REEB ORBITS, CONVEXITY AND MINIMAL DISCS

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Abstract. Let γ be a periodic Reeb orbit of self-linking number $\mathrm{lk}(\gamma)$ on the boundary of a compact strictly convex body $C \subset \mathbb{C}^2$. We show that the Seifert genus of γ equals $\frac{1}{2}(\mathrm{lk}(\gamma)+1)$. In particular, γ is unknotted if and only if $\mathrm{lk}(\gamma) = -1$.

1. Introduction

Consider the complex two-dimensional vector space $\mathbb{C}^2$ with the standard *symplectic form* defined in standard real coordinates by $\omega_0 = \sum_{i=1}^{2} dx_i \wedge dy_i$. This symplectic form is the differential of the one-form

$$\lambda_0 = \frac{1}{2}\sum_{i=1}^{2}(x_i dy_i - y_i dx_i).$$

For every compact convex body $C \subset \mathbb{C}^2$ containing the origin in its interior, with smooth boundary Σ, the restriction λ of λ_0 to Σ defines a smooth *contact form* on Σ. This means that $\lambda \wedge d\lambda$ is a volume form on Σ.

The *Reeb vector field* of the contact structure λ is the smooth vector field X on Σ defined by $\lambda(X) = 1$ and $d\lambda(X, \cdot) = 0$. The *Reeb flow* on Σ generated by the Reeb vector field X admits periodic orbits [Rab78]. In fact, Hofer, Wysocki and Zehnder showed that the Reeb flow on Σ either admits precisely two or infinitely many periodic orbits (Theorem 1.1 of [HWZ98]). If the Reeb flow admits two periodic orbits then these orbits are unknotted.

To each periodic Reeb orbit γ on Σ we can associate its *self-linking number* $\mathrm{lk}(\gamma)$ which is defined as follows. Let $S \subset \Sigma$ be a *Seifert surface* for γ, i.e. S is a smooth embedded oriented surface in Σ whose oriented boundary equals γ. Since γ is a Reeb orbit, there is a natural identification of the restriction to γ of the oriented normal bundle of S in Σ with a real line subbundle N_S of the contact bundle $\xi = \ker(\lambda)$. Since ξ is oriented, N_S defines a trivialization of $\xi|\gamma$. The self-linking number $\mathrm{lk}(\gamma)$ of γ is the winding number with respect to N_S of a trivialization of ξ over γ which extends to a trivialization of ξ on Σ. Eliashberg [Eli92] showed that the self-linking number of a periodic Reeb orbit on Σ is always an odd integer.

In [HH09] the self-linking number of a periodic Reeb orbit γ on Σ is related to the symplectic geometry of the compact convex body C. Namely, let $D \subset \mathbb{C}$ be the closed unit disc with boundary ∂D. Define the *tangential index* $\tan(f)$ of a smooth

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immersion $f:(D,\partial D)\to(C,\gamma)$ with only interior transverse self-intersection points to be the number of such transverse self-intersection points, counted with signs and multiplicities. Then

$$\mathrm{lk}(\gamma)=2\mathrm{tan}(f)-1$$

for any such immersed *symplectic* disc in C with boundary γ, i.e. an immersion $f:(D,\partial D)\to(C,\gamma)$ with the additional property that the pull-back $f^*\omega_0$ of the symplectic form ω_0 on $\mathbb{C}^2$ vanishes nowhere and defines the canonical orientation on D (Theorem 1 of [HH09]). Moreover, such an immersed symplectic disc with boundary γ always exists.

The *Seifert genus* $g(\eta)$ of a knot η is defined to be the smallest genus of a Seifert surface for η. Eliashberg [Eli92] showed that $\mathrm{lk}(\gamma)\le 2g(\gamma)-1$ for every periodic Reeb orbit γ on Σ. We show

thm1 **Theorem.** *Let γ be a periodic Reeb orbit on the boundary Σ of a compact strictly convex body $C\subset\mathbb{C}^2$. Then the Seifert genus of γ equals* $(\mathrm{lk}(\gamma)+1)/2$.

As an immediate corollary of the theorem we obtain

corollary1 **Corollary 1.** *A periodic Reeb orbit on Σ is unknotted if and only if its self-linking number equals -1.*

For a period Reeb orbit γ on Σ there is another invariant, the *Maslov index*. Hofer, Wysocki and Zehnder [HWZ98] proved that the Maslov index of γ is at least three. Moreover, there exists a periodic Reeb orbit of Maslov index three and self-linking number -1. Theorem 2 of [HH09] and Corollary 1 imply the following result of Hainz [H07].

corollary2 **Corollary 2.** *If the principal curvatures of Σ are pointwise $1/4$-pinched then a periodic Reeb orbit γ of Maslov index 3 is unknotted.*

Eliashberg [Eli92] constructed for every $k>0$ a transverse unknot on the standard three-sphere S^3 whose self-linking number equals $-2k-1$. Our result shows that such a knot can not occur as a Reeb orbit on the boundary Σ of a compact convex body in $\mathbb{C}^2$ containing the origin in its interior (note that the radial diffeomorphism $S^3\to\Sigma$ maps the contact distribution on S^3 to the contact distribution on Σ and preserves self-linking numbers of transverse knots).

The main idea for the proof of the theorem is as follows. Let $\hat J$ be a smooth almost complex structure on $\mathbb{C}^2$ which is compatible with the symplectic form, i.e. such that $\omega_0(\cdot,\hat J\cdot)$ is a Riemannian metric on $\mathbb{C}^2$. Assume that there is a $\hat J$-holomorphic disc $f:(D,\partial D)\to(C,\gamma)$ bounding γ. With a small perturbation one can guarantee that f is an immersion [McD91]. Then f is symplectic, and every self-intersection point of f is positive. In particular, there are precisely $\frac{1}{2}(\mathrm{lk}(\gamma)+1)$ such self-intersection points. From this it easily follows that the *slice genus* of γ (which is not bigger than the Seifert genus) does not exceed $\frac{1}{2}(\mathrm{lk}(\gamma)+1)$.

For the construction of such an almost complex structure $\hat J$ and a $\hat J$-pseudo-holomorphic disc with boundary γ we use minimal discs whose existence is always guaranteed. The major part of this work is devoted to control topological properties of such minimal discs.

In Section 3 we establish a sufficient condition for an immersed minimal disc in $\mathbb{C}^2$ to be symplectic. This condition restricts the behavior of the disc along its boundary and requires the vanishing of a topological invariant, the *winding* of the disc. In Section 4 we construct for any Reeb orbit γ on the boundary of a compact convex body C a minimal disc of vanishing winding with boundary γ. This disc is symplectic and has $\frac{1}{2}(\mathrm{lk}(\gamma)+1)$ positive self-intersection points, counted with multiplicities. This implies that the slice genus of γ does not exceed $\frac{1}{2}(\mathrm{lk}(\gamma)+1)$. In Section 5, we introduce a glueing procedure for links. The results of Section 4 and Section 5 are used in Section 6 to control the Seifert genus of γ.

2. Self-linking and winding

In this section we summarize some constructions and results from [HH09] in the form needed for the proof of the main result.

Let $D \subset \mathbb{C}$ be the closed unit disc with connected boundary $\partial D = S^1$.

boundaryreg **Definition 2.1.** A smooth map $f : D \to \mathbb{C}^2$, i.e. a map which is smooth up to and including the boundary, is called *boundary regular* if the singular points of f are contained in the interior of D, i.e. if there is a neighborhood A of ∂D in D such that the restriction of f to $f^{-1}(f(A))$ is an embedding. The map f is called *locally boundary regular* if there is a neighborhood A of ∂D in D such that the restriction of f to A is an embedding.

Clearly a boundary regular map is locally boundary regular.

Let J be the standard complex structure on $\mathbb{C}^2$ and let $\langle , \rangle$ be the (real) euclidean inner product which is J-invariant. Call a real two-dimensional subspace $V \subset \mathbb{C}^2$ *admissible* if $JV \cap V^\perp = \{0\}$ where $V^\perp$ denotes the orthogonal complement of V. If V is admissible then for each $0 \neq X \in V$ the pair $(X, \pi(JX))$ is a basis of V where $\pi : \mathbb{C}^2 \to V$ is the orthogonal projection. The thus defined orientation of V does not depend on X and will be called *canonical.* An admissible subspace $V \subset \mathbb{C}^2$ is symplectic.

A complex structure $\hat{J}$ on $\mathbb{C}^2$ is called *compatible* with the euclidean inner product $\langle , \rangle$ if $\hat{J}$ preserves $\langle , \rangle$. The complex structure is called *positive* if it defines the orientation on $\mathbb{C}^2$ determined by the standard complex structure J. The following simple observation relates admissible planes to compatible complex structures.

compatible **Lemma 2.2.** *For every admissible plane $V \subset \mathbb{C}^2$ there is a unique positive complex structure J_V on $\mathbb{C}^2$ which is compatible with the euclidean inner product such that V is J_V-invariant and that the orientation of V induced by J_V is the canonical orientation. For every $0 \neq X \in \mathbb{C}^2$ the non-oriented angle between JX and $J_V X$ is smaller than $\pi/2$.*

Proof. There is a unique complex structure J_0 on V which is compatible with the restriction of the euclidean inner product and which defines the canonical orientation. Since the orthogonal complement $V^\perp$ of V is also admissible, J_0 can uniquely be extended to a complex structure J_V on $\mathbb{C}^2$ which is compatible with the euclidean inner product and which induces the canonical orientation on $V^\perp$. This complex structure is necessarily positive.

To show the last part of the lemma, note that if the claim does not hold true then by continuity there is a vector $0 \neq X \in \mathbb{C}^2$ such that $\langle J_V X, JX/rangle = 0$. Since both J_V, J are compatible, this implies that $\langle J(J_V X), J_V(J_V X)\rangle = 0 = \langle J_V(JX), J(JX)\rangle$. Now the vectors $X, JX, J_V X, JJ_V X$ span $\mathbb{C}^2$ and hence $\langle JY, J_V Y\rangle = 0$ for all Y. However, this is impossible because J_V preserves the admissible plane V. □

admissible **Definition 2.3.** A smooth locally boundary regular map $f : D \to \mathbb{C}^2$ is called *boundary holomorphic* (or *admissible*) if for each $z \in \partial D$ the tangent plane of $f(D)$ at $f(z)$ is J-invariant (or admissible) and if its canonical orientation coincides with the orientation induced from the orientation of D.

For every smooth embedded oriented curve γ in $\mathbb{C}^2$ the orthogonal complement N of the complex line subbundle of $T\mathbb{C}^2|\gamma$ spanned by the tangent of γ is a complex line bundle over γ. This bundle admits a *preferred trivialization* ρ which is determined as follows. Let $f : (D, \partial D) \to (\mathbb{C}^2, \gamma)$ be any boundary regular boundary holomorphic map. We require that the restriction of f to ∂D is an orientation preserving homeomorphism onto γ. Glue N to the trivial bundle $D \times \mathbb{C}$ over D with the trivialization ρ. The resulting (locally defined) 4-manifold W_ρ admits an almost complex structure extending the complex structure J of $\mathbb{C}^2$. Denote by S_0 the closed oriented 2-sphere obtained by glueing two copies of D along the boundary with an orientation reversing diffeomorphism. There is a natural extension $f_0 : S_0 \to W_\rho$ of f. We require that the evaluation on S_0 of the first Chern class of the tangent bundle of W_ρ equals 2. This does not depend on the choice of a boundary regular boundary holomorphic map $(D, \partial D) \to (\mathbb{C}^2, \gamma)$ (see Section 2 of [HH09] for a detailed discussion).

In fact, let $M : \mathbb{C}^2 \to \mathbb{C}^2$ be the complex anti-linear map $(z_1, z_2) \to (-\bar{z_2}, \bar{z_1})$. Note that M defines a complex structure on $\mathbb{C}^2$ which preserves the euclidean inner product $\langle , \rangle$. We have

preferred **Lemma 2.4.** *The winding of the preferred trivialization ρ of N with respect to the trivialization defined by $M\gamma'(t)$ equals one.*

Proof. Let $f : (D, \partial D) \to (\mathbb{C}^2, \gamma)$ be any boundary regular boundary holomorphic *immersion.* Let X be a nowhere vanishing vector field on D. Then $(df(X), Mdf(X))$ is a global complex trivialization of the pull-back $f^*T\mathbb{C}^2$ of the tangent bundle of $\mathbb{C}^2$. By definition, this implies that $Mdf(X)|\partial D$ is the preferred trivialization of the normal bundle of $f(D)$ over $f(\partial D)$. Since M is anti-holomorphic, the lemma follows (compare [HH09]). □

For a boundary regular boundary holomorphic map $f : D \to \mathbb{C}^2$ the *self-intersection number* $\mathrm{Int}(f)$ of f is defined as follows. Let ρ be the preferred trivialization of the normal bundle of $f(D)$ over $f(\partial D) = \gamma$. Use ρ to construct the almost complex manifold W_ρ as above and extend f to a map $f_0 : S_0 \to W_\rho$. The self-intersection number $\mathrm{Int}(f)$ of f then is the self-intersection number of $f_0(S_0)$ in W_ρ. Note that the self-intersection number is also defined for boundary regular admissible maps $D \to \mathbb{C}^2$ (see [HH09] for details and compare with [McD91]).

Next we look at topological invariants of boundary regular *immersions*. Namely, assume that the boundary regular immersion $f : D \to \mathbb{C}^2$ has a finite number of self-intersections, each of them transverse and contained in the interior of D. Define the *tangential index* $\tan(f)$ of f to be the number of such self-intersection points counted with signs and multiplicities. If a boundary regular immersion $f : D \to \mathbb{C}^2$ has self-intersection points which are not transverse then it can be perturbed with a homotopy of boundary regular immersions to an immersion with only transverse double points whose tangential index is independent of the perturbation (see e.g. [McD91]).

There are also topological invariants of locally boundary regular boundary holomorphic or admissible immersions. For such an immersion $f : D \to \mathbb{C}^2$, the normal bundle of $f(D)$ is the normal bundle of TD in the pull-back $f^*T\mathbb{C}^2$. Its restriction to γ is naturally identified with the orthogonal complement of the complex line subbundle of $T\mathbb{C}^2|\gamma$ spanned by γ'. Thus the restriction to γ of the normal bundle of $f(D)$ is equipped with the preferred trivialization.

winding **Definition 2.5.** The *winding number* $\mathrm{wind}(f)$ of a locally boundary regular boundary holomorphic (or admissible) immersion $f : (D, \partial D) \to (\mathbb{C}^2, \gamma)$ is the winding number of the preferred trivialization of the normal bundle of $f(D)$ over $f(\partial D) = \gamma$ with respect to a trivialization which extends to a global trivialization of the normal bundle of $f(D)$.

Proposition 3.4 of [HH09] is an *adjunction formula* for immersed boundary regular boundary holomorphic (or admissible) discs in $\mathbb{C}^2$.

intofdiscs **Proposition 2.6.**

$$\mathrm{Int}(f) = \mathrm{wind}(f) + 2\tan(f)$$

for any boundary regular boundary holomorphic (or admissible) immersion $f : D \to \mathbb{C}^2$.

Let $f : D \to \mathbb{C}^2$ be an immersion. A *complex point* for f is a point $x \in D$ so that the tangent plane of $f(D)$ at $f(x)$ is a complex line in $T\mathbb{C}^2$. The complex point is called *holomorphic* if the orientation of T_xD coincides with the orientation given by the complex structure of $\mathbb{C}^2$, and it is called *anti-holomorphic* otherwise. A small deformation of the map f ensures that complex points are isolated in the interior of D.

The *index* of an isolated complex point x in the interior of D is defined as follows. The bundle $f^*T\mathbb{C}^2$ contains the tangent bundle TD of D as a real two-dimensional subbundle. Denote by V the orthogonal complement of TD in $f^*T\mathbb{C}^2$ with respect to the euclidean metric and let $\pi : f^*T\mathbb{C}^2 \to V$ be the orthogonal projection. If

v is a smooth local section of TD near the isolated complex point x, then $\pi(Jv)$ has an isolated zero at x. The index of x is defined to be the degree of the zero (or the winding number of $\pi(Jv)$ about x) with respect to a smooth trivialization of V near x. This is well defined since the orientation of TD and the orientation of $f^*T\mathbb{C}$ determine an orientation of the bundle V by requiring that the oriented bundle $f^*T\mathbb{C}^2$ coincides with the oriented sum $TD \oplus V$.

Define $d_-(f)$ to be the sum of the indices of the anti-holomorphic points of $f(D)$. The next observation is a version of Proposition 3.4 of [HH09].

lkcompute **Proposition 2.7.** *Let $f : D \to \mathbb{C}^2$ be a locally boundary regular boundary holomorphic (or admissible) immersion with only isolated anti-holomorphic points. Then*

$$\mathrm{wind}(f) = 2d_-(f).$$

Proof. Let $f : D \to \mathbb{C}^2$ be a locally boundary regular boundary holomorphic immersion with only isolated anti-holomorphic points. Choose a global nowhere vanishing section X of TD. If $M : \mathbb{C}^2 \to \mathbb{C}^2$ is the complex anti-linear map used in Lemma 2.4 then $df(X), Mdf(X)$ define a global complex trivialization of $f^*T\mathbb{C}^2$. This trivialization in turn defines a trivialization $f^*\mathcal{G} = D \times G(2,4)$ of the bundle $f^*\mathcal{G}$ over D whose fibre at a point x is the Grassmannian $G(2,4)$ of oriented real two-dimensional linear subspaces of the fibre of $f^*T\mathbb{C}^2 = \mathbb{C}^2$ at x. The map which associates to $x \in D$ the oriented tangent plane of D at x defines a section of this bundle and hence it defines a continuous map $Gf : D \to G(2,4)$. Since f is boundary holomorphic, the image under f of the boundary of D is a single complex line. Thus Gf determines a map into $G(2,4)$ from the two-sphere obtained from D by collapsing the boundary to a single point. As a consequence, it defines an element $[Gf]$ in the second homology group $H_2(G(2,4),\mathbb{Z})$ of the Grassmannian $G(2,4)$.

Since $G(2,4) = S^2 \times S^2$, the integral homology group $H_2(G(2,4),\mathbb{Z})$ decomposes as $Z_1 \oplus Z_2$ where Z_i is infinite cyclic ($i = 1,2$). Here the group Z_1 is generated by the complex projective line $\mathbb{C}P^1$ of all oriented complex lines in $\mathbb{C}^2$, and Z_2 is generated by the tangent bundle of the two-sphere $S^2 \subset \mathbb{R}^3 \subset \mathbb{C}^2$ (see the beginning of Section 3 of [HH09]). Let $\mathcal{C}_2(Gf)$ be the component of the homology class $[Gf]$ in the subgroup Z_2. By Proposition 3.4 of [HH09], if we view $\mathcal{C}_2(Gf)$ as an element in $\mathbb{Z}$ then we have

$$\mathrm{wind}(f) = 2\mathcal{C}_2(Gf).$$

On the other hand, $\mathcal{C}_2(Gf)$ is the number of intersection points of TD with the bundle of anti-holomorphic complex lines in $f^*T\mathbb{C}^2$ counted with signs and multiplicities (i.e. complex lines with the orientation induced by the complex structure $-J$, see the discussion in Section 3 of [HH09]). Now each anti-holomorphic point of f is contained in the interior of D. Moreover, if f has only isolated anti-holomorphic points, each of index ± 1, then the anti-holomorphic points are precisely the intersection points of TD with the bundle of anti-holomorphic lines in $T\mathbb{C}^2$, and the sign of the intersection of each such point is just the sign of the anti-holomorphic point (see the discussion at the end of Section 3 in [CT97]). In other words, we have $\mathcal{C}_2(Gf) = d_-(S)$. This shows the proposition for locally boundary regular boundary holomorphic maps.

