COUNTING PERIODIC ORBITS IN THE THIN PART OF STRATA

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Abstract. Let S be a closed oriented surface of genus $g \geq 0$ and let $\mathcal{Q}$ be a connected component of a stratum in the moduli space $\mathcal{Q}(S)$ of area one meromorphic quadratic differentials on S with $n \geq 0$ simple poles or in the moduli space $\mathcal{H}(S)$ of abelian differentials on S. For a compact subset K of $\mathcal{Q}(S)$ or of $\mathcal{H}(S)$, we show that the asymptotic growth rate of the number of periodic orbits for the Teichmüller flow Φ^t on $\mathcal{Q}$ which are entirely contained in $\mathcal{Q} - K$ equals the entropy of the action of Φ^t on $\mathcal{Q}$ minus one provided that $3g - 3 + n \geq 5$.

1. Introduction

For a closed oriented surface S of genus $g \geq 0$, the moduli space $\mathcal{Q}(S)$ of area one meromorphic quadratic differentials with at most simple poles which are not squares of holomorphic one-forms decomposes into *strata*. Such a stratum is the subset of $\mathcal{Q}(S)$ which consists of all quadratic differentials with the same number $n \geq 0$ of simple poles and the same number $\ell \geq 0$ of zeros of the same order m_i $(1 \leq i \leq \ell)$. A stratum $\mathcal{Q}$ is a real hypersurface in a complex algebraic orbifold of complex dimension

$$h(\mathcal{Q}) = 2g - 2 + \ell + n.$$

Similarly, for $g \geq 2$ the moduli space $\mathcal{H}(S)$ of area one *abelian differentials* on S decomposes into strata. A stratum is the subset of $\mathcal{H}(S)$ which consists of holomorphic one-forms with the same number $s > 0$ of zeros of the same oder k_i. A stratum $\mathcal{Q}$ is a real hypersurface in a complex algebraic orbifold of complex dimension

$$h(\mathcal{Q}) = 2g - 1 + s.$$

The *Teichmüller flow* Φ^t acts on $\mathcal{Q}(S)$ and $\mathcal{H}(S)$, and this action preserves the strata. Each stratum contains periodic orbits, and these orbits can be counted: Namely, for a subset A of $\mathcal{Q}(S)$ or of $\mathcal{H}(S)$ and a number $R > 0$ denote by $n_A(R)$ the number of period orbits in A of length at most R. Then for any component $\mathcal{Q}$ of a stratum we have [H13]

$$n_{\mathcal{Q}}(R) \sim \frac{1}{h(\mathcal{Q})R} e^{h(\mathcal{Q})R} \quad (R \to \infty).$$

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This result rests on the work of Eskin, Mirzakhani and Rafi [EMR12], see also [H11], who showed that for every $\epsilon > 0$ there is a compact subset K of $\mathcal{Q}$ with the property that the growth rate of the number of periodic orbits in $\mathcal{Q}$ which are entirely contained in $\mathcal{Q} - K$ is at most $h(\mathcal{Q}) - 1 + \epsilon$.

The goal of this work is to show that this estimate is sharp.

Theorem 1. *Let $\mathcal{Q}$ be a component of a stratum of area one meromorphic quadratic differentials with n poles on a closed surface of genus $g \geq 0$ where $3g - 3 + n \geq 5$ or of a stratum of area one abelian differentials on a surface of genus $g \geq 3$. Then for every compact set $K \subset \mathcal{Q}(S)$ we have*

$$\liminf_{r \to \infty} \frac{1}{r} \log n_{\mathcal{Q}-K}(r) \geq h(\mathcal{Q}) - 1.$$

Note that the Teichmüller flow on the space $\mathcal{Q}(1;-1)$ of meromorphic quadratic differentials with a single pole on a torus T^2 can be identified with the geodesic flow on the unit tangent bundle of the modular surface $SL(2,\mathbb{Z})\backslash\mathbf{H}^2$. Thus in this case, there is a compact set K which is intersected by every periodic orbit for the Teichmüller flow.

The main tool for the proof of Theorem 1 is the construction of combinatorial models for components of strata of abelian or quadratic differentials using train tracks. In [H11] we use these models to construct a new symbolic coding for the Teichmüller flow on strata. In forthcoming work, we use the models to investigate the principal boundary of strata and, more generally, the adherence of strata of abelian differentials in the extension of the sheaf of holomorphic differentials to the Deligne Mumford compactification of moduli space.

Our method can also be applied to construct orbits in a given stratum with an arbitrarily prescribed recursion behavior to compact subsets of moduli space. An example for this is given in the following statement. For its formulation, for a point x in the moduli space $\mathcal{M}(S)$ of Riemann surfaces let $\mathrm{syst}(x)$ be the *systole* of x, i.e. the minimal length of a closed geodesic for the hyperbolic metric determined by x.

Theorem 2. *Let $\mathcal{Q}$ be a component of a stratum of area one meromorphic (or abelian) differentials on a surface of genus $g \geq 0$ with $m \geq 0$ simple poles. If $3g - 3 + m \geq 5$ then there is an orbit $\gamma : [0,\infty) \to \mathcal{T}(S)$ defined by a differential with uniquely ergodic vertical measured geodesic lamination and such that*

$$\limsup_{t \to \infty} \frac{1}{t} \log \mathrm{syst}(\gamma(t)) < 0.$$

The organization of the paper is as follows. Section 2 and Section 3 is devoted to the construction of combinatorial models for components of strata which consist of train tracks associated to strata. In Section 4 we use the classification of components of strata by Kontsevich and Zorich [KZ03] (for strata of abelian differentials) and Lanneau [L08] (for strata of quadratic differentials which are not squares of holomophic one-forms) to construct for each component of a stratum some train tracks with specific properties. These train tracks are used in Section 6 to prove Theorem 1. The short Section 5 collects those dynamical properties of the Teichmüller flow on compact subsets of strata needed in Section 6.

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2. Train tracks and geodesic laminations

In this section we summarize some constructions from [PH92, H09] which will be used throughout the paper. Furthermore, we introduce a class of train tracks which will be important in the later sections, and we discuss some of their properties.

2.1. **Geodesic laminations.** Let S be an oriented surface of genus $g \geq 0$ with $n \geq 0$ punctures and where $3g - 3 + n \geq 2$. A *geodesic lamination* for a complete hyperbolic structure on S of finite volume is a *compact* subset of S which is foliated into simple geodesics. A geodesic lamination λ is called *minimal* if each of its half-leaves is dense in λ. Thus a simple closed geodesic is a minimal geodesic lamination. A minimal geodesic lamination with more than one leaf has uncountably many leaves and is called *minimal arational*. Every geodesic lamination λ consists of a disjoint union of finitely many minimal components and a finite number of isolated leaves. Each of the isolated leaves of λ either is an isolated closed geodesic and hence a minimal component, or it *spirals* about one or two minimal components [CEG87].

A geodesic lamination λ on S is said to *fill up* S if its complementary regions are all topological disks or once punctured monogons or once punctured bigons. Here a once puncture monogon is a once punctured disk with a single cusp at the boundary. A *maximal* geodesic lamination is a geodesic lamination whose complementary regions are all ideal triangles or once punctured monogons.

Definition 2.1. A geodesic lamination λ is called *large* if λ fills up S and if moreover λ can be approximated in the *Hausdorff topology* by simple closed geodesics.

Since every minimal geodesic lamination can be approximated in the Hausdorff topology by simple closed geodesics [CEG87], a minimal geodesic lamination which fills up S is large. However, there are large geodesic laminations with finitely many leaves.

The *topological type* of a large geodesic lamination ν is a tuple

$$(m_1, \dots, m_\ell; -m, p) \text{ where } 1 \leq m_1 \leq \dots \leq m_\ell,\ \sum_i m_i = 4g - 4 + m,\ m + p = n.$$

Here $\ell \geq 1$ is the number of complementary regions which are topological disks, and these disks are $m_i + 2$-gons ($i \leq \ell$). There are m once punctured monogons and p once punctured bigons. Let

$$\mathcal{LL}(m_1, \dots, m_\ell; -m, p)$$

be the space of all large geodesic laminations of type $(m_1, \dots, m_\ell; -m, p)$ equipped with the restriction of the Hausdorff topology for compact subsets of S.

A *measured geodesic lamination* is a geodesic lamination λ together with a translation invariant transverse measure. Such a measure assigns a positive weight to each compact arc in S with endpoints in the complementary regions of λ which intersects λ nontrivially and transversely. The geodesic lamination λ is called the *support* of the measured geodesic lamination; it consists of a disjoint union of minimal components. The space $\mathcal{ML}$ of all measured geodesic laminations on S equipped with the weak*-topology is homeomorphic to $S^{6g-7+2n} \times (0,\infty)$. Its projectivization is the space $\mathcal{PML}$ of all *projective measured geodesic laminations.*

The measured geodesic lamination $\mu \in \mathcal{ML}$ *fills up* S if its support fills up S. This support is then necessarily connected and hence minimal, and for some tuple $(m_1,\dots,m_\ell;-m,p)$, it defines a point in the set $\mathcal{LL}(m_1,\dots,m_\ell;-m,p)$. The projectivization of a measured geodesic lamination which fills up S is also said to fill up S.

There is a continuous symmetric pairing $\iota : \mathcal{ML}\times\mathcal{ML}\to[0,\infty)$, the so-called *intersection form*, which extends the geometric intersection number between simple closed curves.

2.2. **Train tracks.** A *train track* on S is an embedded 1-complex $\tau \subset S$ whose edges (called *branches*) are smooth arcs with well-defined tangent vectors at the endpoints. At any vertex (called a *switch*) the incident edges are mutually tangent. Through each switch there is a path of class C^1 which is embedded in τ and contains the switch in its interior. A simple closed curve component of τ contains a unique bivalent switch, and all other switches are at least trivalent. The complementary regions of the train track have negative Euler characteristic, which means that they are different from disks with $0,1$ or 2 cusps at the boundary and different from annuli and once-punctured disks with no cusps at the boundary. We always identify train tracks which are isotopic. Throughout we use the book [PH92] as the main reference for train tracks.

A train track is called *generic* if all switches are at most trivalent. For each switch v of a generic train track τ which is not contained in a simple closed curve component, there is a unique half-branch b of τ which is incident on v and which is *large* at v. This means that every germ of an arc of class C^1 on τ which passes through v also passes through the interior of b. A half-branch which is not large is called *small*. A branch b of τ is called *large* (or *small*) if each of its two half-branches is large (or small). A branch which is neither large nor small is called *mixed.*

Remark: As in [H09], all train tracks are assumed to be generic. Unfortunately this leads to a small inconsistency of our terminology with the terminology found in the literature.

A *trainpath* on a train track τ is a C^1-immersion $\rho : [k,\ell] \to \tau$ such that for every $i < \ell - k$ the restriction of ρ to $[k+i,k+i+1]$ is a homeomorphism onto a branch of τ. More generally, we call a C^1-immersion $\rho : [a,b] \to \tau$ a *generalized* *trainpath*. A trainpath $\rho : [k, \ell] \to \tau$ is *closed* if $\rho(k) = \rho(\ell)$ and if precisely one of the half-branches $\rho[k, k+1/2], \rho[\ell - 1/2, \ell]$ is large.

A generic train track τ is *orientable* if there is a consistent orientation of the branches of τ such that at any switch s of τ, the orientation of the large half-branch incident on s extends to the orientation of the two small half-branches incident on s. If C is a complementary polygon of an oriented train track then the number of sides of C is even. In particular, a train track which contains a once punctured monogon component is not orientable (see p.31 of [PH92] for a more detailed discussion).

A train track or a geodesic lamination η is *carried* by a train track τ if there is a map $F : S \to S$ of class C^1 which is homotopic to the identity and maps η into τ in such a way that the restriction of the differential of F to the tangent space of η vanishes nowhere; note that this makes sense since a train track has a tangent line everywhere. We call the restriction of F to η a *carrying map* for η. Write $\eta \prec \tau$ if the train track η is carried by the train track τ. Then every geodesic lamination ν which is carried by η is also carried by τ.

A train track *fills up* S if its complementary components are topological discs or once punctured monogons or once punctured bigons. Note that such a train track τ is connected. Let $\ell \geq 1$ be the number of those complementary components of τ which are topological disks. Each of these disks is an $m_i + 2$-gon for some $m_i \geq 1$ $(i = 1, \dots, \ell)$. The *topological type* of τ is defined to be the ordered tuple $(m_1, \dots, m_\ell; -m, p)$ where $1 \leq m_1 \leq \cdots \leq m_\ell$ and m (or p) is the number of once punctured monogons (or once punctured bigons); then $\sum_i m_i = 4g - 4 + m$ and $m + p = n$. If τ is orientable then $m = 0$ and m_i is even for all i. A train track of topological type $(1, \dots, 1; -m, 0)$ is called *maximal*. The complementary components of a maximal train track are all trigons, i.e. topological disks with three cusps at the boundary, or once punctured monogons.

A *transverse measure* on a generic train track τ is a nonnegative weight function μ on the branches of τ satisfying the *switch condition*: for every trivalent switch s of τ, the sum of the weights of the two small half-branches incident on s equals the weight of the large half-branch. The space $\mathcal{V}(\tau)$ of all transverse measures on τ has the structure of a cone in a finite dimensional real vector space, and it is naturally homeomorphic to the space of all measured geodesic laminations whose support is carried by τ. The train track is called *recurrent* if it admits a transverse measure which is positive on every branch. We call such a transverse measure μ *positive*, and we write $\mu > 0$ (see [PH92] for more details).

A *subtrack* σ of a train track τ is a subset of τ which is itself a train track. Then σ is obtained from τ by removing some of the branches, and we write $\sigma < \tau$. If b is a small branch of τ which is incident on two distinct switches of τ then the graph σ obtained from τ by removing b is a subtrack of τ. We then call τ a *simple extension* of σ. Note that formally to obtain the subtrack σ from $\tau - b$ we may have to delete the switches on which the branch b is incident.

Lemma 2.2. (1) *A simple extension τ of a recurrent non-orientable connected train track σ is recurrent. Moreover,*

$$\dim \mathcal{V}(\tau) = \dim \mathcal{V}(\sigma) + 1.$$

(2) *An orientable simple extension τ of a recurrent orientable connected train track σ is recurrent. Moreover,*

$$\dim \mathcal{V}(\tau) = \dim \mathcal{V}(\sigma) + 1.$$

Proof. If τ is a simple extension of a train track σ then σ can be obtained from τ by the removal of a small branch b which is incident on two distinct switches s_1, s_2. Then s_i is an interior point of a branch b_i of σ $(i = 1, 2)$.

If σ is connected, non-orientable and recurrent then there is a trainpath $\rho_0 : [0, t] \to \tau - b$ which begins at s_1, ends at s_2 and such that the half-branch $\rho_0[0, 1/2]$ is small at $s_1 = \rho_0(0)$ and that the half-branch $\rho_0[t - 1/2, t]$ is small at $s_2 = \rho_0(t)$. Extend ρ_0 to a closed trainpath ρ on $\tau - b$ which begins and ends at s_1. This is possible since σ is non-orientable, connected and recurrent. There is a closed trainpath $\rho' : [0, u] \to \tau$ which can be obtained from ρ by replacing the trainpath ρ_0 by the branch b traveled through from s_1 to s_2. The counting measure of ρ' on τ satisfies the switch condition and hence it defines a transverse measure on τ which is positive on b. On the other hand, every transverse measure on σ defines a transverse measure on τ. Thus since σ is recurrent and since the sum of two transverse measures on τ is again a transverse measure, the train track τ is recurrent as well. Moreover, we have $\dim \mathcal{V}(\tau) \geq \dim \mathcal{V}(\sigma) + 1$.

Let p be the number of branches of τ. Label the branches of τ with the numbers $\{1, \dots, p\}$ so that the number p is assigned to b. Let $e_1, \dots, e_p$ be the standard basis of $\mathbb{R}^p$ and define a linear map $A : \mathbb{R}^p \to \mathbb{R}^p$ by $A(e_i) = e_i$ for $i \leq p - 1$ and $A(e_p) = \sum_i \nu(i) e_i$ where ν is the weight function on $\{1, \dots, p - 1\}$ defined by the trainpath ρ_0. The map A is a surjection onto a linear subspace of $\mathbb{R}^p$ of codimension one, moreover A preserves the linear subspace V of $\mathbb{R}^p$ defined by the switch conditions for τ. In particular, the corank of $A(V)$ in V is at most one. But $A(V)$ is contained in the space of solutions of the switch conditions on σ and consequently its corank in V is at least one.

Together we obtain that indeed $\dim \mathcal{V}(\tau) = \dim \mathcal{V}(\sigma) + 1$. This completes the proof of the first part of the lemma. The second part follows in exactly the same way, and its proof will be omitted. □

As a consequence we obtain

Corollary 2.3. (1) $\dim \mathcal{V}(\tau) = 2g - 2 + \ell + m + p$ *for every non-orientable recurrent train track τ of topological type $(m_1, \dots, m_\ell; -m, p)$.*

(2) $\dim \mathcal{V}(\tau) = 2g - 1 + \ell + p$ *for every orientable recurrent train track τ of topological type $(m_1, \dots, m_\ell; 0, p)$.*

Proof. The disk components of a non-orientable recurrent train track τ of topological type $(m_1, \dots, m_\ell; -m, p)$ can be subdivided in $4g - 4 + m - \ell$ steps into trigons by successively adding small branches. The once punctured bigon components can be subdivided into a trigon and a once punctured monogon. A repeated application of the first part of Lemma 2.2 shows that the resulting train track η is maximal and recurrent. Since for every maximal recurrent train track η on a surface with

$n = m + p$ punctures we have $\dim\mathcal{V}(\eta) = 6g - 6 + 2n$ (see [PH92]), the first part of the corollary follows from the formula in the first part of Lemma 2.2.

