a handle, then the parity of the spin structure of $\tau$ can be calculated from the parity of the spin structure of $\eta$ as follows. There is a primitive vertex cycle $\alpha_{1}$ for $\tau$ which is disjoint from $\eta$ (it goes around the handle). This vertex cycle $\alpha_{1}$ satisfies $r_{\eta}\left(\alpha_{1}\right)=0$ since up to homotopy, it has a unique intersection point with $\tau$ which is contained in one of the small branches adjacent to the primitive vertex cycle $\alpha_{1}$. Then this intersection point is contained in two consecutive sides of the complementary component of $\tau$.

There is a second curve $\beta_{1}$ in the handle which intersects $\alpha_{1}$ in a single point, and it intersects $\tau$ in a single point $q$ as well. Let $s_{i}, s_{j}(i<j)$ be the sides of the complementary component $C$ of $\eta$ at which branches of $\tau-\eta$ are attached. If we choose $s_{j}=s_{i}+1$ then Figure H shows that $r_{\eta}\left(\beta_{1}\right)=0$.

The curves on $S$ used to calculate the parity of the spin structure for $\eta$ can be chosen to be disjoint from $\tau-\eta$ viewed as a subgraph of the complementary component $C$. Then the indices of the curves used for $\eta$ do not change $\bmod 2$ and hence the parity of the spin structure of $\tau$ is opposite to the parity of the spin structure for $\eta$. In particular, attaching a handle to the train track shown in Figure $G$ results in a train track in special form for the component of $\mathcal{H}(4)$ with odd spin structure. Together with the construction for hyperelliptic components, we obtain a train track in special form for each of the two components of $\mathcal{H}(2)$.

Using again Proposition 4.8 , for $k \geq 3$ the two different non-hyperelliptic components of $\mathcal{H}(2 k)$ are distinguished by the parity of the spin structure they define. It follows from the above discussion that attaching a handle to a train track in special form for a component of $\mathcal{H}(2 k-2)$ with even (or odd) spin structure is a train track in special form for a component of $\mathcal{H}(2 k)$ with odd (or even) spin structure. Now it is easy to see that a train track for a hyperelliptic component can only arise by this construction from a train track for a hyperelliptic component. Since the parity of the spin structure of a hyperelliptic component is even, none of the two train tracks arising from attaching a handle to one of the train tracks in special form for a component of $\mathcal{H}(4)$ is a train track for a hyperelliptic component. Thus by induction beginning with $\mathcal{H}(4)$, we obtain in this way for each $k \geq 1$ and for each non-hyperelliptic component $\mathcal{Q}$ of $\mathcal{H}(2 k)$ a train track in special form for $\mathcal{Q}$.

Step 3: Strata of quadratic differentials with a single zero and at most two poles.
By the classification of Lanneau [L08], strata of quadratic differentials with a single zero and at most two poles are connected.

To obtain a train track in special form for this stratum on a surface of genus $g=2$ with $m=2$ punctures, attach to the train track shown in Figure G a circle as shown in Figure $D$ enclosing two once-puncture monogons. Similarly, to obtain a train track in special form for a stratum with a single zero and a single pole on a surface of genus 3, attach a train track in special form on a surface of genus 3 a once punctured monogon.

A train track in special form for a stratum in higher genus can be obtained by attaching handles to the train track for genus 2 or 3 .

Step 4: Subdividing complementary components.
Following [L08], we say that a component $\mathcal{Q}$ of a stratum in $\mathcal{Q}(S)$ for a surface $S$ of genus $g \geq 2$ is adjacent to a component $\mathcal{Q}_{0}$ of another stratum if $\mathcal{Q}_{0}$ is contained in the closure $\overline{\mathcal{Q}}$ of $\mathcal{Q}$ in $\mathcal{Q}(S)$. Here we allow that poles merge with zeros and disappear.

Lanneau [L08] showed that with the exception of one sporadic component in each of the strata $\mathcal{Q}(9 ;-1), \mathcal{Q}(3,6 ;-1), \mathcal{Q}(3,3,3 ;-1)$, any non-hyperelliptic component of a stratum with at least two distinct types of zeros or poles is adjacent to $\mathcal{Q}(4 g-4)$. For such a component, train tracks in special form can be obtained from train tracks in special form for components of strata with a single zero by subdivision of complementary components.

For the completion of the proof of the proposition we are left with the investigation of the sporadic components in genus $g=3,4$ as listed in the classification of Lanneau [L08].

The sporadic component for $g=4$ is a component of $\mathcal{Q}(12)$ which can be checked explicitly. The sporadic component of $\mathcal{Q}(3,3,3 ;-1)$ is adjacent to the sporadic component of $\mathcal{Q}(3,6 ;-1)$, and the sporadic component of $\mathcal{Q}(3,6 ;-1)$ is adjacent to the sporadic component of $\mathcal{Q}(9 ;-1)$ [L08]. Using Step 2 above, it is therefore enough to construct a train track with the required properties which belongs to the sporadic component of $\mathcal{Q}(9 ;-1)$. However, the sporadic component of $\mathcal{Q}(9 ;-1)$ admits a quadratic differential with a two-cylinder-decomposition which can be used to construct a train track as required (compare the table in [L08]). This completes the proof of the proposition.

Remark 4.10. Although we use the classification result of Kontsevich-Zorich and of Lanneau in our construction, the construction can be used to give a alternative proof for the classification.

Remark 4.11. The computation of the parity of the spin structure in the proof of Proposition 4.9 is similar to the computation of parities in the transition of component of a stratum to a component of its principal boundary in [EMZ03]. Unfortunately we can not use these computations for our needs.

## 5. Coding of pseudo-Anosov mapping classes

In this section we combine the results from Sections 1-4 and from [H13] and [H22] to set up the main properties needed for the proof of Theorem 1. We continue to use the notations from sections 1-4.

The number $m>0$ of branches of a large train track $\tau \in \mathcal{L T}\left(m_{1}, \ldots, m_{\ell} ;-n, p\right)$ only depends on the topological type of $\tau$. A numbering of the branches of $\tau$ defines an embedding of the cone $\mathcal{V}(\tau)$ of transverse measures on $\tau$ onto a closed convex cone in $\mathbb{R}^{m}$ determined by the switch conditions. For the standard basis $e_{1}, \ldots, e_{m}$ of $\mathbb{R}^{m}$, this embedding associates to a measure $\mu \in \mathcal{V}(\tau)$ the vector $\sum_{i} \mu(i) e_{i} \in \mathbb{R}^{m}$ where we identify a branch of $\tau$ with its number. If $\sigma \prec \tau$ then the transformation $\mathcal{V}(\sigma) \rightarrow \mathcal{V}(\tau)$ induced by a carrying map $\sigma \rightarrow \tau$ is linear in these coordinates.

The mapping class group $\operatorname{Mod}(S)$ acts on marked train tracks by precomposition of marking. A train track expansion of a mapping class $\varphi \in \operatorname{Mod}(S)$ is a train track $\tau$ such that $\varphi(\tau) \prec \tau$. Then the composition of the isomorphism $\mathcal{V}(\tau) \rightarrow \mathcal{V}(\varphi \tau)=$ $\varphi(\mathcal{V}(\tau))$ with a carrying map $\mathcal{V}(\varphi \tau) \rightarrow \mathcal{V}(\tau)$ is given by a linear map

$$
A(\varphi, \tau): \mathbb{R}^{m} \rightarrow \mathbb{R}^{m}
$$

The matrix describing this map with respect to the standard basis of $\mathbb{R}^{m}$ has nonnegative entries.

By the Perron Frobenius theorem, an $(m, m)$-matrix $A$ with non-negative entries admits an eigenvector with non-negative entries. The corresponding eigenvalue $\alpha$ is positive. If some power of $A$ is positive, then the generalized eigenspace for $\alpha$ is one-dimensional, and $\alpha$ is bigger than the absolute value of any other eigenvalue
of $A$. We call an eigenvector with nonnegative entries for the eigenvalue $\alpha$ of $A$ a Perron Frobenius eigenvector.

We next collect some information on the relation between a pseudo-Anosov mapping class $\varphi$, a train track expansion $\tau$ for $\varphi$ and the linear map $A(\varphi, \tau)$. We begin with Corollary 3.2 of [P88]. For its formulation, recall that a pseudo-Anosov mapping class $\varphi$ admits an invariant flow line for the Teichmüller flow on the Teichmüller space of area one abelian or quadratic differentials, which is the unit cotangent line of an axis of $\varphi$. The axis is prescribed by a pair of projective measured geodesic laminations $\left(\left[\lambda^{h}\right],\left[\lambda^{v}\right]\right) \in \mathcal{P} \mathcal{M} \mathcal{L}^{2}$ which are the attracting and repelling fixed points for the action of $\varphi$ on $\mathcal{P} \mathcal{M L}$, respectively. Note also that the cone $\mathcal{V}(\tau)$ of all nonnegative solutions to the switch condition on $\tau$ can be identified with a subset of $\mathcal{M} \mathcal{L}$ which is invariant under scaling.
Lemma 5.1. Let $\tau \in \mathcal{L T}\left(m_{1}, \ldots, m_{\ell} ;-n, p\right)$ and let $\varphi \in \operatorname{Mod}(S)$ be such that $\varphi(\tau) \prec \tau$ and that the matrix $A(\varphi, \tau)$ is positive. Then $\varphi$ is pseudo-Anosov. The unit cotangent line of its axis intersects $\mathcal{Q}(\tau)$. The horizontal measured geodesic lamination of a differential in this cotangent line is a Perron Frobenius eigenvector of the linear map $A(\varphi, \tau)$.

Proof. It follows from Corollary 3.2 of [P88] that $\varphi$ is pseudo-Anosov and that the attracting fixed point for its action on $\mathcal{P} \mathcal{M} \mathcal{L}$ is the projectivization of a PerronFrobenius eigenvector $\lambda$ of the matrix $A(\varphi, \tau)$. This eigenvector is positive and unique up to scale.

As $\varphi(\tau) \prec \tau$, we have $\varphi^{-1}\left(\tau^{*}\right) \prec \tau^{*}$ and hence $\mathcal{V}^{*}\left(\varphi^{-1}(\tau)\right) \subset \mathcal{V}^{*}(\tau)$. Since $\varphi$ acts with north-south dynamics on $\mathcal{P} \mathcal{M} \mathcal{L}$, we conclude that the attracting fixed point of $\varphi^{-1}$ is a measured geodesic lamination whose support is carried by $\tau^{*}$. This then implies that the unit cotangent line of the axis of $\varphi$ intersects $\mathcal{Q}(\tau)$.

The following lemma gives a more geometric approach to the study of periodic orbits in a component $\mathcal{Q}$ of a stratum. In its formulation, we assume that the surface $S$ is equipped with a complete finite volume hyperbolic metric. The notion of an $a$-long train track which $\epsilon$-follows a geodesic lamination was introduced at the end of Section 3. We denote as before by $\tilde{\mathcal{Q}}$ the preimage of $\mathcal{Q}$ in the Teichmüller space of marked differentials.
Lemma 5.2. If $q \in \tilde{\mathcal{Q}}$ is a point on the cotangent line of an axis of the pseudoAnosov mapping class $\varphi$ and if $q \in \mathcal{Q}(\tau)$ for some $\tau \in \mathcal{L} \mathcal{T}(\tilde{\mathcal{Q}})$, then for sufficiently large $k>0$, the train track $\varphi^{k}(\tau)$ is a-long and $\epsilon$-follows the support of the attracting fixed point $[\mu]$ for the action of $\varphi$ on $\mathcal{P} \mathcal{M L}$.