If f is locally boundary regular and admissible then f can be deformed to a locally boundary regular boundary holomorphic immersion f' with $\mathrm{wind}(f') = \mathrm{wind}(f)$ and the additional property that the numbers of anti-holomorphic points counted with signs and multiplicities of f, f' coincide. The proposition follows. □

Call a smooth almost complex structure $\hat{J}$ on $\mathbb{C}^2 = \mathbb{R}^4$ *compatible* with the euclidean inner product $\langle , \rangle$ if $\langle , \rangle$ is fibre-wise $\hat{J}$-invariant. A two-dimensional linear subspace $V \subset T\mathbb{C}^2$ is called *admissible for* $\hat{J}$ if $\hat{J}V \cap V^\perp = \{0\}$ where as before, $V^\perp$ is the orthogonal complement of V. $\hat{J}$-boundary holomorphic or $\hat{J}$-admissible locally boundary regular maps are naturally defined as well, and the above discussion carries over without modification to $\hat{J}$-admissible maps. In particular, for a locally boundary regular $\hat{J}$-admissible immersion $f : D \to \mathbb{C}^2$ the $\hat{J}$*-winding* is defined. Note that if f is admissible for the usual complex structure J as well then this $\hat{J}$-winding may be distinct from $\mathrm{wind}(f)$, the winding of f for J (compare [HH09]).

3. Minimal admissible discs

The strategy for the proof of the theorem from the introduction is to show that minimal discs in $\mathbb{C}^2$ which bound a given Reeb orbit on the boundary of a compact convex body $C \subset \mathbb{C}^2$ have the same topological properties as holomorphic discs. In this section we establish some first properties of such minimal discs. In particular, we establish a sufficient condition for such a disc to be symplectic.

As in Section 2, let $D \subset \mathbb{C}$ be the closed unit disc. A minimal disc whose boundary is a smooth embedded oriented simple closed curve γ in $\mathbb{C}^2$ is a continuous map $f : (D, \partial D) \to (\mathbb{C}^2, \gamma)$ satisfying the minimal surface equation and such that the boundary ∂D of D is mapped with an orientation preserving homeomorphism onto γ. There is no reference to a specific parametrization of γ. Particular examples of such minimal discs are discs which minimize the area among all discs with oriented boundary γ.

The parametrization of a minimal disc $f : (D, \partial D) \to (\mathbb{C}^2, \gamma)$ with smooth boundary γ is smooth up to the boundary [S88, Theorem 5.1]. However, a minimal disc $f : (D, \partial D) \to (\mathbb{C}^2, \gamma)$ may have isolated branch points and may even have branch points on the boundary. For the remainder of this section, we only consider locally boundary regular minimal discs, i.e. minimal discs which bound smooth embedded curves and which do not have branch points on the boundary.

Even though a locally boundary regular minimal disc $f : D \to \mathbb{C}^2$ may have interior branch points, it admits well defined tangent planes everywhere varying smoothly with $p \in D$ [DHKW91]. Therefore the tangent bundle TD of D can naturally be identified with a subbundle of the pull-back $f^*T\mathbb{C}^2$. In particular, if f is admissible then the winding number $\mathrm{wind}(f)$ of f is defined.

minissymp **Proposition 3.1.** *A locally boundary regular admissible minimal disc* $f : D \to \mathbb{C}^2$ *with* $\mathrm{wind}(f) = 0$ *is symplectic.*

Proof. Let $f:(D,\partial D)\to(\mathbb{C}^2,\gamma)$ be a locally boundary regular admissible minimal disc with $\mathrm{wind}(f)=0$. Our goal is to show that f is symplectic.

Recall from Section 2 the definition of a holomorphic and an antiholomorphic complex point of an immersed disc $f:D\to\mathbb{C}^2$. By [Web84, Proposition 2], a minimal surface either is holomorphic, anti-holomorphic or has only isolated complex points. As an admissible minimal disc is symplectic near its boundary it can not be anti-holomorphic. Holomorphic discs are necessarily symplectic. Thus it suffices to consider the case that f has only isolated complex points.

As in Section 2, for a complex point p of f the *index* $\mathrm{ind}(p)$ of p is defined. For a minimal disc, the index is always negative. At a complex branch point, the index coincides with the negative of the branching order [Web84]. By Proposition 2.7 we have $\mathrm{wind}(f)=2\sum_p \mathrm{ind}(p)$ where the sum is over all anti-holomorphic points of f. Thus since $\mathrm{wind}(f)=0$ by assumption, f does not have any anti-holomorphic complex points.

We use the arguments of [CW83] and [Wol89]. Namely, let j be the standard complex structure on the tangent bundle TD of the disc D. We also define a (1,0)-form $\Phi\in\Gamma(T^*D\otimes\mathbb{C})$, so that

$$ds^2=\Phi\circ\overline{\Phi}$$

is the metric on TD induced by f. As in [CW83], we choose a unitary $(1,0)$-coframe $\{\omega_1,\omega_2\}$ in $T^*\mathbb{C}^2\otimes\mathbb{C}$ such that on TD we have

$$f^*\omega_1=\cos\frac{\alpha}{2}\Phi,\quad f^*\omega_2=\sin\frac{\alpha}{2}\overline{\Phi},$$

for some function $\alpha:D\to\mathbb{R}$. This function is differentiable away from complex points and continuous on D. Then we have

$$ds^2=f^*(\omega_1\circ\overline{\omega_1}+\omega_2\circ\overline{\omega_2})$$

Furthermore, the symplectic form ω_0 and the induced volume form $\frac{i}{2}\Phi\wedge\overline{\Phi}$ on TD satisfy

$$f^*\omega_0=f^*\left(\frac{i}{2}(\omega_1\wedge\overline{\omega_1}+\omega_2\wedge\overline{\omega_2})\right)=\frac{i}{2}\cos(\alpha)\Phi\wedge\overline{\Phi}.$$

Since f is conformal it suffices to show that $\cos(\alpha)>0$, i.e. that $\alpha\in(-\frac{\pi}{2},\frac{\pi}{2})$. As in [Wol89] we define

$$\begin{array}{rcl} u:\mathbb{R} & \to & \mathbb{R} \\ \alpha & \mapsto & \ln\left(\tan\left(\frac{\alpha}{2}\right)\right). \end{array}$$

Let $\{p_i\}$ be the complex points of the minimal disc. According to Wolfson, on $D-\{p_i\}$ we have

$$\Delta(u\circ\alpha)\Phi\wedge\overline{\Phi}=\partial\overline{\partial}(u\circ\alpha)=-i\mathrm{Ric}=0,$$

where Ric is the Ricci-form on $\mathbb{C}^2$ and hence vanishes. Therefore

$$u\circ\alpha:D-\{p_i\}\to\mathbb{R}$$

is a harmonic function. Since the disc is symplectic near the boundary and since by the above observation there are no anti-holomorphic points, we have $\alpha|_{S^1\cup\{p_i\}}\in$

$(-\frac{\pi}{2},\frac{\pi}{2})$. Since u is monotone in α and since $u\circ\alpha$ attains its maximum and its minimum on the boundary of $D-\{p_i\}$, the extrema of α lie on the boundary as well. Therefore $\alpha\in(-\frac{\pi}{2},\frac{\pi}{2})$ on D. This proves that every tangent plane of the disc f is an admissible and hence symplectic subspace of the tangent space of $\mathbb{C}^2$. ☐

By Proposition minissymp 3.1, for a minimal locally boundary regular admissible disc $f:(D,\partial D)\to(\mathbb{C}^2,\gamma)$ with $\text{wind}(f)=0$ and for every $z\in D$, the tangent plane $df(T_zD)$ of $f(D)$ at $f(z)$ is admissible, and its canonical orientation coincides with the orientation induced from the orientation of D.

We use this observation to show that a boundary regular admissible minimal immersion $f:D\to\mathbb{C}^2$ with $\text{wind}(f)=0$ has only positive transverse self-intersection points. The following lemma is the main technical tool for this purpose.

signofselfint **Proposition 3.2.** *Let $s\to f_s:D\to\mathbb{C}^2$ $(s\in[a,b])$ be an arc of minimal immersions which is continuous in the C^3-topology. Assume that for each s the map f_s is locally boundary regular and admissible, that f_a is an embedding with* $\text{wind}(f_a)=0$ *and that f_b is boundary regular. Then every self-intersection point of f_b is positive.*

Proof. Self-intersection points of minimal immersed discs in $\mathbb{C}^2$ are transverse and hence isolated. Therefore the number of self-intersection points (counted without sign but with multiplicity) of an immersed locally boundary regular minimal disc in $\mathbb{C}^2$ is finite.

Let
$$s\to f_s:D\to\mathbb{C}^2\,(s\in[a,b])$$
be an arc of minimal immersions as in the lemma. In particular, both f_a and f_b are boundary regular, and every disc f_s is locally boundary regular and admissible. Then for every s the winding number $\text{wind}(f_s)$ of f_s is defined. Since the discs f_s depend continuously on s in the C^3-topology, $\text{wind}(f_s)$ depends continuously on s. Now $\text{wind}(f_a)=0$ by assumption and therefore $\text{wind}(f_s)=0$ for all s. Proposition minissymp 3.1 then shows that each of the discs f_s is symplectic.

Let $D^0\subset\mathbb{C}$ be the open unit disc with closure D. By transversality, if $t\in[a,b]$ and if $x_0\neq y_0\in D^0$ are such that $f_t(x_0)=f_t(y_0)$ then there is a connected neighborhood U of t in $[a,b]$ and there are unique continuous maps
$$x:U\to D,y:U\to D$$
with $x(t)=x_0,y(t)=y_0$ such that $x(s)\neq y(s)$ and $f_s(x(s))=f_s(y(s))$ for all $s\in U$. The sign of the corresponding self-intersection point of the disc f_s does not depend on s.

Let $r\in[a,b]$ be the infimum of all numbers $s\in[a,b]$ so that the curves x,y are defined on the interval $[s,t]$. We claim that up to exchanging x_0 and y_0, as $s\searrow r$ we have $|x(s)|\to 1$.

Namely, otherwise we can find a sequence $s_j\searrow r$ so that $|x(s_j)|\to\rho_1<1$ and $|y(s_j)|\to\rho_2<1$. After passing to a subsequence we may assume that $x(s_j)\to\tilde{x}\in D^0,y(s_j)\to\tilde{y}\in D^0$. By continuity, we have $f_r(\tilde{x})=f_r(\tilde{y})$. Since each of the discs f_s is a locally boundary regular immersion depending continuously on

s in the C^3-topology, necessarily $\tilde{x} \neq \tilde{y}$. In particular, f_r has a self-intersection point and hence $r > a$ since f_a is an embedding by assumption. Then there is a connected neighborhood $V \subset [a, b]$ of r such that the points $\tilde{x}, \tilde{y}$ can be developed into continuous arcs $\tilde{x}(s), \tilde{y}(s)$ $(s \in V)$ so that $f_s(\tilde{x}(s)) = f_s(\tilde{y}(s))$ for all $s \in V$. Since self-intersection points of minimal immersed discs are isolated, these arcs have to contain the points $x(s_j), y(s_j)$ for sufficiently large j. However, this violates the definition of r.

Extending the self-intersection arcs $x(s), y(s)$ in the same way as s increases we conclude the following. If $x_0 \neq y_0 \in D^0$ and $t \in (a, b]$ are such that $f_t(x_0) = f_t(y_0)$ then up to exchanging x_0 and y_0, there are unique continuous maps

$$s \to x(s) \in D, s \to y(s) \in D \, (s \in [\alpha, \beta] \subset (a, b])$$

through $x(t) = x_0, y(t) = y_0$ with the following properties.

i) $x(\alpha) \in \partial D$ and $x(s) \neq y(s) \in D^0$ for all $s \in (\alpha, \beta)$.
ii) $f_s(x(s)) = f_s(y(s))$ for all s.
iii) Either $\beta = b$ or $x(\beta) \in \partial D$ or $y(\beta) \in \partial D$.

We call the pair (x, y) of maps $x, y : [\alpha, \beta] \to D$ a *pair of maximal self-intersection arcs*. For each fixed s, there are only finitely many pairs of maximal self-intersection arcs passing through s.

For $r \in (0, 1)$ let $D_r \subset D$ be the closed disc of radius r in $\mathbb{C}$. By continuous dependence of the discs f_s on s in the C^3-topology and compactness, for each $\nu > 0$ there are only finitely many pairs (x, y) of maximal self-intersection arcs so that both x, y intersect $D_{1-\nu}$.

If $f : D \to \mathbb{C}^2$ is a minimal disc and if $E \subset D$ is an embedded subdisc with smooth boundary then the composition of $f|E$ with a uniformizing biholomorphic map $D \to E$ is a minimal disc. This disc is uniquely determined by f and a three-point condition for the uniformizing map. Since moreover boundary regular and admissible discs form an open set of discs in the C^3-topology, we can deform the arc of minimal discs $s \to f_s$ by pushing the boundary of D slightly inside D with a deformation depending smoothly on s which equals the identity for $s = a, s = b$ and such that the resulting arc of minimal discs, again denoted by f_s $(s \in [a, b])$, has the properties stated in the proposition together with the following additional properties.

a) There are only finitely many pairs of maximal self-intersection arcs.
b) For every $s \in [a, b]$, every self-intersection of f_s is contained in a pair of maximal self-intersection arcs.
c) For every $s \in [a, b]$, there is at most one pair of maximal self-intersection arcs which has an endpoint at s.

We now show the statement of the proposition for arcs of minimal discs which satisfy the assumption in the proposition as well as properties a),b),c) above. For this we proceed by induction on the number of pairs of maximal self-intersection arcs. If there is no such pair of arcs then f_b is an embedding and there is nothing to show, so assume that the statement holds true whenever there are at most $k-1$ pairs of maximal self-intersection arcs for some $k \geq 1$.

Let $s \to f_s : D \to \mathbb{C}^2$ ($s \in [a,b]$) by an arc of minimal discs satisfying the assumptions in the proposition as well as properties a),b),c) which contains k pairs of maximal self-intersection arcs. If f_b is an embedding then there is nothing to show, so assume that there are $m \geq 1$ pairs of maximal self-intersection arcs ending at b. In particular, f_b has exactly m double points. Let $(x_1,y_1),\dots,(x_m,y_m)$ be these pairs, ordered in such a way that for $j > i$, the starting points s_i, s_j of the pairs $(x_i,y_i),(x_j,y_j)$ satisfy $s_j > s_i$. The sign of the self-intersection point $f_s(x_i(s)) = f_s(y_i(s))$ does not depend on s.

By properties a),b),c) above, there is a number $\beta > s_m$ such that for every $t \in (s_m, \beta]$, the disc f_t is boundary regular. Let $n \geq m$ be the number of double points of f_β. By the choice of s_m, there is an injection of the set of self-intersection points of f_b into the set of self-intersection points of f_β preserving signs. Thus it suffices to show that each self-intersection point of f_β is positive. In other words, for the induction step it suffices to consider the case that the interval (s_m, b) does not contain an endpoint of a pair of maximal self-intersection arcs. In the sequel we assume that this is indeed the case.

Using again the properties a),b) above, there is a number $\sigma < s_m$ such that for each $s \in [\sigma, s_m)$ the disc f_s is boundary regular. Then for $s \in [\sigma, s_m)$, the disc f_s contains precisely $m-1$ transverse double points. By the induction hypothesis, each of these double points is positive. Since each transverse self-intersection point of a boundary regular immersion contributes to the tangential index, we have $\tan(f_b) = m$ if the self-intersection point $f_b(x_m(b)) = f_b(y_m(b))$ is positive, and $\tan(f_b) = m-2$ otherwise.

Since the discs f_s are locally boundary regular and depend continuously on s in the C^3-topology, there is a number $\nu_0 > 0$ so that for each s the restriction of f_s to $D - D_{1-2\nu_0}$ is an embedding. By continuity and compactness, there is a number $\nu < \nu_0/2$ which is sufficiently small that $x_i(s), y_i(s) \in D_{1-4\nu}$ for all $s \in [\sigma, b]$ and all $i \leq m-1$. We also require that the pair (x_m, y_m) of maximal self-intersection arcs is such that $x_m(s_m) \in \partial D$ and $y_m[s_m, b] \subset D_{1-4\nu}$.

By Proposition 3.1 and and Lemma 2.2, there is a unique positive complex structure $\hat{J}$ on $\mathbb{C}^2$ such that the tangent plane $df(T_{y_m(s_m)}D)$ is $\hat{J}$-invariant. Since f_s is admissible for each s, Lemma 2.2 shows that for all s and all $z \in \partial D$ the non-oriented angle between the inner normal of $f_t(D)$ at $f_t(z)$ and $\hat{J}(Z)$ where Z is the oriented tangent of the curve $f_t(\partial D)$ at $f_t(z)$ is strictly smaller than π.

For $s \in [a,b]$ let $R(s)$ be the ruled surface defined by the smooth curve $f_s(\partial D)$ and the lines whose direction at $f_s(u)$ ($u \in \partial D$) is the image of the oriented tangent of $f_s(\partial D)$ at $f_s(u)$ under the complex structure $\hat{J}$. With respect to the natural parametrization, these ruled surfaces $R(s)$ depend continuously on s in the C^2-topology. By the implicit function theorem, $R(s)$ contains an embedded $\hat{J}$-boundary holomorphic annulus $A(s)$ which depends continuously on s in the C^2-topology. A neighborhood of $A(s)$ in $\mathbb{C}^2$ is naturally diffeomorphic to a neighborhood of $A(s)$ in its normal bundle in $\mathbb{C}^2$.

Since the discs f_s are admissible and depend continuously on s in the C^3-topology, up to perhaps making ν smaller we may assume that for each s the annulus $f_s(D - D_{1-2\nu})$ can be represented as a graph over the embedded $\hat{J}$-boundary holomorphic annulus $A(s)$. This implies that there is a deformation F_t of the disc f_t ($t \in [\sigma, b]$) obtained by flattening the graph over $A(s)$ near the outer boundary circle $f_s(\partial D)$ with the following properties.

(1) $F_\sigma = f_\sigma, F_b = f_b$.
(2) For each t, F_t is a locally boundary regular admissible immersion which depends continuously on t in the C^2-topology.
(3) For all t the restriction of F_t to $D - D_{1-2\nu}$ is an embedding.
(4) $F_t|D_{1-\nu} = f_t|D_{1-\nu}$ and $F_t(\partial D) = f_t(\partial D)$ for all t.
(5) There is an open connected neighborhood U of s_m in (σ, b) such that for every $t \in U$ the map F_t is $\hat{J}$-boundary holomorphic.

Properties (3) and (4) and the choice of ν imply in particular that for $s \in [\sigma, b] - \{s_m\}$ the map F_s is a boundary regular immersion. Moreover, there is a single point $x_m(s_m) \in \partial D$ such that $F_{s_m}(x_m(s_m)) \in F_{s_m}(D^0)$, more precisely we have

$$F_{s_m}(x_m(s_m)) = F_{s_m}(y_m(s_m)) = f_{s_m}(x_m(s_m)) = f_{s_m}(y_m(s_m))$$

where $y_m(s_m) \in D_{1-4\nu}$.

Since for $s \in [\sigma, s_m)$ the map F_s is a boundary regular immersion depending continuously on s in the C^2-topology, the tangential index $\tan(F_s)$ is defined and does not depend on s. In fact, we have

$$\tan(F_s) = \tan(F_\sigma) = \tan(f_\sigma) = m - 1$$

for $s \in [\sigma, s_m)$. Similarly, for $t \in (s_m, b]$ the tangential index of F_s is defined and

$$\tan(F_t) = \tan(F_b) = \tan(f_b).$$

As a consequence, we have $\tan(F_s) = \tan(F_\sigma) \pm 1$ for $s > s_m$. This shows that the self-intersection point $F_{s_m}(x_m(s_m)) = F_{s_m}(y_m(s_m))$ contributes to the tangential index of F_b and hence of f_b according to its sign (and provided that it is transverse).