To show the second part of the corollary, let τ be an orientable recurrent train track of type $(m_1, \dots, m_\ell; 0, p)$. Then m_i is even for all i. Add a branch b_0 to τ which cuts some complementary component of τ into a trigon and a second polygon with an odd number of sides. The resulting train track η_0 is not recurrent since a trainpath on η_0 can only pass through b_0 at most once. However, we can add to η_0 another small branch b_1 which cuts some complementary component of η_0 with at least 4 sides into a trigon and a second polygon such that the resulting train track η is non-orientable and recurrent. The inward pointing tangent of b_1 is chosen in such a way that there is a trainpath traveling through both b_0 and b_1. The counting measure of any simple closed curve which is carried by η gives equal weight to the branches b_0 and b_1. But this just means that $\dim\mathcal{V}(\eta) = \dim\mathcal{V}(\tau) + 1$ (see the proof of Lemma 2.2 for a detailed argument). By the first part of the corollary, we have $\dim\mathcal{V}(\eta) = 2g - 2 + \ell + p + 2$ and consequently $\dim\mathcal{V}(\tau) = 2g - 1 + \ell + p$ as claimed. ☐

Definition 2.4. A train track τ of topological type $(m_1, \dots, m_\ell; -m, p)$ is *fully recurrent* if τ carries a large minimal geodesic lamination $\nu \in \mathcal{LL}(m_1, \dots, m_\ell; -m, p)$.

Note that by definition, a fully recurrent train track is connected and fills up S. The next lemma gives some first property of a fully recurrent train track τ. For its proof, recall that there is a natural homeomorphism of $\mathcal{V}(\tau)$ equipped with the euclidean topology onto the closed subspace of $\mathcal{ML}$ of all measured geodesic laminations carried by τ.

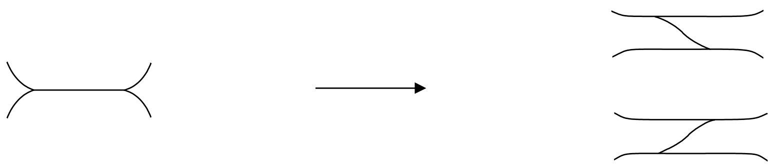
Lemma 2.5. *A fully recurrent train track τ is recurrent.*

Proof. A fully recurrent train track τ of type $(m_1, \dots, m_\ell; -m, p)$ carries a minimal large geodesic lamination $\nu \in \mathcal{LL}(m_1, \dots, m_\ell; -m, p)$. The carrying map $\nu \to \tau$ induces a bijection between the complementary components of τ and the complementary components of ν. In particular, a carrying map $\nu \to \tau$ is surjective. Now a minimal geodesic lamination supports a transverse measure, and such a transverse measure defines a positive transverse measure on τ. In other words, τ is recurrent. ☐

There are two simple ways to modify a fully recurrent train track τ to another fully recurrent train track. Namely, if b is a mixed branch of τ then we can *shift* τ along b to a new train track τ'. This new train track carries τ and hence it is fully recurrent since it carries every geodesic lamination which is carried by τ [PH92, H09].

Similarly, if e is a large branch of τ then we can perform a right or left *split* of τ at e as shown in Figure A below. The new small branch in the split track is called the *diagonal* of the split. A (right or left) split τ' of a train track τ is carried by τ. If τ is of topological type $(m_1, \dots, m_\ell; -m, p)$, if $\nu \in \mathcal{LL}(m_1, \dots, m_\ell; -m, p)$ is minimal and is carried by τ and if e is a large branch of τ, then there is a unique choice of a right or left split of τ at e such that the split track η carries ν. In particular, η is fully recurrent. Note however that there may be a split of τ at e such that the split track is not fully recurrent any more (see Section 2 of [H09] for details).

Figure A



The following observation is useful for the understanding of fully recurrent train tracks.

Lemma 2.6. (1) *Let e be a large branch of a fully recurrent non-orientable train track τ. Then no component of the train track σ obtained from τ by splitting τ at e and removing the diagonal of the split is orientable.*
(2) *Let e be a large branch of a fully recurrent orientable train track τ. Then the train track σ obtained from τ by splitting τ at e and removing the diagonal of the split is connected.*

Proof. We begin with the proof of the first part of the lemma. Thus let τ be a fully recurrent non-orientable train track of topological type $(m_1, \dots, m_\ell; -m, p)$. Let e be a large branch of τ and let v be a switch on which the branch e is incident. Let σ be the train track obtained from τ by splitting τ at e and removing the diagonal branch of the split. The train tracks τ_1, τ_2 obtained from τ by a right and left split at e, respectively, are simple extensions of σ.

We argue by contradiction and we assume that σ contains an orientable connected component σ_1 (not necessarily distinct from σ). Let $b_i \subset \tau_i - \sigma$ be the diagonal of the split which modifies τ to τ_i $(i = 1, 2)$. Since τ is fully recurrent, it can be split at e to a fully recurrent train track and hence at least one of the train tracks τ_i is recurrent. Assume that this holds true for τ_1. Then there is a trainpath $\rho : [0, k] \to \tau_1$ with $\rho[0, 1] = b_1$ and $\rho[1, 2] \in \sigma_1$ which recurs to b_1. Let $j \geq 2$ be the smallest number bigger than one such that $\rho[j, j + 1] = b_i$. Then $\rho[j - 1, j]$ equals the branch $\rho[1, 2]$ traveled through in opposite direction, moreover we have $\rho[1, j] \subset \sigma_1$. Since σ_1 is orientable, this is impossible. This contradiction shows the first part of the lemma.

The second part of the lemma follows from the same argument since a split of an orientable train track is orientable. □

Example: 1) Figure B shows a non-orientable recurrent train track τ of type $(4; 0, 0)$ on a closed surface of genus two. The train track obtained from τ by a split at the large branch e and removal of the diagonal of the split track is orientable and hence τ is not fully recurrent. We show in Section 3 that this corresponds to the following result of Masur and Smillie [MS93]: Every quadratic differential with a single zero and no pole on a surface of genus 2 is the square of a holomorphic one-form.

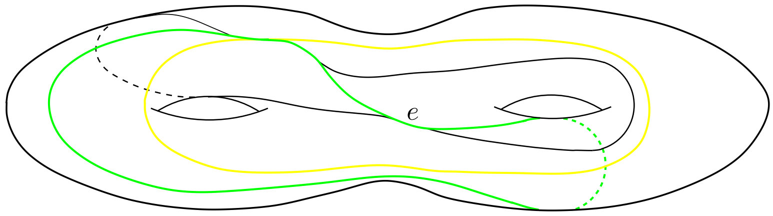


Figure B

2) To construct an orientable recurrent train track of type $(m_1, \ldots, m_\ell; 0, 0)$ which is not fully recurrent let S_1 be a surface of genus $g_1 \geq 2$ and let τ_1 be an orientable fully recurrent train track on S_1 with $\ell_1 \geq 1$ complementary components. Choose a complementary component C_1 of τ_1 in S_1, remove from C_1 a disk D_1 and glue two copies of $S_1 - D_1$ along the boundary of D_1 to a surface S of genus $2g_1$. The two copies of τ_1 define a recurrent disconnected oriented train track τ on S which has an annulus complementary component C.

Choose a branch b_1 of τ in the boundary of C. There is a corresponding branch b_2 in the second boundary component of C. Glue a compact subarc of b_1 contained in the interior of b_1 to a compact subarc of b_2 contained in the interior of b_2 so that the images of the two arcs under the glueing form a large branch e in the resulting train track η. The train track η is recurrent and orientable, and its complementary components are topological disks. However, by Lemma 2.6 it is not fully recurrent.

To each train track τ which fills up S one can associate a *dual bigon track* τ^* (Section 3.4 of [PH92]). There is a bijection between the complementary components of τ and those complementary components of τ^* which are not *bigons*, i.e. disks with two cusps at the boundary. This bijection maps a component C of τ which is an n-gon for some $n \geq 3$ to an n-gon component of τ^* contained in C, and it maps a once punctured monogon or bigon C to a once punctured monogon or bigon contained in C. If τ is orientable then the orientation of S and an orientation of τ induce an orientation on τ^*, i.e. τ^* is orientable.

There is a notion of carrying for bigon tracks which is analogous to the notion of carrying for train tracks. Measured geodesic laminations which are carried by the bigon track τ^* can be described as follows. A *tangential measure* on a train track τ of type $(m_1, \ldots, m_\ell; -m, p)$ assigns to a branch b of τ a weight $\mu(b) \geq 0$ such that for every complementary k-gon of τ or once punctured bigon with consecutive sides $c_1, \ldots, c_k$ and total mass $\mu(c_i)$ (counted with multiplicities) the following holds true.

(1) $\mu(c_i) \leq \mu(c_{i-1}) + \mu(c_{i+1})$.
(2) $\sum_{i=j}^{k+j-1} (-1)^{i-j} \mu(c_i) \geq 0$, $j = 1, \ldots, k$.

The complementary once punctured monogons define no constraint on tangential measures. Our definition of tangential measure on τ is stronger than the definition given on p.22 of [PH92] and corresponds to the notion of a *metric* as defined on p.184 of [P88]. We do not use this terminology here since we find it unfortunate.

The space of all tangential measures on τ has the structure of a convex cone in a finite dimensional real vector space. By Lemma 2.1 of [P88], every tangential measure on τ determines a simplex of measured geodesic laminations which *hit* τ *efficiently*. The supports of these measured geodesic laminations are carried by the bigon track τ^*, and every measured geodesic lamination which is carried by τ^* can be obtained in this way. The dimension of this simplex equals the number of complementary components of τ with an even number of sides. The train track τ is called *transversely recurrent* if it admits a tangential measure which is positive on every branch.

In general, a measured geodesic lamination ν which hits τ efficiently does not determine uniquely a tangential measure on τ either. Namely, let s be a switch of τ and let a, b, c be the half-branches of τ incident on s and such that the half-branch a is large. If β is a tangential measure on τ and if ν is a measured geodesic lamination in the simplex determined by β then it may be possible to drag the switch s across some of the leaves of ν and modify the tangential measure β on τ to a tangential measure $\mu \neq \beta$. Then $\beta - \mu$ is a multiple of a vector of the form $\delta_a - \delta_b - \delta_c$ where δ_w denotes the function on the branches of τ defined by $\delta_w(w) = 1$ and $\delta_w(a) = 0$ for $a \neq w$.

Definition 2.7. Let τ be a train track of topological type $(m_1, \dots, m_\ell; -m, p)$.

(1) τ is called *fully transversely recurrent* if its dual bigon track τ^* carries a large geodesic lamination $\nu \in \mathcal{LL}(m_1, \dots, m_\ell; -m, p)$.
(2) τ is called *large* if τ is fully recurrent and fully transversely recurrent.

For a large train track τ let $\mathcal{V}^*(\tau) \subset \mathcal{ML}$ be the set of all measured geodesic laminations whose support is carried by τ^*. Each of these measured geodesic laminations corresponds to a family of tangential measures on τ. With this identification, the pairing

$$(\nu, \mu) \in \mathcal{V}(\tau) \times \mathcal{V}^*(\tau) \to \sum_b \nu(b)\mu(b) \tag{1}$$

is just the restriction of the intersection form on measured lamination space (Section 3.4 of [PH92]). Moreover, $\mathcal{V}^*(\tau)$ is naturally homeomorphic to a convex cone in a real vector space. The dimension of this cone coincides with the dimension of $\mathcal{V}(\tau)$.

The following observation is an immediate consequence of Lemma 2.2 and the above discussion.

Proposition 2.8. (1) *A simple extension of a large non-orientable train track is large.*
(2) *An orientable simple extension of a large orientable train track is large.*

From now on we denote by $\mathcal{LT}(m_1,\dots,m_\ell;-m,p)$ the set of all isotopy classes of large train tracks on S of type $(m_1,\dots,m_\ell;-m,p)$.

Section 3 of [H09] contains a method to construct large train tracks. Namely, for a fixed choice of a complete hyperbolic metric on S of finite volume and numbers $a>0,\epsilon>0$ there is a notion of *a-long* train track which *ϵ-follows* a large geodesic lamination λ. The following is an immediate consequence of Lemma 3.2 of [H09].

Lemma 2.9. *Let $\lambda\in\mathcal{LL}(m_1,\dots,m_\ell;-m,p)$; then for sufficiently small ϵ, an a-long train track τ which ϵ-follows λ is contained in $\mathcal{LT}(m_1,\dots,m_\ell;-m,p)$.*

3. Strata

For a closed oriented surface S of genus $g\geq 0$ with $n\geq 0$ marked points (punctures) let $\tilde{\mathcal{Q}}(S)$ be the bundle of marked area one holomorphic quadratic differentials with either a simple pole or a regular point at each of the marked points and no other pole over the *Teichmüller space* $\mathcal{T}(S)$ of marked complex structures on S.

Fix a complete hyperbolic metric on S of finite area. An area one quadratic differential $q\in\tilde{\mathcal{Q}}(S)$ is determined by a pair (λ^+,λ^-) of measured geodesic laminations which *jointly fill up* S (i.e. we have $\iota(\lambda^+,\mu)+\iota(\lambda^-,\mu)>0$ for every measured geodesic lamination μ) and such that $\iota(\lambda^+,\lambda^-)=1$. The *vertical* measured geodesic lamination λ^+ for q corresponds to the equivalence class of the *vertical measured foliation* of q. The *horizontal* measured geodesic lamination λ^- for q corresponds to the equivalence class of the *horizontal measured foliation* of q.

For $m\leq n$, $p=n-m$ and $\ell\geq 1$, an ℓ-tuple $(m_1,\dots,m_\ell)$ of positive integers $1\leq m_1\leq\dots\leq m_\ell$ with $\sum_i m_i=4g-4+m$ defines a *stratum* $\tilde{\mathcal{Q}}(m_1,\dots,m_\ell;-m,p)$ in $\tilde{\mathcal{Q}}(S)$. This stratum consists of all marked area one quadratic differentials with m simple poles, p regular marked points and ℓ zeros of order $m_1,\dots,m_\ell$. We require that these differentials are not squares of holomorphic one-forms. The stratum is a real hypersurface in a complex manifold of dimension

$$h=2g-2+\ell+m+p. \tag{2}$$

The closure in $\tilde{\mathcal{Q}}(S)$ of a stratum is a union of components of strata. Strata are invariant under the action of the mapping class group $\mathrm{Mod}(S)$ of S and hence they project to strata in the moduli space $\mathcal{Q}(S)=\tilde{\mathcal{Q}}(S)/\mathrm{Mod}(S)$ of quadratic differentials on S. We denote the projection of the stratum $\tilde{\mathcal{Q}}(m_1,\dots,m_\ell;-m,p)$ by $\mathcal{Q}(m_1,\dots,m_\ell;-m,p)$. The strata in moduli space need not be connected, but their connected components have been identified by Lanneau [L08]. A stratum in $\mathcal{Q}(S)$ has at most two connected components. The number of components of the stratum $\mathcal{Q}(m_1,\dots,m_\ell;-m,p)$ equals the number of components of $\mathcal{Q}(m_1,\dots,m_\ell;-m,0)$.

Similarly, let $\tilde{\mathcal{H}}(S)$ be the bundle of marked area one holomorphic one-forms over Teichmüller space $\mathcal{T}(S)$ of S. Each of the marked points of S is required to be a regular point for the differential. In particular, the bundle is non-empty only if $g\geq 1$. For an ℓ-tuple $k_1\leq\dots\leq k_\ell$ of positive integers with $\sum_i k_i=2g-2$, the stratum $\tilde{\mathcal{H}}(k_1,\dots,k_\ell;n)$ of marked area one holomorphic one-forms on S with

ℓ zeros of order k_i $(i = 1, \ldots, \ell)$ and n regular marked points is a real hypersurface in a complex manifold of dimension

$$h = 2g - 1 + \ell + n. \tag{3}$$

It projects to a stratum $\mathcal{H}(k_1, \ldots, k_\ell; n)$ in the moduli space $\mathcal{H}(S)$ of area one holomorphic one-forms on S. Strata of holomorphic one-forms in moduli space need not be connected, but the number of connected components of a stratum is at most three [KZ03].

We continue to use the assumptions and notations from Section 2. For a large train track $\tau \in \mathcal{LT}(m_1, \ldots, m_\ell; -m, p)$ let

$$\mathcal{V}_0(\tau) \subset \mathcal{V}(\tau)$$

be the set of all measured geodesic laminations $\nu \in \mathcal{ML}$ whose support is carried by τ and such that the total weight of the transverse measure on τ defined by ν equals one. Let

$$\mathcal{Q}(\tau) \subset \tilde{\mathcal{Q}}(S)$$

be the set of all marked area one quadratic differentials whose vertical measured geodesic lamination is contained in $\mathcal{V}_0(\tau)$ and whose horizontal measured geodesic lamination is carried by the dual bigon track τ^* of τ. By definition of a large train track, we have $\mathcal{Q}(\tau) \neq \emptyset$.

The next observation relates $\mathcal{Q}(\tau)$ to components of strata.