Proof. Let $\tau \in \mathcal{L T}(\tilde{\mathcal{Q}})$ and assume that $q \in \mathcal{Q}(\tau)$ for a differential $q \in \tilde{\mathcal{Q}}$ on the contangent line of the axis of the pseudo-Anosov mapping class $\varphi$. Denote by $\hat{\mu}=\operatorname{supp}([\mu]), \hat{\nu}=\operatorname{supp}([\nu])$ the supports of the attracting and repelling fixed points for the action of $\varphi$ on $\mathcal{P} \mathcal{M L}$, respectively. Then $\hat{\nu}$ is carried by $\tau$, furthermore $\hat{\nu}$ is minimal and its topological type coincides with the topological type of $\tau$. In particular, the carrying map $\hat{\nu} \rightarrow \tau^{*}$ induces a bijection between the complementary polygons of $\hat{\nu}$ and the complementary components of $\tau$.

Represent $\hat{\mu}, \hat{\nu}$ by geodesics for the hyperbolic metric $h^{\prime}$ in the conformal class of $q$. Then up to homotopy, we may assume that the switches of $\tau$ are contained in the intersection $S-(\hat{\mu} \cup \hat{\nu})$ of the complementary components of $\hat{\mu}, \hat{\nu}$. Furthermore, the weight deposited by $\hat{\nu}$ on any branch of $\tau$ is positive.

Following the proof of Lemma 6.2 of [CB88], define an equivalence relation $\sim$ on $S$ by $x \sim y$ if either
(i) $x, y$ are in the closure of the same component of $\hat{\mu}-\hat{\nu}$ or
(ii) $x, y$ are in the closure of the same component of $\hat{\nu}-\hat{\mu}$ or
(iii) $x=y$.

Lemma 6.2 of [CB88] shows that $S / \sim$ is a finite type suface homeomorphic to $S$, and the measured laminations $\mu, \nu$ project to transverse singular measured foliations $\mathcal{F}^{s}, \mathcal{F}^{u}$ on $S$ which up to isotopy equal the horizontal and the vertical measured foliations of the differential $q$.

Let $b$ be a branch of $\tau$. Then the endpoints of $b$ project to points $x, y \in S / \sim$, and $b$ defines the homotopy class of a geodesic arc for the singular euclidean metric $h$ defined by $q$ which connects $x$ to $y$. Since the measure deposited by $\nu$ on $b$ is positive, this geodesic arc is transverse to the vertical foliation of $q$. Doing this construction with each of the branches of $\tau$ yields an embedded graph $\mathcal{G} \subset S$ with the following properties.
(1) $\mathcal{G}$ is topologically isotopic to $\tau$ (this means that the isotopy mapping $\mathcal{G}$ to $\tau$ is not necessarily smooth).
(2) $\mathcal{G}$ is disjoint from the singular set of $q$.
(3) The edges of $\mathcal{G}$ are geodesic segments for $h$ whose directions are uniformly bounded away from the vertical direction.

Note that $\mathcal{G}$ may not admit a tangent at its vertices and hence in general, $\mathcal{G}$ is not a train track.

Now $\varphi$ can be represented by a homeomorphism of $S$ which is smooth outside the singular set of $q$ and which expands the horizontal foliation of $q$ and contracts the vertical one. Thus the image of $\mathcal{G}$ under $\varphi$ is a piecewise geodesic graph in the singular euclidean surface $(S, h)$ whose edges have directions which are closer to the horizontal direction than the directions of the edges of $\mathcal{G}$ with respect to the conformal structure on $S$ defined by $q$ or $h$. Iteration then yields that as $k \rightarrow \infty$, up to passing to a subsequence the graphs $\varphi^{k}(\mathcal{G})$ converge in the Hausdorff topology to a closed subset of the horizontal foliation for the flat metric $h$. Since the edges of $\varphi^{k} \mathcal{G}$ are geodesics for $h$ whose lengths tend to infinity with $k$, this set is leaf saturated, that is, it is a union of leaves. By minimality of the horizontal foliation of $q$, this limit equals the horizontal foliation of $q$.

The horizontal geodesic lamination $\hat{\mu}$ of $q$ is obtained from the horizontal foliation by cutting $S$ open along the separatrices of the foliation and straightening the complement with respect to the hyperbolic metric $h^{\prime}$ on $S$. This implies that up to isotopy, the hyperbolic straightening of $\varphi^{k}(\mathcal{G})$, obtained by replacing each edge
by a geodesic segment for the hyperbolic metric $h^{\prime}$, converges as $k \rightarrow \infty$ in the Hausdorff topology to $\hat{\mu}$. This shows the lemma.

The proof of Lemma 5.2 relied on the fact that given a pseudo-Anosov mapping class $\varphi$ and an abelian or quadratic differential $q$ on the cotangent line of an axis of $\varphi$, the mapping class $\varphi$ can be represented by a homeomorphism of $S$ which is smooth outside the singular points of $q$ and preserves the vertical and the horizontal measured foliation of the singular euclidean metric defined by $q$. However, $\varphi$ may permute zeros of $q$ of the same order, it may permute poles, and if $\varphi$ fixes a zero, then is may permute vertical and horizontal separatrices coming out of the zero and only preserve their cyclic order.

We say that $\varphi$ preserves the combinatorics if it fixes each singular point of $q$ and preserves each horizontal and vertical separatrix coming out of a zero. This property does not depend on the choice of the differential $q$ on the cotangent line of an axis of $\varphi$. As the number of singular points of $q$, counted with multiplicity, is a topological invariant, each pseudo-Anosov mapping class $\varphi$ has a positive power which preserves the combinatorics, and the degree of this power is bounded from above by a constant only depending on the Euler characteristic of $S$. Note that similar constraints arise in the context of zippered rectangles which are used by Veech [V86] to study the Teichmüller flow.

For a pseudo-Anosov mapping class $\varphi \in \operatorname{Mod}(S)$ and a differential $q$ on the cotangent line of its axis, contained in the preimage $\tilde{\mathcal{Q}}$ in $\tilde{\mathcal{Q}}(S)$ of a component $\mathcal{Q}$ of a stratum of quadratic or abelian differentials, define

$$
\mathcal{L T}(\varphi)=\{\tau \in \mathcal{L} \mathcal{T}(\tilde{\mathcal{Q}}) \mid q \in \mathcal{Q}(\tau)\}
$$

Note that this only depends on $\varphi$ but not on the choice of $q$.
The following observation is the main technical result of this section. For the purpose of its proof and later use, define a splitting and shifting sequence of a train track $\tau$ to be a sequence of modifications of $\tau$ by splitting or shifting moves. If $\eta$ is carried by $\tau$ then $\tau$ can be connected to $\eta$ by a splitting and shifting sequence [PH92].

Lemma 5.3. If $\varphi$ is a pseudo-Anosov mapping class which preserves the combinatorics, with cotangent line in $\tilde{\mathcal{Q}}$, then

$$
\mathcal{L T}(\varphi)=\{\tau \in \mathcal{L T}(\tilde{\mathcal{Q}}) \mid \varphi(\tau) \prec \tau\} .
$$

Proof. It is shown in [P88] that every pseudo-Anosov mapping class $\varphi$ has a train track expansion $\tau$. Furthermore, we may assume that $\tau \in \mathcal{L} \mathcal{T}(\tilde{\mathcal{Q}})$ where $\tilde{\mathcal{Q}}$ is a component of a stratum in the Teichmüller space of abelian or quadratic differentials which contains the cotangent line of the axis of $\varphi$.

We show first that such a train track $\tau$ is contained in $\mathcal{L T}(\varphi)$. To see this note that by induction, we have $\varphi^{k}(\tau) \prec \tau$ and hence $\varphi^{k} \mathcal{V}(\tau) \subset \mathcal{V}(\tau)$ for all $k \geq 1$. Since $\mathcal{V}(\tau)$ is a cone over a compact complex polyhedron and hence its projectivization $P \mathcal{V}(\tau)$ is homeomorphic to a closed topological ball, and since the action of $\varphi$ on $\mathcal{V}(\tau)$ commutes with rescaling and hence descends to an action on $P \mathcal{V}(\tau)$, the

Brouwer fixed point theorem yields that $\varphi$ has a fixed point in $P \mathcal{V}(\tau)$. But $\varphi$ acts on $\mathcal{P} \mathcal{M L} \supset P \mathcal{V}(\tau)$ with north-south dynamics and therefore this fixed point has to be the attracting fixed point of $\varphi$. As a consequence, the support $\mu$ of the horizontal measured geodesic lamination of a point $q \in \tilde{\mathcal{Q}}$ on the cotangent line of the axis of $\varphi$ is carried by $\tau$.

Reversing the role of $\tau$ and its dual bigon track and replacing $\varphi$ to $\varphi^{-1}$, this argument also shows that the vertical measured geodesic lamination of $q$ is carried by the dual bigon track $\tau^{*}$ of $\tau$ and hence $\tau \in \mathcal{L T}(\varphi)$ as claimed.

The inclusion of the set of train track expansions of $\varphi$ into $\mathcal{L T}(\varphi)$ holds true for all pseudo-Anosov mapping classes. To show the reverse inclusion we assume from now on that the pseudo-Anosov mapping class $\varphi$ preserves the combinatorics. Our goal is to show that $\varphi(\tau) \prec \tau$ for every $\tau \in \mathcal{L} \mathcal{T}(\varphi)$.

To this end let as before $\mu$ be the support of the horizontal measured geodesic lamination of the differential $q$. Then $\mu$ is minimal and filling, and its combinatorial type coincides with the combinatorial type of any $\tau \in \mathcal{L} \mathcal{T}(\varphi)$. If $\eta$ is obtained from $\tau \in \mathcal{L T}(\varphi)$ by a $\mu$-split, that is, by a split with the property that $\eta$ carries $\mu$, then $\eta \in \mathcal{L} \mathcal{T}(\varphi)$.

The infinite cyclic subgroup $\Gamma$ of $\operatorname{Mod}(S)$ generated by $\varphi$ acts on the set $\mathcal{L} \mathcal{T}(\varphi) \subset$ $\mathcal{L} \mathcal{T}(\tilde{\mathcal{Q}})$ as a group of permutations. Let $\mathcal{E} \subset \mathcal{L T}(\varphi)$ be the subset of all train track expansions of $\varphi$. We noted above that this set is non-empty. Moreover, it is clearly $\Gamma$-invariant and invariant under the shift operation on train tracks. We claim that if $\tau \in \mathcal{E}$ and if $\eta$ is obtained from $\tau$ by a $\mu$-split, then $\eta \in \mathcal{E}$.

Thus let $\tau \in \mathcal{E}$ and let $e$ be a large branch of $\tau$. We know that $\varphi(\tau) \prec \tau$. As a consequence, if $\varphi(\tau)$ is carried by the $\mu$-split $\sigma$ of $\tau$ at $e$, then since the $\mu$-split $\varphi(\sigma)$ of $\varphi(\tau)$ at the branch $\varphi(e)$ is carried by $\varphi(\tau)$, we have $\varphi(\sigma) \prec \sigma$ and we are done. So assume that $\varphi(\tau)$ is not carried by the $\mu$-split of $\tau$ at $e$.

Let $A$ be a foliated neighborhood of $\tau$ in $S$; this is a neighborhood of $\tau$ which is foliated by compact arcs, called ties, which are transverse to $\tau$ and intersect $\tau$ in precisely one point. There is a collapsing map $F: A \rightarrow \tau$ which collapses each of these ties to a point. Since $\varphi(\tau) \prec \tau$, the train track $\varphi(\tau)$ can be isotoped to be embedded in $A$ and transverse to the ties. The restriction of the collapsing map $F: A \rightarrow \tau$ to $\varphi(\tau)$ is a carrying map. We may assume that no switch of $\varphi(\tau)$ is mapped by $F$ to a switch of $\tau$.