However, since self-intersections of minimal discs are always transverse, the planes $dF_{s_m}(T_{x_m(s_m)}D), dF_{s_m}(T_{y_m(s_m)}D)$ are $\hat{J}$-complex lines which do not coincide. In particular, they are transverse. Since $\hat{J}$ is positive, this implies that the self-intersection point $F_{s_m}(x_m(s_m)) = F_{s_m}(y_m(s_m)) = f_{s_m}(x_m(s_m)) = f_{s_m}(y_m(s_m))$ is positive. This completes the induction step and shows the proposition. □

We use Proposition minissymp 3.1 and Proposition signofselfint 3.2 to show

likeholo **Corollary 3.3.** *Let $f : D \to \mathbb{C}^2$ be a boundary regular admissible minimal immersion with* $\text{wind}(f) = 0$ *and only transverse double points. Then $f(D)$ has precisely* $\tan(f)$ *self-intersection points.*

Proof. Let $f : D \to \mathbb{C}^2$ be a boundary regular minimal immersion as in the proposition. Then f has a finite number of interior self-intersection points. For $r \leq 1$ let $D_r \subset D \subset \mathbb{C}$ be the closed disc of radius r. After perhaps precomposing f with a biholomorphic automorphism of D we may assume that whenever $x, y \in D$ are such that $f(x) = f(y)$ then $|x| \neq |y|$. Then by the discussion following the proof of Proposition 3.1, for every $r \in (0,1]$ the minimal disc $f_r : D \to \mathbb{C}^2$ defined by $f_r(x) = f(rx)$ is locally boundary regular and admissible. Moreover f_r depends continuously on r in the C^3-topology.

For sufficiently small r, say for all $r \leq r_0$, the map f_r is an embedding with $\text{wind}(f_r) = 0$, moreover $f_1 = f$ is boundary regular by assumption. Thus the arc of discs $s \to f_s$ ($s \in [r_0, 1]$) satisfies the assumptions in Proposition 3.2. The corollary follows. $\square$

4. Minimal discs bounding Reeb orbits

The main goal of this section is to construct for a periodic Reeb orbit γ on the boundary Σ of a compact strictly convex body C a minimal admissible disc $f : (D, \partial D) \to (C, \gamma)$ with $\text{wind}(f) = 0$. We obtain such a disc as the endpoint of an arc of harmonic maps which are all locally boundary regular and admissible and with vanishing winding number.

Assume that C contains 0 in its interior. The restriction λ to Σ of the *radial one-form* λ_0 on $\mathbb{C}^2$ defined as in the introduction by $(\lambda_0)_p(Y) = \frac{1}{2}\langle Jp, Y\rangle$ ($p \in \mathbb{C}^2, Y \in T_p\mathbb{C}^2$) vanishes nowhere and defines a smooth *contact structure* on Σ.

The differential $d\lambda_0$ of λ_0 is just the usual symplectic form ω_0 on $\mathbb{C}^2$. Let N be the outer normal field of $\Sigma \subset \mathbb{C}^2$. The *Reeb vector field* X on Σ is given by

$$X(p) = \varphi(p) JN(p)$$

where

$$\varphi(p) = \frac{2}{\langle p, N(p)\rangle} > 0.$$

Namely, for $p \in \Sigma$ we have

$$d\lambda_p(X, \cdot) = \varphi(p)\omega_0(JN(p), \cdot) = -\varphi(p)\langle N(p), \cdot\rangle = 0$$

on $T_p\Sigma$ and

$$\lambda_p(X) = \frac{1}{2}\langle Jp, X\rangle = \frac{1}{2}\varphi(p)\langle Jp, JN(p)\rangle = 1.$$

In particular, a boundary regular map $f : D \to \mathbb{C}^2$ whose oriented boundary $f(\partial D)$ is a periodic Reeb orbit on Σ, which meets Σ transversely along $f(\partial D)$ and maps a neighborhood of ∂D into C is admissible.

For an oriented Jordan curve γ, we denote by $f : (D, \partial D) \to (\mathbb{C}^2, \gamma)$ a minimal disc whose boundary $f|\partial D$ is an orientation preserving parametrization of γ. Since the boundary Σ of the compact convex body C is smooth, a periodic Reeb orbit γ on Σ is smooth as well. Thus the existence of a minimal disc $f : (D, \partial D) \to (\mathbb{C}^2, \gamma)$ with boundary γ is guaranteed by a classical general existence result (Theorem 4.10 of [S88]; we refer to Struwe's book for more information and for references) which can be stated as follows. Every rectifiable Jordan curve in $\mathbb{R}^n$ ($n \geq 3$) is the boundary of an absolute area minimizing minimal disc in $\mathbb{R}^n$. By the maximum principle, $f(D - \partial D)$ is contained in the interior of C.

The next lemma shows that a minimal disc which bounds a Reeb orbit γ on the boundary Σ of a compact strictly convex body C is boundary regular, i.e. it does not have branch points on the boundary [MW95].

branchpoints **Lemma 4.1.** *Let $\gamma \subset \Sigma$ be a smooth knot. Then a minimal disc $f : (D, \partial D) \to (C, \gamma)$ does not have branch points on ∂D, and it intersects Σ transversely along γ.*

Proof. We follow the proof of the corollary after Theorem 4.5 in [MW95]. Namely, by the maximum principle, f maps the interior of D into the interior of C. Moreover, f is smooth up to and including the boundary (see [S88]). Since C is strictly convex, there is a smooth regular function $u : C \to \mathbb{R}$ which vanishes on Σ, which is convex near Σ and negative in the interior of C. Then $v = u \circ f$ is subharmonic near ∂D, vanishes on ∂D and is negative in the interior of D. By Lemma 3.4 of [GT83], the differential dv vanishes nowhere along ∂D. Since f is conformal, this implies that f is an embedding near ∂D which intersects Σ transversely along γ. □

For the remaineder of this section denote by $\hat{J}$ a smooth almost complex structure on $\mathbb{C}^2$ which is compatible with the euclidean inner product. Call a smooth knot ξ on the boundary Σ of a compact convex body C a *$\hat{J}$-Reeb orbit* if $\hat{J}(\xi'(t))$ is (up to scale) an inner normal for Σ at $\xi(t)$. We reserve the terminology Reeb orbit for a Reeb orbit in the usual definition. If $f : (D, \partial D) \to (\mathbb{C}^2, \xi)$ is admissible for $\hat{J}$ then we denote the winding of f with respect to $\hat{J}$ by $\mathrm{wind}_{\hat{J}}(f)$. As an immediate consequence of Lemma 4.1 we obtain

minisadmis **Corollary 4.2.** *Let γ be a periodic $\hat{J}$-Reeb orbit on the boundary Σ of the compact stricly convex body C. Then a minimal disc $f : (D, \partial D) \to (C, \gamma)$ is boundary regular and admissible for $\hat{J}$.*

By Corollary 4.2, if γ is a periodic Reeb orbit on Σ then the winding number of every minimal disc $f : (D, \partial D) \to (C, \gamma)$ is defined.

To construct a minimal disc $f : (D, \partial D) \to (C, \gamma)$ with vanishing winding number, we need some technical facts about minimal surfaces in $\mathbb{C}^2$. For $m \geq 1$ denote by $C^m(S^1, \mathbb{C}^2)$ the Banach space of maps $S^1 \to \mathbb{C}^2$ of class C^m, equipped with the Banach norm $\| \|_{C^m}$. Denote by the same symbol $\| \|_{C^m}$ the Banach norm on the space of maps $D \to \mathbb{C}^2$ of class C^m. The conformal group $PSL(2, \mathbb{R})$ naturally acts on the space of minimal discs with boundary γ by precomposition. Since this action is triply transitive on the boundary ∂D of D, a representative of an orbit under this action can be chosen by a *three-point-condition* (see [S88] for details).

closeofmin **Lemma 4.3.** *Let $m \geq 1$, let $\gamma \in C^{m+1}(S^1, \mathbb{C}^2)$ be an embedded closed curve and let $\epsilon > 0$. Then there is a number $\delta > 0$ with the following property. If $\hat{\gamma} \in C^{m+1}(S^1, \mathbb{C}^2)$ satisfies $\|\gamma - \hat{\gamma}\|_{C^{m+1}} \leq \delta$ then for every minimal disc $\hat{f} : D \to \mathbb{C}^2$ with boundary $\hat{\gamma}$ which is normalized by a three-point condition there is a minimal disc $f : D \to \mathbb{C}^2$ with boundary γ and $\|f - \hat{f}\|_{C^m} \leq \epsilon$.*

Proof. For the proof of the lemma we argue by contradiction and we assume that there is some embedded closed curve $\gamma \in C^{m+1}(S^1, \mathbb{C}^2)$ for which the lemma does not hold. Then there is a number $\epsilon > 0$ and sequence of curves $\{\gamma_i\} \subset C^{m+1}(S^1, \mathbb{C}^2)$

which converge as $i \to \infty$ to γ with the following properties. For every i there is a minimal disc $f_i : D \to \mathbb{C}^2$ with boundary γ_i which is normalized by a three-point condition and such that $\|f_i - f\|_{C^m} \geq \epsilon$ for every minimal disc $f : D \to \mathbb{C}^2$ with boundary γ.

By Theorem 5.1 of S88 [S88], for every $\alpha \in (0,1)$ the $C^{m,\alpha}$-norms of the discs f_i are uniformly bounded. Thus by the Arzela-Ascoli theorem, for a fixed number $\alpha \in (0,1)$ we may assume up to passing to a subsequence that the discs f_i converge as $i \to \infty$ in $C^{m,\alpha}(D, \mathbb{C}^2)$ to a disc $f : D \to \mathbb{C}^2$. This disc has to satisfy the minimal surface equation, and its boundary equals γ. However, this is a contradiction. □

As a corollary we conclude

immersionenough **Corollary 4.4.** *Let γ be a periodic $\hat{J}$-Reeb orbit on the boundary Σ of a compact strictly convex body C. Then for every $m \geq 1, \epsilon > 0$ there is a number $\delta > 0$ with the following property. If $\tilde{\gamma} \in C^{m+1}(S^1, \mathbb{C}^2)$ satisfies $\|\gamma - \tilde{\gamma}\|_{C^{m+1}} \leq \delta$ then a minimal disc $\tilde{f} : (D, \partial D) \to (\mathbb{C}^2, \tilde{\gamma})$ is boundary regular and admissible for $\hat{J}$. Moreover, if $\tilde{f}$ is normalized by a 3-point condition then there is a minimal disc $f : (D, \partial D) \to (C, \gamma)$ with $\|f - \tilde{f}\|_{C^m} \leq \epsilon$ and* $\mathrm{wind}_{\hat{J}}(\tilde{f}) = \mathrm{wind}_{\hat{J}}(f)$.

Proof. Fix a number $m \geq 1$. The set of all minimal discs $f : (D, \partial D) \to (C, \gamma)$ which are normalized by a 3-point condition is compact in the C^m-topology. By Corollary minisadmis 4.2, each of these discs is boundary regular and admissible. Therefore there is a number $\sigma > 0$ such that for each minimal disc $f : (D, \partial D) \to (C, \gamma)$ and for each $z \in \partial D$ the following holds true. Let Z be the (oriented) derivative of γ at z. Then the non-oriented angle between the inner normal of $f(D)$ at $f(z)$ and $\hat{J}Z$ is at most $\pi/2 - \sigma$. Thus there is a number $\epsilon > 0$ such that every minimal disc $\tilde{f}$ with $\|\tilde{f} - f\|_{C^m} \leq \epsilon$ for some minimal disc $f : (D, \partial D) \to (\mathbb{C}^2, \gamma)$ is boundary regular and admissible for $\hat{J}$. Moreover, its winding with respect to $\hat{J}$ coincides with the winding of f. The corollary now follows from Lemma closeofmin 4.3. □

A minimal disc $f : D \to \mathbb{C}^2$ is a *harmonic map* which is moreover conformal. A harmonic map $D \to \mathbb{C}^2$ is uniquely determined by its boundary values. In fact, for any smooth parametrized curve $\gamma : S^1 \to \mathbb{C}^2$ there is a unique harmonic map $f : D \to \mathbb{C}^2$ with $f|\partial D = \gamma$. We observe

arcsofharmonic **Lemma 4.5.** *There is a number $m \geq 2$ with the following property. Let $t \to f_t : D \to \mathbb{C}^2$ $(t \in [0,1])$ be an arc of harmonic maps which is continuous in the C^m-topology. Assume that there is an arc $t \to J_t$ of smooth almost complex structures on $\mathbb{C}^2$ depending continuously on t such that the following holds true.*

(1) *For each t, f_t is locally boundary regular and admissible for J_t.*
(2) *f_0 is an immersion whose winding with respect to J_0 vanishes.*
(3) *f_1 is an immersion.*

Then the winding of f_1 with respect to J_1 vanishes.

Proof. Harmonic maps depend smoothly on their boundary curves. Thus if $f : D \to \mathbb{C}^2$ is a harmonic locally boundary regular map which is admissible for a smooth almost complex structure $\hat{J}$ on $\mathbb{C}^2$ then whenever $\tilde{f} : D \to \mathbb{C}^2$ is harmonic, with $\tilde{f}|\partial D$ sufficiently close to $f|\partial D$ in the C^2-topology, then $\tilde{f}$ is locally boundary regular and admissible for $\hat{J}$. Moreover, if f is an immersion and if $\hat{J} = J$, then $\tilde{f}$ is an immersion with $\mathrm{wind}(\tilde{f}) = \mathrm{wind}(f)$.

Each coordinate function of a harmonic map $h : D \to \mathbb{C}^2$ is harmonic. Thus we can write $h = (h^1, h^2)$ where $h^i : D \to \mathbb{C}$ is a harmonic map. Such a harmonic map is uniquely determined by its parametrized boundary curve $h^i(\partial D)$.

A *branch point* of a harmonic map $u : D \to \mathbb{C}$ is a point $x \in D$ such that the differential $du(x)$ vanishes. By Theorem 1.19 of [BT81], paths of harmonic maps $D \to \mathbb{C}$ which are continuous in the C^m-topology for sufficiently large m can be approxiated arbitrarily closely in the C^3-topology by paths of harmonic maps without branchpoints which are continuous in the C^3-topology. (What is shown precisely is that the set of maps with branchpoints is of codimension 2 in any Sobolev space of maps $D \to \mathbb{C}$ involving weak derivatives of high enough order.)

As a consequence of this discussion, for the purpose of the lemma we may assume without loss of generality that each of the harmonic maps f_t in the statement of the lemma can be written in the form $f_t = (f_t^1, f_t^2)$ where the maps $f_t^i : D \to \mathbb{C}$ are harmonic without branch points and depend continuously on t in the C^3-topology. In particular, there is a number $c > 0$ such that for all t and for each $x \in D$ the biggest eigenvalue of the self-adjoint operator $(df_t^i)^* \circ df_t^i(x)$ is at least c $(i = 1, 2)$.

For $x \in D$ and $t \in [0, 1]$ let $\rho_t(x)$ be the smallest eigenvalue of the self-adjoint operator $(df_t)^* \circ df_t(x)$. The function $(t, x) \to \rho_t(x)$ is continuous. Since each of the maps f_t is locally boundary regular, by perhaps making c smaller we may assume that there is a neighborhood A of ∂D in D so that for each $x \in A$ and each t we have $\rho_t(x) \geq c$. We may moreover assume that $\rho_0(x) \geq c, \rho_1(x) \geq c$ for every $x \in D$ (recall that the harmonic maps f_0, f_1 are immersions by assumption).

Choose a smooth function $\varphi : \mathbb{R} \to [0, \pi/2]$ such that $\varphi(s) = \pi/2$ for $s \leq c/8$ and $\varphi(s) = 0$ for $s \geq c/4$. For $u \in [0, \pi/2]$ let $O(u)$ be the counter-clockwise rotation in $\mathbb{C}$ by the angle u.

Let $t \in [0, 1]$, $x \in D$ be such that $\rho_t(x) \leq c/4$. Then $x \in D - A$ and there is a unit tangent vector $X \in T_x D$ with

$$\langle df_t(X), df_t(X)\rangle = \langle df_t^1(X), df_t^1(X)\rangle + \langle df_t^2(X), df_t^2(X)\rangle \leq c/4.$$

Since the largest eigenvalue of the map $(df_t^i)^* \circ df_t^i(x)$ is at least c, this means that the unoriented angle between X and an eigenvector for $(df_t^i)^* \circ df_t^i(x)$ for the largest eigenvalue is at least $\pi/4$ $(i = 1, 2)$. Thus for all t, x the linear map

$$L_t(x) = (df_t^1(x), df_t^2(x) \circ O(\varphi(\rho_t(x))) : T_x D \to T_{f(x)}\mathbb{C}^2$$

has the property that the smallest eigenvalue of $(L_t(x))^* \circ L_t(x)$ is not smaller than $c/8$ independent of t, x. The assignment $(t, x) \to L_t(x)$ is continuous.

As a consequence, for each $t \in [0, 1]$ the map $x \to L_t(x)$ defines a section $\alpha_t : D \to f_t^* T\mathbb{C}^2$ of the bundle $f_t^* T\mathbb{C}^2$ by associating to $x \in D$ the plane $\alpha_t(x) =$

$L_t(x)T_xD$. This section depends continuously on t, and it coincides with the tangent plane map of f_t within the annulus A. In particular, for each t this section is admissible for the almost complex structure J_t. This means that the winding of α_t can be defined as the winding of the preferred trivialization of the normal bundle of f_t over $f_t(\partial D)$ for the almost complex structure J_t with respect to a trivialization which extends to a global trivialization of the normal bundle of the subbundle of $f_t^*T\mathbb{C}^2$ defined by the section α_t. By continuity of the maps $t \to \alpha_t$ and $t \to J_t$, this winding in turn depends continuously on t. Since α_0, α_1 is just the tangent plane map of f_0, f_1 and since by the assumption in the lemma the winding of f_0 with respect to J_0 vanishes we conclude that the winding of f_1 with respect to J_1 vanishes as claimed. $\square$

In Lemma 4.5, we assumed the existence of a continuous arc $t \to J_t$ of smooth almost complex structures so that for each t the harmonic map f_t is locally boundary regular and admissible for J_t. In our main application, the existence of such an arc of almost complex structures will be a consequence of the following criterion. For its formulation, let $D^0 \subset \mathbb{C}$ be the open unit disc. A smooth local hypersurface containing a smooth Jordan curve $\gamma : S^1 \to \mathbb{C}^2$ is a smooth embedding $\Gamma : S^1 \times D^0 \to \mathbb{C}^2$ such that $\Gamma(s,0) = \gamma(s)$ for all $s \in S^1$. For $m \geq 2$ the set of such embeddings can be equipped with the C^m-topology. In Lemma 4.6 below, a normal field of a hypersurface in $\mathbb{C}^2$ is a vector field along the hypersurface which is everywhere orthogonal to the tangent bundle of the hypersurface.

localhyper

Lemma 4.6. *For $m \geq 2$ let $t \to \Gamma_t \in C^m(S^1 \times D^0, \mathbb{C}^2)$ be an arc of smooth local hypersurfaces ($t \in [0,1]$) which is continuous in the C^m-topology. For each t write $\gamma_t(s) = \Gamma_t(s,0)$. Assume that for each t there is a unit normal field $n(t)$ of $\Gamma_t(S^1 \times D^0)$ depending continuously on t such that for all t, s the unoriented angle between $J\gamma_t'(s)$ and $n_t(\gamma_t(s))$ is strictly smaller than π. Then there is an arc $t \to J_t$ of smooth almost complex structures on $\mathbb{C}^2$ depending continuously on t which are compatible with $\langle , \rangle$ and such that for all t, s the vector $J_t\gamma_t'(s)$ is orthogonal to $\Gamma_t(S^1 \times D^0)$. Moreover, if for some $t \in [0,1]$ and all s the vector $J\gamma_t'(s)$ is orthogonal to $\Gamma_t(S^1 \times D^0)$ then $J_t = J$.*

Proof. Let $E \subset D^0$ be the open disc of radius $1/2$. Since for each t the map $\Gamma_t : S^1 \times D^0 \to \mathbb{C}^2$ is a smooth embedding depending continuously on t in the C^m-topology, there is an arc of embeddings $t \to \Psi_t \in S^1 \times E \times (-1,1) \to \mathbb{C}^2$ which is continuous in the C^m-topology and such that for each t the restriction of Ψ_t to $S^1 \times E \times \{0\}$ coincides with the restriction of Γ_t.