Lemma 3.1. *Let $\tau \in \mathcal{LT}(m_1, \ldots, m_\ell; -m, p)$ and let $q \in \mathcal{Q}(\tau)$. If the support of the vertical measured geodesic lamination is contained in $\mathcal{LL}(m_1, \ldots, m_\ell; -m, p)$ then $q \in \tilde{\mathcal{Q}}(m_1, \ldots, m_\ell; -m, p)$, and q is an abelian differential if and only if τ is orientable.*

Proof. A marked area one quadratic differential $z \in \tilde{\mathcal{Q}}(S)$ defines a singular euclidean metric on S of area one. A singular point for z is a zero or a pole or one of the $p \geq 0$ regular marked points. A *saddle connection* for z is a geodesic segment for this singular euclidean metric which connects two singular points and does not contain a singular point in its interior. A *separatrix* is a maximal geodesic segment or ray which begins at a singular point and does not contain a singular point in its interior.

The complex structure on S determines a complete finite area hyperbolic metric g on S. Let ξ be the support of the vertical measured geodesic lamination of z. By [L83], the geodesic lamination ξ can be obtained from the vertical foliation of z by cutting S open along each vertical separatrix and straightening the remaining leaves so that they become geodesics for g. In particular, up to homotopy, a vertical saddle connection s of z is contained in the interior of a complementary component C of ξ which is uniquely determined by s.

Let $\tau \in \mathcal{LT}(m_1, \ldots, m_\ell; -m, p)$ be non-orientable. Let $q \in \mathcal{Q}(\tau)$, with vertical measured geodesic lamination $\mu \in \mathcal{V}_0(\tau)$ whose support $\mathrm{supp}(\mu)$ is contained in $\mathcal{LL}(m_1, \ldots, m_\ell; -m, p)$. Then $\mathrm{supp}(\mu)$ is non-orientable since otherwise τ inherits an orientation from $\mathrm{supp}(\mu)$. Since $\mathrm{supp}(\mu) \in \mathcal{LL}(m_1, \ldots, m_\ell; -m, p)$, the orders of the zeros of the quadratic differential q are obtained from the orders $m_1, \ldots, m_\ell$

by subdivision. Moreover, $q \in \tilde{\mathcal{Q}}(m_1, \dots, m_\ell; -m, p)$ if and only if this subdivision is trivial.

By slightly moving the regular marked points in S we may assume that no regular marked point is the endpoint of any saddle connection. Then the subdivision is trivial if and only if q does not have any vertical saddle connection.

The choice of the hyperbolic metric on S identifies the universal covering of S with the hyperbolic plane $\mathbf{H}^2$, and it identifies the fundamental group $\pi_1(S)$ of S with a group of isometries of $\mathbf{H}^2$. Assume to the contrary that q has a vertical saddle connection s. Let $\tilde{q}$ be the lift of q to a quadratic differential on $\mathbf{H}^2$ and let $\tilde{s} \subset \mathbf{H}^2$ be a preimage of s.

The preimage $\zeta \subset \mathbf{H}^2$ of $\mathrm{supp}(\mu)$ is a closed $\pi_1(S)$-invariant set of geodesic lines in $\mathbf{H}^2$. Since μ fills up S, the complementary components of ζ are finite area ideal polygons and half-planes which are the components of the preimages of the once punctured monogons and the once punctured bigons of $S - \mu$. As discussed above, up to homotopy the saddle connection $\tilde{s}$ of $\tilde{q}$ is contained in a complementary component $\tilde{C}$ of ζ which is an ideal polygon with finitely many sides, and it is determined by $\tilde{s}$.

Remove a small open disk D_i about each of the punctures u_i of S such that the closures of these disks are pairwise disjoint and do not contain any zero of q. A vertical or horizontal geodesic arc γ on S not passing through any of the points u_i can be homotoped with fixed endpoints to a path entirely contained in $K = S - \cup_i D_i$ whose length with respect to the singular euclidean metric defined by q equals the length of γ up to a universal multiplicative constant. On K, the singular euclidean metric is uniformly equivalent to the hyperbolic metric g. Therefore a lift to $\mathbf{H}^2$ of a biinfinite vertical or horizontal geodesic is a uniform quasi-geodesic for the hyperbolic metric. Such a quasi-geodesic has well defined endpoints in the ideal boundary $\partial\mathbf{H}^2$ of $\mathbf{H}^2$ (see also [L83, PH92]).

Choose an orientation for the saddle connection $\tilde{s}$. There are two oriented vertical geodesic lines α_0, β_0 for the metric defined by $\tilde{q}$ which contain the saddle connection $\tilde{s}$ as a subarc and which are contained in a bounded neighborhood of a side α, β of $\tilde{C}$. The geodesics α_0, β_0 are determined by the requirement that their orientation coincides with the given orientation of $\tilde{s}$ and that moreover at every singular point x, the angle at x to the left of α_0 (or to the right of β_0) for the orientation of the geodesic and the orientation of $\mathbf{H}^2$ equals π (see [L83] for details of this construction).

The ideal boundary of the closed half-plane of $\mathbf{H}^2$ which is bounded by α (or β) and which is disjoint from the interior of $\tilde{C}$ is a compact subarc a (or b) of $\partial\mathbf{H}^2$ bounded by the endpoints of α (or β). The arcs a, b are disjoint (or, equivalently, the sides α, β of $\tilde{C}$ are not adjacent). A horizontal geodesic line for $\tilde{q}$ which intersects the interior of the saddle connection $\tilde{s}$ is a quasi-geodesic in $\mathbf{H}^2$ with one endpoint in the interior of the arc a and the second endpoint in the interior of the arc b. Since the length of $\tilde{s}$ is positive, the weight placed on $\tilde{s}$ by the transverse measure for the horizontal foliation of $\tilde{q}$ is positive. This means that the support of the horizontal measured geodesic lamination of $\tilde{q}$ contains geodesics with one endpoint in the arc a and the second endpoint in b.

Since the topological types of the support of μ and of τ coincide, a carrying map $F : \mathrm{supp}(\mu) \to \tau$ is surjective and induces a bijection between the complementary components of $\mathrm{supp}(\mu)$ and the complementary components of τ. In particular, the projections to S of the geodesics α, β determine two non-adjacent sides of a complementary component C_τ of τ.

On the other hand, by construction of the dual bigon track τ^* of τ (see [PH92]), if $\rho : (-\infty, \infty) \to \tau^*$ is any trainpath which intersects the complementary component C_τ of τ then every component of $\rho(-\infty, \infty) \cap C_\tau$ is a compact arc with endpoints on adjacent sides of C_τ. In particular, a lift to $\mathbf{H}^2$ of such a trainpath is a quasi-geodesic in $\mathbf{H}^2$ whose endpoints meet at most one of the two arcs $a, b \subset \partial\mathbf{H}^2$. Now the support of the horizontal measured geodesic lamination ν of q is carried by τ^*. Therefore every leaf of the support of ν determines a biinfinite trainpath on τ^*. A lift to $\mathbf{H}^2$ of such a leaf does not connect the arcs $a, b \subset \partial\mathbf{H}^2$. But we observed above that the support of the horizontal measured geodesic lamination of $\tilde{q}$ contains geodesics connecting a to b. This is a contradiction and shows that indeed $q \in \tilde{\mathcal{Q}}(m_1, \dots, m_\ell; -m, p)$.

We are left with the case that $\tau \in \mathcal{LT}(m_1, \dots, m_\ell; 0, p)$ is orientable. If μ is a geodesic lamination which is carried by τ, then μ inherits an orientation from an orientation of τ. The orientation of τ together with the orientation of S determines an orientation of the dual bigon track τ^* (see [PH92]). Therefore any geodesic lamination carried by τ^* admits an orientation, and if (μ, ν) jointly fill up S and if μ is carried by τ, ν is carried by τ^* then the orientations of μ, ν determine the orientation of S and the quadratic differential q of (μ, ν) is the square of a holomorphic one-form. Together with the first part of this proof, the lemma follows. □

As in the introduction, let Φ^t be the Teichmüller flow on $\tilde{\mathcal{Q}}(S)$ and on $\mathcal{Q}(S)$. We use Lemma 3.1 to show

Proposition 3.2. (1) *For every large non-orientable train track $\tau \in \mathcal{LT}(m_1, \dots, m_\ell; -m, p)$ there is a component $\tilde{\mathcal{Q}}$ of the stratum $\tilde{\mathcal{Q}}(m_1, \dots, m_\ell; -m, p)$ such that for every $\delta > 0$ the set $\{\Phi^t q \mid q \in \mathcal{Q}(\tau), t \in [-\delta, \delta]\}$ is the closure in $\tilde{\mathcal{Q}}(S)$ of an open subset of $\tilde{\mathcal{Q}}$.*

(2) *For every large orientable train track $\tau \in \mathcal{LT}(m_1, \dots, m_\ell; 0, n)$ there is a component $\tilde{\mathcal{Q}}$ of the stratum $\tilde{\mathcal{H}}(m_1/2, \dots, m_\ell/2, n)$ such that for every $\delta > 0$ the set $\{\Phi^t q \mid q \in \mathcal{Q}(\tau), t \in [-\delta, \delta]\}$ is the closure in $\tilde{\mathcal{H}}(S)$ of an open subset of $\tilde{\mathcal{Q}}$.*

Proof. Let $\tau \in \mathcal{LL}(m_1, \dots, m_\ell; -m, p)$ and let $\mu \in \mathcal{V}_0(\tau)$, with support $\mathrm{supp}(\mu) \in \mathcal{LL}(m_1, \dots, m_\ell; -m, p)$. If $\nu \in \mathcal{V}^*(\tau)$ then the measured geodesic laminations μ, ν jointly fill up S (since the support of ν is different from the support of μ and $\mathrm{supp}(\mu)$ fills up S) and hence if ν is normalized in such a way that $\iota(\mu, \nu) = 1$ then the pair (μ, ν) defines a point $q \in \mathcal{Q}(\tau)$. By Lemma 3.1, we have $q \in \tilde{\mathcal{Q}}(m_1, \dots, m_\ell; -m, p)$.

Define the *strong unstable manifold* $W^{su}(q)$ of a quadratic differential $q \in \tilde{\mathcal{Q}}(S)$ to consist of all quadratic differentials whose horizontal measured foliation coincides with the horizontal measured foliation of q up to isotopy preserving the regular marked points and Whitehead moves.

The *strong stable manifold* $W^{ss}(q)$ is defined to be the image of $W^{su}(-q)$ under the *flip* $\mathcal{F} : q \to -q$. For a component $\tilde{\mathcal{Q}}$ of a stratum in $\tilde{\mathcal{Q}}(S)$ and for $q \in \tilde{\mathcal{Q}}$, define the strong unstable (or strong stable) manifold $W^{su}_{\tilde{\mathcal{Q}}}(q)$ (or $W^{ss}_{\tilde{\mathcal{Q}}}(q)$) to be the connected component containing q of the intersection $W^{su}(q) \cap \tilde{\mathcal{Q}}$ (or $W^{ss}(q) \cap \tilde{\mathcal{Q}}$). Then $W^i_{\tilde{\mathcal{Q}}}(q)$ is a manifold of dimension $h(\tilde{\mathcal{Q}}) - 1$ ($i = ss, su$). The manifolds $W^{su}_{\tilde{\mathcal{Q}}}(q)$ (or $W^{ss}_{\tilde{\mathcal{Q}}}(q)$) define a foliation of $\tilde{\mathcal{Q}}$ which is called the *strong unstable* (or the *strong stable*) foliation.

Let $\mathcal{P}(\mu) \subset \mathcal{PML}$ be the open set of all projective measured geodesic laminations whose support is distinct from the support of μ. Then the assignment ψ which associates to a projective measured geodesic lamination $[\nu] \in \mathcal{P}(\mu)$ the area one quadratic differential $q(\mu, [\nu])$ with vertical measured geodesic lamination μ and horizontal projective measured geodesic lamination $[\nu]$ is a homeomorphism of $\mathcal{P}(\mu)$ onto a strong stable manifold in $\tilde{\mathcal{Q}}(S)$.

By Corollary 2.3 and the fact that the dimension of $\mathcal{V}^*(\tau)$ coincides with the dimension of $\mathcal{V}(\tau)$, the projectivization $P\mathcal{V}^*(\tau) \subset \mathcal{PML}$ of $\mathcal{V}^*(\tau)$ is homeomorphic to a closed ball in a real vector space of dimension $h(\tilde{\mathcal{Q}}) - 1$, and this is just the dimension of a strong stable manifold in a component of $\tilde{\mathcal{Q}}(m_1, \dots, m_\ell; -m, p)$. Therefore by Lemma 3.1 and invariance of domain, there is a component $\tilde{\mathcal{Q}}$ of the stratum $\tilde{\mathcal{Q}}(m_1, \dots, m_\ell; -m, p)$ such that the restriction of the map ψ to $P\mathcal{V}^*(\tau)$ is a homeomorphism of $P\mathcal{V}^*(\tau)$ onto the closure of an open subset of a strong stable manifold $W^{ss}_{\tilde{\mathcal{Q}}}(q) \subset \tilde{\mathcal{Q}}$.

The above argument also shows that if $z \in \mathcal{Q}(\tau)$ is defined by $\zeta \in \mathcal{V}_0(\tau), \nu \in \mathcal{V}^*(\tau)$ and if the support of ν is contained in $\mathcal{LL}(m_1, \dots, m_\ell; -m, p)$ then we have $z \in \tilde{\mathcal{Q}}(m_1, \dots, m_\ell; -m, p)$. If $\tilde{\mathcal{P}}$ denotes the component of $\tilde{\mathcal{Q}}(m_1, \dots, m_\ell; -m, p)$ containing z then for every point $[\beta]$ in the projectivization $P\mathcal{V}(\tau)$ of $\mathcal{V}(\tau)$, the pair $([\beta], \nu)$ defines a quadratic differential which is contained in the strong unstable manifold $W^{su}_{\tilde{\mathcal{P}}}(z)$. The set of these quadratic differentials equals the closure of an open subset of $W^{su}_{\tilde{\mathcal{P}}}(z)$.

The set of quadratic differentials q with the property that the support of the vertical (or of the horizontal) measured geodesic lamination of q is minimal and of type $(m_1, \dots, m_\ell; -m, p)$ is dense and of full Lebesgue measure in $\tilde{\mathcal{Q}}(m_1, \dots, m_\ell; -m, p)$ [M82, V86]. Moreover, this set is saturated for the strong stable (or for the strong unstable) foliation. Thus by the above discussion, the set of all measured geodesic laminations which are carried by τ (or τ^*) and whose support is minimal of type $(m_1, \dots, m_\ell; -m, p)$ is dense in $\mathcal{V}(\tau)$ (or in $\mathcal{V}^*(\tau)$). As a consequence, the set of all pairs $(\mu, \nu) \in \mathcal{V}(\tau) \times \mathcal{V}^*(\tau)$ with $\iota(\mu, \nu) = 1$ which correspond to a quadratic differential $q \in \tilde{\mathcal{Q}}(m_1, \dots, m_\ell; -m, p)$ is dense in the set of all pairs $(\mu, \nu) \in \mathcal{V}(\tau) \times \mathcal{V}^*(\tau)$ with $\iota(\mu, \nu) = 1$. As closures of distinct components of $\tilde{\mathcal{Q}}(m_1, \dots, m_\ell; -m, p)$ are disjoint, this implies that the set $\mathcal{Q}(\tau)$ is contained in the closure of a component $\tilde{\mathcal{Q}}$ of the stratum $\tilde{\mathcal{Q}}(m_1,\dots,m_\ell;-m,p)$. Moreover, by reasons of dimension, $\{\Phi^t q \mid q\in\mathcal{Q}(\tau), t\in[-\delta,\delta]\}$ contains an open subset of this component. This completes the proof of the proposition. □

The next proposition is a converse to Proposition 3.2 and shows that train tracks can be used to define coordinates on strata.

Proposition 3.3. (1) *For every $q\in\tilde{\mathcal{Q}}(m_1,\dots,m_\ell;-m,p)$ there is a large non-orientable train track $\tau\in\mathcal{LT}(m_1,\dots,m_\ell;-m,p)$ and a number $t\in\mathbb{R}$ so that $\Phi^t q$ is an interior point of $\mathcal{Q}(\tau)$.*

(2) *For every $q\in\tilde{\mathcal{H}}(k_1,\dots,k_s;n)$ there is a large orientable train track $\tau\in\mathcal{LT}(2k_1,\dots,2k_s;0,n)$ and a number $t\in\mathbb{R}$ so that $\Phi^t q$ is an interior point of $\mathcal{Q}(\tau)$.*

Proof. Let $q\in\tilde{\mathcal{Q}}(m_1,\dots,m_\ell;-m,p)$ and assume first that q does not have any horizontal cylinders.

Let $\Sigma=\{u_1,\dots,u_s\}$ $(s=\ell+m+p)$ be the singular set of q, i.e. the union of the zeros and poles and marked regular points. For each i choose a closed disk neighborhood D_i of u_i with smooth boundary in such a way that these disks are pairwise disjoint. We may assume that the vertical separatrices issuing from u_i intersect ∂D_i transversely. Then the connected component containing u_i of the intersection with D_i of each such separatrix is a compact connected arc with one endpoint u_i and the second endpoint on ∂D_i. The union of these arcs is a connected graph G_i embedded in D_i which either is a compact arc with one endpoint on ∂D_i (in the case that u_i is a pole of q) or a compact arc with two endpoints on ∂D_i which contains u_i in its interior (if u_i is a marked regular point of q) or a tree with a single non-univalent vertex (in the case that u_i is a zero of q).