Let $v$ be an endpoint of the large branch $e$. A cutting arc for $\varphi(\tau)$ and $v$ is an embedded arc $\gamma:[0, d] \rightarrow F^{-1}(e)$ beginning at $\gamma(0)=v$ which is transverse to the ties of $A$, which is disjoint from $\varphi(\tau)$ except possibly at its endpoints and such that the length of $F(\gamma) \subset e$ is maximal among arcs with these properties. The maximality condition implies that either $F(\gamma[0, d])=e$, that is, $\gamma$ crosses through the foliated rectangle $F^{-1}(e)$, or that $\gamma(d)$ is a switch of $\varphi(\tau)$ contained in the interior of $F^{-1}(e)$, and the component of $S-\varphi(\tau)$ which contains $v$ has a cusp at $\gamma(d)$. In particular, since $\varphi$ preserves the combinatorics by assumption, if $\gamma(d)$ is a switch contained in the interior of $F^{-1}(e)$, then we have $\gamma(d)=\varphi(v)$ and $\gamma(d)$ is an endpoint of the large branch $\varphi(e)$.

By Lemma A. 2 of [H09], if $\varphi(\tau)$ is not carried by a split of $\tau$ at $e$, then the following holds true. Let $v, v^{\prime}$ be the endpoints of $e$ and let $\gamma:[0, d] \rightarrow F^{-1}(e)$, $\gamma^{\prime}:\left[0, d^{\prime}\right] \rightarrow F^{-1}(e)$ be the cutting arcs for $\varphi(\tau)$ and $v, v^{\prime}$. Then $\gamma(d), \gamma^{\prime}\left(d^{\prime}\right)$ are switches of $\varphi(\tau)$ contained in the interior of $F^{-1}(e)$, and there is a trainpath $\rho:[0, m] \rightarrow \varphi(\tau) \cap F^{-1}(e)$ connecting $\rho(0)=\gamma(d)$ to $\rho(1)=\gamma^{\prime}\left(d^{\prime}\right)$.

Since $\varphi$ preserves the combinatorics, it follows from the above discussion that the trainpath $\rho[0, m]$ consists of the single large branch $\varphi(e)$. Then a collision $\beta$ of $\tau$ at $e$, that is, a split followed by removal of the diagonal, carries $\varphi(\beta)$. As a consequence, this collision is a train track expansion of $\varphi$.

On the other hand, the combinatorial type of $\beta$ does not coincide with the combinatorial type of $\tau$. More precisely, $\beta$ does not carry any geodesic lamination whose topological type equals the topological type of the support of the horizontal measured geodesic lamination of $q$. Together this is a contradiction to the beginning of this proof.

To summarize, we showed so far that if $\tau \in \mathcal{E}$ then so is any $\mu$-split of $\tau$. To complete the proof of the lemma, it now suffices to show the following. Let $\eta \in \mathcal{L} \mathcal{T}(\varphi)$ be arbitrary; then there exists $\tau \in \mathcal{E}$ such that $\eta \prec \tau$. Namely, by [PH92], in this case $\tau$ can be connected to $\eta$ by a splitting and shifting sequence, and since $\eta$ carries $\mu$, by induction and what we have established so far, any train track in the sequence is contained in $\mathcal{E}$.

For $\tau \in \mathcal{E}$ we have $\varphi^{-k+1}(\tau) \prec \varphi^{-k}(\tau)$ and hence $\varphi^{-k}(\tau) \in \mathcal{E}$ for all $k$. By invariance under the action of $\Gamma$, it thus suffices to show that there is some $k>0$ such that $\varphi^{k} \eta \prec \tau$. By Lemma 3.2 of [H09], this follows if for a complete finite volume hyperbolic metric on $S$, a given number $\epsilon>0$ and all sufficiently large $k$, the train track $\varphi^{k} \eta$ is $a$-long and $\epsilon$-follows $\mu$ where $a>0$ only depends on the hyperbolic metric. That this is indeed the case was established in Lemma 5.2.

For a component $\tilde{\mathcal{Q}}$ of a stratum $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-n, p\right)$ we write as before $\tau \in$ $\mathcal{L T}(\tilde{\mathcal{Q}})$ if $\tau \in \mathcal{L} \mathcal{T}\left(m_{1}, \ldots, m_{\ell} ;-n, p\right)$ and if moreover the set $\mathcal{Q}(\tau) \subset \tilde{\mathcal{Q}}(S)$ is contained in the closure of $\tilde{\mathcal{Q}}$. Denote by $\operatorname{Stab}(\tilde{\mathcal{Q}})$ the stabilizer of $\tilde{\mathcal{Q}}$ in $\operatorname{Mod}(S)$.

Lemma 5.4. For every component $\mathcal{Q}$ of a stratum there exists a number $\kappa=$ $\kappa(\mathcal{Q})>0$ with the following property. If $\tau, \sigma \in \mathcal{L} \mathcal{T}(\tilde{\mathcal{Q}})$ and if $\sigma \prec \tau$ then there exists some $\varphi \in \operatorname{Stab}(\tilde{\mathcal{Q}})$ such that $\varphi(\tau) \prec \sigma$ and that $\sigma$ can be connected to $\varphi(\tau)$ by a shifting and splitting sequence of length at most $\kappa$.

Proof. Let $\tau, \sigma \in \mathcal{L} \mathcal{T}(\tilde{\mathcal{Q}})$ with $\sigma \prec \tau$. Then $\mathcal{Q}(\tau) \cap \mathcal{Q}(\sigma)$ contains an open subset of $\tilde{\mathcal{Q}}$. Since cotangent lines of axes of pseudo-Anosov mapping classes are dense in $\tilde{\mathcal{Q}}$, there exists a pseudo-Anosov mapping class $\varphi \in \operatorname{Stab}(\tilde{\mathcal{Q}})$ and a differential $q \in \mathcal{Q}(\tau) \cap \mathcal{Q}(\sigma)$ on the contangent line of an axis of $\varphi$. Lemma 5.2 and Lemma 3.2 of [H09] together show that there exists some $k \geq 1$ such that $\varphi^{k}(\sigma) \prec \tau$. Since the number of $\operatorname{Mod}(S)$-orbits of train tracks in $\mathcal{L T}(\tilde{\mathcal{Q}})$ is finite, by invariance this suffices for the proof of the lemma.

Let $\mathcal{T}(S)$ be the Teichmüller space of $S$. The translation length of a pseudoAnosov element $\varphi \in \operatorname{Mod}(S)$ is the minimal Teichmüller distance between a point $x \in \mathcal{T}(S)$ and its image under $\varphi$. The translation length of $\varphi$ only depends on the conjugacy class of $\varphi$, and it equals the length of the periodic orbit of the Teichmüller flow which is the projection of the cotangent line of an axis of $\varphi$ to the moduli space of quadratic or abelian differentials.

Our goal is to construct periodic orbits in the thin part of a stratum of abelian or quadratic differentials by deforming periodic orbits in components of the principal boundary of the stratum. To implement this idea we evoke counting results in components of the principal boundary. We begin with the setup suitable for our goal.

The moduli space of abelian or quadratic differentials is the quotient of the Teichmüller space of abelian or quadratic differentials by the action of the mapping class group $\operatorname{Mod}(S)$ of $S$. To avoid some technical difficulties, we replace $\operatorname{Mod}(S)$ by a torsion free finite index subgroup and replace each component of a stratum by the corresponding finite cover in the sense of orbifolds. Such a component is a smooth manifold, and every periodic orbit for the Teichmüller flow in such a cover, again denoted by $\mathcal{Q}$, defines a nontrivial free homotopy class of the stratum. No two distinct such orbits defined the same free homotopy class.

There is a further finite cover of a component $\mathcal{Q}$ defined as follows. Number all zeros or poles for a differential $q \in \mathcal{Q}$ which occur with multiplicity greater than one in an arbitrary way. Number furthermore the horizontal separatrices of a differential $q$ at each singular point in a counterclockwise order. This construction defined a fiber bundle over $\mathcal{Q}$ with finite fiber. A point in the fiber over $q$ determines a numbering of the zeros and poles of $q$ and a horizontal separatrix coming out of each zero. If we denote by $\hat{\mathcal{Q}}$ a connected component of this fiber bundle, then $\hat{\mathcal{Q}} \rightarrow \mathcal{Q}$ is a finite connected covering. The Teichmüller flow $\Phi^{t}$ naturally lifts to a flow on this covering, and the flow preserves the lift of the Masur Veech measure. Furthermore, a periodic orbit for $\Phi^{t}$ in $\hat{\mathcal{Q}}$ is defined by a pseudo-Anosov mapping class which preserves the combinatorics and, in particular, fixes singular points. From now on we always denote by $\hat{\mathcal{Q}}$ this finite covering (in the orbifold sense) of the component $\mathcal{Q}$.

Call a point $q \in \hat{\mathcal{Q}}$ recurrent if for every neighborhood $U$ of $q$ there is a sequence of times $t_{i} \rightarrow \infty, s_{j} \rightarrow-\infty$ such that $\Phi^{t_{i}} q \in U, \Phi^{s_{j}} q \in U$. The set of such points has full measure for every $\Phi^{t}$-invariant probability measure on $\hat{\mathcal{Q}}$.

Denote by $\lambda$ the normalized lift of the Masur Veech measure on $\hat{\mathcal{Q}}$. Given a contractible set $U \subset \hat{\mathcal{Q}}$, a point $u \in U$ and a number $T \gg 0$ such that $\Phi^{T} u \in U$, a characteristic curve of the pseudo-orbit $\left(u, \Phi^{T} u\right)$ is a closed curve containing the orbit segment $\cup_{0 \leq t \leq T} \Phi^{t} u$ which is obtained by connecting $\Phi^{T} u$ to $u$ with an arc which is entirely contained in $U$. The following result is Proposition 4.5 of [H22].
Proposition 5.5. Let $q \in \hat{\mathcal{Q}}$ be a recurrent point. Then for every neighborhood $U$ of $q$ and for all $\delta>0, \eta>0$ there are contractible closed neighborhoods $Z_{1} \subset Z_{2} \subset U$ of $q$ and there is a Borel set $Z_{0} \subset Z_{1}$ and a number $R_{0}>0$ with the following properties.
(1) $\lambda\left(Z_{2}\right) \leq(1-\delta)^{-1} \lambda\left(Z_{0}\right)$.
(2) For some integer $m>1 / \delta$, a $\Phi^{t}$-orbit intersects $Z_{1}, Z_{2}$ in arcs of length $2 t_{0}<\eta / m$.
(3) The set

$$
Z_{3}=\cup_{-t_{0}(m-2) \leq t \leq t_{0}(m-2)} \Phi^{t} Z_{1}
$$

is contained in the interior of

$$
Z_{4}=\cup_{-t_{0} m \leq t \leq t_{0} m} \Phi^{t} Z_{2} \subset U .
$$

(4) Let $z \in Z_{0}$ and let $T>R_{0}$ be such that $\Phi^{T} z \in Z_{3}$. Then there exists a path connected set $B(z) \subset Z_{2}$ containing $z$, with

$$
\Phi^{T} B(z) \subset Z_{4} \text { and } \lambda(B(z)) \in\left[(1-\delta) e^{-h T} \lambda\left(Z_{1}\right),(1-\delta)^{-1} e^{-h T} \lambda\left(Z_{1}\right)\right]
$$

There is a periodic orbit $\gamma$ for $\Phi^{t}$ of length contained in $\left[T-m t_{0}, T+m t_{0}\right]$ such that for each $u \in B(z)$, the characteristic curve of the pseudo-orbit ( $u, \Phi^{T} u$ ) with endpoints in $Z_{4}$ determines the same component $\gamma(z, T)$ of $\gamma \cap Z_{4}$. If $u \in Z_{0}-B(z)$ and if $\Phi^{T} u \in Z_{3}$, then the arc $\gamma(u, T)$ is disjoint from $\gamma(z, T)$.
Remark 5.6. In the formulation of Proposition 4.5 of [H22], the point $q$ is required to be a good recurrent point. The additional constraint comes from the difficulty that a component of a stratum is not a manifold. In the finite manifold cover $\hat{\mathcal{Q}}$ of $\mathcal{Q}$, every point is good.