By assumption, for each t, s the angle between the vector $J\gamma_t'(s)$ and the normal $n(t,s) = n_t(\gamma_t(s))$ is strictly smaller than π. As a consequence, there is an arc $u \to Y_{t,s}(u)$ of complex structures on $T_{\gamma_t(s)}\mathbb{C}^2$ ($u \in [0,1]$) such that the following holds true.

(1) $Y_{t,s}(u)$ is compatible with the euclidean inner product $\langle , \rangle$.
(2) $Y_{t,s}(u)$ depends continuously on t, s, u.
(3) $Y_{t,s}(0) = J$ for all t, s.
(4) For all t, s, the linear span of $\gamma_t'(s)$ and $n(t,s)$ is an $Y_{t,s}(1)$-complex line.

For the construction of such a deformation, connect $J\gamma_t'(s)/\|J\gamma_t'(s)\|$ to $n(t,s)$ in the sphere of unit vectors in $T_{\gamma_t(s)}\mathbb{C}^2$ orthogonal to $\gamma_t'(s)$ by the unique shortest geodesic parametrized proportional to arc length on $[0,1]$. This arc depends continuously on t,s and determines an arc of complex structures on $T_{\gamma_t(s)}\mathbb{C}^2$ which are compatible with $\langle , \rangle$ and depend continuously on s,t,u. Namely, such a complex structure Y is uniquely determined by a Y-invariant two-dimensional linear subspace, an orientation on this subspace and the requirement that the orientation induced on $T_{\gamma_t(s)}\mathbb{C}^2$ by Y is the standard orientation (see the proof of Lemma 2.2).

Let $\psi : E \times (-1,1) \to [0,1]$ be a smooth compactly supported function with $\psi(0) = 1$. For $t \in [0,1]$ and for $x \in E \times (-1,1)$ define $J_t(\Psi_t(s,x)) = Y_{t,s}(\psi(x))$. Then $t \to J_t$ is an arc of almost complex structures as required. □

The following corollary is the version of Lemma 4.5 which we are going to use. For its formulation, call a minimal disc $f : D \to \mathbb{C}^2$ *generic* if f can be approximated in the C^2-topology by minimal *immersions*.

Corollary 4.7. *There is a number $m \geq 2$ with the following property. Let γ be a periodic $\hat{J}$-Reeb orbit on the boundary Σ of a compact convex body C. Assume that there is an arc $t \to h_t$ $(t \in [0,1])$ of harmonic maps $D \to \mathbb{C}^2$ and an arc $t \to J_t$ of smooth almost complex structures on $\mathbb{C}^2$ depending continuously on t with the following properties.*

(1) *h_t depends continuously on t in the C^m-topology.*
(2) *Each almost complex structure J_t is compatible with the euclidean inner product, and $J_1 = \hat{J}$.*
(3) *Each of the maps h_t is locally boundary regular and admissible for the almost complex structure J_t.*
(4) *h_0 is an embedding, with vanishing winding for J_0, and $h_1|\partial D$ is an orientation preserving parametrization of γ.*

Then there is a generic minimal disc $f : (D, \partial D) \to (C, \gamma)$ with $\mathrm{wind}_{\hat{J}}(f) = 0$.

Proof. By Theorem 4.14 of [BT81], for all sufficiently large $m \geq 3$ the set of embedded curves $\tilde{\gamma}$ of class C^{m+1} with the property that every minimal disc with boundary $\tilde{\gamma}$ is an immersion is open and dense in the C^{m+1}-topology. (This statement is inasmuch incorrect as Böhme and Tromba use Sobolev spaces to define the topology on the space of Jordan curves. However, the number of weak derivatives which are controlled by the Sobolev norm used can be chosen to be arbitrarily large so that the density statement also applies to the C^{m+1}-topology which is all what we need. Alternatively, we could use Sobolev spaces directly which would not change anything but has some notational disadvantages.) Let γ be a periodic Reeb orbit on the boundary Σ of a compact convex body C. Let $\delta > 0$ is as in Corollary 4.4, let $k > 1/\delta$ and let γ_k be such a curve with the property that $\|\gamma_k - \gamma\|_{C^{m+1}} < 1/k < \delta$.

By perhaps decreasing δ we may assume that there is a continuous arc $t \to \zeta_t \in C^{m+1}(S^1, \mathbb{C}^2)$ $(t \in [1,2])$ which connects $\gamma = \zeta_1$ to $\gamma_k = \zeta_2$ and such that for each t, the curve ζ_t is contained in the boundary Σ_t of a compact strictly convex body depending continuously on t in the C^2-topology. The hypersurface can be represented as the graph of a smooth function $\Sigma \to \mathbb{R}$ with sufficiently small derivatives of order $\leq m$. We may moreover assume that for each t there is a smooth almost complex structure J_t with $J_1 = J$ which is compatible with the euclidean inner product $\langle , \rangle$ and such that for each $s \in S^1$ and all t the vector $J_t\zeta_t'(s)$ is orthogonal to Σ_t (compare the discussion in the proof of Lemma 4.6). Then every minimal disc with boundary γ_k is admissible for J_2. By the choice of γ_k, it is also admissible for J. By perhaps making δ even smaller we may assume that the winding of such a minimal disc with respect to J_2 coincides with its winding with respect to J.

Let $h_1 : D \to \mathbb{C}^2$ be the harmonic map as in 3) of the corollary and let $h_2 = f_k : (D, \partial D) \to (\mathbb{C}^2, \gamma_k)$ be a minimal disc. Connect h_1 to h_2 by an arc $t \to h_t$ ($t \in [1,2]$) of harmonic maps whose boundaries are smooth parametrizations of the curves ζ_t depending continuously on t in the C^{m+1}-topology. Since for each t the curve ζ_t is contained in the boundary Σ_t of a compact strictly convex body, by the maximum principle each of the maps h_t is boundary regular and admissible for J_t (compare Lemma 4.1 and its proof). The concatenation of this arc of harmonic maps with the arc $u \to h_u$ ($u \in [0,1]$) whose existence is assumed in the corollary is an arc of harmonic maps which is continuous in the C^m-topology. This arc connects h_0 to the minimal disc $h_2 = f_k$ and satisfies the assumptions in Lemma 4.5. Thus by Lemma 4.5, we have $\mathrm{wind}(f_k) = 0$.

Now by passing to a subsequence, we may assume that the minimal discs f_k converge as $k \to \infty$ in $C^2(D, \mathbb{C}^2)$ to a minimal disc $f : (D, \partial D) \to (\mathbb{C}^2, \gamma)$. By definition, this disc is generic, and by Corollary 4.4, we have $\mathrm{wind}(f) = 0$ as claimed. □

The following proposition is the main technical result of this note.

windingnb **Proposition 4.8.** *Let γ be a periodic Reeb orbit on the boundary Σ of a compact strictly convex body C. Then there is a generic minimal disc $f : (D, \partial D) \to (C, \gamma)$ with boundary γ and with* $\mathrm{wind}(f) = 0$.

Proof. Let γ be a periodic Reeb orbit on the boundary Σ of a compact strictly convex body C. Our goal is to construct an arc of harmonic maps with the properties stated in Corollary 4.7 for γ_k. We divide this construction into four steps.

Step 1:

In a first preliminary step, we slightly deform the Reeb orbit γ near $\gamma(0), \gamma(\pi)$ to move it into a suitable normal form which is convenient for technical reasons.

Reparametrize γ on $S^1 = [-\pi, \pi]/\sim$ proportional to arc length. Then the second derivative of γ points inside of C. Let $P_0, P_\pi \subset \mathbb{C}^2$ be the affine plane through $\gamma(0), \gamma(\pi)$ whose tangent space at $\gamma(0), \gamma(\pi)$ is spanned by $\gamma'(0), \gamma''(0)$ and $\gamma'(\pi), \gamma''(\pi)$. Since γ is a Reeb orbit and Σ is strictly convex, these planes are admissible.

There is an arc $u \to \gamma_u$ of smooth embedded curves in $\mathbb{C}^2$ through $\gamma_0 = \gamma$ which is continuous in the C^2-topology and with the following additional property. For each $u > 0$ there is a number $\delta(u) > 0$ so that $\gamma_u[-2\delta(u), 2\delta(u)]$ and $\gamma_u[\pi - 2\delta(u), \pi + 2\delta(u)]$ are circular arcs in P_0, P_π. We may also assume that there is a deformation $u \to \Sigma_u$ which is continuous in the C^2-topology (so that each of the hypersurfaces Σ_u is a graph over Σ of a smooth function which varies continuously with u in the C^2-topology) and such that $\gamma_u \subset \Sigma_u$.

By Lemma 4.6 we may assume that there is a continuous family $u \to J_u$ of smooth almost complex structures which are compatible with $\langle , \rangle$ so that $J_0 = J$ and that for each u and each $s \in S^1$ the vector $J_u\gamma_u'(s)$ is normal to Σ_u. For sufficiently small u the hypersurface Σ_u is the boundary of a compact strictly convex body in $\mathbb{C}^2$. Thus by convexity, a minimal disc $\tilde{f}$ with boundary γ_u is admissible for the almost complex structure J_u.

By Corollary 4.4, for sufficiently small u a minimal disc $\tilde{f}$ with boundary γ_u is admissible for the complex structure J, and its winding coincides with the winding of a minimal disc f with boundary γ. Moreover, $\mathrm{wind}(\tilde{f})$ also coincides with the winding of $\tilde{f}$ with respect to J_u. As a consequence, it suffices to construct for sufficiently small u a minimal disc $\tilde{f}$ with boundary γ_u whose winding with respect to J_u vanishes. In other words, we may replace γ by γ_u, Σ by Σ_u and J by J_u for small u we may assume without loss of generality that there is a number $\delta_0 > 0$ such that $\gamma[-2\delta_0, 2\delta_0]$ and $\gamma[\pi - 2\delta_0, \pi - 2\delta_0]$ are circular arcs in admissible affine planes P_0, P_π. Write $\hat{\gamma} = \gamma_u$ and $\hat{J} = J_u$ for some fixed small $u > 0$ which will be decrased several times in the course of this argument. The first such adjustment is as follows.

If γ is a Reeb orbit then for $t \in (0, \pi)$ the tangents $\gamma'(t), \gamma'(-t)$ of γ at $t, -t$ satisfy

$$\langle J\gamma'(t), X_t \rangle > 0, \langle J\gamma'(-t), X_t \rangle < 0 \tag{1}$$

(compare the proof of Lemma 4.1 of [HH09]). By choosing u sufficiently small we may assume that inequality (1) also holds true for the curve $\hat{\gamma}$, with the standard complex structure J.

Step 2:

For the construction of the arc of harmonic maps as in Corollary 4.7 we first construct an arc of Jordan curves $t \to \nu_t$ ($t \in [\delta_0, \pi - \delta_0]$) in $\mathbb{C}^2$ which are piecewise smooth, with two breakpoints, and whose smooth pieces depend continuously on t in the C^{m+1}-topology. The curve ν_{δ_0} is a smooth circle in the admissible affine plane P_0, and $\nu_{\pi - \delta_0} = \gamma$ up to parametrization. These curves will be modified in Step 3 to curves which serve as boundary arcs for the harmonic maps we are looking for.

For $t \in [\delta_0, \pi - \delta_0]$ let ℓ_t be the oriented line segment connecting $\hat{\gamma}(t)$ to $\hat{\gamma}(-t)$. To simplify the notations, we do not specify a parametrization of ℓ_t at the moment. The unit tangent $X_t \in T\mathbb{C}^2$ of ℓ_t depends smoothly on t. By the choice of $\hat{\gamma}$, the two-dimensional linear subspace of $T_{\hat{\gamma}(t)}\mathbb{C}^2$ spanned by $\hat{\gamma}'(t), X_t$ is admissible, and similarly for the two-dimensional linear subspace of $T_{\hat{\gamma}(-t)}\mathbb{C}^2$ spanned by $\hat{\gamma}'(-t), X_t$.

For $t \in [\delta_0, 2\delta_0]$ the line segment ℓ_t is contained in the admissible plane P_0. Moreover, since the arc
hatgamma$[-2\delta_0, 2\delta_0] \subset P_0$ is a segment of a circle, we have $X_s = X_t$ for $s, t \in [\delta_0, 2\delta_0]$ (as vectors with respect to the canonical trivialization of $T\mathbb{C}^2$). Let $Y \in TP_0$ be the unit normal of ℓ_{δ_0} in P_0 which is determined by the requirement that the basis X_{δ_0}, Y of TP_0 defines the canonical orientation. Since P_0 is admissible, the angle between Y and JX_{δ_0} is smaller than $\pi/2$. Thus Y can be connected to $JX_{2\delta_0} = JX_{\delta_0}$ by a unique geodesic segment of minimal length in the two-dimensional sphere of all unit vectors in $\mathbb{C}^2$ which are orthogonal to X_{δ_0}. The span of $X_{2\delta_0}$ with each point on this geodesic segment is an admissible plane. Let $t \to Y_t$ $(t \in [\delta_0, 2\delta_0])$ be a smooth map connecting $Y = Y_{\delta_0}$ to $Y_{2\delta_0} = JX_{2\delta_0}$ which is constant near its endpoints and whose trace equals the trace of this geodesic segment. Similarly, define an arc $t \to Y_t$ $(t \in [\pi - 2\delta_0, \pi - \delta_0])$ which connects the vector $Y_{\pi-2\delta_0} = JX_{\pi-2\delta_0}$ to the oriented normal $Y_{\pi-\delta_0}$ of $X_{\pi-\delta_0}$ in the admissible plane P_π. For $t \in [2\delta_0, \pi - 2\delta_0]$ define $Y_t = JX_t$. Then $t \to Y_t$ is a smooth arc of unit vectors.

The hyperplane $H_t \subset \mathbb{C}^2$ which contains the line segment ℓ_t and is orthogonal to Y_t intersects $\hat\gamma$ transversely at the points $\hat\gamma(t), \hat\gamma(-t)$. The hyperplane H_t depends smoothly on t. It decomposes the convex body C into two convex subsets whose closures C_t^-, C_t^+ depend continuously on t in the Hausdorff topology for compact subsets of $\mathbb{C}^2$. Here we denote by C_t^- the compact convex body whose boundary ∂C_t^- contains the points $\hat\gamma(s), \hat\gamma(-s)$ for $s < t$ sufficiently close to t. Since $\langle \hat\gamma'(t), JX_t\rangle < 0$, by definition of the hyperplanes H_t the vector Y_t is the unit normal of H_t which points inside of the compact convex body C_t^-.

For $t \in [\delta_0, \pi - \delta_0]$ let $x(t) \in H_t \cap C$ be the midpoint of the line segment $\ell_t \subset H_t$ connecting $\hat\gamma(t)$ to $\hat\gamma(-t)$. Note that $x(t)$ depends smoothly on t. For $r > 0, s \in \mathbb{R}$ let $S_{r,s,t} \subset \mathbb{C}^2$ be the distance sphere of radius r about the point $x(t) + sY_t$ (where by abuse of notation we write $x(t) + sY_t$ to denote the point of oriented distance s from $x(t)$ on the oriented line through $x(t)$ whose tangent equals Y_t). For every sufficiently large $r > 0$, say for all $r \geq r_0$ independent of $t \in [\delta_0, \pi - \delta_0]$, there is a unique number $\sigma(r, t) > 0$ such that $S_{r,\sigma(r,t),t}$ contains the points $\hat\gamma(t), \hat\gamma(-t)$. We also assume that for $r \geq r_0$ the intersection $S_{r,\sigma(r,t),t} \cap \Sigma$ is a smooth 2-sphere (i.e. this intersection is transverse). This is possible since as $r \to \infty$, the hypersurfaces $S_{r,\sigma(r,t),t}$ converge locally uniformly in the C^k-topology to the hyperplane H_t for any $k > 0$.

Choose a smooth function $r : [\delta_0, \pi - \delta_0] \to (0, \infty)$ so that $r(\delta_0)$ is the radius of the circle containing $\hat\gamma[-2\delta_0, 2\delta_0]$ in the affine plane P_0, that r is strictly increasing on $[\delta_0, 2\delta_0]$ and that $r(t) \geq r_0$ for all $t \in [2\delta_0, \pi - \delta_0]$. Then there is a unique continuous function $t \in [\delta_0, \pi - \delta_0] \to \sigma(t) \in \mathbb{R}$ with the following properties.

(1) The sphere $S(t) = S_{r(t),\sigma(t),t}$ contains the points $\gamma(-t), \gamma(t)$.
(2) $S(\delta_0)$ contains the circular arc $\gamma[-2\delta_0, \delta_0]$.
(3) $\sigma(t) > 0$ for $t \in [2\delta_0, \pi - 2\delta_0]$.

For $t \in [\delta_0, \pi - \delta_0]$ let $B(t)$ be the compact ball of radius $r(t)$ about $x(t) + \sigma(t)Y_t$ with boundary sphere $S(t) = S_{r(t),\sigma(r(t),t),t}$. The intersection

$$B(t) \cap C = \tilde{C}_t$$

is a compact strictly convex body which depends continuously on t in the Hausdorff topology for compact subsets of $\mathbb{C}^2$.

For $t \in [2\delta_0, \pi - \delta_0]$, the intersection $S(t) \cap \Sigma$ is a smooth two-dimensional sphere which is a smooth submanifold of both $S(t)$ and Σ. The singular 2-sphere $S(t) \cap \Sigma$ decomposes $\partial \tilde{C}_t$ into two smooth 3-balls. One of these 3-balls is contained in $S(t)$, the other is contained in Σ. We may assume that there is a number $\epsilon > 0$ such that $\gamma[-t, -t + \epsilon] \cup \gamma[t - \epsilon, t] \subset \partial \tilde{C}_t$ for all $t \in [2\delta_0, \pi - 2\delta_0]$. We may moreover assume that $\gamma[-t, t] \subset \partial \tilde{C}_t$ for $t \in [\delta_0, 2\delta_0]$ and for $t \in [\pi - 2\delta_0, \pi - \delta_0]$.

For $t \in [2\delta_0, \pi - \delta_0]$ the points $\gamma(-t), \gamma(t)$ are contained in an open hemisphere of $S(t)$. Let $\tilde{\ell}_t \subset S(t)$ be the segment of a great circle in $S(t)$ connecting $\gamma(t)$ to $\gamma(-t)$ which is contained in this hemisphere. Then $\tilde{\ell}_t \subset \tilde{C}_t$ by construction, moreover $\tilde{\ell}_t$ is a segment of a Reeb orbit on $S(t)$. For suitable parametrizations, the arcs $\tilde{\ell}_t$ depend smoothly on t. Moreover, this arc extends to an arc defined on all of $[\delta_0, \pi - \delta_0]$. For $t \in [\delta_0, 2\delta_0]$, $\tilde{\ell}_t$ is contained in the intersection of $S(t)$ with an admissible plane through the center of the ball $B(t)$, i.e. it is a great circle transverse to the two-dimensional J-invariant subbundle of $TS(t)$. In particular, for all t the vector field $J\tilde{\ell}_t'$ along $\tilde{\ell}_t$ is transverse to $S(t)$.

Let $\nu_t \subset S(t) \cup \Sigma$ be the oriented Jordan curve which is composed of $\gamma[-t, t]$ and $\tilde{\ell}_t$. Up to parametrization, the curve ν_{δ_0} is a smooth circle in the plane P_0. For each t, the curve ν_t is smooth away from the points $\gamma(t), \gamma(-t)$. The one-sided tangents of ν_t at the two breakpoints are contained in an admissible plane in the tangent space of $\mathbb{C}^2$. Parametrize ν_t on S^1 (represented as the interval $[0, 2\pi]$ or $[-\pi, \pi]$ with endpoints identified) in such a way that the restriction of ν_t to $[-\pi/2, \pi/2]$ is a parametrization of $\tilde{\ell}_t$ proportional to arc length. We also require that for $m \geq 2$ as above, the arc $\nu_t[\pi/2, 3\pi/2]$ depends continuously on t in the C^{m+1}-topology and that the norm of its tangent at an endpoint equals the norm of the tangent of $\nu_t[-\pi/2, \pi/2]$. There is a number $\epsilon < \pi/8$ such that $\nu_t[-\pi/2 - 2\epsilon, \pi/2 + 2\epsilon] \subset \partial \tilde{C}_t$ for all t.