For each i replace the graph G_i by a connected train track η_i in D_i with stops on ∂D_i as shown in Figure C. We require that η_i is transverse to the horizontal foliation. There is a single complementary component C_i of $D_i-\eta_i$ whose closure is contained in the interior of D_i. This component contains u_i. The cusps of the component are the trivalent vertices of η_i. For each stop of η_i, the branch containing the stop is contained in G_i and connects the stop to a cusp of C_i. The component C_i is a once punctured monogon if u_i is a pole, a once punctured bigon if u_i is a regular marked point, or an m_i+2-gon if u_i is a zero of order m_i.

Let η be the union of the train tracks with stops η_i; this union consists of $\ell+m+p$ connected components, and it contains $\sum_i(m_i+2)+m+2p$ univalent vertices. The graph η is transverse to the horizontal foliation of q. Note that η also is transverse to the straight line foliation on S defined by any direction which is sufficiently close to the horizontal direction.

Let w be a stop of the train track with stops η and let ζ be a neighborhood of w in the branch e of η containing w. We orient ζ so that it goes from the stop w to its second endpoint which is an interior point of e. By construction, ζ is contained in a vertical separatrix of q. Each point on ζ is the starting point of a horizontal geodesic arc which locally lies to the left of ζ. These arcs are contained

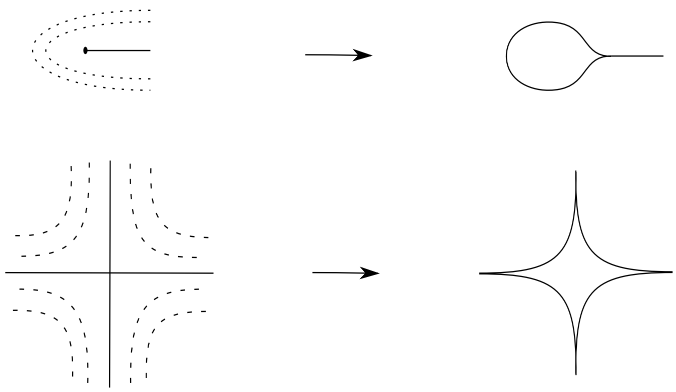


Figure C

in a horizontal strip R, i.e. a rectangle foliated by horizontal arcs. Extend the strip horizontally until its first intersection point with η as shown in Figure D. This is possible since $\cup_i C_i$ is a neighborhood of the critical set of q, the area of q equals one and the width of R is positive. As η is transverse to the horizontal foliation, via a small deformation we may assume that either the side s of R opposite to ζ is contained in a component of η, or that the interior of the side s contains a stop of η which divides s into two subarcs, one arc contained in η and the second arc contained in $S - \eta$.

Assume first that the side s is contained in the interior of a component of η and that it is disjoint from ζ. Then s is contained in the interior of an embedded trainpath of η. Deform the arc ζ within R and glue it to s in such a way that the resulting path is smooth and transverse to the horizontal foliation as shown in Figure D.

If the side s contains a stop v of η in its interior but is disjoint from ζ then there is a subarc ζ' of s which is a neighborhood of the stop v in the branch f of η containing v. The above construction, applied to ζ' with reversed orientation, yields a subrectangle of R with one side a compact subarc of ζ contained in in the interior of ζ. Now simply exchanges the roles of $s \cap \eta$ and ζ and glue ζ' to a subarc of ζ as before.

If s intersects ζ then note that since there are no horizontal cylinders, the endpoint w of ζ which is a stop either is mapped to an interior point of ζ, or it is mapped to a point disjoint from ζ. In the first case we simply decrease the size of ζ and assure in this way that the side opposite to ζ of the new rectangle is disjoint from ζ. Otherwise we reverse the roles of the side s and of ζ as before.

Inductively we decrease in this way the number of univalent vertices of the train track η. After finitely many such steps we obtain a bigon track $\hat{\tau}$ on S without stops. Such a bigon track has all properties of a train track except that there may be complementary components which are bigons, i.e. disks with two cusps

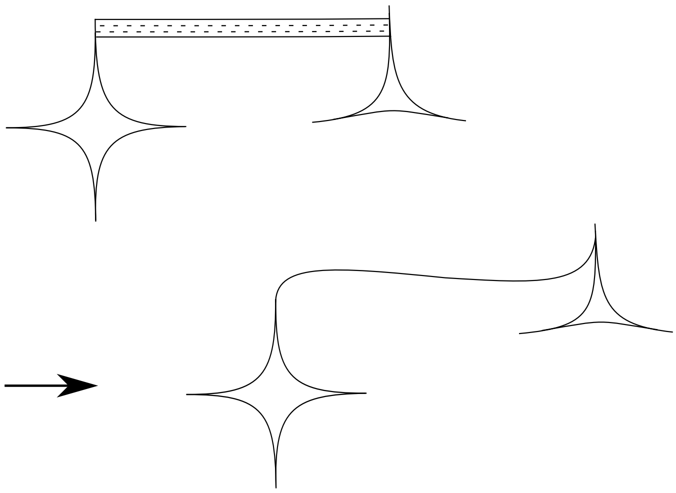


Figure D

on the boundary. By construction, the horizontal measured lamination of q hits $\hat{\tau}$ efficiently, and it is transverse to $\hat{\tau}$.

We claim that $\hat{\tau}$ carries the vertical measured geodesic lamination λ of q. Namely, the vertical measured foliation of q is transverse to the horizontal rectangles. This implies that two vertical geodesics for the flat metric defined by q which pass through the same horizontal rectangle used in the construction bound an embedded foliated rectangle up to the first intersection point of a vertical arc in the germ of a rectangle with a singular point. By construction, this implies that the lamination λ is carried by $\hat{\tau}$.

We collapse the bigon track $\hat{\tau}$ to a train track τ as follows. Let α, β be two sides of a closed bigon B in $\hat{\tau}$, oriented in such a way that they have the same starting point. Assume that the bigon lies to the right of α with respect to the orientation of α and the orientation of S. Note that this makes sense although the interior of β may intersect the interior of α and B may not be homeomorphic to a disk. For $x \in \alpha$ follow the horizontal trajectory through x to the right until its first intersection point $\psi(x)$ with the boundary of B; then $\psi(x) \in \beta$. Since the horizontal foliation of q does not have cylinders, we have $\psi(x) \neq x$ and hence the horizontal arc connecting x to $\psi(x)$ is not a loop. Thus we can collapse this arc to a single point, and this collapsing process can be made to depend continuously on x. Simultaneously collapsing all of these arcs defines a contraction of the bigon B to a single arc.

Successively do this with every bigon; once again, this is possible since there are no horizontal cylinders. The result is a train track τ on S. The horizontal measured geodesic lamination of q hits τ efficiently, and the vertical measured geodesic lamination of q is carried by τ. Each complementary component of τ contains precisely one singular point of q, and the component is a $k + 2$-gon if and only if the singular point is a zero of order k. This yields that τ is large, of type $(m_1,\dots,m_\ell;-m,p)$ and implies the proposition in the case that the horizontal foliation of q does not contain cylinders.

Finally consider a differential q whose horizontal foliation contains cylinders. Recall a direction for q which is sufficiently close to the horizontal foliation is transverse to the initial train track with stops η. Now the set of directions for q such that the foliation of q in this direction is minimal and filling is dense. Thus we can replace the horizontal foliation in the above construction by the foliation in a nearby direction without cylinders. The resulting train track then has all the required properties. ☐

We summarize the discussion in this section as follows.

Let $\mathcal{Q}$ be a component of a stratum $\mathcal{Q}(m_1,\dots,m_\ell;-m,p)$ of $\mathcal{Q}(S)$ (or of a stratum $\mathcal{H}(m_1/2,\dots,m_\ell/2;p)$ of $\mathcal{H}(S)$). Then there is a collection

$$\mathcal{LT}(\tilde{\mathcal{Q}})\subset\mathcal{LT}(m_1,\dots,m_\ell;-m,p)$$

of large marked train tracks τ of the same topological type as $\mathcal{Q}$ such that for every $\tau\in\mathcal{LT}(\tilde{\mathcal{Q}})$ the set

$$\hat{\mathcal{Q}}(\tau)=\cup_t\Phi^t\mathcal{Q}(\tau)$$

contains an open subset of the preimage $\tilde{\mathcal{Q}}$ of $\mathcal{Q}$ in $\tilde{\mathcal{Q}}(S)$ (or in $\tilde{\mathcal{H}}(S)$). A quadratic differential $\tilde q\in\tilde{\mathcal{Q}}(S)$ is contained in the closure of $\tilde{\mathcal{Q}}$ if and only if there a train track $\tau\in\mathcal{LT}(\tilde{\mathcal{Q}})$ and a number $t\in\mathbb{R}$ such that $\Phi^t\tilde q\in\mathcal{Q}(\tau)$.

The set $\mathcal{LT}(\tilde{\mathcal{Q}})$ is invariant under the action of the mapping class group. Its quotient $\mathcal{LT}(\mathcal{Q})$ under this action is finite and is called the *set of combinatorial models for $\mathcal{Q}$*.

Lemma 3.4. *Let $\mathcal{Q}$ be a component of a stratum, with preimage $\tilde{\mathcal{Q}}$ in $\tilde{\mathcal{Q}}(S)$, let $\tau\in\mathcal{LT}(\tilde{\mathcal{Q}})$ and let η be a large train track of the same topological type as τ which is carried by τ. Then $\eta\in\mathcal{LT}(\tilde{\mathcal{Q}})$.*

Proof. A point in $\mathcal{Q}(\tau)$ is defined by a pair (λ,ν) where $\lambda\in\mathcal{V}_0(\tau)$ and where ν is a measured geodesic lamination which is carried by the dual bigon track τ^* of τ. If we choose λ in such a way that its support supp(λ) is of the same topological type as τ and such that λ is carried by the train track η, then up to rescaling, (λ,ν) defines a differential in $\tilde{\mathcal{Q}}(\eta)$. ☐

Define

$$\mathcal{LL}(\tilde{\mathcal{Q}})\subset\mathcal{LL}(m_1,\dots,m_\ell;-m,p)$$

to be the set of all large geodesic laminations of the same topological type as $\tilde{\mathcal{Q}}$ which are carried by some train track $\tau\in\mathcal{LT}(\tilde{\mathcal{Q}})$. The set $\mathcal{LL}(\tilde{\mathcal{Q}})$ is invariant under the action of the mapping class group, and we denote its quotient space by $\mathcal{LL}(\mathcal{Q})$.

We conclude this section with relating train tracks $\tau\in\mathcal{LT}(\tilde{\mathcal{Q}})$ to laminations $\lambda\in\mathcal{LL}(\tilde{\mathcal{Q}})$. To this end choose a complete hyperbolic metric on S of finite volume. Using this metric, for any geodesic lamination λ on S and all $a>0,\epsilon>0$, there is a notion of an a-long train track τ on S which ϵ-*follows* λ (see Section 3 of [H09]). The topological type of τ coincides with the topological type of λ. If λ contains a minimal component which is a simple closed curve c, then τ contains c as an embedded subtrack.

The following is now fairly immediate from the above discussion.

Corollary 3.5. *Let $\tau \in \mathcal{LT}(\tilde{\mathcal{Q}})$ and let λ be a large geodesic lamination of the same topological type as τ which is carried by τ. Then for sufficiently small $\epsilon > 0$, an a-long train track η which ϵ-follows λ is contained in $\mathcal{LT}(\tilde{\mathcal{Q}})$.*

Proof. By Lemma 3.3 of [H09], for sufficiently small ϵ an a-long train track η which ϵ-follows λ is carried by τ, moreover it is large. The corollary now follows from Lemma 3.4. □

4. Components of strata: Combinatorial models

Section 3 can be viewed as a construction of a combinatorial model for every component $\mathcal{Q}$ of a stratum in $\mathcal{Q}(S)$ or in $\mathcal{H}(S)$. The purpose of this section is to refine this construction and obtain models for specific types of degenerations of the stratum.

We begin with introducing the degenerations we are interested in. The framework for these degenerations is in the spirit of the "you see what you get" partial compactification of strata introduced in [MW15]. Although we will not make use of the work in [MW15], we reproduce Definition 2.2 of [MW15]. In its formulation, Σ_i is the singular set of the quadratic differential q_i on the Riemann surface X_i.

Definition 4.1. Say that (X_j, q_j, Σ_j) *converges* to (X, q, Σ) if there are decreasing neighborhoods $U_j \subset X$ with $\cap U_j = \Sigma$ such that the following holds. There are maps $g_j : X - U_j \to X_j$ that are diffeomorphisms onto their range, such that

(1) $g_j^*(q_j)$ converges to q in the compact open topology on $X - \Sigma$.
(2) The injectivity radius at points not in the image of g_j goes to zero uniformly in j.

With this definition, we allow to erase zero area components of a limiting surface with nodes.

Next we introduce two specific types of degenerating sequences which will be used in the sequel.

1) The shrinking half-pillowcase:

Let q be a quadratic differential on a surface S of genus $g \geq 0$ with $n \geq 0$ marked points and vertical foliation $\mathcal{F}$. Choose a point p on S and cut S open along a subarc α of $\mathcal{F}$ issuing from p of length $s > 0$. There are $u + 2$ choices of such arcs where u is the multiplicity of the zero at p (i.e. $u = -1$ means that p is a simple pole, and $u = 0$ means that p is a regular point). The cut open surface has a vertical circle as boundary. Glue a foliated cylinder C to this circle whose opposite boundary is divided into two arcs of the same length which are identified to form half of a pillowcase as shown in Figure E. This does not change the genus of S, but it adds

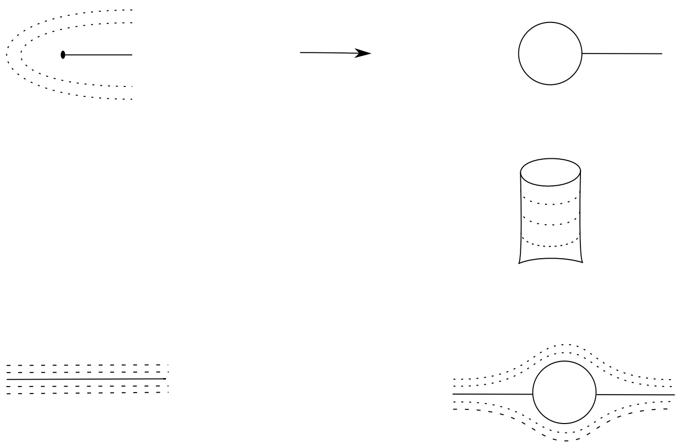


Figure E

two punctures to S, and it increases the multiplicity of each of the endpoints of α by one. Note that for the fixed point p and the fixed (vertical) direction at p, the half-pillowcase is described by three real parameters: Its width (the circumference of the cylinder, or, equivalently, twice the length of the cutting arc), its height and the position of one of the simple poles on the top of the half-pillowcase determined by the choice of a point on the (oriented) cutting arc α.

We call a sequence of quadratic differentials containing a half-pillowcase whose widths tend to zero and which degenerate in the sense of Definition 4.1 to the surface with the half-pillowcase removed a *shrinking half-pillowcase*. We require that the areas of the half-pillowcases (which is the product of the width and the height) tend to zero.

2) The shrinking cylinder:

Choose again a quadratic differential with vertical foliation $\mathcal{F}$. Cut the surface open along two vertical arcs of the same length and glue the two boundary circles to the two boundary circles of a flat cylinder. The genus of the resulting surface S' equals the genus of S plus one. The core curve of the cylinder in S' is non-separating.

We call a sequence of surfaces containing a cylinder which degenerates to the surface with nodes obtained by shrinking the width and the area of the cylinder to zero a *shrinking cylinder*.

Our goal is to construct combinatorial models for quadratic differentials which are suited to describe these two types of degenerations. These models will then be used to construct periodic orbits in the thin part of moduli space.

As in the previous sections, we denote by S a closed oriented surface of genus $g \geq 0$ with n marked points. Fix a number $m \leq n$ so that $3g-3+m \geq 5$. When discussing quadratic differentials on S we will distinguish between m simple poles (punctures) and $p = n - m$ regular marked points.

Definition 4.2. An essential simple closed curve c on S is called *elementary* if either

a) c is non-separating or
b) $m \geq 2$ and c decomposes S into a surface S_0 and a twice punctured disk.

An *elementary pair* is a pair (c_1, c_2) consisting of disjoint elementary curves c_1, c_2 on S. If both c_1 and c_2 are non-separating then we require that $S - (c_1 \cup c_2)$ is connected.

Since $3g-3+m \geq 5$ by assumption, the complement $S-(c_1 \cup c_2)$ of an elementary pair in S contains a (unique) component which is not a three holed sphere. In the sequel we tacitly identify a complementary component of a curve c in S (or any curve system) with its metric completion, i.e. we view $S - c$ as a surface with boundary. We hope that this simplification of notation does not lead to confusion.

Definition 4.3. A *primitive vertex cycle* for a large train track τ is a simple closed curve c embedded in τ which consists of a large branch and a small branch.

If c is a primitive vertex cycle in τ then there are two half-branches incident on the two switches of τ in c which are not contained in c. Since τ is large by assumption, these two half-branches lie on the two different sides of c in an annulus neighborhood of c in S. Namely, otherwise there is a complementary component of τ containing a simple closed curve which is neither contractible nor homotopic into a puncture.

We call a primitive vertex cycle of a train track τ *clean* if its underlying simple closed curve c is elementary and if moreover a branch b which is incident on a switch in c and which is not contained in c satisfies one of the two following conditions.

(1) b is a small branch not contained in the boundary of a bigon with one marked point.
(2) $m \geq 2$, c is separating and b is contained in the twice punctured disk component of $S - c$.