For a set $\mathcal{P} \mathcal{A}$ of conjugacy classes of pseudo-Anosov elements in $\operatorname{Mod}(S)$ and for $R>0$ let $n(\mathcal{P} \mathcal{A}, R)$ be the number of elements in $\mathcal{P} \mathcal{A}$ consisting of conjugacy classes of translation length at most $R$. We use Proposition 5.5 to establish a counting result for periodic orbits in components of strata with combinatorial constraints.
Proposition 5.7. For any component $\tilde{\mathcal{Q}}$ of the stratum $\tilde{\mathcal{Q}}\left(m_{1}, \ldots, m_{\ell} ;-m, p\right)$ there are numbers $c=c(\tilde{\mathcal{Q}})>0, R_{0}=R_{0}(\tilde{\mathcal{Q}})>0$ with the following properties. Let $\tau \in \mathcal{L} \mathcal{T}(\tilde{\mathcal{Q}})$ and let $\mathcal{P} \mathcal{A}(\tau, c)$ be the set of all conjugacy classes of pseudo-Anosov elements $\varphi \in \operatorname{Stab}(\tilde{\mathcal{Q}})$ with the following properties.
a) $\varphi(\tau) \prec \tau$.
b) The matrix $A(\varphi, \tau)$ is positive, and the ratios of the entries of $A(\varphi, \tau)$ are bounded from above by $c$.
c) $\varphi$ preserves the combinatorics.

Then we have

$$
n(\mathcal{P A}(\tau, c), R) \geq \frac{1}{\operatorname{ch}(\tilde{\mathcal{Q}}) R} e^{h(\tilde{\mathcal{Q}}) R}
$$

for all $R \geq R_{0}$.

Proof. Let $\tau \in \mathcal{L} \mathcal{T}(\tilde{\mathcal{Q}})$. We claim that we can find a pseudo-Anosov mapping class $\varphi \in \operatorname{Mod}(S)$ with the following properties.
(1) $\varphi(\tau) \prec \tau$.
(2) The image of any branch of $\varphi(\tau)$ under a carrying map $\varphi(\tau) \rightarrow \tau$ is surjective.
(3) $\varphi$ preserves the combinatorics.

To see that such a mapping class exists, note that we can always find a splitting sequence starting at $\tau$ whose endpoint $\eta$ has the property that the restriction of the carrying map to each branch of $\eta$ is onto $\tau$ (see [H09] for more information). By Lemma 5.4, there is some $\varphi \in \operatorname{Mod}(S)$ such that $\varphi(\tau) \prec \eta$, and this mapping class $\varphi$ has the required properties. Note that $\varphi$ is pseudo-Anosov by Lemma 5.1 since $A(\varphi, \tau)$ is defined by a positive matrix. By perhaps passing to a finite power, we may assume that $\varphi$ preserves the combinatorics.

Denote by $\mathcal{Q}_{0}(\tau)$ the set of quadratic differentials $q \in \mathcal{Q}(\tau)$ whose horizontal measured geodesic lamination deposits the weight one on $\tau$; then $\mathcal{Q}_{0}(\tau)$ is a local cross section for the Teichmüller flow on $\tilde{\mathcal{Q}}$. Let $\hat{\mathcal{U}} \subset \mathcal{Q}_{0}(\tau)$ be the set of all differentials $v \in \mathcal{Q}_{0}(\tau)$ whose horizontal measured geodesic lamination is carried by $\varphi(\tau)$ and whose vertical measured geodesic lamination is carried by the dual $\varphi^{-1}\left(\tau^{*}\right)$ of $\varphi^{-1}(\tau)$. By possibly replacing $\varphi$ by a positive power, we may assume that $\hat{\mathcal{U}} \subset \tilde{\mathcal{Q}}$, that is, no point in $\hat{\mathcal{U}}$ is contained in stratum of the boundary of $\tilde{\mathcal{Q}}$.

To be more precise, let $\mu, \nu$ be the attracting and repelling fixed point for the action of $\varphi$ on $\mathcal{P} \mathcal{M} \mathcal{L}$, respectively. If $v \in \hat{\mathcal{U}}$ then the horizontal measured geodesic lamination of $v$ is contained in $\varphi \mathcal{V}(\tau)$, and the vertical measured geodesic lamination is contained in $\varphi^{-1} \mathcal{V}^{*}(\tau)$. As $\cap_{k \geq 0} \varphi^{k} \mathcal{V}(\tau)$ equals the line whose projective class equals $\mu$, and $\cap_{k \geq 0} \varphi^{-k} \mathcal{V}^{*}(\tau)$ equals the line whose projective class equals $\nu$, and as $\tilde{\mathcal{Q}}$ is an open subset of $\mathcal{Q}(\tau)$, we obtain that by perhaps replacing $\varphi$ by $\varphi^{k}$ for a suitably chosen $k>0$, we may assume that $\hat{\mathcal{U}} \subset \tilde{\mathcal{Q}}$.

For every $\epsilon>0, \tilde{\mathcal{U}}(\epsilon)=\cup_{t \in(-\epsilon, \epsilon)} \Phi^{t} \hat{\mathcal{U}} \cap \tilde{\mathcal{Q}}$ is an open subset of $\tilde{\mathcal{Q}}$. By perhaps decreasing $\tilde{\mathcal{U}}(\epsilon)$ we may assume that $\tilde{\mathcal{U}}(\epsilon)$ projects homeomorphically to an open subset $\mathcal{U}(\epsilon)$ of the finite cover $\hat{\mathcal{Q}}$ of $\mathcal{Q}$. The unit cotangent line of the axis of the mapping class $\varphi$ descends to a periodic orbit of $\Phi^{t}$ in $\hat{\mathcal{Q}}$.

Let $q \in \mathcal{U}(\epsilon)$ be a point on the periodic orbit defined by $\varphi$. Choose an open neighborhood $V \subset \mathcal{U}(\epsilon)$ of $q$ which is sufficiently small that the following holds true. Let $T>0$ be the period of the periodic point $q$. If $w \in V$ then the $\Phi^{t}$ orbit of $w$ intersects $\mathcal{U}(\epsilon)$ in $\Phi^{T} w, \Phi^{2 T} w, \Phi^{-T} w$ and $\Phi^{-2 T} w$. Furthermore, there is an open tubular neighborhood $N \subset \hat{\mathcal{Q}}$ of the periodic orbit through $q$ which is homeomorphic to a ball bundle over a circle and such that for every $w \in V$, we have $\cup_{0 \leq t \leq T} \Phi^{T} w \in N, \cup_{-T \leq t \leq 0} \Phi^{t} w \in N$. Let $U=V \cap \Phi^{T} V \cap \Phi^{-T} V \subset \mathcal{U}(\epsilon)$. Then $U$ is an open neighborhood of $q$ in $\mathcal{U}(\epsilon)$.

Let $Z_{0} \subset Z_{1} \subset Z_{2} \subset U$ and $Z_{3} \subset Z_{4} \subset U$ and $R_{0}>0$ be as in Proposition 5.5. If $w \in Z_{0}$ and if $\Phi^{R} w \in Z_{3}$ for some $R>\max \left\{4 T, R_{0}\right\}$ then Proposition 5.5 shows that the pseudo-orbit ( $w, \Phi^{R} w$ ) determines a periodic orbit for $\Phi^{t}$, and this orbit defines up to conjugation a pseudo-Anosov mapping class $\psi$. We may choose a point $w(\psi) \in \mathcal{Q}_{0}(\tau)$ on the cotangent line of the axis of $\psi$. By the construction of $\hat{\mathcal{Q}}$, the mapping class $\psi$ preserves the combinatorics. Thus by Lemma 5.3, we have $\psi(\tau) \prec \tau$.

Consider the pseudo-orbit ( $\left.\Phi^{-T} w, \Phi^{T+R} w\right)$ with endpoints in $V$. Let $\beta \subset \tilde{\mathcal{Q}}$ be a lift of this pseudo-orbit to $\tilde{\mathcal{Q}}$ and let as before $P: \tilde{\mathcal{Q}}(S) \rightarrow \mathcal{T}(S)$ be the canonical projection. Up to replacing all mapping classes by conjugates and renaming, by the
choice of $V$, the mapping class $\varphi \circ \psi \circ \varphi$ maps $P \beta(-T)$ into a small neighborhood of $P \beta(R+T)$. Up to adjusting the set $N$ which entered the definition of the open set $U$ and taking into account that a torsion free finite index subgroup of $\operatorname{Mod}(S)$ acts properly discontinuously and freely on $\mathcal{T}(S)$, the mapping class $\varphi \circ \psi \circ \varphi$ is unique with this property. Moreover, the translation length of $\varphi \circ \psi \circ \varphi$ does not exceed $R+2 T+\chi$ where $\chi>0$ is a constant only depending on the choice of $U$. Namely, the translation length of $\varphi \circ \psi \circ \varphi$ is the infimum of the Teichmüller distance between a point in $\mathcal{T}(S)$ and its image under $\varphi \circ \psi \circ \varphi$, and this distance does not exceed $2 T+R+\chi$ for a universal constant $\chi$ by the above discussion.

The map $\psi$ has $\tau$ as train track expansion, and the same holds true for $\varphi$. Concatenation shows that $\varphi \circ \psi \circ \varphi$ has $\tau$ as train track expansion as well. We claim that there is a number $c>0$ not depending on $\psi$ such that the ratios of the entries of the matrix

$$
A=A(\varphi, \tau) A(\psi, \tau) A(\varphi, \tau)
$$

are bounded from above by $c$.
Namely, let $\ell>0$ be the maximum of the ratios of the entries of the matrix $A(\varphi, \tau)$. Then up to a factor of at most $\ell$, the entries in each line of the matrix $A(\psi, \tau) A(\varphi, \tau)$ coincide with a fixed multiple of the sum of the entries of the matrix $A(\psi, \tau)$ in the same line. In particular, since $A(\psi, \tau)$ can not have a line all of whose entries vanish, the matrix $A(\psi, \tau) A(\varphi, \tau)$ is positive, and the ratios of its entries in a fixed line are bounded from above by $\ell$.

Similarly, up to a factor of at most $\ell$, the entries in each row of the matrix $A$ coincide with a fixed multiple of the sum of the entries of $A(\psi, \tau) A(\varphi, \tau)$ in the same row. By the discussion in the previous paragraph, this implies that the ratios of the entries of the positive matrix $A$ are bounded from above by $\ell^{2}$. Lemma 5.1 shows that $\varphi \circ \psi \circ \varphi$ is pseudo-Anosov. Furthermore, we have $\varphi \circ \psi \circ \varphi(\mathcal{V}(\tau)) \subset \varphi(\mathcal{V}(\tau)$ and $\varphi^{-1} \psi^{-1} \varphi^{-1} \mathcal{V}^{*}(\tau) \subset \varphi^{-1} \mathcal{V}^{*}(\tau)$. Using once more the Brouwer fixed point theorem, this implies that the attracting fixed point for the action of $\varphi \circ \psi \circ \varphi$ on $\mathcal{P} \mathcal{M} \mathcal{L}$ is contained in the projectivization of $\varphi \mathcal{V}(\tau)$, and the repelling fixed point is contained in the projectivization of $\varphi^{-1} \mathcal{V}^{*}(\tau)$. As a consequence, the cotangent line of an axis of $\varphi \circ \psi \circ \varphi$ is contained in $\tilde{\mathcal{Q}}$. Thus any mapping class of the form $\varphi \circ \psi \circ \varphi$ constructed from a pseudo-orbit ( $u, \Phi^{R} u$ ) with $R>R_{0}, u \in Z_{0}$ and $\Phi^{R} u \in Z_{3}$ has properties (a) -(c) stated in the proposition.