Step 3:

Since the compact convex body C is fixed and since the spheres $H_t \cap \Sigma$ intersect the Reeb orbit γ at the points $\gamma(t), \gamma(-t)$ transversely, for every $a > 0$ there is a number $\kappa(a) > 0$ with the following property.

Let $t \in [\delta_0, \pi - \delta_0]$ and let ζ be a connected subarc of $\nu_t[-\pi/2 - 2\epsilon, -\pi/2]$ or of $\nu[\pi/2, \pi/2 + 2\epsilon]$ of length a. Assume that ζ is parametrized by arc length (but not necessarily respecting the orientation) on the interval $[0, a]$. Let $X \in T_{\zeta(a)}\Sigma$ be a unit tangent vector whose non-oriented angle to the vector $\zeta'(a)$ is smaller than $\kappa(a)$. Then there is a deformation of $\zeta[0, a]$ to a smooth arc $\tilde{\zeta} : [0, a] \to \Sigma \cap \partial \tilde{C}_t$ on Σ with the same endpoints which coincides with ζ near $\zeta(0)$ and such that $\tilde{\zeta}'(a) = X$.

Moreover, the vector field $J\tilde{\zeta}'(s)$ is transverse to Σ for all s. We may assume that $\tilde{\zeta}$ depends smoothly on ζ, X.

Similarly, by making $\kappa(a)$ smaller we may assume that for every $t \in [\delta_0, \pi - \delta_0]$, for every subarc $\zeta : [0, a] \to \tilde{\ell}_t$ of length a parametrized by arc length and for every unit tangent vector $X \in T_{\zeta(a)}S(t)$ there is a deformation $\tilde{\zeta} : [0, a] \to S(t) \cap C$ of ζ with the same endpoints which is transverse to the canonical contact structure and such that $\tilde{\zeta}'(a) = X$.

Choose $a > 0$ small enough that for each $t \in [\delta_0, \pi - \delta_0]$ the length of each of the arcs

$$\nu_t[-\pi/2 - 2\epsilon, -\pi/2 - \epsilon], \nu_t[-\pi/2 + \epsilon, -\pi/2 + 2\epsilon],$$
$$\nu_t[\pi/2 - 2\epsilon, \pi/2 - \epsilon], \nu_t[\pi/2 + \epsilon, \pi/2 + 2\epsilon]$$

is at least a.

For each $t \in (\delta_0, \pi - \delta_0]$ the one-sided tangents at $\nu_t(-\pi/2), \nu_t(\pi/2)$ of the arc ν_t span an admissible plane in $T_{\nu_t(\pi/2)}\mathbb{C}^2, T_{\nu_t(\pi)}\mathbb{C}^2$. For t sufficiently close to δ_0, this plane is just the tangent plane of the affine plane P_0. There is a number $\sigma_0 < \epsilon/2$ such that for each $\sigma \in (0, \sigma_0]$ the affine plane $P(t) \subset \mathbb{C}^2$ (or $Q(t) \subset \mathbb{C}^2$) which passes through the points $\nu_t(-\pi/2 - \sigma), \nu_t(-\pi/2), \nu_t(-\pi/2 + \sigma)$ (or through the points $\nu_t(\pi/2 - \sigma), \nu_t(\pi/2), \nu_t(\pi/2 + \sigma)$) is admissible. Note that we have $P(t) = Q(t) = P_0$ for t sufficiently close to δ_0. As $\sigma \to 0$, these affine planes converge to planes whose tangent space at $\nu_t(-\pi/2), \nu_t(\pi/2)$ contain the one-sided tangents of ν_t.

As a consequence, for sufficiently small σ the angle at $\nu_t(-\pi/2 - \sigma), \nu_t(\pi/2 + \sigma)$ between the tangent of ν_t and the tangent of $P(t) \cap \Sigma, Q(t) \cap \Sigma$ is at most $\delta(a)$, and the angle at $\nu_t(-\pi/2 + \sigma), \nu_t(\pi/2 - \sigma)$ between the tangent of ν_t and the tangent of $P(t) \cap S(t), Q(t) \cap S(t)$ is at most $\delta(a)$. The planes $P(t), Q(t)$ are equipped with a canonical orientation, and they depend smoothly on t.

Let $\hat{C}_t \subset C$ be a compact strictly convex body with smooth boundary $\partial\hat{C}_t$ obtained by pushing the singular sphere $S(t) \cap \Sigma$ slightly inside $\tilde{C}_t$. We assume that the support of this deformation is small enough that

$$\partial\hat{C}_t \supset \nu_t([-\pi/2 - 2\epsilon, \pi/2 + 2\epsilon] - [-\pi/2 - \sigma/2, -\pi/2 + \sigma/2] - [\pi/2 - \sigma/2, \pi/2 + \sigma/2]).$$

We also assume that $\hat{C}_t$ depends smoothly on t.

The intersections

$$\hat{D}_t = \hat{C}_t \cap P(t),\ \hat{E}_t = \hat{C}_t \cap Q(t)$$

are strictly convex discs in the oriented planes $P(t), Q(t)$ with smooth oriented boundary $\partial\hat{D}_t, \partial\hat{E}_t$. The circle $\partial\hat{D}_t$ contains a smooth oriented arc

$$\xi_t : [-\pi/2 - \sigma, -\pi/2 + \sigma] \to P(t)$$

connecting $\xi_t(-\pi/2 - \sigma) = \nu_t(-\pi/2 - \sigma)$ to $\xi_t(-\pi/2 + \sigma) = \nu_t(-\pi/2 + \sigma)$, and the circle $\partial\hat{E}_t$ contains a smooth oriented arc

$$\eta_t : [\pi/2 - \sigma, \pi/2 + \sigma] \to Q(t)$$

connecting $\nu_t(\pi/2 - \sigma)$ to $\nu_t(\pi/2 + \sigma)$. The circles ξ_t, η_t depend smoothly on t. By the choice of σ, the arcs ξ_t, η_t extend smoothly to smooth local deformations of

$\nu_t[-\pi/2-2\epsilon,-\pi/2-\sigma],\nu_t[\pi/2+\sigma,\pi+2\epsilon]$ with the same endpoints which coincide with ν_t near $\nu_t(-\pi/2-2\epsilon),\nu_t(\pi/2+2\epsilon)$. By the above discussion, we may assume that these deformed arcs are contained in $\partial\hat{C}_t$, that they are transverse to the two-dimensional J-invariant subbundle of $T\hat{C}_t$ away from the planar arcs ξ_t,ν_t and that they depend continuously on t in the C^{m+1}-topology. The concatenation of these deformed arcs with the arc $\nu_t([\pi/2+2\epsilon,3\pi/2-2\epsilon])$ is a smooth arc $\hat{\nu}_t$ on $\partial\hat{C}_t\cup\Sigma$ depending continuously on t in the C^m-topology. Moreover by construction, for all t we have

$$\hat{\nu}_t[-\pi/2-2\epsilon,\pi+2\epsilon]\subset\partial\hat{C}_t,\ \hat{\nu}_t[\pi/2+\epsilon,3\pi/2-\epsilon]\subset\gamma\subset\Sigma$$

and $\hat{\nu}_{\delta_0}\subset P_0$ up to parametrization.

Up to modifying the parametrization, we also have

$$\hat{\nu}_{\pi-\delta_0}[-\pi/2-\epsilon,\pi/2+\epsilon]\subset P_\pi$$

and $\hat{\nu}_{\pi-\delta_0}[\pi/2+\epsilon,3\pi/2-\epsilon]\subset\gamma(-\pi+2\delta_0,\pi-2\delta_0)$. In other words, $\hat{\nu}_{\pi-\delta_0}$ is obtained from γ by replacing the subarc $\gamma[\pi-2\delta_0,\pi+2\delta_0]$ by a smooth arc which is contained in P_π and which is strictly convex with respect to the canonical orientation. Moreover, this arc is contained in $C\cap P_\pi$. As a consequence, there is a smooth extension $t\in[\pi-\delta_0,\pi]\to\hat{\nu}_t$ such that for each t the curve $\hat{\nu}_t$ is a smooth deformation of $\hat{\nu}_{\pi-\delta_0}$ with the following properties.

i) $\hat{\nu}_t(s)=\hat{\nu}_{\pi-\delta_0}(s)$ for $s\in[\pi/2+\epsilon,3\pi/2-\epsilon]$.
ii) $\hat{\nu}_t|[-\pi/2-\epsilon,\pi/2+\epsilon]$ is a smooth strictly convex arc in P_π.
iii) $\hat{\nu}_\pi=\gamma$ up to parametrization.

We claim that there is an arc $t\to J_t$ of smooth almost complex structures on $\mathbb{C}^2$ depending continuously on $t\in[\delta_0,\pi]$ with the following properties.

a) $J_\pi=\hat{J}$.
b) For each t and all $s\in S^1$, $J_t\hat{\nu}_t'(s)$ is orthogonal to $\partial\hat{C}_t$.
c) The winding with respect to J_{δ_0} of an embedding $h:D\to P_{\delta_0}$ with boundary $h(\partial D)=\hat{\nu}_{\delta_0}$ vanishes.

By construction, away from perhaps some compact subset in the interior of the arcs ξ_t,η_t the vector field $J\hat{\nu}_t'[-\pi/2-2\epsilon,\pi/2+2\epsilon]$ is transverse to $\partial\hat{C}_t$. However, the arcs ξ_t,η_t are subarcs of the boundary of the strictly convex disc $P(t)\cap\hat{C}_t,Q(t)\cap\hat{C}_t$, and the planes $P(t),Q(t)\subset\mathbb{C}^2$ are admissible. Therefore the angle between $J\xi_t',J\eta_t'$ and the inner normal of ξ_t,η_t in $P(t),Q(t)$ is strictly smaller than $\pi/2$. Moreover, the inner normals of ξ_t,η_t in $P(t),Q(t)$ are transverse to $\partial\hat{C}_t$ and therefore the angles between these inner normals in $P(t),Q(t)$ and the inner normal of $\partial\hat{C}_t$ is strictly smaller than π.

Lemma localhyper 4.6 now shows that there is a continuous arc $t\to\hat{J}_t$ of almost complex structures on $\mathbb{C}^2$ which are compatible with $\langle,\rangle$ and such that $\hat{J}_t\hat{\nu}_t'$ is orthogonal to $\hat{C}_t$ for all t. Moreover, we may assume that $J_\pi=J$. By the explicit construction and the fact that $\hat{\nu}_{\delta_0}$ is a smooth circle in the admissible plane P_0, property c) above holds true as well.

Step 4:

In the last step, we construct a family of smooth parametrizations of the curves $\hat{\nu}_t$ which serve as boundary curves for an arc of harmonic maps with the properties in Corollary 4.7.

Let $\psi_i : S^1 \to S^1$ $(i \geq 0)$ be a sequence of smooth orientation preserving diffeomorphisms beginning with the identity ψ_0. We require that for each $i \geq 0$, the restriction of ψ_i to $[-\pi/2-\epsilon, \pi/2+\epsilon]$ is the identity and that $\psi_i^{-1}[-\pi/2-2\epsilon, \pi/2+2\epsilon] \subset \psi_{i+1}^{-1}[-\pi/2-2\epsilon, \pi/2+2\epsilon]$. Moreover, we require that $\cup_i \psi_i^{-1}[-\pi/2-2\epsilon, \pi/2+2\epsilon] = (-\pi, \pi)$.

For $t \in [\delta_0, \pi]$ and for $i \geq 0$ let $\alpha_{t,i} : D \to \mathbb{C}^2$ be the unique harmonic map with $\alpha_{t,i}(\partial D) = \hat{\nu}_t \circ \psi_i$ as a parametrized curve. For $z \in D - \partial D$, we have

poisson
$$\alpha_{t,i}(z) = \int_{\partial D} \hat{\nu}_t \circ \psi_i(x) d\lambda_z(x) \tag{2}$$

where λ_z is a measure on ∂D in the Lebesgue measure class depending smoothly on z, and $\lambda_z \to \delta_x$ weakly as $z \to x \in \partial D$ where δ_x is the Dirac mass at x. In particular, $\alpha_{t,i}$ is smooth in the interior of D and continuous up to and including the boundary. For fixed i and fixed $m \geq 2$ as above, the arc $t \to \alpha_{t,i}$ is continuous as an arc in the Banach space $C^m(D, \mathbb{C}^2)$. By the maximum principle, $\alpha_{t,i}(D) \subset C$ for all t, i.

Using once more the maximum principle, as $i \to \infty$ the maps $\alpha_{t,i}$ converge uniformly on compact subsets of the union of the interior of D with the boundary arc $(-\pi/2-\epsilon, \pi/2+\epsilon)$ to a map whose image in contained in $\hat{C}_t$, and this convergence is uniform in t. Since $\hat{C}_t$ is strictly convex, this implies that there is a number $i > 0$ and there is a neighborhood U of $[-\pi/2-\epsilon, \pi/2+\epsilon]$ in D which is mapped by $\alpha_{t,i}$ into $\hat{C}_t$ for each $t \in [\delta_0, \pi]$. Namely, for each $s \in [-\pi/2-\epsilon, \pi/2+\epsilon]$ there is a linear functional $\psi_s : \mathbb{C}^2 \to \mathbb{R}$ with the property that $\psi_s(\hat{\nu}_t(s)) > \psi_s(v)$ for all $v \in \hat{C}_t - \{\nu_t(s)\}$. The claim then follows from the Poisson formula (2). Since $\hat{\nu}_t[\pi/2+\epsilon, 3\pi/2-\epsilon]$ is contained in the boundary of the compact convex body C for all t, by the maximum principle this implies that the harmonic map $\alpha_{t,i}$ is J_t-admissible for all t.

For $t = \delta_0$ the curve $\hat{\nu}_t \circ \psi_i$ is a smooth parametrization of a smooth round circle in the affine plane P_0. This implies that the map h_{δ_0} is a diffeomorphism onto the disc in P_0 bounded by this circle. Thus by property c) above, the winding of h_{δ_0} with respect to J_{δ_0} vanishes. The boundary of the map h_π is a parametrization of the Reeb orbit γ. As a consequence, the arc of harmonic maps $t \to h_t$ satisfies the requirements in Corollary 4.7. The proposition is proven. □

As a corollary we obtain

tan **Corollary 4.9.** *Let γ be a periodic Reeb orbit on the boundary of a compact strictly convex body $C \subset \mathbb{C}^2$. Then γ bounds a symplectic disc with precisely $\frac{1}{2}(\mathrm{lk}(\gamma)+1)$ positive self-intersections, counted with multiplicities.*

Proof. By Proposition 4.8 and Corollary 4.2 there is a generic minimal admissible disc $f : (D, \partial D) \to (C, \gamma)$ with $\mathrm{wind}(f) = 0$. Since f is generic, it can be perturbed to an admissible minimal disc $\tilde{f}$ with vanishing winding and only transverse double points. By Proposition 3.1, the disc $\tilde{f}$ is symplectic.

Proposition 3.3 shows that $\tilde{f}$ has precisely $\tan(f)$ self-intersection points counted with multiplicities. By Proposition 2.6 and invarianc under perturbation we have $\tan(\tilde{f}) = \frac{1}{2}(\mathrm{lk}(\gamma)+1)$. Now $\tilde{f}$ can be slightly deformed near the boundary to a symplectic disc with boundary γ and $\tan(f)$ transverse double points which shows the corollary. □

linkone **Corollary 4.10.** *If* $\mathrm{lk}(\gamma) = -1$ *then* γ *bounds an embedded minimal symplectic disc* $f : (D, \partial D) \to (C, \gamma)$.

As another corollary, we obtain

slicegenus **Corollary 4.11.** *The slice genus of* γ *does not exceed* $\frac{1}{2}(\mathrm{lk}(\gamma)+1)$.

Proof. By Corollary 4.9, a periodic Reeb orbit on Σ bounds a boundary regular symplectic disc $f : (D, \partial D) \to (C, \gamma)$ with $\tan(f) = \frac{1}{2}(\mathrm{lk}(\gamma)+1)$ self-intersection points counted with multiplicities. Each of these self-intersection points has positive self-intersection index. A small perturbation of f resolving the branch points to transverse double points and resolving multiple self-intersection points to transverse double points yields an immersed disc with precisely $\frac{1}{2}(\mathrm{lk}(\gamma)+1) = \tan(f)$ simple transverse positive double points. Each of these double points can be removed with a standard surgery. Such a surgery consists in removing a small neighborhood of the double point which is homeomorphic to two discs intersecting transversely at their midpoints and connecting the two boundary components of the resulting surface by an embedded annulus. This amounts to adding for each transverse double point a single handle to the minimal disc and changing the surface only in an arbitrarily small neighborhood of the self-intersection point. With $\tan(f)$ such surgeries we obtain an embedded surface $S \subset C$ of genus $\tan(f)$ with boundary γ as claimed. □

Open questions: Is the slice genus of a periodic Reeb orbit γ on the boundary Σ of a compact convex domain $C \subset \mathbb{C}^2$ equal to $(\mathrm{lk}(\gamma)+1)/2$ (and hence coincides with its Seifert genus)? Is such a periodic Reeb orbit concordant to an iterated torus knot? Note that by a result of Kronheimer and Mrowka [KM93], for torus knots the slice genus and the Seifert genus coincide.

5. Glueing links

The main idea for the proof of the theorem in the introduction is as follows. Let γ be a periodic Reeb orbit on the boundary Σ of a compact strictly convex body $C \subset \mathbb{C}^2$. We use a minimal disc constructed in Section 4 which is bounded by a small perturbation of γ and has $\frac{1}{2}(\mathrm{lk}(\gamma)+1)$ positive transverse self-intersection points to decompose the knot γ into links which can be investigated with a Morse theory type construction. This decomposition is of geometric nature and not directly related to the prime decomposition of γ, nor does it define an invariant of the knot.

The purpose of this section is to introduce a glueing procedure for links (the inverse of the decomposition procedure) and to establish the elementary tools needed for an application of this procedure in Section 6.

We begin with a simple observation about the linking of two knots. Two disjoint knots $\alpha, \beta \subset \Sigma \sim S^3$ are *unlinked* if β is contractible in $\Sigma - \alpha$.

linkingnb

Lemma 5.1. *Let $\alpha, \beta \subset \Sigma$ be disjoint knots. If α is an unknot and if α, β bound disjoint embedded discs in C then α, β are unlinked.*

Proof. Let α, β be disjoint knots in Σ. Assume that α is an unknot. Then the fundamental group of $\Sigma - \alpha$ is infinite cyclic and coincides with its first homology group $H_1(\Sigma - \alpha, \mathbb{Z}) = \mathbb{Z}$. Up to a choice of sign, the linking number of α and β is just β viewed as an element in this homology group. Therefore α, β are unlinked if and only if the linking number of α and β vanishes. By a classical result in knot theory (p. 136 of [R76]), this is the case if α, β bound disjoint embedded discs in C. The lemma follows. $\square$

Call a link ζ in Σ *oriented* if each component of ζ is oriented. We define an *oriented sum* $\eta \sharp \zeta$ of an oriented link η with an oriented link ζ in Σ as follows.

Call an embedded arc α in an embedded three-dimensional compact ball $Q \subset \Sigma$ with endpoints on the boundary ∂Q of Q *unknotted* if α is contained in a smooth embedded disc $S \subset Q$ with boundary $\partial S \subset \partial Q$. Two unknotted disjoint arcs α_1, α_2 in Q with endpoints on ∂Q are called *unlinked* if α_1, α_2 are contained in a common smooth embedded disc $(S, \partial S) \subset (Q, \partial Q)$. Two unknotted unlinked arcs in Q with interior in the interior of Q define a *trivial* $(2,2)$*-tangle* in Q (see [Mu96]).

Let $\eta, \zeta \subset \Sigma$ be oriented links. Assume that η and ζ are disjoint (and hence $\eta \cup \zeta$ is a link). Let $Q_1, \dots, Q_m \subset \Sigma$ be pairwise disjoint closed balls with the following properties.