As there are two types of elementary curves, there are two types of clean vertex cycles. To relate these types to the degeneration of quadratic differentials, note that removing a clean vertex cycle c and all its adjacent branches from a train track τ yields a train track τ' on the complementary component S_0 of $S - c$ which is not a three holed sphere.

Type I: The shrinking half-pillowcase.

If $m \geq 2$ and if c is a separating clean vertex cycle of τ then there is a complementary component C for the train track τ' on S_0 which is an annulus whose core curve is homotopic to c. There is a component γ of ∂C embedded in τ', and this component contains at least one cusp (since otherwise τ has a complementary component which is a bigon).

Let S_0' be the surface obtained from S_0 by replacing the boundary circle of S_0 by a marked point (puncture). The genus of S_0' coincides with the genus of S, and the number of marked points has decreased by one. If the component γ of ∂C contains at most two cusps then τ' is a large train track on S_0' which determines a stratum of differentials on $S' = S_0'$. Note that this case corresponds to a shrinking half-pillowcase obtained by opening a vertical arc which is not adjacent to a zero in the limiting differential. If γ contains at least three cusps then remove from S_0' the marked point enclosed by γ and denote the resulting surface by S'. In both cases, τ' is a large train track on the surface S'.

Type II: The shrinking cylinder.

If c is a non-separating clean vertex cycle of τ then then both branches incident on c are small. The genus of the surface with boundary S_0 obtained from S by cutting open along c equals $g-1$. Let S_0' be obtained from S_0 by replacing the two boundary circles (which are copies of c) by a marked point. As before, τ' defines a large train track on the surface S' which either coincides with S_0' or is obtained from S_0' by removing one or both of the special marked points. These choices depend on the number of cusps of the complementary components of c in τ'.

Let $\mathcal{Q}$ be a component of a stratum of quadratic or abelian differentials. Call a train track τ *in special form for* $\mathcal{Q}$ if $\tau \in \mathcal{LT}(\mathcal{Q})$ and if there is an elementary pair (c_1, c_2) for S with the following additional property.

($*$) τ contains each of the curves c_1, c_2 as a clean vertex cycle.

The rest of this section is devoted to the construction of train tracks in special form for all components of strata. Note first that a component of a stratum $\mathcal{Q}(m_1, \ldots, m_\ell; -m, p)$ is obtained from a component of $\mathcal{Q}(m_1, \ldots, m_\ell; -m, 0)$ by marking p regular points. In particular, for fixed p the forgetful map

$$\mathcal{Q}(m_1, \ldots, m_\ell; -m, p) \to \mathcal{Q}(m_1, \ldots, m_\ell; -m, 0)$$

induces a bijection of connected components. Moreover, by induction, a train track in special form for a component $\mathcal{Q}$ of $\mathcal{Q}(m_1, \ldots, m_\ell; -m, p)$ can be obtained from a train track in special form η for a component $\mathcal{Q}'$ of $\mathcal{Q}(m_1, \ldots, m_\ell; -m, p-1)$ by cutting η open along a compact subarc in the interior of a branch b which is disjoint from the two elementary primitive vertex cycles and any adjacent branch and inserting a marked point in the resulting bigon. This shows

Lemma 4.4. *Assume that for each component of $\mathcal{Q}(m_1, \ldots, m_\ell; -m, 0)$ and each component of $\mathcal{H}(k_1, \ldots, k_s; 0)$ there is a train track in special form. Then for every $p \geq 0$ and each component of $\mathcal{Q}(m_1, \ldots, m_\ell; -m, p)$ and each component of $\mathcal{H}(k_1, \ldots, k_s; p)$ there is a train track in special form.*

As a consequence, we only have to analyze components without regular marked points.

For simplicity, write $\mathcal{Q}(m_1, \ldots, m_\ell; -m)$ instead of $\mathcal{Q}(m_1, \ldots, m_\ell; -m, 0)$, and write $\mathcal{H}(k_1, \ldots, k_s)$ instead of $\mathcal{H}(k_1, \ldots, k_s; 0)$.

The idea is to start with explicit train tracks for components of small strata and modify these train tracks to train tracks for all strata. The first type of modification consists in subdividing complementary components as follows.

Let C be a complementary component of a train track η which is a disk with $k \geq 4$ cusps on its boundary ∂C. Then C can be subdivided into two components by adding a small branch which connects two non-adjacent sides of the component. The resulting train track τ is a simple extension of η as defined in Section 2. If η is orientable and if the number of cusps of ∂C at least six then this subdivision can be done in such a way that the components have an even number of cusps and that τ is orientable as well. By Proposition 2.8, if τ, η are either both orientable or both non-orientable then τ is large if and only if this holds true for η. In the sequel we always choose subdivisions of complementary components of orientable train tracks in such a way that the resulting train track is orientable.

Following [L08], strata of quadratic differentilals with at least three simple poles are connected. We use this fact to observe

Lemma 4.5. *Components of strata of differentials on the two-sphere S^2 with $m \geq 6$ punctures or on the two-torus T^2 with $m \geq 4$ punctures admit a train track in special form.*

Proof. Figure F shows large train tracks in special form for strata of differentials on S^2 with a single zero and six or seven simple poles. To construct train track in special from for strata of differentials with a single zero and at least 8 simple poles just attach more copies of a circle enclosing two punctures and containing a monogon to one of the two train tracks shown in Figure F. Train tracks in special form for arbitrary strata of differentials on S^2 with at least five simple poles are obtained from the train tracks for strata with a single zero by subdivision of complementary components.

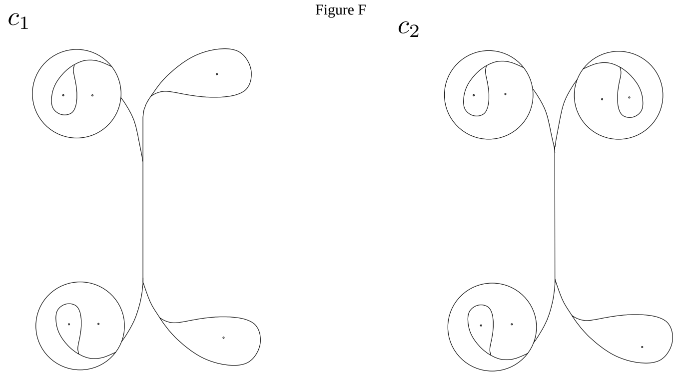
Figure G shows large train tracks containing at least two primitive vertex cycles, one of which is clean, for the three and four punctured torus. To construct train tracks in special form for a torus with at least four punctures, attach more copies of a circle enclosing two punctures and containing a monogon. As before, train tracks in special form for arbitrary strata of differentials on the torus with at least four simple poles are constructed from these train tracks by subdivision of complementary components. □

Call a component of a stratum of abelian or quadratic differentials *hyperelliptic* if it consists of differentials on hyperelliptic surfaces which are invariant under the hyperelliptic involution. Lemma 4.5 is used to show

Lemma 4.6. *Let $\mathcal{Q}$ be a hyperelliptic component of a stratum of quadratic or abelian differentials on a surface of genus at least three. Then there is a train track τ in special form for $\mathcal{Q}$.*

Proof. Let $\mathcal{Q}$ be a hyperelliptic component of a stratum of quadratic differentials on a surface S of genus $g \geq 3$ with $m \geq 0$ punctures. Such a hyperelliptic component

Figure F



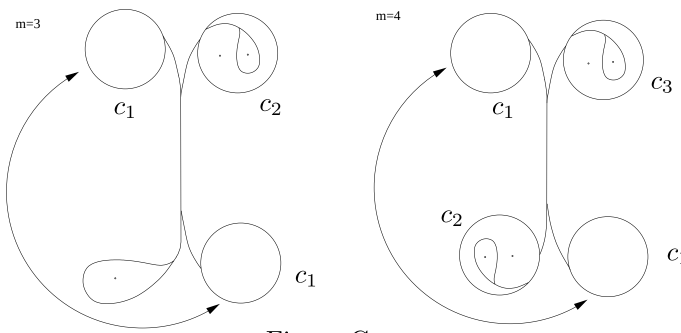


Figure G

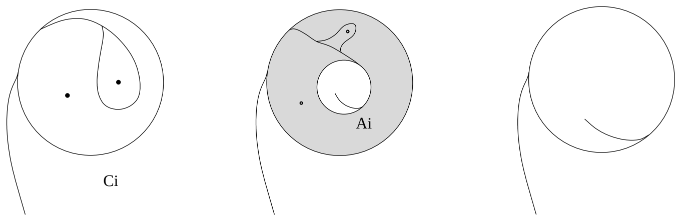
is obtained by pull-back of a stratum $\hat{\mathcal{Q}}$ of quadratic differentials on the sphere S^2 with a double branched cover.

By the main result of [L04] (see also Theorem 1.2 of [L08]), the component $\hat{\mathcal{Q}}$ consists of differentials with at least 6 poles, and the cover is ramified at at all or at all but one of the poles.

Let η be a train track in special form for $\hat{\mathcal{Q}}$ as constructed in Lemma 4.5. Then η contains two clean vertex cycles c_1, c_2 which cut from the punctured sphere two twice punctured disks P_1, P_2.

Choose the branched covering in such a way that it is ramified at each of the punctures in P_1, P_2. The preimage of η under this covering is an embedded graph $\hat{\eta}$ in the surface S. The preimage of the vertex cycle c_i consists of two embedded simple closed curves which bound an embedded annulus A_i. The annulus A_i contains the preimage in $\hat{\eta}$ of the two punctures in the pair of pants with boundary c_i as shown in the middle part of Figure H. It is subdivided into two bigons with an interior marked point by the preimage of the intersection of η with the pair of pants-component of $S - c_i$. The marked point is a preimage of one of the ramification points. Remove these marked points and the branches in the interior of the annulus A_i and identify the two boundary circles of A_i as shown in Figure H so that they form a single simple closed curve v_i $(i = 1, 2)$.

Figure H



Collapse each remaining bigon in $\hat{\eta}$ containing a single preimage of a ramification point to a single arc as described above. The resulting graph τ is a large train track in S which is contained in $\mathcal{LT}(\tilde{\mathcal{Q}})$. The train track τ contains the curves v_i as primitive vertex cycles. By construction, the curves v_i are non-separating and do not form a bounding pair, i.e they define an elementary curve system. Moreover, since c_i is a clean primitive vertex cycle for η, the primitive vertex cycle v_i is clean for τ.

The same reasoning also applies for hyperelliptic components of abelian differentials. Namely, in this case the branched cover defining the component is ramified at each of the simple poles on the two-sphere, i.e. at at least 8 simple poles. The above argument then shows that there is a train track in special form for the component. □

To treat non-hyperelliptic components we construct from a large train track η of topological type $(m_1, \dots, m_\ell; -m)$ on a surface of genus $g \geq 0$ with m punctures a train track τ of type $(m_1, \dots, m_\ell + 4; -m)$ on a surface of genus $g + 1$ by *attaching a handle* as follows.

The train track η has a complementary polygon P with $m_\ell + 2$ sides. Attach two arcs b_1, b_2 of class C^1 to the interior of two branches of η which are contained in two different sides of the polygon P in such a way that b_1, b_2 are disjoint and embedded in P. Attach a simple closed curve $c_i \subset P$ of class C^1 to the arc b_i which meets b_i

only at its free endpoint and is tangent to b_i $(i = 1, 2)$. We require that the curves c_1, c_2 are disjoint and bound disjoint embedded disks D_1, D_2 in the interior of P.

Remove the interiors of the disks D_1, D_2 from P. The boundary of the resulting surface consists of the curves c_1, c_2. Glue c_1 to c_2 with a diffeomorphism which reverses the boundary orientation of D_i. The result is a surface of genus $g+1$ with m punctures which carries a train track τ of topological type $(m_1, \dots, m_\ell + 4; -m)$. It contains the image of the curves c_i under a glueing map as a clean vertex cycle. Note that if η is orientable, then for a suitable choice of the arcs b_1, b_2 the train track τ is orientable as well (see Figure J). In the sequel we always assume that the construction preserves orientability if applicable. We then call τ the train track obtained from η by *attaching a handle*.

Lemma 4.7. *The train track τ obtained from η by attaching a handle is large. Moreover, it is orientable if and only if this holds true for η.*

Proof. By construction, the train track τ is orientable if and only if this holds true for η. Moreover, η can be viewed as a subtrack of τ.

Now η is a large train track and hence it carries a minimal large geodesic lamination of type $(m_1, \dots, m_\ell; -m)$. This geodesic lamination defines a minimal geodesic lamination λ_0 on τ. The train track τ contains a primitive vertex cycle c_0 which is disjoint from λ_0 and which is the image of the curves c_1, c_2 under the glueing process. The union $\lambda_0 \cup c_0$ is a geodesic lamination carried by τ. This lamination is not large, but it is a sublamination of a large geodesic lamination which is the union of $\lambda_0 \cup c_0$ with two isolated leaves which pass through the two branches of τ connecting c_0 to the subtrack η and which spiral from one side about λ_0, from the other side about c_0. Thus τ carries a large geodesic lamination. The same argument also shows that the dual bigon track τ^* carries a large geodesic lamination. In other words, τ is large. □

For the construction of train tracks in special form for all components of strata we use the classifiction of components due to Kontsevich and Zorich [KZ03] (for abelian differentials) and Lanneau [L08] (for quadratic differentials).

Proposition 4.8. (1) *For every $g \geq 4$ the stratum $\mathcal{H}(2g-2)$ has three connected components. One of these components is hyperelliptic, the other two are distinguished by the parity of the spin structure they define.*
(2) *The stratum $\mathcal{H}(4)$ has two components. One of the components is hyperelliptic, the other consists of abelian differentials defining an odd spin structure.*
(3) *$\mathcal{H}(2)$ is connected.*
(4) *For every $g \neq 3, 4$ and every $m \geq 0$ the stratum $\mathcal{Q}(4g-4+m; -m)$ is connected.*
(5) *The strata $\mathcal{Q}(12; 0)$ and $\mathcal{Q}(9; -1)$ have two connected components, and $\mathcal{Q}(4; 0) = \emptyset$,*

We are now ready to show

Proposition 4.9. *Let $\mathcal{Q}$ be a non-hyperelliptic component of a stratum of quadratic or abelian differentials on S where $3g-3+m \geq 5$. Then there is a train track τ in special form for $\mathcal{Q}$.*

Proof. We divide the proof of the proposition into four steps. The case $g=0$ and $g=1$ is covered by Lemma 4.5.

Step 1: Strata of quadratic differentials with at least three poles.

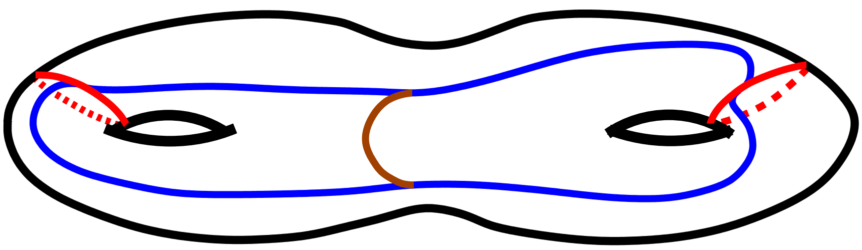
By the classification of Lanneau [L08], every stratum in moduli space consisting of meromorphic quadratic differentials with at least three poles is connected. Thus for $m \geq 3$ and any $g \geq 2$, we can construct a large train track in special form for the stratum $\mathcal{Q}(4g-4+m;-m)$ by attaching handles to train tracks in special form on the torus as described in Lemma 4.7. These train tracks can be subdivided to train tracks in special form for any stratum of quadratic differentials with at least three poles.

Step 2: Strata of abelian differentials with a single zero.

The moduli space $\mathcal{H}(2)$ of abelian differentials with a single zero on a surface of genus 2 is connected. It consists of differentials which define an *even spin structure* Figure I below shows a large train track $\eta \in \mathcal{LT}(\mathcal{H}(2))$.

(see [KZ03]).

Figure I



For $g=3$, the stratum $\mathcal{H}(4)$ consists of two components. One of these components is hyperelliptic. The two components are distinguished by the *parity of the spin structure* they define [KZ03]. The parity of the spin structure for the hyperelliptic component is even. By Lemma 4.6 it suffices to show that we can attach to the train track η in Figure I a handle in such a way that the resulting train track belongs to a component with odd spin structure. To this end we compute from a large train track $\tau \in \mathcal{LT}(\mathcal{Q})$ the parity of the spin structure of the component $\mathcal{Q}$.

The parity of the spin structure defined by an abelian differential ω can be calculated as follows (see p.643 of [KZ03]). For a smooth simple closed curve α on S not passing through a zero of ω define $ind_\alpha \in \mathbb{Z}$ to be the total change of angle between the tangent of α and the vector tangent to the vertical foliation of ω. Let

$\{\alpha_i, \beta_i \mid i = 1, \dots, g\}$ be any system of $2g$ smooth simple closed curves which define a symplectic basis for $H_1(S, \mathbb{Z})$ with the above property. Then

$$\varphi(\omega) = \sum_{i=1}^{g} (ind_{\alpha_i} + 1)(ind_{\beta_i} + 1)(\text{mod } 2).$$

This formula enables us to calculate the parity using a train track. Namely, a large orientable train track τ of type $(4g-4;0)$ has a single complementary component C which is a $4g$-gon. Let α be a smooth simple closed curve on S which intersects τ transversely in finitely many points contained in the interior of some branches of τ. Define the *index* $r_\tau(\alpha) \in \mathbb{Z}/2\mathbb{Z}$ of α as follows.