We are left with counting the number of such periodic orbits. Put $h=h(\tilde{\mathcal{Q}})$. Note that by Proposition 5.5, a point $w \in Z_{0}$ with $\Phi^{R} w \in Z_{3}$ determines a subsegment of a periodic orbit passing through $Z_{4}$ of uniform length $2 t_{0}$, and points which determine the same segment are contained in a subset $B \subset Z_{2}$ with $\Phi^{R}(B) \subset Z_{4}$ whose measure is bounded from above by $(1-\delta)^{-1} e^{-h R} \lambda\left(Z_{1}\right)$. Thus each such set accounts for at most the volume $(1-\delta)^{-1} e^{h R} \lambda\left(Z_{1}\right)$ from $\lambda\left(\Phi^{R} Z_{0} \cap Z_{3}\right)$ and hence there are at least $(1-\delta) e^{h R} \lambda\left(\Phi^{R} Z_{0} \cap Z_{3}\right) / \lambda\left(Z_{1}\right)$ such orbits. As a consequence, to complete the proof of the proposition it suffices to show that for sufficiently large $R \gg T$, the Masur Veech measure of $\Phi^{R} Z_{0} \cap Z_{3}$ is bounded from below by a universal constant. Namely, as the length of an intersection component of an orbit with the set $Z_{3}$ equals a fixed number $2 t_{0}>0$, a periodic orbit of length $R$ can not intersect $Z_{3}$ in more than $R / 2 t_{0}$ components.

Now the normalized Masur Veech measure $\lambda$ is mixing for the Teichmüller flow $\Phi^{t}$ on $\hat{\mathcal{Q}}$ and hence for a given number $\epsilon>0$ we have

$$
\lambda\left(\Phi^{R} Z_{0} \cap Z_{3}\right) \geq(1-\epsilon) \lambda\left(Z_{0}\right) \lambda\left(Z_{3}\right)
$$

for all sufficiently large $R$, say for all $R \geq R_{0}$. Then the statement of the proposition holds true for $c\left((1-\epsilon) \lambda\left(Z_{0}\right) \lambda\left(Z_{3}\right)\right)$ and all $R \geq R_{0}$.

## 6. Periodic orbits in the thin part of strata

In this section we only consider strata of differentials without marked regular points. This means that strata of abelian differentials are defined on surfaces without punctures, and a puncture of a surface $S$ corresponds to a simple pole of a quadratic differential. Our main goal is to prove Theorem 1 from the introduction. Strata of differentials with marked regular points will be used in the proof.

Let $S$ be a surface of genus $g \geq 0$ with $n \geq 0$ punctures and $3 g-3+n \geq 5$. Recall that a marked quadratic (or abelian) differential $q \in \tilde{\mathcal{Q}}(S)$ defines a marked singular euclidean metric of area one on the surface $S$ with singularities at the zeros and at the poles of the differential. There is a unique finite volume complete hyperbolic metric on $S$ for the underlying conformal structure.

Define the systole of this hyperbolic metric to be the smallest length of a simple closed geodesic. We write $\mathcal{T}(S)_{\epsilon} \subset \mathcal{T}(S)$ for the space of all marked complete hyperbolic metrics on $S$ of finite volume whose systole is at least $\epsilon$. The mapping class group $\operatorname{Mod}(S)$ acts properly and cocompactly on $\mathcal{T}(S)_{\epsilon}$.

Given a marked abelian or quadratic differential $q \in \tilde{\mathcal{Q}}(S)$, the $q$-length of an essential simple closed curve $c$, that is, a simple closed curve which is not contractible and not freely homotopic into a puncture, is defined to be the infimum of the lengths with respect to the singular euclidean metric of any curve which is freely homotopic to $c$.

The following observation is a fairly easy consequence of invariance under the action of the mapping class group and cocompactness. A much stronger and more precise version is due to Rafi [R14]. For its formulation, let

$$
P: \tilde{\mathcal{Q}}(S) \rightarrow \mathcal{T}(S)
$$

be the canonical projection which associates to a marked area one quadratic (or abelian) differential its underlying marked hyperbolic metric.

Lemma 6.1. For every $\chi>0$ there is a number $\delta=\delta(S, \chi)>0$ with the following property. Let $q \in \tilde{\mathcal{Q}}(S)$ and assume that there is an essential simple closed curve on $S$ of $q$-length at most $\delta$; then $P q \notin \mathcal{T}(S)_{\chi}$.

Proof. By the collar lemma for hyperbolic surfaces, every cusp of a hyperbolic surface has a standard embedded neighborhood. Such a neighborhood is homeomorphic to a punctured disk. Up to a geometric adjustment of the neighborhoods, the hyperbolic distance between any two such cusp neighborhoods is bounded from below by a universal positive constant.

By Lemma 3.3 of [Mi94], for every $\chi>0$ there is a number $L=L(\chi)>1$ such that for every $q \in \tilde{\mathcal{Q}}(S)$ with $P q \in \mathcal{T}(S)_{\chi}$, the singular euclidean metric defined by $q$ is $L$-bilipschitz equivalent to the hyperbolic metric on the compact set $K$. By the choice of the standard neighborhoods of the cusps, every essential simple closed curve on $S$ either is contained in $K$, or it intersects the set $K$ in a union of arcs whose hyperbolic length is bounded from below by some fixed number $c \in(0, \chi)$ only depending on the topological type of $S$ and on $\chi$. Then the $q$-length of any essential simple closed curve on $S$ is not smaller than $c / L$. This shows the lemma.

Remark 6.2. Lemma 6.1 does not state that an essential simple closed curve of short $q$-length has short hyperbolic length for the hyperbolic metric underlying the complex structure of $q$. In fact, this is not true in general [R14].

For $\chi>0$ define

$$
\tilde{\mathcal{Q}}(S)_{\chi}=\left\{q \in \tilde{\mathcal{Q}}(S) \mid P q \in \mathcal{T}(S)_{\chi}\right\}
$$

The sets $\tilde{\mathcal{Q}}(S)_{\chi}$ are invariant under the action of $\operatorname{Mod}(S)$ on $\tilde{\mathcal{Q}}(S)$. Their projections

$$
\mathcal{Q}(S)_{\chi}=\tilde{\mathcal{Q}}(S)_{\chi} / \operatorname{Mod}(S) \subset \mathcal{Q}(S)
$$

to $\mathcal{Q}(S)$ are compact and satisfy $\mathcal{Q}(S)_{\chi} \subset \mathcal{Q}(S)_{\delta}$ for $\chi>\delta$ and $\cup_{\chi>0} \mathcal{Q}(S)_{\chi}=\mathcal{Q}(S)$.

For a component $Q$ of a stratum of quadratic or abelian differentials and for $\chi>0, R>0$ let

$$
n(\mathcal{Q}, R)^{<\chi} \geq 0
$$

be the number of periodic orbits for the Teichmüller flow $\Phi^{t}$ of length at most $R$ which are contained in $\mathcal{Q}-\mathcal{Q}(\chi)$. Our goal is to show that for every $\chi>0$ and for sufficiently large $R$ depending on $\chi$, the number of such orbits is at least $d e^{(h(\mathcal{Q})-1) R} / R$ for a number $d=d(\mathcal{Q}, \chi)>0$.

The strategy is to use a train track $\tau$ in special form for $\mathcal{Q}$ as a combinatorial tool to lift periodic orbits on components of the principal boundary of $\mathcal{Q}$ into the thin part of $\mathcal{Q}$. We recall the important properties of such a train track $\tau$.
(1) $\tau$ contains two clean vertex cycles $c_{1}, c_{2}$. Either the surface $S-c_{1}-c_{2}$ is connected, or it consists of a connected component $N$ which is different from a sphere with at most three holes or a torus with at most one hole and one or two additional components which are twice punctured disks.
(2) Let $S_{0}$ (or $S_{1}, S_{2}$ ) be the surface which is obtained from the component of $S-c_{1}-c_{2}$ (or of $S-c_{1}, S-c_{2}$ ) different from a sphere with at most three holes by replacing the boundary circles by punctures. Then the graph $\sigma_{0}$ on $S_{0}$ (or $\sigma_{1}, \sigma_{2}$ on $S_{1}, S_{2}$ ) obtained by removing from $\tau$ all branches which are incident on a switch in $c_{1} \cup c_{2}$ (or incident on a switch in $c_{1}, c_{2}$ ) is a connected large train track, either on $S_{i}$ or on the surface obtained from $S_{i}$ by removing one or two of the newborn punctures.

We showed in Section 4 that for $i=1,2$ there is a component $\mathcal{Q}_{i}$ of a stratum of differentials for the surface $S_{i}$ (possibly with marked regular points or with some of the newborn punctures removed) such that $\sigma_{i} \in \mathcal{L} \mathcal{T}\left(\tilde{\mathcal{Q}}_{i}\right)$ where $\tilde{\mathcal{Q}}_{i}$ is the preimage of $\mathcal{Q}_{i}$. The component $\mathcal{Q}_{i}$ is determined as follows.

Case 1: $n \geq 2$ and the simple closed curve $c_{i}$ is separating.
Then $c_{i}$ bounds a twice punctured disk. There is a simple closed curve $\alpha$ embedded in the train track $\sigma_{i}$ (however with cusps) which is freely homotopic to $c_{i}$. Viewing $\sigma_{i}$ as a train track on the surface $S_{i}$, the curve $\alpha$ encloses the added marked point on $S_{i}$ (which replaces the circle $c_{i}$ ). We require that a differential in $\mathcal{Q}_{i}$ has

- a simple pole at this marked point in the case that $\alpha$ has a single cusp,
- a regular point if $\alpha$ is a bigon or
- a zero if $\alpha$ has at least three cusps (in which case we remove the marked point).

Let $h\left(\mathcal{Q}_{i}\right)$ be the dimension of the complex algebraic orbifold containing $\mathcal{Q}_{i}$ as a real hypersurface. We have $h\left(\mathcal{Q}_{i}\right)=h(\mathcal{Q})-2$. A degeneration of differentials in $\mathcal{Q}$ to a differential in $\mathcal{Q}_{i}$ corresponds to a shrinking half-pillowcase.

Case 2: The simple closed curve $c_{i}$ is non-separating
Then $S-c_{i}$ has two boundary components. Each of these boundary components is contained in a complementary component of $\sigma_{i}$ (these components may coincide). If there are two distinct such components then each of these components is an annulus. One boundary component of such an annulus is the curve $c_{i}$, and the second boundary component $\alpha$ is contained in $\sigma_{i}$. As before, put a simple pole on the puncture of $S_{i}$ which replaces the curve $c_{i}$ if $\alpha$ has a single cusp, view the puncture of $S_{i}$ as a regular marked point if $\alpha$ is a bigon, and remove the puncture and view is as a zero if $\alpha$ contains at least three cusps. Once again, $h\left(\mathcal{Q}_{i}\right)=h(\mathcal{Q})-2$. A degeneration of differentials in $\mathcal{Q}$ to a differential in $\mathcal{Q}_{i}$ corresponds to a shrinking cylinder. If both boundary components of $S-c_{i}$ are contained in the same complementary component of $\sigma_{i}$ then we proceed in exactly the same way.

Our goal is to use the growth estimate from Section 5 for periodic orbits in $\mathcal{Q}_{i}$ which are defined by pseudo-Anosov mapping classes preserving the combinatorics, with train track expansion $\sigma_{i}$. Such mapping classes can be lifted to reducible mapping classes on $S$. We then concatenate suitably chosen such lifts to a pseudoAnosov element for $S$ with train track expansion $\tau$. We show that this can be accomplished in such a way that the axis of this pseudo-Anosov mapping class is contained in $\mathcal{T}(S)-\mathcal{T}(S)_{\chi}$ where $\chi>0$ is an arbitrarily chosen constant.

Recall from Proposition 5.7 the counting constants $d_{i}=d\left(\mathcal{Q}_{i}\right)>0(i=0,1,2)$. Choose elements $\varphi_{i} \in \operatorname{Mod}\left(S_{i}\right)$ with the properties stated in Proposition 5.7 for the numbers $d_{i}$. Each of these mapping classes has $\sigma_{i}$ as a train track expansion and fixes each of the complementary components of $\sigma_{i}$. These mapping classes can be lifted to the mapping class group of the bordered surface $S-c_{i}$ (or of the bordered
surface $S-\left(c_{1} \cup c_{2}\right)$ if $\left.i=0\right)$ which consists of isotopy classes of diffeomorphisms fixing the curve $c_{i}$ (or $c_{1} \cup c_{2}$ ) pointwise. Such an extension in turn can be viewed as a reducible element of $\operatorname{Mod}(S)$.