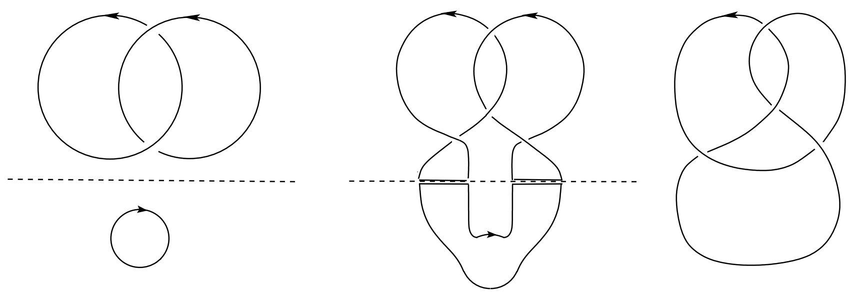
(1) For each i, the intersection of $\eta \cup \zeta$ with Q_i consists of two unknotted unlinked arcs $\alpha_i \subset \eta, \beta_i \subset \zeta$.
(2) Let G be the finite graph whose vertices are the components of $\eta \cup \zeta$ and whose edges are defined as follows. For each $i \leq m$ there is an edge in G which connects the component p of η containing α_i to the component q of ζ containing β_i. Then G does not have cycles (i.e. G is a disjoint union of trees). We call G the *glueing graph* of the oriented sum.

For each i let ξ_i^1 be an unknotted arc in Q_i which connects the endpoint of α_i to the starting point of β_i, and let ξ_i^2 be an unknotted arc in Q_i which connects the endpoint of β_i to the starting point of α_i. Assume that the arcs ξ_i^1, ξ_i^2 are unlinked and that the knot defined as the concatenation $\alpha_i \circ \xi_i^1 \circ \beta_i \circ \xi_i^2$ (read from the left to the right) is the unknot. Here starting point and endpoint of α_i, β_i are determined by the orientation of η, ζ (see [Mu96]).

For each i remove the arcs α_i, β_i from η, ζ. Connect the endpoint of $\eta - \alpha_i$ (which is the starting point of α_i) to the starting point of $\zeta - \beta_i$ by the inverse of the arc ξ_i^2, and connect the endpoint of $\zeta - \beta_i$ to the starting point of $\zeta - \alpha_i$ by the inverse of ξ_i^1. The resulting link $\eta\sharp\zeta$ in Σ is obtained from $\eta\cup\zeta$ by m surgeries and will be called an oriented sum of η and ζ. It depends on many choices made in the construction, but it is independent of the choice of the arcs ξ_i^1,ξ_i^2 and of the order in which the m surgeries are performed on $\eta\cup\zeta$. If each component ζ_i of ζ intersects precisely one of the balls Q_i then we call $\eta\sharp\zeta$ a *ζ-injective oriented sum.* If η is a knot then a ζ-injective oriented sum $\eta\sharp\zeta$ is a knot. The usual connected sum of two oriented knots is a special case of a ζ-injective oriented sum.

Example: Figure A below shows that both the figure eight knot (which is prime) and the unknot can be obtained as a ζ-injective oriented sum of the unknot η with the Hopf link ζ. More generally, any twisted double of the unknot can be obtained in this way.

Figure A



The following simple observation is the main basic tool for our purpose. For its formulation, call two disjoint links $\eta,\zeta\subset\Sigma$ *unlinked* (or splittable) if there are disjoint compact balls $B_1,B_2\subset\Sigma$ such that $\eta\subset B_1,\zeta\subset B_2$. A link ζ in Σ is *trivial* if ζ is a union of pairwise unlinked unknots. If $m\geq 1$ is the number of components of ζ then ζ has a regular link diagram which consists of m disjoint circles in $\mathbb{R}^2$.

trivialadd **Lemma 5.2.** *Let ζ be a trivial oriented link and let η be an oriented link which is unlinked with ζ.*

(1) *If η is a knot then a ζ-injective oriented sum $\eta\sharp\zeta$ is isotopic to η.*
(2) *If $\eta\cup\zeta$ is a trivial link then an oriented sum $\eta\sharp\zeta$ is a trivial link.*

Proof. Let $\zeta_1,\dots,\zeta_m$ be the components of the trivial link ζ. Assume first that η is a knot and that $\eta\sharp\zeta$ is a ζ-injective oriented sum. Number the balls Q_i in the definition of the oriented sum in such a way that $Q_i\cap\zeta_i\neq\emptyset$ for $1\leq i\leq m$. Assume that $\alpha_i=Q_i\cap\eta,\beta_i=Q_i\cap\zeta_i$ and that β_i meets the boundary ∂Q_i of Q_i only at its endpoints. Since ζ is a union of pairwise unlinked unknots and η is unlinked with ζ, the component ζ_i of ζ can be isotoped in $\Sigma-\eta-\cup_{j\neq i}(\zeta_j\cup Q_j)$ to a loop ζ_i' which is contained in Q_i and which intersects the interior of Q_i in the interior of the arc β_i. The closure of $\zeta_i'-\beta_i$ is an arc in ∂Q_i which is unlinked with α_i. Replacing the subarc α_i of η as in the definition of the oriented sum $\eta\sharp\zeta_i$ results in a knot which is isotopic to η. Thus attaching successively in m such steps the components of ζ to η yields a knot $\eta\sharp\zeta$ which is isotopic to η. This shows the first part of the lemma.

To show the second part of the lemma, we proceed by induction of the number n of components of $\eta \cup \zeta$. If $n = 2$ then η, ζ are unlinked unknots, and by the second requirement in the definition of an oriented sum (no cycles for the glueing graph G), $\eta\sharp\zeta$ is obtained from $\eta \cup \zeta$ by at most one surgery. Then either $\eta\sharp\zeta$ is a trivial link with two components (if there is no surgery) or an unknot by the observation in the previous paragraph. Thus assume that the statement is known for oriented sums of unlinked trivial links with at most $n_0 - 1 \geq 2$ components.

Let η, ζ be unlinked trivial links so that $\eta \cup \zeta$ has n_0 components and let $\eta\sharp\zeta$ be an oriented sum of η and ζ. We may assume that there is at least one surgery in the construction of $\eta\sharp\zeta$ from $\eta \cup \zeta$. Let G be the glueing graph of the oriented sum $\eta\sharp\zeta$ whose vertices are the components of $\eta \cup \zeta$. Let η_i, ζ_i be components of η, ζ which are connected by an edge in G (where we view η_i, ζ_i as vertices of G) and such that one of these components, say ζ_i, is a univalent vertex of G. Such a component exists since G is a disjoint union of trees. Let Q_i be the ball as in the definition of an oriented sum which intersects ζ_i and η_i. Then ζ_i can be isotoped in $\Sigma - \eta - \cup_{j\neq i}(\zeta_j \cup Q_j)$ to a loop ζ_i' contained in Q_i. By the observation in the first paragraph of this proof, the link obtained from $\eta \cup \zeta$ by a single surgery as in the definition of an oriented sum which joins ζ_i' to the component η_i of η is a union $\eta' \cup \zeta'$ of pairwise unlinked unknots with $n_0 - 1$ connected components. The oriented sum $\eta\sharp\zeta$ coincides with an oriented sum $\eta'\sharp\zeta'$ whose glueing graph G' is obtained from G by removing the vertex corresponding to ζ_i and the edge incident on this vertex. In particular, G' does not have cycles. The claim now follows from the induction hypothesis and the fact that the surgeries in the construction of $\eta\sharp\zeta$ can be performed in an arbitrary order. □

Remark: Without the second requirement in the definition of an oriented sum of two oriented links, Lemma trivialadd 5.2 does not hold. For example, a pretzel knot can be obtained from two unlinked unknots by a surgery construction whose graph has two vertices and three edges connecting them.

Define the *Euler characteristic* $\chi(\eta)$ of an oriented link $\eta \subset \Sigma$ to be the largest Euler characteristic of an oriented (possibly disconnected) embedded surface $S \subset \Sigma$ without closed components and with oriented boundary η. As an example, the Euler characteristic of a trivial link ζ with $\ell \geq 1$ components equals $\chi(\zeta) = \ell$. The next lemma relates the Euler characteristic of a particular oriented sum to the Euler characteristic of its components. For its formulation, define the *Hopf link* in $S^3 \sim \Sigma$ to be the intersection with S^3 of two distinct complex lines in $\mathbb{C}^2$.

hopflinkadd **Lemma 5.3.** *Let ζ be an oriented Hopf link and let η be an oriented link which is unlinked with ζ. Then*

$$\chi(\eta\sharp\zeta) \geq \chi(\eta) - 2$$

for any ζ-injective oriented sum $\eta\sharp\zeta$.

Proof. Since η, ζ are unlinked by assumption, we can find a two-sphere $S^2 \subset \Sigma$ which separates Σ into two balls B_1, B_2 so that $\eta \subset B_1, \zeta \subset B_2$. This implies that there is an oriented surface S_1 contained in B_1 with oriented boundary η and Euler characteristic $\chi(S_1) = \chi(\eta)$. Since ζ is the Hopf link, there is an annulus $S_2 \subset B_2$ with boundary ζ.

Let $\eta\sharp\zeta$ be a ζ-injective oriented sum. Then $\eta\sharp\zeta$ is obtained by surgery on two proper compact subarcs α_1, α_2 and β_1, β_2 of η, ζ which are contained in embedded disjoint balls $Q_1, Q_2 \subset \Sigma$. Or, put differently, by connecting the arc α_i to β_i by an embedded band $E_i \subset \Sigma$ with oriented boundary (such a band can also be viewed as a rectangle) contained in Q_i and by removing the interior of E_i as well as the interior of the sides α_i, β_i contained in η, ζ. The bands E_i $(i = 1, 2)$ can be chosen in such a way that they intersect the surfaces S_1, S_2 only in $E_i \cap (\eta \cup \zeta)$. The union $S_1 \cup S_2 \cup E_1 \cup E_2$ is then an embedded surface S in Σ whose oriented boundary equals $\eta\sharp\zeta$.

For a suitable choice of a triangulation of S_i with v_i vertices, e_i edges and f_i faces $(i = 1, 2)$, the surface S admits a triangulation with $v_1 + v_2$ vertices, $e_1 + e_2 + 6$ edges and $f_1 + f_2 + 4$ faces. Thus the Euler characteristic $\chi(S)$ of S is

$$\chi(S) = \chi(S_1) + \chi(S_2) - 2.$$

Since $\chi(S_2) = 0$ this shows that $\chi(\eta\sharp\zeta) \geq \chi(S_1) - 2 = \chi(\eta) - 2$ as claimed in the lemma. □

Example: Both the unknot of Euler characteristic 1 and the Figure eight knot of Euler characteristic -1 are oriented sums of an unknot and an unlinked Hopf link. This shows that unlike in the case of connected sums of knots, in general equality does not hold in Lemma 5.3.

6. The Seifert genus of periodic Reeb orbits

Using the assumptions and notations from the previous sections, the goal of this section is to show that the Seifert genus of a periodic Reeb orbit γ on the boundary Σ of a compact strictly convex body $C \subset \mathbb{C}^2$ does not exceed $\frac{1}{2}(\mathrm{lk}(\gamma) + 1)$.

Let $H \subset \mathbb{C}^2$ be a hyperplane which intersects the interior of C. Then H divides C into two disjoint sets whose closures C_1, C_2 are compact convex bodies. We call these compact convex bodies the components of the H*-cut* of C. The boundaries $\partial C_1, \partial C_2$ of C_1, C_2 are smooth away from the hypersphere $\Sigma \cap H$. Then $\partial C_1, \partial C_2$ admit a natural PL-structure which we use without further comments. An embedded two-sphere in such a 3-dimensional PL–sphere M is a two-sphere S^2 which is embedded in M as a PL-submanifold. It divides M into two standard 3-balls and has an open neighborhood homeomorphic to $S^2 \times \mathbb{R}$.

Let η be an oriented link on ∂C_1 and let ζ be an oriented link on ∂C_2. We define an *oriented sum* $\eta\sharp\zeta$ of η, ζ as follows. Choose a point x contained in the interior of $H \cap C$ which is disjoint from both η, ζ. Remove from $H \cap C$ a small open ball B centered at x whose boundary is a smooth embedded two-sphere and whose closure is disjoint from η, ζ and contained in the interior of $H \cap C$. Glue the holed 3-spheres $\partial C_1 - B, \partial C_2 - B$ along their boundaries with the obvious identification map. The resulting PL-sphere $\tilde{\Sigma}$ (which is not embedded in $\mathbb{C}^2$) contains an oriented link whose isotopy class is independent of the choice of x and whose oriented components are the components of the links η, ζ. Note that in $\tilde{\Sigma}$, the link η is separated from ζ by an embedded 2-sphere and hence η and ζ are unlinked. An oriented sum $\eta\sharp\zeta$ is defined to be an oriented sum of the oriented links η, ζ in $\tilde{\Sigma}$.

The next lemma is the basic tool for the decomposition of the Reeb orbit γ as an oriented sum. It is more generally valid for piecewise linear knots and if the boundary Σ of the compact convex body C is only piecewise smooth provided that the hyperplane H in the statement of the lemma intersects Σ in an embedded two-sphere and intersects γ transversely as PL-manifolds.

Hcut **Lemma 6.1.** *Let $\gamma : S^1 = \mathbb{R}/\mathbb{Z} \to \Sigma$ be a smooth oriented knot. Let $H \subset \mathbb{C}^2$ be a hyperplane which intersects Σ transversely in a two-sphere $H \cap \Sigma$. Assume that γ intersects $H \cap \Sigma$ in $2m$ points $\gamma(s_i), \gamma(t_i)$ for some $s_1 < t_1 < \cdots < s_m < t_m \subset S^1$ and that these intersection points are transverse. For each $i \leq m$ let $\ell_i \subset H \cap C$ be a smooth embedded oriented arc connecting $\gamma(s_i)$ to $\gamma(t_i)$ which intersects $H \cap \Sigma$ transversely at $\gamma(s_i), \gamma(t_i)$ and which does not have any other intersection point with $H \cap \Sigma$. Assume also that the arcs ℓ_i are pairwise disjoint. The consecutive concatenation of $\gamma[t_i, s_{i+1}]$ with ℓ_{i+1} $(0 \leq i \leq m-1$ and indices are taken modulo $m)$ defines an oriented knot η on the boundary ∂C_1 of a component C_1 of the H-cut of C, and the concatenations of the arcs $\gamma[s_i, t_i]$ with the inverses of the arcs ℓ_i $(1 \leq i \leq m)$ define an oriented link ζ on the boundary ∂C_2 of the second component C_2 of the H-cut of C. The knot γ is isotopic to a ζ-injective oriented sum $\eta\sharp\zeta$.*

Proof. Using the assumptions and notations in the lemma, assume that the arcs $\ell_i \subset H \cap C$ are parametrized on $[0, 1]$, with $\ell_i(0) = \gamma(s_i)$. The arcs ℓ_i are smooth, pairwise disjoint and embedded and hence they have open tubular neighborhoods in H whose closures are pairwise disjoint and which are diffeomorphic to a three-ball each. Since ℓ_i meets $H \cap \Sigma$ transversely at $\gamma(s_i)$, there is for every $\delta \in (0, 1)$ an open (topological) ball $B_\delta \subset H$ with smooth boundary which is contained in the interior of $H \cap C$, whose closure $\overline{B_\delta}$ contains $\ell_i[0, \delta]$ for each i and whose boundary is tangent to $H \cap \Sigma$ at $\gamma_i(s_i)$. Moreover, we can choose B_δ in such a way that $B_\delta \cap \ell_i = \ell_i(0, \delta)$ and that the arcs $\ell_i[0, \delta]$ are unknotted and unlinked in $\overline{B_\delta}$ (see the discussion in Section 5). Such a ball can be constructed as a thickening of a finite connected graph embedded in $H \cap B$ with vertices $\gamma_i(s_i), \ell_i(\delta)$ and $2m - 1$ edges containing the arcs $\ell_i[0, s_i]$ as well as for each i a smooth embedded arc connecting $\gamma(s_i)$ to $\gamma(s_{i+1})$ whose interior is contained in the interior of $H \cap C$.

Let C_1, C_2 be the components of the H-cut of C. Form the connected sum $\tilde{\Sigma}_\delta$ of ∂C_1 and ∂C_2 by removing the ball B_δ from $\partial C_1, \partial C_2$ and by identifying the boundaries of $\partial C_1 - B_\delta, \partial C_2 - B_\delta$ with the obvious identification map. The gluing map identifies the point $\gamma(s_i)$ in ∂C_1 with the point $\gamma(s_i)$ in ∂C_2, and it identifies the point $\ell_i(\delta)$ in ∂C_1 with the point $\ell_i(\delta)$ in ∂C_2. The three-balls $\partial C_1 - B_\delta, \partial C_2 - B_\delta$ contain $2m$ arcs $\eta - \cup_i \ell_i(0, \delta), \zeta_i - \ell_i(0, \delta) (i = 1, \dots, m)$ whose endpoints lie on the boundary of $\partial C_1 - B_\delta, \partial C_2 - B_\delta$ and are identified pairwise by the glueing map. The resulting knot ξ_δ on $\tilde{\Sigma}_\delta$ is a ζ-injective oriented sum of the oriented knot η on ∂C_1 with the oriented link ζ on ∂C_2.

Now the isotopy class of ξ_δ does not depend on δ, morever ξ_δ is clearly isotopic to γ for δ sufficiently close to 1. This shows the lemma. $\square$

To apply Lemma 6.1 for an estimate of the Seifert genus of a Reeb orbit γ on Σ we have to characterize unknots in a way which is suitable for our purpose. The following lemma provides such a description of the unknot. It is motivated by the work of Milnor [Mil50] on the *crookedness* of knots. For its formulation, note that for smooth function $f : D \to \mathbb{R}$ (i.e. a function which is smooth up to and including the boundary) a boundary point $z \in \partial D$ is critical if the restriction of f to ∂D has a critical point at z and if moreover $df(n(z)) = 0$ where $n(z)$ is the inner normal of D at z. In the sequel, a critical point of a smooth function on D may be a critical boundary point.

Lemma 6.2. *Let γ be a smooth knot on the boundary Σ of a compact strictly convex body $C \subset \mathbb{C}^2$. Assume that γ bounds a smooth embedded disc $S \subset C$ which is transverse to Σ along γ. Assume moreover that there is a (real) linear functional $\varphi : \mathbb{C}^2 \to \mathbb{R}$ whose restriction to γ has a single minimum and a single maximum and no additional critical point, and whose restriction to S does not have critical points. Then γ is unknotted.*

Proof. Let $\varphi : \mathbb{C}^2 \to \mathbb{R}$ be a linear functional whose restriction to γ has a single maximum and a single minimum and no additional critical points, and whose restriction φ_S to a smooth embedded disc $S \subset C$ with boundary $\partial S = \gamma$ does not have critical points. Let $\varphi(S) = [a, b]$ for some $a < b$. By assumption, every $s \in (a, b)$ is a regular value for φ_S, and $\varphi_S^{-1}(s)$ consists of a single smooth arc $\ell_s \subset S$ connecting two points on $\gamma = \partial S$. Choose the orientation of ℓ_s in such a way that the oriented normal of ℓ_s in the disc S points inside $\varphi_S^{-1}(s, \infty)$. For $s = a, b$ the set $\varphi_S^{-1}(s)$ consists of a single point.

For each $s \in (a, b)$ the hyperplane $\varphi^{-1}(s)$ intersects C in a compact ball B_s with smooth boundary ∂B_s. The arc $\varphi_S^{-1}(s)$ is contained in B_s and it intersects ∂B_s only at its endpoints. We claim that for each $s \in (a, b)$ the arc $\varphi_S^{-1}(s) \subset B_s$ is unknotted (see Section 5 for the definition of an unknotted arc in a smooth 3-ball with endpoints on the boundary). To see that this is indeed the case note first that since S is smooth by assumption and since φ_S does not have a critical point at $\varphi_S^{-1}(b)$, the claim holds true for all s sufficiently close to b. Thus it suffices to show that for all $s \in (a, b)$ there is a neighborhood U of s in (a, b) such that for $t, u \in U$ the arcs $\varphi_S^{-1}(t) \subset B_t, \varphi_S^{-1}(u) \subset B_u$ are isotopic as $(1, 1)$-tangles (see [Mu96]). However, this can be seen as follows.