Choose a numbering of the sides of the complementary region of τ in counterclockwise order. Choose also an orientation of α. A transverse intersection point $p \in \alpha \cap \tau$ is contained in precisely two sides s_1, s_2 of C. Write $r(p) = s_2 - s_1 + 1\,(\text{mod } 2) = s_1 - s_2 + 1\,(\text{mod } 2)$ and define

$$r_\tau(\alpha) = \sum_p r(p) \in \mathbb{Z}/2\mathbb{Z}.$$

Note that if α' is isotopic to α with an isotopy which moves some subarc of α across a switch then this number is unchanged, and the same holds true if α is fixed and τ is modified by a split.

Choose smooth simple closed curves $\{\alpha_i, \beta_i \mid i = 1, \dots, g\}$ which define a symplectic basis of $H_1(S, \mathbb{Z})$. Assume that each of the curves α_i intersects τ in finitely many points which are contained in the interior of some branch of τ. Define

$$\varphi(\tau) = \sum_{i=1}^{g} (r_\tau(\alpha_i) + 1)(r_\tau(\beta_i) + 1) \in \mathbb{Z}/2\mathbb{Z}$$

and call this number the *parity of the spin structure* of τ.

Recall that τ is a large train track with a single complementary component C. If the train track η is obtained from τ by attaching a handle then the parity of the spin structure of η can be calculated from the parity of the spin structure of τ as follows. There is a primitive vertex cycle α_1 for η which is disjoint from τ (it goes around the handle). This vertex cycle α_1 satisfies $r_\eta(\alpha_1) = 0$ since up to homotopy, it has a unique intersection point with η which is contained in one of the small branches adjacent to the primitive vertex cycle α_1. Then this intersection point is contained in two consecutive sides of the complementary component of η.

There is a second curve β_1 in the handle which intersects α_1 in a single point, and it intersects η in a single point q as well. Let s_i, s_j $(i < j)$ be the sides of the complementary component C of τ at which branches of $\eta - \tau$ are attached. If we choose $s_j = s_i + 1$ then Figure J shows that $r_\eta(\beta_1) = 0$.

The curves on S used to calculate the parity of the spin structure for τ can be chosen to be disjoint from $\eta - \tau$ viewed as a subgraph of the complementary component C. Then the indices of the curves used for τ do not change mod 2 and hence the parity of the spin structure of η is opposite to the parity of the spin structure for τ. In particular, attaching a handle to the train track shown in Figure

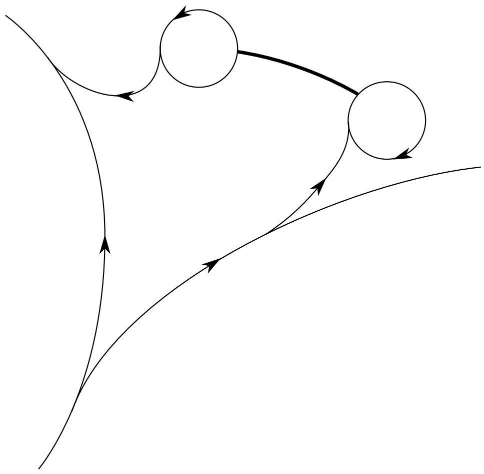


Figure J

I results in a train track in special form for the component of $\mathcal{H}(4)$ with odd spin structure. Thus together with the construction for hyperelliptic components, we obtain a train track in special form for each of the two components of $\mathcal{H}(2)$.

Using again Proposition 4.8, for $k \geq 3$ the two different non-hyperelliptic components of $\mathcal{H}(2k)$ are distinguished by the parity of the spin structure they define. It follows from the above discussion that attaching a handle to a train track in special form for a component of $\mathcal{H}(2k-2)$ with even (or odd) spin structure is a train track in special form for a component of $\mathcal{H}(2k)$ with odd (or even) spin structure. Now it is easy to see that a train track for a hyperelliptic component can only arise by this construction from a train track for a hyperelliptic component. Since the parity of the spin structure of a hyperelliptic component is even, none of the two train tracks arising from attaching a handle to one of the train tracks in special form for a component of $\mathcal{H}(4)$ is a train track for a hyperelliptic component. Thus by induction beginning with $\mathcal{H}(4)$, we conclude in this way that for each $k \geq 1$ and for each non-hyperelliptic component of $\mathcal{H}(2k)$ there is a train track in special form.

Step 3: Strata of quadratic differentials with a single zero and at most two poles.

By the classification of Lanneau [L08], strata of quadratic differentials with a single zero and at most two poles are connected.

To obtain a train track in special form for this stratum on a surface of genus $g = 2$ with $m = 2$ punctures, attach to the train track shown in Figure I a circle as shown in Figure F enclosing two once-puncture monogons. Similarly, to obtain a train track in special form for a stratum with a single zero and a single pole on a surface of genus 3, attach a train track in special form on a surface of genus 3 a once punctured monogon.

A train track in special form for a stratum in higher genus can be obtained by attaching handles to the train track for genus 2 or 3.

Step 4: Subdividing complementary components.

Following [L08], we say that a component $\mathcal{Q}$ of a stratum in $\mathcal{Q}(S)$ for a surface S of genus $g \geq 2$ is *adjacent* to a component $\mathcal{Q}_0$ of another stratum if $\mathcal{Q}_0$ is contained in the closure $\overline{\mathcal{Q}}$ of $\mathcal{Q}$ in $\mathcal{Q}(S)$. Here we allow that poles merge with zeros and disappear.

Lanneau [L08] showed that with the exception of one sporadic component in each of the strata $\mathcal{Q}(9;-1)$, $\mathcal{Q}(3,6;-1)$, $\mathcal{Q}(3,3,3;-1)$, any non-hyperelliptic component of a stratum with at least two distinct types of zeros or poles is adjacent to $\mathcal{Q}(4g-4)$. For such a component, train tracks in special form can be obtained from train tracks in special form for components of strata with a single zero by subdivision of complementary components.

For the completion of the proof of the proposition we are left with the investigation of the sporadic components in genus $g = 3, 4$ as listed in the classification of Lanneau [L08].

The sporadic component for $g = 4$ is a component of $\mathcal{Q}(12)$ which can be checked explicitly. The sporadic component of $\mathcal{Q}(3,3,3;-1)$ is adjacent to the sporadic component of $\mathcal{Q}(3,6;-1)$, and the sporadic component of $\mathcal{Q}(3,6;-1)$ is adjacent to the sporadic component of $\mathcal{Q}(9;-1)$ [L08]. Using Step 2 above, it is therefore enough to construct a train track with the required properties which belongs to the sporadic component of $\mathcal{Q}(9;-1)$. However, the sporadic component of $\mathcal{Q}(9;-1)$ admits a quadratic differential with a two-cylinder-decomposition which can be used to construct a train track as required (compare the table in [L08]). This completes the proof of the proposition. □

Remark 4.10. Although we use the classification result of Kontsevich-Zorich and of Lanneau in our construction, the construction can be used to give a alternative proof for the classification.

5. Periodic orbits in compact subsets of strata

In this short section we collect some results from [H13] in a form needed in Section 6. We continue to use the notations from sections 1-4.

The number $k > 0$ of branches of a large train track $\tau \in \mathcal{LT}(m_1, \ldots, m_\ell; -m, p)$ only depends on the topological type of τ. A numbering of the branches of τ defines an embedding of the cone $\mathcal{V}(\tau)$ of transverse measures on τ onto a closed convex cone in $\mathbb{R}^k$ determined by the switch conditions. For the standard basis $e_1, \ldots, e_k$ of $\mathbb{R}^k$, this embedding associates to a measure $\mu \in \mathcal{V}(\tau)$ the vector $\sum_i \mu(i) e_i \in \mathbb{R}^k$ where we identify a branch of τ with its number. If $\sigma \prec \tau$ then the transformation $\mathcal{V}(\sigma) \to \mathcal{V}(\tau)$ induced by a carrying map $\sigma \to \tau$ is linear in these coordinates.

The mapping class group $\mathrm{Mod}(S)$ acts on marked train tracks by precomposition of marking. If $\varphi \in \mathrm{Mod}(S)$ is such that $\varphi(\tau) \prec \tau$ then the composition of the isomorphism $\mathcal{V}(\tau) \to \mathcal{V}(\varphi\tau) = \varphi(\mathcal{V}(\tau))$ with a carrying map $\mathcal{V}(\varphi\tau) \to \mathcal{V}(\tau)$ is given by a linear map
$$A(\varphi, \tau) : \mathbb{R}^k \to \mathbb{R}^k.$$

By the Perron Frobenius theorem, a (k, k)-matrix A with non-negative entries admits an eigenvector with non-negative entries. The corresponding eigenvalue α is positive. If some power of A is positive, then the generalized eigenspace for α is one-dimensional, and α is bigger than the absolute value of any other eigenvalue of A. We call an eigenvector with nonnegative entries for the eigenvalue α of A a *Perron Frobenius eigenvector.*

The following lemma is Corollary 3.2 of [P88]. For its formulation, recall that a pseudo-Anosov mapping class admits an invariant cotangent line in the Teichmüller space of abelian or quadratic differentials.

Lemma 5.1. *Let $\tau \in \mathcal{LT}(m_1, \dots, m_\ell; -m, p)$ and let $\varphi \in \mathrm{Mod}(S)$ be such that $\varphi(\tau) \prec \tau$ and that the matrix $A(\varphi, \tau)$ is positive. Then φ is pseudo-Anosov. The unit cotangent line of its axis is contained in the stratum $\tilde{\mathcal{Q}}(m_1, \dots, m_\ell; -m, p)$, and it intersects $\mathcal{Q}(\tau)$. The vertical measured geodesic lamination of φ is a Perron Frobenius eigenvector of the matrix $A(\varphi, \tau)$.*

Proof. It follows from Corollary 3.2 of [P88] that φ is pseudo-Anosov and that the attracting fixed point for its action on $\mathcal{PML}$ is the projectivization of a Perron-Frobenius eigenvector λ of the matrix $A(\varphi, \tau)$. Moreover, the unit cotangent line of the axis of φ intersects $\mathcal{Q}(\tau)$. Since $A(\varphi, \tau)$ is positive by assumption, the Perron Frobenius eigenvector λ is unique up to scale and positive. The support of λ (viewed as a measured geodesic lamination) is minimal and of type $(m_1, \dots, m_\ell; -m, p)$. Then a quadratic differential contained in the unit cotangent line of the axis of φ is contained in $\tilde{\mathcal{Q}}(m_1, \dots, m_\ell; -m, p)$. ☐

In general, it is not easy to detect whether or not for two large train tracks $\tau, \eta \in \mathcal{LT}(m_1, \dots, m_\ell; -m, p)$ there is some $\varphi \in \mathrm{Mod}(S)$ so that $\varphi(\eta) \prec \tau$, even if the stratum $\mathcal{Q}(m_1, \dots, m_\ell; -m, p)$ in moduli space is connected. We need the following simple technical lemma to overcome this difficulty. For its formulation, as in Section 2, for a component $\tilde{\mathcal{Q}}$ of a stratum $\tilde{\mathcal{Q}}(m_1, \dots, m_\ell; -m, p)$ we write $\tau \in \mathcal{LT}(\tilde{\mathcal{Q}})$ if $\tau \in \mathcal{LT}(m_1, \dots, m_\ell; -m, p)$ and if moreover the set $\mathcal{Q}(\tau) \subset \tilde{\mathcal{Q}}(S)$ is contained in the closure of $\tilde{\mathcal{Q}}$. Recall also from Section 2 the definition of the set $\mathcal{LL}(\tilde{\mathcal{Q}})$ and of an a-long train track which ϵ-follows a lamination $\lambda \in \mathcal{LL}(\tilde{\mathcal{Q}})$.

Denote by $\mathrm{Stab}(\tilde{\mathcal{Q}})$ the stabilizer of $\tilde{\mathcal{Q}}$ in $\mathrm{Mod}(S)$. A *splitting and shifting sequence* of a train track τ is a sequence of modifications by splitting or shifting moves. If η is carried by τ then τ can be connected to η by a splitting and shifting sequence [PH92].

Lemma 5.2. *There are numbers $\kappa > 0, \epsilon > 0$ with the following property. Let $\tilde{\mathcal{Q}}$ be a component of a stratum in $\tilde{\mathcal{Q}}(S)$ and let $\lambda \in \mathcal{LL}(\tilde{\mathcal{Q}})$. Let $\tau \in \mathcal{LT}(\tilde{\mathcal{Q}})$ be an a-long train track which ϵ-follows λ and let $\sigma \in \mathcal{LT}(\tilde{\mathcal{Q}})$ be carried by τ. Then there is some $\varphi \in \mathrm{Stab}(\tilde{\mathcal{Q}})$ such that $\varphi(\tau) \prec \sigma$ and such that σ can be connected to $\varphi(\tau)$ by a splitting and shifting sequence of length at most κ.*

Proof. Define the *combinatorial type* of a large train track τ on S to be an orbit of τ under the action of the mapping class group (compare Section 2). Let $\tilde{\mathcal{Q}}$ be a component of $\tilde{\mathcal{Q}}(m_1, \dots, m_\ell; -m, p)$ and let $\mathcal{LT}(\mathcal{Q})$ be the finite set of all combinatorial types of large train tracks $\tau \in \mathcal{LT}(\tilde{\mathcal{Q}})$. Denote by $[\tau]$ the element of $\mathcal{LT}(\mathcal{Q})$ which is represented by the train track $\tau \in \mathcal{LT}(\tilde{\mathcal{Q}})$.

For $[\sigma], [\tau] \in \mathcal{LT}(\mathcal{Q})$ define $[\sigma] < [\tau]$ if there is a representative of $[\sigma]$ which is carried by a representative of $[\tau]$. By construction and equivariance under the action of the mapping class group, if $[\eta] < [\sigma]$ and $[\sigma] < [\tau]$ then $[\eta] < [\tau]$. We clearly also have $[\tau] < [\tau]$ for all $[\tau]$ and hence $<$ is a partial order.

Let $\lambda \in \mathcal{LL}(\tilde{\mathcal{Q}})$ and let $\mathcal{C}(\lambda) \subset \mathcal{LT}(\mathcal{Q})$ be the set of all combinatorial types of large train tracks admitting a representative which carries λ. Let $[\tau] \in \mathcal{C}(\lambda)$ be a combinatorial type so that the number n of elements $[\sigma] \in \mathcal{C}(\lambda)$ with $[\sigma] < [\tau]$ is minimal among all elements with $[\tau] < [\eta]$. Now if $[\sigma] < [\tau]$ and if $[\tau] \not< [\sigma]$, then by minimality, there are at least n elements in $\mathcal{C}(\lambda)$ distinct from $[\tau]$ which are smaller than $[\tau]$. This is impossible and consequently $[\tau] < [\sigma]$.

Choose representatives $\sigma \prec \tau$ of the topological types $[\sigma], [\tau]$. Then there is some $\varphi \in \mathrm{Mod}(S)$ so that $\varphi(\tau) \prec \sigma$. By invariance under the action of the mapping class group and the fact that the number of elements in $\mathcal{LT}(\mathcal{Q})$ is finite, the length of a splitting and shifting sequence connecting σ to $\varphi(\tau)$ can be chosen to be uniformly bounded.

To summarize, if $[\tau]$ is a minimal combinatorial type as above, then any representative of $[\tau]$ has the properties stated in the lemma and hence the same holds true for any representative of a combinatorial type $[\sigma]$ with $[\sigma] < [\tau]$. On the other hand, as λ is carried by τ, by Lemma 3.3 of [H09] there is an a-long large train track η which carries λ and ϵ-follows λ and which is carried by τ. Then $[\eta] < [\tau]$ and therefore η has the properties stated in the lemma. □

We call a train track $\tau \in \mathcal{LT}(\tilde{\mathcal{Q}})$ with the properties stated in Lemma 5.2 *essential* for the component $\tilde{\mathcal{Q}}$. It seems likely that every $\eta \in \mathcal{LT}(\tilde{\mathcal{Q}})$ is essential for $\tilde{\mathcal{Q}}$, however we do know this, and we will not need this in the sequel.

Let $\mathcal{T}(S)$ be the Teichmüller space of S. The *translation length* of a pseudo-Anosov element $\varphi \in \mathrm{Mod}(S)$ is the minimal Teichmüller distance between a point $x \in \mathcal{T}(S)$ and its image under φ. The translation length of φ only depends on the conjugacy class of φ.

For a set $\mathcal{A}$ of conjugacy classes of pseudo-Anosov elements in $\mathrm{Mod}(S)$ define the *growth* $\mathrm{gr}(\mathcal{A})$ as follows. For $R > 0$ let $n(\mathcal{A}, R)$ be the number of elements in $\mathcal{A}$ consisting of conjugacy classes of translation length at most R and let

$$\mathrm{gr}(\mathcal{A}) = \lim\inf_{R\to\infty} \frac{1}{R} \log n(\mathcal{A}, R).$$

The following statement is now an easy consequence of the results in [H13]. For its formulation, let $\tau \in \mathcal{LT}(\tilde{\mathcal{Q}})$ and let $\varphi \in \mathrm{Mod}(S)$ be a pseudo-Anosov element with $\varphi(\tau) \prec \tau$ so that the cotangent line of the axis of φ is contained in $\tilde{\mathcal{Q}}$. Then there is a quadratic differential $q \in \mathcal{Q}(\tau)$ which is contained in the cotangent line of the axis of φ. There is a bijection between the complementary components of τ and the singular points of the quadratic differential q. In general, φ permutes these singular points, but a fixed multiple of φ fixes each of the singular points. Or, equivalently, φ fixes each of the complementary components of τ.