For a bordered surface $X$ with a distinguished boundary component $c$ and the surface $X^{\prime}$ obtained from $X$ by replacing $c$ by a puncture, there exists an exact sequence

$$
1 \rightarrow \mathbb{Z} \rightarrow \operatorname{Mod}(X) \rightarrow \operatorname{Mod}\left(X^{\prime}\right) \rightarrow 1
$$

where the infinite cyclic group $\mathbb{Z}$ is the group of Dehn twists about the boundary curve $c$. Thus a choice of an extension of $\varphi_{i}$ as described in the previous paragraph is by no means unique. However, as $\varphi_{i}$ is required to preserve the combinatorics, we can use the train track $\tau$ to construct from $\varphi_{i}$ a unique such extension.

Lemma 6.3. There is a natural choice of an extension of $\varphi_{i}$ to an element of $\operatorname{Mod}(S)$, again denoted by $\varphi_{i}$, which fulfills $\varphi_{i}(\tau) \prec \tau$ and is such that $\varphi_{i}(\tau)$ is obtained from $\tau$ by a splitting and shifting sequence of $\tau$ which does not involve $a$ split at any branch in $\tau-\sigma_{i}$.

Proof. Since $\varphi\left(\sigma_{i}\right) \prec \sigma_{i}$, there exists a splitting and shifting sequence connecting $\sigma_{i}$ to $\varphi_{i}\left(\sigma_{i}\right)$. Our goal is to extend this sequence of a splitting and shifting sequence of $\tau$.

By construction, there are at most two small branches $b_{1}, b_{2}$ of $\tau$ which connect $\sigma_{i}$ to the primitive vertex cycle $c_{i}$ of $\tau$. Let $e$ be a large branch of $\sigma_{i}$. If none of the two small branches $b_{1}, b_{2}$ of $\tau$ has an endpoint in $e$, then $e$ is a large branch of $\tau$ and a split of $\sigma_{i}$ at $e$ can be viewed as a split of $\tau$ at $e$. The split track contains $c_{i}$ as a clean vertex cycle.

If an endpoint of one of the branches $b_{1}, b_{2}$ is contained in $e$ then there is a modification of $\tau$ by a sequence of shifts and splits at branches of $\tau$ contained in $e$ such that the modified train track $\tau^{\prime}$ is large, it contains $e$ as a large branch and the images of $b_{1}, b_{2}$ in $\tau^{\prime}$ are small branches in $\tau^{\prime}$. These splits and shifts move an endpoint of $b_{i}$ contained in $e$ across an endpoint of $e$. In this way we obtain from a splitting and shifting sequence connecting $\sigma_{i}$ to $\varphi_{i}(\sigma)$ a splitting and shifting sequence which connects $\tau$ to a train track $\hat{\varphi}_{i}(\tau)$ containing $c_{i}$ as a primitive vertex cycle and such that the restriction of a carrying map $\hat{\varphi}_{i}(\tau) \rightarrow \tau$ to $\hat{\varphi}_{i}\left(\sigma_{i}\right)$ coincides with the carrying map $\varphi_{i}\left(\sigma_{i}\right) \rightarrow \sigma_{i}$.

Since the positions of the endpoints of the small branches $b_{i}$ are moved along sides of complementary components of $\sigma_{i}$, the train track $\hat{\varphi}_{i}(\tau)$ may not be isomorphic to $\tau$. However, since $\varphi$ preserves the combinatorics by assumption, it maps the complementary component $C$ of $\sigma_{i}$ which contains the curve $c_{i}$ to the complementary component $\varphi_{i}(C)$ of $\varphi_{i}\left(\sigma_{i}\right)$ containing $c_{i}$, and it maps a side of $C$ containing an endpoint of one of the small branches $b_{i}$ to the side of $\varphi_{i}(C)$ containing the endpoint of the same branch (with the canonical identification). Thus there is a unique way to slide the endpoints of $b_{1}, b_{2}$ along these sides in such a way that the extension of $\varphi_{i}\left(\sigma_{i}\right)$ constructed in this way is isomorphic to $\tau$ with an isomorphism (that is, a mapping class), which preserves $c_{i}$ and restricts to $\varphi_{i}$ on $\sigma_{i}$. This is what we wanted to show.

In the sequel we will always identify the map $\varphi_{i}$ with this particular extension, that is, we view $\varphi_{i}$ as a reducible mapping class for $S$ with $\varphi_{i}(\tau) \prec \tau$ and such that there is a carrying map $\varphi_{i}(\tau) \rightarrow \tau$ which is the identity on $\tau-\sigma_{i}$. The map $\varphi_{0}$ is viewed as a reducible element of $\operatorname{Mod}(S)$ fixing $c_{1} \cup c_{2}$ pointwise.

In the statement of the following lemma, o means composition from the left, that is, $a \circ b$ represents the mapping class obtained by applying $b$ first followed by an application of $a$. This amounts to using the notational convention that $(\varphi \circ \psi)(\tau)$ is the train track obtained from $\tau$ by first changing the marking with $\psi$ and afterwards with $\varphi$.

Lemma 6.4. For every $k>0$ the mapping class

$$
\zeta(k)=\left(\varphi_{0}^{k} \circ \varphi_{2} \circ \varphi_{0}^{k}\right) \circ\left(\varphi_{0}^{k} \circ \varphi_{1} \circ \varphi_{0}^{k}\right)
$$

is pseudo-Anosov.

Proof. Note first that $\zeta(k)(\tau) \prec \tau$ for all $k$. Namely, by assumption on $\varphi_{i}$ we have $\varphi_{i}(\tau) \prec \tau$ (see the discussion) in Lemma 6.3 and its proof) and hence by invariance of the carrying relation under the action of the mapping class group and induction, we conclude that

$$
\zeta(k)(\tau)=\left(\varphi_{0}^{k} \circ \varphi_{2} \circ \varphi_{0}^{2 k} \circ \varphi_{1} \circ \varphi_{0}^{k}(\tau)\right) \prec \tau
$$

Let as before $m$ be the number of branches of $\tau$ and let $\mathbb{R}^{m}$ be the real vector space with basis the branches of $\tau$. Write $\mathbb{R}^{m}=\mathbb{R}^{\ell_{0}} \oplus \mathbb{R}^{\ell_{1}} \oplus \mathbb{R}^{\ell_{2}}$ where $\mathbb{R}^{\ell_{0}}$ is the real vector space with basis the branches of $\tau$ contained in $\sigma_{0}$ and where for $i=1,2$ the vector space $\mathbb{R}^{\ell_{i}}$ has a basis consisting of the branches of $\tau-\sigma_{i+1}$. Then $\mathbb{R}^{\ell_{0}} \oplus \mathbb{R}^{\ell_{i}}$ is the real vector space with basis the branches of $\sigma_{i}$.

Since $\zeta(k)^{2}$ is pseudo-Anosov if and only if this is true for $\zeta(k)$, by Lemma 5.1, it suffices to show that the linear self map

$$
A\left(\zeta(k)^{2}, \tau\right)=A\left(\varphi_{0}^{k}, \tau\right) \cdots A\left(\varphi_{1}, \tau\right) A\left(\varphi_{0}^{k}, \tau\right)
$$

of $\mathbb{R}^{m}$ is given with respect to the above basis by a positive matrix. This is equivalent to stating that a carrying map $\zeta(k)^{2} \tau \rightarrow \tau$ maps every branch of $\zeta(k)^{2} \tau$ onto $\tau$.

For $i=1,2$ define

$$
A_{i}=A\left(\varphi_{0}^{k} \circ \varphi_{i} \circ \varphi_{0}^{k}, \tau\right)
$$

The linear map $A_{1}$ preserves the decomposition $\mathbb{R}^{m}=\mathbb{R}^{\ell_{0}+\ell_{1}} \oplus \mathbb{R}^{\ell_{2}}$ and therefore it is given by a matrix in block form. The square matrix which corresponds to the restriction of $A_{1}$ to $\mathbb{R}^{\ell_{0}+\ell_{1}}$ is positive, and the square matrix which corresponds to the action on $\mathbb{R}^{\ell_{2}}$ is the identity. The same holds true for the linear map $A_{2}$, with the roles of $\ell_{1}$ and $\ell_{2}$ exchanged.

As a consequence, the image of a basis vector in $\mathbb{R}^{\ell_{0}+\ell_{1}}$ under the matrix $A_{2} A_{1}$ is a positive vector in $\mathbb{R}^{m}$. Similarly, the image of a basis vector in $\mathbb{R}^{\ell_{0}+\ell_{2}}$ under the matrix $A_{1} A_{2}$ is a positive vector in $\mathbb{R}^{m}$ and hence the matrix $A_{2} A_{1} A_{2} A_{1}$ is indeed positive. This is what we wanted to show.

For the proof of Theorem 1, we count periodic orbits in the $\chi$-thin part of a component $\mathcal{Q}$ of a stratum by counting orbits which are defined by pseudo-Anosov classes of the form described in Lemma 6.4 where we let $\varphi_{1}$ vary, and we fix $\varphi_{0}, \varphi_{2}$ and some large enough $k$. The next lemma shows that indeed, such mapping classes give rise to periodic orbits in the thin part of moduli space. Recall that the axis of a pseudo-Anosov mapping class $\varphi$ is the Teichmüller geodesic which is invariant under $\varphi$.

Lemma 6.5. For every $\chi>0$ there exists a number $k_{0}=k_{0}(\chi)>0$ with the following property. If $k \geq k_{0}$ then the axis of each of the mapping classes $\zeta(k)$ as in Lemma 6.4 does not intersect $\mathcal{T}(S)_{\chi}$.

Proof. By Lemma 6.1, it suffices to show the following. For every $\epsilon>0$, there exists a number $k=k(\epsilon)>0$ such that if $k>k(\epsilon)$ and if $q \in \tilde{\mathcal{Q}}$ is contained in the cotangent line of an axis of $\zeta(k)$, then there exists an essential simple closed curve $c$ on $S$ of $q$-length at most $\epsilon$.

Using the notations from Lemma 6.4, the carrying map $\varphi_{i}(\tau) \rightarrow \tau$ can be chosen in such a way that it maps each branch of $\varphi_{i}\left(\sigma_{i}\right)$ onto $\sigma_{i}$, and it induces the identity on $\tau-\sigma_{i}(i=0,1,2)$.

For a transverse measure $\mu \in \mathcal{V}(\tau)$ and a subset $B$ of the branches of $\tau$ denote by $\mu(B)$ the total mass deposited by $\mu$ on the branches in $B$. Clearly $\mu(B) \geq 0$ for all $\mu, B$.

By Proposition 5.7 and the construction, for $i=1,2$ there is a number $a_{i}>0$ with the following property. Let $\mu_{i}$ be a measured geodesic lamination on $S$ which is carried by $\varphi_{i}\left(\sigma_{i}\right)$ and which defines the transverse measure $\mu_{i} \in \mathcal{V}\left(\varphi_{i}\left(\sigma_{i}\right)\right) \subset$ $\mathcal{V}\left(\sigma_{i}\right) \subset \mathcal{V}(\tau)$; then $\mu_{i}\left(b_{1}\right) / \mu_{i}\left(b_{2}\right) \leq a_{i}$ for any two branches $b_{1}, b_{2}$ of $\sigma_{i}$.