Let $x \in \varphi_S^{-1}(s)$ be an interior point and let $\rho : [-\epsilon, \epsilon] \to C$ be a compact line segment through $\rho(0) = x$ which is orthogonal to the hyperplanes $\varphi^{-1}(u)$. Assume that $\rho[-\epsilon, \epsilon]$ is contained in the interior of C. Let $B(3) \subset \varphi^{-1}(0)$ be the standard unit ball in the hyperplane $\varphi^{-1}(0)$ which is bounded by the standard two-sphere S^2. Since $C \cap \varphi^{-1}(\rho(u))$ is convex for all $u \in [-\epsilon, \epsilon]$ there is a natural radial diffeomorphism $\psi_u : B(3) \to \varphi^{-1}(\rho(u)) \cap C$ depending continuously on u in the C^m-topology for any $m > 0$. This diffeomorphism is determined by the basepoint $\rho(u)$ and fixed standard coordinates in $\varphi^{-1}(0)$. The coordinates translate to coordinates on the affine hyperplane $\varphi^{-1}(u)$ with the zero at $\rho(u)$. For each u the arc $\psi_u^{-1}(\varphi_S^{-1}(\rho(u)))$ is a $(1, 1)$-tangle in $B(3)$. Since the surface S is smooth, this tangle depends smoothly on u and hence all these tangles are isotopic.

For $\delta > 0$ and $s \in [a, b-\delta]$ let $\eta_{s,\delta}$ be the oriented knot on the boundary of the compact convex body $C \cap \varphi^{-1}[s, s+\delta]$ which consists of the arc ℓ_s, the inverse of $\ell_{s+\delta}$ and the two components of $\gamma \cap \varphi^{-1}(s, s+\delta)$. Note that $\eta_{a,b-a}$ is isotopic to γ. The knot $\eta_{s,\delta}$ bounds the disc $\varphi_S^{-1}[s, s+\delta] \subset S$. Since γ is smooth, $S \subset C$ is a smooth embedded disc and the arcs $\varphi_S^{-1}(u) \subset \varphi^{-1}(u) \cap C$ are unknotted there is a number $\epsilon > 0$ such that for every $s \in [a, b-\epsilon]$, the knot $\eta_{s,\epsilon}$ is the unknot. (In fact, it is easy to see that for sufficiently small $\epsilon > 0$ the knot $\eta_{s,\epsilon}$ is the connected sum of a knot K obtained from the arc ℓ_s by connecting its endpoints with an unknotted arc and the knot $-K$ obtained from K by reversing the orientation).

By Lemma 6.1, applied to the knot $\eta_{a,2\epsilon}$ on the boundary of $C \cap \varphi^{-1}[a, a+2\epsilon]$ (which is piecewise smooth) and the hyperplane $\varphi^{-1}(a+\epsilon)$, the knot $\eta_{a,2\epsilon}$ is an oriented sum of the unlinked unknots $\eta_{a,\epsilon}$ and $\eta_{a+\epsilon,\epsilon}$ and hence by Lemma 5.2, $\eta_{a,2\epsilon}$ is an unknot. Inductively we conclude in this way that for each $k \geq 0$ the knot $\eta_{a,k\epsilon}$ is an unknot. This implies that indeed $\gamma = \eta_{a,b-a}$ is unknotted. ☐

With the help of Lemma 6.2 we can relate properties of a knot γ on Σ to geometric properties of a disc in C with boundary γ. This idea is exploited in the next proposition which is the main remaining step toward the proof of the theorem from the introduction. For its formulation, for some $p \geq 2$ define a *$2p$-pronged singularity* of a smooth real-valued function φ on a disc $D \subset \mathbb{C}$ to be a singularity x for φ contained in the interior of D with the following property. There is an open neighborhood U of x in D and there is a diffeomorphism $\psi : U \subset D \to \psi(U) \subset \mathbb{C}$ with $\psi(x) = 0$ and such that $\varphi \circ \psi^{-1}(z) = \mathrm{Re}(z^p)$ for z near 0. We have

nolink **Proposition 6.3.** *Let γ be a smooth knot on the boundary Σ of a compact strictly convex body $C \subset \mathbb{C}^2$. Assume that γ bounds a smooth embedded disc $S \subset C$ which is transverse to Σ along γ. Assume moreover that there is a (real) linear functional $\varphi : \mathbb{C}^2 \to \mathbb{R}$ with the following properties.*

(1) *The restriction φ_S of φ to S has only finitely many critical points, each contained in the interior of S.*
(2) *Each interior critical point of φ_S is a $2p$-pronged singularity for some $p \geq 2$.*
(3) *The only critical points of the restriction of φ to γ are non-degenerate local minima and non-degenerate local maxima.*

Then γ is unknotted.

Proof. Let Σ be the smooth boundary of a compact strictly convex body $C \subset \mathbb{C}^2$. Let γ be a smooth knot on Σ which bounds a smooth embedded disc $S \subset C$ meeting Σ transversely along γ. Let $\varphi : \mathbb{C}^2 \to \mathbb{R}$ be a linear functional with the properties stated in the proposition. Our goal is to show that γ is unknotted.

For this the idea is as follows. The restriction $\varphi_S = \varphi|S$ of φ to S has only finitely many critical points. Each critical point is an interior $2p$-pronged singularity. The critical points of the restriction of φ to γ are non-degenerate local maxima and non-degenerate local minima. Call $c \in \mathbb{R}$ a *regular value* for φ_S if $\varphi^{-1}(c)$ neither contains a critical point of φ_S nor a critical point of the restriction of φ to γ. Then for every regular value c for φ_S, the hyperplane $H = \varphi^{-1}(c)$ decomposes the disc S into finitely many subdiscs. The oriented boundary of each of these subdiscs is composed of a nonempty finite collection of oriented subarcs of γ and a finite collection of arcs which are embedded in S, with endpoints on γ. Discs whose boundaries intersect (i.e. which contain the same component of $\varphi_S^{-1}(c)$ in their boundary) are contained in distinct components of the H-cut of C. Thus the oriented boundaries of the components of $S - \varphi^{-1}(c)$ define two oriented links η, ζ on the two components of the H-cut of C. These links can be analyzed separately using Lemma 5.1, Lemma 6.1 and Lemma 6.2. Up to isotopy, the oriented boundary γ of S can be represented as an oriented sum of η, ζ.

In the sequel we call a point $z \in S$ *degenerate* for φ_S if either z is a critical point for φ_S or if $z \in \partial S$ is a local maximum or a local minimum for the restriction of φ to $\gamma = \partial S$. Let k be the number of degenerate points of φ_S counted with multiplicities. We proceed by induction on k. In the case $k = 2$ there is a single minimum and a single maximum of the restriction of φ to γ and no critical point for φ_S and hence this case is covered by Lemma 6.2. Thus assume that the statement of the proposition holds true whenever there is a linear functional $\varphi : \mathbb{C}^2 \to \mathbb{R}$ with the properties stated in the proposition such that for some $k \geq 3$, $\varphi_S = \varphi | S$ has at most $k - 1$ degenerate points counted with multiplicities.

Let γ be a knot on Σ bounding a smooth embedded disc $S \subset C$. Assume that there is a linear functional $\varphi : \mathbb{C}^2 \to \mathbb{R}$ whose restriction φ_S to S has k degenerate points counted with multiplicities which are of the form described in the proposition. With a small deformation of S we may assume that for each critical value s of φ_S there is a single degenerate point $z \in \varphi_S^{-1}(s)$ and that each critical point of φ_S is a 4-pronged singularity in the interior of S. Let $[a, b] = \varphi(S)$.

For every regular value $s \in [a, b]$ for φ_S let

$$\zeta_s = \partial(\varphi_S^{-1}[s, \infty)), \eta_s = \partial(\varphi_S^{-1}(-\infty, s]).$$

Then ζ_s is an oriented link on the boundary ∂C_s^+ of the compact convex body

$$C_s^+ = C \cap \varphi^{-1}[s, \infty),$$

and η_s is an oriented link on the boundary ∂C_s^- of the compact convex body

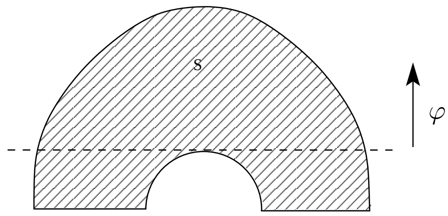
$$C_s^- = C \cap \varphi^{-1}(-\infty, s].$$

The links ζ_s, η_s are composed of an even number of arcs. For each component of ζ_s, η_s these arcs alternate between subarcs of γ and components of $\varphi_S^{-1}(s)$.

Call a local maximum (or a local minimum) $z \in \gamma$ for φ_S (by this we mean that z is a local maximum or minimum for the restriction of φ to γ) *of type I* if there is a neighborhood U of z in S so that $\varphi_S(y) < \varphi_S(z)$ (or $\varphi_S(y) > \varphi_S(z)$) for every $y \in U - \{z\}$. Since φ_S does not have critical points at the boundary, z is a local maximum of type I if and only if $d\varphi(n(z)) < 0$ where $n(z)$ is an inner normal of S at z. A local maximum or local minimum for φ_S which is not of type I is called *of type II.* If z is a local maximum of type II then we have $d\varphi(n(z)) > 0$. Since by assumption a critical point for the restriction of φ to γ is non-degenerate, a global maximum for φ_S is of type I. Figure B below shows a local maximum of type II.

If $z \in \gamma$ is any local maximum for φ_S of type I with $\varphi(z) = s_0$ then the following holds true. Let $s < s_0$ be such that every degenerate point y of φ_S with critical

Figure B



value $\varphi_S(y) \in [s, s_0]$ is contained in γ and is a local maximum of type I. Then the component of $\varphi_S^{-1}[s, \infty) \subset S$ containing z is a subdisc of S not containing any critical point of φ_S in its interior. The restriction of φ to the boundary of this subdisc has a single local maximum.

Let $c < b$ be the largest critical value for φ_S which is *not* the value of a local maximum of type I. Let $m \geq 1$ be the number of degenerate points for φ_S contained in the interval (c, ∞). Let $s > c$ be such that there is no critical value for φ_S in the interval $(c, s]$. By the discussion in the previous paragraph, the link ζ_s has the following properties.

a) ζ_s consists of m components $\zeta_s^1, \dots, \zeta_s^m$.
b) Each of the components ζ_s^i bounds a subdisc S^i of S which does not contain any critical point of φ_S in its interior.
c) For each i the restriction of φ to ζ_s^i has a single local maximum.

Since the restriction of φ to ζ_s^i has a single local maximum, the subset $\zeta_s^i \cap \varphi^{-1}(s)$ of ζ_s^i on which φ assumes its minimum is connected. Then $\zeta_s^i \cap \gamma$ is connected as well. The closed discs $S^i \subset S$ bounded by ζ_s^i are pairwise disjoint. Together this implies that $S - \cup_i S^i$ is connected and hence the boundary η_s of $S - \cup_i S^i$ is connected. The intersection $\eta_s \cap \varphi^{-1}(s)$ consists of m connected components.

The disc S intersects the hyperplane $\varphi^{-1}(s)$ transversely. Thus with a small deformation of the compact convex body C_s^+ near $C_s^+ \cap \varphi^{-1}(s)$ (by pushing the boundary subset $\partial C_s^+ \cap \varphi^{-1}(s)$ slightly outward so that the resulting compact convex body is strictly convex, with smooth boundary) and of the knots ζ_s^i we may assume that for each i the knot ζ_s^i on ∂C_s^+ satisfies the assumptions in Lemma 6.2. Thus by Lemma 6.2, ζ_s^i is an unknot in the PL-sphere ∂C_s^+. By Lemma 5.1, the unknots ζ_s^i $(1 \leq i \leq m)$ are pairwise unlinked and hence ζ_s is a trivial link. By Lemma 6.1, the knot γ is a ζ_s-injective oriented sum of η_s with the unlinked trivial link ζ_s and hence by the first part of Lemma 5.2, γ is isotopic to η_s. Note that $s > c$ can be chosen arbitrarily close to c.

Let $z \in S$ be the critical point for φ_S with critical value c. We distinguish three cases.

Case 1: $z \in \gamma$ is a local maximum for φ_S.

Then z is a local maximum for φ_S of type II.

Let $\delta_0 > 0$ be sufficiently small that c is the only critical value for φ_S in the interval $[c-\delta_0, c+\delta_0]$. For $u \in [c-\delta_0, c]$ let V_u be the component of $\varphi_S^{-1}[u,\infty)$ which contains z. Since z is a local maximum of type II and φ_S does not have a critical point at z, the component V_c of $\varphi_S^{-1}[c,\infty)$ is an embedded subdisc of S whose boundary ∂V_c intersects the hyperplane $\varphi^{-1}(c)$ in an embedded arc containing z in its interior. To see this, extend the disc S smoothly beyond z so that the restriction of φ to this extended disc S' does not have a critical point at z. Then c is a regular value for the restriction of φ to S', and each degenerate point for the restriction of φ to S' with critical value contained in $[c,\infty)$ is a local maximum of type I, see Figure B. For $u \in [c-\delta_0, c)$ the component V_u is a subdisc of S whose boundary is composed of four arcs. Two of these arcs are disjoint subarcs of γ. They are connected by the two components β_u^1, β_u^2 of $\partial V_u \cap \varphi_S^{-1}(u)$.

Let again S' be a smooth enlargement of the disc S near z. Since S' is smooth there is a number $\delta < \delta_0$ such that the $(1,1)$-tangle $V_{c-\delta} \cap \varphi^{-1}(c+\delta)$ in the ball $C \cap \varphi^{-1}(c+\delta)$ is isotopic to the tangle $V_{c-\delta} \cap \varphi^{-1}(c)$ and hence the latter tangle is trivial. Moreover, this tangle is isotopic to a component of the intersection of S' with $\varphi^{-1}(c-\delta)$. Now the disc S' can be chosen so that its boundary is contained in the boundary of a compact convex body $C' \supset C$ which is a smooth deformation of C. As a consequence, for sufficiently small δ the boundary of the component of $S' \cap \varphi^{-1}[c-\delta, c+\delta] \cap C'$ which intersects V_c is an unknot. On the other hand, for sufficiently small δ is knot is isotopic to the boundary ξ of $V_{c-\delta} \cap \varphi^{-1}[c-\delta, c+\delta]$. Now by Lemma 6.1 and the above discussion, $\partial V_{c-\delta}$ is a knot in $\partial C_{c-\delta}^+$ which is an oriented sum of two unlinked unknots, namely the unknot ξ and the boundary of $V_{c+\delta}$, and hence it is an unknot by Lemma 5.2.

Every component of the link $\zeta_{c-\delta}$ which is distinct from $\partial V_{c-\delta}$ bounds an embedded subdisc of S in $C_{c-\delta}^+$ so that the restriction of φ to this subdisc has a single local maximum and no interior critical point. By Lemma 6.2 (see the above discussion), such a component is an unknot. Lemma 5.1 then implies as above that for sufficiently small $\delta > 0$ the link $\zeta_{c-\delta}$ in $C_{c-\delta}^+$ is a trivial link with m components where as before, $m \geq 1$ is the number of critical points of φ_S with critical value strictly bigger than c.

The two components $\beta_{c-\delta}^1, \beta_{c-\delta}^2$ of $\partial V_{c-\delta} \cap \varphi^{-1}(c-\delta)$ decompose the disc S into three closed subdiscs $W_1, V_{c-\delta}, W_2$ (see Figure B). The disc which contains both arcs $\beta_{c-\delta}^1, \beta_{c-\delta}^2$ in its boundary is the disc $V_{c-\delta}$. For sufficiently small δ, the link $\eta_{c-\delta}$ on $\partial C_{c-\delta}^-$ has two connected components $\eta_{c-\delta}^1, \eta_{c-\delta}^2$. The component $\eta_{c-\delta}^1$ is contained in the subdisc W_1 of S, and $\eta_{c-\delta}^2$ is contained in the subdisc W_2 of S.

Each of the knots $\eta_{c-\delta}^i$ $(i = 1,2)$ intersects the hyperplane $\varphi^{-1}(c-\delta)$ in $m_i \leq m$ connected components. One of these components is the arc $\beta_{c-\delta}^i$. Any other component is the intersection with $\varphi_S^{-1}(c-\delta)$ of a component of the link $\zeta_{c-\delta}$ which is distinct from $\partial V_{c-\delta}$. Deform $C_{c-\delta}^-$ to a compact strictly convex body with smooth boundary by pushing $\partial C_{c-\delta}^- \cap \varphi^{-1}(c-\delta)$ slightly outward in such a way that the $m+1$ deformed components of $\varphi_S^{-1}(c-\delta) \subset \partial C_{c-\delta}^- \cap \varphi^{-1}(c-\delta)$ are embedded arcs in S so that φ assumes a single local maximum on each of these deformed arcs. This local maximum is of type I.

After this deformation, the knots $\eta^i_{c-\delta}$ $(i = 1, 2)$ on the boundary $\partial C^-_{c-\delta}$ of the compact convex body $C^-_{c-\delta}$ satisfy the assumptions in the proposition for a subdisc S^i of S and the restriction of φ to this subdisc. Each critical point of $\varphi|S^i$ is a critical point for φ_S. Each critical point for the restriction of φ to the boundary of S^i which is not one of the m_i local maxima of type I produced in the deformation process is a critical point for the restriction of φ to γ. Thus the restriction of φ to S^i has at most $k-1$ degenerate points, and each of these degeneratel points is of the form described in the proposition. Hence by the induction hypothesis, $\eta^i_{c-\delta}$ is an unknot. Since the subdiscs S^1, S^2 of S which are bounded by $\eta^1_{c-\delta}, \eta^2_{c-\delta}$ are disjoint, Lemma 5.1 shows that the unknots $\eta^1_{c-\delta}, \eta^2_{c-\delta}$ on $\partial C^-_{c-\delta}$ are unlinked. Therefore $\eta_{c-\delta}$ is a trivial link with two components.

By Lemma 6.1 (more precisely, by its obvious modification which allows for η to be disconnected), for small $\delta > 0$ the knot γ is an oriented sum of the links $\zeta_{c-\delta}$ and $\eta_{c-\delta}$ with the following glueing graph G. The vertex $\partial V_{c-\delta} \subset \zeta_{c-\delta}$ of G is connected to each of the two vertices $\eta^1_{c-\delta}, \eta^2_{c-\delta}$ by an edge. Any component of $\zeta_{c-\delta}$ different from $\partial V_{c-\delta}$ is connected to either $\eta^1_{c-\delta}$ or $\eta^2_{c-\delta}$ by a single edge, and there are no other edges in G. In particular, the glueing graph G is a tree. Now $\eta_{c-\delta}, \zeta_{c-\delta}$ are trivial links and hence by the second part of Lemma 5.2, γ is an unknot. This completes the induction step in the case that z is a local maximum for φ_S.

Case 2: $z \in \gamma$ is a local minimum for φ_S.

We claim that in this case z is a local minimum of type II.

To see this assume otherwise. Let Q be the component of $\varphi_S^{-1}[c, \infty)$ containing z. Since by assumption the restriction of φ to γ is non-degenerate, the point z is contained in the interior of a subarc of $\gamma \cap \partial Q$. By assumption, all critical points of φ_S contained in $\varphi^{-1}(c, \infty)$ are local maxima of type I and therefore a local maximum for the restriction of φ_S to Q is unique. Moreover, z is the only local minimum for the restriction of φ_S to Q, and there are no interior critical points. Then for each s the intersection of Q with $\varphi^{-1}(s)$ is connected and hence $Q = S$. This implies that φ_S has a single local maximum and a single local minimum and no other critical points. In particular, the number of critical points of φ_S equals 2 which violates the assumption that this number is at least 3. Thus indeed, z is a local minimum for φ_S of type II.