Proposition 5.3. *Let $\tilde{\mathcal{Q}}$ be a component of the stratum $\tilde{\mathcal{Q}}(m_1,\dots,m_\ell;-m,p)$. For $\tau\in\mathcal{LT}(\tilde{\mathcal{Q}})$ and $c>0$ let $\mathcal{A}(\tau,c)$ be the set of all conjugacy classes of pseudo-Anosov elements $\varphi\in\mathrm{Stab}(\tilde{\mathcal{Q}})$ with the following properties.*

a) $\varphi(\tau)\prec\tau$.
b) *The matrix $A(\varphi,\tau)$ is positive, and the ratios of the entries of $A(\varphi,\tau)$ are bounded from above by c.*
c) *φ fixes each of the complementary components of τ.*

There is a number $c=c(\tilde{\mathcal{Q}})>0$ such that whenever τ is essential for $\tilde{\mathcal{Q}}$, then the growth of $\mathcal{A}(\tau,c)$ is not smaller than $h(\tilde{\mathcal{Q}})$.

Proof. Let τ be essential for $\tilde{\mathcal{Q}}$. Then $U=\cup_{t\in\epsilon,\epsilon)}\Phi^t\mathcal{Q}(\tau)\cap\tilde{\mathcal{Q}}$ is an open subset of $\tilde{\mathcal{Q}}$. Choose a point $\zeta\in U$ which projects to a periodic point for the Teichmüller flow on moduli space. This periodic point is defined by a mapping class φ with $\varphi(\tau)\prec\tau$. We may assume that the matrix $A(\varphi,\tau)$ is positive. In particular, if $\beta\in\mathrm{Mod}(S)$ is such that $\beta\tau\prec\tau$ then by Lemma 5.1, $\varphi\circ\beta$ is pseudo-Anosov, with positive matrix $A(\varphi,\tau)A(\beta,\tau)$. The matrix $A(\beta,\tau)$ is non-negative.

We claim that there is a number $c>0$ not depending on β such that the ratios of the entries of the matrix

$$A=A(\varphi,\tau)A(\beta,\tau)A(\varphi,\tau)$$

are bounded from above by c.

Namely, let $\ell>0$ be the maximum of the ratios of the entries of the matrix $A(\varphi,\tau)$. Then up to a factor of at most ℓ, the entries in each line of the matrix $A(\beta,\tau)A(\varphi,\tau)$ coincide with a fixed multiple of the sum of the entries of the matric $A(\beta,\tau)$ in the same line. In particular, the matrix $A(\beta,\tau)A(\varphi,\tau)$ is positive, and the ratios of its entries in a fixed line are bounded from above by ℓ.

Similarly, up to a factor of at most ℓ, the entries in each row of the matrix A coincide with a fixed multiple of the sum of the entries of $A(\beta,\tau)A(\varphi,\tau)$ in the same row. By the discussion in the previous paragraph, this implies that the ratios of the positive matrix A are bounded from above by ℓ^2.

On the other hand, as $\cup_{t\in(-\delta,\delta)}\Phi^t\mathcal{Q}(\tau)$ is an open subset of $\tilde{\mathcal{Q}}$ for any $\delta>0$, the main result of [H13] shows that the growth rate of periodic orbits through $\mathcal{Q}(\tau)$ equals the dimension $h(\tilde{\mathcal{Q}})$. The above calculation shows that concatentation at the beginning and the end with the fixed orbit defined by φ adjusts the ratios of the entries of the corresponding matrices without changing the growth rate.

With the same argument, we can also achieve that the complementary components of τ are all fixed by the pseudo-Anosov mapping classes we found. $\square$

6. Periodic orbits in the thin part of strata

In this section we only consider strata of differentials without marked regular points. Our main goal is to prove Theorem 1 from the introduction. Thus strata of abelian differentials are defined on surfaces without punctures. Strata of differentials with marked regular points will however be used in the proof.

Let S be a surface of genus $g \geq 0$ with $m \geq 0$ punctures where $3g-3+m \geq 5$. Recall that a quadratic (or abelian) differential $q \in \tilde{\mathcal{Q}}(S)$ defines a singular euclidean metric of area one on the surface S with singularities at the zeros and at the poles of the differential. There is a unique finite volume complete hyperbolic metric on S for the underlying conformal structure.

Define the *systole* of this hyperbolic metric to be the smallest length of a simple closed geodesic. We write $\mathcal{T}(S)_\epsilon \subset \mathcal{T}(S)$ for the set of all marked complete hyperbolic metrics on S of finite volume whose systole is at least ϵ. The mapping class group acts properly and cocompactly on $\mathcal{T}(S)_\epsilon$.

The *q-length* of an *essential* simple closed curve c (i.e. a simple closed curve which is not contractible and not freely homotopic into a puncture) is defined to be the infimum of the lengths with respect to the singular euclidean metric of any curve which is freely homotopic to c.

The following observation is an easy consequence of invariance under the action of the mapping class group and cocompactness. A much stronger and more precise version is due to Rafi [R14]. For its formulation, let

$$P : \tilde{\mathcal{Q}}(S) \to \mathcal{T}(S)$$

be the canonical projection which associates to a marked quadratic differential its underlying marked hyperbolic metric.

Lemma 6.1. *For every $\epsilon > 0$ there is a number $\delta = \delta(S, \epsilon) > 0$ with the following property. Let $q \in \tilde{\mathcal{Q}}(S)$ and assume that there is an essential simple closed curve on S of q-length at most δ; then $Pq \notin \mathcal{T}(S)_\epsilon$.*

Proof. By the collar lemma for hyperbolic surfaces, every cusp of a hyperbolic surface has a standard embedded neighborhood. Such a neighborhood is homeomorphic to a punctured disk. The hyperbolic distance between any two such cusp neighborhoods is bounded from below by a universal positive constant. Every simple closed geodesic for the hyperbolic metric is contained in the compact complement K of the union of these neighborhoods. For every $\epsilon > 0$ and every surface in $\mathcal{T}(S)_\epsilon$, the diameter of K is bounded from above by a constant only depending on ϵ and the topological type of S.

By Lemma 3.3 of [Mi94], for every $\epsilon > 0$ there is a number $L = L(\epsilon) > 1$ such that for every $q \in \tilde{\mathcal{Q}}(S)$ with $Pq \in \mathcal{T}(S)_\epsilon$, the singular euclidean metric defined by q is L-bilipschitz equivalent to the hyperbolic metric on the compact set K. By the choice of the standard neighborhoods of the cusps, for every hyperbolic surface whose systole is at least ϵ, every essential simple closed curve on S intersects the set K in a union of arcs whose hyperbolic length is bounded from below by some fixed number $c > 0$ only depending on the topological type of S and on ϵ. Then the q-length of any essential simple closed curve on S is not smaller than c/L. This shows the lemma. $\square$

For $\epsilon > 0$ define

$$\tilde{\mathcal{Q}}(\epsilon) = \{q \in \tilde{\mathcal{Q}}(S) \mid Pq \in \mathcal{T}(S)_\epsilon\}.$$

The sets $\tilde{\mathcal{Q}}(\epsilon)$ are invariant under the action of $\mathrm{Mod}(S)$ on $\tilde{\mathcal{Q}}(S)$. Their projections

$$\mathcal{Q}(\epsilon) = \tilde{\mathcal{Q}}(\epsilon)/\mathrm{Mod}(S) \subset \mathcal{Q}(S)$$

to $\mathcal{Q}(S)$ are compact and satisfy $\mathcal{Q}(\epsilon) \subset \mathcal{Q}(\delta)$ for $\epsilon > \delta$ and $\cup_{\epsilon>0}\mathcal{Q}(\epsilon) = \mathcal{Q}(S)$.

For a component Q of a stratum of quadratic or abelian differentials and for $\epsilon > 0$ let

$$n_\epsilon(\mathcal{Q}) \geq 0$$

be the asymptotic growth rate of the number of closed orbits for the Teichmüller geodesic flow which are contained in $\mathcal{Q}$ and which do not intersect $\mathcal{Q}(\epsilon)$. Our goal is to show that $n_\epsilon(\mathcal{Q}) \geq h(\mathcal{Q}) - 1$ for all ϵ.

The strategy is to use a combinatorial control on an invariant subset of the Teichmüller flow with a train track τ in special form for $\mathcal{Q}$. We recall the important properties of τ.

(1) τ contains two clean vertex cycles c_1, c_2. Either the surface $S - c_1 - c_2$ is connected, or it consists of a connected component N which is different from a sphere with at most three holes or a torus with at most one hole and one or two additional components which are twice punctured disks.
(2) Let S_0 (or S_1, S_2) be the surface which is obtained from the component of $S - c_1 - c_2$ (or of $S - c_1, S - c_2$) different from a sphere with at most three holes by replacing the boundary circles by punctures. Then the graph σ_0 on S_0 (or σ_1, σ_2 on S_1, S_2) obtained by removing from τ all branches which are incident on a switch in $c_1 \cup c_2$ (or incident on a switch in c_1, c_2) is a connected train track with at least one large branch.

We showed in Section 4 that there is a component $\mathcal{Q}_1$ of a stratum for the surface S_1 (possibly with marked regular points) such that $\sigma_1 \in \mathcal{LT}(\tilde{\mathcal{Q}}_1)$ where $\tilde{\mathcal{Q}}_1$ is the preimage of $\mathcal{Q}_1$. In particular, $\cup_t \Phi^t \mathcal{Q}(\sigma_1)$ contains an open subset of $\tilde{\mathcal{Q}}_1$. The component $\mathcal{Q}_1$ is determined as follows.

Case 1: $m \geq 2$ and the simple closed curve c_1 is separating.

Then c_1 bounds a twice punctured disk. There is a simple closed curve α embedded in the train track σ_1 (however with cusps) which is freely homotopic to c_1. Viewing σ_1 as a train track on the surface S_1, the curve α encloses the added marked point on S_1 (which replaces the circle c_1). We require that a differential in $\mathcal{Q}_1$ has

- a simple pole at this marked point in the case that α has a single cusp,
- a regular point if α is a bigon or

- a zero if α has at least three cusps (in which case we remove the marked point).

Recall from the introduction the dimension $h(\mathcal{Q}_1)$ of the complex algebraic orbifold containing $\mathcal{Q}_1$ as a real hypersurface. We have $h(\mathcal{Q}_1) = h(\mathcal{Q}) - 2$. A degeneration of differentials in $\mathcal{Q}$ to a differential in $\mathcal{Q}_1$ corresponds to a shrinking half-pillowcase.

Case 2: The curve c_1 is non-separating

Then $S - c_1$ has two boundary components. Each of these boundary components is contained in a complementary component of σ_1 (these components may coincide). If there are two distinct such components then each of them is an annulus. One boundary component of such an annulus is the curve c_1, and the second boundary component α is contained in σ_1. As before, put a simple pole on the marked point which replaces the curve c_1 if α is a mongon, a regular marked point if α is a bigon and a zero if α contains at least three cusps. Once again, $h(\mathcal{Q}_1) = h(\mathcal{Q}) - 2$. A degeneration of differentials in $\mathcal{Q}$ to a differential in $\mathcal{Q}_1$ corresponds to a shrinking cylinder.

Our goal is to use the growth estimate from Section 5 for periodic orbits in $\mathcal{Q}_1$ which are defined by pseudo-Anosov mapping classes with train track expansion σ_1. Such orbits can be thought of as defined by reducible mapping classes on S. We then concatenate these mapping classes in a controlled way with a pseudo-Anosov element for S with train track expansion τ. The next technical observation is fairly immediate from Lemma 5.2.

Lemma 6.2. *τ can be chosen such that the train track σ_1 is essential for $\mathcal{Q}_1$.*

Proof. Let η be in special form for $\mathcal{Q}$, with clean primitive vertex cycles c_1, c_2, and let λ be a large geodesic lamination of the same topological type as τ which is carried by τ and contains c_1, c_2 as minimal components. We may assume that λ has three minimal components, and that one of the components fills S_0. Then the lamination obtained from λ by removing the minimal component c_1 and the leaves spiraling about c_1 is a large lamination on S_1 which is contained in $\mathcal{LL}(\mathcal{Q}_1)$.

By Lemma 5.2, for sufficiently small $\epsilon > 0$ an a-long train track which ϵ-follows λ has the required property. $\square$

For $i = 0, 1, 2$ and for the train tracks σ_i choose elements $\varphi_i \in \mathrm{Mod}(S_i)$ with the properties stated in Proposition 5.3 for the numbers $c_i = c(\mathcal{Q}_i)$. Since these elements are required to fix each of the complementary components of σ_i, for $i = 1, 2$ they can be extended to elements of the mapping class group of $S - c_i$ fixing c_i pointwise. Such an extension in turn can be viewed as a reducible element of $\mathrm{Mod}(S)$.

Now the mapping class group of a punctured surface is a quotient of the mapping class group of a surface with boundary, so a choice for an extension as described in the previous paragraph is by no means unique. We use the ambient train track τ to construct a specific extension as follows.

Lemma 6.3. *There is a natural choice of an extension of φ_i to an element of* $\mathrm{Mod}(S)$ *defined by a splitting and shifting sequence of τ which does not involve a split at any branch in $\tau - \sigma_i$.*

Proof. By construction, there are at most two small branches b_1, b_2 which connect σ_i to the primitive vertex cycle c_i.

Now let e be a large branch of σ_i. Then e defines a trainpath on τ of length at most three. There is a modification of τ by a sequence of shifts and splits at branches contained in σ_i such that the modified train track τ' is large and contains e as a large branch. Then a split of σ_i at e can be viewed as a split of τ' at e, and the split track contains c_i as a clean vertex cycle. By induction, we obtain a splitting and shifting sequence of τ as required. □

In the sequel we will always identify the map φ_i with this particular extension, i.e. we simply view φ_i as a reducible mapping class with $\varphi_i\tau \prec \tau$ and such that there is a carrying map $\varphi_i\tau \to \tau$ which is the identity on $\tau - \sigma_i$. The map φ_0 is viewed as a reducible element of $\mathrm{Mod}(S)$ fixing $c_1 \cup c_2$ pointwise.

In the statement of the following lemma, $\circ$ means composition, i.e. $a \circ b$ represents the mapping class obtained by applying b first followed by an application of a. The mapping class group acts by precomposition of marking from the *right* on the space of marked train tracks. Thus with this convention, $(\varphi \circ \psi)(\tau)$ is the train track obtained from τ by first changing the marking with ψ^{-1} and afterwards with φ^{-1}.

As we will need this many times in the sequel, we explain now how the composition $\varphi \circ \psi$ acts on the cone $\mathcal{V}(\tau)$ of transverse measures on τ. Namely, view as before $\mathcal{V}(\tau)$ as a cone in a linear subspace of the vector space $\mathbb{R}^k$ with basis the branches of τ. Since $\psi\tau \prec \tau$, the map ψ induces a linear mapping $A(\psi, \tau)$ of $\mathbb{R}^k$ preserving $\mathcal{V}(\tau)$. Then the action of $\varphi \circ \psi$ is defined by the product matrix $A(\varphi, \tau)A(\psi, \tau)$.

Lemma 6.4. *For every $k > 0$ the mapping class*

$$\zeta(k) = (\varphi_0^k \circ \varphi_2 \circ \varphi_0^k) \circ (\varphi_0^k \circ \varphi_1 \circ \varphi_0^k)$$

is pseudo-Anosov.

Proof. Note first that $\zeta(k)(\tau) \prec \tau$ for all k. Namely, by assumption on φ_i we have $\varphi_i(\tau) \prec \tau$ (see the above discussion) and hence by invariance of the carrying relation under the action of the mapping class group and induction, we conclude that

$$\zeta(k)(\tau) = (\varphi_0^k \circ \varphi_2 \circ \varphi_0^{2k} \circ \varphi_1)(\varphi_0^k(\tau)) \prec \varphi_0^k(\tau) \prec \tau.$$

Now $\zeta(k)^2$ is pseudo-Anosov if and only if this is true for $\zeta(k)$. Thus by Lemma 5.1, it suffices to show that the matrix

$$A(\zeta(k)^2, \tau) = A(\varphi_0^k, \tau) \cdots A(\varphi_1, \tau)A(\varphi_0^k, \tau)$$

is positive. This is equivalent to stating that a carrying map $\zeta(k)^2\tau \to \tau$ maps every branch of $\zeta(k)^2\tau$ *onto* τ.

Let k be the number of branches of τ and let $\mathbb{R}^k$ be the real vector space spanned by the branches of τ. Write $\mathbb{R}^k = \mathbb{R}^{\ell_0} \oplus \mathbb{R}^{\ell_1} \oplus \mathbb{R}^{\ell_2}$ where $\mathbb{R}^{\ell_0}$ is spanned by the branches of τ contained in σ_0 and where for $i = 1, 2$ the vector space $\mathbb{R}^{\ell_i}$ is spanned by the branches of $\sigma_i - \sigma_0$.

For $i = 1, 2$ define

$$A_i = A(\varphi_0^k \circ \varphi_i \circ \varphi_0^k, \tau).$$

The matrix A_1 preserves the decomposition $\mathbb{R}^k = \mathbb{R}^{\ell_0+\ell_1} \oplus \mathbb{R}^{\ell_2}$ and therefore with respect to the basis consisting of the branches of τ, it is in block form. The square matrix which describes the action of A_1 on $\mathbb{R}^{\ell_0+\ell_1}$ is positive, and the square matrix which describes the action on $\mathbb{R}^{\ell_2}$ is the identity. The same holds true for A_2, with the roles of ℓ_1 and ℓ_2 exchanged.