This implies the existence of a number $a>0$ with the following property. Let $\mu$ be any measured geodesic lamination which is carried by $\varphi_{i}(\tau)$. Assume that the transverse measure on $\varphi_{i}(\tau)$ defined by $\mu$ is not supported in $\varphi_{i}(\tau)-\varphi_{i}\left(\sigma_{i}\right)$. Let $\hat{\mu}$ be the transverse measure on $\tau$ induced from $\mu$ by a carrying map $\varphi_{i}(\tau) \rightarrow \tau$; then

$$
\begin{equation*}
\hat{\mu}\left(b_{i}\right) \geq 2 a \hat{\mu}\left(\sigma_{i}\right) \tag{4}
\end{equation*}
$$

for every branch $b_{i}$ of $\sigma_{i}(i=1,2)$.
Let $\mathcal{V}_{0}(\tau) \subset \mathcal{V}(\tau)$ be the set of all transverse measures on $\tau$ of total mass one. Note that $\mathcal{V}_{0}(\tau)$ is naturally homeomorphic to the projectivization of $\mathcal{V}(\tau)$. For $\epsilon \in\left(0, \frac{1}{2}\right)$ and $i=0,1,2$ let $C_{i}(\tau, \epsilon)$ be the closed subset of $\mathcal{V}_{0}(\tau)$ containing all transverse measures $\nu$ with the following properties.
(1) $\nu\left(\tau-\sigma_{i}\right) \leq \epsilon$.
(2) For any branch $b_{i}$ of $\sigma_{i}<\tau$ we have $\nu\left(b_{i}\right) \geq a$.

Note that the set $C_{i}(\epsilon)$ is not empty by the above choice of the number $a$. Also, by the second property above, we have $\nu\left(\sigma_{0}\right) \geq 2 a$ for every $\nu \in C_{i}(\tau, \epsilon)$ and $i=1,2$ (recall that $\sigma_{0}$ contains at least two branches).

As in the proof of Lemma 6.4, let

$$
\mathbb{R}^{m}=\mathbb{R}^{\ell_{0}} \oplus \mathbb{R}^{\ell_{1}} \oplus \mathbb{R}^{\ell_{2}}
$$

be the vector space with basis the branches of $\tau$. The subspace $\mathbb{R}^{\ell_{0}}$ is spanned by the branches of $\tau$ contained in $\sigma_{0}$. A point $\nu \in C_{i}(\tau, \epsilon)(i=0,1,2)$ can be viewed as a non-negative vector $v(\nu) \in \mathbb{R}^{m}$ with the property that the coefficients of the basis elements in $\mathbb{R}^{\ell_{0}} \oplus \mathbb{R}^{\ell_{i}}$ are bounded from below by $a>0$ and that the sum of the coordinates equals one.

Denote as before by $A\left(\varphi_{i}, \tau\right)$ the linear map which describes the action of $\varphi_{i}$ on $\mathcal{V}(\tau)$, represented by a matrix with respect to the basis of $\mathbb{R}^{m}$ given by the branches of $\tau$. There is an induced action on $\mathcal{V}_{0}(\tau)$ by rescaling of the total mass; we denote this action by $\hat{A}\left(\varphi_{i}, \tau\right)$. We claim that there is a constant $u>0$ only depending on $\varphi_{0}$ such that for every $\epsilon>0$, for every $k \geq-u \log \epsilon$ and for $i=1,2$ we have

$$
\hat{A}\left(\varphi_{0}, \tau\right)^{k}\left(C_{i}(\tau, \epsilon)\right) \subset C_{0}(\tau, \epsilon)
$$

To show the claim note that the linear map $A\left(\varphi_{0}, \tau\right)$ preserves the decomposition $\mathbb{R}^{m}=\mathbb{R}^{\ell_{0}} \oplus \mathbb{R}^{\ell_{1}} \oplus \mathbb{R}^{\ell_{2}}$ and hence its matrix is in block form. The square matrix $A_{0}$ which defines the action on $\mathbb{R}^{\ell_{0}}$ is positive and integral, and the square matrix defining the action on $\mathbb{R}^{\ell_{1}} \oplus \mathbb{R}^{\ell_{2}}$ is the identity. Since $\ell_{0} \geq 2$, the entries of the matrix $A_{0}^{k}$ are bounded from below by $2^{k-1}$ and hence

$$
A\left(\varphi_{0}, \tau\right)^{k}(\nu)\left(\sigma_{0}\right) \geq 2^{k-1} \nu\left(\sigma_{0}\right) \text { for all } \nu \in \mathcal{V}(\tau)
$$

Now if $\nu \in C_{i}(\tau, \epsilon)$ then we have $\nu\left(\sigma_{0}\right) \geq 2 a$ and therefore

$$
A\left(\varphi_{0}, \tau\right)^{k}(\nu)\left(\sigma_{0}\right) \geq a 2^{k}
$$

moreover $A\left(\varphi_{0}, \tau\right)^{k}(\nu)\left(\tau-\sigma_{0}\right) \leq 1-2 a$ for all $k \geq 1$. Thus if

$$
k \geq \hat{k}(\epsilon)=(\log (2(1-a))-\log (a \epsilon)) / \log (2)
$$

then it holds $A\left(\varphi_{0}, \tau\right)^{k}(\nu)\left(\sigma_{0}\right) \geq A\left(\varphi_{0}, \tau\right)^{k}(\nu)\left(\tau-\sigma_{0}\right) / \epsilon$ and therefore

$$
A\left(\varphi_{0}, \tau\right)^{k}(\nu) /\left(A\left(\varphi_{0}, \tau\right)^{k}(\nu)(\tau)\right) \in C_{0}(\tau, \epsilon)
$$

Together with the estimate (4), this shows $\hat{A}\left(\varphi_{0}, \tau\right)^{k}\left(C_{i}(\tau, \epsilon)\right) \subset C_{0}(\tau, \epsilon)$ for all $k \geq \hat{k}(\epsilon)$.

By definition of the maps $\varphi_{i}$, we also have $\hat{A}\left(\varphi_{i}, \tau\right)\left(C_{0}(\tau, \epsilon)\right) \subset C_{i}(\tau, \epsilon)$ for $i=$ 1, 2. Together we deduce the existence of a number $k(\epsilon) \sim-\log \epsilon>0$ such that for $k>k(\epsilon)$, the set $C_{0}(\tau, \epsilon)$ is invariant under the map which assigns to a measured geodesic lamination $0 \neq \mu \in \mathcal{V}(\tau)$ the normalized image of $\zeta(k) \mu \in \mathcal{V}(\zeta(k) \tau)$ under a carrying map $\mathcal{V}(\zeta(k) \tau) \rightarrow \mathcal{V}(\tau)$.

Let $\mathcal{Q}_{0}(\tau)$ be the set of all differentials $q \in \mathcal{Q}(\tau)$ with the property that the total mass deposited on $\tau$ by the horizontal measured geodesic lamination of $q$ equals one. Then if $k>k(\epsilon)$ and if $q \in \mathcal{Q}_{0}(\tau)$ is contained in the $\zeta(k)$-invariant flow line of the Teichmüller flow, with horizontal measured geodesic lamination $\mu$, then

$$
\mu \in C_{0}(\tau, \epsilon)
$$

Namely, the open cone $C_{0}(\tau, \epsilon) \subset \mathcal{V}_{0}(\tau)$ is non-empty and invariant under the action of the map $\zeta(k)$, viewed as a map $\mathcal{V}_{0}(\tau) \rightarrow \mathcal{V}_{0}(\tau)$. Since the closure of $C_{0}(\tau, \epsilon)$ is homeomorphic to closed a ball and we have $\zeta(k) C_{0}(\tau, \epsilon) \subset C_{0}(\tau, \epsilon)$, the
$\operatorname{map} \zeta(k)$ has a fixed point in $C_{0}(\tau, \epsilon)$ by the Brouwer fixed point theorem. As $\zeta(k)$ is pseudo-Anosov, the projectivization of this fixed point is the attracting fixed point for the action of $\zeta(k)$ on $\mathcal{P} \mathcal{M} \mathcal{L}$, that is, it equals the horizontal projective measured lamination of $q$.

The $q$-length of the simple closed curve $c_{i}$ is contained in the interval

$$
\left[\frac{1}{\sqrt{2}}\left(\iota\left(\mu, c_{i}\right)+\iota\left(\nu, c_{i}\right)\right), \iota\left(\mu, c_{i}\right)+\iota\left(\nu, c_{i}\right)\right]
$$

where $\mu, \nu$ is the horizontal and the vertical measured geodesic lamination of $q$, respectively [R14]. The intersection numbers $\iota\left(\mu, c_{i}\right), \iota\left(\nu, c_{i}\right)$ can be estimated as follows.

By Lemma 2.5 of [H06], we have

$$
\iota\left(\mu, c_{i}\right) \leq \mu\left(\tau-\sigma_{i}\right)
$$

and hence as $\mu \in C_{0}(\tau, \epsilon)$, it holds

$$
\begin{equation*}
\iota\left(\mu, c_{i}\right) \leq \epsilon \tag{5}
\end{equation*}
$$

The support of the vertical (or horizontal) measured geodesic lamination of a quadratic differential $q$ is invariant under the action of the Teichmüller flow $\Phi^{t}$, and its transverse measure scales with the scaling constant $e^{t / 2}$ (or with the scaling constant $e^{-t / 2}$ - note that this means that the length of the horizontal measured geodesic lamination is decrasing along a flow line of the Teichmüller flow). Recall that $\varphi_{0}^{k} \varphi_{1} \varphi_{0}^{k}\left(c_{1}\right)=c_{1}$ for all $k$. The estimate (5) thus implies that $\iota\left(e^{-t / 2} \mu, c_{1}\right)<\epsilon$ for all $t \geq 0$. In other words, for $t \geq 0$, the intersection between the horizontal measured geodesic lamination of $\Phi^{t} q$ and $c_{1}$ does not exceed $\epsilon$.

Let $\kappa>0$ be such that $\Phi^{2 \kappa} q \in \mathcal{Q}_{0}\left(\varphi_{0}^{k} \varphi_{1} \varphi_{0}^{k}(\tau)\right)$, that is, the total weight deposited by $e^{\kappa} \mu$ on $\varphi_{0}^{k} \varphi_{1} \varphi_{0}^{k}(\tau)$ equals one. Our goal is to show that the $\Phi^{t} q$-length of $c_{1}$ is smaller than $2 \epsilon$ for $0 \leq t \leq \kappa$. To this end it suffices to show that $\iota\left(c_{1}, e^{\kappa} \nu\right)<\epsilon$. To facilitate the notations, we show this by replacing $\zeta(k)=\left(\varphi_{0}^{k} \circ \varphi_{2} \circ \varphi_{0}^{k}\right) \circ\left(\varphi_{0}^{k} \circ \varphi_{1} \circ \varphi_{0}^{k}\right)$ by its conjugate

$$
\hat{\zeta}(k)=\left(\varphi_{0}^{k} \circ \varphi_{1} \circ \varphi_{0}^{k}\right) \circ\left(\varphi_{0}^{k} \circ \varphi_{2} \circ \varphi_{0}^{k}\right) .
$$

This conjugate admits $\tau$ as train track expansion. Let $\hat{\nu}$ be the vertical measured geodesic lamination of the differential $\hat{q} \in \mathcal{Q}_{0}(\tau)$ on the cotangent line of $\hat{\zeta}(k)$. We claim that $\iota\left(c_{1}, \hat{\nu}\right)<\epsilon$ which is equivalent to stating that $\iota\left(c_{1}, e^{\kappa} \nu\right)<\epsilon$.

To see that this is indeed the case recall that by construction, the carrying map $\varphi_{2} \circ \varphi_{0}^{k}(\tau) \prec \varphi_{0}^{k}(\tau)$ maps every branch of $\varphi_{2}\left(\varphi_{0}^{k}\left(\sigma_{2}\right)\right)$ onto $\varphi_{0}^{k}\left(\sigma_{2}\right)$. More precisely, the following holds true. Let $\hat{\mu} \in \mathcal{V}_{0}(\tau)$ be the horizontal measured geodesic lamination of $\hat{q}$ and let $\chi=\chi(k)<1$ be such that the total weight of the measured geodesic lamination $\chi \hat{\mu}$ on $\varphi_{0}^{k}(\tau)$ equals one. Then the $\chi \hat{\mu}$-weight of every branch in $\varphi_{0}^{k}\left(\sigma_{2}\right)<\varphi_{0}^{k}(\tau)$ is bounded from below by the number $a$ introduced in the beginning of this proof.