As a consequence, there is a component α of $\varphi_S^{-1}(c)$ containing z in its interior. The arc α intersects the boundary of two components S_1, S_2 of $\varphi_S^{-1}(c, \infty)$. The components S_1, S_2 are subdiscs of S which come together at z (see Figure B).

Let $\delta_0 > 0$ be sufficiently small that z is the only critical point for φ_S in $\varphi_S^{-1}[c - \delta_0, c + \delta_0]$. For $\delta \leq \delta_0$ there is a unique component $V_{c-\delta}$ of $\varphi_S^{-1}[c - \delta, \infty)$ which contains $S_1 \cup S_2$. As in the proof of Lemma 6.2, the knot type of the boundary $\partial V_{c-\delta}$ of $V_{c-\delta}$ does not depend on $\delta \leq \delta_0$. However, it follows as in Case 1 above that this knot is an oriented sum of the boundary of S_1 with the boundary of S_2. Now the boundaries of S_1, S_2 are unlinked unknots and hence $\partial V_{c-\delta}$ is an unknot. As a consequence, the link $\zeta_{c-\delta}$ on $\partial C^+_{c-\delta}$ is trivial, with $m - 1 \geq 1$ components where $m \geq 2$ is the number of critical points of φ_S contained in $\varphi^{-1}(c, \infty)$. The link $\eta_{c-\delta}$ on $\partial C^-_{c-\delta}$ is connected, and it intersects $\varphi_S^{-1}(c-\delta)$ in $m-1$ connected components. Moreover, γ is isotopic to $\eta_{c-\delta}$.

Deform $C^-_{c-\delta}$ slightly to a compact strictly convex body with smooth boundary by pushing $\partial C^-_{c-\delta}\cap\varphi^{-1}(c-\delta)$ slighty outward and deform $\eta_{c-\delta}$ accordingly in such a way that each of the components of $\varphi_S^{-1}(c-\delta)$ is replaced by an arc in the disc S containing a single local maximum for φ. The resulting knot on the boundary of a compact convex body (which is isotopic to $\eta_{c-\delta}$) satisfies the assumptions in the proposition for a subdisc S_0 of S and such that the restriction of φ to S_0 has $k-2$ critical points counted with multiplicities. By the induction hypothesis, we conclude that $\eta_{c-\delta}$ and hence γ is unknotted.

Case 3: The critical point z is contained in the interior of S.

As before, we assume that z is the only critical point with critical value c, and that z is a 4-pronged singularity for φ_S. Then $\varphi_S^{-1}(c)$ consists of 4 smooth arcs connecting z to $\partial S=\gamma$ and perhaps an additional collection of pairwise disjoint smooth arcs with endpoints on the boundary of S. The arcs which end at z divide a small neighborhood of z in S into 4 regions. The values of the function φ are alternating bigger and smaller than c in these regions. Thus for sufficiently small $\delta>0$, the link $\zeta_{c+\delta}$ on $C^+_{c+\delta}=C\cap\varphi^{-1}[c+\delta,\infty)$ has 2 components α_1,α_2 which come together at z as $\delta\to 0$. Moreover, the link $\zeta_{c+\delta}$ is trivial.

There is a unique component ξ of the link $\zeta_{c-\delta}$ on $C\cap\varphi^{-1}[c-\delta,\infty)$ which is an oriented sum of the components α_1,α_2 (in the sense discussed before). Since the components α_i are unlinked unknots, the knot ξ is an unknot. Moreover, ξ is unlinked with the remaining components of $\zeta_{c-\delta}$.

As in Case 1) above, the knot ξ bounds a subdisc V of S. The closure of $S-V$ consists of two disjoint closed subdiscs S^1,S^2 of S. The link $\eta_{c-\delta}$ has two components $\eta^1_{c-\delta}\subset S^1,\eta^2_{c-\delta}\subset S^2$. After a small deformation, the knot $\eta^i_{c-\delta}$ satisfies the hypothesis in the proposition for a subdisc of the disc S^i with the linear functional φ whose restriction to S^i has at most $k-1$ critical points. By the induction hypothesis, the knot $\eta^i_{c-\delta}$ is an unknot $(i=1,2)$. As in Case 1) above, this shows that γ is an oriented sum of mutually unlinked unknots and hence γ is an unknot. This completes the proof of the proposition. □

Proposition nolink 6.3 can be extended as follows.

knotdiagram **Proposition 6.4.** *Let γ be a smooth knot on the boundary Σ of a compact convex body $C\subset\mathbb{C}^2$. Assume that there is a smooth boundary regular immersion $f:(D,\partial D)\to(C,\gamma)$ which is transverse to Σ along γ and whose singular set consists of $k\geq 0$ double points. Assume moreover that there is a (real) linear functional $\varphi:\mathbb{C}^2\to\mathbb{R}$ with the following properties.*

(1) *$\varphi\circ f$ has only finitely many critical points.*
(2) *The critical points of the restriction of φ to γ are non-degenerate local minima and maxima.*
(3) *Any interior critical point for $\varphi\circ f$ is a $2p$-pronged singularity for some $p\geq 2$.*

Then the Seifert genus of γ is at most k.

Proof. Recall from Section 5 the definition of the Euler characteristic of an oriented link ζ in Σ. We show the following slight extension of the statement in the proposition.

Let $\gamma \subset \Sigma$ be any link with components $\gamma_1, \dots, \gamma_\ell$. Assume that for every $i \in \{1, \dots, \ell\}$ there is a boundary regular immersion $f_i : (D, \partial D) \to (C, \gamma_i)$ with only transverse double points. Let $\coprod_{i=1}^{\ell} D_i$ be the disjoint union of ℓ copies of the disc D and let $f : \coprod_i D_i \to \mathbb{C}^2$ be the map whose restriction to the i-th copy D_i of D is the map f_i. Assume that f has only finitely many transverse double points (i.e. that for all i the intersection $f_i(D) \cap (\bigcup_{j \neq i} f_j(D))$ consists of finitely many transverse double points). Let $k \geq 0$ be the total number of all transverse double points of the map f. Assume that there is a linear functional $\varphi : \mathbb{C}^2 \to \mathbb{R}$ such that for each i the function $\varphi \circ f_i$ only has finitely many critical points, and that each of these critical points either is a non-degenerate local maximum or minimum on the boundary of D or a $2p$-pronged singularity in the interior of D for some $p \geq 2$. We claim that there is a Seifert surface S for the link γ of Euler characteristic

$$\chi(S) = \ell - 2k.$$

The case $\ell = 1$ is exactly the statement of the proposition.

Let $m \geq 2\ell$ be the total number of critical points of $\varphi \circ f$ counted with multiplicities. We proceed by induction on $k + 2m - \ell \geq 3$. In the case $k + 2m - \ell = 3$ we necessarily have $\ell = 1$ and $m = 2, k = 0$ and the claim follows from Lemma 6.2. Thus assume that the claim holds true whenever $k + 2m - \ell \leq s_0 - 1$ for some $s_0 \geq 4$.

Let $\gamma = \cup_{i=1}^{\ell} \gamma_i$ be a link with the above properties for $k + 2m - \ell = s_0$. Proposition 6.3 and Lemma 5.1 imply that if the map f does not have double points then γ is a trivial link with ℓ components and hence γ bounds a Seifert surface of Euler characteristic ℓ. Thus we may assume without loss of generality that f has double points.

We proceed as in the proof of Proposition 6.3. Note first that with a small deformation of the maps f_i we may assume that each interior critical point of $\varphi \circ f$ is a 4-pronged singularity and that for each critical value of $\varphi \circ f$ there is a unique critical point for this value. We also assume that for any double point $f(x) = f(y)$ $(x \neq y \in \coprod_i D_i)$ for f the value $\varphi(f(x))$ is not critical.

Assume that $\varphi \circ f(\coprod_i D_i) = [a, b]$. Using the notations from the proof of Proposition 6.3, let $c_0 < b$ be the largest critical value for $\varphi \circ f$ which is not a local maximum of type I. Let moreover $c_1 < b$ be the maximum of the values of φ on the double points of f and let $c = \max\{c_0, c_1\}$. For $s \in (a, b)$ write

$$C_s^+ = C \cap \varphi^{-1}[s, \infty),\ C_s^- = C \cap \varphi^{-1}(-\infty, s].$$

If s is a regular value for $\varphi \circ f$ which is not the value of a double point then the immersed the immersed surface $f(\coprod_i D_i)$ intersects the boundary ∂C_s^+ of C_s^+ in a link ζ_s, and it intersects the boundary ∂C_s^- of C_s^- in a link η_s.

There are two cases.

Case 1: c is a critical value for $\varphi \circ f$.

Then there is a unique point $x \in \coprod_i D_i$ so that $\varphi \circ f(x) = c$ and that x is critical for $\varphi \circ f$. Assume for simplicity of notation that x is a critical point for $\varphi \circ f_1$. By Proposition 6.3 and Lemma 5.1, if $\delta > 0$ is sufficiently small that $\varphi^{-1}[c-\delta, \infty)$ does not contain a double point for f then the link $\zeta_{c-\delta}$ on $\partial C^+_{c-\delta}$ is trivial.

Using the notations from the proof of Proposition 6.3, we distinguish four subcases.

Subcase 1a: $\varphi \circ f$ has a local maximum at z.

Then c is a local maximum of type II and the link $\eta_{c-\delta}$ has $\ell+1$ components. Two of these components, say the components α_1, α_2, bound subdiscs of $f_1(D)$. Up to a small deformation, the link $\eta_{c-\delta}$ satisfies the hypotheses listed above with the same number of double points and the same number of critical points for the restriction of φ to the subdiscs of $f(\coprod_i D_i)$ which are bounded by the components of $\eta_{c-\delta}$. In other words, $\eta_{c-\delta}$ satisfies the assumptions in the proposition for $s_0 - 1$ (compare the discussion in the proof of Proposition 6.3). Now $\eta_{c-\delta}$ has $\ell + 1$ components and hence by the induction hypothesis, there is a Seifert surface for $\eta_{c-\delta}$ of Euler characteristic $\ell + 1 - 2k$.

By the discussion in the proof of Proposition 6.3, up to isotopy the link γ is an oriented sum of $\eta_{c-\delta}$ with a single unlinked unknot ζ. The glueing graph has two edges which connect the unknot ζ with two distinct components α_1, α_2 of $\eta_{c-\delta}$. Thus there is a Seifert surface V for γ which can be obtained from a Seifert surface $V_{c-\delta}$ for $\eta_{c-\delta}$ by attaching two opposite sides of a rectangle (i.e. a topological discs with four distinguished points on the boundary) to two subarcs of α_1, α_2 in such a way that the interior of the rectangle is disjoint from $V_{c-\delta}$. Adding the rectangle to $V_{c-\delta}$ decreases the Euler characteristic by one. This then shows that there is a Seifert surface for γ of Euler characteristic $\ell - 2k$ as claimed.

Subcase 1b: $\varphi \circ f$ has a local minimum at z of type II.

In this case it follows from the discussion in the proof of Proposition 6.3 that for small enough δ the link $\eta_{c-\delta}$ is isotopic to γ. Moreover, $\eta_{c-\delta}$ bounds a union of ℓ subdiscs of $f(\coprod_i D_i)$ so that the restriction of φ to these subdiscs has only $m - 2$ critical points counted with multiplicities. Thus the induction hypothesis can be applied to $\eta_{c-\delta}$ and yields that there is a Seifert surface for γ of Euler characteristic $\ell - 2k$.

Subcase 1c: $\varphi \circ f$ has a local minimum at z of type I.

In this case the proof of Proposition 6.3 shows that the component γ_1 of γ is an unknot which is unlinked with $\cup_{i\geq 2}\gamma_i$. By induction hypothesis, applied to the link $\cup_{i\geq 2}\gamma_i$, there is a Seifert surface V for $\cup_{i\geq 2}\gamma_i$ of Euler characteristic $\ell - 1 - 2k$. Since γ_1 is an unknot which is unlinked with $\cup_{i\geq 2}\gamma_i$, there is a Seifert surface W for γ which is the disjoint union of V with a disc. The Euler characteristic of W then equals $\ell - 2k$.

Subcase 1d: z is an interior critical point of $\varphi \circ f$.

If z is an interior critical point of $\varphi \circ f$ then the reasoning in the discussion of the first subcase together with the proof of Proposition 6.3 shows the claim as in Subcase 1a) above.

Case 2: $\varphi^{-1}(c)$ contains a double point of f.

Let $\delta > 0$ be sufficiently small that there are no double points for f and no critical points for $\varphi \circ f$ in $\varphi^{-1}[c-\delta, c)$. Consider the link $\zeta_{c-\delta}$ on $\partial C^+_{c-\delta}$ with components $\zeta^1_{c-\delta}, \dots, \zeta^p_{c-\delta}$ $(p \geq 1)$. It follows from the discussion in the proof of Proposition 6.3 that for each component $\zeta^j_{c-\delta}$ of $\zeta_{c-\delta}$ there is an embedded disc $E^j_{c-\delta} \subset \coprod_i D_i$ such that $f(\partial E^j_{c-\delta}) = \zeta^j_{c-\delta}$. Moreover, the restriction of $\varphi \circ f$ to each of the subdiscs $E^j_{c-\delta}$ has a single maximum and no interior critical point. There are two of these discs (not nexcessarily distinct) which contains distinct interior points $x \neq y$ with $f(x) = f(y)$.

We distinguish again two subcases.

Subcase 2a: $x \in E^i_{c-\delta}$, $y \in E^j_{c-\delta}$ for $i \neq j$.

Assume without loss of generality that $i = 1, j = 2$. Then each of the disc $f(E^j_{c-\delta}) \subset C^+_{c-\delta}$ $(j = 1, \dots, \ell)$ is embedded. The components $\zeta^1_{c-\delta}, \dots, \zeta^p_{c-\delta}$ are all unknots. The link $\cup_{j=3}^p \zeta^j_{c-\delta}$ is trivial, and it is unlinked with both $\zeta^1_{c-\delta}, \zeta^2_{c-\delta}$. The link γ is a $\zeta_{c-\delta}$-injective oriented sum of the links $\eta_{c-\delta}$ and $\zeta_{c-\delta}$.

We claim that up to orientation, $\zeta^1_{c-\delta} \cup \zeta^2_{c-\delta}$ is the Hopf link on $\partial C^+_{c-\delta}$. Namely, the discs $f(E^1_{c-\delta}), f(E^2_{c-\delta})$ intersect transversely in the single point z and hence the linking number between $\zeta^1_{c-\delta}, \zeta^2_{c-\delta}$ equals 1 up to a change of orientation. Since $\pi_1(\partial C^+_{c-\delta} - \zeta^1_{c-\delta})$ is infinite cyclic, this means that $\zeta^2_{c-\delta}$ is freely homotopic in $\partial C^+_{c-\delta} - \zeta^1_{c-\delta}$ to a meridian of a solid torus neighborhood of $\zeta^1_{c-\delta}$. Since $\zeta^2_{c-\delta}$ is an unknot, it is indeed isotopic to such a meridian. This shows that $\zeta^1_{c-\delta} \cup \zeta^2_{c-\delta}$ is the Hopf link in $\partial C^+_{c-\delta}$ as claimed.

By Lemma 6.1, the link γ is an oriented sum of the unlinked oriented links $\eta_{c-\delta}, \zeta_{c-\delta}$. The discussion in the previous paragraph shows that $\zeta_{c-\delta}$ is a union of a trivial link $\cup_{j=3}^p \zeta^j_{c-\delta}$ with $p-2$ components and an unlinked Hopf link $\zeta''_{c-\delta}$.

Now up to a small deformation, the link $\eta_{c-\delta}$ has ℓ components (i.e. as many components as γ). It bounds a union of ℓ subdiscs of $f(\coprod_i D_i)$. The number of double points of these subdiscs equals $k-1$. The restriction of φ to the union of these subdiscs has as many critical points as there are for $\varphi \circ f$. In other words, the link $\eta_{c-\delta}$ satisfies the assumption in the proposition for $s_0 - 1$. Thus by the induction hypothesis, there is a Seifert surface for $\eta_{c-\delta}$ of Euler characteristic $\ell - 2k + 2$. Lemma 5.3 and Lemma 5.2 then immediately imply that there is a Seifert surface for γ of Euler characteristic $\ell - 2k$ as claimed.

Subcase 2b: x and y are contained in the same disc.

Assume that $x, y \in E^1_{c-\delta}$. The components $\zeta^j_{c-\delta}$ of $\zeta_{c-\delta}$ are mutually unlinked. Moreover, for $j \geq 2$ the component $\zeta^j_{c-\delta}$ is an unknot. The above discussion applied to $\zeta^1_{c-\delta}$ shows that $\zeta^1_{c-\delta}$ is an oriented sum of a Hopf link with an unknot. By Lemma 5.3, there is a Seifert surface for $\zeta^1_{c-\delta}$ of Euler characteristic -1.

As in Subcase 2a, the induction hypothesis can be applied to the link $\eta_{c-\delta}$ and shows that there is a Seifert surface for $\eta_{c-\delta}$ of Euler characteristic $\ell - 2k + 2$. Since γ is up to isotopy an oriented sum of the unlinked links $\eta_{c-\delta}$ and $\zeta^1_{c-\delta}$ (compare the discussion in the proof of Proposition 6.3), this implies that there is a Seifert surface for γ of Euler characteristic $\ell - 2k$ as claimed. This completes the proof of the proposition. ☐

Now we are ready to show

unknot **Corollary 6.5.** *The Seifert genus of a periodic Reeb orbit on the boundary Σ of a compact strictly convex body $C \subset \mathbb{C}^2$ equals $\frac{1}{2}(\mathrm{lk}(\gamma)+1)$. In particular, if $\mathrm{lk}(\gamma) = -1$ then γ is unknotted.*

Proof. Let γ be a periodic Reeb orbit on the boundary Σ of a compact strictly convex body C. By Corollary 4.10, there is a minimal immersed disc $f : (D, \partial D) \to (C, \gamma)$ with boundary γ and $\frac{1}{2}(\mathrm{lk}(\gamma) + 1)$ transverse positive self-intersection points counted with multiplicity.

Choose a linear functional $\varphi : \mathbb{C}^2 \to \mathbb{R}$ such that the restriction of φ to γ only has non-degenerate critical points. This means in particular that each such critical point either is a local minimum or a local maximum for $\varphi|\gamma$. Since f is the real part of a holomorphic map $D \to \mathbb{C}^2 \otimes \mathbb{C}$ and since φ is linear, the pull-back $\varphi_S = \varphi \circ f$ is the real part of a holomorphic function on D. In particular, φ_S is harmonic and hence it neither has local minima nor local maxima in the interior of D. Moreover, the number of its singular points is finite, and each interior singular point is a standard $2p$-pronged singularity for some $p \geq 1$.

An application of Proposition 6.4 now shows that the Seifert genus of γ does not exceed $\frac{1}{2}(\mathrm{lk}(\gamma) + 1)$. On the other hand, Eliashberg [Eli92] showed that the Seifert genus of γ is at least $\frac{1}{2}(\mathrm{lk}(\gamma) + 1)$. ☐

For a periodic Reeb orbit γ on Σ the *Maslov index* $\mu(\gamma)$ of γ is defined. Since Σ is strictly convex by assumption, this Maslov index is not smaller than 3 [HWZ98]. As an immediate consequence of Proposition 6.5 and Theorem 2 of [HH09] we obtain.

maslov **Corollary 6.6.** *Let $\Sigma \subset \mathbb{C}^2$ be the boundary of a compact convex body. Assume that the principal curvatures $a \geq b \geq c$ of Σ satisfy the inequality $a \leq b + c$ pointwise. Then a periodic Reeb orbit γ on Σ of Maslov index 3 is unknotted.*

Proof. By Theorem 2 of [HH09], the self-linking number of a periodic Reeb orbit γ on Σ of Maslov-index 3 equals -1. ☐

We complete this section with a conjecture.

Conjecture: Let γ be a periodic Reeb orbit on the boundary $\Sigma \subset \mathbb{C}^2$ a compact strictly convex body. Then the Maslov index of γ is not smaller than $\mathrm{lk}(\gamma) + 4$.

It makes also sense to extend the above conjecture to dynamically convex energy surfaces in the sense of [HWZ98].

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