Since $\ell_0 > 0$, this implies that the image of a basis vector of $\mathbb{R}^{\ell_0+\ell_1}$ under the matrix A_2A_1 is a positive vector in $\mathbb{R}^k$. Similarly, the image of a basis vector in $\mathbb{R}^{\ell_0+\ell_2}$ under the matrix A_1A_2 is a positive vector in $\mathbb{R}^k$ and hence the matrix $A_2A_1A_2A_1$ is indeed positive. This is what we wanted to show. □

The periodic orbits in the ϵ-thin part of the strata we are going to count are defined by pseudo-Anosov classes of the form described in Lemma 6.4 where we let φ_1 vary and fix k and φ_0, φ_2. This construction is carried out in the next proposition which completes the proof of Theorem 1 from the introduction.

Proposition 6.5. *Let S be a closed surface of genus $g \geq 0$ with $m \geq 0$ punctures and $3g - 3 + m \geq 5$. Then for every component $\mathcal{Q}$ of a stratum in $\mathcal{Q}(S)$ or $\mathcal{H}(S)$ we have*

$$n_\epsilon(\mathcal{Q}) \geq h(\mathcal{Q}) - 1.$$

Proof. Using the notations from Lemma 6.4, the carrying map $\varphi_i\tau \to \tau$ can be chosen in such a way that it maps each branch of $\varphi_i\sigma_i$ onto σ_i, and it induces the identity on $\tau - \sigma_i$.

By Proposition 5.3 and the construction, for $i = 1, 2$ there is a number $a_i > 0$ with the following property. Let μ be a measured geodesic lamination on S which is carried by $\varphi_i(\sigma_i)$ and which defines the transverse measure $\mu_i \in \mathcal{V}_0(\sigma_i)$; then $\mu_i(b_1)/\mu_i(b_2) \leq a_i$ for any two branches b_1, b_2 of σ_i.

This implies the existence of a number $a > 0$ with the following property. Let μ be *any* measured geodesic lamination which is carried by $\varphi_i\tau$. Assume that the transverse measure on $\varphi_i\tau$ defined by μ is not supported in $\varphi_i\tau - \varphi_i\sigma_i$. Let μ_0 be the measure on τ induced from μ by a carrying map $\varphi_i\tau \to \tau$ and let $\mu_0(\sigma_i)$ be the total weight of the restriction of μ_0 to σ_i. Then

$$\mu_0(b_i) \geq 2a\mu_0(\sigma_i)$$

for every branch b_i of σ_i $(i = 1, 2)$.

For $\epsilon \in (0, \frac{1}{2})$ and $i = 0, 1, 2$ let $C_i(\epsilon)$ be the closed subset of $\mathcal{V}_0(\tau)$ containing all transverse measures ν with the following properties.

(1) The sum of the ν-weights over all branches of τ which are *not* contained in σ_i is at most ϵ.
(2) For any branch b_i of $\sigma_i < \tau$ we have $\nu(b_i) \geq a$.

Note that the set $C_i(\epsilon)$ is not empty by the above choice of the number a. Also, by the second property above, we have $\nu(\sigma_0) \geq a$ for every $\nu \in C_i(\epsilon)$ and $i = 1, 2$.

As in the proof of Lemma 6.4, let

$$\mathbb{R}^k = \mathbb{R}^{\ell_0} \oplus \mathbb{R}^{\ell_1} \oplus \mathbb{R}^{\ell_2}$$

be the vector space generated by the branches of τ. The subspace $\mathbb{R}^{\ell_0}$ is generated by the branches of τ contained in σ_0. A point $\nu \in C_i(\epsilon)$ $(i = 0, 1, 2)$ is a non-negative vector $v(\nu) \in \mathbb{R}^k$ with the property that the coordinates of the basis elements in $\mathbb{R}^{\ell_0} + \mathbb{R}^{\ell_i}$ are bounded from below by $a > 0$ and that the sum of the coordinates equals one.

Let as before $A(\varphi_0, \tau)$ be the matrix which describes the action of φ_0 on $\mathcal{V}(\tau)$. There is an induced action on $\mathcal{V}_0(\tau)$ by rescaling of the total mass; we denote this action by $\hat{A}(\varphi_0, \tau)$. We claim that there is a constant $u > 0$ only depending on φ_0 such that for every $\epsilon > 0$, for every $k \geq -u \log \epsilon$ and for $i = 1, 2$ we have

$$\hat{A}(\varphi_0, \tau)^k (C_i(\epsilon)) \subset C_0(\epsilon).$$

We show the claim for $i = 1$, the claim for $i = 2$ follows in exactly the same way. Thus let $\nu \in C_1(\epsilon)$. Recall first that by the choice of the number $a > 0$, we have $\nu(\sigma_0) \geq a$.

The matrix $A(\varphi_0, \tau)$ preserves the decomposition $\mathbb{R}^k = \mathbb{R}^{\ell_0} \oplus \mathbb{R}^{\ell_1} \oplus \mathbb{R}^{\ell_2}$ and hence it is in block form. The square matrix A_0 which defines the action on $\mathbb{R}^{\ell_0}$ is positive, and the square matrix defining the action on $\mathbb{R}^{\ell_1} \oplus \mathbb{R}^{\ell_2}$ is the identity. Let $\|A_0\|$ be the operator norm of A_0 with respect to the norm $|v| = \sum_i |v_i|$. By positivity, there is a number $\delta > 0$ such that the smallest entry of A_0 is not smaller than $\delta \|A_0\|$. Note that $1/\delta$ is just the largest ratio of any two entries of A_0.

Let $\alpha > 1$ be the Perron Frobenius eigenvalue of A_0. Since $\nu(\sigma_0) \geq a$, by eventually decreasing δ, for $k \geq 1$ we have

$$\varphi_0^k(\nu)(\sigma_0) \geq a\delta\alpha^k,$$

moreover $\varphi_0^k(\nu)(\tau - \sigma_0) \leq 1 - a$. Therefore for

$$k \geq (\log(2(1-a)) - \log(a\delta\epsilon))/\log(\alpha)$$

we have $\varphi_0^k(\nu)(\sigma_0) \geq 2\varphi_0^k(\nu)(\tau - \sigma_0)/\epsilon$. It now follows from the assumptions on φ_0 that indeed $\varphi_0^k(\nu)/\varphi_0^k(\nu)(\tau) \in C_0(\epsilon)$.

By definition of the maps φ_i, we also have $\hat{A}(\varphi_i, \tau)(C_0(\epsilon)) \subset C_i(\epsilon)$ for $i = 1, 2$. Together we deduce the existence of a number $k(\epsilon) \sim -\log \epsilon > 0$ such that for $k > k(\epsilon)$, the set $C_0(\epsilon)$ is invariant under the map which assigns to a measured geodesic lamination $0 \neq \mu \in \mathcal{V}(\tau)$ the normalized image of $\zeta(k)\mu \in \mathcal{V}(\zeta(k)\tau)$ under a carrying map $\mathcal{V}(\zeta(k)\tau) \to \mathcal{V}(\tau)$. In particular, if $k > k(\epsilon)$ and if $q \in \mathcal{Q}(\tau)$ is contained in the $\zeta(k)$-invariant flow line of the Teichmüller flow then the vertical measured geodesic lamination λ of q is contained in $C_0(\epsilon)$ by invariance.

The q-length of the simple closed curve c_i is contained in the interval

$$[\frac{1}{\sqrt{2}}(\iota(\lambda,c_i)+\iota(\nu,c_i)),\iota(\lambda,c_i)+\iota(\nu,c_i)]$$

where λ,ν is the vertical and the horizontal measured geodesic lamination of q, respectively. The intersection numbers $\iota(\lambda,c_i),\iota(\nu,c_i)$ can be estimated as follows.

By Lemma 2.5 of [H06], the intersection of λ with the simple closed curve on S defined by the embedded trainpath $c_i\subset\tau-\sigma_i$ on τ is bounded from above by the sum of the λ-weights of the branches of $\tau-\sigma_i$. As $\lambda\in C_0(\epsilon)$, this intersection is at most ϵ.

The support of the vertical (or horizontal) measured geodesic lamination of a quadratic differential q is invariant under the action of the Teichmüller flow Φ^t, and its transverse measure scales with the scaling constant $e^{t/2}$ (or with the scaling constant $e^{-t/2}$- note that this means that the length of the vertical measured geodesic lamination is decrasing along a flow line of the Teichmüller flow).

Let $\kappa>1$ be such that $\kappa\lambda\in\mathcal{V}_0(\varphi_0^k\varphi_i\varphi_0^k(\tau))$. By the above argument, we have $k\lambda(\varphi_0^k\varphi_1\varphi_0^k(\tau-\sigma_1))<\epsilon$ and therefore

$$\iota(\kappa\lambda,\varphi_0^k\varphi_1\varphi_0^k(c_1))\leq\epsilon.$$

Since $\varphi_0^k\varphi_1\varphi_0k(c_1)=c_1$, for every $s\leq 2\log\kappa$ the intersection number between c_1 and the vertical measured geodesic lamination of $\Phi^s q$ is at most ϵ.

As the transverse measure of the horizontal measured geodesic lamination for a quadratic differential is decreasing along the Teichmüller flow, to show that the $\Phi^s q$-length of c_1 is at most 2ϵ for every $0\leq s\leq 2\log\kappa$ it now suffices to show that $\iota(\nu,c_1)\leq\epsilon$ where ν is the horizontal measured geodesic lamination of q.

By construction, the carrying map $\varphi_2\tau\prec\tau$ maps every branch of $\varphi_2(\sigma_2)$ *onto* σ_2. This implies the following. Let $\chi<1$ be such that the total weight of the measured geodesic lamination $\chi\lambda$ on $\varphi_2^{-1}\varphi_0^{-k}\tau=\eta$ equals one. Then the $\chi\lambda$-weight of every branch in $\varphi_2^{-1}c_1=\varphi_2^{-1}\varphi_0^{-k}(c_1)$ is bounded from *below* by the number a introduced in the beginning of this proof.

The intersection number $\iota(\chi\lambda,\chi^{-1}\nu)=\iota(\lambda,\nu)$ can be calculated as

$$\sum_b\omega(\chi\lambda,b)\omega^*(\chi^{-1}\nu,b)$$

where the sum is over all branches b of $\varphi_2^{-1}\varphi_0^{-k}(\tau)=\eta$ and $\omega(\chi\lambda,b)$ and $\omega^*(\chi^{-1}\nu,b)$ are the weights of b for the transverse or tangential measure determined by $\chi\lambda,\chi^{-1}\nu$ [PH92]. As this intersection number equals one, we have $\omega^*(\chi^{-1}\nu,b)\leq/\ a$ for every branch b of the subtrack $\varphi_2^{-1}\varphi_0^{-k}(\sigma_2)$. Now c_1 is embedded in σ_2 and therefore the total weight of c_1 with respect to the tangential measure on τ defined by ν is at most $2\chi/a$ (recall that c_1 is an embedded subtrack of τ consisting of precisely two branches). As $\chi=\chi(k)\to 0$ $(k\to\infty)$, for sufficiently large k the intersection number between ν and c_1 is smaller than ϵ.

To summarize, by the scaling properties for the transverse measures of the vertical and horizontal measured geodesic laminations under the Teichmüller flow, for sufficiently large but fixed $k > 0$, for the number $\kappa > 0$ determined above and for $0 \le t \le 2\log\kappa$, the $\Phi^t q$-length of the curve c_1 does not exceed 2ϵ.

Let $u > 0$ be such that the total weight of $u\lambda$ on $\zeta(k)\tau$ equals one. The same argument as above shows that the $\Phi^t q$-length of $\varphi_0^k\varphi_1\varphi_0^k(c_2)$ is less than ϵ for $2\log\kappa \le t \le 2\log u$. Together we conclude that indeed, the orbit defined by $\zeta(k)$ is entirely contained in the set of differentials which admit an essential simple closed curve of length at most 2ϵ.

We are left with estimating the number of periodic orbits of the form $\zeta(k)$ as above where we fix φ_0 and φ_2 and vary φ_1. To this end let $T(\varphi_1) > 0$ be the translation length of the pseudo-Anosov map φ_1 acting on the surface $S - c_1$. We claim that there is a constant $\beta > 0$ only depending on k (and the choice of φ_0, φ_2) such that the translation length of $\zeta(k)$ is contained in the interval $[0, T(\varphi_1) + \beta]$.

Namely, the translation length of $\zeta(k)$ is the logarithm of the Perron Frobenius eigenvalue of the matrix $A(\tau, \zeta(k))$ which determines $\zeta(k)$. Now this Perron Frobenius eigenvalue does not exceed the operator norm $\|A(\zeta(k), \tau)\|$ of the matrix $A(\zeta(k), \tau)$. This operator norm in turn is bounded from above by

$$\|A(\tau, \zeta(k))\| \le \|A_1\|\|B\|$$

where $\|A_1\|$ is the operator norm of the matrix $A_1 = A(\varphi_1, \sigma_1)$ defining the map φ_1 (which coincides with the operator norm of $A(\varphi_1, \tau)$) and where $\|B\|$ is the operator norm of the linear map $A(\tau, \varphi_0^{2k}\varphi_2\varphi_0^{2k})$. On the other hand, by the choice of φ_1 there is a universal constant $\kappa > 0$ such that

$$|T(\varphi_1) - \log\|A_1\|| \le \kappa.$$

The claim follows.

By Proposition 5.3, the asymptotic growth of the number of conjugacy classes of pseudo-Anosov elements in $\mathrm{Mod}(S_1)$ which satisfy the requirements in the proposition equals $h(\mathcal{Q}) - 2$. Let D be the Dehn twist about the curve c_1 in the direction determined by τ. By construction, we have $D^k\tau \prec \tau$ for all $k \ge 0$. If $m \le e^{T(\varphi_1)}$ then the above discussion shows that the map $\varphi_0^k \circ \varphi_2 \circ \varphi_0^{2k} \circ D^m \circ \varphi_1 \circ \varphi_0^k$ satisfies the requirements in the proposition.

This implies that simultaneous twisting about c_1 adds one to the counting of the orbits constructed above and completes the proof of the proposition. □

Remark 6.6. The *stable length* of a pseudo-Anosov element $g \in \mathrm{Mod}(S)$ on the *curve graph* $\mathcal{C}(S)$ of S is defined to be

$$sl(g) = \lim_{k\to\infty} \frac{1}{k} d(g^k c, c).$$

This does not depend on the choice of $c \in \mathcal{C}(S)$.

Bowditch [Bw08] showed that there is an integer $\ell > 0$ only depending on the topological type of S such that the stable length on the curve graph of every pseudo-Anosov element φ is rational with denominator ℓ. The stable length of each of the (infinitely many) pseudo-Anosov elements $\zeta(k)$ constructed in the proof of Proposition 6.5 is at most 2.

A similar argument also yields Theorem 2 from the introduction.

Proposition 6.7. *For every component $\mathcal{Q}$ of a stratum as in Proposition 6.5 there is a Teichmüller geodesic with uniquely ergodic vertical measured geodesic lamination whose projection to moduli space escapes with linear speed to infinity.*

Proof. We argue as in the proof of Proposition 6.5. Namely, choose simple closed curves c_1, c_2 as in the proof of Proposition 6.5 and fix pseudo-Anosov elements φ_i of S_i with the properties stated in the proof.

Let τ be a large train track as in the proof of Proposition 6.5, with subtracks σ_i. Let $A_i = A(\varphi_i, \sigma_i)$ and denote by $\|A_i\|$ the operator norm of A. Choose a sequence of numbers (k_i) such that for each i,

$$k_i \geq 2 \sum_{j \leq i-1} k_j.$$

Let $\psi_i = \varphi_1 \circ \varphi_0^{k_i} \circ \varphi_2 \circ \varphi_0^{k_i}$. Then for each i we have $\psi_i \tau \prec \tau$. Write $\zeta_k = \psi_k \circ \cdots \circ \psi_1$. We claim that $\cap_k \zeta_k \mathcal{V}(\tau)$ consists of a single ray.

To see that this is the case, note from the proof of Proposition 6.5 that for each i a carrying map $(\psi_{i+1} \circ \psi_i)\tau \to \tau$ maps every branch of $(\psi_{i+1} \circ \psi_i)\tau$ *onto* τ and its normalization contracts distances in the cone $\mathcal{V}_0(\tau)$ with a factor which is independent of i. This implies immediately that $\cap_k \zeta_k \mathcal{V}(\tau)$ consists of a single ray. In particular, a point on this ray is a uniquely ergodic measured geodesic lamination which fills up S.

To show linear escape in moduli space, let $\lambda \in \mathcal{V}_0(\tau)$ be the normalized measured geodesic lamination contained in this ray and let ν be a measured geodesic lamination which fills and hits τ efficiently. Then the pair (λ, ν) determines a quadratic differential q. Let $\ell > 0$ and let $a > 0$ be such that the measure $e^{a/2}\lambda$ on $\varphi_0^{k_\ell} \psi_{\ell-1}(\tau)$ is normalized. Then the arguments in the proof of Lemma 6.5 show that the intersection of the curve $\varphi^{k_\ell} \psi^{\ell-1} c_1$ with the lamination $e^{a/2}\lambda$ is at most $ce^{-a/2}$, and similarly for the intersection with $e^{-a/2}\nu$. This yields the proposition. ☐

Remark 6.8. By the main result of [CE07], there is a number $\epsilon > 0$ so that if a Teichmüller geodesic in moduli space escapes into the cusp with a speed of at most $\epsilon \log t$, then the vertical measured geodesic lamination of a differential on the geodesic is uniquely ergodic. The above example implies that one can construct differentials with uniquely ergodic vertical measured laminations and arbitrarily prescribed excursions into the cusp.

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