The intersection number $1=\iota\left(\chi \hat{\mu}, \chi^{-1} \hat{\nu}\right)=\iota(\hat{\mu}, \hat{\nu})$ can be calculated as

$$
\iota\left(\chi \hat{\mu}, \chi^{-1} \hat{\nu}\right)=\sum_{b} \chi \hat{\mu}(b) \chi^{-1} \hat{\nu}^{*}(b)
$$

where the sum is over all branches $b$ of $\varphi_{0}^{k}(\tau)=\eta$ and $\chi \hat{\mu}(b)$ and $\chi^{-1} \hat{\nu}^{*}(b)$ are the weights of $b$ for the transverse or tangential measure determined by $\chi \hat{\mu}, \chi^{-1} \hat{\nu}$ [PH92]. As this intersection number equals one, we have

$$
\chi^{-1} \hat{\nu}^{*}(b) \leq 1 / a
$$

for every branch $b$ of the subtrack $\varphi_{0}^{k}\left(\sigma_{2}\right)$. As $c_{1} \subset \varphi_{0}^{k}(\tau)$ is an embedded subtrack consisting of precisely two branches, we obtain

$$
\iota\left(\chi^{-1} \hat{\nu}, c_{1}\right) \leq 2 / a
$$

It follows from the above discussion that $\chi(k) \rightarrow \infty$ as $k \rightarrow \infty$. Thus for sufficiently large $k$, say for $k \geq k(\epsilon)$, we indeed have $\iota\left(c_{1}, \hat{\nu}\right)<\epsilon$.

We showed so far the following. Let $k \geq k(\epsilon)$, let $\zeta(k)$ be as in the statement of the lemma and let $q \in \mathcal{Q}_{0}(\tau)$ be contained in the unit contangent line of an axis of $\zeta(k)$. Let $\kappa_{1}>0$ be such that $\Phi^{\kappa_{1}} q \in \mathcal{Q}_{0}\left(\varphi_{0}^{k} \circ \varphi_{1} \circ \varphi_{0}^{k}(\tau)\right)$; then for every $t \in\left[0, \kappa_{1}\right]$, the $\Phi^{t} q$-length of the curve $c_{1}$ is at most $2 \epsilon$. The same argument also shows that if $\kappa_{2}>0$ is such that $\Phi^{\kappa_{2}} q \in \mathcal{Q}_{0}(\zeta(k)(\tau))$ then the $\Phi^{t} q$-length of $\varphi_{0}^{k} \varphi_{1} \varphi_{0}^{k}\left(c_{2}\right)$ is at most $2 \epsilon$ for $\kappa_{1} \leq t \leq \kappa_{2}$. But $\left\{\Phi^{t} q \mid 0 \leq t \leq \kappa_{2}\right\}$ is a fundamental domain for the action of $\zeta(k)$ on the contangent line of its axis and hence the proof of the lemma is completed.

We are now ready to complete the proof of Theorem 1.
Proposition 6.6. Let $S$ be a closed surface of genus $g \geq 0$ with $n \geq 0$ punctures and $3 g-3+n \geq 5$. Then for every component $\mathcal{Q}$ of a stratum in $\mathcal{Q}(S)$ or $\mathcal{H}(S)$ and for every $\epsilon>0$, there exists a number $c=c(\mathcal{Q}, \epsilon)>0$ such that

$$
n(\mathcal{Q}, R)^{<\epsilon} \geq e^{(h(\mathcal{Q})-1) R} / c R
$$

for all sufficiently large $R>0$.

Proof. For the proof of the proposition, we consider periodic orbits which are defined by pseudo Anosov mapping classes $\zeta(k)=\left(\varphi_{0}^{k} \circ \varphi_{2} \circ \varphi_{0}^{k}\right)\left(\varphi_{0}^{k} \circ \varphi_{1} \circ \varphi_{0}^{k}\right)$ as constructed in Lemma 6.4 where we fix $\varphi_{0}$ and $\varphi_{2}$ and vary $\varphi_{1}$.

Let $T\left(\varphi_{i}\right)>0$ be the translation length of the pseudo-Anosov mapping class $\varphi_{i}$ acting on the surface $S-c_{i}(i=1,2)$. We claim that there is a constant $\chi>0$ only depending on $k$ (and the choice of $\varphi_{0}$ ) such that the translation length of $\zeta(k)$ is contained in the interval $\left(0, T\left(\varphi_{1}\right)+T\left(\varphi_{2}\right)+\chi\right]$.

Namely, the translation length of $\zeta(k)$ is the logarithm of the Perron Frobenius eigenvalue of the linear map $A(\tau, \zeta(k))$ determined by $\zeta(k)$. This Perron Frobenius eigenvalue does not exceed the operator norm $\|A(\zeta(k), \tau)\|$ of the linear map $A(\zeta(k), \tau)$ with respect to the norm $\|\nu\|=\sum_{b}|\nu(b)|$ on $\mathbb{R}^{m}$. We have

$$
\|A(\tau, \zeta(k))\| \leq\left\|A\left(\sigma_{1}, \varphi_{1}\right)\right\|\left\|A\left(\sigma_{2}, \varphi_{2}\right)\right\|\|B\|^{4 k}
$$

where $\left\|A\left(\sigma_{i}, \varphi_{i}\right)\right\|$ is the operator norm of the linear map $A\left(\sigma_{i}, \varphi_{i}\right)$ defining the $\operatorname{map} \varphi_{i}$ (which coincides with the operator norm of $\left.A\left(\tau, \varphi_{i}\right)\right)$ and where $\|B\|$ is the operator norm of the linear map $A\left(\tau, \varphi_{0}\right)$.

On the other hand, by the choice of $\varphi_{i}$, there is a constant $\kappa>0$ such that

$$
\left|T\left(\varphi_{i}\right)-\log \left\|A\left(\sigma_{i}, \varphi_{i}\right)\right\|\right| \leq \kappa
$$

The claim follows.

By Proposition 5.7, there exist numbers $R_{0}>0, d>0$ such that for $T \geq R_{0}$, the number of mapping classes $\varphi_{i} \in \operatorname{Mod}\left(S_{1}\right)$ of translation length at most $R \geq R_{0}$ which can be used in the construction of the mapping classes $\zeta(k)$ is bounded from below by $e^{h(\mathcal{Q})-2} / d R$. Then the number of elements in $\operatorname{Mod}(S)$ of the form

$$
\zeta(k)=\varphi_{0}^{k} \circ \varphi_{2} \circ \varphi_{0}^{2 k} \circ \varphi_{1} \circ \varphi_{0}^{k}
$$

whose translation length is at most $R \geq R_{0}+\chi$ is at least $a e^{(h(\mathcal{Q})-2) R} / R$ where $a>0$ is computed from $\chi$ and from the constant $d$. In particular, the asymptotic growth rate of such periodic orbits is at least $h(\mathcal{Q})-2$.

Let $D$ be the Dehn twist about the curve $c_{1}$ in the direction determined by $\tau$. By this we mean that we require $D \tau \prec \tau$ and hence $D^{k} \tau \prec \tau$ for all $k \geq 0$. If $p \leq u e^{T\left(\varphi_{1}\right)}$ for a suitable choice of $u>0$ depending on $\epsilon$, then it follows from the above discussion that the mapping class $\varphi_{0}^{k} \circ \varphi_{2} \circ \varphi_{0}^{2 k} \circ D^{m} \circ \varphi_{1} \circ \varphi_{0}^{k}$ also has the required properties. This implies that simultaneous twisting about $c_{1}$ adds one to the exponent in the counting of the orbits constructed above and completes the proof of the proposition.

Remark 6.7. The stable length of a pseudo-Anosov element $g \in \operatorname{Mod}(S)$ on the curve graph $\mathcal{C}(S)$ of $S$ is defined to be

$$
s l(g)=\lim _{k \rightarrow \infty} \frac{1}{k} d\left(g^{k} c, c\right)
$$

This does not depend on the choice of $c \in \mathcal{C}(S)$.

Bowditch [Bw08] showed that there is an integer $\ell>0$ only depending on the topological type of $S$ such that the stable length on the curve graph of every pseudoAnosov element $\varphi$ is rational with denominator $\ell$. The stable length of each of the (infinitely many) pseudo-Anosov elements $\zeta(k)$ constructed in the proof of Proposition 6.6 is at most 2 . We expect that this is approximately sharp, which means that the asymptotic growth rate of all pseudo-Anosov mapping classes of stable length at most 2 is not bigger than $h(\mathcal{Q}(S))-1$.

A similar argument also yields Theorem 2 from the introduction.
Proposition 6.8. For every component $\mathcal{Q}$ of a stratum as in Proposition 6.6 there is a Teichmüller geodesic with uniquely ergodic vertical measured geodesic lamination whose projection to moduli space escapes with linear speed to infinity.

Proof. We argue as in the proof of Proposition 6.6. Namely, choose simple closed curves $c_{1}, c_{2}$ as in the proof of Proposition 6.6 and fix pseudo-Anosov elements $\varphi_{i}$ of $S_{i}$ with the properties stated in the proof.

Let $\tau$ be a large train track as in the proof of Proposition 6.6, with subtracks $\sigma_{i}$. Let $A_{i}=A\left(\varphi_{i}, \sigma_{i}\right)$ and denote by $\left\|A_{i}\right\|$ the operator norm of $A$. Choose a sequence of numbers $\left(k_{i}\right)$ such that for each $i$,

$$
k_{i} \geq 2 \sum_{j \leq i-1} k_{j}
$$

Let $\psi_{i}=\varphi_{1} \circ \varphi_{0}^{k_{i}} \circ \varphi_{2} \circ \varphi_{0}^{k_{i}}$. Then for each $i$ we have $\psi_{i} \tau \prec \tau$. Write $\zeta_{k}=\psi_{k} \circ \cdots \circ \psi_{1}$. We claim that $\cap_{k} \zeta_{k} \mathcal{V}(\tau)$ consists of a single ray.

To see that this is the case, note from the proof of Proposition 6.6 that for each $i$ a carrying map $\left(\psi_{i+1} \circ \psi_{i}\right) \tau \rightarrow \tau$ maps every branch of $\left(\psi_{i+1} \circ \psi_{i}\right) \tau$ onto $\tau$ and its normalization contracts distances in the cone $\mathcal{V}_{0}(\tau)$ with a factor which is independent of $i$. This implies immediately that $\cap_{k} \zeta_{k} \mathcal{V}(\tau)$ consists of a single ray. In particular, a point on this ray is a uniquely ergodic measured geodesic lamination which fills up $S$.

To show linear escape in moduli space, let $\lambda \in \mathcal{V}_{0}(\tau)$ be the normalized measured geodesic lamination contained in this ray and let $\nu$ be a measured geodesic lamination which fills and hits $\tau$ efficiently. Then the pair $(\lambda, \nu)$ determines a quadratic differential $q$. Let $\ell>0$ and let $a>0$ be such that the measure $e^{a / 2} \lambda$ on $\varphi_{0}^{k_{\ell}} \psi_{\ell-1}(\tau)$ is normalized. Then the arguments in the proof of Lemma 6.6 show that the intersection of the curve $\varphi^{k_{\ell}} \psi^{\ell-1} c_{1}$ with the lamination $e^{a / 2} \lambda$ is at most $c e^{-a / 2}$, and similarly for the intersection with $e^{-a / 2} \nu$. This yields the proposition.

Remark 6.9. By the main result of [CE07], there is a number $\epsilon>0$ so that if a Teichmüller geodesic in moduli space escapes into the cusp with a speed of at most $\epsilon \log t$, then the vertical measured geodesic lamination of a differential on the geodesic is uniquely ergodic. The above example implies that one can construct differentials with uniquely ergodic vertical measured laminations and arbitrarily prescribed excursions into the cusp.

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